International Game Theory Review

DYNAMIC BARGAINING AND TIME-CONSISTENCY IN LINEAR-STATE AND HOMOGENEOUS LINEAR-QUADRATIC COOPERATIVE DIFFERENTIAL GAMES

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Received (Day Month Year)
Revised (Day Month Year)

Three different solution concepts are reviewed and computed for linear-state and homogeneous linear-quadratic cooperative differential games with asymmetric players. Discount rates can be nonconstant and/or different. Special attention is paid to the issues of time-consistency, agreeability and subgame-perfectness, both from the viewpoint of sustainability of cooperation and from the credibility of the announced equilibrium strategies.

Keywords: Cooperative differential games; asymmetric players; time-consistency; dynamic bargaining; linear-state games; linear-quadratic games.

1. Introduction

This paper deals the problem of dynamic individual rationality (DIR) in cooperative differential games with asymmetric players. Players can be asymmetric in their utility functions and/or their discount functions. We address the problem for constant and nonconstant discount rates. Two notions of DIR have been widely studied in the literature: time-consistency (Petrosyan and Zenkevich [1996], Petrosyan [1997]) and agreeability (Kaitala and Pohjola (1990)). Suppose that players can make an agreement in order to coordinate their strategies in the future, but this agreement is not binding. An agreement (or cooperative solution) is said to be time-consistent if, at any future moment, along the optimal trajectory, all players find if optimal to stick to their parts in the agreement. If this property is satisfied, not just along the optimal trajectory, but for any trajectory, the cooperative solution is said to be agreeable.

Consider a cooperative differential game with equal and constant discount rates for all players. Pareto optimal solutions can be found by maximizing a weighted

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sum of the payoffs. Conditions for time-consistency and agreeability were studied, for linear-state differential games, in Jørgensen *et al.* [2003]; and for linear-quadratic differential games, in Jørgensen *et al.* [2005]. In the case of TU cooperative differential games, if players maximize their joint payoffs, this sum will be non lower than the sum of payoffs obtained under non cooperation. In that case, side payments schemes can be introduced to guarantee that, in any subgame starting along the cooperative trajectory, all players will receive higher (nonlower) payoffs in the cooperative solution than in the disagreement solution (see e.g. Jørgensen and Zaccour [2001] for a 2-player game, Yeung and Petrosyan [2004] for payoff distribution procedures in N-player differential games, or Petrosyan and Zaccour [2003] for the introduction of allocation schemes via the Shapley value). For NTU cooperative differential games, among all the Pareto optimal solutions, one could look for those guaranteeing the stability of cooperation. This issue was addressed in Yeung and Petrosyan [2005] (see also Petrosyan and Yeung [2014] and references therein). More recently, the possibility of introducing nonconstant weights for players in the search of time-consistent (agreeable) solutions was proposed in Marín-Solano [2014] (for differential games) and Yeung and Petrosyan [2015] (in a discrete time setting). In Castañer *et al.* [2020] such nonconstant weights were obtained as the result of repeated bargainings along time.

If players discount the future at different rates, or with nonconstant rates, in the computation of the "optimal" decision rules, a different problem of time-inconsistency arises: what is optimal for the coalition at time t is not optimal at time s, for $s > t$. We refer to Marín-Solano and Shevkoplyas [2011] for the study of differential games with time-inconsistent preferences, and to de-Paz *et al.* [2013] and Ekeland *et al.* [2013] for a detailed analysis of the case in which players have different discount rates (in the former paper the problem with different discount functions and nonconstant discount rates was also briefly addressed). The solution concept proposed in those papers is time-consistent and subgame perfect according to the definition in noncooperative differential games (see e.g. Dockner *et al.* [2000] or Haurie *et al.* [2012]). Indeed, in the computation of the solution, each coalition solves a sequential game, in which the coalition at different times is seen as a different player. However, it is not Pareto optimal, in general. As a result, joint payoffs under time-consistent cooperation can be lower than under noncooperation (Marín-Solano, [2015]). A way to avoid this problem is to assign nonconstant (state-dependent) weights to the players. As mentioned above, Castañer *et al.* [2020] proposed a way to fix such nonconstant weights as the result of a dynamic bargaining procedure. The corresponding solution, that makes use of memory (trigger) strategies, satisfies the property of DIR (i.e., it is time-consistent and agreeable in the language of NTU cooperative differential games). Alternatively, for memoryless strategies, a time-consistent dynamic bargaining procedure has been recently introduced in Castañer *et al.* [2021]. However, this solution concept, that is time-consistent and subgame perfect in the language of noncooperative differential games, linked to the issue of credibility of the announced strategy, may not satisfy the property of DIR.

In the present paper, all the previous issues are discussed for general differential games with possibly nonconstant and/or different discount rates, by paying special attention to two classes of differential games. First, linear-state differential games are studied. These games admit constant strategies and linear value functions, which allows us to develop a more detailed analysis on both the different cooperative solution concepts and the issues related to DIR. Next, homogeneous linear-quadratic differential games are considered.

The paper is structured as follows. Section 2 describes the general model and provides the basic definitions. Section 3 presents a review of two new solution concepts, based on bargaining theory, recently proposed in the literature of NTU cooperative differential games. Section 4 is devoted to the study of linear-state differential games with general discount functions, by paying special attention to conditions on time-consistency and agreeability of the different solutions. Homogeneous linear-quadratic differential games are analyzed in Section 5. Section 6 concludes the paper.

2. Preliminaries

Consider an N-player differential game played on the time interval $[0,\infty)$. Let $x \in X \subset \mathbf{R}$ be the state variable. For each player $i \in \{1,\dots,N\}$, let $u_i \in U_i \subset \mathbf{R}$ be her control variable (each player has a scalar control), $u = (u_1,\dots,u_n)$ the corresponding vector of decision rules, $L_i(x,u)$ the instantaneous utility function, and $\theta_i(s)$ the discount function. The payoff functional of player i at time t is

$$J_i(x_t;u_1,\dots,u_n;t) = \int_t^\infty \theta_i(s-t)L_i(x(s),u_1(s),\dots,u_n(s))\,ds\ , \quad \text{with} \tag{1}$$

$$\dot{x}(s) = f(x(s),u_1(s),\dots,u_N(s))\ ,\ x(t) = x_t\ . \tag{2}$$

We will assume that functions L_i, f and θ_i are, at least, continuously differentiable in all their arguments. We will also assume that, along the admissible control trajectories, equation (2) admits a unique solution and the integral in (1) converges. The extension to the multidimensional state and control case is straightforward.

Note that Problem (1)-(2) is autonomous. In the present paper we will center our attention in stationary solutions, so that equilibrium strategies and value functions will not depend explicitly of time, but only on the state of the system.

First, we recall the definitions of time-consistency and subgame perfectness in noncooperative differential games (see, e.g., Dockner *et al.* [2000] or Haurie *et al.* [2012]), linked to the issue of credibility of the announced equilibrium strategies.

Definition 1. *Let $\Gamma(x_0,0)$ denote a game played along $[0,\infty)$, with initial state $x_0 \in X$, and let $\Gamma(x,t)$ be the corresponding subgame defined on the time interval $[t,\infty)$ with initial state $x(t) = x \in X$.*

(1) Let $(\phi_1,\phi_2,\dots,\phi_N)$ be a Markovian solution for the game $\Gamma(x_0,0)$, and denote by $x^*(t)$ the unique state trajectory generated by the solution to the game. The

solution is time consistent if, for each $t \in [0, \infty)$*, the subgame* $\Gamma(x^*(t), t)$ *admits a Markovian solution* $(\psi_1, \psi_2, \ldots, \psi_N)$ *such that* $\psi_i(y, s) = \phi_i(y, s)$ *holds for all* $i \in \{1, \ldots, N\}$ *and all* $(y, s) \in X \times [t, \infty)$*.*

(2) If, for each $(x, t) \in X \times [0, \infty)$*, the subgame admits a Markovian solution* $(\psi_1, \psi_2, \ldots, \psi_N)$ *such that* $\psi_i(y, s) = \phi_i(y, s)$ *holds for all* $i \in \{1, \ldots, N\}$ *and all* $(y, s) \in X \times [t, \infty)$*, the solution* $(\phi_1, \phi_2, \ldots, \phi_N)$ *is said to be subgame perfect.*

Next, we recall the definitions of time-consistent and agreeable cooperative solutions. Let us denote by $V_i(x)$ the payoff of player i under some cooperation or agreement with all the other players. In a noncooperative setting, let $W_i(x)$ represent the corresponding payoffs of all players. In the present paper, $W_i(x)$ will denote the payoffs in the Markov Perfect Nash Equilibria (MPNE).

Definition 2. *Let* $x^c(\tau)$ *be the trajectory along the cooperative solution. A cooperative solution is time-consistent at* $(x_0, 0)$ *if, at any position* $(x^c(\tau), \tau)$ *and for all* $\tau \in [0, \infty)$*, it holds that*

$$V_i(x^c(\tau)) \geq W_i(x^c(\tau)) \quad \textit{for all} \quad i = 1, \ldots, n\ . \tag{3}$$

If inequality (3) is satisfied, not just along the cooperative trajectory, but for all possible deviations of it, then we have:

Definition 3. *A cooperative solution is agreeable at* $(x_0, 0)$ *if, for any feasible position* $(x(\tau), \tau)$*, and for all* $\tau \in [0, \infty)$*, it holds that*

$$V_i(x^c(\tau)) \geq W_i(x^c(\tau)) \quad \textit{for all} \quad i = 1, \ldots, n\ . \tag{4}$$

In Definitions 2 and 3 it is costumary to take as value functions $V_i(x)$ the payoffs of players obtained from joint maximization. If the players discount the future at the same constant discount rate, this solution is time-consistent subgame perfect in the sense of Definition 1 (but not, in general, time-consistent as a cooperative solution, i.e., according to Definition 2). If discount rates are nonconstant and/or different, a new solution concept, the t-cooperative equilibrium rule, has been proposed in the last decade, that is time-consistent and subgame perfect in the sense of Definition 1. Next we present this solution concept for the general case of (possibly) unequal and nonconstant weights. Following Marín-Solano (2014), in a cooperative setting we can aggregate preferences as

$$J^c(x_t, u, t) = \sum_{i=1}^{N} \lambda_i(x_t) J_i(x_t, u, t) = \sum_{i=1}^{N} \lambda_i(x) \int_t^\infty \theta(s - t) L_i(x(s), u(s))\, ds\ . \tag{5}$$

Coefficients $\lambda_i(x_t) \geq 0$ represent the weight of agent i at state x_t and time t. In this paper, since we are interested in stationary solutions, we assume that weights do not depend explicitly on time t, and $\lambda_i(x)$ are continuously differentiable.

If $u^*(s) = \phi(x(s))$ is a continuously differentiable equilibrium rule for Problem (5) subject to (2), by denoting $x_t = x$, the corresponding value function is

$V(x) = \sum_{i=1}^{N} \lambda_i(x) V_i(x)$, where $V_i(x) = \int_t^{\infty} \theta_i(s-t) L_i(x(s), \phi(x(s)))\, ds$. For $\epsilon > 0$ and $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_N)$, $\bar{u}_i \in U_i \subset \mathbf{R}$, let

$$u_\epsilon(s) = \begin{cases} \bar{u} & \text{if } s \in [t, t+\epsilon)\ , \\ \phi(x(s)) & \text{if } \quad s \geq t+\epsilon\ . \end{cases} \tag{6}$$

Let $J(x, u_\epsilon, t) = \sum_{i=1}^{N} \lambda_i(x) J_i(x, u_\epsilon, t)$, where

$$J_i(x, u_\epsilon, t) = \left\{ \int_t^{t+\epsilon} \theta_i(s-t) L_i(x(s), \bar{u})\, ds + \int_{t+\epsilon}^{\infty} \theta_i(s-t) L_i(x(s), \phi(x(s)))\, ds \right\} . \tag{7}$$

If we expand $J(x, u_\epsilon, t)$ in ϵ, we obtain $J(x, u_\epsilon, t) = V(x) + P(x, \phi, \bar{u})\epsilon + o(\epsilon)$.

Definition 4. *A decision rule $u^{tc}(s) = \phi^{tc}(x(s))$ is a t-cooperative equilibrium with nonconstant weights $\lambda_i(x)$, $i = 1, \ldots, N$, if function $P(x, \phi, \bar{u})$ attains its maximum for $\bar{u} = \phi^{tc}(x)$. In the case with $\lambda_1 = \cdots = \lambda_N$ we simply say that the strategy is a t-cooperative equilibrium.*

In Theorem 1 in Marín-Solano (2014) (for the case of constant weights, see de-Paz *et al.* (2013) and Ekeland *et al.* (2013)) it is proved that, under certain regularity conditions, t-cooperative equilibrium can be obtained by solving

$$(\phi_1^{tc}, \ldots, \phi_N^{tc}) \in \text{argmax}_{\{\phi_1, \ldots, \phi_N\}} \left\{ \sum_{i=1}^{N} \lambda_i(x) \left(L_i(x, u) + V_i'(x) \cdot f(x, u) \right) \right\} . \tag{8}$$

3. Dynamic bargaining

3.1. *The agreeable dynamic bargaining solution*

In Castañer *et al.* (2020) a new cooperative solution concept for NTU differential games, the agreeable dynamic bargaining solution, was proposed. Its extension to general discount functions is straightforward. The idea is to work within a Nash bargaining setting, but taking into account the possibility that players can renegotiate agreements achieved at time t at every future moment $s > t$ (and, in particular, immediately later). In case of disagreement, the threat point is to receive the non-cooperative outcome given by a Markov Perfect Nash Equilibrium at perpetuity. This threat point will be credible as long as, in case of disagreement, players will not have the possibility to renegotiate in the future. From this perspective, memory strategies are considered. Since in the trigger strategy (punitive mode of play) the threat is an equilibrium, it becomes credible.

Let us consider the generalized Nash welfare function with strictly positive bargaining powers $\eta_1, \ldots, \eta_N$

$$\prod_{i=1}^{N} \left[J_i(x, c_1, \ldots, c_N, t) - W_i(x, t) \right]^{\eta_i} . \tag{9}$$

Without loss of generality, we normalize bargaining powers so that $\sum_{i=1}^{N} \eta_i = 1$. For $\eta_1 = \cdots = \eta_N = 1/N$ we recover the classical Nash bargaining solution.

For a decision rule $(u_1, \ldots, u_N) = (\phi_1(x), \ldots, \phi_N(x))$ and a set of threat value functions $W(x,t) = (W_1(x,t), \ldots, W_N(x,t))$ satisfying $J_i(x, \phi_1(x), \ldots, \phi_N(x), t) - W_i(x,t) \geq 0$, for all $i = 1, \ldots, N$, let

$$\Pi(x,t) = \prod_{i=1}^{N} \left[J_i(x, \phi_1(x), \ldots, \phi_N(x), t) - W_i(x,t) \right]^{\eta_i} . \tag{10}$$

If we take the variations (6), let $J_i(x, u_\epsilon, t)$ given as in (7). The corresponding Nash product becomes

$$\Pi(x, u_\epsilon, t) = \prod_{i=1}^{N} \left[J_i(x, u_\epsilon, t) - W_i(x,t) \right]^{\eta_i} = \Pi(x,t) + \Pi_1(x, \phi, \bar{u}, t)\epsilon + o(\epsilon) , \tag{11}$$

with

$$\Pi_1(x, \phi, \bar{u}, t) = \lim_{\epsilon \to 0^+} \frac{\Pi(x, u_\epsilon, t) - \Pi(x,t)}{\epsilon} .$$

Definition 5. *A strategy $\phi^{db}(x) = (\phi_1^{db}(x), \ldots, \phi_N^{db}(x))$, with payments $V^{db}(x) = (V_1^{db}(x, \phi^{db}(x)), \ldots, V_N^{db}(x, \phi_N^{db}(x)))$, is said to be an agreeable dynamic bargaining solution if*

$$\phi^{db}(x,t) = \arg\max_{\{\bar{u}\}} \Pi_1(x, \phi^{db}, \bar{u}) ,$$

where the threat point $(W_1(x), \ldots, W_N(x))$ in case of disagreement at time t, with $x(t) = x$, is given as follows: For $s \in [t, \infty)$, players apply strategies $\phi^{nc} = (\phi_1^{nc}, \ldots, \phi_N^{nc})$ such that

$$W_i(x) = J_i(x, \phi^{nc}) \geq J_i(x, \phi_{-i}^{nc}, \sigma_i) , \quad \forall i, \sigma_i , \quad i = 1, \ldots, N ,$$

where $\sigma_i : X \to U_i \subset \mathbf{R}$ is any possible admissible Markovian strategy for player i.

In Definition 5 we have omitted the temporal argument in the different expressions. The reaon is that, in our autonomous problem, both the threat strategies and the agreeable dynamic bargaining strategies are stationary, and the corresponding value functions depend just on the state of the system. The following proposition characterizes interior agreeable dynamic bargaining solutions.

Proposition 1. *Assume that an agreeable dynamic bargaining solution exists satisfying the constraints $V_i^{db} - W_i > 0$, for all $i = 1, \ldots, N$. Then it is the solution to (8) with weight functions given by*

$$\lambda_i^{db}(x) = \frac{\eta_i}{V_i^{db}(x, \phi^{db}(x)) - W_i(x)} .$$

Proof. It is similar to the proof of Proposition 1 in Castañer *et al.* (2020) for the case of constant discount rates. □

Note that, by construction, the agreeable dynamic bargaining solution is agreeable and, therefore, time-consistent, as a cooperative solution.

3.2. *The time-consistent dynamic bargaining solution*

Whereas the agreeable dynamic bargaining solution is time-consistent (and agreeable) in the language of NTU cooperative differential games, it makes use of memory strategies: in case of disagreement, cooperation will not be allowed forever. If, on the contrary, new negotiations can take place in the future (something that is not unusual in real life situations), the threat losses credibility. The time-consistent dynamic bargaining solution proposed in Castañer *et al.* (2021) addresses this issue, by taking into account that, in case of disagreement, new negotiations can take place at every future moment and, in particular, immediately later. As a result, the actual threat is to act in a noncooperative way just during a very small period of time. In this section we extend the results in Castañer *et al.* (2021) to the case of general discount functions.

As in the previous section, let us consider the generalized Nash welfare function (9) or (10) for a particular decision rule. By taking variations (6), then we obtain (11) with $J_i(x, u_\epsilon, t)$ given as in (7). The objective is, as in the agreeable dynamic bargaining solution, to maximize the first order term $\Pi_1(x, \phi, \bar{u}, t)$ for the new threat point, in such a way that $\bar{u}^* = \phi(x(t), t)$. However, the nature of the new threat point implies that we can not proceed as in Proposition 1 to characterize the solution.

Let us assume that, for $s \in [t, t+\epsilon)$, the decision rule in case of non cooperation is given by $\phi^{\epsilon,nc}(x(s), s)$ and, for $s \geq t+\epsilon$, players follow $\phi^b(x(s), s)$. More precisely,

Definition 6. *Assume that, at time* t*, players take as given the future decision rule* $\phi^b(x(s), s)$*, for* $s \geq t + \epsilon$*. The threat point in case of disagreement during the time interval time* $[t, t+\epsilon)$*, with* $x(t) = x$*, is given as follows:*

$$W_i^\epsilon(x,t) = J_i(x, \phi^{\epsilon,nc}(x,t), t) \geq J_i(x, \phi_{-i}^{\epsilon,nc}(x,t), \sigma_i(x,t), t) \ , \quad \forall i, \sigma_i \ , \quad i = 1, \ldots, N \ ,$$

where

- $\sigma_i \colon X \times [t, \infty) \to U_i \subset \mathbf{R}$*,* $i = 1, \ldots, N$*, is given by*

$$\sigma_i(x(s), s) = \begin{cases} \bar{\sigma}_i(x(s), s) & \text{if } t \leq s < t+\epsilon \ , \\ \phi_i^b(x(s), s) & \text{if } s \geq t+\epsilon \ ; \end{cases}$$

- $\bar{\sigma}_i \colon X \times [t, t+\epsilon) \to U_i \subset \mathbf{R}$*,* $i = 1, \ldots, N$*, is any possible admissible feedback law for player* i *for the problem with planning horizon* $[t, t+\epsilon)$*; and*
- $\phi_j^{\epsilon,nc} \colon X \times [t, \infty) \to U_j \subset \mathbf{R}$*,* $j = 1, \ldots, N$*, is such that* $\phi_j^{\epsilon,nc}(x(s), s) = \phi_j^b(x(s), s)$*, for* $s \geq t+\epsilon$*.*

Hence, the threat point is given by

$$W_i^\epsilon(x,t) = \int_t^{t+\epsilon} \theta_i(s-t) L_i(x^\epsilon(s), \phi^{\epsilon,nc}(x^\epsilon(s), s)) \, ds$$

$$+ \int_{t+\epsilon}^\infty \theta_i(s-t) L_i(x^\epsilon(s), \phi^b(x^\epsilon(s), s)) \, ds \ ,$$

for $i \in \{1, \ldots, N\}$. Let $x^\epsilon(s)$ be the solution to $\dot{x}^\epsilon(s) = f(x^\epsilon(s), \phi^{\epsilon,nc}(x^\epsilon(s)))$, with $x^\epsilon(t) = x$, for $s \in [t, t+\epsilon)$ and $x^\epsilon(t) = x$; $\bar{x}(s)$ the solution to $\dot{\bar{x}}(s) = f(\bar{x}(s), \bar{u})$, with $\bar{x}(t) = x$, for $s \in [t, t+\epsilon)$ and $\bar{x}(t) = x$; and $\dot{x}_\epsilon(s) = f(x_\epsilon(s), \phi^b(x_\epsilon(s)))$ for $s \geq t+\epsilon$, with the initial condition $x_\epsilon(t+\epsilon)$ derived by continuity. From (7) for $\phi = \phi^b$ we obtain

$$J_i(x, u_\epsilon, t) - W_i^\epsilon(x, t)$$
$$= \int_t^{t+\epsilon} \theta_i(s-t) \left[L_i(\bar{x}(s), \bar{u}) - L_i(x^\epsilon(s), \phi^{\epsilon,nc}(x^\epsilon(s), s))\right] ds$$
$$+ \int_{t+\epsilon}^\infty \theta_i(s-t) \left[L_i(x_\epsilon(s), \phi^b(x_\epsilon(s), s)) - L_i(x^\epsilon(s), \phi^b(x^\epsilon(s), s))\right] ds \, .$$

The first integral becomes $\left[L_i(x, \bar{u}) - L_i(x, \phi^{0,nc}(x, t))\right] \epsilon + o(\epsilon)$. In addition, if the discount rate is defined by $\rho_i(\tau) = -\frac{\dot{\theta}_i(\tau)}{\theta_i(\tau)}$ so that $\theta_i(s) = \exp\left(-\int_0^s \rho_i(\tau)\, d\tau\right)$, then $\theta_i(s-t) = \theta_i(\epsilon) \cdot \theta_i(s-t-\epsilon)$ and, after some calculations, the second integral simplifies to $\frac{\partial V_i^b(x,t)}{\partial x} \left[f(x, \bar{u}) - f(x, \phi^{0,nc}(x, t))\right] \epsilon + o(\epsilon)$ where, for $x(t) = x$, $V_i^b(x, t) = \int_t^\infty \theta_i(s-t) L_i(x(s), \phi_i^b(x(s), s))\, ds$. By simplifying, the Nash welfare function becomes

$$\prod_{i=1}^N \left[J_i(x, u_\epsilon, t) - W_i^\epsilon(x, t)\right]^{\eta_i} = \Pi_1(x, \phi^b, \bar{c}, t)\epsilon + o(\epsilon) \, , \quad \text{where}$$

$$\Pi_1 = \prod_{i=1}^N \left[L_i(x, \bar{u}) - L_i(x, \phi^{0,nc}(x, t)) + \frac{\partial V_i^b(x, t)}{\partial x} \left(f(x, \bar{u}) - f(x, \phi^{0,nc}(x, t))\right)\right]^{\eta_i} . \quad (12)$$

Definition 7. *Let*

$$\phi_i^\epsilon(x(s), s) = \begin{cases} \phi_i^{\epsilon,nc}(x(s), s) & \text{if } t \leq s < t+\epsilon \, , \\ \phi_i^b(x(s), s) & \text{if } s \geq t+\epsilon \, ; \end{cases}$$

be the decision rule followed by player i, for $i = 1, \ldots, N$, in case of disagreement during the time period $[t, t+\epsilon)$. Let $\phi^0(x, t) = \lim_{\epsilon \to 0^+} \phi^\epsilon(x, t)$. We define the time-consistent dynamic bargaining solution (TCB) as

$$\phi^b = \arg\max_{\{\bar{u}\}} \Pi_1(x, \phi^b, \bar{u}, t) \, , \quad (13)$$

where $\Pi_1(x, \phi^b, \bar{u}, t)$ is given by (12).

Note that, since there is no commitment and players can bargain again at any possible future moment $\tau > t$, ϵ can be arbitrarily small. Hence, the threat point is given by $\phi^{0,nc}(x, t) = \lim_{\epsilon \to 0^+} \phi^{\epsilon,nc}(x, t)$, with $\phi^{\epsilon,nc}(x, t)$ given by Definition 6. Since our problem is autonomous, in the limit $\epsilon \to 0^+$, strategies become stationary and value functions do not depend on time. Hence, in the following, we will omit the temporal argument in the expressions of the strategies and the value functions.

If interior solutions to equation (13) exist, the first order optimality conditions are given by

$$\sum_{i=1}^{N} \frac{\eta_i}{L_i(x,\bar{u}) - L_i(x,\phi^{0,nc}(x)) + \big(V_i^b(x)\big)' \left[f(x,\bar{u}) - f(x,\phi^{0,nc}(x))\right]} \Bigg(\frac{\partial L_i(x,\bar{u})}{\partial \bar{u}_j} + \big(V_i^b(x)\big)' \frac{\partial f(x,\bar{u})}{\partial \bar{u}_j} \Bigg) = 0 \,, \tag{14}$$

for $j = 1, \ldots, N$. It is important to realize that the TCB solution obtained from Definitions 7 and 6 is, by construction, time-consistent and subgame perfect in the sense of Definition 1. On the contrary, time-consistency and agreeability according to Definitions 2 and 3 is not guaranteed, in general.

4. A linear-state differential game

Consider the N-player differential game

$$J_i(x_t; u_1, \ldots, u_n; t) = \int_t^{\infty} \theta_i(s-t) \left[h_i(u_1(s), \ldots, u_n(s)) - \varphi_i(s)\right] ds \,, \quad \text{with} \tag{15}$$

$$\dot{x}(s) = g(u_1(s), \ldots, u_N(s)) - \delta x(s) \,, \; x(t) = x_t \,, \tag{16}$$

i.e., in (1) and (2), $L_i(x, u_1, \ldots, u_N) = h_i(u_1, \ldots . u_N) - \varphi_i x$ and $f(x, u_1, \ldots, u_N) = g(u_1, \ldots, u_N) - \delta x$, for $i = 1, \ldots, N$. Then we have a linear-state differential game.

In the case with constant discount rates, it is well-known that linear-state differential games have the property that open-loop Nash equilibria are Markov perfect. The same property is preserved in the case with nonconstant discounting. Indeed, in a linear-state differential game with symmetric players and nonconstant discount rates, Karp [2017] showed that the limit equilibrium[a] is unique, independent of the state variable, and dominant, but there are many other differentiable state dependent Markov Perfect equilibria. In this section we show that, for the four solution concepts studied in the paper (MPNE, t-cooperative equilibrium, agreeable Nash bargaining solution and time-consistent dynamic bargaining solution), constant strategies exist, with corresponding linear value functions. In that case, time-consistent cooperative solutions are also agreeable. Conditions for the time-consistency (or agreeability) of the cooperative solutions are stated.

4.1. *The general case*

As in Section 3, we confine our interest to stationary solutions, as is standard in autonomous differential games in infinite horizon.

[a]The limit equilibrium is the limit of the sequence of equilibria of finite horizon models, as the planning horizon goes to infinity.

Proposition 2. *In the linear-state differential game (15)-(16), for the MPNE, the t-cooperative solution, the agreeable dynamic bargaining solution, and the time-consistent dynamic bargaining solution, constant decision rules exist. Along those solutions, the corresponding value functions are linear in the state variable and are such that* $V_i^J(x) = a_i x + b_i^J$, *for* $J \in \{tc, db, b\}$, *and* $W_i(x) = a_i x + b_i^{nc}$, *with*

$$a_i = -\varphi_i \int_0^\infty \theta_i(\tau) e^{-\delta\tau}\, d\tau \, .$$

Proof. It follows from the computation of the four solutions.

Markov Perfect Nash Equilibrium. Let $u_j^{nc} = \phi_j^{nc}(x)$, $j = 1, \ldots, N$, be a candidate to noncooperative MPNE. Then, as in the standard case (see Marín-Solano and Shevkoplyas [2011]), Player i has to solve

$$\max_{\{u_i\}} \left\{ h_i(u_i, \phi_{-i}^{nc}) - \varphi_i x + (W_i(x))' \left(g(u_i, \phi_{-i}^{nc}) - \delta x \right) \right\} \, .$$

From the first order optimality conditions, for all $i = 1, \ldots, N$,

$$\frac{\partial h_i(u_i, \phi_{-i}^{nc})}{\partial u_i} + (W_i(x))' \, \frac{\partial g(u_i, \phi_{-i}^{nc})}{\partial u_i} = 0 \, . \tag{17}$$

By guessing $W(x) = a_i^{nc} x + b_i^{nc}$, $(W_i(x))' = a_i^{nc}$ that implies that u_i^{nc} are constant. The coefficients a_i^{nc} and b_i^{nc} are derived by solving the value function. Note that

$$W_i(x) = \int_t^\infty \theta_i(s-t) \left[h_i(u_1^{nc}, \ldots, u_N^{nc}) - \varphi_i x(s) \right] ds \, , \tag{18}$$

where

$$\dot{x}(s) = g(u_1^{nc}, \ldots, u_N^{nc}) - \delta x(s) \quad \text{with} \quad x(t) = x \, . \tag{19}$$

The solution to (19) is

$$x(s) = x e^{-\delta(s-t)} + \frac{g(u^{nc})}{\delta} \left(1 - e^{-\delta(s-t)} \right) \, . \tag{20}$$

By substituting (20) in (18) we obtain $W_i(x) = a_i^{nc} x + b_i^{nc}$, with

$$a_i^{nc}(x) = -\varphi_i \int_0^\infty \theta_i(\tau) e^{-\delta\tau}\, d\tau \, , \tag{21}$$

$$b_i^{nc} = \int_0^\infty \theta_i(\tau) \left[h_i(u_1^{nc}, \ldots, u_N^{nc}) - \frac{\varphi_i g(u_1^{nc}, \ldots, u_N^{nc})}{\delta} \left(1 - e^{-\delta\tau} \right) \right] d\tau \, . \tag{22}$$

t-cooperative equilibrium. For the calculation of the the t-cooperative decision we solve (8) for $\lambda_i = 1$:

$$\max_{\{u_1, \ldots, u_N\}} \left\{ \sum_{i=1}^N \left(h_i(u_1, \ldots, u_N) - \varphi_i x \right) + \left(\sum_{i=1}^N \left(V_i^{tc}(x) \right)' \right) \left(g(u_1, \ldots, u_N) - \delta x \right) \right\} \, ,$$

hence

$$\sum_{i=1}^{N} \frac{\partial h_i(u_1,\ldots,u_N)}{\partial u_j} + \sum_{i=1}^{N} \left(V_i^{tc}(x)\right)' \frac{\partial g(u_1,\ldots,u_N)}{\partial u_j} = 0 \,, \tag{23}$$

for $j=1,\ldots,N$. By guessing $V_i^{tc}(x) = a_i^{tc}x + b_i^{tc}$, the corresponding decision rules are constant. The coefficients a_i^{tc} and b_i^{tc} are derived as in the noncooperative case. By substituting the upper indices $\{nc\}$ by $\{tc\}$ in (18), (19) and (20), we obtain that a_i^{tc} is given by (21) and

$$b_i^{tc} = \int_0^\infty \theta_i(\tau) \left[h_i(u_1^{tc},\ldots,u_N^{tc}) - \frac{\varphi_i g(u_1^{tc},\ldots,u_N^{tc})}{\delta} \left(1-e^{-\delta\tau}\right) \right] d\tau \,. \tag{24}$$

Agreeable dynamic bargaining solution. In this case, we have to compute the t-cooperative equilibria for weights given as in Proposition 1. By using (8), solutions to

$$\max_{\{u_1,\ldots,u_N\}} \left\{ \sum_{i=1}^{N} \left(\lambda_i^{db}(x) h_i(u) - \varphi_i x\right) + \left(\sum_{i=1}^{N} \lambda_i^{db}(x) \left(V_i^{db}(x)\right)' \right) (g(u)-\delta x) \right\} \tag{25}$$

satisfy

$$\sum_{i=1}^{N} \lambda_i^{db}(x) \frac{\partial h_i(u_1,\ldots,u_N)}{\partial u_j} + \sum_{i=1}^{N} \lambda_i^{db}(x) \left(V_i^{db}(x)\right)' \frac{\partial g(u_1,\ldots,u_N)}{\partial u_j} = 0 \,. \tag{26}$$

By guessing $V_i^{db}(x) = a_i^{db}x + b_i^{db}$, from Proposition 1, weight functions are given by

$$\lambda_i^{db}(x) = \frac{\eta_i}{(a_i^{db}-a_i^{nc})x + (b_i^{db}-b_i^{nc})} \,,$$

with a_i^{nc} and b_i^{nc} given by (21) and (22), respectively.

It is clear that equation (26) have constant solutions (and hence the corresponding value function will be linear in the state variable) if $a_i^{db} = a_i^{nc}$. Otherwise, decision rules will be typically nonconstant, and the corresponding value functions will be nonlinear, in contradiction with our hypothesis. Therefore, for a_i^{db} given by (21) we obtain a solution to (25) with linear value functions, whose coefficients b_i^{db} are given by

$$b_i^{db} = \int_0^\infty \theta_i(\tau) \left[h_i(u_1^{db},\ldots,u_N^{db}) - \frac{\varphi_i g(u_1^{db},\ldots,u_N^{db})}{\delta} \left(1-e^{-\delta\tau}\right) \right] d\tau \,. \tag{27}$$

Time-consistent dynamic bargaining solution. We guess $V_i^b(x) = a_i^b x + b_i^b$. By denoting $u^b = (u_1^b,\ldots,u_N^b)$, from (14) we obtain

$$\sum_{i=1}^{N} \frac{\eta_i}{h_i(u^b) - h_i(u^{0,nc}) + a_i^b\left(g(u^b) - g(u^{0,nc})\right)} \left(\frac{\partial h_i(u^b)}{\partial u_j^b} + a_i \frac{\partial g(u^b)}{\partial u_j^b} \right) = 0 \,, \tag{28}$$

for $j=1,\ldots,N$. This equation system does not depend on the state variable. Since, in case of disagreement, threat strategies $u^{0,nc} = (u_1^{0,nc},\ldots,u_N^{0,nc})$ are constant, its solutions are also constant strategies. As in the previous solutions, the corresponding value function is linear, with a_i^b given as in (21) and

$$b_i^b = \int_0^\infty \theta_i(\tau) \left[h_i(u_1^b, \ldots, u_N^b) - \frac{\varphi_i g(u_1^b, \ldots, u_N^{tc})}{\delta} \left(1 - e^{-\delta\tau}\right) \right] d\tau \, . \qquad (29)$$

□

Corollary 1. *In the linear-state differential game (15)-(16),*

(1) If a t-cooperative equilibrium is time consistent in the sense of Definition 2, then it is also agreeable.
(2) If a time-consistent dynamic bargaining solution is time consistent in the sense of Definition 2, then it is also agreeable.

Proof. It follows from the fact that $V_i^J(x) - V_i^{nc}(x) = b_i^J - b_i^{nc}$, for all $i = 1, \ldots, N$ and $J \in \{tc, b\}$. □

For the calculation of the MPNE, we should solve the equation system (17), with $(W_i(x))' = a_i^{nc} = a_i$ given by (21). The t-cooperative equilibrium can be computed by solving (23) with $(V_i^{tc}(x))' = a_i$. Agreeable dynamic bargaining strategies are the solutions to (26) with $(V_i^{db}(x))' = a_i$ and weights

$$\lambda_i^{db} = \frac{\eta_i}{b_i^{db} - b_i^{nc}} \, ,$$

with b_i^{nc} given by (22). The calculation of the time-consistent dynamic bargaining solution is more complicated because, before solving (28), we should compute first the threat strategy $u^{0,nc}$. We will do it in the example analyzed in Section 4.2.

In the case of constant discount rates, $\theta_i(t) = e^{-\rho_i t}$,

$$a_i = -\frac{\varphi_i}{\rho_i + \delta} \, .$$

In addition, from (22), (24), (27) and (29), for $J \in \{nc, tc, db, b\}$,

$$b_i^J = \frac{1}{\rho_i} \left[h_i(u_1^J, \ldots, u_N^J) - \frac{\varphi_i}{\rho_i + \delta} g(u_1^J, \ldots, u_N^J) \right] \, . \qquad (30)$$

4.2. *Application: An environmental model of pollution control*

As an application, consider an environmental problem where N countries can agree in their pollution strategies. If the emissions of country i, $E_i(t)$, are proportional to its production, the revenue function can be expressed as a function of the emissions, say $R_i(E_i) = \gamma_i \log(\alpha_i E_i)$, for $\gamma_i, \alpha_i > 0$. If the damage function is a linear function on the stock of the pollution $S(t)$, then $D_i(S) = \varphi_i S$, for $\varphi_i > 0$. Hence, the payoff functionals are given by

$$J_i(S, t) = \int_t^\infty \theta_i(s - t) \left[\gamma_i \log\left(\alpha_i E_i(s)\right) - \varphi_i S(s) \right] ds \, . \qquad (31)$$

As for the state dynamics, the evolution of the stock of pollution is governed by the differential equation

$$\dot{S}(s) = \sum_{i=1}^{N} \beta_i E_i(s) - \delta S(s) \ , \tag{32}$$

where $\beta_i \geq 0$ is a positive transformation parameter of emissions into pollution stock and δ is the natural absorption rate of pollution. For the case of a unique and constant discount rate, the time-consistency and agreeability of the standard cooperative solution was studied in Jørgensen *et al.* [2003]. In our more general setting, we need to compute the value of the parameters b_i^J, for $i = 1, \ldots, N$ and $J \in \{nc, tc, db, b\}$, and check if $b_i^{tc} \geq b_i^{nc}$ (for the agreeability of the t-cooperative equilibrium) and $b_i^d \geq b_i^{nc}$ (for the agreeability of the time-consistent dynamic bargaining solution). Since the agreeable dynamic bargaining solution is time-consistent and agreeable, $b_i^{db} \geq b_i^{nc}$, for all i.

In the following, we compute the solution to (31)-(32) for the diffferent solution concepts.

4.2.1. *Noncooperative MPNE*

From (17), we can easily compute the emissions of each country under noncooperation, that are given by

$$E_i^{nc} = \frac{\gamma_i}{\varphi_i \beta_i \int_0^\infty \theta_i(t) e^{-\delta t}\, dt} \ ,$$

for $i = 1, \ldots, N$.

In particular, in the case of constant discount rates, $\theta_i(t) = e^{-\rho_i t}$,

$$E_i^{nc} = \frac{\gamma_i(\rho_i + \delta)}{\varphi_i \beta_i} \ .$$

Note that emissions are increasing with the discount rate, i.e., more impatient countries pollute more, as expected. Concerning the payoffs, by substituting in (30) we obtain that, in this case,

$$b_i^{nc} = \frac{1}{\rho_i} \left[\gamma_i \log\left(\frac{\alpha_i \gamma_i (\rho_i + \delta)}{\varphi_i \beta_i} \right) - \frac{\varphi_i}{\rho_i + \delta} \sum_{j=1}^{N} \frac{\gamma_j(\rho_j + \delta)}{\varphi_j} \right] \ .$$

4.2.2. *t-cooperative equilibrium*

By using (23), the emissions of each player under t-cooperative behavior becomes

$$E_i^{tc} = \frac{\gamma_i}{\beta_i \sum_{j=1}^{N} \varphi_j \int_0^\infty \theta_j(t) e^{-\delta t}\, dt} \ ,$$

for $i = 1, \ldots, N$.

As in the case with constant and equal discount rates, we obtain the classical result $E_i^{nc} > E_i^{tc}$.

If discount rates ρ_i are constant,

$$E_i^{tc} = \frac{\gamma_i}{\beta_i \sum_{j=1}^N \frac{\varphi_j}{\rho_j+\delta}} \,.$$

Unlike the noncooperative case, if $\gamma_1 = \cdots = \gamma_N$ and $\beta_1 = \cdots = \beta_N$, emissions coincide and do not depend on the discount rate of each particular country. Therefore, in that case, it is beneficial for the society that all countries pollute the same, independently of their differences in their discount rates.

Finally, in order to check the agreeability (or time-consistency) of the t-cooperative decision rule, we should check if, for the values of the parameters of a particular model, $b_i^{tc} \geq b_i^{nc}$, where b_i^{tc} is obtained by substituting in (30),

$$b_i^{tc} = \frac{1}{\rho_i}\left[\gamma_i \log\left(\frac{\alpha_i\gamma_i}{\beta_i \sum_{j=1}^N \frac{\varphi_j}{\rho_j+\delta}}\right) - \frac{\varphi_i}{\rho_i+\delta}\sum_{j=1}^N \frac{\gamma_j}{\sum_{k=1}^N \frac{\varphi_k}{\rho_k+\delta}}\right] \,.$$

4.2.3. *Agreeable dynamic bargaining solution*

By solving (26) for the constant weights $\lambda_i^{db} = \eta_i/(b_i^{db} - b_i^{nc})$, we easily obtain

$$E_i^{db} = \frac{\gamma_i\eta_i}{(b_i^{db} - b_i^{nc})\beta_i \sum_{j=1}^N \frac{\varphi_j\eta_j}{b_j^{db}-b_j^{nc}} \int_0^\infty \theta_j(t) e^{-\delta t}\, dt} \,,$$

for $i = 1, \ldots, N$. Again, we obtain that emissions under the agreeable dynamic bargaining procedure are lower than under noncooperation, $E_i^{db} < E_i^{nc}$.

In the case of constant discount rates ρ_i, emissions become

$$E_i^{db} = \frac{\gamma_i\eta_i}{(b_i^{db} - b_i^{nc})\beta_i \sum_{j=1}^N \frac{\varphi_j\eta_j}{(b_j^{db}-b_j^{nc})(\rho_j+\delta)}} \,.$$

Unfortunately, we can not derive an explicit expression for coefficients b_i^b in the value function. Instead, we will have to solve numerically the highly nonlinear equation system

$$b_i^{db} = \frac{1}{\rho_i}\left[\gamma_i \log\left(\frac{\alpha_i\gamma_i\eta_i}{(b_i^{db} - b_i^{nc})\beta_i \sum_{j=1}^N \frac{\varphi_j\eta_j}{(b_j^{db}-b_j^{nc})(\rho_j+\delta)}}\right)\right.$$
$$\left. - \frac{\varphi_i}{\rho_i+\delta}\sum_{j=1}^N \frac{\gamma_j\eta_j}{(b_j^{db} - b_j^{nc})\sum_{k=1}^N \frac{\varphi_k\eta_k}{(b_k^{db}-b_k^{nc})(\rho_k+\delta)}}\right] \,,$$

for $i = 1, \ldots, N$.

4.2.4. *Time-consistent dynamic bargaining solution*

According to the time-consistent dynamic bargaining procedure, decision rules are obtained from (28), i.e.,

$$\frac{\eta_i}{\gamma_i\left(\log E_i^b - \log E_i^{0,nc}\right) + a_i\beta_i\left(E_i^b - E_i^{0,nc}\right)} \frac{\gamma_i}{E_i^b} = \qquad (33)$$

$$-\sum_{j=1}^{N}\frac{\eta_j a_j}{\gamma_j\left(\log E_j^b-\log E_j^{0,nc}\right)+a_j\beta_j\left(E_j^b-E_j^{0,nc}\right)}\beta_i\ ,$$

for $i=1,\dots,N$, where $a_i=-\varphi_i/(\rho_i+\delta)$.

Concerning the threat strategy, $E_i^{0,nc}$, we have:

Proposition 3. *In Problem (31)-(32), time-consistent bargaining solutions solve the system of nonlinear equations (33) with threat strategies given by*

$$\phi_i^{0,nc}=\frac{\gamma_i}{\varphi_i\beta_i\int_o^\infty\theta_i(s)e^{-\delta s}\,ds}\ . \tag{34}$$

Proof. First note that, from Definition 6, we have to compute a MPNE in a differential game played along the time interval $[t,t+\epsilon]$. If $\phi_j^{\epsilon,nc}(t)$ are the equilibrium strategies for players $j\neq i$, player i solves

$$\max_{\{E_i\}}\left\{\gamma_i\log(\alpha_iE_i)-\varphi_iS+\frac{\partial W_i^\epsilon}{\partial S}\left(\beta_iE_i+\sum_{j\neq i}\beta_j\phi_j^{\epsilon,nc}(t)-\delta S\right)\right\}\ .$$

Therefore, $\dfrac{\partial W_i^\epsilon}{\partial S}=-\gamma_i\dfrac{\gamma_i}{\beta_i\phi_i^{\epsilon,nc}(t)}$, so $W_i^\epsilon(S,t)=-\dfrac{\gamma_i}{\beta_i\phi_i^{\epsilon,nc}(t)}S+b_i^{\epsilon,nc}(t)$ and, therefore,

$$W_i^0(S,t)=-\frac{\gamma_i}{\beta_i\phi_i^{0,nc}(t)}S+b_i^{0,nc}(t)\ . \tag{35}$$

Next, from Definition 6,

$$\begin{aligned}W_i^\epsilon(S,t)=&\int_t^{t+\epsilon}\theta_i(s-t)\left[\gamma_i\log(\alpha_iE_i^{\epsilon,nc}(s))-\varphi_iS(s)\right]ds\\&+\int_{t+\epsilon}^\infty\theta_i(s-t)\left[\gamma_i\log\left(\alpha_iE_i^b(s)\right)-\varphi_iS(s)\right]ds\ ,\end{aligned} \tag{36}$$

with

$$\dot S(s)=\sum_{i=1}^N\beta_i\phi_i^{\epsilon,nc}(s)-\delta S(s)\ ,\quad S(t)=S\ ,\quad \text{for}\quad s\in[t,t+\epsilon)\ ,\quad\text{and} \tag{37}$$

$$\dot S(s)=\sum_{i=1}^N\beta_i\phi_i^b(s)-\delta S(s)\ ,\quad S(t+\epsilon)=S_{t+\epsilon}\ ,\quad\text{for}\quad s\geq t+\epsilon\ , \tag{38}$$

where $S_{t+\epsilon}$ is the limit when $s\to t+\epsilon$ of the solution to (37). By solving (38) and substituting the value of $S(s)$ for $s\geq t+\epsilon$ in (36),

$$W_i^\epsilon(S,t)=\int_t^{t+\epsilon}\theta_i(s-t)\left[\gamma_i\log(\alpha_iE_i^{\epsilon,nc}(s))-\varphi_iS(s)\right]ds+$$

$$+\gamma_i\int_{t+\epsilon}^\infty\theta_i(s-t)\log\left(\alpha_iE_i^b(s)\right)ds-\varphi_i\int_{t+\epsilon}^\infty\theta_i(s-t)\left(\int_{t+\epsilon}^s e^{-\delta(s-z)}\sum_{j=1}^N\beta_j\phi_j^b(z)\,dz\right)ds$$

$$-\varphi_i S_{t+\epsilon} \int_{t+\epsilon}^{\infty} \theta_i(s-t) e^{-\delta(s-t-\epsilon)}\, ds \,. \tag{39}$$

Next, by solving $S(s)$ in (37), computing

$$S_{t+\epsilon} = \lim_{s\to t+\epsilon^-} S(s) = S_t + \int_t^{t+\epsilon} e^{-\delta(t+\epsilon-\tau)} \sum_{j=1}^{N} \beta_j \phi_j^{\epsilon,nc}(\tau)\, d\tau \,,$$

substituting in (39), and taking the limit $\epsilon \to 0$, after several calculations we obtain

$$W_i^0(S,t) = -\left(\varphi_i \int_0^{\infty} \theta_i(\tau) e^{-\delta\tau}\, d\tau\right) S + b_i^{0,nc} \,. \tag{40}$$

Finally, by identifying (35) and (40), we obtain that the threat strategy is given by (34). □

Remark 1. The threat strategy to be used for the calculation of the time-consistent dynamic bargaining solution coincides, in this problem, with the standard MPNE. This result is a consequence of the simple structure of the model. Indeed, a similar result was also obtained in Castñer *et al.* [2021] in a simple nonrenewable resource model with logarithmic utilities. However, for other power utilities, that paper showed that MPNE are not the right threat point for this bargaining solution. In deed, this is also the case for homogeneous linear-quadratic differential games, as we show in Section 5.

Finally, once the strategies are determined by using numerical techniques, coefficients b_i^b are easily obtained from (29) or, for the case of constant discount rates, from (30).

4.3. *The case of quadratic revenues*

It is customary, in enviromental models of pollution control, to consider that revenues are described by second degree polynomials of the form $R_i(E_i) = E_i\left(\gamma_i - \frac{1}{2}E_i\right)$. Calculations for the different solution concepts can be replicated. For example, for the case of constant discount rates, it is easy to check that emissions are given by

$$E_i^{nc} = \gamma_i - \frac{\varphi_i \beta_i}{\rho_i + \delta} \,,$$

$$E_i^{tc} = \gamma_i - \beta_i \sum_{j=1}^{N} \frac{\varphi_j}{\rho_j + \delta} \,,$$

$$E_i^{db} = \gamma_i - \frac{\beta_i}{\eta_i}\left(b_i^{db} - b_i^{nc}\right) \sum_{j=1}^{N} \frac{\varphi_j \eta_j}{\left(b_j^{db} - b_j^{nc}\right)(\rho_j + \delta)} \,,$$

and

$$\frac{\eta_i}{\left(1-\frac{\varphi_i}{\rho_i+\delta}\right)\left(E_i^b-E_i^{0,nc}\right)}\left(E_i^b-\gamma_i\right)-\beta_i\sum_{j=1}^N\frac{\frac{\eta_j\varphi_j}{\rho_j+\delta}}{\left(1-\frac{\varphi_j}{\rho_j+\delta}\right)\left(E_j^b-E_j^{0,nc}\right)}=0\ ,$$

with $E_i^{0,nc}=E_i^{nc}$.

Coefficients b_i^J, for $J\in\{nc,tc,db,b\}$, are obtained from (30). As in the case with logarithmic utilities, the calculation of the two dynamic bargaining solutions has to be performed numerically.

5. An homogeneous linear-quadratic differential game

As a second example, we study an homogeneous linear-quadratic differential game. Homogeneous linear-quadratic differential games are characterized by the lack of linear terms in the objective functionals of all players. They have some well-known properties that simplify their analysis: stationary MPNE are linear in the state variable and value functions are quadratic. As we will see, these properties are important in the search of manageable agreeable and time-consistent dynamic bargaining solutions. Otherwise, the search of these solutions can become extremely complicated, in general. For the case of equal and constant discount rates, conditions for the sustainability of cooperation for this class of differential games were studied in Jørgensen *et al.* (2005).

In the present section, we study in detail the following simple homogeneous game:

$$J_i=\int_t^\infty\theta_i(s-t)\left[m_ix^2+g_iu_i^2\right]\,ds\ , \tag{41}$$

$$\dot{x}(s)=\sum_{j=1}^N\beta_ju_j-\delta x\ , \tag{42}$$

with $g_i\neq 0$.

For the computation of the different solution concepts in Problem (41)-(42), we proceed as in the previous section.

5.1. *Noncooperative MPNE*

From the first order optimality conditions of

$$\max_{\{u_i\}}\left\{m_ix^2+g_ix^2+(W_i(x))'\left(\beta_iu_i+\sum_{j=1,j\neq i}^N\beta_j\phi_j^{nc}(x)-\delta x\right)\right\}$$

we obtain $u_i^{nc}=-\dfrac{\beta_i\,(W_i(x))'}{2g_i}$. By guessing $V_i^{nc}(x)=a_i^{nc}x^2$, the noncooperative strategy of Player i becomes

$$u_i^{nc}(s)=-\frac{\beta_ia_i^{nc}}{g_i}x(s)\ . \tag{43}$$

For the calculation of a_i^{nc} we solve, for $s \in [t, \infty)$, the state equation (42) for the strategy (43) and initial condition $x(t) = x$, that is given by

$$x(s) = e^{-\left(\sum_{j=1}^N \frac{\beta_j^2 a_j^{nc}}{g_j} + \delta\right)(s-t)} x \, .$$

By substituting in the value function and using (43),

$$W_i(x) = a_i^{nc} x^2 = \int_t^\infty \theta_i(s-t) \left[m_i (x(s))^2 + g_i (u_i^{nc}(s))^2 \right] ds =$$

$$\left(m_i + \frac{\beta_i^2 + (a_i^{nc})^2}{g_i} \right) \left(\int_0^\infty \theta_i(\tau) e^{-\left(\sum_{j=1}^N \frac{\beta_j^2 a_j^{nc}}{g_j} + \delta\right)\tau} d\tau \right) x^2 \, .$$

Therefore,

$$a_i^{nc} = \left(m_i + \frac{\beta_i^2 + (a_i^{nc})^2}{g_i} \right) \left(\int_0^\infty \theta_i(\tau) e^{-\left(\sum_{j=1}^N \frac{\beta_j^2 a_j^{nc}}{g_j} + \delta\right)\tau} d\tau \right) \, .$$

In particular, for the case of contant discount rates ρ_i, coefficients a_i^{nc} satisfy the following system of algebraic Riccati equations:

$$m_i - a_i^{nc}(\rho_i + 2\delta) + \frac{\beta_i^2}{g_i} (a_i^{nc})^2 - 2 a_i^{nc} \sum_{j=1}^N \frac{\beta_j^2}{g_j} a_j^{nc} = 0 \, .$$

The above non-linear equation system is highly coupled and coefficients a_i^{nc} must be determined numerically in order to check time-consistency and agreeability.

5.2. *t-cooperative equilibrium*

We have to solve (8) for $\lambda_i = 1$, i.e.,

$$\max_{\{u_1,\ldots,u_N\}} \left\{ \sum_{i=1}^N \left(m_i x^2 + g_i x^2 \right) + \left(\sum_{i=1}^N \left(V_i^{tc}(x) \right)' \right) \left(\sum_{j=1}^N \beta_j u_j - \delta x \right) \right\} \, .$$

From the first order conditions, $u_i^{tc} = -\frac{\beta_i}{2g_i} \sum_{j=1}^N \left(V_j^{tc}(x) \right)'$. If $V_i^{tc}(x) = a_i^{tc} x^2$, the t-cooperative strategy of Player i is

$$u_i^{tc}(s) = -\frac{\beta_i}{g_i} \sum_{j=1}^N a_j^{tc} x(s) \, . \qquad (44)$$

Note that, if $\beta_i = \beta$ and $g_i = g$, for all $i \in \{1, \ldots, N\}$, then $u_j^{tc} = u_k^{tc}$, for all j, k.

For the calculation of a_i^{nc}, the solution to (42), with initial condition $x(t) = x$, for the strategy (44) is

$$x(s) = e^{-\left[\left(\sum_{j=1}^N \frac{\beta_j^2}{g_j}\right)\left(\sum_{k=1}^N a_k^{tc}\right) + \delta\right](s-t)} x \, .$$

By substituting in the value function, using (44) and proceeding as in the calculation of the MPNE, we easily derive

$$a_i^{tc} = \left(m_i + \frac{\beta_i^2}{g_i} \left(\sum_{j=1}^{N} a_j^{tc} \right)^2 \right) \left(\int_0^\infty \theta_i(\tau) e^{-2\left[\left(\sum_{j=1}^{N} \frac{\beta_j^2}{g_j} \right) \left(\sum_{k=1}^{N} a_k^{tc} \right) + \delta \right] \tau} d\tau \right) .$$

In particular, in the case of constant discount rates ρ_i, coefficients a_i^{tc} satisfy the following system of nonlinear algebraic equations:

$$a_i^{tc} \left(\rho_i + 2\delta + 2 \left(\sum_{j=1}^{N} \frac{\beta_j^2}{g_j} \right) \left(\sum_{k=1}^{N} a_k^{tc} \right) \right) = m_i + \frac{\beta_i^2}{g_i} \left(\sum_{j=1}^{N} a_j^{tc} \right)^2 . \tag{45}$$

In the case of equal discount rates ($\rho_i = \rho$), by taking the sum in i of all a_i^{tc}, we obtain a second degree equation in $A^{tc} = \sum_{j=1}^{N} a_j^{tc}$. By substituting its solution in (45), parameters a_i^{tc} can be computed. In the case of different discount rates, this procedure does not work, but we can still find algebraic solutions if there are at most three players with different discount rates. Note that we can solve equation (45) by expressing a_i^{tc} as a function of A^{tc}, i.e., $a_i^{tc} = \psi_i(A^{tc})$, where

$$\psi_i(A^{tc}) = \frac{m_i + \frac{\beta_i^2}{g_i} \left(A^{tc} \right)^2}{\rho_i + 2\delta + 2 \left(\sum_{j=1}^{N} \frac{\beta_j^2}{g_j} \right) A^{tc}} .$$

Next, if we take $\sum_{i=1}^{N} a_i^{tc} = A^{tc} = \sum_{i=1}^{N} \psi_i(A^{tc})$, we will obtain an algebraic equation of order $m + 1$, where m is the number of different discount rates. As a result, for $m \leq 3$, we can obtain algebraic solutions. On the contrary, for $m \geq 4$, the t-cooperative solution has to be computed numerically, in general.

5.3. *Agreeable dynamic bargaining solution*

As in the t-cooperative solution, we have to solve (8) but, for $V_i^{db}(x) = a_i^{db} x^2$, weights are now given by

$$\lambda_i^{db}(x) = \left(\frac{\eta_i}{a_i^{db} - a_i^{nc}} \right) \frac{1}{x^2} .$$

By substituting in (8) and solving the first order conditions, we obtain

$$u_i^{db}(s) = -\frac{\beta_i \left(a_i^{db} - a_i^{nc} \right)}{\eta_i g_i} \sum_{j=1}^{N} \frac{\eta_j a_j^{db}}{a_j^{db} - a_j^{nc}} x(s) .$$

By proceeding as in the previous cases, after some calculations we obtain that, in the case of constant discount rates ρ_i, coefficients a_i^{db} satisfy the following system of nonlinear equations:

$$a_i^{db}\left[\rho_i + 2\delta + 2\left(\sum_{j=1}^N \frac{\beta_j^2(a_j^{db}-a_j^{nc})}{\eta_j g_j}\right)\left(\sum_{k=1}^N \frac{\eta_k a_k^{db}}{a_k^{db}-a_k^{nc}}\right)\right]$$
$$= m_i + \frac{\beta_i^2\left(a_i^{db}-a_i^{nc}\right)^2}{g_i\eta_i^2}\left(\sum_{j=1}^N \frac{\eta_j a_j^{db}}{a_j^{db}-a_j^{nc}}\right)^2 . \tag{46}$$

In order to compute the solution, the equation system (46) has to be solved numerically.

5.4. *Time-consistent dynamic bargaining solution*

According to the TCB solution, interior decision rules are obtained from (14). By guessing $V_i^b(x) = a_i^b x^2$, for threat linear decision rules $u_i^{0,nc} = \alpha_i^{0,nc}x$, the equation system (14) has linear solutions $u_i^b = \alpha_i^b x$ giving rise to a quadratic value function, in agreement with the hypothesis. Coefficients α_i^b solve the equation system

$$\sum_{j=1}^N \frac{\eta_j a_j^b}{g_j\left(\left(\alpha_j^b\right)^2 - \left(\alpha_j^{0,nc}\right)^2\right) + 2a_j^b\sum_{k=1}^N \beta_k\left(\alpha_k^b - \alpha_k^{0,nc}\right)}\beta_i \tag{47}$$
$$+\frac{\eta_i g_i \alpha_i^b}{g_i\left(\left(\alpha_i^b\right)^2 - \left(\alpha_i^{0,nc}\right)^2\right) + 2a_i^b\sum_{k=1}^N \beta_k\left(\alpha_k^b - \alpha_k^{0,nc}\right)} ,$$

for $i = 1, \ldots, N$, together with

$$a_i^b = \left(m_i + \frac{\beta_i^2\left(a_i^b\right)^2}{g_i}\right)\int_0^\infty \theta_i(\tau e^{-2\left(\delta-\sum_{j=1}^N\beta_k\alpha_j^b\right)\theta}\, d\tau . \tag{48}$$

In particular, if discount rates are constant,

$$a_i^b\left(\rho_i + 2\delta - 2\sum_{j=1}^N \beta_j\alpha_j^b\right) = m_i + \frac{\beta_i^2\left(a_i^b\right)^2}{g_i} .$$

It remains to compute the threat strategy $\alpha^{0,nc} = (\alpha_1^{0,nc}, \ldots, \alpha_N^{0,nc})$. The following proposition summarizes the equations that must be satisfied by candidates to interior time-consistent dynamic bargaining solutions.

Proposition 4. *In Problem (41)-(42), time-consistent bargaining solutions solve the system of nonlinear equations (47)-(48) with threat strategies given by* $u_i^{0,nc} = \alpha_i^{0,nc}x$*, where*

$$\alpha_i^{0,nc} = -\frac{\beta_i}{g_i}\left(m_i + g_i\left(\alpha_i^b\right)^2\right)\int_0^\infty \theta_i(s)e^{2\left(\sum_{j=1}^N\beta_j\alpha_j^b-\delta\right)s}\, ds . \tag{49}$$

Proof. As in Proposition 3 we compute first a MPNE in a differential game played along the time interval $[t, t+\epsilon]$. If $\phi_j^{\epsilon,nc}(x,t) = \alpha_j^{\epsilon,nc}(t)x$ are the equilibrium strategies for players $j \neq i$, player i solves

$$\max_{\{u_i\}} \left\{ m_i x^2 + g_i u_i^2 + \frac{\partial W_i^\epsilon}{\partial x} \left(\beta_i u_i + \sum_{j\neq i} \beta_j \alpha_j^{\epsilon,nc}(t) x - \delta x \right) \right\} .$$

Therefore, $\dfrac{\partial W_i^\epsilon}{\partial x} = -\dfrac{2g_i}{\beta_i}\alpha_i^{\epsilon,nc}(t)x$, so $W_i^\epsilon(x,t) = -\dfrac{g_i}{\beta_i}\alpha_i^{\epsilon,nc}(t)x^2 + b_i^{\epsilon,nc}(t)$ and, therefore,

$$W_i^0(x,t) = -\frac{g_i}{\beta_i}\alpha_i^{0,nc}(t)x^2 + b_i^{0,nc}(t) . \tag{50}$$

Next, from Definition 6,

$$\begin{aligned} W_i^\epsilon(x,t) = & \int_t^{t+\epsilon} \theta_i(s-t) \left[m_i(x(s))^2 + g_i \left(\alpha_i^{\epsilon,nc}(s)\right)^2 (x(s))^2 \right] ds \\ & + \int_{t+\epsilon}^\infty \theta_i(s-t) \left[m_i(x(s))^2 + g_i \left(\alpha_i^b\right)^2 (x(s))^2 \right] ds , \end{aligned} \tag{51}$$

with

$$\dot{x}(s) = \sum_{i=1}^N \beta_i \alpha_i^{\epsilon,nc}(s)x(s) - \delta x(s) , \quad x(t) = S , \quad \text{for} \quad s \in [t, t+\epsilon) , \quad \text{and} \tag{52}$$

$$\dot{x}(s) = \sum_{i=1}^N \beta_i \alpha_i^b x(s) - \delta x(s) , \quad S(t+\epsilon) = S_{t+\epsilon} , \quad \text{for} \quad s \geq t+\epsilon , \tag{53}$$

where $x_{t+\epsilon}$ is the limit when $s \to t+\epsilon$ of the solution to (52). By solving (53) and substituting the value of $x(s)$ for $s \geq t+\epsilon$ in (51), we obtain

$$\begin{aligned} W_i^\epsilon(x,t) = & \int_t^{t+\epsilon} \theta_i(s-t) \left[m_i(x(s))^2 + g_i \left(\alpha_i^{\epsilon,nc}(s)\right)^2 (x(s))^2 \right] ds \\ & + \left(m_i + g_i \left(\alpha_i^b\right)^2 \right) \left(\int_{t+\epsilon}^\infty \theta_i(s-t) e^{2\left(\sum_{j=1}^N \beta_j \alpha_j^b - \delta\right)(s-t-\epsilon)} ds \right) (x_{t+\epsilon})^2 . \end{aligned} \tag{54}$$

Next, by solving $x(s)$ in (52), computing

$$x_{t+\epsilon} = \lim_{s \to t+\epsilon^-} x(s) = x e^{\int_t^{t+\epsilon} \left(\sum_{j=1}^N \beta_j \alpha_j^{\epsilon,nc}(\tau) - \delta \right) d\tau} ,$$

substituting in (54), and taking the limit $\epsilon \to 0$, after several calculations we obtain

$$W_i^0(x,t) = \left(m_i + g_i \left(\alpha_i^b\right)^2 \right) \left(\int_t^\infty \theta_i(s-t) e^{2\left(\sum_{j=1}^N \beta_j \alpha_j^b - \delta\right)(s-t)} ds \right) x^2 . \tag{55}$$

Finally, by identifying (50) and (55), we obtain that the threat strategy is given by (49). □

Note that, unlike the case of linear-state differential games, the threat point in the time-consistent dynamic bargaining solution for homogeneous linear-quadratic differential games is not the MPNE.

5.5. *Dicussion*

As in the case with a unique constant discount rate, the t-cooperative equilibrium is time consistent as a cooperative solution (i.e., in the sense of Definition 2) if, and only if, $a_i^{tc} \geq a_i^{nc}$, for all $i = 1, \ldots, N$. In a similar way, the time-consistent dynamic bargaining solution is time-consistent as a cooperative solution if, and only if, $a_i^b \geq a_i^{nc}$, for all $i = 1, \ldots, N$. Since these inequalities do not depend on time and the state of the system, time-consistency implies agreeability for the homogeneous linear-quadratic differential game given by Problem (41)-(42). Unfortunately, in order to test the time consistency and agreeability of the solutions, we must perform numerical calculations for each particular problem.

In any case, for this differential game, we can look for linear decision rules for all the solution concepts. If we consider more general linear-quadratic differential games, as the one in Section 4.3, but with a quadratic damage cost, $D_i(S) = \varphi_i S^2$, linear (affine) strategies can exist (provided that the corresponding system of nonlinear equations has a solution) for the noncooperative MPNE and the t-cooperative equilibrium. However, this will not be the case, in general, for the two dynamic bargaining solutions. In the agreeable dynamic bargaining solution, weight functions given by as in Proposition 1 will not be constant (as in the linear-state differential games) nor a constant times a common function (as in the homogeneous linear-quadratic game studied in the present section). On the contrary, they will have a more complicated structure that will make much more difficult the search of solutions. An inspection to equation (14) suggests that the time-consistent bargaining solutions will be also nonlinear, in general.

6. Concluding remarks

In this paper we have studied noncooperative MPNE and three cooperative solutions for linear-state and a class of homogeneous differential games with heterogeneous, possibly nonconstant, discount rates. We have seen that, for these classes of differential games and the three cooperative solution concepts, time consistency (Definition 2) and agreeability (Definition 3) are equivalent.

Concerning the t-cooperative equilibrium, it coincides with the standard cooperative solution, obtained by maximizing the sum of payoffs of all players, for the case of constant and equal discount rates. For more general discount rates, this solution concept has the nice property that it can be easily computed for the two differential game models studied in the paper. For the linear-state differential games solved in Section 4, there is a unique constant equilibrium, and conditions for the time consistency or agreeability can be easily checked. In the case of the homogeneous differential game, we can look for linear equilibria, but the test of time consistency has to be performed analytically.

In the two dynamic bargaining solutions studied in the paper, for the derivation of constant strategies in linear-state differential games, and linear strategies in the homogeneous differential game, we have to solve numerically a system of highly coupled nonlinear equations. As a result, the time consistency or agreeability of the time-consistent dynamic bargaining solution has to be checked numerically. The agreeable dynamic bargaining solution is time consistent and agreeable by construction.

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