Universitat de Barcelona
Facultat de Matemàtiques i Informàtica

GRAU DE MATEMÀTIQUES

Treball final de grau

ITERATION OF TRANSCENDENTAL FUNCTIONS

Autor: Àlex Rodríguez Reverter

Directora: Dra. Núria Fagella Rabionet
Realitzat a: Departament de Matemàtiques i Informàtica

Barcelona, June 20, 2021

Contents

Abstract

In this project we analyze the behavior of transcendental functions under iteration i.e., those with an essential singularity at ∞. We emphasize the general case of meromorphic transcendental functions with the aim of understanding the dynamical consequences of the presence of poles.

Finally, we apply these results and techniques to study, on the one hand, the dynamics of the exponential family $E_\lambda(z) = \lambda e^z$, and on the other hand, the family of meromorphic maps

$$f_\lambda(z) = \lambda \left(\frac{e^z}{z+1} - 1 \right).$$

In this last part, which is original work, we prove that under certain conditions, the basin of attraction of $z = 0$ is infinitely connected.

Abstract en Català

En aquest projecte estudiem el comportament de funcions transcendents sota iteració, és a dir, aquelles que tenen una singularitat essencial a ∞. Fem èmfasi en el cas general de les funcions transcendents meromorfes amb la intenció d'entendre les conseqüències dinàmiques de la presència de pols.

Finalment, apliquem els resultats i tècniques desenvolupades per estudiar, per una part, el comportament dinàmic de la família exponencial $E_\lambda(z) = \lambda e^z$, i per l'altra, la família de funcions meromorfes

$$f_\lambda(z) = \lambda \left(\frac{e^z}{z+1} - 1 \right).$$

En aquesta darrera part, la qual és treball original, demostrem que sota certes condicions, la conca d'atracció de $z = 0$ és infinitament connexa.

Acknowlegments

First, I must start with my family and all my friends from my native land, for being always an unconditional support.

Thanks to my degree colleagues and friends, Guillem Tena, Marc Maset, Oriol Reig, Bernat López and Eduard Antentas. What a pleasure getting to know each one of you!

Thanks to all the amazing professors and researchers that we have in the Faculty of Mathematics and Computer Science of UB. I would like to thank specially the ones that "suffered" me the most: Àlex Haro, Carmen Cascante, Joan Carles Naranjo, Arturo Vieiro, Xavier Massaneda, Artur Travesa, Carlos d'Andrea, Susana Romano and Joan Carles Tatjer.

Thanks to all the amazing professionals that I got to know this semester, all the team of the Holomorphic Dynamics seminar and Albert García.

And finally, I would like to express my most sincere gratitude to my mentor, Núria Fagella, this work started two years ago when you gave us a brief introduction to Holomorphic Dynamics in Mathematical Models and Dynamical Systems. It has been a real pleasure having you as a professor and specially as the director of my bachelor's thesis. For all the patience and all the hours, gràcies per tot Núria.

2020 Mathematics Subject Classification. 37F10, 30D45, 30D30

Introduction

Iteration theory appears in daily problems, often from a mathematical model regarded as a dynamical system. In many cases, methods from numerical analysis require iteration, for example the Newton method has the aim of approximating the solutions, real or complex, of $f(z) = 0$ by considering the iterative function

$$N_f(z) = z - \frac{f(z)}{f'(z)}.$$

Given an initial condition $z \in \mathbb{C}$ we consider the sequence

$$z \longmapsto N_f(z) \longmapsto N_f(N_f(z)) \longmapsto \cdots$$

and we wish to determine under which conditions this sequence converges, and if it does so, whether it converges to a zero of f or not.

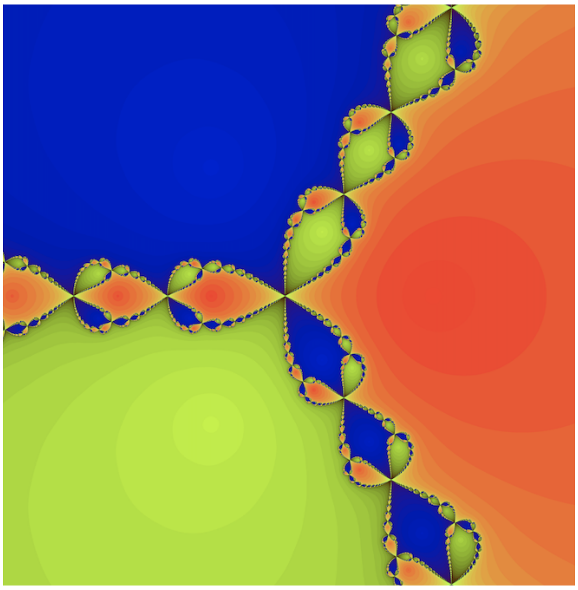

Figure 1: Newton method applied to the cubic polynomial $P(z) = z^3 - 1$. Points of the same color converge to the same root of P under iteration.

The mathematical area that aims to study the iteration of general holomorphic functions of one-complex variable is known as Holomorphic Dynamics. The first important results came in the nineteenth century from the work of Schröder, who was mainly motivated by Newton's method, and the work of Koenigs, who studied functional equations.

The key ingredient for the study of iteration came in the beginning of the twentieth century with the notion of normal family introduced by Montel A.2. Julia [Jul] and Fatou [Fat], who set the basis of what is known today as Holomorphic Dynamics, based his approach in Montel's theory for the case of rational maps. Fatou, in fact, in 1926 extended some results to the case of transcendental entire functions (entire functions with infinitely many terms in their series expansions), but he did not consider the more general case of transcendental meromorphic functions.

In this work we focus on transcendental meromorphic functions, i.e., functions with an essential singularity at ∞, which are allowed to have poles. The interest for these functions is double; The essential singularity, on the one hand, adds a lot of chaos to the dynamical system, mainly because of Picard's Theorem (Theorem A.62), which states that in each punctured neighborhood of ∞, these functions assume each value of the Riemann Sphere $\mathbb{C}_\infty$, with at most two exceptions, infinitely often. Hence, given a point $z \in \mathbb{C}$, if its orbit

$$\mathcal{O}_f^+(z) = \{f^n(z) \colon n \in \mathbb{N}\}$$

is near ∞ at some moment, after one iteration it can land at almost any place of the plane. On the other hand, the presence of poles allows for more generality, since ∞ is not required to be an omitted value.

More precisely, we shall consider f a meromorphic function in $\Omega \subset \mathbb{C}$ an open set, and we write $f \in \mathcal{M}(\Omega)$, if f is holomorphic in Ω except for a countable set of isolated singularities which are poles (more details provided in A.1). Given $f \in \mathcal{M}(\mathbb{C})$, we denote

$$f^n(z) = (f \circ \overset{(n)}{\cdots} \circ f)(z)$$

which is well-defined for all $z \in \mathbb{C}$ except for the countable set of the poles of $f, \dots, f^{n-1}$.

The study of iteration of holomorphic maps, as it has been already pointed out, requires the notions of normality introduced by Montel. Although Montel's theory is covered in the (optional) course of Functions of One Complex Variable, any interested reader is encouraged to see A.2 for definitions and many fundamental results, like Montel's Theorem, Marty's Theorem and Picard's Theorem.

With the tool of Montel's theory at hand, Chapter 1 is devoted to the study of the local dynamical behavior of a function. A quite remarkable result is displayed, for example, Lemma 1.16 where a characterization of the convergence near what is known as a parabolic fixed point is given. Problems that require Value Distribution Theory are also faced, for example the existence of periodic points for an entire transcendental function.

In Chapter 2 the global study of iteration of holomorphic functions is addressed. The dynamical plane splits into two completely invariant sets, the Fatou set, where the sequence of iterates $\{f^n\}$ have Montel's normality, and its complement, the Julia set. Some celebrated results are proved. For example, Theorem 2.28 is a result by Fatou which concerns the limit functions that we can obtain under iteration in a periodic component of the Fatou set. Other important properties concerning the limit functions can be found also in this Chapter.

The results developed on the previous chapters allows us, in Chapter 3, to study two families of transcendental functions. We start with the well-known Exponential

Family $E_\lambda(z) = \lambda e^z$, for which we describe the remarkable topological properties of its Julia set (a Cantor set of curves). Afterwards, we study a new function in 3.2, namely

$$f_\lambda(z) = \lambda\left(\frac{e^z}{z+1} - 1\right).$$

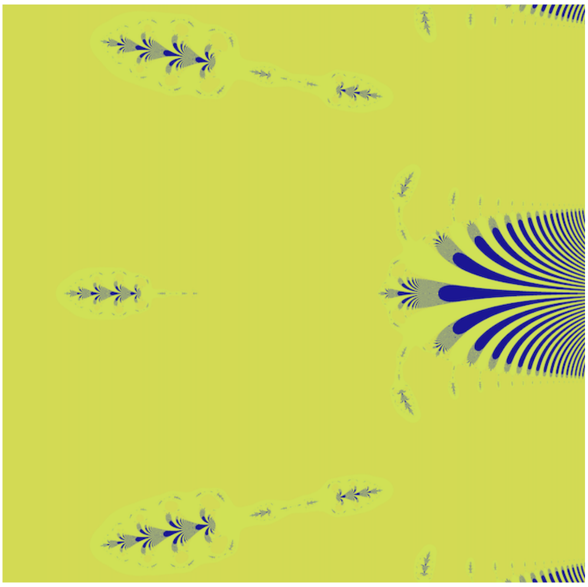

Figure 2: In green, the Julia Set of f_λ for $\lambda = 0.89$.

This, previously unexplored, family of maps is interesting since it is the simplest meromorphic map with two singularities of f^{-1}: $z = 0$, which is a fixed critical point, and $-\lambda$, which is a free asymptotic value. It has also one single pole, $z = -1$, which is not omitted except for $\lambda = 1$. This family is the meromorphic analogue to the well-known Milnor family of cubic polynomials $P_a(z) = z^2(z-a)$ [Mil1] or its entire version $\lambda z^2 e^z$ [FG2].

Opposed to these two cases, the basins of attraction of f_λ are not simply connected and the relation between this topological property and the dynamics of f_λ promises to be a source of interesting problems. In this thesis we prove the following result.

Theorem. *Consider f_λ and let $\mathcal{A}_\lambda(0)$ denote the basin of attraction of 0. Then,*

(a) If $-\lambda$ belongs to the connected component of $\mathcal{A}_\lambda(0)$ which contains $z = 0$, then $\mathcal{A}_\lambda(0)$ is connected.

(b) If $-\lambda$ belongs to the connected component of $\mathcal{A}_\lambda(0)$ which contains $z = 0$, then $\mathcal{A}_\lambda(0)$ is infinitely connected.

Moreover, $D(0, 1/2) \setminus \{0\} \subset \{\lambda \in \mathbb{C}^* : -\lambda \in \mathcal{A}_\lambda(0)\}$.

The proof of this result can be found in Chapter 3 (see Theorem 3.25, Theorem 3.27 and Corollary 3.23).

Chapter 1

Periodic Points and Local Theory

The aim of this chapter is to study the local dynamical behavior of a function to obtain properties that lead to the global theory. The fundamental objects of this section are the periodic points. The main references for this section are [BB, Ber1, CG, HY, Mil2].

From now on, unless we do not state the opposite, f denotes a non-constant meromorphic function, $f \in \mathcal{M}(\mathbb{C})$, as defined in the introduction.

Definition 1.1 (Forward and Backward Orbit, Periodic and Preperiodic Point)**.** *Let $z_0 \in \mathbb{C}$, then*

- *The **forward orbit** of z_0 is the set $\mathcal{O}^+(z_0) = \{z_n = f^n(z_0) \colon n \in \mathbb{N}\}$.*
- *The **backward orbit** of z_0 is the set $\mathcal{O}^-(z_0) = \{z \colon f^n(z) = z_0\ ,\ n \in \mathbb{N}\}$.*
- *z_0 is called **periodic** if exists $n \in \mathbb{N}$ such that $z_n = z_0$, $p = \min\{n \in \mathbb{N} \colon z_n = z_0\}$ is called its **period**. If $p = 1$ we say that z_0 is a **fixed point**.*
- *z_0 is called **preperiodic** if $f^k(z_0)$ is periodic for some $k \in \mathbb{N}$ and **strictly preperiodic** if it is preperiodic but not periodic.*

1.1 Periodic Points

For a periodic point z_0 of period p, we define its **multiplier** as $\lambda = (f^p)'(z_0)$. Using the chain rule, it can be verified that

$$\lambda = \prod_{n=0}^{p-1} f'(f^n(z_0)) = \prod_{n=0}^{p-1} f'(z_n)$$

and therefore, the multiplier is the same for every periodic point of the orbit. Hence, we regard it as the multiplier of the orbit.

Periodic points can be classified according to their multiplier.

Definition 1.2 (Classification of Fixed Points)**.** *Given a periodic point z_0 of period p, the cycle $\mathcal{O}_f^+(z_0)$ is called:*

- ***Attracting** iff $0 < |\lambda| < 1$ and **super-attracting** iff $\lambda = 0$.*

- ***Repelling*** *iff* $|\lambda| > 1$.
- ***Indifferent*** *iff* $|\lambda| = 1$. *We have two possibilities:*
  - ***Rationally indifferent*** *iff* $\lambda^m = 1$ *for some* $m \in \mathbb{N}$*, i.e.,* $\lambda = e^{2\pi i j/m}$ *for some* $j \in \mathbb{Z}$ *(also called a* ***parabolic*** *cycle).*
  - ***Irrationally indifferent*** *iff* $\lambda = e^{2\pi i\theta}$*, for* $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

1.1.1 Existence of Periodic Points

Before we address the dynamical relevance of the different types of periodic points we comment on their existence.

If f is a rational function, $f(z) = p(z)/q(z)$, the fixed points of f^p are the solutions of

$$p^n(z) = zq^n(z)$$

Since both p and q are polynomials, from the Fundamental Theorem of Algebra we obtain a finite number of solutions.

Definition 1.3 (Transcendental Function)**.** *We say that a meromorphic function* $f\colon \mathbb{C} \to \mathbb{C}_\infty$ *is* ***transcendental*** *if it has an essential singularity at* $z = \infty$*, that is,* $g(w) = f(1/w)$ *has an essential singularity at* $w = 0$.

Note that not every transcendental map has fixed points, as we can see from the example $f(z) = e^z + z$. Nevertheless, the case is different when the period is larger or equal than 2.

Theorem 1.4. *Let* f *be a transcendental entire function, then for each* $n \in \mathbb{N}$*,* $n \geq 2$*,* f *has infinitely many periodic points of period* n.

The proof can be found in [BB]. Although weaker, we can prove the following result.

Proposition 1.5. *Let* f *be a transcendental entire function. Then the equation* $f^2(z) = z$ *has a solution.*

Proof. Consider

$$g(z) = \frac{f^2(z) - z}{f(z) - z}.$$

Suppose that $f^2(z) = z$ does not have any solution. Then f does not have any fixed point and $g(z)$ is an entire function that omits 0 and 1, due to Picard's Little Theorem, $g \equiv c$ for $c \in \mathbb{C}$, i.e.,

$$f(f(z)) - z = c(f(z) - z).$$

- If $c = 0$, then $f^2(z) = z$ which contradicts our assumption.
- If $c = 1$, then $f(f(z)) = f(z)$, which also contradicts our assumption.

- If $c \notin \{0, 1\}$, then

$$f'(f(z))f'(z) - 1 = cf'(z) - c \Longrightarrow f'(z)(f'(f(z)) - c) = 1 - c.$$

Since $c \neq 1$, f' omits the value 0 and $f' \circ f$ omits $c \neq 0$. So, $f' \circ f$ omits $\{0, c\}$, thus by Theorem A.63 (Picard's Little Theorem), $f' \circ f$ is constant. Since f is transcendental, it cannot be constant, which implies that f' is constant and this also contradicts that f is transcendental.

So the equation $f^2(z) = z$ has at least a solution. □

Bergweiler in [BB] proved the following related result.

Theorem 1.6. *Let f be a transcendental entire function, then for any $n \in \mathbb{N}$, $n \geq 2$, f^n has infinitely many repelling fixed points.*

In Chapter 2 there is a similar result when f is a transcendental meromorphic function with two or more poles, or one pole which is not an omitted value.

1.2 Normal Forms

We are concerned now with the study of the dynamical behavior of a function near a periodic point. Observe first that since periodic points of f are fixed points of f^n for a given n, without loss of generality we may assume that they are fixed points.

To accomplish this goal, we want to represent our function in the simplest possible way, the normal form. To do so, we introduce the concept of conjugacy.

Definition 1.7 (Conformal Conjugacy)**.** *We say that a function $f\colon U \to U$ is **(conformally) conjugate** to a function $g\colon V \to V$ if and only if there is a conformal one-to-one map $\varphi\colon U \to V$ such that*

$$\varphi(f(z)) = g(\varphi(z))$$

i.e., the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{f} & U \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ V & \xrightarrow{g} & V \end{array}$$

Two conjugate functions have the same dynamics. Indeed, the iterates of f are also conjugate by the same map φ since $g^n = \varphi \circ f^n \circ \varphi^{-1}$. The inverses, f^{-1} and g^{-1}, whenever well-defined are also related by φ, i.e., $g^{-1} = \varphi \circ f^{-1} \circ \varphi^{-1}$.

It can also be verified that conjugacies send orbits to orbits, fixed points to fixed points, periodic orbits of period p to periodic orbits of period p, attracting points to attracting points, etc.

The goal now is, depending on the multiplier of a fixed point, to obtain the normal form of a function near a fixed point. To do so, assume that $z = 0$ is a fixed point of $f(z)$ with multiplier λ.

First, we address the cases in which we can conjugate our function to a linear one. This is known as the **linearization problem**, and consists in finding under which conditions there exists a conformal map φ such that f is φ-conjugate to the linear map $w \mapsto \lambda w$, i.e.,

$$\varphi(f(z)) = \lambda\varphi(z) \Longrightarrow f(\varphi^{-1}(w)) = \varphi^{-1}(\lambda w)$$

also known as the **Schröder equation**.

1.2.1 Attracting and Repelling Fixed Points

It turns out that the linearization problem has different solutions depending on the type of fixed point we consider. In the attracting and repelling case Theorem 1.9 (Koenigs Theorem) gives us an affirmative answer.

First, we justify the definition of "attracting" fixed point.

Proposition 1.8. *A fixed point p of a meromorphic function f is attracting, if and only if, there exists $r > 0$ such that $f(D(p,r)) \subset D(p,r)$ and for all $z_0 \in D(p,r)$,*

$$f^n(z_0) \xrightarrow[n\to\infty]{} p.$$

Proof. Suppose that p is an attracting fixed point with multiplier λ, then there exists $c > 0$ such that $|\lambda| < c < 1$, and $r > 0$ such that $|f'(z_0)| < c$ for all $z_0 \in D(p,r)$ and $f \in \mathcal{H}(D(p,r))$. Then

$$|f(z_0) - p| = |f(z_0) - f(p)| = \left| \int_{[p,z_0]} f'(\xi)d\xi \right| \leq c|z_0 - p|.$$

Therefore,

$$|f^n(z_0) - p| \leq c^n|z_0 - p| < c^n r \xrightarrow[n\to\infty]{} 0.$$

Conversely, there exists $0 < \varepsilon < 1$ and $n \in \mathbb{N}$ such that $f^n(D(p,\varepsilon)) \subset D(p,\varepsilon)$. Hence by Schwarz's Lemma, $|(f^n)'(p)| = |\lambda^n| < 1$, and therefore $|\lambda| < 1$. □

We now prove Koenigs Linearization Theorem.

Theorem 1.9 (Koenigs Linearization Theorem)**.** *Let f be holomorphic in some neighborhood of $z = 0$, a fixed point of f with multiplier λ. If $|\lambda| \neq 0, 1$, then there exists a local conformal change of coordinate $w = \varphi(z)$ with $\varphi(0) = 0$ such that*

$$\varphi \circ f \circ \varphi^{-1} \colon w \mapsto \lambda w$$

for all w in some neighborhood of the origin. Furthermore, φ is unique up to multiplication by a nonzero constant.

Proof. We prove first the existence of such conjugation for the attracting and repelling case, finally we deal with uniqueness.

- Let $z = 0$ be an attracting fixed point (not super-attracting by hypothesis), then exists $c < 1$ such that $c^2 < |\lambda| < c$ and exists r small enough such that
$$|f(z)| \leq c|z| \quad , \quad z \in D(0,r)$$
For any $z_0 \in D(0,r)$,
$$|z_n| = |f^n(z_0)| \leq c^n|z_0| \leq c^n r \xrightarrow[n\to\infty]{} 0$$
so the orbit $\mathcal{O}_f^+(z_0)$ converges to the origin.
By Taylor's theorem, $f(z) = \lambda z + O(z^2)$ as $z \to 0$, i.e., (considering r smaller if necessary)
$$|f(z) - \lambda z| \leq C|z|^2 \quad , \quad z \in D(0,r)$$
for some constant $C > 0$, and then
$$|z_{n+1} - \lambda z_n| = |f(z_n) - \lambda z_n| \leq C|z_n|^2 \leq Cr^2c^{2n}.$$
If we set $k = Cr^2/|\lambda|$, then $w_n = \varphi_n(z) = f^n(z)/\lambda^n$ satisfies
$$|\varphi_{n+1}(z) - \varphi_n(z)| = |\lambda|^{-(n+1)}|z_{n+1} - \lambda z_n| \leq Cr^2c^{2n}|\lambda|^{-(n+1)} = k(c^2/|\lambda|)^n.$$
Since $c^2 < |\lambda|$,
$$\sum_{n\geq 1} k(c^2/|\lambda|)^n < \infty$$
so $\varphi_n(z)$ converges to $\varphi(z)$ uniformly for $z \in D(0,r)$ and $\varphi_n'(0) = \lambda^{-n}(f^n)'(0) = 1$, then $\varphi'(0) = 1$ and thus its a local change of coordinate.
By the definition of φ_n,
$$\varphi_n(f(z)) = f^{n+1}(z)/\lambda^n = \lambda\varphi_{n+1}(z).$$
Hence, $\varphi(f(z)) = \lambda\varphi(z)$.
- If $|\lambda| > 1$, then f^{-1} is locally well-defined in a neighborhood of $z = 0$ by the Inverse Function Theorem, $z = 0$ is also a fixed point of f^{-1} and its multiplier is $1/\lambda$, so we can apply the argument above.
- Finally we prove uniqueness (up to multiplication by a nonzero constant).
If there are two maps φ, ϕ such that
$$(\varphi \circ f \circ \varphi^{-1})(w) = \lambda w = (\phi \circ f \circ \phi^{-1})(w)$$
then,
$$\begin{aligned}(\varphi \circ \phi^{-1})(\lambda w) &= (\varphi \circ \phi^{-1})((\phi \circ f \circ \phi^{-1})(w)) = \\ &= (\varphi \circ f \circ \varphi^{-1})((\varphi \circ \phi^{-1})(w)) = \lambda(\varphi \circ \phi^{-1})(w).\end{aligned}$$
Since $(\varphi\circ\phi^{-1})(0) = 0$, then $(\varphi\circ\phi^{-1})(w) = b_1 w + b_2 w^2 + \cdots$ So $\lambda b_n = \lambda^n b_n$ for all $n \geq 1$. Since $|\lambda| \neq 0, 1$, in particular λ is not a root of the unity, therefore $b_n = 0$ for all $n \geq 2$, i.e.,
$$(\varphi \circ \phi^1)(w) = b_1 w \Longrightarrow \varphi(\phi^{-1}(w)) = b_1\phi(z) \Longrightarrow \varphi = b_1\phi$$
and we obtain the uniqueness.

□

1.2.2 Super-Attracting Fixed Points

The proof of Theorem 1.9 (Koenigs) shows that the arguments used cannot be extended to the case of a super-attracting fixed point (where $\lambda = 0$).

In this case the answer to the linearization problem is negative. However, we can prove that the map is locally conjugated to $w \mapsto w^p$, where p is the local degree.

Theorem 1.10 (Böttcher)**.** *Suppose f has a super-attracting fixed point at $z = 0$, so that in a neighborhood of $z = 0$, the function can be written as*

$$f(z) = a_p z^p + a_{p+1} z^{p+1} + \cdots \qquad \text{for some } p \geq 2.$$

Then there is a conformal map $w = \varphi(z)$ defined in a neighborhood of $z = 0$ onto a neighborhood of $w = 0$ that conjugates $f(z)$ to $w \mapsto w^p$. Furthermore, φ is unique up to multiplication by a $(p-1)$-th root of unity.

Proof.

- Existence:

  For a neighborhood $D(0,r)$ of $z = 0$ with r small enough, there exists $C > 1$ such that
  $$|f(z)| \leq C|z|^p \quad , \quad \forall z \in D(0,r).$$
  Using induction we obtain that:
  $$|f^n(z)| \leq C^{\sum_{j=0}^{n-1} p^j} |z|^{p^n} \leq (C|z|)^{p^n}$$
  and we can choose r even smaller if necessary, so that $Cr < 1$ and then $f^n(z) \to 0$ for all $z \in D(0,r)$.

  If we change the variables by setting $w = \phi(z) = bz$, where $b^{p-1} = 1/a_p$, then:
  $$\phi^{-1}(f(\phi(z))) = \frac{1}{b}\left(a_p (bz)^p + \cdots\right) = a_p b^{p-1} z^p + \cdots = z^p + \cdots$$
  so we can assume without loss of generality that $a_p = 1$.

  Consider, for $n \in \mathbb{N}$
  $$\varphi_n(z) = (f^n(z))^{p^{-n}} = (z^{p^n} + \cdots)^{p^{-n}} = z(1 + \cdots)^{p^{-n}}$$
  which are well defined in a neighborhood of $z = 0$ and
  $$\varphi_n(f(z)) = (f^{n+1}(z))^{p^{-n}} = \varphi_{n+1}(z)^p.$$
  So, if we prove that $\varphi_n \to \varphi$, then $\varphi \circ f = \varphi^p$ and $\varphi'(0) \neq 0$ (because $\varphi_n'(0) = 1$), obtaining a solution φ.

  Recall that,
  $$\frac{\varphi_{n+1}}{\varphi_n} = \left(\frac{\varphi_1 \circ f^n}{f^n}\right)^{p^{-n}}$$

and

$$\frac{\varphi_1 \circ f^n}{f^n} = \frac{(f^n(z)^p + a_{p+1} f^n(z)^{p+1} + \cdots)^{1/p}}{f^n(z)} =$$
$$= (1 + a_{p+1} f^n(z) + \cdots)^{1/p} = 1 + O(|f^n|) =$$
$$= 1 + O(|z|^{p^n}) \quad \text{for} \quad |z| \le r.$$

Therefore,

$$\frac{\varphi_{n+1}}{\varphi_n} = (1 + O(|z|^{p^n}))^{p^{-n}} = 1 + O(p^{-n})$$

for $|z| \le r$. Thus, the product

$$\prod_{n=1}^{\infty} \frac{\varphi_{n+1}}{\varphi_n}$$

converges uniformly on $|z| \le r$ and then $\{\varphi_n\}_n$ converges.

Hence φ exists.

- Uniqueness:

  Suppose that $\exists \varphi, \phi$ such that

  $$(\varphi \circ f \circ \varphi^{-1})(w) = w^p = (\phi \circ f \circ \phi^{-1})(w)$$

  then

  $$(\varphi \circ \phi^{-1})(w^p) = (\varphi \circ \phi^{-1})((\phi \circ f \circ \phi^{-1})(w)) =$$
  $$= (\varphi \circ f \circ \varphi^{-1})((\varphi \circ \phi^{-1})(w)) = (\varphi \circ \phi^{-1})(w)^p.$$

  Since $(\varphi \circ \phi^{-1})(0) = 0$, then $(\varphi \circ \phi^{-1})(z) = c_1 z + \cdots$. The condition above tells us that $c_1^{p-1} = 1$ and the other coefficients are 0.

  Therefore, $\varphi = c_1 \phi$ where c_1 is a $(p-1)$-th root of unity.

□

This result is especially relevant in the context of polynomials, since ∞ is always a super-attracting fixed point for this type of maps.

The local change of variables given by Theorem 1.10 is known as the **Böttcher coordinates** around the super-attracting fixed point.

1.2.3 Rationally Indifferent Fixed Points

We analyze here the case where $z = 0$ is a rationally indifferent fixed point of f, i.e., $\lambda = f'(0)$ is a root of unity. Our goal is to characterize the dynamics of f in a neighborhood of the origin. First, we need to introduce some preliminary concepts.

Since $\lambda \neq 0$, then $f'(0) \neq 0$ and f^{-1} is locally well-defined. Hence, we can choose a neighborhood N of $z = 0$ that is small enough so that f maps N conformally onto some neighborhood N_0 of the origin.

Definition 1.11 (Attracting and Repelling Petals)**.** *A connected open set U, with compact closure $\bar{U} \subset N \cap N_0$ is called*

- *An **attracting petal** for f at the origin if*

$$f(\bar{U}) \subset U \cup \{0\} \qquad \bigcap_{k \geq 0} f^k(\bar{U}) = \{0\}.$$

- *A **repelling petal** for f at the origin if U is an attracting petal for f^{-1}.*

We study first the special case where $\lambda = 1$, because if $\lambda = e^{2\pi i p/q}$, then f^q has multiplier 1. Along this section we suppose that f is a meromorphic function with a rationally indifferent fixed point at $z = 0$ with multiplier 1. We have:

$$f(z) = z(1 + az^n + (\text{H.O.T.})) \quad a \neq 0, n \geq 1.$$

We call $n + 1$ the **multiplicity** of the fixed point.

Definition 1.12 (Repulsion and Attraction Vectors)**.** *A vector $v \in \mathbb{C}$ is called a **repulsion vector** for f at the origin if $nav^n = +1$, and an **attraction vector** if $nav^n = -1$.*

Thus, there are n equally spaced attraction vectors at the origin, separated by n equally spaced repulsion vectors.

We number these vectors as $v_0, v_1, \ldots, v_{2n-1}$, where v_0 is repelling and $v_j = e^{\pi ij/n} v_0$, so

$$nav_j^n = nav_0^n e^{\pi ij} = (-1)^j.$$

Remark 1.13. *In order to understand the definition of the v_j, we look at its local behavior under iteration. We have $av_j^n = (-1)^j/n$, therefore*

$$f(v_j) = v_j(1 + (-1)^j/n) + (H.O.T.)$$

and then

$$f^k(v_j) = v_j(1 + (-1)^j/n)^k + (H.O.T.).$$

On one side if j is even, v_j is repelled by the origin in the direction v_j, on the other side if j is odd, v_j is attracted by the origin in the direction v_j.

Recall that in some neighborhood of the origin f^{-1} is well-defined and holomorphic, therefore Remark 1.13 shows us that the repulsion vectors of f are the attraction vectors of f^{-1} and vice-versa.

Definition 1.14 (Nontrivial Convergence)**.** *We say that an orbit $\mathcal{O}_f^+(z_0)$ converges to zero **nontrivially** if $z_k \xrightarrow[k \to \infty]{} 0$ but $z_k \neq 0$.*

Definition 1.15 (Directional Convergence)**.** *If an orbit $\mathcal{O}_f^+(z_0)$ under f converges to zero, with $z_k \sim v_j/\sqrt[n]{k}$ (where j is necessarily odd), then we say that this orbit $\{z_k\}_k$ tends to zero **from the direction** v_j.*

Now we can finally prove the main result of this section, which will lead us to the well-known Leau-Fatou Flower Theorem.

Lemma 1.16 (Convergence Directions)**.** *If an orbit $\mathcal{O}_f^+(z_0)$ converges to zero nontrivially, then z_k is asymptotic to $v_j/\sqrt[n]{k}$ as $k \to \infty$ for one of the n attraction vectors v_j. In other words, the limit $\lim_k z_k \sqrt[n]{k} = v_j$. Similarly, if an orbit $\mathcal{O}_f^-(z_0')$ under f converges to zero nontrivially, then z_k' is asymptotic to $v_j/\sqrt[n]{k}$, where v_j is one of the n repulsion vectors, with j even.*

Proof. Consider the change of coordinates $w = \varphi(z) = c/z^n$, where $nac = -1$ and $Re(w) = Re\varphi(z) = Re(c/z^n)$.

In the special case of an attraction or repulsion vector we have

$$\varphi(v_j) = Re\varphi(v_j) = (-1)^{j+1}$$

We are interested in the behavior of f when $|z|$ is small, i.e., $|w|$ is large.

Let $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_- = (-\infty, 0]$. We want to label the different branches of $\varphi^{-1}(w) = \sqrt[n]{c/w}$. First of all we cover $\mathbb{C} \setminus \{0\}$ by $2n$ open sectors Δ_j with angle $2\pi/n$:

$$\Delta_j = \{re^{i\theta}v_j \in \mathbb{C} \colon r > 0 \ , \ |\theta| < \pi/n\}$$

Then,

$$\varphi(re^{i\theta}v_j) = -\frac{1}{na}\frac{1}{r^n e^{in\theta}v_j^n} = \frac{1}{-nav_j^n}\frac{1}{r^n}e^{-in\theta} = (-1)^{j+1}\frac{1}{r^n}e^{-in\theta}.$$

Since $|\theta| < \pi/n$ and $r > 0$, φ maps Δ_j biholomorphically onto:

- $\mathbb{C} \setminus \mathbb{R}_+$ if j is even (v_j is repelling).
- $\mathbb{C} \setminus \mathbb{R}_-$ if j is odd (v_j is attracting).

Hence, there is a uniquely defined branch ψ_j of φ^{-1} with

$$\psi_j \colon \mathbb{C} \setminus \mathbb{R}_{(-1)^j} \longrightarrow \Delta_j.$$

Recall that $\Delta_j \cap \Delta_{j+1} = \{re^{i\theta}v_j \in \mathbb{C} \colon r > 0 \ , \ 0 < \theta < \pi/n\}$ and

$$\varphi(\Delta_j \cap \Delta_{j+1}) = \begin{cases} \{Im(z) > 0\} & \text{if j is even.} \\ \{Im(z) < 0\} & \text{if j is odd.} \end{cases}$$

We then have

$$f(z) = z(1 + az^n + o(z^n)) \quad \text{as} \quad z \to 0.$$

We want to understand the behavior of this map for z close to 0 in Δ_j, we use the transformation:

$$w \longmapsto F_j(w) = (\varphi \circ f \circ \psi_j)(w)$$

defined outside a large disk in $\mathbb{C} \setminus \mathbb{R}_{(-1)^j}$. We have:

$$(f \circ \psi_j)(w) = \sqrt[n]{c/w}\left(1 + a\frac{c}{w} + o\left(\frac{1}{w}\right)\right) \quad \text{as} \quad |w| \to \infty$$

and using Taylor

$$F_j(w) = w\left(1 + a\frac{c}{w} + o\left(\frac{1}{w}\right)\right)^{-n} = w\left(1 + \frac{-nac}{w} + o\left(\frac{1}{w}\right)\right).$$

Since $-nac = 1$ we can write it as

$$F_j(w) = w + 1 + o(1) \quad \text{as} \quad |w| \to \infty. \tag{1.1}$$

We can improve it using that

$$f(z) = z(1 + az^n + O(z^{n+1}))$$

and then

$$F_j(w) = w\left(1 + \frac{1}{w} + O\left(\frac{1}{\sqrt[n]{w^{n+1}}}\right)\right) \quad \text{as} \quad |w| \to \infty$$

i.e.,

$$F_j(w) = w + 1 + O\left(\frac{1}{\sqrt[n]{w}}\right) \quad \text{as} \quad |w| \to \infty. \tag{1.2}$$

By (1.1) we can choose $R > 0$ such that

$$|F_j(w) - (w+1)| < 1/2 \quad \text{for} \quad |w| > R \tag{1.3}$$

which is represented in Figure 1.1.

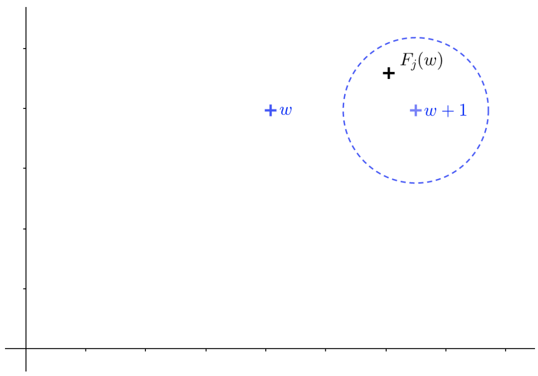

Figure 1.1: Graphic representation of (1.3)

Therefore,

$$Re(F_j(w)) > Re(w) + 1/2 \quad \text{for} \quad |w| > R \tag{1.4}$$
$$|Im(F_j(w) - w)| < Re(F_j(w) - w) \quad \text{for} \quad |w| > R \tag{1.5}$$

and then taking $w = \varphi(z)$, we also obtain that

$$Re\varphi(f(z)) > Re\varphi(z) + 1/2 \quad \text{for} \quad |z| \to 0. \tag{1.6}$$

Let $\mathbb{H}_R = \{Re(w) > R\}$ and

$$\mathcal{P}_j(R) = \psi_j(\mathbb{H}_R) = \{\psi_j(w) \colon Re(w) > R\} = \{z \in \Delta_j \colon Re\varphi(z) > R\}.$$

Recall that j must be odd in order to have $\mathcal{P}_j(R) = \psi_j(\mathbb{H}_R)$ well defined. (1.4) implies that $F_j(\mathbb{H}_R) \subset \mathbb{H}_R$ and (1.6) implies that $f(\mathcal{P}_j(R)) \subset \mathcal{P}_j(R)$. Furthermore, the successive iterates of f restricted to $\mathcal{P}_j(R)$ converge uniformly to the constant map 0 (because F_j^k converges uniformly to ∞).

Consider an orbit $\mathcal{O}_f^+(z_0)$ which converges to zero nontrivially. For all k large enough we have $Re\varphi(z_{k+1}) > Re\varphi(z_k)+1/2$, therefore exists m large enough so that $Re\varphi(z_m) > R$, i.e., $z_m \in \mathcal{P}_j(R) \subset \Delta_j$. Remark 1.13 shows us that z_m must belong to one of the attracting petals (j odd). Since $f(\mathcal{P}_j(R)) \subset \mathcal{P}_j(R)$, $z_k \in \mathcal{P}_j(R))$ for all $k \geq m$.

If we consider now the sequence $w_k = \varphi(z_k)$. Then $w_k \in \mathbb{H}_R$ and

$$w_{k+1} = \varphi(f^{k+1}(z_0)) = \varphi(f(\psi_j(\varphi(f^k(z_0))))) = (\varphi \circ f \circ \psi_j)(w_k) = F_j(w_k).$$

Since $Re(w_k) \to \infty$, then $|w_k| \to \infty$, (3.1) gives us that $F_j(w_k) - w_k = 1 + o(1)$ for k large enough. Hence,

$$w_{k+1} - w_k \xrightarrow[k\to\infty]{} 1$$

therefore,

$$\frac{w_k - w_0}{k} = \frac{1}{k}\sum_{j=0}^{k+1}(w_{j+1} - w_j) \xrightarrow[k\to\infty]{} 1$$

and then $w_k/k \xrightarrow[k\to\infty]{} 1$, i.e., $w_k \sim k$ as $k \to \infty$.

Since $1/w_k = -naz_k^n$, we obtain that naz_k^n is asymptotic to $-1/k$, the equality $nav_j^n = -1$ shows us that

$$z_k^n \sim v_j^n/k.$$

If we extract the n-th root (we can because $z_k \in \mathcal{P}_j(R)$), we obtain:

$$z_k \sim v_k/\sqrt[n]{k}.$$

$\square$

A consequence of Lemma 1.16 is the following well-known result.

Theorem 1.17 (Leau-Fatou Flower Theorem). *Let*

$$f(z) = z + az^{n+1} + (H.O.T.) \quad \textit{with} \quad a \neq 0 \ , \ n \geq 1$$

be holomorphic in some neighborhood of the origin, then there exist $2n$ petals $\mathcal{P}_j$, where $\mathcal{P}_j$ is either repelling or attracting depending to whether j is even or odd. Furthermore, we can choose those petals so that

$$\{0\} \cup \mathcal{P}_0 \cup \cdots \mathcal{P}_{2n-1}$$

is an open neighborhood of $z = 0$. When $n > 1$, each $\mathcal{P}_j$ intersects each of its two immediate neighborhoods in a simply connected region $\mathcal{P}_j \cap \mathcal{P}_{j\pm1}$ but is disjoint from the remaining $\mathcal{P}_k$ (we consider j modulo $2n$).

The proof can be found in [Mil2, CG].

The following result introduces the Fatou coordinates, gives us the normal form and a negative answer to the linearization problem. In fact, a consequence of Lemma 1.16 is that the behavior of f near a rationally indifferent fixed point is not compatible with being linearizable.

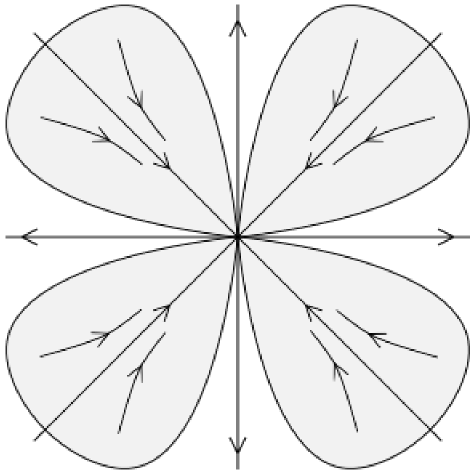

Figure 1.2: Flower with four attracting petals.

Theorem 1.18 (Parabolic Linearization). *Let*

$$f(z) = \lambda z + az^{n+1} + (H.O.T.) \quad \textit{with} \quad a \neq 0 \ , \ n \geq 1$$

be holomorphic in some neighborhood of the origin, where λ is a primitive q-th root of unity. Then for any attracting or repelling petal $\mathcal{P}$, there is one and, up to composition with a translation of $\mathbb{C}$, only one conformal one-to-one map $\varphi\colon \mathcal{P} \to \mathbb{C}$ such that $\varphi(f(z)) = 1 + \varphi(z)$ for all $z \in \mathcal{P} \cap f^{-1}(\mathcal{P})$.

The proof can be found in [Mil2].

1.2.4 Irrationally Indifferent Fixed Points

Finally, we study the remaining case. Consider a map of the form

$$f(z) = \lambda z + \sum_{k\geq 2} a_k z^k$$

which is defined in some neighborhood of $z = 0$, with multiplier:

$$\lambda = e^{2\pi i\theta} \quad , \quad \theta \in \mathbb{R} \setminus \mathbb{Q}$$

We want to give sufficient conditions so that we can find a solution of the Schröder equation, i.e., a conjugacy φ that gives us an affirmative answer to the linearization problem

$$\varphi(f(z)) = \lambda\varphi(z),$$

normalized by $\varphi'(0) = 1$. For $h = \varphi^{-1}$ we have

$$f(h(\xi)) = h(\lambda\xi) \quad , \quad h'(0) = 1.$$

We will not prove all the results related to this question since they are too many to fit in this project. Nevertheless, we prove some of them and state others that are relevant in Chapter 2.

Theorem 1.19. *A solution h to the Schröder equation is univalent in any disk $\{|\xi| < r\}$.*

Proof. Suppose $h(\xi_1) = h(\xi_2)$. Then $f(h(\xi_1)) = f(h(\xi_2))$ which implies that $h(\lambda\xi_1) = h(\lambda\xi_2)$, therefore $h(\lambda^n\xi_1) = h(\lambda^n\xi_2)$ for all $n \in \mathbb{N}$.

Since $\{\lambda^n : n \in \mathbb{N}\}$ is dense in $\{|z| = 1\}$, by analytic continuation we obtain that $h(\xi_1 z) = h(\xi_2 z)$ for $|z| < 1$. If we take the derivative, using $h'(0) = 1$ we obtain that $\xi_1 = \xi_2$. □

Theorem 1.20. *A solution h to the Schröder equation exists, if and only if, the sequence of iterates $\{f^n\}$ is uniformly bounded in some neighborhood of the origin.*

Proof. On one hand, if h exists then we can write $f^n(z) = f^n(h(\xi)) = h(\lambda^n\xi) = h(\lambda^n h^{-1}(z))$, which is uniformly bounded.

On the other hand, if $|f^n| \leq M$ for all $n \in \mathbb{N}$, we can define

$$\varphi_n(z) = \frac{1}{n}\sum_{j=0}^{n-1} \lambda^{-j} f^j(z).$$

Then $\{\varphi_n\}$ is a uniformly bounded sequence of holomorphic functions in some neighborhood of the origin (bounded by the same constant M):

$$|\varphi_n(z)| \leq \frac{1}{n}M\sum_{j=0}^{n-1}|\lambda^{-j}| = M$$

so it contains a convergent subsequence (due to Theorem A.47 (Montel)). Observe that

$$\begin{aligned}\varphi_n(f(z)) &= \frac{\lambda}{n}\sum_{j=0}^{n-1}\lambda^{-(j+1)} f^{j+1}(z) = \\ &= \lambda\left(\frac{1}{n}\sum_{j=0}^{n-1}\lambda^{-j} f^j(z)\right) + \frac{1}{n}\left(\lambda^{-n} f^n(z) - \lambda z\right)\end{aligned}$$

and

$$\left|\frac{1}{n}\left(\lambda^{-n} f^n(z) - \lambda z\right)\right| \leq \frac{2M}{n} = O\left(\frac{1}{n}\right).$$

Therefore $\varphi_n \circ f = \lambda\varphi_n + O(1/n)$, thus any limit φ of the φ_n's satisfies $\varphi \circ f = \lambda\varphi$. Since $\lambda = f'(0)$, we also have that

$$\varphi_n'(0) = \frac{1}{n}\sum_{j=0}^{n-1}\lambda^{-j}\lambda^j = 1 \Longrightarrow \varphi'(0) = 1.$$

Hence, $h = \varphi^{-1}$ is a solution of the Schröder equation. □

The following result shows us that such linearization in not always possible.

Theorem 1.21. *There exists $\lambda = e^{2\pi i\theta}$ so that the Schröder equation has no solution for any polynomial f.*

Definition 1.22 (Cremer and Siegel Points)**.** *We say that an irrationally indifferent fixed point is a*

- ***Cremer point*** *if a local linearization is not possible.*
- ***Siegel point*** *if a local linearization is possible.*

The goal is now giving sufficient conditions on θ so that the Schröder equation has solution.

Definition 1.23 (Diophantine Number)**.** *We say that* $\theta \in \mathbb{R}$ *is* ***Diophantine*** *if it is badly approximable by rational numbers, in the sense that there exists* $k < \infty$ *and* $\varepsilon > 0$ *so that*

$$\left| \theta - \frac{p}{q} \right| \geq \frac{\varepsilon}{q^k} \quad \textit{for all} \quad p/q \in \mathbb{Q}$$

The following result is relevant because shows us that the set of Diophantine points in not empty.

Theorem 1.24 (Liouville)**.** *Every algebraic irrational number is Diophantine.*

Finally, we give an affirmative answer to the linearization problem when θ is Diophantine.

Theorem 1.25 (Siegel)**.** *If* θ *is Diophantine, and if* f *has a fixed point at* $z = 0$ *with multiplier* $e^{2\pi i\theta}$*, then there exists a solution of the Schröder equation.*

All the proofs of these results can be found in [Mil2, CG].

Sharper conditions are known. In fact, J.C. Yoccoz received a Fields Medal in 1994 for his contributions to the linearization problem. Nevertheless, as of today, the complete solution to this question is still an open problem.

Chapter 2

The Julia and Fatou Sets

The goal of this chapter is to obtain global properties of the dynamics of meromorphic functions. We use here the local theory previously studied in Chapter 1 and Montel's normality to accomplish this goal. Furthermore, the limit functions that we can have under iteration in some sets are also studied and we are also concerned with some topological properties of this sets.

The references for this chapter are: [BKL1, BKL2, BKL3, BKL4, Bea, BF, Ber1, Ber2, BE, BF2, CG, Fat, Mil2, Ripp, Sch].

We shall introduce first the concept of normality[1].

Definition 2.1 (Normal Convergence). *We say that a sequence $\{f_k(z)\}_k$ of meromorphic functions on a domain D* ***converges normally*** *to $f(z)$ on D, if the sequence converges uniformly on compact subsets of D to $f(z)$ in the spherical metric.*

Definition 2.2 (Normal Family). *A family $\mathcal{F}$ of meromorphic functions on a domain D is a* ***normal family*** *if every sequence in $\mathcal{F}$ has a subsequence that converges normally on D.*

Our applications to transcendental dynamics are based on the following theorem, which is proved in A.2.

Theorem 2.3 (Montel).

(a) Suppose that $\mathcal{F}$ is a family of holomorphic functions on a domain D such that $\mathcal{F}$ is uniformly bounded on each compact subset of D. Then $\mathcal{F}$ is a normal family.

(b) A family $\mathcal{F}$ of meromorphic functions on a domain D that omits three values is normal.

From now on f denotes a transcendental function unless we state the opposite.

The dynamical of f plane splits into two sets, the study of which is the main topic in the document.

Definition 2.4 (Fatou and Julia Sets). *We define the* ***Fatou set*** *as:*

$$F(f) = \{z \in \mathbb{C}_\infty \colon \{f^n(z) \colon n \in \mathbb{N}\} \text{ is well-defined and normal in some neighborhood of } z\}$$

[1] See A.2 for further details.

and the ***Julia set*** $J(f) = \mathbb{C}_\infty \setminus F(f)$*, where* $\mathbb{C}_\infty$ *is the Riemann Sphere, i.e., the Alexandroff compactification of the complex plane* $\mathbb{C}$*.*

Notice that the condition of being "well-defined" is necessary, since orbits are truncated if they land on a pole of f.

In our case, we always have $\infty \in J(f)$. From the definition it follows that $F(f)$ is an open set and $J(f)$ a closed set.

Some special classes of transcendental functions will be relevant for us.

$$\begin{aligned} E &= \{f \colon f \text{ is transcendental entire}\}, \text{ or} \\ P &= \{f \colon f \text{ is transcendental meromorphic, with exactly one pole which} \\ &\qquad \text{is an omitted value}\}. \end{aligned}$$

In the first case, for every $z \in \mathbb{C}$ the iterates are always defined. In the second case, every function of the class can be expressed according to the following result.

Lemma 2.5. *If* $f \in P$*, then* f *has the form*

$$f(z) = z_0 + \frac{e^{g(z)}}{(z - z_0)^m} \quad \textit{where } g \in \mathcal{H}(\mathbb{C}) \textit{ and } m \in \mathbb{N}.$$

Proof. By hypothesis $f(z) - z_0$ omits the value 0 and has a pole of order m at z_0, therefore $h(z) = (z - z_0)^m (f(z) - z_0) \in \mathcal{H}(\mathbb{C})$ and omits the value 0. Then $h'/h \in \mathcal{H}(\mathbb{C})$ and we can take an holomorphic primitive $H \in \mathcal{H}(\mathbb{C})$ of h'/h. Then,

$$\frac{d}{dz}(e^{-H(z)} h(z)) = h'(z) e^{-H(z)} - \frac{h'(z)}{h(z)} h(z) e^{-H(z)} = 0$$

which means that $e^{-H(z)} h(z) \equiv C \neq 0$ constant, i.e.,

$$h(z) = C e^{H(z)} = e^{H(z) + \log(C)} = e^{g(z)}$$

with $g \in \mathcal{H}(\mathbb{C})$. $\square$

Since the pole is omitted, the sequence of iterates is defined for every initial condition in $\mathbb{C} \setminus \{z_0\}$. Hence, if $f \in P$ we have $\{z_0, \infty\} \subset J(f)$.

The most general class of meromorphic maps is.

$$\begin{aligned} M = \{f \colon f &\text{ is transcendental meromorphic, with at least two poles or} \\ &\text{exactly one pole which is not an omitted value}\}. \end{aligned}$$

In this case $\mathcal{O}_f^-(\infty)$ is an infinite set. In fact $f^{-3}(\infty)$ is actually an infinite set by Picard's Theorem:

- If f has two poles a, b, then $f^{-1}(\infty) = \{a, b\}$ and by Picard's Theorem $f^{-2}(\infty)$ is an infinite set.
- If f has just one pole (which. is not an omitted value), then $f^{-1}(\infty) = \{a\}$, then $f^{-2}(\infty)$ can be a finite set (and not empty), but by Picard's Theorem $f^{-3}(\infty)$ is an infinite set.

The largest set where all iterates are defined is

$$\mathbb{C}_\infty \setminus \overline{\mathcal{O}_f^-(\infty)}.$$

Since $f(\mathbb{C}_\infty \setminus \overline{\mathcal{O}_f^-(\infty)}) \subset \mathbb{C}_\infty \setminus \mathcal{O}_f^-(\infty)$, is a set that misses (far) more than three points, Theorem 2.3 (Montel) tells us that

$$F(f) = \mathbb{C}_\infty \setminus \overline{\mathcal{O}_f^-(\infty)} \quad \text{and} \quad J = \overline{\mathcal{O}_f^-(\infty)}.$$

2.1 Basic Properties

The aim of this section is to show some classical results concerning properties of the Julia and Fatou sets.

Definition 2.6 (Forward, Backward and Completely Invariant)**.**

- *We say that a set S is* ***forward invariant*** *under f if $z \in S$ implies $f(z) \in S$ or $f(z)$ is undefined.*
- *We say that S is* ***backward invariant*** *under f if $z \in S$ implies that $w \in S$ for all w such that $f(w) = z$.*
- *We say that S is* ***completely invariant*** *if it is both forward and backward invariant under f.*

Lemma 2.7 (Invariance Lemma)**.** *The Julia set and the Fatou set of a transcendental meromorphic function f are completely invariant.*

Proof. We just have to show that $F(f)$ is completely invariant.

Let $z \in F(f)$ such that the iterates are well-defined and a neighborhood $z \in U \subset F(f)$, then $f \in \mathcal{H}(U)$. Since $\{f_n\}_n$ is a normal family on U, for every $f^{n_k} \vdash f^n$ exists $f^{n_{k_j}} \vdash f^{n_k}$ that converges normally on U.

The Open Mapping Theorem gives us that $f(U)$ is open, therefore $\{f^{n_{k_j}-1}\}_j$ converges normally on $f(U)$, and then $\{f^{n-1}\}_n$ is normal on $f(U)$, which implies that $f(z) \in F(f)$.

If we now consider $\{f^{n_{k_j}+1}\}_j$ and $f^{-1}(U)$ we obtain that any $w \in \mathbb{C}$ such that $f(w) = z$ also lies in $F(f)$.

Therefore $F(f)$ is completely invariant, and then $J(f) = \mathbb{C}_\infty \setminus F(f)$ is also completely invariant. □

Lemma 2.8 (Iteration Lemma)**.** *For any $q \in \mathbb{N}$, $J(f^q) = J(f)$.*

Proof. We prove $F(f) = F(f^q)$ which is equivalent to our statement.

$F(f) \subset F(f^q)$: Suppose that $z \in F(f)$, then exists an open set $U \ni z$ such that $\{f^n\}_n$ is normal in U. Which means that for every $f^{n_k} \vdash f^n$, exists $f^{n_{k_j}} \vdash f^{n_k}$ that converges normally on U.

Consider a sequence $f^{qn_j} \vdash f^{qn}$, then $f^{qn_j} \vdash f^n$, since $\{f^n\}$ is normal, exists $f^{qn_{j_k}} \vdash f^{qn_j}$ that converges normally on U, i.e., $\{f^{qn}\}_n$ is normal in U. Therefore $z \in F(f^q)$.

$F(f^q) \subset F(f)$: Suppose that $z \in F(f^q)$, then exists an open set $U \ni z$ such that $\{f^{qn}\}_n$ is normal on U. Consider a sequence $f^{n_l} \vdash f^n$,

- Suppose that $\#\{l\colon n_l \equiv 0 \pmod q\} = \infty$, then we can consider the subsequence $q\tilde{n}_j \vdash n_l$ defined by; for all $j \geq 1$ exists $l \geq 1$ such that $q\tilde{n}_j = n_l$), then:
$$f^{q\tilde{n}_j} \vdash f^{qn}$$
since $\{f^{qn}\}$ is normal on U, exists $f^{q\tilde{n}_{j_k}} \vdash f^{q\tilde{n}_j}$ that converges normally on U, but
$$f^{q\tilde{n}_{j_k}} \vdash f^{q\tilde{n}_j} \vdash f^{n_l}$$
and then $\{f^n\}$ is normal on U, which implies that $z \in F(f)$.

- Suppose now that $\#\{l\colon n_l \equiv 0 \pmod q\} < \infty$, then exists $m \in \{1, \ldots, q-1\}$ such that $\#\{l\colon n_l \equiv -m \pmod q\} = \infty$.

  So we can apply the argument above to the sequence $\{f^{n_l+m}\}_l$ to conclude that $\{f^{n+m}\}_n$ is normal on U, i.e., $\{f^n\}_n$ is normal on U, which implies that $z \in F(f)$.

□

The following is a consequence of Theorem 2.3 (Montel).

Proposition 2.9. *If $z \in J(f)$ and U is a neighborhood of z, then $\bigcup_{n\in\mathbb{N}} f^n(U)$ covers $\mathbb{C}_\infty$ with at most two exceptions.*

Proof. If $\cup_{n\in\mathbb{N}} f^n(U)$ misses three points of $\mathbb{C}_\infty$, by Theorem 2.3 (Montel) $z \in F(f)$, a contradiction. □

Definition 2.10 (Exceptional point)**.** *We say that z_0 is **exceptional** if $\mathcal{O}_f^-(z_0)$ is finite.*

Proposition 2.11. *If $z_0 \in J(f)$ is not exceptional, then $\mathcal{O}_f^-(z_0)$ is dense in $J(f)$.*

Proof. Consider $w \in J(f)$ and U, any neighborhood of w.

We have to show that U contains a point $\tilde{w}$ such that $f^N(\tilde{w}) = z_0$. Proposition 2.9 tells us that
$$\bigcup_{n\in\mathbb{N}} f^n(U) \supset \mathbb{C}_\infty \setminus \{a, b\}.$$
Since z_0 is not exceptional, exists $N > 0$ such that $z_0 \in f^N(U)$, i.e., $\exists \tilde{w} \in U$ such that $f^N(\tilde{w}) = z_0$. The neighborhood U of w can be as small as necessary, therefore $\mathcal{O}_f^-(z_0)$ is dense in $J(f)$. □

Theorem 2.12. *If $J(f)$ has an interior point, then $J(f) = \mathbb{C}_\infty$. In other words, either $J(f) = \mathbb{C}_\infty$ of $J(f)$ has empty interior.*

Proof. Suppose that exists $U \subset J(f)$, U open, by Proposition 2.9,
$$\bigcup_{n\in\mathbb{N}} f^n(U) \supset \mathbb{C}_\infty \setminus \{a, b\}$$
Lemma 2.7 (Invariance Lemma) shows us that
$$\mathbb{C}_\infty \setminus \{a, b\} \subset \bigcup_{n\in\mathbb{N}} f^n(U) \subset J(f)$$

and then

$$\mathbb{C}_\infty = \overline{\mathbb{C}_\infty \setminus \{a,b\}} \subset \overline{J(f)} = J(f) \subset \mathbb{C}_\infty$$

i.e., $J(f) = \mathbb{C}_\infty$. □

Remark 2.13. *We show later that the case $J(f) = \mathbb{C}_\infty$ is possible, for example $J(2\pi i e^z) = \mathbb{C}_\infty$.*

Corollary 2.14. *If $J(f) \neq \mathbb{C}_\infty$, then $F(f)$ is unbounded.*

Proof. Suppose that $F(f)$ is bounded, then $J(f)$ has interior points, by the previous theorem, $J(f) = \mathbb{C}_\infty$ which is a contradiction. □

Now we want to translate the local properties that we have seen in Chapter 1 into global properties.

Definition 2.15 (Basin of Attraction)**.** *If $\mathcal{O}_f$ is an attracting periodic orbit of period m, we define the **basin of attraction** to be the open set $\mathcal{A} \subset \mathbb{C}_\infty$ consisting of all points $z \in \mathbb{C}_\infty$ for which the successive iterates converge to some point of $\mathcal{O}_f$.*

Proposition 2.16. *Every attracting periodic orbit is contained in $F(f)$. In fact, the entire basin of attraction $\mathcal{A}$ for an attracting periodic orbit is contained in $F(f)$. However, every repelling periodic orbit is contained in the $J(f)$.*

Proof. Consider a fixed point z_0 with multiplier λ.

- If $|\lambda| > 1$, then no subsequence of iterates of f can converge uniformly near z_0, because
$$(f^n)'(z_0) = \lambda^n \xrightarrow[n\to\infty]{} \infty$$
and then $z_0 \in J(f)$.
- If $|\lambda| < 1$, we have already seen (Proposition 1.8) that for $|z - z_0| < \delta$ with δ small enough the iterates of f converge uniformly to the constant map $z \mapsto z_0$, so $z_0 \in F(f)$.

The case of an attracting periodic orbit of order m follows directly considering f^m instead of f and the Lemma 2.8 (Iteration Lemma).

Finally, in any compact subset of $\mathcal{A}$, the successive iterates of f converge uniformly to the constant map $z \mapsto z_0$ and thus $\mathcal{A} \subset F(f)$. □

Proposition 2.17. *$F(f)$ contains all Siegel points of f and their linearizing neighborhoods. Instead, $J(f)$ contains all rationally indifferent fixed points and Cremer fixed points.*

Proof. We have seen in Theorem 1.20 that a solution to the Schröder equation exists, if and only if, the sequence of iterates $\{f_n\}_n$ is uniformly bounded in some neighborhood of the origin, which is equivalent, by Theorem 2.3 (Montel), to $\{f_n\}_n$ being normal in some neighborhood of the fixed point. Therefore, all Siegel points of f are in $F(f)$ and all Cremer points of f are in $J(f)$.

If we have a parabolic fixed point (we can assume without loss of generality that is $z = 0$), we obtain a power series of the form

$$f^m : z \mapsto z + az^p + \text{H.O.T.} \quad (a \neq 0, p \geq 2)$$

and then $f^{mk} = z + kaz^p + \text{H.O.T.}$. The p-th derivative of f^{mk} at $z = 0$ is then

$$p!ka \xrightarrow[k\to\infty]{} \infty.$$

Therefore, no subsequence can converge normally.

Together with $J(f^m) = J(f)$, we obtain that rationally indifferent fixed points of f are in the Julia Set. □

2.1.1 Repelling Periodic Points and Baker's Theorem

In this subsection we focus on the Julia set. The results stated here hold for all transcendental functions, but the proofs are different for each class. In Chapter 3 we study a function $f \in M$ in detail, therefore proofs are only given for this class.

It has been pointed out that the presence of poles in a transcendental function requires a careful study, however here we show that in many cases it could also work in our favor.

Definition 2.18 (Perfect set)**.** *We say that a set is **perfect** if it is closed, nonempty and does not contain isolated points.*

Theorem 2.19. *Let f be a transcendental meromorphic function, then $J(f)$ is perfect.*

Proof. If $f \in M$, we have already seen that $J(f) = \overline{\mathcal{O}_f^-(\infty)}$. The discussion at the beginning of this chapter also shows us that there exists $z \in \mathcal{O}_f^-(\infty)$ that is not exceptional, therefore by Proposition 2.11 $J(f)$ is perfect. □

As a direct consequence we have

Corollary 2.20. *$J(f)$ is an infinite set.*

The goal is proving a well-known result known as Baker's Theorem and a result concerning the periodic points of a transcendental meromorphic functions, to do so, we need the following result of Ahlfors.

Theorem 2.21 (Ahlfors' Five Islands Theorem)**.** *Let f be a transcendental meromorphic function, and let $D_1, \dots, D_5$ be five simply connected domains in $\mathbb{C}$ with disjoint closures. Then there exists $j \in \{1, \dots, 5\}$ and, for any $R > 0$, a simply connected domain $G \subset \{z \in \mathbb{C} : |z| > R\}$ such that f is a conformal map of G onto D_j. If f has only finitely many poles, then "five" may be replaced by "three".*

A weaker version proved can be found in [Ber2]. This result is the key ingredient for the following accumulation lemma.

Lemma 2.22. *Suppose that $f \in M$ and that $z_1, z_2, \dots, z_5 \in \mathcal{O}_f^-(\infty) \setminus \{\infty\}$ are distinct. Define n_j by $f^{n_j}(z_j) = \infty$. Then there exists $j \in \{1, 2, \dots, 5\}$ such that z_j is a limit point of repelling periodic points of minimal period $n_j + 1$. If f has only finitely many poles, then "five" may be replaced by "three".*

Proof. Consider D_j disks around z_j where the radii is small enough so that:

- $D_j \setminus \{z_j\}$ does not contain any critical point of f^{n_j}.
- The closures of the D_j are pairwise disjoint.

There exists $R > 0$ such that $f^{n_j}(D_j) \supset \{z\colon |z| > R\} \cup \{\infty\}$, we now consider j and G according to Ahlfors' Five Islands Theorem.

We can find a region $H \subset D_j \setminus \{z_j\}$ such that $f^{n_j}\colon H \to G$ is conformal. Since $f\colon G \to D_j$ is conformal so it is $f^{n_j+1}\colon H \to D_j$ and then we have that $f^{-n_j-1}\colon D_j \to H$ is also conformal. Since $\overline{H} \subset D_j$, f^{-n_j-1}, by Corollary A.31 it has an attracting fixed point in D_j, therefore f^{n_j+1} has a repelling fixed point in D_j. Since the D_j can be chosen arbitrarily small, we obtain a sequence $\{w_l\}_l$ of repelling fixed points of f^{n_j+1} that accumulate at z_j.

We show now that, in fact, the $\{w_l\}$ are repelling periodic points of f of period n_j+1. Suppose the opposite, i.e., for every w_l there exists $k_l \in \{1, \ldots, n_j\}$ such that $f^{k_l}(w_l) = w_l$. Since $\{1, \ldots, n_j\}$ is a finite set, we can find a subsequence $w_{l_s} \vdash w_l$ and $k \in \{1, \ldots, n_j\}$ such that for every $s \in \mathbb{N}$, $f^k(w_{l_s}) = w_{l_s}$ and

$$w_{l_s} \xrightarrow[s\to\infty]{} z_j$$

then,

$$z_j = \lim_{s\to\infty} w_{l_s} = \lim_{s\to\infty} f^k(w_{l_s}) = f^k\left(\lim_{s\to\infty} w_{l_s}\right) = f^k(z_j).$$

Therefore, $\infty = f^{n_j}(z_j) = f^{n_j+k}(z_j)$, which is a contradiction because $f(\infty)$ is not defined.

Hence, z_j is a limit point of repelling periodic points of period n_j+1. □

We can finally prove the results concerning the existence of infinitely repelling periodic points.

Theorem 2.23. *If $f \in M$ and $n \geq 4$, then f has infinitely many repelling periodic points of minimal period n.*

Proof. We have seen at the beginning of this chapter that if $f \in M$ and $n \geq 4$ then $f^{-n+1}(\infty)$ is an infinite set, and then Lemma 2.22 gives us the result. □

Finally, we prove the well-known Baker's theorem.

Theorem 2.24 (Baker)**.** *Let f be a meromorphic function. Then $J(f)$ is the closure of the set of repelling periodic points of f.*

Proof. Consider $a \in J(f)$ and $\varepsilon > 0$, we want to see that there exists a repelling periodic point w of f such that $|a - w| < \varepsilon$. Since $J(f) = \overline{\mathcal{O}_f^-(\infty)}$ is perfect, there exists $z_1, \ldots, z_n \in \mathcal{O}_f^-(\infty)$ different such that $|z_j - a| < \varepsilon/2$.

By Lemma 2.22, there exists $j \in \{1, \ldots, 5\}$ and a repelling periodic point w of f such that $|z_j - w| < \varepsilon/2$.

Therefore,

$$|a - w| \leq |a - z_j| + |z_j - w| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

and the theorem is proved. □

2.2 Components of the Fatou Set

The aim in this section is to understand the possible limit functions that we can obtain under iteration in the set of normality of the iterates of f. To do so we introduce the components of the Fatou set.

Definition 2.25 (Fatou component). *A* ***component*** *U of $F(f)$ is a maximal connected domain of normality of the iterates of f.*

Lemma 2.26. *Every component of $F(f)$ contains at most one periodic point of f.*

Proof. Suppose that U is a component of $F(f)$ and that $a, b \in U$ are periodic points of f. If we consider f^k instead of f, for k big enough we can assume that both a and b are fixed points of f.

- If a is attracting or super-attracting, by Analytic Continuation any limit of the $\{f_n\}_n$ is constant in U and therefore since $f(a) = a$ and $f(b) = b$, $a = b$.
- If a is an indifferent fixed point, since $a \in F(f)$, by Theorem 1.20 f in U is conjugate to an irrational rotation, which has only one fixed point and then $a = b$.

□

We want to understand now the behavior of f in the different components of the Fatou set. To do so we distinguish two different cases.

Definition 2.27 (Preperiodic and Wandering components). *Given a component U of $F(f)$, then by the Invariance Lemma $f^n(U)$ is contained in a component of $F(f)$ that we denote U_n.*

- *U is called* ***preperiodic*** *if there exists $n > m \geq 0$ such that $U_n = U_m$. If $m = 0$ we say that U is* ***periodic*** *with period n and $\{U, U_1, \ldots, U_{n-1}\}$ is called a* ***cycle*** *of components. The smallest n with this property is called the* ***minimal period*** *of U.*
- *If U is not preperiodic, U is called a* ***wandering domain****.*

Wandering Domains are discussed in Section 2.4.

2.2.1 Dynamics on Periodic Fatou Components

The behavior of the successive iterates of f on periodic components is well understood. The following celebrated result, originally stated by Fatou, summarizes the different possibilities that we can have.

Theorem 2.28 (Classification Theorem for periodic components). *Let U be a periodic component of period p. Then we have one of the following possibilities:*

(a) U contains an attracting periodic point z_0 of period p. Then

$$f^{np}(z) \xrightarrow[n\to\infty]{} z_0 \quad \forall z \in U$$

and U is called the ***immediate attractive basin*** *of z_0.*

(b) ∂U *contains a periodic point* z_0 *of period* p *and*

$$f^{np}(z) \xrightarrow[n\to\infty]{} z_0 \quad \forall z \in U$$

Then $(f^p)'(z_0) = 1$ *if* $z_0 \in \mathbb{C}$*.* U *is called a* ***Leau domain****.*

(c) Exists $\phi\colon U \to \mathbb{D}$ *conformal such that* $\phi(f^p(\phi^{-1}(z))) = e^{2\pi i\alpha}z$ *for some* $\alpha \in \mathbb{R}\setminus\mathbb{Q}$*.* U *is called a* ***Siegel disk****.*

(d) Exists $\phi\colon U \to A$ *conformal where* $A = \{z\colon 1 < |z| < r\}, r > 1$ *is an annulus such that* $\phi(f^p(\phi^{-1}(z))) = e^{2\pi i\alpha}z$ *for some* $\alpha \in \mathbb{R}\setminus\mathbb{Q}$*.* U *is called a* ***Herman ring****.*

(e) Exists $z_0 \in \partial U$ *such that*

$$f^{np}(z) \xrightarrow[n\to\infty]{} z_0 \quad \forall z \in U$$

but $f^p(z_0)$ *is not defined. In this case* U *is called a* ***Baker domain****.*

This result is partially proved in this document. Observe that we just have to restrict ourselves to the case $p = 1$ (otherwise we consider f^p instead of f). To do so, we need to define what we understand as a limit function.

Definition 2.29 (Limit Function)**.** *We say that* ϕ *is a* ***limit function*** *of* $\{f^n\}$ *on a Fatou component* U *if for some* $f^{n_k} \vdash f^n$*, there exists* $f^{n_{k_j}} \vdash f^{n_k}$ *that converges normally on* U *to* ϕ*. We denote by* $\mathcal{L}(U)$ *the set of all limit functions.*

The first step is showing that when we have a forward invariant component of the Fatou set, either $\mathcal{L}(U)$ contains constant limit functions or non-constant limit functions.

Theorem 2.30. *Suppose that* U *is a forward invariant Fatou component and that* $\mathcal{L}(U)$ *contains some non-constant limit functions. Then,*

(a) f *is conformal in* U*.*

(b) The identity map $id_U \in \mathcal{L}(U)$*.*

(c) Any non-constant limit function is conformal in U*.*

(d) $\mathcal{L}(U)$ *does not contain any constant limit function.*

Proof. We begin the proof with a simple observation due to Theorem A.42 (Hurwitz):

Let $\phi(z) \in \mathcal{L}(U)$ be a non-constant limit function, then exists $n_j \vdash n$ such that $f^{n_j} \to \phi(z)$ normally on U.

For any $w \in U$, the zeros of $\phi(z)-\phi(w)$ are isolated. By Hurwitz's theorem, exists $j_0 \geq 1$ such that $f^{n_j}(z) - \phi(w)$ has zeros for all $j \geq j_0$, therefore exists $\tilde{z} \in U$ such that $f^{n_j}(\tilde{z}) = \phi(w)$, since $f(U) \subset U$, then $\phi(w) \in U$, which implies that $\phi(U) \subset U$.

And now we can prove the result:

- By passing to a subsequence if necessary, we can assume that

· $m_j = n_j - n_{j-1} \underset{j}{\to} \infty$

· f^{m_j} converges normally on U to a limit function ψ.

Then:

$$\psi(\phi(z)) = \lim_{j\to\infty} f^{m_j}(f^{n_{j-1}}(z)) = \lim_{j\to\infty} f^{n_j}(z) = \phi(z)$$

Since $\phi(z)$ is non-constant, by the Open Mapping Theorem $\phi(U) \subset U$ is open. Observe that $\psi(w) = w$ for every $w \in \phi(U)$ and $\phi(U)$ has limit points, hence by Analytic Continuation ψ is the identity map and we obtain (b).

- Suppose that exist $z, w \in U$ such that $f(z) = f(w)$, then

$$z = \lim_{j\to\infty} f^{m_j}(z) = \lim_{j\to\infty} f^{m_j-1}(f(z)) = \lim_{j\to\infty} f^{m_j-1}(f(w)) = \lim_{j\to\infty} f^{m_j}(w) = w$$

then $z = w$ and therefore f is conformal, hence (a) is proved). Recall that we can also prove that f is bijective, in fact, if $w \in U$, then $id(z) - w$ is non-constant and by Hurwitz's theorem exists $j_0 \geq 1$ such that $f^{m_j}(z) - w$ has zeros for all $j \geq j_0$, therefore exists $\tilde{z} \in U$ such that $f^{m_j}(\tilde{z}) = w$, which implies that $w \in f(U)$.

- Since f is a conformal and bijective, we can consider $g\colon U \to U$ the inverse of f. By passing to a subsequence of the n_j if necessary, we can assume that $g^{n_j} \to \varphi$ normally on U. Then:

$$\varphi(\phi(z)) = \lim_{j\to\infty} g^{n_j}(f^{n_j}(z)) = z$$

which implies that both ϕ and φ are not constant and (d) is proved.

- Furthermore, in the limit functions above ϕ is injective (because $\phi(z) = \phi(w)$ implies that $z = \varphi(\phi(z)) = \varphi(\phi(w)) = w$) and φ is surjective.

  Finally, $(\phi \circ \varphi)(\phi(z)) = \phi(\varphi(\phi(z))) = \phi(z)$, which means that $\phi \circ \varphi$ is the identity in $\phi(U)$ and by Analytic Continuation $\phi \circ \varphi = id_U$, therefore ϕ is surjective and φ is injective. (c) has also been proved.

□

The goal now is to prove Theorem 2.28 in the case of constant limit functions, which in fact are the ones that concerns us in Chapter 3.

As it has already been said, it is enough to restrict ourselves to the forward invariant component case. First, we need a technical lemma.

Lemma 2.31. *Let $U \subset F(f)$ be a forward invariant component. If there exists a constant limit function ξ, then either ξ is a fixed point of f or $\xi = \infty$.*

Proof. If $\xi \neq \infty$, then exists $n_k \vdash n$ such that $f^{n_k}(z)$ converges normally on U to ξ and

$$f(\xi) = f\left(\lim_{k\to\infty} f^{n_k}(z)\right) = \lim_{k\to\infty} f^{n_k}(f(z))$$

since by hypothesis $f(z) \in U$, then

$$f(\xi) = \lim_{k\to\infty} f^{n_k}(f(z)) = \xi.$$

$\square$

We also need to show that in a forward invariant Fatou component with constant limit functions, in fact, we only have one limit function.

Theorem 2.32. *Suppose that U is a forward invariant component of $F(f)$ such that $\mathcal{L}(U)$ only contains constant limit functions. Then $\mathcal{L}(U)$ contains exactly one function ξ such that $f^n \to \xi$ normally on U.*

Proof. Since we are assuming that $\mathcal{L}(U)$ only contains constant limit functions, we regard $\xi \in \mathcal{L}(U)$ as a (constant) function and as a value of $\mathbb{C}_\infty$.

Suppose that a limit function $\xi \in U$, then $f(\xi) = \xi$ and exists $f^{n_j} \vdash f^n$ such that $f^{n_j} \to \xi$ normally on U.

Consider a disk $D(\xi, r)$ such that $\overline{D(\xi, r)} \subset U$. Our hypothesis assures us that there is $j_0 \geq 1$ such that

$$f^{n_{j_0}}(\overline{D(\xi, r)}) \subset D(\xi, r)$$

Since ξ is a fixed point, by Theorem A.13 (Schwarz's Lemma) $|(f^{n_{j_0}})'(\xi)| < 1$, and then $|f'(\xi)| < 1$, therefore ξ is attracting and in a neighborhood of ξ the sequence f^n converges normally to ξ, then by Analytic Continuation $f^n \to \xi$ normally on U and there is just one limit function.

Suppose that every constant limit function $\xi \in \mathcal{L}(U)$ satisfies $\xi \in \partial U$. Consider a connected compact set $K \subset U$, enlarging K if necessary we can assume that K contains a pair of points z and $f(z)$, then $f(K)$ meets K, $f^2(K)$ meets $f(K)$, etc. Thus for any n_0, the set

$$\bigcup_{n=n_0}^{\infty} f^n(K)$$

is connected.

Since the fixed points of f are isolated (they are the zeros of $f(z) - z$), we can consider a collection of pairwise disjoint open neighborhoods $\{V_j\}_j$ of the fixed points.

Suppose that exists $n_j \vdash n$ such that each $f^{n_j}(K)$ meets

$$\mathbb{C} \setminus \bigcup_{j=1}^{\infty} V_j,$$

then no subsequence can converge uniformly on K to one of the fixed points (which it has to), therefore exists $n_0 \geq 1$ such that

$$\bigcup_{n=n_0}^{\infty} f^n(K) \subset \bigcup_{j=1}^{\infty} V_j.$$

Since $\bigcup_{n=n_0}^{\infty} f^n(K)$ is connected it lies on exactly one V_j, hence the fixed point $\xi_j \in V_j$ is the only limit function in $\mathcal{L}(U)$.

Finally, since for every $n_k \vdash n$ exists $n_{k_j} \vdash n_k$ that converges normally to ξ_j on U, the whole sequence f^n converges normally to ξ_j on U. $\square$

Theorem 2.32 already proves the cases (a) and (e) of Theorem 2.28. Finally, we prove that in the remaining case we have a parabolic domain.

Theorem 2.33. *Suppose that U is a forward invariant component of $F(f)$ and that there is a constant limit function $\xi \in \mathcal{L}(U)$ such that $\xi \in \partial U \cap \mathbb{C}$. Then ξ is a rationally indifferent fixed point of f such that:*

(a) $f^n \to \xi$ *normally on* U.

(b) $f'(\xi) = 1$.

Proof. Theorem 2.32 shows us that $\{\xi\} = \mathcal{L}(U)$, therefore since for all $n_k \vdash n$, exists $n_{k_j} \vdash n_k$ such that

$$f^{n_{k_j}} \xrightarrow[j\to\infty]{} \xi \in \partial U,$$

then $f^n \xrightarrow[j\to\infty]{} \xi$ (in particular the fixed point cannot be repelling).

Since ξ cannot be neither repelling nor attracting, we must have $|f'(\xi)| = 1$, we just have to show that $f'(\xi) = 1$.

Without loss of generality we can suppose that $\xi = 0$ and that $W \cap D(0, r)$ is forward invariant. Consider $z_0 \in W$ and define the function:

$$\varphi_n(z) = \frac{f^n(z)}{f^n(z_0)}$$

The normality of $\{f^n\}$ implies that $\{\varphi_n\}_n$ is normal on W, hence some $\varphi_{n_j} \vdash \varphi_n$ converges normally to $\varphi \in \mathcal{H}(W)$.

Observe that

$$\varphi_n(f(z)) = \frac{f^{n+1}(z)}{f^n(z_0)} = \varphi_n(z)\frac{f^{n+1}(z)}{f^n(z)} = \varphi_n(z)\frac{f(f^n(z)) - f(0)}{f^n(z) - 0} \xrightarrow[n\to\infty]{} \lambda\varphi(z)$$

Hence $\varphi(f(z)) = \lambda\varphi(z)$.

Since $\lambda \neq 0$ we can assume that f is injective in W and then φ_n is also injective in W. Thus by Theorem A.42 (Hurwitz) either φ is constant or injective.

- If φ is constant, then $\varphi_n(z_0) = 1$ implies that $\varphi \equiv 1$. Hence $\lambda = 1$, because we have $1 = \varphi(f(z)) = \lambda\varphi(z) = \lambda$.
- If φ is injective, it has an inverse φ^{-1} which maps $\varphi(W)$ onto W, and then

  $$\varphi(f^n(z_0)) = \lambda^n\varphi(z_0) = \lambda^n.$$

  Since $|\lambda| = 1$, there exists an increasing sequence $m_j \to \infty$ of integers such that $\lambda^{m_j} \to 1$[2] and then $\varphi(f^{m_j}(z_0)) \to 1$.

  But $\varphi(W)$ contains the point $1 = \varphi(z_0)$, and thus for j large enough $\varphi(f^{m_j}(z_0)) \in \varphi(W)$, for these j then

  $$f^{m_j}(z_0) = \varphi^{-1}(\lambda^{m_j}) \to \varphi^{-1}(1) = z_0$$

  which contradicts that $f^n \to 0$. So φ is constant and $\lambda = 1$.

□

[2] Trivial in the case that λ is a root of the unity. If not, then $\{\lambda^n : n \in \mathbb{N}\}$ is dense in $\{|z| = 1\}$ and it also holds.

2.2.2 Topological Properties

The topological properties of the different Fatou components play a fundamental role in the dynamics of our functions. The results introduced before and some classical results of Complex Analysis are, in fact, crucial at this point.

Definition 2.34 (Winding number)**.** *We denote by $ind(\gamma, a)$ the **index** of a closed curve $\gamma \subset \mathbb{C}$ with respect to a point a (also known as the **winding number** of γ about a).*

The key lemma, which together with Theorem 2.28 leads us to our goal is:

Lemma 2.35. *Let $f \in E$ and U be a multiply connected component of $F(f)$. Suppose that γ is a Jordan curve that is not contractible in U. Then:*

(a) $f^n \to \infty$ normally on U.

(b) $ind(f^n(\gamma), 0) > 0$ for n large enough.

Proof.

(a) Consider a compact subset $K \subset U$ that is multiply connected and suppose that exists $n_k \vdash n$ such that $|f^{n_k}| \leq M < \infty$ on a Jordan curve Γ (with $\Gamma^* \subset K$) that is not contractible in U. By the maximum principle:

$$|f^{n_k}(z)| \leq M \qquad \forall z \in int(\Gamma)$$

but that's not possible because $int(\Gamma) \cap J(f) \neq \emptyset$. In fact, if $z \in int(\Gamma) \cap J(f)$ and $D \ni z$ is a neighborhood of z, $D \subset int(\Gamma)$, then $\bigcup_{n \in \mathbb{N}} f^n(D)$ covers $\mathbb{C}_\infty$ with at most two exceptions, in particular, exists $n_0 \in \mathbb{N}$ such that

$$\sup_{\substack{z \in D \\ n \leq n_0}} |f^n(z)| \geq M + 1$$

which is a contradiction. Therefore $f^n \to \infty$ normally on U.

(b) Suppose that exists $n_k \vdash n$ such that $ind(f^{n_k}(\gamma), 0) = 0$ then by the Argument Principle f^{n_k} does not have zeros in $int(\gamma)$, by the Minimum Principle f^{n_k} attains its minimum in γ, since by (a) $f^{n_k} \to \infty$ uniformly on γ, we obtain that $f^{n_k} \to \infty$ on $int(\gamma)$. But again, since $int(\gamma) \cap J(f) \neq \emptyset$, this is a contradiction (using again Proposition 2.9).

□

Lemma 2.35 has a great variety of important applications. It characterizes the behavior of an entire function in a multiply connected component of the Fatou set.

Theorem 2.36. *If $f \in E$, then $F(f)$ does not have Herman rings.*

Proof. Lemma 2.35 shows us that if a component is multiply connected and f is entire, then $f^n \to \infty$ normally on U, which is not compatible with being conjugate to an irrational rotation as in Theorem 2.28 (d). □

Observe that, in fact, to have Herman rings we need poles.

The following result is important in the study of the Exponential Family in Chapter 3.

Proposition 2.37. *Suppose that $f \in E$ is bounded on some curve Γ going to ∞, then all components of $F(f)$ are simply connected.*

Proof. Suppose that there exists $U \subset F(f)$ that is multiply connected and consider a Jordan curve γ that is not contractible in U, then by Lemma 2.35, $\exists n_0 \geq 1$ such that $f^n(\gamma) \cap \Gamma \neq \emptyset$ for all $n \geq n_0$. Consider $z_n \in f^n(\gamma) \cap \Gamma$ and $\{z_n\}_{n \geq n_0}$, since

$$f(z_n) \in f^{n+1}(\gamma) \xrightarrow[n\to\infty]{} \infty$$

then $f(z_n) \xrightarrow[n\to\infty]{} \infty$, which is a contradiction with the hypothesis. □

Example 2.38. *If we consider $\Gamma(t) = -t$, then for $E_\lambda(z) = \lambda e^z$, $\lambda \neq 0$ the curve $E_\lambda(\Gamma(t))$ is bounded. Therefore all components of $F(E_\lambda)$ are simply connected.*

We end this subsection with a couple of results concerning the connectivity of the components of the Fatou set.

Theorem 2.39. *Let $f \in E$, then any multiply connected component of $F(f)$ is a wandering domain. In other words, any preperiodic component of $F(f)$ is simply connected.*

Proof. Lemma 2.35 tells us that given a multiply connected component U of $F(f)$ and a Jordan curve γ contained in U, the successive iterates of U converges normally to ∞, furthermore for n large enough the winding number of $f^n(\gamma)$ (which is a closed curve) is nonzero and $f^n(\gamma) \to \infty$. Therefore U cannot be preperiodic and the claim follows. □

Example 2.40. *The exponential family $E_\lambda(z) = \lambda e^z$ does not have Wandering domains. This follows from Example 2.38 and the previous Theorem.*

Theorem 2.41. *Let f be a meromorphic function and U be an invariant component of $F(f)$. Then the connectivity of U has value either $1, 2$ or ∞. The value 2 only takes places when U is a Herman ring.*

The proof can be found in [BKL3].

2.3 Singularities of the Inverse

The points where the inverse function is not well-defined play a fundamental role in the dynamical behavior of our function. In fact, they are closely related with the different Fatou components that we can have.

We consider $f \in \mathcal{M}(\mathbb{C})$ unless we state the opposite.

Definition 2.42 (Singular value)**.** *We say that $a \in \mathbb{C}$ is a **singular value** if some branch of f^{-1} is not well-defined (holomorphic and injective) in a neighborhood of $a \in \mathbb{C}$.*

The goal is to analyze the different singular values that we can have.

Let $a \in \mathbb{C}_\infty$ and denote by $D(a,r)$ the disk of radius $r > 0$, in the spherical metric[3], centered at a. For every $r > 0$, choose a connected component $U(r)$ of $f^{-1}(D(a,r))$ such that $r_1 < r_2$ implies $U(r_1) \subset U(r_2)$. Obtaining a function:

$$U : r \mapsto U(r).$$

We have two possibilities:

(i) $\cap_{r>0} U(r) = \{z\}$, $z \in \mathbb{C}$. Then $a = f(z)$. If $a \in \mathbb{C}$ and $f'(z) \neq 0$ or if $a = \infty$ and z is a simple pole, then we say that z is an **ordinary or regular point**.

If $a \in \mathbb{C}$ and $f'(z) = 0$ or if $a = \infty$ and z is a multiple pole of f, then z is called a **critical point** and a is called a **critical value**.

(ii) $\cap_{r>0} U(r) = \emptyset$. Then we say that our choice $r \mapsto U(r)$ defines a (transcendental) **singularity** (for simplicity we just call such U a singularity). For every $r > 0$, the open set $U(r) \subset \mathbb{C}$ is called a **neighborhood of the singularity** U. So if $z_k \in \mathbb{C}$, we say that $z_k \to U$ if for every $\varepsilon > 0$, $\exists k_0$ such that $z_k \in U(\varepsilon)$ for all $k \geq k_0$.

Meanwhile the first type is elementary to determine with this characterization, the second is not, so we are urged to obtain a criterion to decide when we have a singularity.

Definition 2.43 (Asymptotic Value)**.** *We say that $a \in \mathbb{C}$ is an **asymptotic value** if there exists a curve γ such that*

$$\gamma(t) \xrightarrow[t\to\infty]{} \infty \quad \text{and} \quad f(\gamma(t)) \xrightarrow[t\to\infty]{} a.$$

*We call γ an **asymptotic path or curve** of a.*

Proposition 2.44. *A point $a \in \mathbb{C}$ is an asymptotic value of f, if and only if, there is a singularity U (as in (ii)).*

Proof. Let γ be an asymptotic curve, on which $f(\gamma(t)) \to a$. Then, for every $r > 0$, the tail of γ where $f(\gamma(t)) \in D(a,r)$ belongs to $f^{-1}(D(a,r))$. We define $U(r)$ as the connected component of $f^{-1}(D(a,r))$ that contains this tail. So U is a singularity.

We want to construct an explicit asymptotic curve. Consider a sequence $\{r_k\}_k$, which decreases to 0 and a sequence $\{z_k\}$ such that $z_k \in U(r_k)$. Since for every $n \geq k$, $z_n \in U(r_k)$ (because $U(r_k) \supset U(r_n)$) and $U(r_k)$ is connected, we can connect z_k to z_{k+1} by a curve γ_k such that $\gamma_k^* \subset U(r_k)$. Then $\gamma = \cup_k \gamma_k$ is an asymptotic curve (since $\cap_{r>0} U(r) = \emptyset$, then $\lim_k z_k = \infty$ and $\cup_{n \geq k}\{z_k\} \subset f^{-1}(D(a,r_k))$, so $\lim_k f(z_k) = a$). □

Example 2.45. *Consider $E_\lambda(z) = \lambda e^z$, where $\lambda \in \mathbb{C}^*$, then*

- *Let $\gamma(t) = -t$, then $E_\lambda(\gamma(t)) = \lambda e^{-t} \to 0$ as $t \to \infty$. So 0 is an asymptotic value of E_λ.*

[3] See A.2.1 for the definition and main properties.

- *Let $\alpha(t) = t$, then $E_\lambda(\alpha(t)) = \lambda e^t \to \infty$ as $t \to \infty$. So ∞ is an asymptotic value of E_λ.*

The behavior near a critical point is important in holomorphic dynamics, but in this document, we focus mainly in asymptotic values. However, any reader interested is encouraged to read A.1.3.

2.3.1 Picard Values

In some cases we can detect asymptotic values without constructing an asymptotic curve. We know that omitted points of a function play an important role due to Theorem A.62 (Picard's Big Theorem) in the behavior of a function, in fact, they are also asymptotic values.

We regard f as a transcendental meromorphic function.

Definition 2.46 (Picard Value)**.** *We say that $a \in \mathbb{C}$ is a **Picard exceptional value** or **Picard value** of f if $a \notin f(\mathbb{C})$. We write $a \in PV(f)$.*

Due to Theorem A.61 (Picard's Big Theorem), the set $PV(f)$ has at most one or two finite points, depending on if f is entire or meromorphic respectively.

Theorem 2.47. *Every $a \in PV(f)$ is an asymptotic value of f.*

See [Sch] for a proof.

Example 2.48. *If $f(z) = a + (z-a)^{-m} e^{g(z)}$, for $m \in \mathbb{N}$ and $g \in \mathcal{H}(\mathbb{C})$. Then:*

- *f has an isolated singularity at $z = a$, which is a pole of order m.*
- *f does not assume the value a in $\mathbb{C}$ ($a = f(z) \iff e^{g(z)} = 0$).*

So $a \in PV(f)$.

2.3.2 The Role of the Singularities of the Inverse

This subsection is devoted to showing why the singularities of the inverse play a fundamental role in the dynamics of a given function. Again, we focus on the case of periodic components with constant limit functions.

Definition 2.49 (Singular Set)**.** *We define the set of **singular values***

$$S(f) = \overline{\{\textit{critical and asymptotic values}\}}.$$

*A **postsingular point** is a point on the orbit of a singular value.*

The next result shows the relevance of $S(f)$ in the case of attracting and parabolic domains, which are the two cases that concern us in Chapter 3.

Theorem 2.50 (Role of the Singular Values)**.** *Let f be a meromorphic function.*

(a) Suppose that f has an attracting fixed point or cycle. Then there is at least one singular value in the immediate basin of attraction of this point.

(b) *Suppose that f has a parabolic fixed point. Then there is at least one singular value in the immediate basin of attraction of this point.*

Proof. We argue it by contradiction.

(a) Suppose that z_0 is an attracting fixed point. It can be assumed that $f'(z_0) \neq 0$ (otherwise the result is trivial).

Let U_0 be a bounded neighborhood of z_0 contained in $\mathcal{A}$, the immediate basin of attraction of z_0, such that $f(U_0) \subset U_0$. It can also be assumed that f is one-to-one on U_0.

Now we pull back by a branch of f^{-1}, more precisely, we consider U_1 as the preimage of U_0 that contains U_0. Since we are supposing that we do not have singular values in the basin, U_1 cannot be an unbounded set (otherwise an asymptotic path can be obtained), so U_1 is bounded. Again, since by hypothesis there are no critical points in $\mathcal{A}$, we can assume that $f_{|U_1}$ is one-to-one.

Iterating this procedure, a sequence of open bounded subsets $\{U_n\}_n$ is obtained such that $U_{n+1} \subset U_n$ and $f\colon U_{n+1} \to U_n$ is one-to-one.

Let $W = \cup_{n=0}^{\infty} U_n$, then

$$f\colon W \to W$$

is one-to-one. Which means that

$$f^{-1}\colon W \to W$$

is well-defined and holomorphic.

$J(f)$ is infinite and $W \subset \mathbb{C} \setminus J(f)$, therefore the family $\{f^{-1}\}_{n\in\mathbb{N}}$ misses (far) more than two points. By Montel's Theorem $\{f^{-n}\}_n$ is normal in W, but since z_0 is attracting for f, it is repelling for f^{-1} and $z_0 \in W$, which contradicts the normality of $\{f^{-n}\}_n$.

(b) The construction in this case is similar. Consider the n attracting petals provided by Lemma 1.16,

$$\mathcal{P}_1, \ldots, \mathcal{P}_n.$$

Consider V_0 an open bounded neighborhood of z_0 such that

$$U_0 = V_0 \cap (\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_n)$$

satisfies:

- $f_{|U_0}$ is one-to-one.
- $f(U_0) \subset U_0$.

The procedure is the same; consider U_1 as the preimage of U_0 that contains U_0. Since we do not have singular values in $\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_n$, U_1 cannot be unbounded and since we do not have critical values, we can assume that $f_{|U_1}$ is one-to-one.

If we keep iterating this procedure, we obtain a sequence $\{U_n\}_{n\geq 0}$ of open bounded sets such that.

- $U_n \subset U_{n+1}$.
- $f_{|U_n}$ is one-to-one.
- $f(U_{n+1}) \subset U_n$.

Then $W = \cup_{n \geq 0} U_n$ is such that

$$f\colon W \to W$$

is one-to-one. Hence $f^{-1}\colon W \to W$ is well-defined and holomorphic. Again $W \subset F(f)$ misses (far) more than three points, thus $\{f^{-n}\}_n$ is a normal family. We also know that $\{f^{-n}\}_n$ is normal on the attracting petals of f^{-1} (which are the repelling for f). Hence by the Theorem 1.17 (Leau-Fatou Flower Theorem) $\{f^{-n}\}_n$ is normal on a punctured neighborhood of z_0. Since $f(z_0) = z_0$ is defined, then $\{f^{-n}\}_n$ is normal on a neighborhood of z_0 (by the Maximum Modulus Principle and Montel's Theorem), which contradicts that $z_0 \in J(f)$ (because is parabolic).

□

Remark 2.51. *As we have already seen in the proof, in Theorem 2.50 we need a **finite** singular value in both cases (they are the ones that play a dynamical role).*

The next result shows that the singular values also play a fundamental role when we have Siegel disks or Herman rings.

Theorem 2.52. *Let f be a meromorphic function, and let $C = \{U_0, U_1, \ldots, U_{p-1}\}$ be a periodic cycle of components of $F(f)$. If C is a cycle of Siegel disks or Herman rings, then*

$$\partial U_j \subset \overline{\mathcal{O}_f^+(S(f))} \text{ for all } j \in \{0, 1, \ldots, p-1\}$$

This result is not proved in this document. See [Ber1] for references in order to find the proof.

The singular values are also relevant for Baker domains. We have the following result.

Theorem 2.53. *Let f be a meromorphic function, and let $\{U_0, \ldots, U_{p-1}\}$ be a periodic cycle of Baker domains of f. Denote by z_j the limit corresponding to U_j and define $z_p = z_0$. Then*

$$z_j \in \bigcup_{n=0}^{p-1} f^{-n}(\infty) \quad \forall j \in \{0, \ldots, p-1\}$$

and $z_j = \infty$ for at least one $j \in \{0, \ldots, p-1\}$. If $z_j = \infty$, then z_{j+1} is an asymptotic value of f.

Proof. The limit z_j corresponding to U_j is such that

$$f^{np}(z) \xrightarrow[n\to\infty]{} z_j \quad \text{for } z \in U_j.$$

It is clear that $z_{j+1} = f(z_j)$ if $z_j \neq \infty$ (and we are setting $z_p = z_0$).

Since $f^p(z_j)$ is not defined, there exists $j \in \{0, \ldots, p-1\}$ such that $z_j = \infty$, and then, for every $j \in \{0, \ldots, p-1\}$ there exists $l = l(j) \in \{0, \ldots, p-1\}$ such that $f^l(z_j) = \infty$, and then

$$\{z_0, \ldots, z_{p-1}\} \subset \bigcup_{n=0}^{p-1} f^{-n}(\infty).$$

Finally, choose $w_0 \in U_0$ and a curve $\sigma \subset U_0$ that joins w_0 and $f^p(w_0)$, defining

$$\gamma_0 = \bigcup_{n=0}^{\infty} f^{np}(\sigma) \text{ and } \gamma_j = f^j(\gamma_0) \ , \ j \in \{0, \ldots, p-1\}$$

we obtain curves γ_j in U_j that tend to z_j such that

$$f^p(\gamma_j) \subset \gamma_j \text{ and } f^p(z) \xrightarrow[z \to z_j \text{ in } \gamma_j]{} z_j$$

hence,

$$f(z) \xrightarrow[z \to z_j \text{ in } \gamma_j]{} z_{j+1}$$

so if $z_j = \infty$, then z_{j+1} is an asymptotic value. □

The following two results are a direct consequence of Theorem 2.53.

Corollary 2.54. *Let f be a meromorphic function and let*

$$\{U_0, \ldots, U_{p-1}\}$$

be a periodic cycle of Baker domains of f. Then exists $j \in \{0, \ldots, p-1\}$ such that

$$\partial U_j \cap S(f) \neq \emptyset$$

Corollary 2.55. *Let f be a meromorphic function and let U be a (forward invariant) Baker domain. Then $\infty \in S(f)$ and U is unbounded.*

2.4 Wandering Domains

This Fatou components have been unnoticed during many years due to their absence in rational maps. The no-Wandering Domains Theorem was first proven by Sullivan in 1985 in his famous paper *Quasiconformal Homeomorphisms and Dynamics I. Solution of the Fatou-Julia Problem on Wandering Domains* (see [Sul]). This paper introduced quasiconformal analysis techniques into holomorphic dynamics, which meant remarkable advances in the field.

However, transcendental functions do have Wandering Domains. For example the function

$$g(z) = z + \sin(z) + 2\pi$$

has a Wandering Domain, which is represented in Figure 2.1.

They are the least understood Fatou components and still subject of current research. Their analysis is out of the scope of this project. However, some special classes of transcendental maps do not have Wandering Domains.

The maps studied in Chapter 3 are some examples, so we give sufficient conditions for not having these Fatou components.

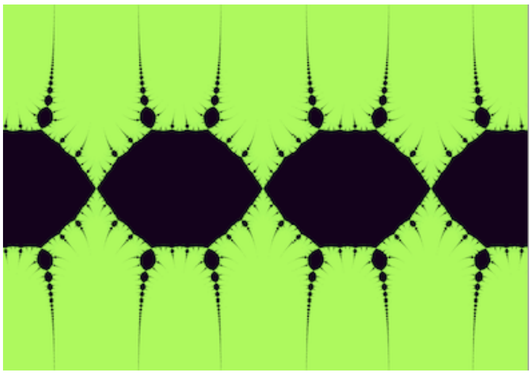

Figure 2.1: In black, the Wandering Domain of $g(z)$.

Definition 2.56 (Classes of Meromorphic Functions)**.** *We define the following classes of meromorphic functions:*

- $S = \{f\colon f$ *has only finitely many critical and asymptotic values*$\}$.
- $F = \{f\colon f(z) = z + r(z)e^{p(z)}$, *where* $r(z)$ *is rational and* $p(z)$ *a polynomial*$\}$.
- $R = \{f\colon f'(z) = r(z)(f(z) - z)^2$ *or* $f'(z) = r(z)(f(z) - z)(f(z) - \tau)$ *where* $r(z)$ *is rational and* $\tau \in \mathbb{C}\}$.

Theorem 2.57. *Functions in* S, F *and* R *do not have Wandering Domains.*

Theorem 2.58. *Functions in* S *do not have Baker domains.*

For references to find the corresponding proofs of this theorems see [Ber1].

We refer to [BKL1, BKL2, BKL3, BKL4] for general further details and to [BEFRS] for a classification of simply connected Wandering Domains.

Chapter 3

Some Families of Functions

The aim of this final chapter is to study two families of transcendental functions, whose maps belong to three different families of transcendental functions, E, P and M introduced in Chapter 2.

3.1 The Exponential Family

We start with the Exponential Family $E_\lambda(z) = \lambda e^z$, for $\lambda \neq 0$, which is the simplest and best understood transcendental entire function. It displays some patterns (Cantor Bouquets) that appear in many families of transcendental functions, a reason for which this family is considered a model. The references used for this section are [Dev, DT].

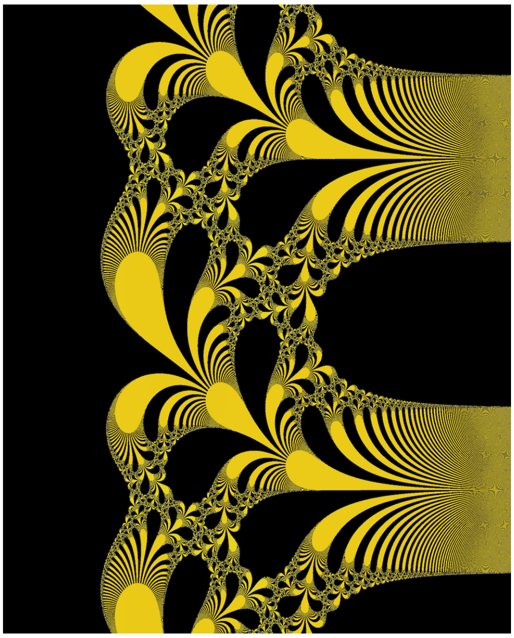

Figure 3.1: $J(E_\lambda)$ in yellow for $\lambda = 0.3 + 6i$.

The following Proposition summarizes several facts we have already discussed.

Proposition 3.1. *Let $E_\lambda(z) = \lambda e^z$. Then,*

(a) $E_\lambda(z)$ has only two asymptotic values, which are 0 and ∞.

(b) E_λ can have at most one attracting cycle.

(c) All the components of $F(E_\lambda)$ are simply connected.

(d) $F(E_\lambda)$ dos not have wandering domains nor Baker domains.

Proof. (a) and (b) are a consequence of Example 2.45 together with the fact that only one of the asymptotic values is finite. (c) is Example 2.38. (d) is a consequence of Example 2.40 and Theorem 2.58 together with (a). □

Definition 3.2 (Escaping set)**.** *We define the* ***escaping set*** *of a meromorphic function f as*

$$I(f) = \left\{ z \in \mathbb{C} \colon f^n(z) \xrightarrow[n\to\infty]{} \infty \right\}.$$

Theorem 3.3. $J(E_\lambda) = \overline{I(E_\lambda)}$[1].

Proof. Since we do not have Baker domains, it is clear that $I(E_\lambda) \subset J(E_\lambda)$ and $I(E) \neq \emptyset$ because $|E_\lambda(z)| = |\lambda| e^{Re(z)}$, and the real dynamical system $g(x) = |\lambda| e^x$ has orbits that tend to infinity.

Since any $x_0 \in \mathbb{R}^+ \cap J(E_\lambda)$ is not exceptional, using Proposition 2.11 we obtain $\overline{I(E_\lambda)} = J(E_\lambda)$. □

Observe that the picture that we obtain is just an approximation of the Julia set. In fact, the essential singularity at infinity puts us in a difficult situation to decide whether an orbit tends to infinity or not.

The importance of the Singular Values has already been shown, recall that in Theorem 2.50 and Theorem 2.52, we actually need a **finite** singular value. In our case the only finite singular value is $z = 0$ and its dynamical behavior plays a fundamental role.

Theorem 3.4. *Let $E_\lambda(z) = \lambda e^z$. Then,*

(a) If the orbit of 0 tends to ∞ or it is preperiodic, then $J(E_\lambda) = \mathbb{C}_\infty$.

(b) If E_λ has an attracting or parabolic periodic orbit, then $J(E_\lambda)$ has empty interior. In fact, $J(E_\lambda) \subset \{z \in \mathbb{C} \colon Re(z) \geq \nu\}$, where $\nu \in \mathbb{R}$.

[1] In order to draw Figure 3.1 we have used the following algorithm:

1. Compute the orbit of $z \in \mathbb{C}$ up to a given transient N.
2. If the orbit of z enters $\{z \in \mathbb{C} \colon Re(z) \geq 50\}$ at an iteration $j \leq N$, then $z \in J(E_\lambda)$ and color z depending on j.
3. If at the iteration N the orbit has not entered $\{z \in \mathbb{C} \colon Re(z) \geq 50\}$, then color z black (and we assume that $z \notin J(E_\lambda)$)

Proof. Due to Theorem 2.50 and Theorem 2.52 in the first two cases $F(E_\lambda) = \emptyset$ because an attracting periodic orbit must have a singular value in its basin of attraction (the same with a Leau domain) and if we have a cycle $\{U_0, \dots, U_{p-1}\}$ of Siegel disks, then

$$\partial U_j \subset \overline{\mathcal{O}^+_{E_\lambda}(\{0\})}$$

but $\overline{\mathcal{O}^+_{E_\lambda}(\{0\})}$ is finite and hence we cannot have Siegel disks.

In the remaining case, we have $F(E_\lambda) \neq \emptyset$ and thus, $J(E_\lambda)$ has empty interior. Also recall that we can find a disk centered at $z = 0$ which is contained in $F(E_\lambda)$, its preimage $\{z \in \mathbb{C} \colon Re(z) < \nu\}$ is also in $F(E_\lambda)$, thus

$$J(E_\lambda) \subset \{z \in \mathbb{C} \colon Re(z) \geq \nu\}$$

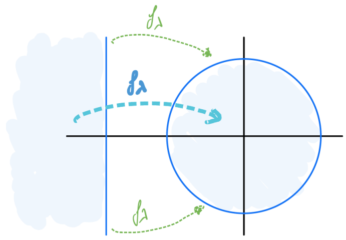

Figure 3.2: Preimage of the disk.

□

Definition 3.5 (Misiurewicz point)**.** *When* 0 *is preperiodic, by Theorem 3.4,* $J(E_\lambda) = \mathbb{C}_\infty$ *and we say that* λ *is a* ***Misiurewicz point****.*

Example 3.6. $\lambda_{1,k} = 2k\pi i$ *and* $\lambda_{2,k} = (2k+1)\pi i$ *for* $k \in \mathbb{Z}$ *are Misiurewicz points, because*

$$0 \mapsto 2k\pi i \mapsto 2k\pi i \ \mapsto \cdots$$

$$0 \mapsto (2k+1)\pi i \mapsto -(2k+1)\pi i \mapsto -(2k+1)\pi i \mapsto \cdots$$

3.1.1 Cantor Bouquets

The goal in this section is to understand how the orbits behave in the Julia Set of E_λ and define a structure that appears in many dynamical systems. To do so, we restrict to the case where the parameter λ is real and $0 < \lambda < 1/e$.

Definition 3.7 (Cantor Set)**.** *We say that a set* $C \subset \mathbb{C}$ *is a* ***Cantor Set*** *if* C *is compact, perfect and totally disconnected.*

If we consider a positive integer N and the set

$$\Sigma_N \coloneqq \{(s_0, s_1, s_2, \dots) \colon s_j \in \mathbb{Z} \ , \ |s_j| \leq N\}$$

then we can define a topology in Σ_N that makes this set a Cantor Set.

In Σ_N we define the shift map

$$\begin{aligned} \sigma \colon \Sigma_N &\longrightarrow \Sigma_N \\ (s_0, s_1, s_2, \dots) &\longmapsto (s_1, s_2, \dots). \end{aligned}$$

Definition 3.8 (Cantor N-Bouquet)**.** *We say that a closed set $C_N \subset \mathbb{C}$ is a* ***Cantor N-Bouquet*** *of a meromorphic function f if:*

- $f(C_N) \subset C_N$.
- *There exists a homeomorphism*

$$h \colon \Sigma_N \times [0, \infty) \longmapsto C_N$$

such that

(a) $(\pi \circ h^{-1} \circ f \circ h)(s,t) = \sigma(s)$ *for all* $t \in [0,\infty]$*, where*

$$\begin{aligned} \pi \colon \Sigma_N \times [0,\infty) &\longrightarrow \Sigma_N \\ (s,t) &\longmapsto s \end{aligned}$$

is the projection on the first component.

(b) $h(s,t) \xrightarrow[t\to\infty]{} \infty$.

(c) $f^n(h(s,t)) \xrightarrow[n\to\infty]{} \infty$ *if* $t > 0$.

Definition 3.9 (Cantor Bouquet)**.** *Given a sequence $\{C_N\}_{N>0}$ of Cantor N-Bouquets such that $C_N \subset C_{N+1}$, we call the set*

$$C_\infty = \overline{\bigcup_{N=1}^{\infty} C_N}$$

a ***Cantor Bouquet****.*

We want to give a sketch of the proof of the following result.

Theorem 3.10. *For $0 < \lambda < 1/e$, $J(E_\lambda)$ is a Cantor Bouquet.*

In order to accomplish this goal, we have to construct a sequence $\{C_N\}_{N>0}$ of Cantor N-Bouquets such that $C_N \subset C_{N+1}$.

Given $N > 0$, we consider $c > 1$ such that

$$E_\lambda(c) > c + (2N+1)\pi$$

and for $j \in \{-N, \dots, 0, \dots, N\}$ the rectangles

$$R_j = \{z \in \mathbb{C} \colon 1 < Re(z) < c \;,\; (2j-1)\pi < Im(z) < (2j+1)\pi\}.$$

Then for each $j \in \{-N, \dots, 0, \dots, N\}$ we have:

$$E_\lambda(R_j) = \{z \in \mathbb{C} \colon \lambda e < |z| < \lambda e^c \;,\; |arg(z)| < \pi\}$$

so by our choice of $c > 1$ we have that for any $j,k \in \{-N, \dots, 0, \dots, N\}$, $R_k \subset E_\lambda(R_j)$.

Now we define

$$B_N := \bigcup_{j=-N}^{N} R_j \quad \text{and} \quad C_N := \{z \in B_N \colon E_\lambda^l(z) \in B_N \text{ for all } l \geq 0\}.$$

The final step is showing that C_N is a Cantor N-Bouquet. To do so, we need to define a homeomorphism

$$h \colon \Sigma_N \times [0, \infty) \longrightarrow C_N$$

we refer to [DT] (*Proposition 2.7*) for the definition of the function and proving that the required conditions hold.

So C_N is a Cantor N-Bouquet and $C_N \subset C_{N+1}$, hence for $0 < \lambda < 1/e$, $J(E_\lambda)$ is a Cantor Bouquet.

This construction has interest for itself, in fact, if we fix an **adress**

$$s = (s_0, s_1, s_2, \dots) \in \Sigma_N$$

then the homeomorphism h gives us a path $h_s(t) = h(s,t) \colon [0, \infty) \to C_N$ such that

$$h_s(t) \xrightarrow[t\to\infty]{} \infty \quad \text{and} \quad E_\lambda^n(h_s(t)) \xrightarrow[n\to\infty]{} \infty$$

we call this path a **ray**.

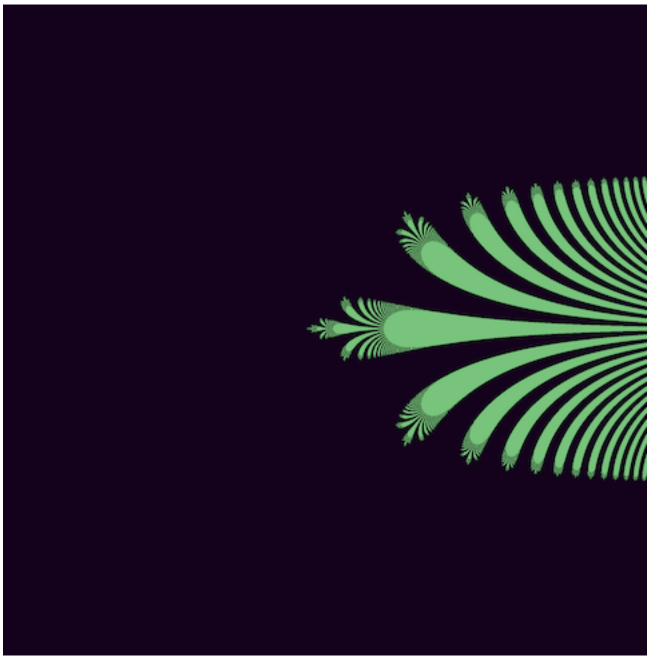

Figure 3.3: $J(E_{0.2})$, in green, is a Cantor Bouquet.

3.2 A Family of Meromorphic Functions

In this section we study the family of functions

$$f_\lambda(z) = \lambda \left(\frac{e^z}{z+1} - 1 \right) \qquad \lambda \neq 0.$$

The results that we present below are not obtained from any written text.

3.2.1 General Properties

We show first that this family covers the two remaining types of transcendental functions.

Proposition 3.11. *$f_\lambda \in M$ for $\lambda \neq 1$ and $f_\lambda \in P$ for $\lambda = 1$.*

Proof. We have $f_\lambda = -1$ (the only pole), if and only if,

$$\frac{e^z}{z+1} = 1 - \frac{1}{\lambda}$$

and this is an omitted value, if and only if, $\lambda = 1$. □

So for $\lambda \neq 1$ we already know that $J(f_\lambda) = \overline{\mathcal{O}^-_{f_\lambda}(\infty)}$, and we expect to have big dynamical changes when λ passes through 1.

The first step is studying the fixed points of this function, which correspond to the solutions of the equation

$$h_\lambda(z) = \frac{\lambda e^z}{(z+1)(z+\lambda)} = 1$$

Since $h_\lambda(z)$ misses 0, by Picard's Theorem we have infinitely many fixed points. Also observe that $f_\lambda(0) = 0$ is always a fixed point and in order to compute the local stability:

$$f'_\lambda(z) = \lambda \left(\frac{e^z}{z+1} - \frac{e^z}{(z+1)^2} \right) = \lambda \frac{ze^z}{(z+1)^2}$$

thus 0 is always a super attracting fixed point and hence we already know that $F(f_\lambda) \neq \emptyset$, which implies that $J(f_\lambda)$ always has empty interior.

Also note that at a fixed point $f_\lambda(z) = z$ we have

$$f'_\lambda(z) = \frac{z}{z+1} \left(\lambda \left(\frac{e^z}{z+1} - 1 \right) + \lambda \right) = \frac{z(z+\lambda)}{z+1}$$

and thus, in the case $z \neq 0$, it would be attracting, if and only if,

$$|z||z+\lambda| < |z+1|$$

The next step is to study the singular values. Their dynamical behavior, as we have seen, plays a crucial role.

- The only critical point is $z = 0$, which is a fixed point.

- We are in a similar situation as in the exponential family, we only have two asymptotic values, which are:
  - * $-\lambda$, with asymptotic paths $\Gamma(t) = -t - 2 + ai$, where $a \in \mathbb{R}$.
$$f_\lambda(\Gamma(t)) = \lambda \frac{e^{-t-2}e^{ia}}{t + ai} - \lambda \xrightarrow[t \to \infty]{} -\lambda$$
  - * ∞, with asymptotic paths $\Gamma(t) = t + ai$, where $a \in \mathbb{R}$.
$$f_\lambda(\Gamma(t)) = \lambda \frac{e^{t}e^{ia}}{t + 1 + ai} - \lambda \xrightarrow[t \to \infty]{} \infty$$

and just one of them is finite. Therefore, by the results in 2.4 Wandering Domains:

Proposition 3.12. *$F(f_\lambda)$ does not have Wandering nor Baker domains.*

As a consequence:

Theorem 3.13. $J(f_\lambda) = \overline{I(f_\lambda)}$.

Proof. It is clear that we have orbits that escape to ∞. Since we do not have Baker domains
$$\emptyset \neq I(f_\lambda) \subset J(f_\lambda)$$
And $I(f_\lambda)$ contains non-exceptional points, by Proposition 2.11 so the result holds. □

Also recall that $0 \in \mathcal{A}(0)$ and by Theorem 2.50 and Theorem 2.52,

- Any attractive basin needs a finite singular value.
- Any rotation domain needs that its boundary is contained in the **postsingular set** $\overline{\mathcal{O}^+_{f_\lambda}(-\lambda)}$.

So we can have at most two periodic cycles of Fatou components[2] for every parameter $\lambda \in \mathbb{C}^*$, one of which is the basin of $z = 0$. The different possibilities for the dynamics of $-\lambda$ play a crucial role.

Since $z = 0$ is a super-attracting fixed point, there is a disk
$$D(0, \varepsilon(\lambda)) \subset \mathcal{A}_\lambda(0)$$
where $\mathcal{A}_\lambda(0)$ is the basin of attraction of 0 of f_λ and necessarily $\varepsilon(\lambda) < 1$, because at $z = -1$ we have a pole.

This simple fact is important in the case that $-\lambda \in \mathcal{A}_\lambda(0)$, for which we only have one periodic Fatou component (the basin of $z = 0$) and its preimages (if any). Hence we can draw an accurate picture of $F(f)$ by considering the points that at a certain iterate land near $z = 0$.

[2]In fact, the other component that we have depending on $\overline{\mathcal{O}^+_{f_\lambda}(-\lambda)}$ can only be an attracting domain or a Siegel disk, i.e., we cannot have Herman rings. That is because Herman rings require at least two singular values in a certain position. The proof of this fact requires some knowledge of Quasiconformal Surgery, which is an important tool in Holomorphic Dynamics. We refer to [BF2] for more details.

Hence it is to our interest to study the set $\Omega_0 = \{\lambda \in \mathbb{C}^*\colon -\lambda \in \mathcal{A}_\lambda(0)\}$.

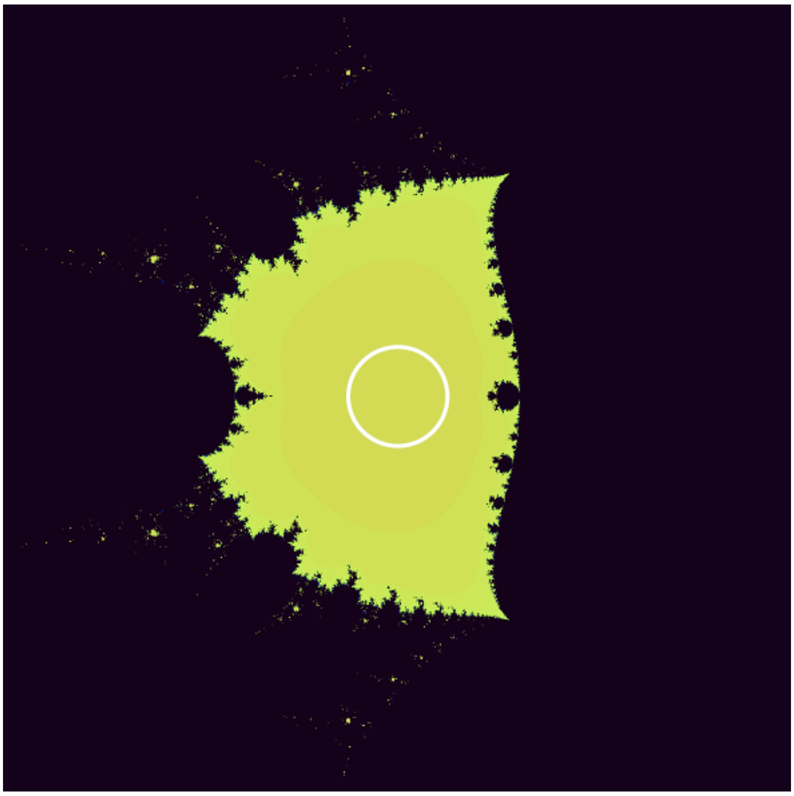

Figure 3.4: In green, Ω_0. $\partial D(0, 1/2)$ is represented in white.

Our first goal is to obtain a lower bound of

$$r_\lambda = \sup\{r > 0\colon D(0,r) \subset \mathcal{A}_\lambda(0)\} < 1.$$

Proposition 3.14. *For every $\lambda \in \mathbb{C}^*$, the function*

$$\varepsilon(\lambda) = \frac{1}{2}\left(2 + |\lambda| - \sqrt{|\lambda|^2 + 4|\lambda|}\right) \in (0,1)$$

gives a lower bound of r_λ, i.e., $\varepsilon(\lambda) \le r_\lambda$.

Proof. For $0 < \varepsilon < 1$ and $|z| < \varepsilon$ we have

$$|f_\lambda(z)| = |f_\lambda(z) - f_\lambda(0)| \le |\lambda| \left(\max_{|z|=\varepsilon} \left|\frac{ze^z}{(z+1)^2}\right|\right) |z|$$

where we have used Theorem A.10 (Maximum Modulus Principle) for f'_λ. For $z = \varepsilon e^{i\theta}$ we have

$$\left|\frac{ze^z}{(z+1)^2}\right| = \frac{\varepsilon e^{\varepsilon\cos(\theta)}}{1 + \varepsilon^2 + 2\varepsilon\cos(\theta)}.$$

If we define the function

$$g_\lambda(\varepsilon, \theta) = |\lambda| \frac{\varepsilon e^{\varepsilon\cos(\theta)}}{1 + \varepsilon^2 + 2\varepsilon\cos(\theta)}$$

the goal is to obtain the maximum ε such that f_λ is a strict contraction in $D(0,\varepsilon)$, because then all points in $D(0,\varepsilon)$ converge to $z=0$ under iteration, i.e., we want to obtain:

$$\sup\{\varepsilon \in (0,1)\colon g_\lambda(\varepsilon,\theta) < 1 \ ,\ \theta \in [0,2\pi)\}$$

since then it follows that $|f_\lambda(z)| < |z|$. We split it in two cases depending on θ.

- For $\theta \in [-\pi/2, \pi/2)$, we have

$$g_\lambda(\varepsilon,\theta) \leq |\lambda| \frac{\varepsilon e^{\varepsilon}}{1+\varepsilon^2} =: g_{\lambda,1}(\varepsilon)$$

- For $\theta \in [\pi/2, 3\pi/2)$, we have

$$g_\lambda(\varepsilon,\theta) \leq |\lambda| \frac{\varepsilon}{(1-\varepsilon)^2} =: g_{\lambda,2}(\varepsilon)$$

Observe now that for $0 < \varepsilon < 1$, we always have $g_{\lambda,1}(\varepsilon) \leq g_{\lambda,2}(\varepsilon)$. Moreover, for $0 < \varepsilon < 1$,

$$|\lambda| \frac{\varepsilon}{(1-\varepsilon)^2} < 1 \iff \varepsilon^2 - (2+|\lambda|)\varepsilon + 1 > 0,$$

and this last polynomial has roots

$$\frac{2+|\lambda| \pm \sqrt{|\lambda|(|\lambda|+4)}}{2}.$$

If we define

$$\varepsilon(\lambda) = \frac{1}{2}\left(2 + |\lambda| - \sqrt{|\lambda|^2 + 4|\lambda|}\right)$$

it is not difficult to check that

- For every $\lambda \in \mathbb{C}^*$, we have $\varepsilon(\lambda) \in (0,1)$.
- For every $\varepsilon \in (0, \varepsilon(\lambda))$, we have $g_{\lambda,2}(\varepsilon) < 1$.

Hence, applying the same argument used in Proposition 1.8, for every $\lambda \in \mathbb{C}^*$ we have $D(0,\varepsilon(\lambda)) \subset \mathcal{A}_\lambda(0)$. $\square$

Corollary 3.15. $D^*(0,1/2) = D(0,1/2) \setminus \{0\} \subset \Omega_0$.

Proof. From the lower bound on r_λ given by Proposition 3.14, we want to study for which λ it holds that $-\lambda \in D(0,\varepsilon(\lambda))$, i.e., $\varepsilon(\lambda) - |\lambda| > 0$, i.e., we want to find $|\lambda|$ such that

$$2 - |\lambda| > \sqrt{|\lambda|^2 + 4|\lambda|}$$

and is immediate to verify that this inequality holds for $|\lambda| < 1/2$. $\square$

Example 3.16. *We have to be careful with this result. For example,* $\lambda = 0.89 \in \Omega_0$ *because*

$$f_{0.89}^{10}(-0.89) \in D(0,0.4) = D(0,\varepsilon(0.89))$$

however, $0.89 \notin D^*(0,1/2)$. *We can draw a clear picture of* $F(f_{0.89}) = \mathcal{A}_{0.89}(0)$ *(see Figure 3.5).*

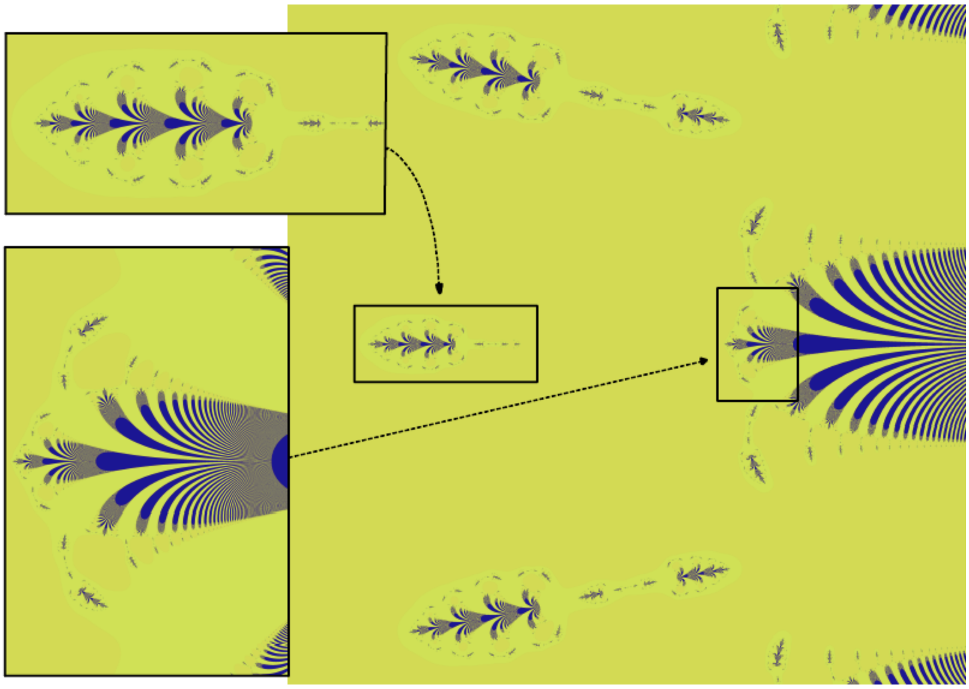

Figure 3.5: In green, $F(f_{0.89})$.

Remark 3.17. *In fact,* $(0, 0.89] \subset \Omega_0$ *and* $0.9 \notin \Omega_0$.

Remark 3.18. *Since* $\overline{f_\lambda(\bar{z})} = f_{\bar{\lambda}}(z)$, *for* $\lambda \in \mathbb{R}^*$ *the Julia set* $J(f_\lambda)$ *is symmetric with respect to the real axis.*

3.2.2 A Journey to ∞

When the orbit of $-\lambda$ goes to ∞, as in the exponential family, we only have one periodic Fatou component (the basin of $z = 0$). Joining this condition with the previous one we obtain the set:

$$\Omega = \Omega_0 \cup \{\lambda \in \mathbb{C}^* : \ -\lambda \in I(f_\lambda)\}$$

and for every $\lambda \in \Omega$, $F(f_\lambda) = \mathcal{A}_\lambda(0)$.

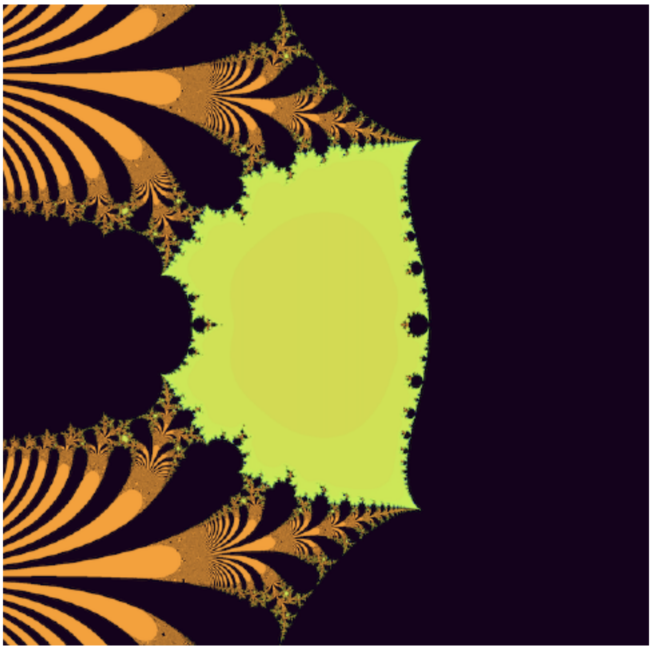

Figure 3.6: In yellow, Ω_0. In orange, the parameters for which $-\lambda \in I(f_\lambda)$.

Example 3.19. *Observe that $\lambda = 1 \in \Omega \setminus \Omega_0$, because $f_\lambda(-1) = \infty$. So for $\lambda = 1$ we have $F(f_1) = \mathcal{A}_1(0)$. Recall that $f_1 \in \mathcal{P}$.*

However in the study of this function we focus on Ω_0 in order to obtain some topological relevant properties.

We have seen in Figure 3.5 some similar patterns with respect to the exponential family in the dynamical plane, at first sight we may say that we have Cantor Bouquets (which can be regarded as the closure of a collection of curves that go to ∞). But in our case we have a richer structure: near every point $w \in \mathcal{O}^-_{f_\lambda}(\infty) \setminus \{\infty\}$ we obtain some "copies" of this "Cantor Bouquet", where the ∞ is now condensed in a single point w.

In Lemma 2.22 we have proved how near a generic point $w \in \mathcal{O}^-_{f_\lambda}(\infty) \setminus \{\infty\}$, where $f^k_\lambda(w) = \infty$ we can find a sequence of repelling periodic points $\{w_l\}_l$, all of minimal period $k+1$, such that

$$w_l \xrightarrow[l \to \infty]{} w.$$

From Theorem 2.19 and Theorem 2.24 (Baker), it makes sense to ask the following question.

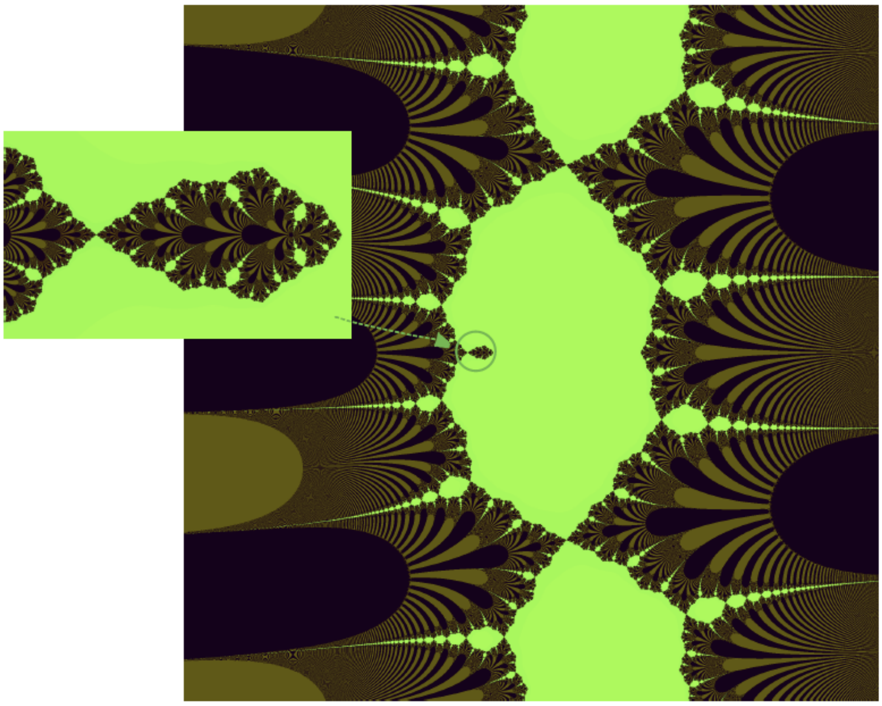

Figure 3.7: In light green, $F(f_1)$.

Question 3.20. *Given $\lambda \in \Omega_0$, ¿is every repelling periodic point in some curve of a Cantor Bouquet?*

We are concerned with the dynamical relevance of Ω_0 and we want to turn the ideas suggested by the numerical exploration into results.

Definition 3.21 (Immediate Basin of Attraction)**.** *We define the* ***immediate basin of attraction*** *of a fixed point $z_0 \in \mathbb{C}$, $\mathcal{A}^*(z_0)$, as the connected component of the basin of attraction of z_0, $\mathcal{A}(z_0)$, that contains z_0.*

Definition 3.22. *We define $\Omega_0^* = \{\lambda \in \mathbb{C}^* \colon -\lambda \in \mathcal{A}_\lambda^*(0)\}$.*

As a consequence of Proposition 3.14 we have:

Corollary 3.23. $D^*(0,1/2) = D(0,1/2) \setminus \{0\} \subset \Omega_0^*$.

The goal now is to prove the result announced in the introduction, we split it in two technical Lemmas and two Theorems.

Lemma 3.24. *Let $\lambda \in \Omega_0^*$, then all asymptotic paths of $-\lambda$ intersect the same component of $F(f_\lambda)$.*

Proof. First of all recall that given an asymptotic path Γ of $-\lambda$, i.e.,

$$\Gamma(t) \xrightarrow[t\to\infty]{} \infty \quad \text{and} \quad f_\lambda(\Gamma(t)) \xrightarrow[t\to\infty]{} -\lambda$$

then $Re(\Gamma(t))$ must be bounded from above, i.e., exists $M_\Gamma < \infty$ such that

$$\sup_{t\geq 0} Re(\Gamma(t)) \leq M_\Gamma.$$

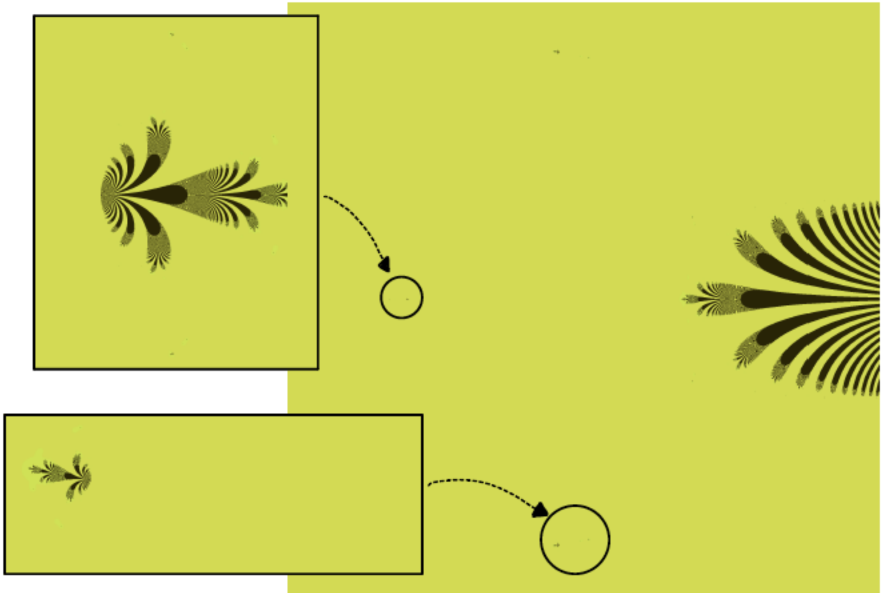

Figure 3.8: In green, $F(f_{1/2})$.

Since $-\lambda \in \mathcal{A}^*_\lambda(0)$, there exists $\varepsilon > 0$ such that

$$D(-\lambda, \varepsilon) \subset \mathcal{A}^*_\lambda(0).$$

Given that ε,

- There exists $\nu < 0$ and a half-plane, $\Pi_\nu = \{z \in \mathbb{C} \colon Re(z) < \nu\}$ such that

$$f_\lambda(\Pi_\nu) \subset D(-\lambda, \varepsilon).$$

- Exists $a \in \mathbb{R}$ and a half line $L_a = \{z \in \mathbb{C} \colon Im(z) = a\ ,\ Re(z) < x_a\}$, where $x_a \geq 0$, such that

$$f_\lambda(L_a) \subset D(-\lambda, \varepsilon)$$

  and $L_a \cap \Gamma \neq \emptyset$.

Since $L_a \cap \Pi_\nu \neq \emptyset$, they are in the same component of $F(f_\lambda) = \mathcal{A}_\lambda(0)$ and the result follows, because for every $\nu_1, \nu_2 \in \mathbb{R}^-$, we have $\Pi_{\nu_1} \cap \Pi_{\nu_2} \neq \emptyset$. □

We now can prove the first of the results announced in the introduction.

Theorem 3.25. *If $\lambda \in \Omega^*_0$, then $\mathcal{A}_\lambda(0) = F(f_\lambda)$ is connected. In particular, $\mathcal{A}_\lambda(0)$ is totally invariant.*

Proof. Suppose that $-\lambda \in \mathcal{A}^*_\lambda(0)$.

From Proposition 3.14, we can consider the disk $U_0 = D(0, \varepsilon(\lambda)) \subset \mathcal{A}^*_\lambda$.

Now we pull-back U_0 in order to obtain the whole immediate basin $\mathcal{A}^*_\lambda(0)$:

Consider, for $N > 0$, U_N as the connected component of $f_\lambda^{-1}(U_{N-1})$ that contains U_{N-1}. This recurrence defines a sequence of subsets $\{U_N\}_{N \geq 0}$ such that:

- $U_N \subset \mathcal{A}^*_\lambda(0)$ for all $N \geq 0$.
- $U_N \subset U_{N+1}$ for all $N \geq 0$.
- $\mathcal{A}^*_\lambda(0) = \bigcup_{N\geq 0} U_N$.

Since $-\lambda \in \mathcal{A}^*_\lambda(0)$, there exists $N > 0$ such that $-\lambda \in U_N$ (i.e., $f^N_\lambda(-\lambda) \in U_0$), and we can find a path $\gamma \subset U_N$ that joins $-\lambda$ and 0.

So U_{N+1} is unbounded, because $-\lambda$ is an asymptotic value (a Picard Value), hence the preimage of γ must contain a path that joins 0 and ∞ (which is contained in U_{N+1}). Using Lemma 3.24 we obtain that, in fact, when $\lambda \in \Omega^*_0$ all asymptotic tracts intersect $\mathcal{A}^*_\lambda(0)$.

Now suppose that $\mathcal{A}_\lambda(0)$ is not connected, then we must have at least two connected components, $\mathcal{A}^*_\lambda(0)$ and U. Furthermore,

$$f_\lambda(U) = \mathcal{A}^*_\lambda(0) \setminus \{-\lambda\}.$$

So U must contain a tail of an asymptotic path, but by Lemma 3.24 and the previous observation, this tail must be contained in $\mathcal{A}^*_\lambda(0)$ and the claim follows. □

We highlighted the relevance of the Böttcher coordinates in Chapter 1. They are the key ingredient for the proof of the following Lemma.

Lemma 3.26. *Let $\lambda \in \Omega^*_0$. Then there exists a closed, simple curve β, contained in $\mathcal{A}_\lambda(0)$, such that $0 \notin \beta$ and $ind(f_\lambda(\beta), 0) = -1$.*

Proof. We construct the curve. Since 0 is super-attracting, we have the Böttcher coordinates (note also that 0 is a zero of f_λ of order 2).

Let U be a neighborhood of $z = 0$ and $\varphi\colon U \to D(0, r)$ be the Böttcher coordinates map which locally conjugates f_λ and $Q_0(w) = w^2$.

Consider $\varepsilon < r < 1$ and define, $D = \varphi^{-1}(D(0, \varepsilon))$ and $D' = \varphi^{-1}(D(0, \varepsilon^2))$. The curves, $\tilde{r}_1(t) = it$, $\tilde{r}_2(t) = -it$ are mapped by Q_0 to $\tilde{r}_0(t^2) = -t^2 = Q_0(\tilde{r}_j(t))$, $j = 1, 2$, where $\tilde{r}_0(t) = -t$. Now set $r_j(t) = \varphi^{-1}(\tilde{r}_j(t))$, $j = 1, 2$.

Since $-\lambda \in \mathcal{A}^*_\lambda(0)$, there exists a disk V centered at $-\lambda$ such that $\bar{V} \subset \mathcal{A}^*_\lambda(0)$ and, by Lemma 3.24, $f^{-1}_\lambda(\bar{V})$ contains a half-plane $\{z \in \mathbb{C} \colon Re(z) < \nu\}$.

Furthermore, since by Theorem 3.25, $\mathcal{A}^*_\lambda(0) = \mathcal{A}_\lambda(0)$ is connected, we can find a simple curve $\alpha_0 \subset \mathcal{A}^*_\lambda(0)$ such that $\alpha_0(0) = 0$, $\alpha_0(1) = -\lambda$, $\alpha_0(\varepsilon) = r_0(\varepsilon)$ and $(r_0)_{|[0,\varepsilon]} = (\alpha_0)_{|[0,\varepsilon]}$.

Define $s \in (\varepsilon, 1)$ such that $\alpha_0(s) \in \partial V$. Observe that the preimage of α_0 by f_λ are two simple curves, α_1, α_2 (because the preimage of $\tilde{r}_0$ by Q_0 are two curves and they are conjugate), which are asymptotic paths, such that.

- $(r_j)_{|[0,\varepsilon]} = (\alpha_j)_{|[0,\varepsilon]}$ for $j = 1, 2$.
- $(\alpha_1)_{|[\varepsilon,s)} \cap (\alpha_2)_{|[\varepsilon,s)} = \emptyset$, that is because $0 \notin f_\lambda((\alpha_1)_{|[\varepsilon,s)}) = f_\lambda((\alpha_2)_{|[\varepsilon,s)}) = (\alpha_0)_{|[\varepsilon,s)}$ and hence, f_λ is conformal for every $z \in (\alpha_1)_{|[\varepsilon,s)} \cup (\alpha_2)_{|[\varepsilon,s)}$.

Now define.

- $\gamma_j = (\alpha_j)_{|[\varepsilon,s]}$ for $j = 0, 1, 2$.

- $\gamma_3 \subset f_\lambda^{-1}(\partial V)$ the simple curve that joins $\gamma_1(s)$ and $\gamma_2(s)$.
- $\tilde{\gamma}_4(t) = \varepsilon e^{-2\pi i t}$ and $\gamma_{4,1} = \varphi^{-1}((\tilde{\gamma}_4)_{|[1/4,3/4]})$, $\gamma_{4,2} = \varphi^{-1}((\tilde{\gamma}_4)_{|[-1/4,1/4]})$.

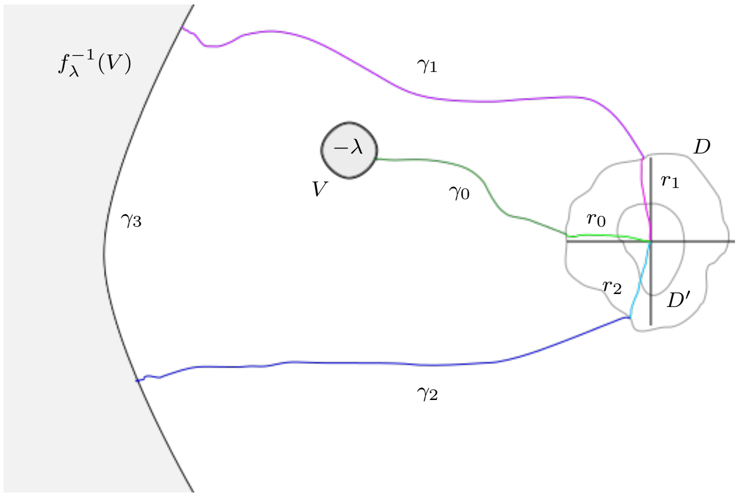

Figure 3.9: Representation of the curves and domains in the proof of Lemma 3.26.

Then we can define the curves,

$$\beta_1 = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_{4,1} \quad \text{and} \quad \beta_2 = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_{4,2}$$

which by construction are closed, simple curves contained in $\mathcal{A}_\lambda(0)$ that omit 0. Observe that β_1 (resp. β_2) can be parametrized as a closed, simple curve preserving the orientation that $\gamma_{4,1}$ (resp. $\gamma_{4,2}$) inherits from $\tilde{\gamma}_4$.

Finally,

$$f_\lambda(\beta_j) = \partial V \cup \gamma_0 \cup \partial D',$$

hence, since $\partial V \cup \gamma_0$ does not contribute to $ind(f_\lambda(\beta_j), 0)$, we have $ind(f_\lambda(\beta_j), 0) = ind(\partial D', 0)$, and we have

$$ind(f_\lambda(\beta_1), 0) = ind(\partial D', 0) = -1 \quad \text{or} \quad ind(f_\lambda(\beta_2), 0) = ind(\partial D', 0) = -1.$$

(we want the curve β_1 or β_2 to be oriented counterclockwise), so we can take $\beta = \beta_1$ or $\beta = \beta_2$ so that $ind(f_\lambda(\beta), 0) = -1$. □

Finally, we prove the remaining part of the theorem.

Theorem 3.27. *If $\lambda \in \Omega_0^*$, then $\mathcal{A}_\lambda(0) = F(f_\lambda)$ is infinitely connected.*

Proof. By Theorem 3.25 we know that $\mathcal{A}_\lambda(0) = \mathcal{A}_\lambda^*(0) = F(f_\lambda)$ is connected.

Consider the closed, simple curve provided by Lemma 3.26. By Theorem A.25 (The Argument Principle),

$$\begin{aligned} ind(f_\lambda(\beta), 0) = -1 &= \sum_{a \in Z(f_\lambda)} m(a) ind(\beta, a) - \sum_{a \in P(f_\lambda)} m(a) ind(\beta, a) = \\ &= \sum_{a \in Z(f_\lambda)} m(a) ind(\beta, a) - ind(\beta, -1) \end{aligned}$$

Since β is a closed, simple curve, then we have $ind(\beta, -1) = 1$ and $ind(\beta, a) = 0$ for every $a \in Z(f_\lambda)$.

So, $\beta \subset \mathcal{A}_\lambda(0) = \mathcal{A}^*_\lambda(0)$ and $-1 \in int(\beta)$. Then, the successive preimages of $\overline{int(\beta)}$ contains points $w \in \mathcal{O}^-_{f_\lambda}(\infty) \subset J(f)$ which lie in the interior of a closed curve contained in $\mathcal{A}_\lambda(0)$. Hence, since $\mathcal{O}^-_{f_\lambda}(\infty)$ is an infinite set, $\mathcal{A}_\lambda(0)$ is infinitely connected. □

3.2.3 Further Study

We have seen several results concerning the properties of the dynamical plane of f_λ. However, there are still much many questions to solve and yet to even ask.

Conjecture 3.28. *If $\lambda \in \Omega^*_0$, then $J(f_\lambda)$ contains Cantor Bouquets.*

If Conjecture 3.28 holds, by Theorem 3.27 we obtain that near every point of $w \in \mathcal{O}^-_{f_\lambda}(\infty) \setminus \{\infty\}$ we have a copy of a Cantor Bouquet.

The next step is studying the cases where $\lambda \notin \Omega^*_0$. In fact, the dynamical behavior of $f_{0.89}$ and $f_{0.9}$ seems to be quite different ($F(f_{0.89})$ is in Figure 3.5).

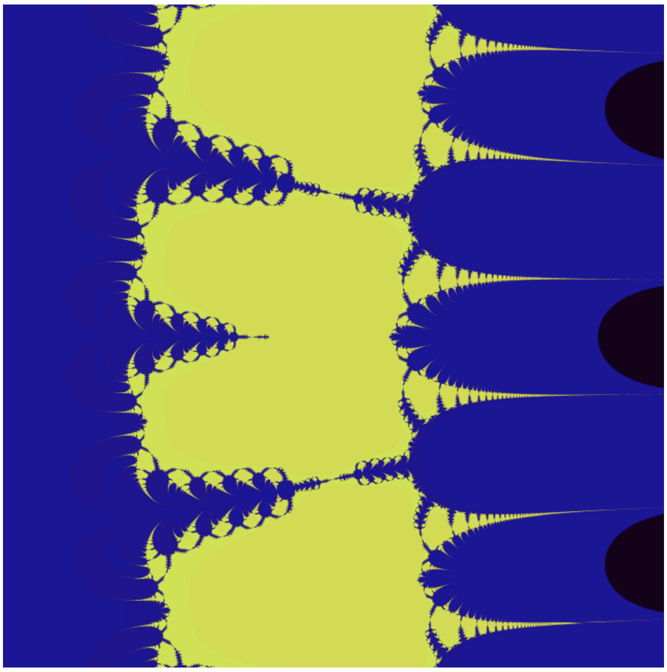

Figure 3.10: In green $\mathcal{A}_{0.9}(0)$ and in blue $I(f_{0.9})$.

For further study in the field, the reader is encouraged to read [BKL1, BKL2, BKL3, BKL4, BF, BF2, DT, Dom, FG1, FG2, Ripp, Sul].

Conclusions

This research developed all the necessary tools required to study transcendental iteration in the general case of meromorphic functions from knowledge in Complex Analysis, Dynamical Systems and Topology, pointing out the multidisciplinary character of Holomorphic Dynamics. To accomplish this goal, some background in Montel's theory has been developed, including all the necessary results, along with the study of the dynamical relevance of periodic points, required to provide a satisfactory description of the Julia and Fatou sets.

The page limit has disallowed presenting complete proofs of the classification theorem and the role of the singular values theorem, as well as proving the results concerning the existence of periodic points for transcendental entire functions. However, the suitable results needed to analyze the families in Chapter 3 have been proved. Likewise, considering the general case where there is presence of poles, the existence of infinitely many repelling periodic points has also been established.

Once achieved, the investigation of two families of maps has been addressed, obtaining results concerning some general properties of the dynamical behavior of the maps and, for some parameter values, results related with topological properties of the dynamical plane. Finally, some references were cited for further study.

Appendix A

Complex Analysis Theory

We are going to list some of the main results of the Complex Analysis theory that we need along this document. Some results will be used in proofs without being formally stated before, for far more details see [Con, Gam, MH].

A.1 Holomorphic functions

Definition A.1 (Holomorphic Function)**.** *Let $f\colon A \to \mathbb{C}$ where $A \subset \mathbb{C}$ is an open set. The function f is said to be differentiable at $z_0 \in A$ if the following limit exists*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

This limit is denoted by $f'(z_0)$.

The next theorem states that a holomorphic function is analytic, and in particular infinitely differentiable.

Theorem A.2. *Let $\Omega \subseteq \mathbb{C}$ be an open set and $f \in \mathcal{H}(\Omega)$. If $D(a, r)$ is a disk with center $a \in C$ and radius $r > 0$ such that $\overline{D(a, r)} \subset \Omega$, then f is analytic on $D(a, r)$, i.e.,*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \ , \ z \in D(a, r),$$

where the coefficients c_n are determined by ***Cauchy's Integral Formula for Derivatives****:*

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_{|\xi - a| = r} \frac{f(\xi)}{(\xi - a)^{n+1}} d\xi.$$

Theorem A.3 (Liouville)**.** *Let $f \in \mathcal{H}(\mathbb{C})$, if f is bounded, then f is constant.*

Definition A.4 (Zeros of an Holomorphic Function)**.** *Given $\Omega \subset \mathbb{C}$ and open set and $f \in \mathcal{H}(\Omega)$, we define the set of zeros of f:*

$$Z(f) = \{z \in \Omega \colon f(z) = 0\}.$$

Theorem A.5. *Let Ω be a connected open set and $f \in \mathcal{H}(\Omega)$, $f \not\equiv 0$. Then for all $z_0 \in Z(f)$, $\exists! m \geq 1$ such that $f(z) = (z - z_0)^m g(z)$ (we say that m is the order of the zero), where g is an holomorphic function such that $g(z_0) \neq 0$.*

Proof. Consider $r > 0$ such that $\overline{D(z_0, r)} \subset \Omega$, then for Theorem 2.2, $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$, $\forall z \in D(z_0, r)$. Let $m = \min\{n \in \mathbb{N} \colon c_m \neq 0\}$, then:

$$f(z) = (z - z_0)^m \underbrace{\left[\sum_{n=m}^{\infty} c_n(z - z_0)^{n-m}\right]}_{g(z)}$$

and g is analytic with $g(z_0) = c_m \neq 0$. □

Corollary A.6. *In the same conditions of Theorem A.5, if $z_0 \in Z(f)$, then z_0 is an isolated point.*

Theorem A.7 (Principle of Analytic Continuation)**.** *Let $\Omega \in \mathbb{C}$ be connected and $f \in \mathcal{H}(\Omega)$, $f \not\equiv 0$. Then $Z(f)$ has no accumulation points.*

Corollary A.8 (Principle of Analytic Continuation - Identity Theorem)**.** *Let $f, g \in \mathcal{H}(\Omega)$. Suppose that there is a sequence $\{z_n\}_{n\in\mathbb{N}} \subset \Omega$ of distinct points converging to $w \in \Omega$, such that $f(z_n) = g(z_n)$, $\forall n \in \mathbb{N}$. Then $f \equiv g$ on Ω.*

Definition A.9 (Maximum Value)**.** *Let $f \in \mathcal{H}(\Omega)$, where Ω is a domain. We say that $a \in \Omega$ is a maximum if $|f(z)| \leq |f(a)|$, $\forall z \in \Omega$.*

Theorem A.10 (Maximum Modulus Principle)**.** *Let $f \in \mathcal{H}(\Omega)$, where Ω is a domain. If exists $a \in \Omega$ maximum, then f is constant.*

The proof follows from the *Mean Value Property*, which is a Corollary of *Cauchy's Integral Formula.*

Corollary A.11. *Let Ω be an open bounded set, and $f \in \mathcal{H}(\Omega) \cap \mathcal{C}(\bar{\Omega})$. Then*

$$\sup_{z\in\Omega} |f(z)| = \sup_{z\in\partial\Omega} |f(z)|$$

i.e., the maximum of $|f|$ is attained on $\partial\Omega$.

Corollary A.12 (Minimum Modulus Principle)**.** *Let $f \in \mathcal{H}(\Omega)$, where Ω is a domain and f is non-constant. If exists $a \in \Omega$ such that $|f(a)| \leq |f(z)|$ for all $z \in \Omega$, then $f(a) = 0$.*

Theorem A.13 (Schwarz's Lemma)**.** *Let $f \colon \mathbb{D} \to \bar{\mathbb{D}}$ holomorphic such that $f(0) = 0$. Then*

(i) $|f(z)| \leq |z|$ *, for $z \in \mathbb{D}$.*

(ii) $|f'(0)| \leq 1$.

Moreover, if $|f(z)| = |z|$ for $z \neq 0$ or $|f'(0)| = 1$, then $f(z) = \lambda z$, for $|\lambda| = 1$.

We conclude this section with a well-known result.

Theorem A.14 (Open Mapping Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be a domain and $f \in \mathcal{H}(\Omega)$ non-constant. Then f is an open mapping, i.e the image of any open set under f is open.*

A.1.1 Singularities

Theorem A.2 enables us to find a convergent power series expansion of a function around a point when the function is holomorphic in a disk around the point. Thus, this theorem does not apply to functions like e^z/z or $1/z^2$ around $a = 0$.

Definition A.15 (Laurent Series)**.** *We call de Laurent series around $a \in \mathbb{C}$ of a function $f(z)$ the series:*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n = \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-a)^n} + \sum_{n=0}^{\infty} c_n(z-a)^n = f_-(z) + f_+(z).$$

This series is holomorphic on $A = \{z \in \mathbb{C} \colon r < |z-a| < R\}$, where $1/r$ is the radius of convergence of $g(w) = f_-(1/w)$ and R is the radius of convergence of $f_+(z)$.

Theorem A.16 (Laurent Expansion Theorem)**.** *Let $a \in \mathbb{C}$, $0 \leq r < R \leq \infty$, and $A = \{z \in \mathbb{C} \colon r < |z-a| < R\}$. If $f \in \mathcal{H}(A)$, then it exists an only Laurent series such that*

$$f(z) = \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-a)^n} + \sum_{n=0}^{\infty} c_n(z-a)^n \quad , \quad r < |z-a| < R$$

where both series converge uniformly in compacts of A. And if $r < \rho < R$ the coefficients are given by

$$c_n = \frac{1}{2\pi i} \int_{|\xi-a|=\rho} \frac{f(\xi)}{(\xi-a)^{n+1}} d\xi \quad , \quad \forall n \in \mathbb{Z}.$$

Definition A.17 (Isolated Singularity)**.** *We say that f has an isolated singularity at $a \in \mathbb{C}$ if exists $r > 0$ such that $f \in \mathcal{H}(D(a,r) \setminus \{a\})$ (hence, the Laurent expansion is valid in this deleted neighborhood).*

Definition A.18 (Classification of Singularities)**.** *We classify an isolated singularity $a \in \mathbb{C}$ of a function f depending on the part $f_-(z)$ of the Laurent expansion:*

(a) Removable singularity: If $c_{-n} = 0$ for all $n \geq 1$.

(b) Pole of order m: If $c_{-m} \neq 0$ and $c_{-n} = 0$ for all $n > m$.

(c) Essential singularity: If there are infinitely many $c_{-n} \neq 0$.

The next proposition states that these three cases cover all possibilities, and they are mutually exclusive.

Proposition A.19. *Let $f \in \mathcal{H}(D(a,r) \setminus \{a\})$. The singularity in $a \in \mathbb{C}$ is:*

(a) Removable $\iff \lim_{z\to a} f(z)(z-a) = 0$.

(b) Pole $\iff \lim_{z\to a} |f(z)| = \infty$.

(c) Essential $\iff \nexists \lim_{z\to a} |f(z)|$.

Corollary A.20 (Riemann's Theorem on Removable Singularities)**.** *Let $f \in \mathcal{H}(D(a,r) \setminus \{a\})$. If f is bounded near a, then f has a removable singularity at $a \in \mathbb{C}$.*

Proof. By hypothesis in a neighborhood of $a \in \mathbb{C}$, exists $M > 0$ such that $|f(z)| < M$, then

$$\lim_{z\to a} |f(z)(z-a)| \leq M \lim_{z\to a} |z-a| = 0$$

and by Proposition A.19 we obtain the result. □

A.1.2 Residue Theory

Let f have an isolated singularity at $a \in \mathbb{C}$, we consider the Laurent expansion around $a \in \mathbb{C}$,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n.$$

Definition A.21 (Residue)**.** *We define the* ***residue of*** f ***at*** $a \in \mathbb{C}$ *as* $Res(f,a) = c_{-1}$.

Let $r > 0$ be such that $f \in \mathcal{H}(D(a,r) \setminus \{a\})$ and γ a smooth closed curve such that $\gamma^* \subset D(a,r) \setminus \{a\}$, then by Laurent Expansion Theorem (Theorem A.16):

$$Res(f,a)ind(\gamma,a) = \frac{1}{2\pi i}\int_\gamma f(z)dz$$

where $ind(\gamma,a) \in \mathbb{Z}$ is the **index** of γ with respect to a (the **winding number**).

Theorem A.22 (Residue Theorem)**.** *Let* $\Omega \subset \mathbb{C}$ *be an open set and let* A *be a countable set without limit points in* Ω. *Let* $f \in \mathcal{H}(\Omega \setminus A)$ *and* Γ *be a smooth closed cycle such that* $ind(\Gamma, z) = 0$ *for* $z \notin \Gamma$. *Then:*

$$\int_\Gamma f(z)dz = 2\pi i \sum_{a\in A} Res(f,a)ind(\Gamma,a).$$

Definition A.23 (Meromorphic Function)**.** *We say that* f *is meromorphic in* $\Omega \subset \mathbb{C}$ *an open set (and we write* $f \in \mathcal{M}(\Omega)$*) if* f *is holomorphic in* Ω *except for isolated singularities which are poles. We usually denote the set of poles as* $P(f)$ *and if* $a \in P(f)$, $m(a)$ *denotes the order of the pole* a.

Proposition A.24. *Let* $\Omega \in \mathbb{C}$ *be an open set, then*

$$\mathcal{M}(\Omega) = \{f/g \colon f, g \in \mathcal{H}(\Omega)\ ,\ g \not\equiv 0\}.$$

Theorem A.25 (The Argument Principle)**.** *Let* Ω *be a domain and* $f \in \mathcal{M}(\Omega)$ *a non-constant function. If* Γ *is a smooth closed cycle such that* $ind(\Gamma, z) = 0$ *for* $z \notin \Gamma$ *and* $\Gamma^* \cap (P(f) \cup Z(f)) = \emptyset$, *then:*

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}dz = \sum_{a\in Z(f)} m(a)ind(\Gamma,a) - \sum_{a\in P(f)} m(a)ind(\Gamma,a).$$

A.1.3 Critical Points

Our aim is to understand the behavior of a holomorphic function near a critical point, and to understand the behavior of the inverse function near the corresponding critical value.

Let $f \in \mathcal{H}(D)$ be a non-constant holomorphic function on a domain D. Suppose that $z_0 \in \mathbb{C}$ is a critical point of f with critical value $f(z_0) = w_0$. We define the **order** of the critical point z_0 to be the order of zero of $f'(z)$ at z_0. The critical points of f are isolated because they are the zeros of the non-constant holomorphic function $f'(z)$.

Suppose that z_0 has order $m-1$ as a critical point. Then $f(z) - w_0$ has a zero of order m at z_0. Considering $\rho > 0$ so that $f(z) - w_0 \neq 0$ for $0 < |z - z_0| \leq \rho$, and $\delta > 0$ such that $|f(z) - w_0| \geq \delta$ for $|z - z_0| = \rho$, the logarithmic integral:

$$N(w) = \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f'(z)}{f(z) - w} dz \quad , |w - w_0| < \delta$$

is well-defined and it depends analytically on w. Since the holomorphic function $N(w)$ is integer-valued, is constant, and $N(w_0) = m$ gives us that $N(w) = m$ for $|w - w_0| < \delta$. Thus, each w satisfying $|w - w_0| < \delta$ is assumed m times counting multiplicity by $f(z)$ in $\{|z - z_0| < \rho\}$, and then for $\rho > 0$ small enough and for w near w_0, there are m points $z_1(w), \ldots, z_m(w)$ (repeated according to multiplicity) such that $f(z)$ attains the value w in the disk $\{|z - z_0| < \rho\}$ (at the points $z_j(w)$, $1 \leq j \leq m$). We want to see how the points $z_j(w)$ depend on w.

We are going to make a change of variable in order to show that the behavior of $f(z)$ near z_0 is the same as the behavior of ξ^m near $\xi = 0$.

Since $f(z) - w_0$ has order m at z_0, using Theorem A.5 exists $h(z)$ holomorphic at z_0 such that $h(z_0) \neq 0$ and

$$f(z) - w_0 = (z - z_0)^m h(z).$$

Near z_0, we can define an holomorphic branch of $h(z)^{1/m}$, if we consider $g(z) = (z - z_0)h(z)^{1/m}$ then

$$f(z) = w_0 + g(z)^m.$$

Since $g(z)$ is holomorphic at z_0 and has a simple zero at z_0, $g'(z_0) \neq 0$ and $g(z)$ is one-to-one near z_0 (univalent). Thus $f(z)$ is represented as the composition of three functions

$$z \mapsto g(z) \mapsto g(z)^m \mapsto w_0 + g(z)^m$$

We know the behavior of

$$\xi \mapsto \xi^m \quad \text{and} \quad \xi \mapsto w_0 + \xi$$

and since $g(z)$ is univalent near z_0 we can explain the behavior of $f(z)$ near z_0.

- The set where ξ^m is real consists of m straight line segments passing through 0, which divide the ξ-plane near $\xi = 0$ into $2m$ sectors of equal aperture. The imaginary part of ξ^m is alternately positive and negative.

- The set where $f(z) - w_0 = g(z)^m$ is real is the image under $g^{-1}(\xi)$ of these line segments, obtaining m curves through z_0. They divide the z-plane near z_0 into $2m$ domains with vertex z_0 on which the imaginary part of $f(z) - w_0$ is alternately positive and negative.

Then a point w near w_0, $w \neq w_0$ has m distinct preimages $z_1(w), \ldots, z_m(w)$ and they are the m branches of $(w - w_0)^{1/m}$ composed with $g^{-1}(\xi)$. If we consider the principal branch $(w - w_0)^{1/m}$ on $\{|w - w_0| < \delta\} \setminus (w_0 - \delta, w_0]$, the other branches are of the form
$$e^{2\pi ij/m}(w - w_0)^{1/m}$$
and then the preimages of w are given by
$$z_j(w) = g^{-1}(e^{2\pi ij/m}(w - w_0)^{1/m}) \quad , 1 \leq j \leq m.$$
Finally, for $w \neq w_0$, the preimages $z_j(w)$ are distinct and each $z_j(w)$ depends analytically on w.

A.1.4 Conformal Mappings

Definition A.26 (Conformal Map)**.** *A map $f \colon A \to \mathbb{C}$ is called **conformal** at z_0 if there exists $\theta \in [0, 2\pi)$ and $r > 0$ such that for any curve $\gamma(t)$ that is differentiable at $t = 0$, for which $\gamma(t) \in A$, $\gamma(0) = z_0$ and $\gamma'(0) \neq 0$, the curve $\sigma(t) = f(\gamma(t))$ is differentiable at $t = 0$ and*
$$|\sigma'(0)| = r|\gamma'(0)| \ , \ arg(\sigma'(0)) = arg(\gamma'(0)) + \theta \mod (2\pi).$$

We usually take the next theorem as the definition:

Theorem A.27. *If $f \in \mathcal{H}(A)$ and $f'(z_0) \neq 0$, then f is conformal at z_0 with $\theta = arg(f'(z_0))$ and $r = |f'(z_0)|$.*

Theorem A.28 (Inverse Function Theorem)**.** *Let $f \in \mathcal{H}(A)$ and assume $f'(z_0) \neq 0$. Then there exists a neighborhood U of z_0 and a neighborhood V of $f(z_0)$ such that $f \colon U \to V$ is a bijection and its inverse function f^{-1} is holomorphic with derivative given by*
$$\frac{d}{dw} f^{-1}(w) = \frac{1}{f'(z)} \qquad where \qquad w = f(z).$$

Proof. If $f(z) = u(z) + iv(z)$, using the Cauchy-Riemann equations then the Jacobian matrix is:
$$J_f = \begin{pmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix}$$
and the determinant is:
$$\det(J_f) = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = |f'(z)|^2.$$
So $\det(J_f(z_0)) \neq 0$ and we can apply the Inverse Function Theorem for real valued functions, which it gives us the neighborhoods stated before.

Finally, if we take $w, w_1 \in U$ and we set $g(w) = f^{-1}(w)$, $g(w_1) = z_1$, the following limit:

$$\lim_{w \to w_1} \frac{g(w) - g(w_1)}{w - w_1} = \lim_{w \to w_1} \frac{z - z_1}{f(z) - f(z_1)} = \frac{1}{f'(z_1)}$$

exists and then f^{-1} is holomorphic. □

Proposition A.29.

(i) If $f\colon A \to B$ is conformal and bijective, then $f^{-1}\colon B \to A$ is also conformal.

(ii) If $f\colon A \to B$ and $g\colon B \to C$ are conformal and bijective, then $g \circ f\colon A \to C$ is conformal and bijective.

Proof. Since f is bijective, the mapping f^{-1} exists. By the inverse function theorem $f^{-1}(w)$ is holomorphic and $(f^{-1})'(w) \neq 0$, thus it is conformal and we obtain (i).

To prove (ii), we just have to apply the chain rule $(g \circ f)'(z) = g'(f(z))f'(z) \neq 0$, so $g \circ f$ is also conformal. □

Theorem A.30 (Riemann Mapping Theorem)**.** *Let A be a simply connected region $A \neq \mathbb{C}$. Then there exists a bijective conformal map $f\colon A \to \mathbb{D}$. Furthermore, for any fixed $z_0 \in A$, we can find f such that $f(z_0) = 0$ and $f'(z_0) > 0$. With such specification, f is unique.*

Proof. We only prove uniqueness, the existence is proved in [Gam]. Suppose f and g are bijective conformal maps of A onto $\mathbb{D}$ with $f(z_0) = g(z_0) = 0$ and $f'(z_0) > 0$, $g'(z_0) > 0$. We define $h\colon \mathbb{D} \to \mathbb{D}$ such that $h(w) = g(f^{-1}(w))$, then $h \in \mathcal{H}(D)$ and $h(0) = g(z_0) = 0$.

By Schwarz's Lemma $|h(w)| \leq |w|$ and the same argument applies to $h^{-1} = f \circ g^{-1}$, so $|h^{-1}(\xi)| \leq |\xi|$. If we take $\xi = h(w)$, these both inequalities gives us that

$$|h^{-1}(h(w))| \leq |h(w)| \leq |w|.$$

So $|h(w)| = |w|$ for all $w \in \mathbb{D}$, i.e., $h(w) = cw$, for $|c| = 1$. Then $cw = g(f^{-1}(w))$, if we take $z = f^{-1}(w)$ we obtain $cf(z) = g(z) \Rightarrow cf'(z) = g'(z)$. Since $f'(z_0)$ and $g'(z_0)$ are real positive values, so is c, and then $c = 1$, i.e., $f(z) = g(z)$. □

Using Schwarz's Lemma, Riemann's Mapping Theorem and a well-known result in Topology called the Brouwer's Fixed Point Theorem we can prove:

Corollary A.31. *Suppose U is a simply connected open set, $U \neq \mathbb{C}$, and that $f(\bar{U}) \subset U$. Then f has a fixed point $z_0 \in U$ and $|f'(z_0)| < 1$.*

Definition A.32 (Conformally Equivalent)**.** *Two regions are called **conformally equivalent** if there exists a bijective conformal map $g\colon A \to B$.*

Corollary A.33. *If A, B are two simply connected regions with $A \neq \mathbb{C} \neq B$, then there is a bijective conformal map $g\colon A \to B$.*

Proof. If $f\colon A \to \mathbb{D}$ and $g\colon B \to D$ are conformal, we can set $g = h^{-1} \circ f$, which is conformal and bijective. □

Remark A.34. *$\mathbb{C}$ cannot be conformally equivalent with $\mathbb{D}$ because $\phi\colon \mathbb{C} \to \mathbb{D}$ due to Liouville's theorem must be constant.*

A.2 Normal families

The results and the proofs here have been obtained from [Gam].

Definition A.35 (Uniform Convergence)**.** *Let $\{f_k\}$ be a sequence of functions defined on $E \subset \mathbb{C}$. We say that the sequence* ***converges uniformly*** *on E to f if for any $\varepsilon > 0$, exists $n_0 \geq 1$ such that for all $n \geq n_0$, $|f_n(z) - f(z)| < \varepsilon$ for all $z \in E$. We write $f_k \rightrightarrows f$.*

Remark A.36. *The definition above can also be stated as: $f_k \rightrightarrows f$, if and only if, $|f_k(z) - f(z)| < \varepsilon_k$ for all $z \in E$, where $\varepsilon_k \to 0$ as $k \to \infty$.*

Theorem A.37 (Weierstrass M-Test)**.** *Suppose $M_k \geq 0$ and $\sum_k M_k < \infty$. If $\{g_k(z)\}$ are functions on $E \subset \mathbb{C}$ such that $|g_k(z)| \leq M_k$ for all $z \in E$, then $\sum_k g_k(z)$ converges uniformly on E.*

Theorem A.38. *If $\{f_k(z)\}$ is a sequence of holomorphic functions on a domain D such that $f_k \rightrightarrows f$ on D, then $f \in \mathcal{H}(D)$.*

Proof. Let E be a closed rectangle contained in D. By Cauchy's theorem $\int_{\partial E} f_k(z)dz = 0$ for each k. Since $f_k \rightrightarrows f$,

$$\int_{\partial E} f(z)dz = \int_{\partial E} \lim_k f_k(z)dz = \lim_k \int_{\partial E} f_k(z)dz = 0.$$

Then, by Morera's theorem $f \in \mathcal{H}(D)$. □

Theorem A.39. *Suppose that $f_k(z)$ is holomorphic for $|z - z_0| \leq R$, and suppose that $f_k \rightrightarrows f$ for $|z - z_0| \leq R$. Then for each $r < R$ and $m \geq 1$*

$$f_k^{(m)} \rightrightarrows f^{(m)} \quad , \text{for } |z - z_0| \leq r.$$

Proof. Suppose $\varepsilon_k \to 0$ are such that $|f_k(z) - f(z)| < \varepsilon_k$ for $|z - z_0| < R$. We fix $s \in (r, R)$, then by the Cauchy Integral Formula for Derivatives:

$$f_k^{(m)}(z) - f^{(m)}(z) = \frac{m!}{2\pi i} \int_{|z-z_0|=s} \frac{f_k(\xi) - f(\xi)}{(\xi - z)^{m+1}} d\xi.$$

If $|\xi - z_0| = s$ and $|z - z_0| \leq r$, then $|\xi - z| \geq |\xi - z_0| - |z_0 - z| = s - r$, so

$$|f_k^{(m)}(z) - f^{(m)}(z)| \leq \frac{m!}{2\pi} \int_{|z-z_0|=s} \left| \frac{f_k(\xi) - f(\xi)}{(\xi - z)^{m+1}} \right| d\xi \leq \frac{m!}{2\pi} \frac{\varepsilon_k}{(s-r)^{m+1}} 2\pi s = \rho_k.$$

and $\rho_k \to 0$ as $k \to \infty$, thus we obtain the desired uniform convergence. □

Definition A.40 (Normal Convergence)**.** *We say that a sequence $\{f_k(z)\}_k$ of holomorphic functions on a domain D* ***converges normally*** *to the holomorphic function $f(z)$ on D, if it converges uniformly to $f(z)$ on each closed disk contained in D.*

A consequence of the previous theorems is:

Theorem A.41. *Suppose that $\{f_k(z)\}_k$ is a sequence of holomorphic functions on a domain D that converges normally on D to the holomorphic function $f(z)$. Then, for each $m \geq 1$, the sequence $\{f_k^{(m)}(z)\}_k$ converges normally to $f^{(m)}(z)$ on D.*

Given a convergent sequence of holomorphic functions, the argument principle allows us to prove that, in some sense, the zeros of the functions in the sequence converge to the zeros of the limit function.

Theorem A.42 (Hurwitz). *Suppose $\{f_k(z)\}_k$ is a sequence of holomorphic functions on a domain D that converges normally on D to $f(z)$, and suppose that $f(z)$ has a zero of order N at z_0. Then there exists $\rho > 0$ such that for k large enough, $f_k(z)$ has exactly N zeros in the disk $\{|z - z_0| < \rho\}$ counting multiplicity, and these zeros converge to z_0 as $k \to \infty$.*

Proof. Let $\rho > 0$ be small enough so that $\{|z - z_0| \leq \rho\} \subset D$ and $f(z) \neq 0$ for $0 < |z - z_0| \leq \rho$. Now we choose $\delta > 0$ such that $|f(z)| \geq \delta$ on $|z - z_0| = \rho$. Since $f_k(z)$ converges uniformly to $f(z)$ for $|z - z_0| \leq \rho$, for k large enough we have $f_k(z) > \delta/2$ for $|z - z_0| = \rho$, and further $f'_k(z)/f_k(z)$ converges uniformly to $f'(k)/f(k)$ for $|z - z_0| = \rho$. Hence the integrals converge:

$$N_k = \frac{1}{2\pi i}\int_{|z-z_0|=\rho} \frac{f'_k(z)}{f_k(z)}dz \xrightarrow[k\to\infty]{} \frac{1}{2\pi i}\int_{|z-z_0|=\rho} \frac{f'(z)}{f(z)}dz = N$$

where due to the argument principle, $N_k \equiv$ number of zeros of $f_k(z)$ in the disk $\{|z - z_0| < \rho\}$ and $N \equiv$ number of zeros of $f(z)$ in $\{|z - z_0| < \rho\}$.

Since $N_k \to N$ and they are both integers, it exists $k_0 \geq 1$ such that $N_k = N$ for all $k \geq k_0$. So, for $k \geq k_0$, $f_k(z)$ has exactly N zeros counting with multiplicity in $\{|z - z_0| < \rho\}$. The same argument works for $\rho > 0$ smaller, so the zeros of $f_k(z)$ must accumulate at z_0. $\square$

Definition A.43 (Equicontinuous Family). *We say that a family $\mathcal{F}$ of functions on $E \subset \mathbb{C}$ is **equicontinuous** at $z_0 \in E$ if for any $\varepsilon > 0$, there is $\delta > 0$ such that if $z \in E$ satisfies $|z - z_0| < \delta$, then $|f(z) - f(z_0)| < \varepsilon$ for all $f \in \mathcal{F}$.*

Definition A.44 (Uniformly Bounded). *We say that a family $\mathcal{F}$ is **uniformly bounded** on E if there exists a constant $M > 0$ such that $|f(z)| \leq M$ for all $z \in E$ and all $f \in \mathcal{F}$.*

Theorem A.45 (Arzelà-Ascoli). *Let $E \subset \mathbb{C}$ be compact set and $\mathcal{F}$ be a family of continuous functions on E that is uniformly bounded. Then the following statements are equivalent:*

(i) $\mathcal{F}$ is equicontinuous at each point of E.

(ii) Each sequence of functions in $\mathcal{F}$ has a subsequence that converges uniformly on E.

Proof. $(i) \implies (ii)$

- Every compact space is separable, i.e., $\exists\{z_j\}_j \subset E$ that is dense in E:

  If we fix $n \in \mathbb{N}$, then
  $$\cup_{z\in E} D(z, 1/n) \supset E$$
  since E is compact, there exists a finite cover, which gives us
  $$\{z_{n,j}\}_{j\in J_n} \subset E \quad , \quad \#J_n < \infty$$

such that $E \subset \cup_{j \in J_n} D(z_{n,j}, 1/n)$. So

$$\{z_j\}_j = \cup_{n\in\mathbb{N}}\{z_{n,j}\}$$

is countable and by construction is dense in E.

- We use a standard diagonalization argument in order to find a subsequence of $\{f_n\}_n \subset \mathcal{F}$ that converges pointwise on $\{z_j\}_j$:

  Since $\{f_n(z_1)\}_n$ is bounded ($\mathcal{F}$ is uniformly bounded), by Bolzano-Weierstrass theorem it has a convergent subsequence $\{f_{1,n}(z_1)\}_n$. Now, $\{f_{1,n}(z_2)\}_n$ is bounded, so it has a convergent subsequence $\{f_{2,n}(z_2)\}_n$. Note that since $f_{2,n} \vdash f_{1,n}$ and $\{f_{1,n}(z_1)\}_n$ converges, then $\{f_{2,n}(z_1)\}_n$ converges. Thus $\{f_{2,n}\}_n$ converges in $\{z_1, z_2\}$.

  Repeating this method we obtain a countable collection of subsequences of $\{f_n\}_n$,

  $$\{\{f_{k,n}\}_n \colon f_{k,n} \vdash f_{k-1,n} \vdash \cdots \vdash f_n\}$$

  such that $\{f_{k,n}\}_n$ converges at $\{z_1, \ldots, z_k\}$. Then, $f_{n,n} \vdash f_n$ is a subsequence that converges for all z_j.

- Finally, if $\varepsilon > 0$, for any $z \in E$:

  - $\mathcal{F}$ is equicontinuous implies that $\exists \delta > 0$ such that

    $$|z - \tilde{z}| < \delta \Rightarrow |f_{n,n}(z) - f_{n,n}(\tilde{z})| < \varepsilon/3.$$

  - $\{z_j\}_j$ is dense in E implies that $\exists j \in \mathbb{N}$ such that $|z - z_j| < \delta$.
  - $\{f_{n,n}\}_n$ converges in $\{z_j\}_j$ implies that $\exists n_0 \in \mathbb{N}$ such that for $n, m > n_0$ $|f_{n,n}(z_j) - f_{m,m}(z_j)| < \varepsilon/3$.

  And then, for all $n, m > n_0$:

  $$\begin{aligned}|f_{n,n}(z) - f_{m,m}(z)| \leq{}& |f_{n,n}(z) - f_{n,n}(z_j)| + |f_{n,n}(z_j) - f_{m,m}(z_j)| + \\ &+ |f_{m,m}(z_j) - f_{m,m(z)}| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

  Thus $f_{n,n} \vdash f_n$ is uniformly Cauchy and then uniformly convergent.

$(ii) \implies (i)$ Assume that $\mathcal{F}$ is compact but not equicontinuous. Then $\exists \varepsilon > 0$ such that $\forall \delta > 0$, $\exists z, w \in E$ and $f \in \mathcal{F}$ such that $|z - w| < \delta$ but $|f(z) - f(w)| > \varepsilon$.

So, we can define a sequence $\delta_n = 1/n$ such that $\exists z_n, w_n \in E$ and $f_n \in \mathcal{F}$ such that

$$|z_n - w_n| < \delta_n = 1/n \quad \text{but} \quad |f_n(z_n) - f_n(w_n)| > \varepsilon \quad (*1)$$

Which gives us a sequence of functions $\{f_n\}_n \subset \mathcal{F}$. No subsequence of $\{f_n\}_n$ can be equicontinuous, but by hypothesis, $\exists f_{n_k} \vdash f_n$ that converges uniformly on E to the continuous function f. Then,

$$|f_{n_k}(z) - f_{n_k}| \leq |f_{n_k} - f(z)| + |f(z) - f(w)| + |f_{n_k}(w) - f(w)| \quad (*2)$$

- The uniform convergence gives us that $\exists k_1 > 0$ such that for any $k > k_1$,

  $$|f_{n_k}(z) - f(z)| < \varepsilon/3 \quad \forall z \in E.$$

- Since f is continuous $\exists k_2 > 0$ such that if $|z - w| < 1/k_2 = \delta_{k_2}$, then $|f(z) - f(w)| < \varepsilon/3$.

If we consider $k_0 = \max\{k_1, k_2\}$, then $\forall k > k_0$, $\forall |z - w| < \delta_{k_0}$, for $(*2)$:

$$|f_{n_k}(z) - f_{n_k}(w)| < 3 \cdot \varepsilon/3 = \varepsilon.$$

Which is a contradiction with $(*1)$. So $\mathcal{F}$ is equicontinuous. □

Definition A.46 (Normal Family). *A family $\mathcal{F}$ of holomorphic functions on a domain D is a **normal family** if every sequence in $\mathcal{F}$ has a subsequence that converges normally on D.*

Our applications to holomorphic dynamics are be based on the following theorem.

Theorem A.47 (Montel). *Suppose that $\mathcal{F}$ is a family of holomorphic functions on a domain D such that $\mathcal{F}$ is uniformly bounded on each compact subset of D. Then $\mathcal{F}$ is a normal family.*

Proof. If $z_0 \in D$, $\exists r > 0$ such that $\bar{D}(z_0, r) \subset D$. $\mathcal{F}$ is uniformly bounded on $\bar{D}(z_0, r)$. By Cauchy Estimates, the derivatives of the functions in $\mathcal{F}$ are uniformly bounded on $\bar{D}(z_0, r)$, i.e., there is $M > 0$ such that $|f'(z)| \leq M$ on $\bar{D}(z_0, r)$ for all $f \in \mathcal{F}$. Then, if $z \in \bar{D}(z_0, r)$ is close enough to z_0,

$$|f(z) - f(z_0)| = \left| \int_{z_0}^{z} f'(\xi) d\xi \right| \leq M|z - z_0|.$$

So $\mathcal{F}$ is equicontinuous at z_0.

Let $E_n = \{z \in D\colon |z| \leq n \ , d(z, \partial D) \geq 1/n\}$ Then E_n is compact, $E_n \to D$ and for every compact subset K of D, exists $n_0 \in \mathbb{N}$ such that $K \ \subset E_n$ for all $n \geq n_0$.

Let $\{f_n\}_n \subset \mathcal{F}$, then by the Arzelà-Ascoli theorem, $\exists f_{1,n} \vdash f_n$ such that $\{f_{1,n}\}_n$ converges uniformly on E_1. Then $\exists f_{2,n} \vdash f_{1,n}$ that converges uniformly on E_2 .... We obtain a diagonal sequence $\{f_{n,n}\}_n$ that converges uniformly on each E_n, hence uniformly on each compact subset of D. □

A.2.1 Compact Families of Meromorphic Functions

Our aim is to obtain a stronger version of Montel's theorem.

To prove further results we work with the spherical metric and we work in the Riemann Sphere, $\mathbb{C}_\infty$, which is the Alexandroff compactification of $\mathbb{C}$.

Definition A.48 (Spherical Length). *If γ is a path in $\mathbb{C}_\infty$, its **length** in the spherical metric is*

$$2 \int_\gamma \frac{|dz|}{1 + |z|^2}.$$

It is not difficult to check that the geodesics are the great circles on the sphere. Using Gauss-Bonet theorem we obtain that the sum of the angles of a geodesic triangle is strictly greater than π.

We have:

Theorem A.49 (Arzelà-Ascoli)**.** *Let $D \subset \mathbb{C}$ be a domain, and let $\mathcal{F}$ be a family of continuous functions from D to $\mathbb{C}_\infty$. Then the following statements are equivalent:*

(i) Any sequence in $\mathcal{F}$ has a subsequence that converges uniformly on compact subsets of D in the spherical metric.

(ii) $\mathcal{F}$ is equicontinuous at each point of D, with respect to the spherical metric.

For $f \in \mathcal{M}(D)$, f is a function from D to $\mathbb{C}_\infty$. We use distances on the sphere to measure distances between function values. $\sigma(z, w)$ denotes the spherical distance from z to w. We have:

- The spherical metric is invariant under rotations of the sphere. Since $z \mapsto 1/z$ is a rotation of π degrees, then $\sigma(z, w) = \sigma(1/z, 1/w)$.
- On any bounded subset of the complex plane, the spherical metric is equivalent to the Euclidean metric.

Definition A.50 (Normal Convergence)**.** *We say that a sequence $\{f_n(z)\}$ of meromorphic functions on a domain D* ***converges normally*** *to $f(z)$ on D, if the sequence converges uniformly on compact subsets of D to $f(z)$ in the spherical metric.*

Since the spherical and the Euclidean metrics are equivalent on bounded subsets of $\mathbb{C}$, the above definition is consistent with the previous one.

Theorem A.51. *Let D be a domain.*

(i) If $\{f_n(z)\}_n \subset \mathcal{M}(D)$ converges normally to $f(z)$, then either $f \in \mathcal{M}(D)$ or $f(z) \equiv \infty$.

(ii) If $\{f_n(z)\}_n \subset \mathcal{H}(D)$ converges normally to $f(z)$, then either $f \in \mathcal{H}(D)$ or $f(z) \equiv \infty$.

Proof.

(i) Recall that $D = \{z \in D \colon f(z) \neq 0\} \cup \{z \in D \colon f(z) \neq \infty\}$.

Every $z \in D$ such that $f(z) \neq \infty$, has a neighborhood on which the f_n are uniformly bounded and converge uniformly in the Euclidean metric. Thus $f(z)$ is holomorphic on the set where $|f(z)| < \infty$ (by Theorem A.38).

We also have that $1/f_n(z)$ converges normally to $1/f(z)$ (because $\sigma(z, w) = \sigma(1/z, 1/w)$), $1/f(z)$ is holomorphic on the set where $f(z) \neq 0$. So, either $1/f(z) \equiv 0$ and then $f(z) \equiv \infty$, or the zeros of $1/f(z)$ are isolated in D and they are poles of $f(z)$, so $f \in \mathcal{M}(D)$.

(ii) If exists $z_0 \in D$ such that $f(z_0) = \infty$, then $1/f_n(z)$ is meromorphic with no zeros near z_0 but $1/f_n(z_0) \xrightarrow[n\to\infty]{} 0$. By Hurwitz's theorem, $1/f_n(z)$ converges uniformly to zero on some neighborhood of z_0, so $1/f(z) \equiv 0$ on that neighborhood. By analytic continuation, $1/f(z) \equiv 0$ on D, and hence $f(z) \equiv \infty$.

$\square$

Definition A.52 (Normal Family)**.** *A family $\mathcal{F}$ of meromorphic functions on D is said to be a* ***normal family*** *if every sequence in $\mathcal{F}$ has a subsequence that converges normally on D.*

Theorem A.51 shows us that a family $\mathcal{F}$ of holomorphic functions on D is a normal family, if and only if, every sequence $\{f_n\}_n \subset \mathcal{F}$ converges normally to an holomorphic function or to $f \equiv \infty$.

Remark A.53. *If $f \in \mathcal{M}(D)$, D domain and $\gamma(t)$ is a curve then:*

$$\textit{spherical length of } f \circ \gamma = \int_\gamma \frac{2|f'(z)|}{1+|f(z)|^2} dz.$$

Definition A.54 (Spherical Derivative)**.** *We define the* ***spherical derivative*** *of f as:*

$$f^{\#}(z) = \frac{2|f'(z)|}{1+|f(z)|^2}.$$

Note that since the spherical metric is invariant under the inversion $z \mapsto 1/z$, then $(1/f)^{\#} = f^{\#}$.

The equality $(1/f)^{\#} = f^{\#}$ allows us to replace $f(z)$ near its poles by $g(z) = 1/f(z)$. Hence, it is enough to consider the case where $f(z)$ is holomorphic.

Lemma A.55. *If $f_k(z) \to f(z)$ normally on D, then $f_k^{\#} \to f^{\#}(z)$ uniformly on compact subsets of D.*

Proof.

- If f is holomorphic at z_0, then $f_k' \to f'$ uniformly in some neighborhood of z_0 (Theorem A.39), so $f_k^{\#} \to f^{\#}(z)$ in some neighborhood of z_0.
- If f is not holomorphic at z_0, then $1/f$ is holomorphic at z_0 and $1/f_k \to 1/f$ normally. So $f_k^{\#} = (1/f_k)^{\#}$ converges uniformly to $f^{\#} = (1/f)^{\#}$ in some neighborhood of z_0.

□

Theorem A.56 (Marty)**.** *A family $\mathcal{F}$ of meromorphic functions is normal on a domain D, if and only if, the spherical derivatives $\{f^{\#} \colon f \in \mathcal{F}\}$ are uniformly bounded on each compact subset of D.*

Proof. $\Longleftarrow$) Suppose that the spherical derivatives are uniformly bounded near z_0. Let's say $f^{\#}(z) \leq C$ for $|z - z_0| < r$ and $f \in \mathcal{F}$. If $|z_1 - z_0| < r$ and γ is the straight line segment from z_0 to z_1, then

$$\sigma(f(z_0), f(z_1)) \leq \int_\gamma f^{\#}(z)|dz| \leq C|z_1 - z_0|.$$

Since this estimate is independent of the function $f \in \mathcal{F}$, then $\mathcal{F}$ is equicontinuous at each $z_0 \in D$. By the Arzelà-Ascoli theorem (Theorem A.45), then $\mathcal{F}$ is a normal family.

$\Longrightarrow$) Suppose that the spherical derivatives of the functions in $\mathcal{F}$ are not uniformly bounded on compact subsets of D, then there are $f_k \in \mathcal{F}$ such that the maximum of $f_k^{\#}$ over some compact set tends to ∞. By the previous Lemma, $\{f_k\}_k$ cannot have a normally convergent subsequence, so $\mathcal{F}$ is not normal. $\square$

The definition of a normal family of meromorphic functions can be extended to include domains D with $\infty \in D$.

Definition A.57 (Normal Family). *A family $\mathcal{F}$ of meromorphic functions on a domain $D \subset \mathbb{C}_\infty$ is a **normal family** of meromorphic functions on D if $\mathcal{F}$ is a normal family on $D \setminus \{\infty\}$ and the functions $g(w) = f(1/w)$, $1/w \in D$ form a normal family of meromorphic functions in some neighborhood of $w = 0$.*

Marty's theorem is still valid for domains $D \subset \mathbb{C}_\infty$.

Theorem A.58 (Zalcman's Lemma). *Suppose $\mathcal{F}$ is a family of meromorphic functions on a domain D that is not normal. Then there are points $z_n \in D$ converging to a point of D, numbers ρ_n converging to 0, and functions $f_n \in \mathcal{F}$ such that the scaled functions $g_n(\xi) = f_n(z_n + \rho_n \xi)$ converge normally to a non-constant meromorphic function $g(\xi)$ on $\mathbb{C}$ satisfying $g^{\#}(0) = 1$ and $g^{\#}(\xi) \leq 1$ for $\xi \in \mathbb{C}$.*

Proof. By Marty's theorem, there are sequences $\{w_n\}_n$ in a compact subset of D and $f_n \in \mathcal{F}$ such that $f_n^{\#}(w_n) \to \infty$. Without loss of generality we can assume that $w_n \to 0 \in D$ and that $\{|z| \leq 1\} \subset D$. Define

$$R_n = \max_{|z| \leq 1} f_n^{\#}(z)(1 - |z|).$$

Since $w_n \to 0$ and $f_n^{\#} \to \infty$, then $R_n \to \infty$. Suppose that $f_n^{\#}(z)(1 - |z|)$ attains its maximum at z_n;

$$R_n = f_n^{\#}(z_n)(1 - |z_n|).$$

Since $f_n^{\#}(z_n) \geq R_n$, then we also have $f^{\#}(z_n) \to \infty$.

We define $\rho_n = 1/f_n^{\#}(z_n)$, then $\rho_n \to 0$. So $D(z_n, 1 - |z_n|) = D(z_n, \rho_n R_n) \subset \{|z| \leq 1\} \subset D$, and it can be parametrized by

$$\xi \mapsto z_n + \rho_n \xi \quad \text{for } |\xi| < R_n.$$

Now we define $g_n(\xi) = f_n(z_n + \rho_n \xi)$, for $|\xi| < R_n$, since $R_n \to \infty$, given any compact set $K \subset \mathbb{C}$, exists n_0 large enough such that for every $n > n_0$, g_n is defined on K. If we set $h_n(\xi) = z_n + \rho_n \xi$, using the chain rule $(f_n \circ h_n)^{\#}(\xi) = f_n^{\#}(h_n(\xi))|h_n'(\xi)|$, so

$$g_n^{\#}(\xi) = \rho_n f_n^{\#}(z_n + \rho_n \xi) \quad \text{for } |\xi| < R_n.$$

If we fix $R > 0$, for n large enough, $R_n > R$, and g_n is defined on $\{|\xi| < R\}$. Since $f_n^{\#}(z_n + \rho_n \xi)(1 - |z_n + \rho_n \xi|) \leq R_n$, we obtain:

$$\begin{aligned} g_n^{\#}(\xi) \leq \rho_n \frac{R_n}{1 - |z_n + \rho_n \xi|} &\leq \frac{\rho_n R_n}{1 - |z_n| - \rho_n R} = \frac{\rho_n R_n}{\rho_n R_n - \rho_n R} = \\ &= \frac{1}{1 - \frac{R}{R_n}} \quad \text{for} |\xi| < R. \end{aligned}$$

By Marty's theorem, the g_n's for n large enough form a normal family on $\{|z| < R\}$. Passing to a subsequence if necessary, we can assume that $\{g_n\}$ converges normally on $\mathbb{C}$ to a meromorphic function $g(\xi)$. Since

$$g_n^{\#} \leq 1/(1 - R/R_n) \to 1$$

for every fixed R, then $g^{\#}(\xi) \leq 1$ for all $\xi \in \mathbb{C}$ and $g_n^{\#}(0) = \rho_n f_n^{\#}(z_n) = 1$ for every n, so $g^{\#}(0) = 1$. □

A.2.2 Montel's and Picard's Theorems

Suppose that $f \in \mathcal{M}(\{0 < |z - z_0| < r\})$.

Definition A.59 (Omitted Value). *We say that $w_0 \in \mathbb{C}_\infty$ is an **omitted value** of $f(z)$ at z_0 if there exists $\delta > 0$ such that $f(z) \neq w_0$ for $0 < |z - z_0| < \delta$*

Theorem A.60 (Montel). *A family $\mathcal{F}$ of meromorphic functions on a domain D that omits three values is normal.*

Proof. Since normality is a local property, we can assume $D = \{|z| < 1\}$. By composing the functions in $\mathcal{F}$ with a fractional linear transformation, we can assume that the omitted values are $0, 1$ and ∞.

Since the functions in $\mathcal{F}$ are then holomorphic and nonzero on D, they have holomorphic roots of all orders. Let $\mathcal{F}_k$ be the family of all 2^kth roots of functions in $\mathcal{F}$, using Marty's theorem and a computation we can show that $\mathcal{F}_k$ is normal if and only if $\mathcal{F}$ is normal.

The functions in $\mathcal{F}_k$ omit the values $0, \infty$ and all 2^kth roots of 1. We argue it by contradiction.

Suppose $\mathcal{F}$ is not normal, then $\mathcal{F}_k$ is not normal. Let $G_k(\xi)$ be the entire function from Zalcman's Lemma, which satisfies $G_k^{\#}(\xi) \leq 1$, $G^{\#}(0) = 1$ and G_k is a limit of restrictions of functions in $\mathcal{F}_k$, appropriately scaled. Since the functions in $\mathcal{F}_k$ omit the 2^kth roots of the unity, so do their scaled restrictions, and from Hurwitz's theorem so does any non-constant limit. Thus G_k omits the 2^kth roots of 1 (G is non-constant because $G^{\#}(0) = 1$).

By Marty's theorem, $\{G_k\}$ is a normal family. Let G be any normal limit of a subsequence of the G_k's, then $G^{\#}(0) = 1$, so that G is non-constant, and by Hurwitz's theorem, G omits all 2^kth roots of 1, for all $k \geq 1$.

Since G is an open mapping, G omits the unit circle (the closure of all the 2^kth roots of 1 is $\{|z| = 1\}$). Thus either $|G| < 1$ or $|G| > 1$. Since G is an entire function, applying Liouville's theorem in the first case with G and in the second case with $1/G$, we conclude that G is constant. Which is a contradiction. □

Theorem A.61 (Picard's Big Theorem). *Suppose $f(z)$ is meromorphic on a punctured neighborhood $\{0 < |z - z_0| < \delta\}$ of z_0. If $f(z)$ omits three values at z_0, then $f(z)$ extends to be meromorphic at z_0.*

Proof. We assume $z_0 = 0$ and that $f(z)$ omits the values 0 and ∞ on the punctured disk. Then $f(z)$ is holomorphic on the punctured disk $\{0 < |z - z_0| < \delta\}$. Let

$\{\varepsilon_n\}_n$ be a sequence that decreases to 0 (we can suppose that $\varepsilon_n < 1$) and define $g_n(z) = f(\varepsilon_n z)$, for $0 < |z| < \delta$.

Then $\{g_n\}$ omits three values, including 0 and ∞. By Montel's theorem, $\{g_n\}$ is a normal family. Passing to a subsequence, we can assume that $g_n(z)$ converges normally to $g(z)$ for $0 < |z| < \delta$.

- If $g(z)$ is not identically ∞, then $g(z)$ is holomorphic for $0 < |z| < \delta$. Fix $\rho \in (0, \delta)$ and choose M such that $|g(z)| < M$ for $|z| = \rho$. Then, for n large enough we have $|g_n(z)| < M$ for $|z| = \rho$, and consequently $|f(z)| < M$ for $|z| = \varepsilon_n \rho$. By the maximum principle, $|f(z)| < M$ for $\varepsilon_n \rho \leq |z| \leq \rho$ (for all n large enough).

  This annulus increases to a punctured neighborhood of 0 on which $|f(z)| < M$. By Riemann's Theorem on Removable Singularities, $f(z)$ extends to be holomorphic at 0.

- If $g(z) \equiv \infty$, we apply the above argument to $1/f(z)$. We obtain that $1/f(z)$ extends to be holomorphic at 0, so f extends meromorphically to have a pole at 0.

□

Theorem A.62 (Picard's Big Thorem)**.** *Suppose that $f(z)$ has an essential singularity at $z_0 \in \mathbb{C}$. Then in each neighborhood of z_0 f assumes each complex number, with at most one exception, infinitely often.*

Proof. In a small enough punctured disk centered in z_0 $A_\delta = \{0 < |z - z_0| < \delta\}$ the function f is holomorphic (it is an isolated singularity). Suppose that it misses two complex values, then $f \colon A_\delta \mapsto \mathbb{C}_\infty$ misses those two complex values and ∞ so we can apply Theorem A.61 to obtain a contradiction.

If there are two complex values which are assumed only a finite number of times, taking A_δ with δ small enough, we obtain a punctured disk in which f does not assume two complex values, which is a contradiction. □

Theorem A.63 (Picard's Little Theorem)**.** *The image of an entire non-constant function misses at most one point of $\mathbb{C}$.*

Proof. Suppose that $f \in \mathcal{H}(\mathbb{C})$.

If $f(z)$ is a polynomial, then for the Fundamental Theorem of Algebra we obtain $f(\mathbb{C}) = \mathbb{C}$.

If $f(z)$ is not a polynomial, then $f(z) = \sum_{n=0}^{\infty} c_n z^n$, with $c_n \neq 0$ for infinitely many $n \geq 0$. Then $g(z) = f(1/z)$ has an essential singularity at $z = 0$. By Picard's (Big) Theorem, $g(\mathbb{C} \setminus \{0\})$ is $\mathbb{C}$ or $\mathbb{C} \setminus \{w\}$, with $w \in \mathbb{C}$. But $g(\mathbb{C} \setminus \{0\}) = f(\mathbb{C} \setminus \{0\})$ and we obtain the result. □

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