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Remarks on Stationary and Uniformly-rotating Vortex Sheets: Rigidity Results

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Abstract: In this paper, we show that the only solution of the vortex sheet equation, either stationary or uniformly rotating with negative angular velocity Ω, such that it has positive vorticity and is concentrated in a finite disjoint union of smooth curves with finite length is the trivial one: constant vorticity amplitude supported on a union of nested, concentric circles. The proof follows a desingularization argument and a calculus of variations flavor.

1. Introduction

A vortex sheet is a weak solution of the 2D Euler equations:

$$v_t + v \cdot \nabla v = -\nabla p, \quad \nabla \cdot v = 0, \tag{1.1}$$

whose vorticity $\omega = \mathrm{curl}(v)$ is a delta function supported on a curve or a finite number of curves $\Gamma_i = z_i(\alpha, t)$, i.e.

$$\omega(x,t) = \sum_i \varpi_i(\alpha,t)\delta(x - z_i(\alpha,t)). \tag{1.2}$$

Here $\varpi_i(\alpha, t)$ is the vorticity strength on Γ_i with respect to the parametrization z_i, and the above equation is understood in the sense that

$$\int_{\mathbb{R}^2} \varphi(x) d\omega(x,t) = \sum_i \int \varphi(z_i(\alpha,t))\varpi_i(\alpha,t)d\alpha$$

for all test functions $\varphi(x) \in C_0^\infty(\mathbb{R}^2)$.

The motivation of the study of the equation (1.1) with vortex sheet initial data comes from the fact that in fluids with small viscosity, flows separate from rigid walls and corners [24,32]. To model it, one may think of a solution to (1.1) with one incompressible fluid

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where the velocity changes sign in a discontinuous (tangential) way across a streamline z. This discontinuity induces vorticity in z.

The equations of motion of ϖ_i and z_i can be derived by means of the Birkhoff–Rott operator [7,21,24,35], namely:

$$BR(z,\varpi)(x,t) = \frac{1}{2\pi} PV \int \frac{(x - z(\beta,t))^{\perp}}{|x - z(\beta,t)|^2} \varpi(\beta,t) d\beta, \tag{1.3}$$

yielding

$$\partial_t z_i(\alpha,t) = \sum_j BR(z_j,\varpi_i)(z_i(\alpha,t)) + c_i(\alpha,t)\partial_\alpha z_i(\alpha,t) \tag{1.4}$$

$$\partial_t \varpi_i(\alpha,t) = \partial_\alpha(c_i(\alpha,t)\varpi_i(\alpha,t)), \tag{1.5}$$

where the term $c_i(\alpha,t)$ accounts for the reparametrization freedom of the curves. See the paper [19] by Izosimov–Khesin where they propose geodesic, group-theoretic, and Hamiltonian frameworks for their description.

The main goal of this paper is to establish radial symmetry properties of stationary/uniformly-rotating vortex sheets to (1.1). To do so, we first define what we mean by a stationary vortex sheet. Assume the initial data ω_0 of (1.2) is supported on a finite number of curves parametrized by $z_i(\alpha)$, with strength $\varpi_i(\alpha)$ (with respect to the parametrization z_i) respectively. If there exists some reparametrization choice $c_i(\alpha)$ such that the right hand sides of (1.4)–(1.5) are both identically zero for every i, it gives that $\omega(\cdot,t)$ is invariant in time, and we say $\omega(\cdot,t) = \omega_0$ is a *stationary vortex sheet*.

For any $x \in \mathbb{R}^2$ and $\Omega \in \mathbb{R}$, let $R_{\Omega t}x$ denote the rotation of x counter-clockwise by angle Ωt about the origin. We say $\omega(x,t) = \omega_0(R_{\Omega t}x)$ is a *uniformly-rotating vortex sheet* with angular velocity Ω if ω_0 is stationary in the rotating frame with angular velocity Ω. (Note that in the special case $\Omega = 0$, the uniformly-rotating sheet is in fact stationary.) In Lemma 2.1, we will derive the equations satisfied by a stationary/rotating vortex sheet.

It is easy to see that if the z_i's are concentric circles with constant ϖ_i (with respect to the constant-speed parametrization) for every i, the solution is stationary, and it is also uniformly-rotating with any $\Omega \in \mathbb{R}$. We would like to understand the reverse implication, namely:

Question 1. Under what conditions must a stationary/uniformly-rotating vortex sheet be radially symmetric?

This type of rigidity question has been very lately understood for different equations and different settings such as in the papers by Koch–Nadirashvili–Sverak [20] for Navier-Stokes, Hamel–Nadirashvili [16–18] for the 2D Euler equation on a strip, punctured disk or the full plane, Gómez-Serrano–Park–Shi–Yao [14] for the 2D Euler and modified SQG in the full plane and Constantin–Drivas–Ginsberg [8] for the 2D and 3D Euler, as well as the 2D Boussinesq and the 3D Magnetohydrostatic (MHS) equations.

The next theorem is the main result of the paper, solving it for the vortex sheet equations:

Theorem 1.1. *Let $\omega(x,t) = \omega_0(R_{\Omega t}x)$ be a stationary/uniformly-rotating vortex sheet with angular velocity Ω. Assume that ω_0 is concentrated on Γ, which is a finite union of smooth curves, and ω_0 has positive vorticity strength on Γ. (See **(H1)–(H3)** in Sect. 2 for the precise regularity and positivity assumptions.)*

If $\Omega \leq 0$, Γ must be a union of concentric circles, and ω_0 must have constant strength along each circle (with respect to the constant-speed parametrization). In addition, if $\Omega < 0$, all circles must be centered at the origin.

Remark 1.2. Note that for any uniformly-rotating solution with $\Omega < 0$, our theorem yields that ω_0 must be concentrated on concentric circles centered at the origin, with constant strength on each circle. Such ω_0 is actually stationary. As a result, there are no uniformly rotating solutions with negative angular velocity that are non-stationary.

We now go first over the history of the equations (1.4)–(1.5), focusing later on the case of steady solutions. The study of those solutions is important due to the ill-posedness of the vortex sheet equation, thus they represent (unstable) structures for which there is global existence.

1.1. Brief history of the dynamical problem. The existence of solutions to (1.4)–(1.5) has been widely studied. The seminal paper of Delort [9] proved global existence of weak solutions of (1.1) for an initial velocity in L^2_{loc} and a vorticity a positive Radon measure. Majda [23] provided a simpler proof. See also the works by Schochet [33,34] and Evans–Muller [13]. All of them use the hypothesis that the vorticity has a definite sign.

If the vorticity does not have a sign, Lopes Filho–Nussenzveig Lopes–Xin proved existence in [22], in the case where the system enjoys reflection symmetry. For the setting in which the curve z_i is not closed and represented as a graph, Sulem–Sulem–Bardos–Frisch [35] proved local existence in the case of analytic initial data.

The first sign of singularities with analytic initial data goes back to Moore [26], where he demonstrated that the curvature may blow up in finite time. Ebin [11] showed ill-posedness in Sobolev spaces when γ has a distinguished sign, and Duchon–Robert [10] proved global existence for a class of initial data in the unbounded setting. Caflisch–Orellana [6] also showed global existence for a class of initial data, as well as ill-posedness in H^s for $s > \frac{3}{2}$ and simplified the analysis of Moore [5]. We also mention here the work of Wu [38], in which she proved the existence of solutions to (1.4)–(1.5) in spaces which are less regular than H^s. Székelyhidi [36] (resp. Mengual–Székelyhidi [25]) constructed infinitely many admissible weak solutions to (1.1) for vortex sheet initial data with (resp. without necessarily) a distinguished sign.

1.2. Stationary and rotating solutions. Relative equilibria are an important family of solutions of fluid equations since their structures persist for long times. This is specially important when the equations of motion are ill-posed. In the particular case of (1.4)–(1.5), our knowledge is very small and only very few explicit cases are known: the circle and the straight line (with constant γ), which are stationary, and the segment of length $2a$ and density

$$\gamma(x) = \Omega\sqrt{a^2 - x^2}, \qquad x \in [-a, a], \tag{1.6}$$

which is a rotating solution with angular velocity Ω [2]. Protas–Sakajo [31] generalized this solution and proved the existence of several others made out of segments rotating about a common center of rotation with endpoints at the vertices of a regular polygon by solving a Riemann–Hilbert problem, even finding some of them analytically.

In the paper [15] we prove the existence of a family of vortex sheet rotating solutions with non-constant vorticity density supported on a non-radial curve, bifurcating from the circle with constant density.

Numerically, some solutions have been computed before. O'Neil [27,28] used point vortices to approximate the vortex sheet and compute uniformly rotating solutions and Elling [12] constructed numerically self-similar vortex sheets forming cusps. O'Neil [29,30] also found numerically steady solutions which are combinations of point vortices and vortex sheets.

1.3. Structure of the proof. The proof is inspired by our recent rigidity result in the paper [14] on stationary and rotating solutions of the 2D Euler equations both in the smooth and vortex patch settings. To prove it, we constructed an appropriate functional and showed, on one hand, that any stationary solution had to be a critical point, and on the other, for any curve which is not a circle there existed a vector field along which the first variation was non-zero. This vector field is defined in terms of an elliptic equation in the interior of the patch. In the case of the vortex sheet, this is not possible anymore. Instead, we desingularize the problem by considering patches of thickness $\sim \varepsilon$ which are tubular neighborhoods of the sheet. The drawback is that we lose the property that any stationary solution has to be a critical point if $\varepsilon > 0$ and very careful, quantitative estimates need to be done to show that indeed the first variation of a stationary solution tends to 0 as $\varepsilon \to 0$. This setup is also reminiscent of the numerical work by Baker–Shelley [1], where they approximate the motion of a vortex sheet by a vortex patch of very small width. In [3], Benedetto–Pulvirenti proved the stability (for short time) of vortex sheet solutions with respect to solutions to 2D Euler with a thin strip of vorticity around a curve. See also the work by Caflisch–Lombardo–Sammartino [4] for more stability results with a different desingularization.

1.4. Organization of the paper. In Sect. 2 the equations for the stationary/rotating vortex sheet are derived, and in Sect. 3 we perform the desingularization procedure. Section 4 is devoted to construct the aforementioned divergence free vector-field along which the first variation is non-zero. Finally in Sect. 5 we conclude the quantitative estimates and prove the symmetry result from Theorem 1.1.

1.5. Notations. For a bounded domain $D \subset \mathbb{R}^2$, we denote $|D|$ by its area (i.e. its Lebesgue measure). For $x \in \mathbb{R}^2$ and $r > 0$, denote by $B(x, r)$ or $B_r(x)$ the open ball centered at x with radius r.

Through Sects 3–5 of this paper, we will desingularize the vortex sheet into a vortex layer with width $\sim \epsilon$, and obtain various quantitative estimates. In all these estimates, we say a term f is $O(g(\epsilon))$ if $|f| \le Cg(\epsilon)$ for some constant C independent of ϵ.

For a domain $U \subset \mathbb{R}^2$, in the boundary integral $\int_{\partial U} \boldsymbol{f} \cdot n d\sigma$, n denotes the outer normal of the domain U.

2. Equations for a Stationary/Rotating Vortex Sheet

Let $\omega(\cdot, t) = \omega_0(R_{\Omega t})$ be a stationary/rotating vortex sheet solution to the incompressible 2D Euler equation, where $\omega_0 \in \mathcal{M}(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$ is a Radon measure. Here $\Omega = 0$ corresponds to a stationary solution, and $\Omega \neq 0$ corresponds to a rotating solution. Assume ω_0 is concentrated on Γ, which is a finite disjoint union of curves. Throughout this paper we assume Γ satisfies the following:

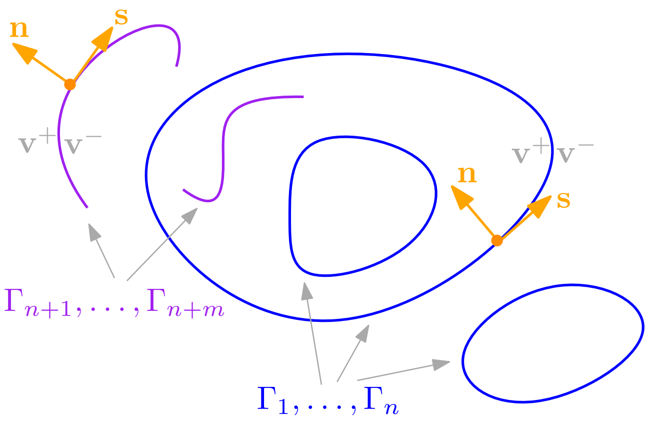

Fig. 1. Illustration of the closed curves $\Gamma_1, \ldots, \Gamma_n$ and the open curves $\Gamma_{n+1}, \ldots, \Gamma_{n+m}$, and the definitions of $\mathbf{n}$, $\mathbf{s}$, $\mathbf{v}^+$ and $\mathbf{v}^-$

(H1) Each connected component of Γ is smooth and with finite length, and it is either a simple closed curve (denote them by $\Gamma_1, \ldots, \Gamma_n$), or a non-self-intersecting curve with two endpoints (denote them by $\Gamma_{n+1}, \ldots, \Gamma_{n+m}$). Here we require $n + m \geq 1$, but allow either n or m to be 0.

Let us denote

$$d_\Gamma := \min_{k \neq i} \operatorname{dist}(\Gamma_i, \Gamma_k), \tag{2.1}$$

which is strictly positive since we assume the curves $\{\Gamma_i\}_{i=1}^{n+m}$ are disjoint. For $i = 1, \ldots, n+m$, denote by L_i the length of Γ_i. Let $z_i : S_i \to \Gamma_i$ denote a constant-speed parameterization of Γ_i (in counter-clockwise direction if Γ_i is a closed curve), where the parameter domain S_i is given by

$$S_i := \begin{cases} \mathbb{R}/\mathbb{Z} & \text{for } i = 1, \ldots, n, \\ [0, 1] & \text{for } i = n+1, \ldots, n+m. \end{cases}$$

Note that this gives $|z_i'| \equiv L_i$, and the arc-chord constant

$$F_\Gamma := \max_{i=1,\ldots,n+m} \sup_{\alpha \neq \beta \in S_i} \frac{|\alpha - \beta|}{|z_i(\alpha) - z_i(\beta)|} \tag{2.2}$$

is finite, since Γ is non-self-intersecting. Let $\mathbf{s} : \Gamma \to \mathbb{R}^2$ be the unit tangential vector on Γ, given by $\mathbf{s}(z_i(\alpha)) := \frac{z_i'(\alpha)}{|z_i'(\alpha)|} = \frac{z_i'(\alpha)}{L_i}$, and $\mathbf{n} : \Gamma \to \mathbb{R}^2$ be the unit normal vector, given by $\mathbf{n} = \mathbf{s}^\perp$. See Fig. 1 for an illustration.

For $i = 1, \ldots, n+m$, let us denote by $\gamma_i(\alpha)$ the vorticity strength at $z_i(\alpha)$ with respect to the arclength parametrization, which is related to $\varpi_i(\alpha)$ by

$$\gamma_i(\alpha) = \frac{\varpi_i(\alpha)}{|z_i'(\alpha)|} \quad \text{for } \alpha \in S_i. \tag{2.3}$$

Throughout this paper we will be working with γ_i, instead of ϖ_i. We impose the following regularity and positivity assumptions on γ_i:

(H2) Assume that $\gamma_i \in C^2(S_i)$ for $i = 1, \ldots, n$ and $\gamma_i \in C^b(S_i) \cap C^1(S_i^\circ)$ for some $b \in (0, 1)$ for $i = n+1, \ldots, n+m$.[1]

(H3) For $i = 1, \ldots, n$, assume $\gamma_i > 0$ in S_i. And for $i = n+1, \ldots, n+m$, assume $\gamma_i > 0$ in S_i°, and $\gamma_i(0) = \gamma_i(1) = 0$.

Note that for a closed curve, **(H3)** implies that γ_i is uniformly positive; whereas for an open curve, γ_i is positive in the interior of S_i but vanishes at its endpoints. This is because any stationary/rotating vortex sheet with continuous γ_i must have it vanishing at the two endpoints of any open curve: if not, one can easily check that $|BR(z_i(\alpha)) \cdot \mathbf{n}(z_i(\alpha))| \to \infty$ as α approaches the endpoint, thus such a vortex sheet cannot be stationary in the rotating frame.

With the above notations of z_i and γ_i, the Birkhoff–Rott integral (1.3) along the sheet can now be expressed as

$$BR(z_i(\alpha)) = \sum_{k=1}^{n+m} BR_k(z_i(\alpha)) := \sum_{k=1}^{n+m} PV \int_{S_k} K_2(z_i(\alpha) - z_k(\alpha'))\, \gamma_k(\alpha')|z_k'(\alpha')|\, d\alpha', \tag{2.4}$$

with the kernel K_2 given by

$$K_2(x) := (2\pi)^{-1}\nabla^\perp \log|x| = \frac{x^\perp}{2\pi|x|^2}, \tag{2.5}$$

and the principal value in (2.4) is only needed for the integral with $k = i$.

Let $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2$ be the velocity field generated by ω_0, given by $\mathbf{v} := \nabla^\perp(\omega_0 * \mathcal{N})$. Note that $\mathbf{v} \in C^\infty(\mathbb{R}^2 \setminus \Gamma)$, but $\mathbf{v}$ is discontinuous across Γ. Let $\mathbf{v}^+, \mathbf{v}^- : \Gamma \to \mathbb{R}^2$ denote the two limits of $\mathbf{v}$ on the two sides of Γ (with $\mathbf{v}^+$ being the limit on the side that $\mathbf{n}$ points into—see Fig. 1 for an illustration), and $[\mathbf{v}] := \mathbf{v}^- - \mathbf{v}^+$ the jump in $\mathbf{v}$ across the sheets. $[\mathbf{v}]$ is related to the vortex-sheet strength γ as follows (see [24, Eq. (9.8)] for a derivation): $[\mathbf{v}] \cdot \mathbf{n} = 0$, and

$$[\mathbf{v}] \times \mathbf{n} = [\mathbf{v}] \cdot \mathbf{s} = \gamma.$$

In addition, the Birkhoff–Rott integral (2.4) is the the average of $\mathbf{v}^+$ and $\mathbf{v}^-$, namely

$$BR(z_i(\alpha)) = \frac{1}{2}(\mathbf{v}^+(z_i(\alpha)) + \mathbf{v}^-(z_i(\alpha))) \quad \text{for all } \alpha \in S_i, i = 1, \ldots, n+m.$$

In the following lemma, we derive the equation that the Birkhoff–Rott integral satisfies for a stationary/rotating vortex sheet.

Lemma 2.1. *Assume $\omega(\cdot, t) = \omega_0(R_{\Omega t}x)$ is a stationary/uniformly-rotating vortex sheet with angular velocity $\Omega \in \mathbb{R}$, and ω_0 is concentrated on $\cup_{i=1}^{n+m}\Gamma_i$, with z_i and γ_i defined as above. Then the Birkhoff–Rott integral BR (2.4) and the strength γ_i satisfy the following two equations:*

$$(BR - \Omega x^\perp) \cdot \mathbf{n} = \mathbf{v}^+ \cdot \mathbf{n} = \mathbf{v}^- \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \tag{2.6}$$

and

$$(BR(z_i(\alpha)) - \Omega z_i^\perp(\alpha)) \cdot \mathbf{s}(z_i(\alpha))\, \gamma_i(\alpha) = \begin{cases} C_i & \text{on } S_i \text{ for } i = 1, \ldots, n, \\ 0 & \text{on } S_i \text{ for } i = n+1, \ldots, n+m. \end{cases} \tag{2.7}$$

[1] For an open curve $i = n+1, \ldots, n+m$, note that **(H2)** does not require γ_i to be C^1 up to the boundary of S_i, and its derivative is allowed to blow up at the endpoints. This is motivated by the fact that in the explicit uniformly-rotating solution (1.6), its strength γ is Hölder continuous in $[-a, a]$ and smooth in the interior, but its derivative blows up at the endpoints.

In particular, the above two equations imply that $BR(z_i(\alpha)) - \Omega z_i^\perp(\alpha) \equiv \mathbf{0}$ *for* $i = n+1, \dots, n+m$.

Proof. By definition of the stationary/uniformly-rotating solutions, ω_0 is a stationary vortex sheet in the rotating frame with angular velocity Ω. In this rotating frame, an extra velocity $-\Omega z_i^\perp$ should be added to the right hand side of (1.4). Therefore the evolution equations (1.4)–(1.5) become the following in the rotating frame (where we also use (2.4)):

$$\partial_t z_i(\alpha, t) = BR(z_i(\alpha, t)) - \Omega z_i^\perp(\alpha, t) + c_i(\alpha, t)\partial_\alpha z_i(\alpha, t) \tag{2.8}$$
$$\partial_t \varpi_i(\alpha, t) = \partial_\alpha(c_i(\alpha, t)\varpi_i(\alpha, t)), \tag{2.9}$$

where the term $c_i(\alpha, t)$ accounts for the reparametrization freedom of the curves. Since ω_0 is stationary in the rotating frame, $z_i(\cdot, t)$ parametrizes the same curve as $z_i(\cdot, 0)$. Therefore $\partial_t z_i(\alpha, t)$ is tangent to the curve Γ_i, and multiplying $\mathbf{n}(z_i(\alpha, t))$ to (2.8) gives

$$0 = \partial_t z_i(\alpha, t) \cdot \mathbf{n}(z_i(\alpha, t)) = (BR(z_i(\alpha, t)) - \Omega z_i^\perp(\alpha, t)) \cdot \mathbf{n}(z_i(\alpha, t)), \tag{2.10}$$

where we use that $\mathbf{n}(z_i(\alpha, t)) \cdot \partial_\alpha z_i(\alpha, t) = 0$. This proves (2.6).

Now we prove (2.7). Towards this end, let us choose

$$c_i(\alpha, t) := -\frac{(BR(z_i(\alpha, t)) - \Omega z_i^\perp(\alpha, t)) \cdot \mathbf{s}(z_i(\alpha, t))}{|\partial_\alpha z_i(\alpha, t)|},$$

so that multiplying $\mathbf{s}(z_i(\alpha, t))$ to (2.8) gives $\partial_t z_i(\alpha, t) \cdot \mathbf{s}(z_i(\alpha, t)) = 0$, and combining it with (2.10) gives $\partial_t z_i(\alpha, t) = 0$. In other words, with such choice of c_i, the parametrization $z_i(\alpha, t)$ remains fixed in time. Since ω_0 is stationary in the rotating frame, we know that with a fixed parametrization $z_i(\alpha, t) = z_i(\alpha, 0)$, the strength $\varpi_i(\alpha, t)$ must also remain invariant in time. Thus (2.9) becomes

$$c_i(\alpha, t)\varpi_i(\alpha, t) \equiv C_i.$$

Plugging the definition of c_i into the equation above and using the fact that z_i is invariant in t, we have

$$\frac{(BR(z_i(\alpha)) - \Omega z_i^\perp(\alpha)) \cdot \mathbf{s}(z_i(\alpha))\varpi_i(\alpha)}{|\partial_\alpha z_i(\alpha)|} \equiv -C_i \quad \text{for all } \alpha \in S_i,$$

and finally the relationship between γ_i and ϖ_i in (2.3) yields (2.7) for $i = 1, \dots, n$.

And for the open curves $i = n+1, \dots, n+m$, note that we do not have any reparametrization freedom at the two endpoints $\alpha = 0, 1$, therefore the endpoint velocity $BR(z_i(0, t)) - \Omega z_i^\perp(0, t)$ must be 0 to ensure that ω_0 is stationary in the rotating frame. This immediately leads to $C_i = 0$ for $i = n+1, \dots, n+m$, finishing the proof of (2.7). □

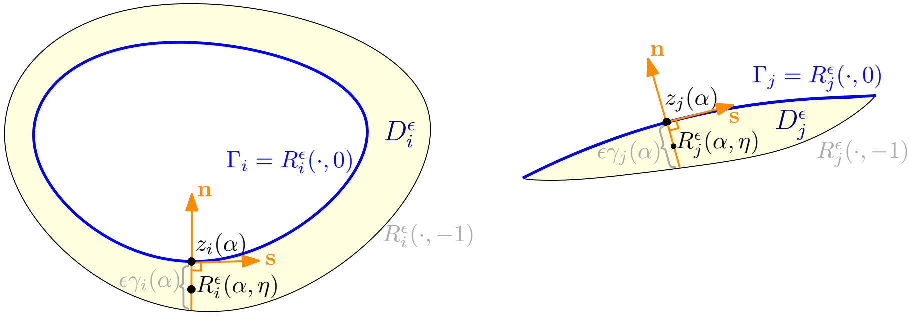

Fig. 2. Illustration of the definitions of R_i^ϵ and D_i^ϵ for a closed curve (left) and an open curve (right)

3. Approximation by a Thin Vortex Layer

Our aim in this section is to desingularize the vortex sheet ω_0. Namely, for $0 < \epsilon \ll 1$, we will construct a vorticity $\omega^\epsilon \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ that only takes values 0 and ϵ^{-1}, and is supported in an $O(\epsilon)$ neighborhood of Γ, such that ω^ϵ weakly converges to ω_0 as $\epsilon \to 0^+$.

For each $i = 1, \dots, n+m$, we will describe a neighborhood of Γ_i using the following change of coordinates: let $R_i^\epsilon : S_i \times \mathbb{R} \to \mathbb{R}^2$ be given by

$$R_i^\epsilon(\alpha, \eta) := z_i(\alpha) + \epsilon\gamma_i(\alpha)\mathbf{n}(z_i(\alpha))\eta, \tag{3.1}$$

and let

$$D_i^\epsilon := \left\{R_i^\epsilon(\alpha, \eta) : \alpha \in S_i^\circ, \eta \in (-1, 0)\right\}.$$

Note that each D_i^ϵ is a connected open set, and for all $\epsilon > 0$ sufficiently small, the sets $(D_i^\epsilon)_{i=1}^{n+m}$ are disjoint. For $i = 1, \dots, n$, the domains D_i^ϵ are doubly-connected with smooth boundary, and its inner boundary coincides with Γ_i; see the left of Fig. 2 for an illustration. And for $i = n+1, \dots, n+m$, the domains D_i^ϵ are simply-connected, and its boundary is smooth except at most two points; see the right of Fig. 2 for an illustration.

In addition, for $\epsilon > 0$ that is sufficiently small, one can check that $R_i^\epsilon : S_i^\circ \times (-1, 0) \to D_i^\epsilon$ is a diffeomorphism. Since $\gamma_i \in C^1(S_i)$ and $z_i \in C^2(S_i)$, we only need to show $R_i^\epsilon : S_i^\circ \times (-1, 0) \to D_i^\epsilon$ is injective. Below we prove this fact in a stronger quantitative version, which will be used later.

Lemma 3.1. *For any $i = 1, \dots, n+m$, assume Γ_i and γ_i satisfy **(H1)–(H2)**. Then the map $R_i^\epsilon : S_i^\circ \times (-1, 0) \to D_i^\epsilon$ given by (3.1) is injective. In addition, there exist some $c_0, \epsilon_0 > 0$ depending on $\|z_i\|_{C^2(S_i)}$, $\|\gamma_i\|_{L^\infty(S_i)}$ and F_Γ, such that for all $\epsilon \in (0, \epsilon_0)$ we have*

$$|R_i^\epsilon(\alpha', \eta') - R_i^\epsilon(\alpha, \eta)| \geq c_0\big(|\alpha' - \alpha| + \epsilon|\gamma_i(\alpha)\eta - \gamma_i(\alpha')\eta'|\big), \tag{3.2}$$

for all $\alpha, \alpha' \in S_i^\circ$, $\eta, \eta' \in (-1, 0)$.[2]

Proof. To begin with, note that (3.2) immediately implies that $R_i^\epsilon : S_i^\circ \times (-1, 0) \to D_i^\epsilon$ is injective, where we used the positivity assumption $\gamma_i > 0$ in S_i° in **(H2)**. Thus it suffices to prove (3.2). Throughout the proof, we fix any $i \in \{1, \dots, n+m\}$, and we

[2] In fact, (3.2) also holds (with a slightly smaller ϵ_0 and c_0) for $\eta, \eta' \in (-2, 2)$, even though such R_i^ϵ may not belong to D_i^ϵ. We will use this fact later in the proof of Lemma 3.5.

will omit the subscript i for notational simplicity. Using the definition (3.1), let us break $R^\epsilon(\alpha', \eta') - R^\epsilon(\alpha, \eta)$ into

$$\begin{aligned} R^\epsilon(\alpha', \eta') - R^\epsilon(\alpha, \eta) = & \underbrace{z(\alpha') - z(\alpha)}_{=:T_1} + \underbrace{\epsilon \left(\gamma(\alpha')\eta' - \gamma(\alpha)\eta\right) \mathbf{n}(z(\alpha'))}_{=:T_2} \\ & + \underbrace{\epsilon\gamma(\alpha)\eta \left(\mathbf{n}(z(\alpha')) - \mathbf{n}(z(\alpha))\right)}_{=:T_3}. \end{aligned} \tag{3.3}$$

For T_1 and T_3, we have

$$\begin{aligned} \left|T_1 - z'(\alpha')(\alpha' - \alpha)\right| & \le \|z\|_{C^2(S)} |\alpha - \alpha'|^2, \\ |T_3| & \le \epsilon\gamma(\alpha)\|z\|_{C^2(S)}|\alpha - \alpha'|. \end{aligned} \tag{3.4}$$

Also, using that $z'(\alpha') = L\mathbf{s}(z(\alpha'))$ is perpendicular to $\mathbf{n}(z(\alpha'))$, we have

$$\begin{aligned} |z'(\alpha')(\alpha' - \alpha) + T_2| & = \left|L(\alpha' - \alpha)\mathbf{s}(z(\alpha')) + \epsilon \left(\gamma(\alpha')\eta' - \gamma(\alpha)\eta\right) \mathbf{n}(z(\alpha'))\right| \\ & \ge \frac{1}{2}L|\alpha' - \alpha| + \frac{1}{2}\epsilon \left|\gamma(\alpha')\eta' - \gamma(\alpha)\eta\right|, \end{aligned}$$

where we use that $\sqrt{x^2 + y^2} \ge \frac{1}{2}(|x| + |y|)$. Combining this with (3.4) gives

$$|T_1 + T_2 + T_3| \ge |\alpha - \alpha'| \left(\frac{L}{2} - \|z\|_{C^2(S)} \left(|\alpha - \alpha'| + \epsilon\gamma(\alpha)\right)\right) + \frac{1}{2}\epsilon|\gamma(\alpha')\eta' - \gamma(\alpha)\eta|,$$

thus

$$|R^\epsilon(\alpha', \eta') - R^\epsilon(\alpha, \eta)| \ge \frac{L}{4}|\alpha - \alpha'| + \frac{1}{2}\epsilon|\gamma(\alpha')\eta' - \gamma(\alpha)\eta| \tag{3.5}$$

for all $0 < \epsilon < L(8\|z\|_{C^2}\|\gamma\|_{L^\infty})^{-1}$ and $|\alpha - \alpha'| \le \frac{L}{8\|z\|_{C^2}}$.

For $|\alpha - \alpha'| > \frac{L}{8\|z\|_{C^2}}$, recall that the definition of F_Γ in (2.2) gives $|z(\alpha') - z(\alpha)| \ge F_\Gamma^{-1}|\alpha' - \alpha|$. Thus a crude estimate gives

$$|R^\epsilon(\alpha', \eta') - R^\epsilon(\alpha, \eta)| \ge |z(\alpha') - z(\alpha)| - 2\epsilon\|\gamma\|_{L^\infty(S)} \ge \frac{1}{2F_\Gamma}|\alpha' - \alpha| + \epsilon|\gamma(\alpha')\eta' - \gamma(\alpha)\eta| \tag{3.6}$$

for $0 < \epsilon < L(64F_\Gamma\|z\|_{C^2}\|\gamma\|_{L^\infty})^{-1}$. (Note that for such ϵ we have $4\epsilon\|\gamma\|_{L^\infty} \le \frac{1}{2F_\Gamma}|\alpha' - \alpha|$ due to our assumption that $|\alpha - \alpha'| > \frac{L}{8\|z\|_{C^2}}$).

Finally, combining (3.5) and (3.6), it follows that (3.2) holds for $c_0 = \min\{\frac{L}{4}, \frac{1}{2F_\Gamma}, \frac{1}{2}\}$ and $\epsilon_0 = \min\{L(8\|z\|_{C^2}\|\gamma\|_{L^\infty})^{-1}, L(64F_\Gamma\|z\|_{C^2}\|\gamma\|_{L^\infty})^{-1}\}$. This finishes the proof. □

In the next lemma we compute the partial derivatives and Jacobian of $R_i^\epsilon(\alpha, \eta)$, which will be useful later.

Lemma 3.2. *For any $i = 1, \ldots, n+m$, let z_i be a constant-speed parameterization of the curve Γ_i (with length L_i), and let R_i^ϵ be given by (3.1). Then its partial derivatives are*

$$\begin{aligned}\partial_\alpha R_i^\epsilon(\alpha,\eta) &= z_i'(\alpha) + \epsilon\left(\gamma_i'(\alpha)\frac{z_i'(\alpha)^\perp}{L_i}\eta + \gamma_i(\alpha)\frac{z_i''(\alpha)^\perp}{L_i}\eta\right),\\ \partial_\eta R_i^\epsilon(\alpha,\eta) &= \epsilon\gamma_i(\alpha)\frac{z_i'(\alpha)^\perp}{L_i}.\end{aligned} \tag{3.7}$$

Moreover, its Jacobian is given by

$$\det(\nabla_{\alpha,\eta}R_i^\epsilon) = \epsilon L_i\gamma_i(\alpha) - \epsilon^2 L_i\gamma_i^2(\alpha)\kappa_i(\alpha)\eta, \tag{3.8}$$

where $\kappa_i(\alpha)$ denotes the signed curvature of Γ_i at $z_i(\alpha)$.

Proof. Since z_i is the constant-speed parameterization of Γ_i (which has length L_i), we have $|z_i'| \equiv L_i$ and $\mathbf{n}(z_i(\alpha)) = z_i'(\alpha)^\perp/L_i$. Taking the α and η partial derivatives of (3.1) directly yields (3.7).

Putting the two partial derivatives into columns of a 2×2 matrix and computing the determinant, we have

$$\begin{aligned}\det(\nabla_{\alpha,\eta}R_i^\epsilon) &= \epsilon\gamma_i(\alpha)\frac{|z_i'(\alpha)|^2}{L_i} + \epsilon^2\gamma_i^2(\alpha)\frac{z_i''(\alpha)^\perp\cdot z_i'(\alpha)}{L_i^2}\eta\\ &= \epsilon L_i\gamma_i(\alpha) - \epsilon^2 L_i\gamma_i^2(\alpha)\kappa_i(\alpha)\eta,\end{aligned}$$

where in the second equality we used that $z_i''(\alpha) = \kappa_i(\alpha)\mathbf{n}(z_i(\alpha))L_i^2$ (recall that z_i has constant speed L_i). This finishes the proof. $\square$

Remark 3.3. We point out that for each $i = 1, \ldots, n+m$, the determinant formula (3.8) immediately gives the following approximation of $|D_i^\epsilon|$, which will be helpful in the proofs later:

$$\frac{|D_i^\epsilon|}{\epsilon} = \frac{1}{\epsilon}\int_{D_i^\epsilon} 1dx = \frac{1}{\epsilon}\int_{S_i}\int_{-1}^0 \det(\nabla_{\alpha,\eta}R_i^\epsilon(\alpha,\eta))\,d\eta d\alpha = L_i\int_{S_i}\gamma_i(\alpha)d\alpha + O(\epsilon), \tag{3.9}$$

where the $O(\epsilon)$ error term has its absolute value bounded by $C\epsilon$, with C only depending on $\|z_i\|_{C^2(S_i)}$ and $\|\gamma_i\|_{L^\infty(S_i)}$.

Finally, let $D^\epsilon := \cup_{i=1}^{n+m} D_i^\epsilon$, and $\omega^\epsilon : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$\omega^\epsilon(x) := \epsilon^{-1}1_{D^\epsilon}(x) = \epsilon^{-1}\sum_{i=1}^{n+m} 1_{D_i^\epsilon}(x),$$

and let

$$\mathbf{v}^\epsilon = \nabla^\perp(\omega^\epsilon * \mathcal{N}) \tag{3.10}$$

be the velocity field generated by ω^ϵ.

In the next lemma we aim to obtain some fine estimate of $\mathbf{v}^\epsilon$ in the thin vortex layer D^ϵ. Our goal is to show that along each cross section of the thin layer (i.e. fix i and α, and let η vary in $[-1, 0]$), the function $\eta \mapsto \mathbf{v}^\epsilon(R_i^\epsilon(\alpha,\eta))$ is almost a linear function in η, with the endpoint values (at $\eta = -1$ and 0) being almost $\mathbf{v}^-(z_i(\alpha))$ and $\mathbf{v}^+(z_i(\alpha))$ respectively.

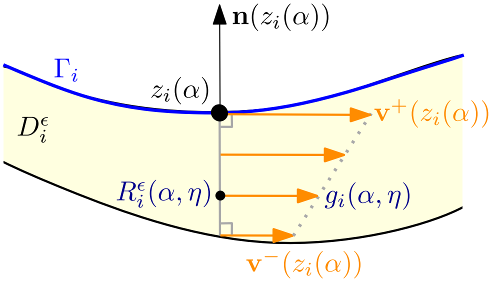

Fig. 3. Illustration of the definition of $g_i(\alpha,\cdot)$ (the orange arrows)

Lemma 3.4. *For $i = 1,\ldots,n+m$, assume Γ_i and γ_i satisfy **(H1)–(H3)**. Let*

$$g_i(\alpha,\eta) := BR(z_i(\alpha)) - \Big(\eta + \frac{1}{2}\Big)[\mathbf{v}](z_i(\alpha)) \quad \text{for } \alpha \in S_i,$$

and note that $g_i(\alpha,0) = \mathbf{v}^+(z_i(\alpha))$ and $g_i(\alpha,-1) = \mathbf{v}^-(z_i(\alpha))$ (see Fig. 3 for an illustration of $g_i(\alpha,\eta)$). Then for all sufficiently small $\epsilon > 0$, for all $i = 1,\ldots,n+m$ we have

$$|\mathbf{v}^\epsilon(R_i^\epsilon(\alpha,\eta)) - g_i(\alpha,\eta)| \le C\epsilon^b|\log\epsilon| \quad \text{for all } \alpha \in S_i, \eta \in [-1,0], \tag{3.11}$$

where $b \in (0,1)$ is as in **(H2)**, *and C depends on b, $\max_i \|z_i\|_{C^2(S_i)}$, $\max_i \|\gamma_i\|_{C^b(S_i)}$, d_Γ and F_Γ.*

Proof. Let i be any fixed index in $1,\ldots,n+m$. We begin with breaking $\mathbf{v}^\epsilon$ into contributions from different components $\{D_k^\epsilon\}_{k=1}^{n+m}$, namely

$$\mathbf{v}^\epsilon(x) = \sum_{k=1}^{n+m} \mathbf{v}_k^\epsilon(x) := \sum_{k=1}^{n+m} \epsilon^{-1}\int_{D_i^\epsilon} K_2(x-y)dy,$$

where the kernel K_2 is given by (2.5). Similarly, we can break $BR(z_i(\alpha))$ into $BR(z_i(\alpha)) = \sum_{k=1}^{n+m} BR_k(z_i(\alpha))$, where BR_k is the contribution from the k-th integral in (2.4), and note that the PV symbol is only needed for $k = i$.

- *Estimates for $k \ne i$ terms.* For any $k \ne i$, we aim to show that

$$|\mathbf{v}_k^\epsilon(R_i^\epsilon(\alpha,\eta)) - BR_k(z_i(\alpha))| \le C\epsilon, \tag{3.12}$$

where C depends on d_Γ, $\max_k \|z_k\|_{C^2}$ and $\max_k \|\gamma_k\|_{L^\infty}$. Applying a change of variable $y = R_k^\epsilon(\alpha',\eta')$, we can rewrite $\mathbf{v}_k^\epsilon$ as

$$\begin{aligned}\mathbf{v}_k^\epsilon(R_i^\epsilon(\alpha,\eta)) &= \epsilon^{-1}\int_{D_k^\epsilon} K_2(R_i^\epsilon(\alpha,\eta) - y)\,dy \\ &= \int_{S_k}\int_{-1}^{0} \underbrace{K_2(R_i^\epsilon(\alpha,\eta) - R_k^\epsilon(\alpha',\eta'))}_{=:T_1}\underbrace{\epsilon^{-1}\det(\nabla_{\alpha',\eta'}R_k^\epsilon(\alpha',\eta'))}_{=:T_2}\,d\eta' d\alpha'.\end{aligned} \tag{3.13}$$

Using the facts that $R_i^\epsilon(\alpha,\eta) - R_k^\epsilon(\alpha',\eta') = z_i(\alpha) - z_k(\alpha') + O(\epsilon)$ as well as $|z_i(\alpha) - z_k(\alpha')| \ge d_\Gamma > 0$ (recall that d_Γ is as given in (2.1)), for all sufficiently small $\epsilon > 0$ we have $T_1 = K_2(z_i(\alpha) - z_k(\alpha')) + O(\epsilon)$. For T_2, the explicit formula (3.8) for the determinant gives $T_2 = L_k\gamma_k(\alpha') + O(\epsilon)$. Plugging these into the above integral yields

$$\mathbf{v}_k^\epsilon(R_i^\epsilon(\alpha,\eta)) = \int_{S_k} K_2(z_i(\alpha) - z_k(\alpha'))L_k\gamma_k(\alpha')\,d\alpha' + O(\epsilon) = BR_k(z_i(\alpha)) + O(\epsilon),$$

finishing the proof of (3.12).

- *Estimates for the $k = i$ term.* It will be more involved to control the $k = i$ term, and our goal is to show that

$$\left|\mathbf{v}_i^\epsilon(R_i^\epsilon(\alpha,\eta)) - BR_i(z_i(\alpha)) + \left(\eta + \frac{1}{2}\right)[\mathbf{v}](z_i(\alpha))\right| \le C\epsilon^b|\log\epsilon|. \tag{3.14}$$

To begin with, we again rewrite $\mathbf{v}_i^\epsilon$ as in (3.13) with $k = i$, and plug in the formula (3.8) for the determinant. This leads to

$$\begin{aligned}\mathbf{v}_i^\epsilon(R_i^\epsilon(\alpha,\eta)) &= \int_{S_k}\int_{-1}^{0} K_2(R_i^\epsilon(\alpha,\eta) - R_i^\epsilon(\alpha',\eta'))\big(L_i\gamma_i(\alpha')\\ &\quad -\epsilon L_i\gamma_i^2(\alpha')\kappa_i(\alpha')\eta'\big)\,d\eta' d\alpha'\\ &=: I_1 + I_2,\end{aligned}$$

where I_1, I_2 are the contributions from the two terms in the last parenthesis respectively. Let us control I_2 first, and we claim that

$$|I_2| \le C\epsilon|\log\epsilon|. \tag{3.15}$$

Using (3.2) of Lemma 3.1 and the fact that $|K_2(x)| \le |x|^{-1}$, we can bound I_2 as

$$\begin{aligned}|I_2| &= \left|\int_{S_k}\int_{-1}^{0} K_2(R_i^\epsilon(\alpha,\eta) - R_i^\epsilon(\alpha',\eta'))\,\epsilon L_i\gamma_i^2(\alpha')\kappa_i(\alpha')\eta'\,d\eta' d\alpha'\right|\\ &\le C\epsilon\int_{S_k}\int_{-1}^{0}\frac{\gamma_i(\alpha')}{|\alpha'-\alpha| + \epsilon|\gamma_i(\alpha')\eta' - \gamma_i(\alpha)\eta|}\,d\eta' d\alpha'\\ &\le C\epsilon\int_{S_k}\int_{-\|\gamma_i\|_\infty}^{\|\gamma_i\|_\infty}\frac{1}{|\alpha'-\alpha| + \epsilon|\theta'|}\,d\theta' d\alpha' \quad (\theta' := \gamma_i(\alpha')\eta' - \gamma_i(\alpha)\eta)\\ &\le C\epsilon\int_{-1/\epsilon}^{1/\epsilon}\int_{-\|\gamma_i\|_\infty}^{\|\gamma_i\|_\infty}\frac{1}{|\beta'| + |\theta'|}\,d\theta' d\beta' \quad (\beta' := \epsilon^{-1}(\alpha'-\alpha))\\ &\le C\epsilon|\log\epsilon|\end{aligned} \tag{3.16}$$

where C depends on $\|z_i\|_{C^2}$ and $\|\gamma_i\|_{L^\infty}$.

In the rest of the proof we focus on estimating $I_1 = \int_{S_k}\int_{-1}^{0} K_2(R_i^\epsilon(\alpha,\eta) - R_i^\epsilon(\alpha',\eta'))$ $L_i\gamma_i(\alpha')\,d\eta' d\alpha'$. For $t \in [0,1]$, let us define

$$\begin{aligned} f(\alpha,\alpha',\eta,\eta';t) &:= R_i^\epsilon(\alpha,\eta - t\eta') - R_i^\epsilon(\alpha',\eta' - t\eta'),\\ J(t) &:= \int_{S_k}\int_{-1}^{0} K_2(f(\alpha,\alpha',\eta,\eta';t))L_i\gamma_i(\alpha')\,d\eta' d\alpha'.\end{aligned} \tag{3.17}$$

Note that in the definition of f, the argument $\eta - t\eta'$ of R_i^ϵ belongs to $[-1, 1]$, instead of $[-1, 0]$ as in the original definition of (3.1). Here $R_i^\epsilon(\alpha, \eta - t\eta')$ is defined as in the formula (3.1), even though it might not belong to D_i^ϵ. Clearly, $J(0) = I_1$. The motivation for us to define such f and $J(t)$ is that at $t = 1$, we have

$$J(1) = \int_{S_k}\int_{-1}^0 K_2(R_i^\epsilon(\alpha, \eta-\eta') - z_i(\alpha'))L_i\gamma_i(\alpha')\,d\eta'd\alpha' = \int_{-1}^0 \mathbf{v}_i(R_i^\epsilon(\alpha, \eta-\eta'))\,d\eta', \tag{3.18}$$

where $\mathbf{v}_i$ is the velocity field generated by the sheet Γ_i. Recall that $\mathbf{v}_i$ has a jump across Γ_i, where we denote its limits on two sides by $\mathbf{v}_i^\pm$. Using Lemma 3.5, which we will prove momentarily, we have

$$\mathbf{v}_i(R_i^\epsilon(\alpha, \eta - \eta')) = \begin{cases} \mathbf{v}_i^+(z_i(\alpha)) + O(\epsilon^b|\log\epsilon|) & \text{if } \eta - \eta' \in (0, 2), \\ \mathbf{v}_i^-(z_i(\alpha)) + O(\epsilon^b|\log\epsilon|) & \text{if } \eta - \eta' \in (-2, 0). \end{cases} \tag{3.19}$$

We can then split the integration domain on the right hand side of (3.18) into $\eta' \in (-1, \eta)$ and $\eta' \in (\eta, 0)$, and use (3.19) to approximate the integrand in each interval. This gives

$$\begin{aligned} J(1) &= (\eta + 1)\mathbf{v}_i^+(z_i(\alpha)) - \eta\mathbf{v}_i^-(z_i(\alpha)) + O(\epsilon^b|\log\epsilon|) \\ &= BR_i(z_i(\alpha)) - \left(\eta + \frac{1}{2}\right)[\mathbf{v}](z_i(\alpha)) + O(\epsilon^b|\log\epsilon|), \end{aligned} \tag{3.20}$$

where in the last step we used that $[\mathbf{v}](z_i(\alpha)) = [\mathbf{v}_i](z_i(\alpha))$, since all other $\mathbf{v}_k$ with $k \neq i$ are continuous across Γ_i.
Finally, it remains to control $|J(0) - J(1)|$. Note that by (3.2), we have

$$f(\alpha, \alpha', \eta, \eta'; t) \geq c_0\big(|\alpha - \alpha'| + \epsilon|\gamma_i(\alpha')\eta' - \gamma_i(\alpha)\eta|\big).$$

In addition, we have

$$\left|\frac{\partial}{\partial t} f(\alpha, \alpha', \eta, \eta'; t)\right| = \left|\epsilon\big(\gamma_i(\alpha)\mathbf{n}(z_i(\alpha)) - \gamma_i(\alpha')\mathbf{n}(z_i(\alpha'))\big)\eta'\right| \leq C\epsilon|\alpha - \alpha'|^b,$$

where the last inequality follows from **(H2)** and the fact that $\mathbf{n}(z_i(\alpha)) \in C^1(S_i)$. Therefore, for any $t \in (0, 1)$, taking the t derivative of (3.17) and using that $|\nabla K_2(x)| \leq |x|^{-2}$, we have

$$\begin{aligned} |J'(t)| &\leq C\int_{S_k}\int_{-1}^0 \frac{\epsilon|\alpha - \alpha'|^b\gamma_i(\alpha')}{\big(|\alpha - \alpha'| + \epsilon|\gamma_i(\alpha')\eta' - \gamma_i(\alpha)\eta|\big)^2}d\eta'd\alpha' \\ &\leq C\epsilon\int_{S_k}\int_{-1}^0 \frac{\gamma_i(\alpha')}{|\alpha - \alpha'|^{1-b}\,(|\alpha - \alpha'| + \epsilon|\gamma_i(\alpha')\eta' - \gamma_i(\alpha)\eta|)}d\eta'd\alpha' \\ &\leq C\epsilon^b\int_{-1/\epsilon}^{1/\epsilon}\int_{-\|\gamma_i\|_\infty}^{\|\gamma_i\|_\infty} \frac{1}{|\beta'|^{1-b}(|\beta'| + |\theta'|)}\,d\theta'd\beta' \\ &\qquad (\theta' := \gamma_i(\alpha')\eta' - \gamma_i(\alpha)\eta,\ \beta' := \epsilon^{-1}(\alpha' - \alpha)) \\ &\leq C\epsilon^b\int_{-1/\epsilon}^{1/\epsilon} |\beta'|^{b-1}\log\left(1 + \frac{\|\gamma_i\|_{L^\infty}}{|\beta'|}\right)d\beta' \\ &\leq C\epsilon^b, \end{aligned}$$

where C depends on b, $\|\gamma_i\|_{C^b(S_i)}$, $\|z_i\|_{C^2(S_i)}$ and F_Γ. This leads to

$$|J(1) - I_1| = |J(1) - J(0)| \le C\epsilon^b |\log \epsilon|.$$

Finally, combining this with (3.20) and (3.15) yields (3.14), finishing the proof of the $k = i$ case. We can then conclude the proof by taking the sum of this estimate with all the $k \neq i$ estimates in (3.12). □

The following lemma proves (3.19). Let $\mathbf{v}_i$ be the velocity field generated by the sheet Γ_i, which is smooth in $\mathbb{R}^2 \setminus \Gamma_i$, and has a discontinuity across Γ_i. It is known that $\mathbf{v}_i$ converges to $\mathbf{v}_i^\pm$ respectively on the two sides of Γ_i [24]. However, we were unable to find a quantitative convergence rate (in terms of the distance from the point to Γ_i) in the literature, especially under the assumption that γ_i is only in $C^b(S_i)$ for the open curves. Below we prove such an estimate.

Lemma 3.5. *For $i = 1, \dots, n+m$, let $\mathbf{v}_i$ be the velocity field generated by the sheet Γ_i, given by*

$$\mathbf{v}_i(x) := \int_{S_i} K_2(x - z_i(\alpha'))\, \gamma_i(\alpha') |z_i'(\alpha')|\, d\alpha' \quad \text{for } x \in \mathbb{R}^2 \setminus \Gamma_i.$$

Then there exist constants $C, \epsilon_0 > 0$ depending on on b (as in **(H2)***), $\|z_i\|_{C^2(S_i)}$, $\|\gamma_i\|_{C^b(S_i)}$ and F_Γ, such that for all $\epsilon \in (0, \epsilon_0)$ and $\eta \in (-2, 2)$ we have*

$$\left|\mathbf{v}_i(R_i^\epsilon(\alpha,\eta)) - \mathbf{v}_i^+(z_i(\alpha))\right| \le C\epsilon^b |\log\epsilon| \quad \text{if } \eta \in (0,2), \tag{3.21}$$

$$\left|\mathbf{v}_i(R_i^\epsilon(\alpha,\eta)) - \mathbf{v}_i^-(z_i(\alpha))\right| \le C\epsilon^b |\log\epsilon| \quad \text{if } \eta \in (-2,0), \tag{3.22}$$

where

$$\mathbf{v}_i^+ = BR_i(z_i(\alpha)) + \frac{\mathbf{n}(z_i(\alpha))^\perp \gamma_i(\alpha)}{2}, \quad \mathbf{v}_i^- = BR_i(z_i(\alpha)) - \frac{\mathbf{n}(z_i(\alpha))^\perp \gamma_i(\alpha)}{2},$$

and BR_i is the contribution from the i-th integral in (2.4).

Proof. We will show (3.21) only since (3.22) can be treated in the same way. From the definition of R_i^ϵ in (3.1), we have

$$\begin{aligned}
\mathbf{v}_i(R_i^\epsilon(\alpha,\eta)) &= \frac{L_i}{2\pi}\int_{S_i} \frac{\big(z_i(\alpha) - z_i(\alpha')\big)^\perp \gamma_i(\alpha')}{|z_i(\alpha) - z_i(\alpha') + \epsilon\eta\mathbf{n}(z_i(\alpha))\gamma_i(\alpha)|^2}\, d\alpha' \\
&\quad + \frac{L_i}{2\pi}\int_{S_i} \frac{\epsilon\eta\mathbf{n}(z_i(\alpha))^\perp \gamma_i(\alpha)\gamma_i(\alpha')}{|z_i(\alpha) - z_i(\alpha') + \epsilon\eta\mathbf{n}(z_i(\alpha))\gamma_i(\alpha)|^2}\, d\alpha' \\
&=: A_1 + A_2.
\end{aligned}$$

We claim that for all $\epsilon > 0$ sufficiently small and $\eta \in [0, 2)$, we have

$$|A_1 - BR_i(z(\alpha))| \le C\epsilon^b |\log\epsilon|, \tag{3.23}$$

$$\left|A_2 - \frac{\mathbf{n}(z(\alpha))^\perp \gamma(\alpha)}{2}\right| \le C\epsilon^b, \tag{3.24}$$

and note that these two claims immediately yield (3.21). From now on, let us fix $i \in \{1, \dots, n+m\}$ and omit it in the notation for simplicity. Throughout this proof, let us denote

$$\mathbf{y}(\alpha,\alpha') := z(\alpha) - z(\alpha') \quad \text{and} \quad \mathbf{c}(\alpha) := \epsilon\eta\mathbf{n}(z(\alpha))\gamma(\alpha),$$

so that

$$A_1 = \frac{L}{2\pi}\int_S \frac{\mathbf{y}^\perp(\alpha,\alpha')\gamma(\alpha')}{|\mathbf{y}(\alpha,\alpha')+\mathbf{c}(\alpha)|^2}d\alpha', \quad A_2 = \frac{L}{2\pi}\int_S \frac{\mathbf{c}^\perp(\alpha)\gamma(\alpha')}{|\mathbf{y}(\alpha,\alpha')+\mathbf{c}(\alpha)|^2}d\alpha'.$$

Note that

$$F_\Gamma^{-1}|\alpha-\alpha'| \le |\mathbf{y}(\alpha,\alpha')| \le \|z\|_{C^1}|\alpha-\alpha'|. \tag{3.25}$$

For the closed curves with $i = 1, \dots, n$, since z has period 1, we can always set $\alpha - \alpha' \in [-\frac{1}{2}, \frac{1}{2})$ in this proof.

Applying (3.2) (with $\eta' = 0$), we have

$$|\mathbf{y}(\alpha,\alpha')+\mathbf{c}(\alpha)|^2 \ge c_0(|\alpha-\alpha'|^2 + \epsilon^2\eta^2\gamma^2(\alpha)) = c_0(|\alpha-\alpha'|^2 + |\mathbf{c}(\alpha)|^2). \tag{3.26}$$

Since $z'(\alpha) = L\mathbf{s}(z(\alpha))$, let us define

$$\tilde{\mathbf{y}}(\alpha,\alpha') := L\mathbf{s}(z(\alpha))(\alpha-\alpha'),$$

which is a close approximation of $\mathbf{y}$ in the sense that

$$|\mathbf{y}(\alpha,\alpha') - \tilde{\mathbf{y}}(\alpha,\alpha')| \le \|z\|_{C^2}(\alpha-\alpha')^2. \tag{3.27}$$

Using $\mathbf{s}(z(\alpha)) \perp \mathbf{n}(z(\alpha))$, we have

$$|\tilde{\mathbf{y}}(\alpha,\alpha')+\mathbf{c}(\alpha)|^2 = L^2|\alpha-\alpha'|^2 + \epsilon^2\eta^2\gamma^2(\alpha) = L^2|\alpha-\alpha'|^2 + |\mathbf{c}(\alpha)|^2. \tag{3.28}$$

From now on, for notational simplicity, we compress the dependence of $\mathbf{y}(\alpha,\alpha')$, $\tilde{\mathbf{y}}(\alpha,\alpha')$, $\mathbf{c}(\alpha)$ on α and α' in the rest of the proof.

- *Estimate (3.23).* Note that $BR_i(z(\alpha))$ can also be written using the above notations as

$$BR_i(z(\alpha)) = \frac{L}{2\pi}PV\int_S \frac{\mathbf{y}^\perp}{|\mathbf{y}|^2}\gamma(\alpha')d\alpha,$$

thus $A_1 - BR_i(z(\alpha))$ can be written as follows:

$$\begin{aligned} A_1 - BR_i(z(\alpha)) &= \frac{L}{2\pi}PV\int_S \underbrace{\left(\frac{\mathbf{y}^\perp}{|\mathbf{y}+\mathbf{c}|^2} - \frac{\mathbf{y}^\perp}{|\mathbf{y}|^2}\right)}_{=:\mathbf{f}(\mathbf{y},\mathbf{c})}\gamma(\alpha')d\alpha' \\ &= \frac{L}{2\pi}\int_S \mathbf{f}(\mathbf{y},\mathbf{c})(\gamma(\alpha')-\gamma(\alpha))d\alpha' + \frac{L\gamma(\alpha)}{2\pi}PV\int_S \mathbf{f}(\mathbf{y},\mathbf{c})d\alpha' \\ &=: A_{11} + A_{12}. \end{aligned}$$

A direct computation gives

$$\mathbf{f}(\mathbf{y},\mathbf{c}) = -\frac{\mathbf{y}^\perp}{|\mathbf{y}|^2}\frac{2\mathbf{y}\cdot\mathbf{c} + |\mathbf{c}|^2}{|\mathbf{y}+\mathbf{c}|^2}. \tag{3.29}$$

Since $\mathbf{y}\cdot\mathbf{c} = (\mathbf{y}-\tilde{\mathbf{y}})\cdot\mathbf{c} \le C|\alpha-\alpha'|^2|\mathbf{c}|$, (where we use $\tilde{\mathbf{y}} \perp \mathbf{n}(z(\alpha))$ and (3.27)), combining this with (3.25) and (3.26) gives a crude bound

$$|\mathbf{f}(\mathbf{y},\mathbf{c})| \lesssim \frac{|\alpha-\alpha'|^2|\mathbf{c}| + |\mathbf{c}|^2}{|\alpha-\alpha'|(|\alpha-\alpha'|^2+|\mathbf{c}|^2)}.$$

Plugging this into A_{11} and using the Hölder continuity of γ, we have

$$\begin{aligned}
|A_{11}| &\lesssim \int_S \frac{|\alpha-\alpha'|^2|\mathbf{c}| + |\mathbf{c}|^2}{|\alpha-\alpha'|(|\alpha-\alpha'|^2+|\mathbf{c}|^2)}|\alpha-\alpha'|^b d\alpha' \\
&\lesssim \int_{|\theta|<|\mathbf{c}|} (|\theta|^{1+b}|\mathbf{c}|^{-1} + |\theta|^{b-1})d\theta + \int_{|\mathbf{c}|\le|\theta|\le 1} (|\mathbf{c}||\theta|^{b-1} \\
&\quad + |\mathbf{c}|^2|\theta|^{b-3})d\theta \quad (\theta := \alpha'-\alpha) \\
&\lesssim |\mathbf{c}|^b \le C\epsilon^b,
\end{aligned}$$

where the last step follows from the fact that $|\mathbf{c}| \le 2\epsilon\|\gamma\|_\infty$. Now let us turn to A_{12}, which requires a more delicate estimate of $\mathbf{f}(\mathbf{y},\mathbf{c})$. Let us break A_{12} as

$$A_{12} = \frac{L\gamma(\alpha)}{2\pi}\int_S (\mathbf{f}(\mathbf{y},\mathbf{c}) - \mathbf{f}(\tilde{\mathbf{y}},\mathbf{c}))d\alpha' + \frac{L\gamma(\alpha)}{2\pi} PV\int_S \mathbf{f}(\tilde{\mathbf{y}},\mathbf{c})d\alpha' =: B_1 + B_2.$$

For B_1, let us take the gradient of $\mathbf{f}(\mathbf{y},\mathbf{c})$ (as in (3.29)) in the first variable. An elementary computation yields that

$$|\nabla_{\mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{c})| \le C|\mathbf{x}|^{-2}\min\left\{1, \frac{|\mathbf{c}|}{|\mathbf{x}|}\right\} \tag{3.30}$$

as long as $\mathbf{x}$ satisfies

$$|\mathbf{x}+\mathbf{c}|^2 \ge c_0(|\mathbf{x}|^2 + |\mathbf{c}|^2). \tag{3.31}$$

We point out that $\mathbf{x} = \xi\mathbf{y} + (1-\xi)\tilde{\mathbf{y}}$ indeed satisfies (3.31) for all $\xi \in [0,1]$: to see this, in the proof of Lemma 3.1, if we replace T_1 in (3.3) by $\xi\mathbf{y} + (1-\xi)\tilde{\mathbf{y}}$, one can easily check the proof still goes through for $\xi \in [0,1]$. In addition, for any $\xi \in [0,1]$ we also have

$$|\xi\mathbf{y} + (1-\xi)\tilde{\mathbf{y}}| \ge c_0|\alpha-\alpha'|. \tag{3.32}$$

Thus the gradient estimate (3.30) together with (3.27) and (3.32) yields

$$|f(\mathbf{y},\mathbf{c}) - f(\tilde{\mathbf{y}},\mathbf{c})| \lesssim \min\{1, |\mathbf{c}||\alpha-\alpha'|^{-1}\} \lesssim \min\{1, \epsilon|\alpha-\alpha'|^{-1}\},$$

and plugging this into B_1 gives

$$|B_1| \lesssim \epsilon + \int_{\epsilon<|\alpha-\alpha'|<1} \epsilon|\alpha-\alpha'|^{-1}d\alpha' \lesssim \epsilon|\log\epsilon|.$$

As for B_2, using the definition of $\tilde{\mathbf{y}}$, the identity (3.28) and the fact that $\tilde{\mathbf{y}}\cdot\mathbf{c} = 0$, we have

$$\begin{aligned}
B_2 &= \frac{L\gamma(\alpha)}{2\pi} PV\int_S -\frac{\tilde{\mathbf{y}}^\perp}{|\tilde{\mathbf{y}}|^2}\frac{|\mathbf{c}|^2}{|\tilde{\mathbf{y}}+\mathbf{c}|^2}d\alpha' \\
&= \frac{L\gamma(\alpha)|\mathbf{c}|^2\mathbf{n}(z(\alpha))}{2\pi L} PV\int_S \frac{\alpha'-\alpha}{|\alpha'-\alpha|^2(L^2|\alpha'-\alpha|^2 + |\mathbf{c}|^2)}d\alpha'.
\end{aligned}$$

For the closed curves $i = 1,\ldots,n$, we immediately have $B_2 = 0$ since $\alpha-\alpha' \in [-\frac{1}{2},\frac{1}{2})$, and the integrand is an odd function of $\alpha'-\alpha$.

For the open curves $i = n+1, \ldots, n+m$, the above integral becomes

$$\begin{aligned} B_2 &= \frac{L\gamma(\alpha)|\mathbf{c}|^2\mathbf{n}(z(\alpha))}{2\pi L} PV \int_{-\alpha}^{1-\alpha} \frac{\theta}{|\theta|^2(L^2|\theta|^2 + |\mathbf{c}|^2)} d\theta \quad (\theta := \alpha' - \alpha) \\ &= \frac{L\gamma(\alpha)|\mathbf{c}|^2\mathbf{n}(z(\alpha))}{2\pi L} \int_{\alpha}^{1-\alpha} \frac{\theta}{\theta^2(L^2\theta^2 + |\mathbf{c}|^2)} d\theta, \end{aligned}$$

where in the second inequality we used that the integral in $[-\alpha, \alpha]$ gives zero contribution to the principal value, since the integrand is odd.

Next we discuss two cases. If $\alpha > |\mathbf{c}|$, we bound the integrand by $C\theta^{-3}$, which gives

$$|B_2| \le C\gamma(\alpha)|\mathbf{c}|^2\alpha^{-2} \le C|\mathbf{c}|^2\alpha^{b-2} \le C|\mathbf{c}|^b \le C\epsilon^b.$$

where the second inequality follows from the assumption $\gamma(0) = 0$ for an open curve in **(H3)**, as well as the Hölder continuity of γ. And if $0 < \alpha \le |\mathbf{c}|$, the integrand can be bounded above by $\theta^{-1}|\mathbf{c}|^{-2}$, which immediately leads to

$$|B_2| \le C\gamma(\alpha)|\log\alpha| \le C|\mathbf{c}|^b|\log|\mathbf{c}|| \le C\epsilon^b|\log\epsilon|.$$

In both cases we have $|B_2| \le C\epsilon^b|\log\epsilon|$, and combining it with the B_1 and A_{11} estimates gives (3.23).

- *Estimate (3.24).* We break A_2 into

$$\begin{aligned} A_2 &= \frac{L\mathbf{c}^\perp}{2\pi}\int_S \frac{\gamma(\alpha') - \gamma(\alpha)}{|\mathbf{y}+\mathbf{c}|^2} d\alpha' + \frac{L\mathbf{c}^\perp\gamma(\alpha)}{2\pi}\int_S \left(\frac{1}{|\mathbf{y}+\mathbf{c}|^2} - \frac{1}{|\tilde{\mathbf{y}}+\mathbf{c}|^2}\right) d\alpha' \\ &\quad + \frac{L\mathbf{c}^\perp\gamma(\alpha)}{2\pi}\int_S \frac{1}{|\tilde{\mathbf{y}}+\mathbf{c}|^2} d\alpha' \\ &=: A_{21} + A_{22} + A_{23}. \end{aligned}$$

For A_{21}, (3.26) and the Hölder continuity of γ immediately lead to

$$|A_{21}| \le C|\mathbf{c}|\int_S \frac{|\alpha - \alpha'|^b}{|\alpha-\alpha'|^2 + |\mathbf{c}|^2} d\alpha' \le |\mathbf{c}|^b \le C\epsilon^b. \tag{3.33}$$

For A_{22}, its integrand can be controlled as

$$\left|\frac{1}{|\mathbf{y}+\mathbf{c}|^2} - \frac{1}{|\tilde{\mathbf{y}}+\mathbf{c}|^2}\right| \le \frac{|\mathbf{y}-\tilde{\mathbf{y}}|(|\mathbf{y}+\mathbf{c}| + |\tilde{\mathbf{y}}+\mathbf{c}|)}{|\mathbf{y}+\mathbf{c}|^2|\tilde{\mathbf{y}}+\mathbf{c}|^2} \le \frac{C|\alpha-\alpha'|^2}{(|\alpha-\alpha'|^2 + |\mathbf{c}|^2)^{3/2}},$$

where the last step follows from (3.26), (3.27) and (3.28). This allows us to control A_{22} as

$$|A_{22}| \le C|\mathbf{c}|\int_{-1}^{1} \frac{\theta^2}{(\theta^2 + |\mathbf{c}|^2)^{3/2}} d\theta \le C|\mathbf{c}|\,\big|\log|\mathbf{c}|\big| \le C\epsilon|\log\epsilon|. \tag{3.34}$$

Finally, for the A_{23} term, (3.28) gives

$$A_{23} = \frac{L\mathbf{c}^{\perp}\gamma(\alpha)}{2\pi}\int_S \frac{1}{L^2|\alpha'-\alpha|^2+|\mathbf{c}|^2}d\alpha' = \frac{\mathbf{n}^{\perp}(\alpha)\gamma(\alpha)}{2\pi}\int_I \frac{1}{\theta^2+1}d\theta$$
$$(\text{set } \theta := \frac{L(\alpha'-\alpha)}{|\mathbf{c}|}),$$

where the integration interval $I=(-\frac{L}{2|\mathbf{c}|}, \frac{L}{2|\mathbf{c}|})$ for $i=1,\ldots,n$, and $I=(-\frac{L\alpha}{|\mathbf{c}|}, \frac{L(1-\alpha)}{|\mathbf{c}|})$ for $i=n+1,\ldots,n+m$, and in the last equality we also used that $\frac{\mathbf{c}^{\perp}}{|\mathbf{c}|}=\mathbf{n}^{\perp}$. For $i=1,\ldots,n$, one can easily check that

$$\left|\int_I \frac{1}{\theta^2+1}d\theta - \pi\right| = 2\int_{\frac{L}{2|\mathbf{c}|}}^{\infty}\frac{1}{\theta^2+1}d\theta \le C|\mathbf{c}| \le C\epsilon,$$

which immediately leads to

$$\left|A_{23} - \frac{\mathbf{n}(z(\alpha))^{\perp}\gamma(\alpha)}{2}\right| = \left|\frac{\mathbf{n}^{\perp}(\alpha)\gamma(\alpha)}{2\pi}\left(\int_I \frac{1}{\theta^2+1}d\theta - \pi\right)\right| \le C\epsilon$$

for $i=1,\ldots,n$. Next we turn to the open curves $i=n+1,\ldots,n+m$, and let us assume $\alpha\in[0,\frac{1}{2}]$ without loss of generality. In this case we have

$$\left|\int_I \frac{1}{\theta^2+1}d\theta - \pi\right| = \int_{-\infty}^{-\frac{L\alpha}{|\mathbf{c}|}}\frac{1}{\theta^2+1}d\theta + \int_{\frac{L(1-\alpha)}{|\mathbf{c}|}}^{\infty}\frac{1}{\theta^2+1}d\theta \le \min\left\{C\frac{|\mathbf{c}|}{\alpha}, \frac{\pi}{2}\right\} + C\epsilon.$$

where we used $1-\alpha>\frac{1}{2}$ to control the second integral by $C\epsilon$. Using the above inequality as well as the fact that $\gamma(\alpha)\le C\alpha^b$ due to **(H3)**, we have

$$\left|A_{23} - \frac{\mathbf{n}(z(\alpha))^{\perp}\gamma(\alpha)}{2}\right| = \frac{\gamma(\alpha)}{2\pi}\left|\int_I \frac{1}{\theta^2+1}d\theta - \pi\right| \le C\alpha^b\min\left\{\frac{|\mathbf{c}|}{\alpha}, 1\right\} + C\epsilon$$
$$\le C(|\mathbf{c}|^b+\epsilon) \le C\epsilon^b$$

for $i=n+1,\ldots,n+m$. Finally, combining the A_{23} estimates together with (3.33) and (3.34) yields (3.24). □

4. Constructing a Divergence-Free Perturbation

In this section, we aim to construct a divergence-free velocity field $\mathbf{u}^{\epsilon}: D^{\epsilon}\to\mathbb{R}^2$, such that $-\mathbf{u}^{\epsilon}$ tends to make each D_i^{ϵ} "more symmetric". Let $\mathbf{u}^{\epsilon}: D^{\epsilon}\to\mathbb{R}^2$ be given by

$$\mathbf{u}^{\epsilon} := x + \nabla p^{\epsilon} \quad \text{in } D^{\epsilon}, \tag{4.1}$$

where the function $p^{\epsilon}: \overline{D^{\epsilon}}\to\mathbb{R}$ is chosen such that

$$\nabla\cdot\mathbf{u}^{\epsilon} = 0 \quad \text{in } D^{\epsilon}, \tag{4.2}$$

and on each connected component l of ∂D^{ϵ}, u^{ϵ} satisfies

$$\int_l \mathbf{u}^{\epsilon}\cdot n\,d\sigma = 0, \tag{4.3}$$

where n is the unit normal of l pointing outwards of D^ϵ. Note that ∂D^ϵ has a total of $2n+m$ connected components: D_i^ϵ is doubly-connected for $i = 1, \dots, n$ (denote its outer and inner boundaries by $\partial D_{i,\text{out}}^\epsilon$ and $\partial D_{i,\text{in}}^\epsilon$; note that $\partial D_{i,\text{in}}^\epsilon$ coincides with Γ_i), whereas it is simply-connected for $i = n+1, \dots, n+m$ (denote its boundary by ∂D_i^ϵ).

Next we show that there indeed exists a function p^ϵ so that $\mathbf{u}^\epsilon$ satisfies (4.2)–(4.3). Clearly, (4.2) requires that p^ϵ satisfies

$$\Delta p^\epsilon = -2 \quad \text{in } D^\epsilon. \tag{4.4}$$

Once (4.2) is satisfied, the divergence theorem yields that $\mathbf{u}^\epsilon$ satisfies (4.3) for each $l = \partial D_i^\epsilon$ for $i = n+1, \dots, n+m$.

Next let us set the boundary conditions as

$$p^\epsilon|_{\partial D_i^\epsilon} = 0 \quad \text{for } i = n+1, \dots, n+m. \tag{4.5}$$

For $i = 1, \dots, n$, let

$$p^\epsilon = \begin{cases} 0 & \text{on } \partial D_{i,\text{out}}^\epsilon \\ c_i^\epsilon & \text{on } \partial D_{i,\text{in}}^\epsilon = \Gamma_i \end{cases} \qquad \text{for } i = 1, \dots, n, \tag{4.6}$$

where $c_i^\epsilon > 0$ is the unique constant such that

$$\int_{\partial U_i} \nabla p^\epsilon \cdot n d\sigma = -2|U_i| \quad \text{for } i = 1, \dots, n, \tag{4.7}$$

where U_i is the domain enclosed by $\partial D_{i,\text{in}}^\epsilon = \Gamma_i$ (thus U_i is independent of ϵ), and n is the outer normal of U_i (thus the inner normal of D_i^ϵ). The existence of c_i^ϵ is guaranteed by [14, Lemma 2.5]. One can then check that $\int_{\partial U_i} \mathbf{u}^\epsilon \cdot n d\sigma = 0$. Applying the divergence theorem in D_i^ϵ then gives us that $\int_{\partial D_{i,\text{out}}^\epsilon} \mathbf{u}^\epsilon \cdot n d\sigma = 0$ as well, thus $\mathbf{u}^\epsilon$ satisfies (4.3) for $i = 1, \dots, n$.

In [14] we proved a rearrangement inequality for such p^ϵ in a similar spirit of Talenti's rearrangement inequality for elliptic equations [37], which we state below.

Lemma 4.1 [14, Proposition 2.6]. *The function $p^\epsilon : \overline{D^\epsilon} \to \mathbb{R}$ defined in (4.4)–(4.7) satisfies the following in each D_i^ϵ for $i = 1, \dots, n+m$:*

$$\sup_{D_i^\epsilon} p^\epsilon \le \frac{|D_i^\epsilon|}{2\pi}, \tag{4.8}$$

and

$$\int_{D_i^\epsilon} p^\epsilon(x) dx \le \frac{|D_i^\epsilon|^2}{4\pi}. \tag{4.9}$$

Moreover, each inequality above achieves equality if and only D_i^ϵ is either a disk or an annulus.

Note that the inequalities (4.8)–(4.9) hold for any domain with $C^{1,\alpha}$ boundary. Even though the inequalities are strict when D_i^ϵ is non-radial, they are not strong enough to rule out non-radial vortex sheets, as we need quantitative versions of strict inequalities that are still valid in the $\epsilon \to 0^+$ limit. As we will see in the proof of Proposition 5.2, the key step is to show that if some Γ_i is either not a circle or does not have a constant γ_i, then the following quantitative version of (4.9) holds: $\epsilon^{-2}\left(\frac{|D_i^\epsilon|^2}{4\pi} - \int_{D_i^\epsilon} p^\epsilon(x)dx\right) \geq c_0 > 0$, where c_0 is independent of ϵ.

In order to upgrade (4.9) into a quantitative version, we need to obtain some fine estimates for p^ϵ that take into account the shape of the thin domains D_i^ϵ. For $i = n+1, \ldots, n+m$, since $p^\epsilon = 0$ on ∂D_i^ϵ, and the domain D_i^ϵ is a thin simply-connected domain with width $\epsilon \ll 1$, intuitively one would expect that $|p^\epsilon| \leq C\epsilon^2$. The next proposition shows that this crude estimate is indeed true, and its proof is postponed to Sect. 4.1.

Proposition 4.2. *For any $i = n+1, \ldots, n+m$, let $p^\epsilon : \overline{D_i^\epsilon} \to \mathbb{R}$ be given by* (4.4)–(4.5). *Then there exist ϵ_1 and C only depending on $\|z_i\|_{C^2(S_i)}$, $\|\gamma_i\|_{L^\infty(S_i)}$ and F_Γ, such that*

$$|p^\epsilon| \leq C\epsilon^2 \quad \text{in } D_i^\epsilon$$

for all $\epsilon \in (0, \epsilon_1)$.

For $i = 1, \ldots, n$, the estimate is more involved, since p^ϵ takes different values c_i^ϵ and 0 on the inner and outer boundaries of D_i^ϵ. Heuristically speaking, since D_i^ϵ is a doubly-connected thin tubular domain with width $\sim \epsilon$, we would expect that p_i^ϵ (in α, η coordinate) changes almost linearly from 0 to c_i^ϵ as η goes from -1 (outer boundary) to 0 (inner boundary). Next we will show that the error between $p^\epsilon(R_i^\epsilon(\alpha, \eta))$ and the linear-in-η function $c_i^\epsilon(1+\eta)$ is indeed controlled by $O(\epsilon^2)$. We will also obtain fine estimates of the gradient of the function $c_i^\epsilon(1+\eta)$, as well as the boundary value c_i^ϵ. Again, its proof is postponed to Sect. 4.1.

Proposition 4.3. *For any $i = 1, \ldots, n$, let $p^\epsilon : \overline{D_i^\epsilon} \to \mathbb{R}$ and $c_i^\epsilon \in \mathbb{R}$ be given by* (4.4) *and* (4.6)–(4.7). *For such p^ϵ, let us define $\tilde{p}^\epsilon, q^\epsilon : \overline{D_i^\epsilon} \mapsto \mathbb{R}$ as follows:*

$$\begin{aligned} \tilde{p}^\epsilon(R_i^\epsilon(\alpha, \eta)) &:= c_i^\epsilon(1+\eta) \quad \text{for } \alpha \in S_i, \eta \in [0, -1], \\ q^\epsilon &:= p^\epsilon - \tilde{p}^\epsilon \qquad \text{in } \overline{D_i^\epsilon}. \end{aligned} \tag{4.10}$$

Also let

$$\beta_i := \frac{2|U_i|}{L_i \int_{S_i} \gamma_i^{-1}(\alpha)d\alpha}. \tag{4.11}$$

Then there exist ϵ_1 and C only depending on $\|z_i\|_{C^3(S_i)}$, $\|\gamma_i\|_{C^2(S_i)}$ and F_Γ, such that for all $\epsilon \in (0, \epsilon_1)$ we have the following:

$$\begin{cases} |q^\epsilon| \leq C\epsilon^2 & \text{in } D_i^\epsilon, \\ q^\epsilon = 0 & \text{on } \partial D_i^\epsilon, \end{cases} \tag{4.12}$$

$$\left|\frac{c_i^\epsilon}{\epsilon} - \beta_i\right| \leq C\epsilon, \tag{4.13}$$

$$\left|\nabla\tilde{p}^\epsilon(R_i^\epsilon(\alpha, \eta)) - \frac{\beta_i}{\gamma_i(\alpha)}\mathbf{n}(z_i(\alpha))\right| \leq C\epsilon \quad \text{for } \alpha \in S_i, \eta \in [0, -1]. \tag{4.14}$$

4.1. Proof of the quantitative lemmas for p^ϵ. In this subsection we aim to prove Propositions 4.2 and 4.3 . We start with a technical lemma on estimating the solution of Poisson's equation (with zero boundary condition) in the domain D_i^ϵ.

Lemma 4.4. *For any $i = 1, \dots, n+m$, assume Γ_i and γ_i satisfy **(H1)–(H3)**. Let $v^\epsilon \in C^2(D_i^\epsilon) \cap C(\overline{D_i^\epsilon})$ solve the Poisson's equation with zero boundary condition:*

$$\begin{cases} \Delta v^\epsilon = -1 & \text{in } D_i^\epsilon, \\ v^\epsilon = 0 & \text{on } \partial D_i^\epsilon. \end{cases} \tag{4.15}$$

Then there exist positive constants $\epsilon_0 = C(\|z_i\|_{C^2(S_i)}, \|\gamma_i\|_{L^\infty(S_i)}, F_\Gamma)$ and $C_1, C_2 = C(\|\gamma_i\|_{L^\infty(S_i)})$, such that for all $\epsilon \in (0, \epsilon_0)$ we have

$$0 \le v^\epsilon \le C_1 \epsilon^2 \quad \text{in } D_i^\epsilon \tag{4.16}$$

and

$$\|\nabla v^\epsilon\|_{L^\infty(\Gamma_i)} \le C_2 \epsilon \quad \text{for } i = 1, \dots, n. \tag{4.17}$$

Proof. Throughout the proof, let $i \in \{1, \dots, n+m\}$ be fixed. For notational simplicity, in the rest of the proof we omit the subscript i in R_i^ϵ, D_i^ϵ, S_i, z_i and γ_i.

Step 1. We start with a simple geometric result that D^ϵ is "flat" in a small neighborhood of any interior $z(\alpha)$. For any $\alpha \in S^\circ$, let $V^\epsilon(\alpha) := D^\epsilon \cap B_{6\epsilon\|\gamma\|_\infty}(z(\alpha))$, where $\|\cdot\|_\infty$ denotes $\|\cdot\|_{L^\infty(S)}$. We will show that any $y \in V^\epsilon(\alpha)$ satisfies

$$\left|(z(\alpha) - y) \cdot \mathbf{n}(z(\alpha))\right| \le 2\epsilon \|\gamma\|_\infty \tag{4.18}$$

for all sufficiently small $\epsilon > 0$ (to be quantified in (4.23)). See Fig. 4a for an illustration.

Since $y \in V^\epsilon(\alpha) \subset D^\epsilon$, there exist $\beta \in S$ and $\eta \in (-1, 0)$ such that $y = R^\epsilon(\beta, \eta) = z(\beta) + \epsilon\gamma(\beta)\mathbf{n}(z(\beta))\eta$. It follows that

$$\begin{aligned} \left|(z(\alpha) - y) \cdot \mathbf{n}(z(\alpha))\right| &\le |(z(\alpha) - z(\beta)) \cdot \mathbf{n}(z(\alpha))| + \epsilon\|\gamma\|_\infty \\ &\le \|z''\|_\infty (\alpha - \beta)^2 + \epsilon\|\gamma\|_\infty, \end{aligned} \tag{4.19}$$

where in the second inequality we used

$$|(z(\alpha) - z(\beta)) - z'(\alpha)(\alpha - \beta)| \le \|z''\|_\infty (\alpha - \beta)^2 \tag{4.20}$$

and $z'(\alpha) \cdot \mathbf{n}(z(\alpha)) = 0$. To bound $\alpha - \beta$ on the right hand side of (4.19), the fact that $y \in B_{6\epsilon\|\gamma\|_\infty}(z(\alpha))$ gives

$$6\epsilon\|\gamma\|_\infty \ge |z(\alpha) - y| \ge |z(\alpha) - z(\beta)| - \epsilon\gamma(\beta), \tag{4.21}$$

which implies $|z(\alpha) - z(\beta)| \le 7\epsilon\|\gamma\|_\infty$. Since the arc-chord constant F_Γ given in (2.2) is finite, this implies

$$|\alpha - \beta| \le 7 F_\Gamma \|\gamma\|_\infty \epsilon. \tag{4.22}$$

Plugging this into the right hand side of (4.19), we know (4.18) holds for all

$$0 < \epsilon \le (49\|z''\|_\infty F_\Gamma^2 \|\gamma\|_\infty)^{-1}. \tag{4.23}$$

Step 2. Next we prove (4.16). Note that v^ϵ is superharmonic in D^ϵ and vanishes on the boundary, thus it follows from the maximum principle that $v^\epsilon \ge 0$ in D^ϵ. Denote $M := \max_{x \in D^\epsilon} v^\epsilon(x)$, and pick $x_0 = R(\alpha_0, \eta_0) \in D^\epsilon$ such that $v^\epsilon(x_0) = M$. Note that

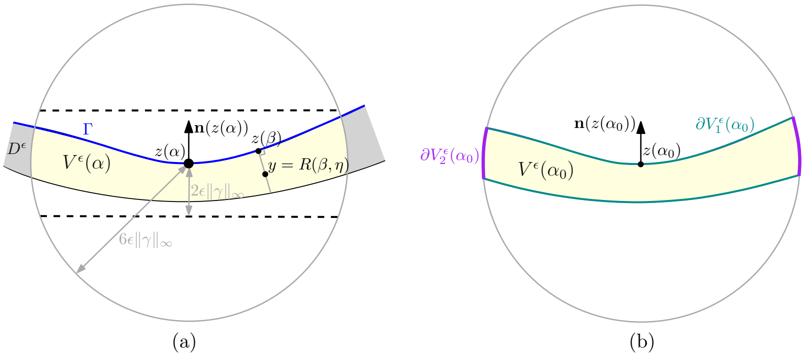

Fig. 4. **a** In Step 1, $V^\epsilon(\alpha)$ (the yellow set) must lie between the two dashed lines for small ϵ. **b** In Step 2, $\partial V^\epsilon(\alpha_0)$ is decomposed into $\partial V_1^\epsilon(\alpha_0)$ (in dark green) and $\partial V_2^\epsilon(\alpha_0)$ (in purple)

$\alpha_0 \in S^\circ$. Without loss of generality, we can assume that $z(\alpha_0) = (0, 0)$ and $\mathbf{s}(z(\alpha_0)) = \mathbf{e}_1 := (1, 0)$, so that $\mathbf{n}(z(\alpha_0)) = (0, 1)$ and $x_0 = (0, \epsilon\gamma(\alpha_0)\eta_0)$. Let us consider a barrier function $b_1 : \mathbb{R}^2 \mapsto \mathbb{R}$ given by

$$b_1(x_1, x_2) = x_2^2 - \frac{x_1^2}{2}.$$

Clearly $\Delta b_1 = 1$, so $v^\epsilon + b_1$ is harmonic in D^ϵ. It then follows from the maximum principle that $\max_{\overline{V^\epsilon(\alpha_0)}}(v^\epsilon + b_1)$ is achieved at some boundary point $\tilde{x}_0 \in \partial V^\epsilon(\alpha_0)$. Let us break $\partial V^\epsilon(\alpha_0)$ into $\partial V_1^\epsilon(\alpha_0) \cup \partial V_2^\epsilon(\alpha_0)$ (see Fig. 4b for an illustration), given by

$$\partial V_1^\epsilon(\alpha_0) := \partial D^\epsilon \cap B_{6\epsilon\|\gamma\|_\infty}(z(\alpha_0)), \quad \partial V_2^\epsilon(\alpha_0) := \overline{D^\epsilon} \cap \partial B_{6\epsilon\|\gamma\|_\infty}(z(\alpha_0)). \tag{4.24}$$

We claim that $\tilde{x}_0 \in \partial V_1^\epsilon(\alpha_0)$. To see this, note that any $y = (y_1, y_2) \in \partial V_2^\epsilon(\alpha_0)$ satisfies $|y| = 6\epsilon\|\gamma\|_\infty$ and $|y_2| \le 2\epsilon\|\gamma\|_\infty$, where the latter follows from (4.18) and our assumptions that $\mathbf{s}(z(\alpha_0)) = \mathbf{e}_1$ and $z(\alpha_0) = (0, 0)$. This implies that $|y_1| \ge 4\epsilon\|\gamma\|_\infty > |y_2|$, thus $b_1(y) < 0$. Using that $v^\epsilon(x_0) = M \ge v^\epsilon(y)$ and $b_1(x_0) = b_1(0, \epsilon\gamma(\alpha_0)\eta_0) \ge 0$, we have $(v^\epsilon + b_1)(y) < (v^\epsilon + b_1)(x_0)$. This shows that $\max_{\overline{V^\epsilon(\alpha_0)}}(v^\epsilon + b_1)$ cannot be achieved on $\partial V_2^\epsilon(\alpha_0)$, finishing the proof of the claim.

Since $\tilde{x}_0 \in \partial V_1^\epsilon(\alpha_0) \subset \partial D^\epsilon$, the boundary condition in (4.15) yields that $v^\epsilon(\tilde{x}_0) = 0$. Thus

$$M + b_1(x_0) = v^\epsilon(x_0) + b_1(x_0) \le v^\epsilon(\tilde{x}_0) + b_1(\tilde{x}_0) = b_1(\tilde{x}_0).$$

Using $b_1(x_0) = b_1(0, \epsilon\gamma(\alpha_0)\eta_0) \ge 0$, the above inequality becomes

$$M \le b_1(\tilde{x}_0) \le |\tilde{x}_0|^2 \le 36\|\gamma\|_\infty^2\epsilon^2, \tag{4.25}$$

where the second inequality follows from the definition of b_1. This proves (4.16) for $C_1 = 36\|\gamma\|_\infty^2$.

Step 3. It remains to prove (4.17). First note that for $i \in \{1, \dots, n\}$, the assumptions **(H1)–(H3)** yield that D_i^ϵ has C^2 boundary, therefore $v^\epsilon \in C^2(D_i^\epsilon) \cap C^1(\overline{D_i^\epsilon})$. Let us fix $i \in \{1, \dots, n\}$ and any $\alpha \in S$, and we aim to show that $|\nabla v^\epsilon(z(\alpha))| \le C_2\epsilon$. Again, without loss of generality we can assume that $z(\alpha) = (0, 0)$ and $\mathbf{s}(z(\alpha)) = \mathbf{e}_1$. Let us consider a new barrier function $b_2 : \mathbb{R}^2 \to \mathbb{R}$

$$b_2(x_1, x_2) := x_2^2 + 4\epsilon\|\gamma\|_\infty x_2 - \frac{x_1^2}{2}, \tag{4.26}$$

which satisfies $b_2(0,0) = 0$, and one can easily check that its zero level set has horizontal tangent at $(0,0)$ (thus tangent to ∂D^ϵ at $z(\alpha)$).

Again, let us decompose $\partial V^\epsilon(\alpha)$ as $\partial V_1^\epsilon(\alpha) \cup \partial V_2^\epsilon(\alpha)$ as in (4.24) (except that α_0 now becomes α). We claim that for all sufficiently small $\epsilon > 0$, the new barrier function b_2 satisfies

$$\Delta b_2 = 1 \quad \text{in } V^\epsilon(\alpha), \tag{4.27}$$

$$b_2 \leq 0 \quad \text{on } \partial V_1^\epsilon, \tag{4.28}$$

$$b_2 \leq -\epsilon^2 \quad \text{on } \partial V_2^\epsilon. \tag{4.29}$$

Let us assume for a moment that (4.27)–(4.29) are true. Then it follows that

$$v^\epsilon + C_2 b_2 \leq 0 \text{ in } V^\epsilon(\alpha), \tag{4.30}$$

where $C_2 := \max\{1, C_1\}$ and C_1 is as in (4.16) (in the end of step 2 we have $C_1 = 36\|\gamma\|_\infty^2$). To show (4.30), note that $v^\epsilon + C_2 b_2$ is subharmonic in $V^\epsilon(\alpha)$ due to (4.27) and the definition of C_2, thus its maximum is attained on its boundary. The boundary conditions in (4.15) and (4.28) yield that $v^\epsilon + C_2 b_2 \leq 0$ on $\partial V_1^\epsilon(\alpha)$; whereas (4.16), (4.29) and the definition of C_2 yield that $v^\epsilon + C_2 b_2 \leq 0$ on $\partial V_2^\epsilon(\alpha)$. Thus $v^\epsilon + C_2 b_2 \leq 0$ on $\partial V_1^\epsilon(\alpha) \cup \partial V_2^\epsilon(\alpha)$, implying (4.30).

However, $v^\epsilon + C_2 b_2$ is actually zero at $z(\alpha) \in \partial V^\epsilon(\alpha)$, therefore Hopf's Lemma implies that $\nabla(v^\epsilon + C_2 b_2)(z(\alpha)) \cdot \boldsymbol{n}(z(\alpha)) > 0$, where $\boldsymbol{n}(z(\alpha))$ is the outer normal of ∂D^ϵ at $z(\alpha)$. Hence

$$|\nabla v^\epsilon(z(\alpha))| = -\nabla v^\epsilon(z(\alpha)) \cdot \boldsymbol{n}(z(\alpha)) < C_2 \nabla b_2(z(\alpha)) \cdot \boldsymbol{n}(z(\alpha)) = 4C_2\|\gamma\|_\infty \epsilon, \tag{4.31}$$

where the first equality follows from the fact that v^ϵ is superharmonic in D^ϵ and constant on ∂D^ϵ, and the second equality is a direct computation of ∇b_2. Thus (4.31) proves (4.17).

To complete the proof, we only need to prove (4.27)–(4.29) for small $\epsilon > 0$. Note that (4.27) follows immediately from computing the Laplacian of b_2. For (4.28), let us pick $y \in \partial V_1^\epsilon(\alpha)$, and we aim to show that $b_2(y) \leq 0$. Note that $y = R^\epsilon(\beta, 0)$ or $R^\epsilon(\beta, -1)$ for some $\beta \in S$. We first deal with the first case.

Let us denote $y = (y_1, y_2)$. Rewriting (4.20) into two inequalities for the two components, and using that $z(\alpha) = (0,0)$ and $z'(\alpha) = L\mathbf{e}_1$ (L is the length of the curve Γ_i), we have

$$|0 - y_1 - L(\alpha - \beta)| \leq \|z''\|_\infty (\alpha - \beta)^2 \tag{4.32}$$

$$|y_2| = |0 - y_2| \leq \|z''\|_\infty (\alpha - \beta)^2. \tag{4.33}$$

Also, (4.22) gives $|\alpha - \beta| \leq 7F_\Gamma \|\gamma\|_\infty \epsilon$. Applying it to (4.32), for all $\epsilon > 0$ sufficiently small we have that

$$|y_1| \geq \frac{L}{2}|\alpha - \beta|. \tag{4.34}$$

Plugging (4.34) and (4.33) into $b_2(y) = -\frac{1}{2}y_1^2 + y_2^2 + 4\epsilon\|\gamma\|_\infty y_2$, we have

$$\begin{aligned} b_2(y) &\leq -\frac{L^2}{8}(\alpha - \beta)^2 + \|z''\|_\infty^2 (\alpha - \beta)^4 + 4\epsilon\|\gamma\|_\infty \|z''\|_\infty (\alpha - \beta)^2 \\ &\leq \left(-\frac{L^2}{8} + C\epsilon^2 + C\epsilon\right)(\alpha - \beta)^2 \leq 0, \end{aligned}$$

for all $\epsilon > 0$ sufficiently small, where the second inequality follows from (4.22). This finishes the proof of (4.28) for the case $y = R^\epsilon(\beta, 0)$.

Before we deal with the case $y = R^\epsilon(\beta, -1)$, let us prove (4.29) first. For any $y = (y_1, y_2) \in \partial V_2^\epsilon(\alpha)$, (4.18) gives $|y_2| \le 2\epsilon\|\gamma\|_\infty$. Combining this with $|y| = 6\epsilon\|\gamma\|_\infty$ yields $|y_1| \ge \sqrt{32}\epsilon\|\gamma\|_\infty$. Thus

$$b_2(y) \le (2\epsilon\|\gamma\|_\infty)^2 + 4\epsilon\|\gamma\|_\infty(2\epsilon\|\gamma\|_\infty) - \frac{(\sqrt{32}\epsilon\|\gamma\|_\infty)^2}{2} \le -4\epsilon^2\|\gamma\|_\infty^2.$$

Finally we turn to the proof of (4.28) for the case $y = R^\epsilon(\beta, -1)$. Note that the curve $\{R^\epsilon(\beta, -1) : \beta \in S\} \cap B_{6\epsilon\|\gamma\|_\infty}(z(\alpha))$ lies in the interior of the region bounded by $\Gamma \cap B_{6\epsilon\|\gamma\|_\infty}(z(\alpha))$ on the top, $\partial B_{6\epsilon\|\gamma\|_\infty}(z(\alpha))$ on the sides, and $y_2 = -2\epsilon\|\gamma\|_\infty$ on the bottom. (The last one follows from (4.18) and our assumption that $\mathbf{s}(z(\alpha)) = \mathbf{e}_1$). We have already shown $b_2 \le 0$ on $\Gamma \cap B_{6\epsilon\|\gamma\|_\infty}(z(\alpha))$ and the lateral boundaries, and it is easy to check that $b_2 \le 0$ on $y_2 = -2\epsilon\|\gamma\|_\infty$. Since the set $\{b_2 \le 0\}$ is simply-connected, it implies that $b_2 \le 0$ in the interior of this region, finishing the proof. □

Note that (4.16) of Lemma 4.4 immediately implies Proposition 4.2. (The only difference is that $\Delta v^\epsilon = -1$ in Lemma 4.4 whereas $\Delta p^\epsilon = -2$ in Proposition 4.2, so the constant C in Proposition 4.2 is twice of that in (4.16)). The lemma also implies the following corollary, which will be helpful in the proof of Proposition 4.3.

Corollary 4.5. *For any $i = 1, \dots, n+m$, assume Γ_i and γ_i satisfy **(H1)–(H3)**. Assume $v^\epsilon \in C^2(D_i^\epsilon) \cap C(\overline{D_i^\epsilon})$ satisfies that*

$$\begin{cases} |\Delta v^\epsilon| \le C_0 & \text{in } D_i^\epsilon, \\ v^\epsilon = 0 & \text{on } \partial D_i^\epsilon, \end{cases}$$

for some constant $C_0 > 0$. Then for the same constants ϵ_0, C_1, C_2 as in Lemma 4.4, the following holds for all $\epsilon \in (0, \epsilon_0)$:

$$|v^\epsilon| \le C_0 C_1 \epsilon^2 \quad \text{in } D_i^\epsilon, \tag{4.35}$$

and if $v^\epsilon \in C^2(D_i^\epsilon) \cap C^1(\overline{D_i^\epsilon})$, we have

$$\|\nabla v^\epsilon\|_{L^\infty(\Gamma_i)} \le C_0 C_2 \epsilon \quad \text{for} \quad i = 1, \dots, n. \tag{4.36}$$

Proof. Let $\tilde{v}$ be a solution to

$$\begin{cases} \Delta \tilde{v} = -C_0 & \text{in } D_i^\epsilon, \\ \tilde{v} = 0 & \text{on } \partial D_i^\epsilon. \end{cases}$$

It is clear that $v^\epsilon + \tilde{v}$ is super-harmonic and $v^\epsilon - \tilde{v}$ is sub-harmonic in D_i^ϵ, and they both vanish on the boundary. Thus the maximum principle implies that

$$-\tilde{v} \le v^\epsilon \le \tilde{v} \quad \text{in } D_i^\epsilon. \tag{4.37}$$

Applying (4.16) of Lemma 4.4 to $\frac{\tilde{v}}{C_0}$, we obtain $0 \le \tilde{v} \le C_0 C_1 \epsilon^2$ in D_i^ϵ for all $\epsilon \in (0, \epsilon_0)$, leading to (4.35). Furthermore, (4.37) and the fact that v^ϵ and v both have zero boundary condition imply that

$$|\nabla v^\epsilon| \le |\nabla \tilde{v}| \quad \text{on } \partial D_i^\epsilon.$$

We then apply (4.17) of Lemma 4.4 to $\frac{\tilde{v}}{C_0}$ and obtain $\|\nabla v^\epsilon\|_{L^\infty(\Gamma_i)} \le C_0 C_2 \epsilon$, which proves (4.36). $\square$

Now we are ready to prove Proposition 4.3.

Proof of Proposition 4.3. Throughout the proof, let $i \in \{1, \dots, n\}$ be fixed. For notational simplicity, in the rest of the proof we omit the subscript i from all terms.

We claim that

$$\left| \nabla \tilde{p}^\epsilon(R^\epsilon(\alpha, \eta)) - \frac{c^\epsilon}{\epsilon \gamma(\alpha)} \mathbf{n}(z(\alpha)) \right| \le C\epsilon \quad \text{for all } \alpha \in S, \eta \in [0, -1], \tag{4.38}$$

$$\|\Delta q^\epsilon\|_{L^\infty(D^\epsilon)} \le C \tag{4.39}$$

for some constant $C > 0$ only depending on $\|z_i\|_{C^3(S_i)}$, $\|\gamma_i\|_{C^2(S_i)}$ and F_Γ. Assuming these are true, let us explain how they lead to (4.12)–(4.14). By (4.6) and (4.10), p^ϵ and $\tilde{p}^\epsilon$ have the same boundary condition, thus $q^\epsilon = 0$ on ∂D^ϵ. This and (4.39) allow us to apply Corollary 4.5 to q^ϵ to obtain the estimate (4.35), implying (4.12).

Due to (4.36) of Corollary 4.5, we also have

$$\|\nabla q^\epsilon\|_{L^\infty(\Gamma)} \le C\epsilon. \tag{4.40}$$

Using (4.7) and $p^\epsilon = \tilde{p}^\epsilon + q^\epsilon$, we have

$$\begin{aligned} -2|U| &= \int_{\partial U} \nabla \tilde{p}^\epsilon \cdot n d\sigma + \int_{\partial U} \nabla q^\epsilon \cdot n d\sigma \\ &= -\frac{c^\epsilon L}{\epsilon} \int_S \gamma^{-1}(\alpha) d\alpha + O(\epsilon), \end{aligned}$$

where the second equality follows from (4.38) for $\eta = 0$, $n(z(\alpha)) = -\mathbf{n}(z(\alpha))$ and $d\sigma = L d\alpha$, as well as (4.40). Rearranging the terms and using the definition of β in (4.11) yields (4.13).

Finally, note that (4.13) and (4.38) directly lead to (4.14), where we are using the fact that γ_i is uniformly positive for $i = 1, \dots, n$, due to **(H3)**.

The rest of the proof is devoted to proving the claims (4.38) and (4.39). For (4.38), we compute the gradient of $\tilde{p}^\epsilon$. Differentiating (4.10) with respect to α and η, we obtain

$$(\nabla_{\alpha,\eta} R^\epsilon(\alpha, \eta))^t \nabla \tilde{p}(R^\epsilon(\alpha, \eta)) = \begin{pmatrix} 0 \\ c^\epsilon \end{pmatrix}, \tag{4.41}$$

where $(\nabla_{\alpha,\eta} R^\epsilon)^t$ denotes the transpose of the Jacobian matrix of R^ϵ. Since $\nabla_{\alpha,\eta} R^\epsilon = (\partial_\alpha R^\epsilon, \partial_\eta R^\epsilon)$, using the formula for inverses of 2×2 matrices, we have

$$\left(\left(\nabla_{\alpha,\eta} R^\epsilon \right)^t \right)^{-1} = \frac{1}{J(\alpha, \eta)} \left(-(\partial_\eta R^\epsilon)^\perp, (\partial_\alpha R^\epsilon)^\perp \right). \tag{4.42}$$

where $J(\alpha, \eta) := \det(\nabla_{\alpha,\eta} R^\epsilon)$. Multiplying the inverse matrix on both sides of (4.41), we have

$$\nabla \tilde{p}^\epsilon(R^\epsilon(\alpha, \eta)) = \frac{1}{J} \left(-(\partial_\eta R^\epsilon)^\perp, (\partial_\alpha R^\epsilon)^\perp \right) \begin{pmatrix} 0 \\ c^\epsilon \end{pmatrix} = \frac{c^\epsilon}{J} (\partial_\alpha R^\epsilon)^\perp. \tag{4.43}$$

Recall that Lemma 3.2 gives $(\partial_\alpha R^\epsilon)^\perp = z'(\alpha)^\perp + O(\epsilon) = L\mathbf{n}(z(\alpha)) + O(\epsilon)$, and $J = \epsilon L\gamma + O(\epsilon^2)$. Plugging these into (4.43) gives

$$\nabla \tilde{p}^\epsilon(R(\alpha,\eta)) = \frac{c^\epsilon}{\epsilon}\left(\frac{\mathbf{n}(z(\alpha))}{\gamma} + O(\epsilon)\right). \tag{4.44}$$

Note that it follows from (4.8) that $c^\epsilon \le \frac{|D^\epsilon|}{2\pi}$, where $|D^\epsilon| \le C\epsilon$ due to (3.9). These imply

$$\frac{c_\epsilon}{\epsilon} \le C, \tag{4.45}$$

and applying it to (4.44) yields (4.38).

To prove (4.39), since $q^\epsilon = p^\epsilon - \tilde{p}^\epsilon$ and $\Delta p^\epsilon = -2$ in D^ϵ, it suffices to show that

$$\left|\Delta \tilde{p}^\epsilon\right| \le C \quad \text{in } D^\epsilon, \tag{4.46}$$

and we will begin with an explicit computation of $\partial_{x_1x_1}\tilde{p}^\epsilon$ and $\partial_{x_2x_2}\tilde{p}^\epsilon$. Let us denote $R^\epsilon =: (R^1, R^2)$. For notational simplicity, in the rest of the proof we will use subscripts on R^ϵ, R^1, R^2 and J to denote their partial derivative, e.g. $R^1_\alpha := \partial_\alpha R^1$.

From (4.43), it follows that

$$\partial_{x_1}\tilde{p}^\epsilon(R^\epsilon(\alpha,\eta)) = -\frac{c^\epsilon}{J}R^2_\alpha.$$

Differentiating in α and η, we get

$$\begin{aligned}\nabla\left(\partial_{x_1}\tilde{p}^\epsilon\right)(R^\epsilon(\alpha,\eta)) &= \left(\left(\nabla_{\alpha,\eta}R^\epsilon\right)^t\right)^{-1}\nabla_{\alpha,\eta}\left(-\frac{c^\epsilon}{J}R^2_\alpha\right)\\ &= \frac{c^\epsilon}{J}\begin{pmatrix} R^2_\eta & -R^2_\alpha \\ -R^1_\eta & R^1_\alpha\end{pmatrix}\begin{pmatrix}\frac{J_\alpha}{J^2}R^2_\alpha - \frac{1}{J}R^2_{\alpha\alpha}\\ \frac{J_\eta}{J^2}R^2_\alpha - \frac{1}{J}R^2_{\alpha\eta}\end{pmatrix},\end{aligned}$$

thus

$$\partial_{x_1x_1}\tilde{p}^\epsilon(R(\alpha,\eta)) = \frac{c^\epsilon}{J}\left(\frac{J_\alpha}{J^2}R^2_\eta R^2_\alpha - \frac{1}{J}R^2_\eta R^2_{\alpha\alpha} - \frac{J_\eta}{J^2}(R^2_\alpha)^2 + \frac{1}{J}R^2_\alpha R^2_{\alpha\eta}\right).$$

Likewise, $\partial_{x_2x_2}\tilde{p}(R(\alpha,\eta))$ takes the same expression except every R^2 is changed into R^1. Adding them together gives

$$\Delta\tilde{p}^\epsilon(R(\alpha,\eta)) = \frac{c^\epsilon}{J}\left(\frac{J_\alpha}{J^2}R^\epsilon_\eta\cdot R^\epsilon_\alpha - \frac{1}{J}R^\epsilon_\eta\cdot R^\epsilon_{\alpha\alpha} - \frac{J_\eta}{J^2}R^\epsilon_\alpha\cdot R^\epsilon_\alpha + \frac{1}{J}R^\epsilon_\alpha\cdot R^\epsilon_{\alpha\eta}\right). \tag{4.47}$$

Using the explicit formulae of R_α, R_η and J in Lemma 3.2, we directly obtain $|R^\epsilon_\alpha|, |R^\epsilon_{\alpha\alpha}| \le C$; $|R^\epsilon_\eta|, |R^\epsilon_{\alpha\eta}|, |J_\alpha| \le C\epsilon$; $|J_\eta| \le C\epsilon^2$; and $J^{-1} \le C\epsilon^{-1}$ when ϵ is sufficiently small, where C depends on $\|z_i\|_{C^3(S_i)}$ and $\|\gamma_i\|_{C^2(S_i)}$. As a result, all the four terms in the parenthesis of (4.47) are bounded by some constant C independent of ϵ. Finally, (4.45) yields $\frac{c_\epsilon}{J} \le C$ as well, thus $|\Delta\tilde{p}^\epsilon| \le C$, and this proves the second claim (4.39).

□

5. Proof of the Symmetry Result

In this section we prove that a stationary vortex sheet with positive vorticity must be radially symmetric up to a translation, and a rotating vortex sheet with positive vorticity and angular velocity $\Omega < 0$ must be radially symmetric. The key idea of the proof is to define the integral

$$
\begin{aligned}
I^\epsilon &:= \int_{D^\epsilon} \epsilon^{-1} \mathbf{u}^\epsilon \cdot \nabla \left(\omega^\epsilon * \mathcal{N} - \frac{\Omega}{2}|x|^2 \right) dx \\
&= \int_{D^\epsilon} \epsilon^{-1}(x + \nabla p^\epsilon) \cdot \nabla \left(\omega^\epsilon * \mathcal{N} - \frac{\Omega}{2}|x|^2 \right) dx,
\end{aligned} \tag{5.1}
$$

and compute it in two different ways. The motivation of the definition is as follows. As discussed in [14, Sect. 2.1], I^ϵ can be thought of as a first variation of an "energy functional"

$$
\mathcal{E}[\omega^\epsilon] := \int \frac{1}{2}\omega^\epsilon(\omega^\epsilon * \mathcal{N}) - \frac{\Omega}{2}\omega^\epsilon |x|^2 \, dx
$$

when we perturb ω^ϵ by a divergence free vector $\mathbf{u}^\epsilon$ in D^ϵ. (This functional $\mathcal{E}$ only serves as our motivation, and will not appear in the proof.) On the one hand, using that ω_0 is stationary in the rotating frame with angular velocity Ω and ω^ϵ is a close approximation of ω_0, we will show in Proposition 5.1 that I^ϵ is of order $O(\epsilon|\log\epsilon|)$, thus goes to zero as $\epsilon \to 0$. On the other hand, using the particular $\mathbf{u}^\epsilon$ that we constructed in Sect. 4, we will prove in Proposition 5.2 that if $\Omega = 0$, I^ϵ is strictly positive independently of ϵ unless all the vortex sheets are nested circles with constant density; and also prove a similar result in Corollary 5.3 for $\Omega < 0$.

Proposition 5.1. *Assume $\omega(\cdot, t) = \omega_0(R_{\Omega t}\cdot)$ is a stationary/uniformly-rotating vortex sheet with angular velocity $\Omega \in \mathbb{R}$, where ω_0 satisfies **(H1)–(H3)**. Then there exists some $C > 0$ only depending on b (as in **(H2)**), $\max_i \|z_i\|_{C^3(S_i)}$, $\max_{i\le n} \|\gamma_i\|_{C^2(S_i)}$, $\max_{i>n} \|\gamma_i\|_{C^b(S_i)}$, d_Γ and F_Γ, such that $|I^\epsilon| < C\epsilon^b|\log\epsilon|$ for all sufficiently small $\epsilon > 0$.*

Proof. Let us decompose $I^\epsilon =: \sum_{i=1}^{n+m} I_i^\epsilon$, where $I_i^\epsilon := \int_{D_i^\epsilon} \epsilon^{-1}(x+\nabla p^\epsilon)\cdot\nabla(\omega^\epsilon * \mathcal{N} - \frac{\Omega}{2}|x|^2)dx$.

We start with showing that $|I_i^\epsilon| \le C\epsilon^b|\log\epsilon|$ for $i = n+1, \ldots, n+m$. For such i, $p^\epsilon = 0$ on ∂D_i^ϵ, thus the divergence theorem (and the fact that $\omega^\epsilon = \epsilon^{-1}$ in D_i^ϵ) gives

$$
I_i^\epsilon = \underbrace{\int_{D_i^\epsilon} \epsilon^{-1} x \cdot \nabla \left(\omega^\epsilon * \mathcal{N} - \frac{\Omega}{2}|x|^2 \right) dx}_{=:T_i^\epsilon} - \int_{D_i^\epsilon} \epsilon^{-1}(\epsilon^{-1} - 2\Omega) p^\epsilon(x) dx.
$$

Using the estimate $|p^\epsilon| \le C\epsilon^2$ in Proposition 4.2 and the fact that $|D_i^\epsilon| \le C\epsilon$ from (3.9), we easily bound the second integral by $C\epsilon$. To control the first integral T_i^ϵ, we rewrite it using the change of variables $x = R_i^\epsilon(\alpha, \eta)$ and the definition $\mathbf{v}^\epsilon := \nabla^\perp(\omega^\epsilon * \mathcal{N})$ in (3.10): (also note that on the right hand side we group ϵ^{-1} with the determinant)

$$
T_i^\epsilon = \int_{S_i}\int_{-1}^{0} R_i^\epsilon(\alpha,\eta) \cdot \underbrace{\left(-(\mathbf{v}^\epsilon)^\perp(R_i^\epsilon(\alpha,\eta)) - \Omega R_i^\epsilon(\alpha,\eta) \right)}_{=:J_i^\epsilon} \underbrace{\epsilon^{-1}\det(\nabla_{\alpha,\eta} R_i^\epsilon(\alpha,\eta))}_{=:K_i^\epsilon} \, d\eta d\alpha,
$$

and recall that the exact expression of the determinant was given in (3.8). Let us take a closer look at the integrand, which is a product of 3 terms. Clearly, the definition of R_i^ϵ gives $R_i^\epsilon(\alpha, \eta) = z_i(\alpha) + O(\epsilon)$. As for the middle term J_i^ϵ, Lemma 3.4 yields

$$J_i^\epsilon(\alpha, \eta) = -BR^\perp(z_i(\alpha)) + \left(\eta + \frac{1}{2}\right)[\mathbf{v}]^\perp(z_i(\alpha)) - \Omega z_i(\alpha) + O(\epsilon^b|\log\epsilon|). \quad (5.2)$$

Using the fact that $BR(z_i(\alpha)) = \Omega z_i^\perp(\alpha)$ for $i = n+1, \dots, n+m$ (which follows from (2.6) and (2.7)), it becomes

$$J_i^\epsilon(\alpha, \eta) = \left(\eta + \frac{1}{2}\right)[\mathbf{v}]^\perp(z_i(\alpha)) + O(\epsilon^b|\log\epsilon|). \quad (5.3)$$

Also it follows from (3.8) that $K_i^\epsilon(\alpha, \eta) = L_i\gamma_i(\alpha) + O(\epsilon)$. Plugging these three estimates into the above integral gives

$$T_i^\epsilon = \int_{S_i}\int_{-1}^0 z_i(\alpha)\cdot\left(\eta + \frac{1}{2}\right)[\mathbf{v}]^\perp(z_i(\alpha))L_i\gamma_i(\alpha)d\eta d\alpha + O(\epsilon^b|\log\epsilon|) = O(\epsilon^b|\log\epsilon|),$$

where the last step follows from the fact that $\int_{-1}^0(\eta + \frac{1}{2})d\eta = 0$. This finishes the proof that $|I_i^\epsilon| \le C\epsilon^b|\log\epsilon|$ for $i = n+1, \dots, n+m$, where C depends on b, $\max_i \|z_i\|_{C^2(S_i)}$, $\max_i \|\gamma_i\|_{C^b(S_i)}$, d_Γ and F_Γ.

In the rest of the proof we aim to show $|I_i^\epsilon| \le C\epsilon^b|\log\epsilon|$ for $i = 1, \dots, n$, which is slightly more involved. Recall that in Proposition 4.3 we defined $\tilde{p}^\epsilon$ and q^ϵ in D_i^ϵ for $i = 1, \dots, n$, where they satisfy $p^\epsilon = \tilde{p}^\epsilon + q^\epsilon$ in D_i^ϵ, and $q^\epsilon = 0$ on ∂D_i^ϵ. This allows us to apply the divergence theorem (to the q^ϵ term only) and decompose I_i^ϵ as

$$\begin{aligned} I_i^\epsilon &= \int_{D_i^\epsilon} \epsilon^{-1}(x + \nabla\tilde{p}_\epsilon)\cdot\nabla\left(\omega^\epsilon * \mathcal{N} - \frac{\Omega}{2}|x|^2\right)dx \\ &\quad - \int_{D_i^\epsilon}\epsilon^{-1}(\epsilon^{-1} - 2\Omega)q^\epsilon(x)dx =: I_{i,1}^\epsilon + I_{i,2}^\epsilon. \end{aligned}$$

We can easily show that $I_{i,2}^\epsilon = O(\epsilon)$: (4.12) of Proposition 4.3 gives $|q^\epsilon| \le C\epsilon^2$, and combining it with $|D_i^\epsilon| \le C\epsilon$ in (3.9) immediately yields the desired estimate.

Next we turn to $I_{i,1}^\epsilon$. Again, the change of variables $x = R_i^\epsilon(\alpha, \eta)$ and the definition $\mathbf{v}^\epsilon := \nabla^\perp(\omega^\epsilon * \mathcal{N})$ gives

$$\begin{aligned} I_{i,1}^\epsilon = \int_{S_i}\int_{-1}^0 &\left(R_i^\epsilon(\alpha,\eta) + \nabla\tilde{p}^\epsilon(R_i^\epsilon(\alpha,\eta))\right)\cdot\underbrace{\left(-(\mathbf{v}^\epsilon)^\perp(R_i^\epsilon(\alpha,\eta)) - \Omega R_i^\epsilon(\alpha,\eta)\right)}_{=:J_i^\epsilon} \\ &\underbrace{\epsilon^{-1}\det(\nabla_{\alpha,\eta}R_i^\epsilon(\alpha,\eta))}_{=:K_i^\epsilon}\,d\eta d\alpha. \end{aligned}$$

For the three terms in the product of the integrand, we will approximate the first term using the definition of R_i^ϵ and (4.14) of Proposition 4.3:

$$R_i^\epsilon(\alpha,\eta) + \nabla\tilde{p}^\epsilon(R_i^\epsilon(\alpha,\eta)) = z_i(\alpha) + \frac{\beta_i}{\gamma_i(\alpha)}\mathbf{n}(z_i(\alpha)) + O(\epsilon),$$

where $\beta_i := \frac{2|U_i|}{L_i \int_{S_i} \gamma_i^{-1}(\alpha)d\alpha}$ is given by (4.11). Lemma 3.4 allows us to approximate the middle term J_i^ϵ as (5.2), however (5.3) no longer holds since for $i = 1, \dots, n$ we do not have $BR(z_i(\alpha)) = \Omega z_i^\perp(\alpha)$. As for K_i^ϵ, we again use (3.8) to approximate it by $K_i^\epsilon(\alpha, \eta) = L_i\gamma_i(\alpha) + O(\epsilon)$. Plugging these three estimates into the integrand of $I_{i,1}^\epsilon$ gives

$$I_{i,1}^\epsilon = \int_{S_i} \left(z_i(\alpha) + \frac{\beta_i}{\gamma_i(\alpha)}\mathbf{n}(z_i(\alpha)) \right) \cdot \Big(-BR^\perp(z_i(\alpha)) - \Omega z_i(\alpha) \Big) L_i\gamma_i(\alpha)d\alpha$$
$$+O(\epsilon^b|\log\epsilon|),$$

where we again use the fact that the $(\eta + \frac{1}{2})$ term gives zero contribution since $\int_{-1}^0(\eta + \frac{1}{2})d\eta = 0$. Next we will show the integral on the right hand side is in fact 0. Since ω is a rotating solution with angular velocity Ω, the conditions (2.6) and (2.7) yield that

$$-BR^\perp(z_i(\alpha)) - \Omega z_i(\alpha) = C_i\gamma_i^{-1}(\alpha)\mathbf{n}(z_i(\alpha)),$$

for some constant C_i. Plugging this into the above integral gives

$$\begin{aligned} I_{i,1}^\epsilon &= C_iL_i \int_{S_i} \left(z_i(\alpha)\cdot\mathbf{n}(z_i(\alpha)) + \frac{\beta_i}{\gamma_i(\alpha)} \right) d\alpha + O(\epsilon^b|\log\epsilon|) \\ &= C_iL_i \left(\int_{S_i} z_i(\alpha)\cdot\mathbf{n}(z_i(\alpha))d\alpha + \frac{2|U_i|}{L_i} \right) + O(\epsilon^b|\log\epsilon|), \end{aligned}$$

where the second step follows from the definition of β_i in (4.11). Let us compute the integral on the right hand side by changing to arclength parametrization and applying the divergence theorem:

$$\int_{S_i} z_i(\alpha)\cdot\mathbf{n}(z_i(\alpha))d\alpha = -\frac{1}{L_i}\int_{\partial U_i} x\cdot n d\sigma = -\frac{2|U_i|}{L_i},$$

which yields $I_{i,1}^\epsilon = O(\epsilon^b|\log\epsilon|)$, and finishes the proof that $|I_i^\epsilon| \le C\epsilon^b|\log\epsilon|$ for $i = 1, \dots, n$, where C depends on b, $\|z_i\|_{C^3(S_i)}$, $\|\gamma_i\|_{C^2(S_i)}$, d_Γ and F_Γ.

Finally, summing the I_i^ϵ estimates for $i = 1, \dots, n+m$ gives $|I^\epsilon| \le C\epsilon^b|\log\epsilon|$ for all sufficiently small $\epsilon > 0$, thus we can conclude. □

Now we will use a different way to compute I^ϵ. Let us first define a new integral $\tilde{I}^\epsilon$ that is the same as I^ϵ except with Ω set to zero:

$$\tilde{I}^\epsilon := \int_{D^\epsilon} \epsilon^{-1}(x + \nabla p^\epsilon)\cdot\nabla\left(\omega^\epsilon * \mathcal{N}\right)dx. \tag{5.4}$$

Next we will prove that $\tilde{I}^\epsilon$ is strictly positive independently of ϵ unless all the vortex sheets are nested circles with constant density. As we will see in the proof, the key step is to show that if some Γ_i is either not a circle or does not have a constant γ_i, then the estimates on p^ϵ in Propositions 4.2–4.3 lead to the following quantitative version of (4.9): $\epsilon^{-2}\left(\frac{|D_i^\epsilon|^2}{4\pi} - \int_{D_i^\epsilon} p^\epsilon(x)dx\right) \ge c_0 > 0$, where c_0 is independent of ϵ.

Proposition 5.2. *Let $\tilde{I}^\epsilon$ be defined as in (5.4). Assume that Γ_i and γ_i satisfy **(H1)–(H3)** for $i = 1, \ldots, n+m$. Then we have $\tilde{I}^\epsilon \geq 0$ for all sufficiently small $\epsilon > 0$.*

*In addition, if Γ is **not** a union of nested circles with constant γ_i's on each connected component, there exists some $c_0 > 0$ independent of ϵ, such that $\tilde{I}^\epsilon > c_0 > 0$ for all sufficiently small $\epsilon > 0$.*

Proof. We start by decomposing $\tilde{I}^\epsilon$ as

$$\tilde{I}^\epsilon = \int_{D^\epsilon} \epsilon^{-1} x \cdot \nabla(\omega^\epsilon * \mathcal{N}) dx + \int_{D^\epsilon} \epsilon^{-1} \nabla p^\epsilon \cdot \nabla(\omega^\epsilon * \mathcal{N}) dx =: I_1^\epsilon + I_2^\epsilon.$$

I_1^ϵ can be easily computed as

$$I_1^\epsilon = \frac{1}{2\pi\epsilon^2} \int_{D^\epsilon} \int_{D^\epsilon} \frac{x \cdot (x-y)}{|x-y|^2} dx dy = \frac{|D^\epsilon|^2}{4\pi\epsilon^2} = \frac{1}{4\pi\epsilon^2} \left(\sum_{i=1}^{n+m} |D_i^\epsilon| \right)^2 \tag{5.5}$$

where the second equality is obtained by exchanging x with y and taking the average with the original integral. As for I_2^ϵ, we have

$$\begin{aligned} I_2^\epsilon &= \frac{1}{\epsilon} \int_{\partial D^\epsilon} p^\epsilon \nabla(\omega^\epsilon * \mathcal{N}) \cdot n d\sigma - \frac{1}{\epsilon} \int_{D^\epsilon} p^\epsilon \omega^\epsilon dx \\ &= -\frac{1}{\epsilon} \sum_{i=1}^{n} c_i^\epsilon \int_{\partial U_i} \nabla(\omega^\epsilon * \mathcal{N}) \cdot n d\sigma - \frac{1}{\epsilon^2} \int_{D^\epsilon} p^\epsilon dx \\ &\geq -\frac{1}{\epsilon^2} \sum_{i=1}^{n} \sum_{j=1}^{n+m} \frac{|D_i^\epsilon|}{2\pi} \int_{U_i} 1_{D_j^\epsilon} dx - \frac{1}{\epsilon^2} \sum_{i=1}^{n+m} \int_{D_i^\epsilon} p^\epsilon dx, \end{aligned} \tag{5.6}$$

where the first equality follows from the divergence theorem, the second equality follows from the boundary conditions (4.5) and (4.6) for p^ϵ (as well as the fact that ∂U_i and ∂D_i^ϵ have opposite outer normals), and the last inequality follows from the divergence theorem as well as the inequality $c_i^\epsilon \leq \sup_{D_i^\epsilon} p \leq \frac{|D_i^\epsilon|}{2\pi}$ due to (4.8).

Let us denote $j \prec i$ if $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n+m\}$, $j \neq i$ and Γ_j lies in the interior of the domain enclosed by Γ_i (that is, $\Gamma_j \subset U_i$). If not, we denote $j \not\prec i$. Note that for sufficiently small $\epsilon > 0$, we have

$$\int_{U_i} 1_{D_j^\epsilon} dx = \begin{cases} |D_j^\epsilon| & \text{if } j \prec i, \\ 0 & \text{otherwise.} \end{cases} \tag{5.7}$$

Applying this to (5.6) yields

$$\begin{aligned} I_2^\epsilon &\geq -\frac{1}{2\pi\epsilon^2} \sum_{i,j=1}^{n+m} \mathbb{1}_{j \prec i} |D_i^\epsilon||D_j^\epsilon| - \frac{1}{\epsilon^2} \sum_{i=1}^{n+m} \int_{D_i^\epsilon} p_i^\epsilon dx \\ &= -\frac{1}{4\pi\epsilon^2} \sum_{i,j=1}^{n+m} (\mathbb{1}_{j \prec i} + \mathbb{1}_{i \prec j}) |D_i^\epsilon||D_j^\epsilon| - \frac{1}{\epsilon^2} \sum_{i=1}^{n+m} \int_{D_i^\epsilon} p_i^\epsilon dx \end{aligned} \tag{5.8}$$

where in the first step we used that the $i = n+1, \ldots, n+m$ terms have zero contribution in the first sum, due to the definition of $j \prec i$.

Adding (5.5) and (5.8) together, we obtain

$$\tilde{I}^\epsilon \geq \sum_{i=1}^{n+m} \underbrace{\frac{1}{\epsilon^2}\left(\frac{|D_i^\epsilon|^2}{4\pi} - \int_{D_i^\epsilon} p_i^\epsilon dx\right)}_{=:A_i^\epsilon} + \sum_{i,j=1}^{n+m} \underbrace{\frac{1}{\epsilon^2}\left(\mathbb{1}_{i\neq j} - \left(\mathbb{1}_{j\prec i} + \mathbb{1}_{i\prec j}\right)\right)\frac{|D_i^\epsilon||D_j^\epsilon|}{4\pi}}_{=:B_{i,j}^\epsilon}, \tag{5.9}$$

From (4.9), it follows that $A_i^\epsilon \geq 0$ for all $i = 1, \ldots, n+m$, with equality achieved if and only if each D_i^ϵ is a disk or an annulus. Note that $B_{i,j}^\epsilon \geq 0$ as well for all i and j, since for any $i \neq j$, at most one of $i \prec j$ and $j \prec i$ can hold. Putting these together yields that $\tilde{I}^\epsilon \geq 0$ for any sufficiently small $\epsilon > 0$.

In the rest of the proof, we assume Γ is **not** a union of nested circles with constant γ_i's on each connected component. Therefore at least one of the following 3 cases must be true. In each case we aim to show that $\tilde{I}_\epsilon \geq c_0 > 0$, where c_0 is independent of ϵ for all sufficiently small $\epsilon > 0$.

Case 1. There exists some open curve Γ_i that is not a loop. In this case D_i^ϵ is simply-connected, and $p^\epsilon = 0$ on ∂D_i^ϵ by (4.5). Applying Proposition 4.2 to p^ϵ in D_i^ϵ, we have $\sup_{D_i^\epsilon} p^\epsilon \leq C\epsilon^2$, where C is independent of ϵ. This leads to $\int_{D_i^\epsilon} p^\epsilon dx \leq C\epsilon^3$, since $|D_i^\epsilon| = O(\epsilon)$ by (3.9). As a result, for the index i we have

$$A_i^\epsilon = \frac{|D_i^\epsilon|^2}{4\pi\epsilon^2} - \epsilon^{-2}\int_{D_i^\epsilon} p_i^\epsilon dx \geq \frac{L_i^2}{4\pi}\left(\int_{S_i}\gamma_i(\alpha)d\alpha\right)^2 - C\epsilon,$$

where we again used (3.9) in the second inequality. This gives that $A_i^\epsilon \geq \frac{L_i^2}{8\pi}(\int_{S_i}\gamma_i(\alpha)d\alpha)^2 > 0$ for all sufficiently small $\epsilon > 0$.

Case 2. There exists some closed curve Γ_i that is either not a circle, or γ_i is not a constant. In this case we aim to show that $A_i^\epsilon \geq c_0 > 0$, and this will be done by finding good approximations (independent of ϵ) for both terms in A_i^ϵ. For the first term $\frac{|D_i^\epsilon|^2}{4\pi\epsilon^2}$, using (3.9) we again have

$$\frac{|D_i^\epsilon|^2}{4\pi\epsilon^2} \geq \frac{L_i^2}{4\pi}\left(\int_{S_i}\gamma_i(\alpha)d\alpha\right)^2 - C\epsilon =: J_i - C\epsilon, \tag{5.10}$$

where $J_i > 0$ is independent of ϵ. For the second term $\epsilon^{-2}\int_{D_i^\epsilon} p_i^\epsilon dx$, rewriting the integral using the change of variables $x = R_i^\epsilon(a, \eta)$ gives

$$\epsilon^{-2}\int_{D_i^\epsilon} p_i^\epsilon dx = \int_{S_i}\int_{-1}^{0} \frac{p^\epsilon(R_i^\epsilon(\alpha,\eta))}{\epsilon}\frac{\det(\nabla_{\alpha,\eta}R_i^\epsilon)}{\epsilon} d\eta d\alpha.$$

Recall that in Proposition 4.3 we defined $\tilde{p}^\epsilon(R_i^\epsilon(\alpha,\eta)) := c_i^\epsilon(1+\eta)$ and q_ϵ such that $p^\epsilon - \tilde{p}_\epsilon = q_\epsilon$. By (4.12) and (4.13), for all $\alpha \in S_i$ and $\eta \in (-1, 0)$ we have

$$\left|\frac{p^\epsilon(R_i^\epsilon(\alpha,\eta))}{\epsilon} - \beta_i(1+\eta)\right| \leq \left|\frac{p^\epsilon(R_i^\epsilon(\alpha,\eta))}{\epsilon} - \frac{c_i^\epsilon}{\epsilon}(1+\eta)\right| + \left|\frac{c_i^\epsilon}{\epsilon} - \beta_i\right| \leq C\epsilon,$$

where $\beta_i := \frac{2|U_i|}{L_i \int_{S_i} \gamma_i^{-1}(\alpha)d\alpha}$ is defined in (4.11). Combining this with the expression of the determinant in (3.8), we have

$$\epsilon^{-2} \int_{D_i^\epsilon} p_i^\epsilon dx = \int_{S_i} \int_{-1}^{0} (\beta_i(1+\eta) + O(\epsilon))(L_i \gamma_i(\alpha) + O(\epsilon)) d\eta d\alpha$$
$$\leq \frac{|U_i|}{\int_{S_i} \gamma_i^{-1}(\alpha)d\alpha} \int_{S_i} \gamma_i(\alpha)d\alpha + C\epsilon =: K_i + C\epsilon,$$

where K_i is independent of ϵ. Putting this together with (5.10) yields the following:

$$\begin{aligned} A_i^\epsilon &\geq J_i - K_i - C\epsilon \\ &= \frac{L_i^2}{4\pi} \frac{\int_{S_i} \gamma_i(\alpha)d\alpha}{\int_{S_i} \gamma_i^{-1}(\alpha)d\alpha} \left(\int_{S_i} \gamma_i^{-1}(\alpha)d\alpha \int_{S_i} \gamma_i(\alpha)d\alpha - \frac{4\pi|U_i|}{L_i^2} \right) - C\epsilon. \end{aligned} \tag{5.11}$$

Let us take a closer look at the two terms inside the parenthesis. For the first term, Cauchy-Schwarz inequality gives

$$\int_{S_i} \gamma_i^{-1}(\alpha)d\alpha \int_{S_i} \gamma_i(\alpha)d\alpha \geq 1,$$

with equality achieved if and only if γ_i is a constant. For the second term, the isoperimetric inequality yields

$$\frac{4\pi|U_i|}{L_i^2} \leq 1,$$

(recall that $L_i = |\partial U_i|$), with equality achieved if and only U_i is a disk. By the assumption of Case 2, at least one of the inequalities must be strict, thus the parenthesis on the right hand side of (5.11) is strictly positive (and independent of ϵ). Therefore there exists some constant $c_0 > 0$ such that $\tilde{I}^\epsilon \geq A_i^\epsilon \geq c_0$ for all sufficiently small ϵ.

Case 3. There exist $i \neq j$ such that $i \not\prec j$ and $j \not\prec i$. Then it is clear that for such i, j, $B_{i,j}^\epsilon$ in (5.9) is given by $B_{i,j}^\epsilon = \frac{|D_i^\epsilon||D_j^\epsilon|}{4\pi\epsilon^2}$. Hence (3.9) gives

$$B_{i,j}^\epsilon \geq L_i L_j \left(\int_{S_i} \gamma_i(\alpha)d\alpha \right) \left(\int_{S_j} \gamma_j(\alpha)d\alpha \right) - C\epsilon,$$

which yields $\tilde{I}^\epsilon \geq \frac{1}{2} L_i L_j (\int_{S_i} \gamma_i d\alpha)(\int_{S_j} \gamma_j d\alpha) > 0$ for all sufficiently small $\epsilon > 0$.

This finishes our discussion on all 3 cases. To conclude, since Γ is not a union of nested circles with constant γ_i's on each connected component, at least one of the 3 cases must hold, and all of them lead to $\tilde{I}^\epsilon \geq c_0 > 0$. □

The above proposition immediately leads to the following corollary for the $\Omega < 0$ case.

Corollary 5.3. *Assume that Γ_i and γ_i satisfy **(H1)–(H3)** for $i = 1, \ldots, n+m$. Let I^ϵ be defined as in (5.1), and assume $\Omega < 0$. Then we have $I^\epsilon \geq 0$ for all sufficiently small $\epsilon > 0$. In addition, if Γ is **not** a union of concentric circles all centered at the origin with constant γ_i's, there exists some $c_0 > 0$ independent of ϵ, such that $I^\epsilon > c_0 > 0$ for all sufficiently small $\epsilon > 0$.*

Proof. Let us decompose I^ϵ as follows (recall the definition of $\tilde{I}^\epsilon$ in (5.4))

$$I^\epsilon = \tilde{I}^\epsilon + (-\Omega)\left(\epsilon^{-1}\int_{D^\epsilon}(|x|^2 + \nabla p^\epsilon \cdot x)dx\right) =: \tilde{I}^\epsilon + \underbrace{(-\Omega)}_{>0} J^\epsilon. \tag{5.12}$$

Recall that Proposition 5.2 gives $\tilde{I}_\epsilon \geq c_0 > 0$ as long as Γ is not a union of nested circles with constant γ_i's. By [14, Lemma 2.11], we have

$$\int_{D_i^\epsilon}(|x|^2 + \nabla p^\epsilon \cdot x)dx \geq 0 \quad \text{for any } i = 1, \ldots, n+m,$$

thus $J^\epsilon \geq 0$. Putting them together, and using the fact that $\Omega < 0$, we know $I^\epsilon \geq c_0 > 0$ if Γ is not a union of nested circles with constant γ_i's.

To finish the proof, we only need to focus on the case that the Γ_i's are nested circles with constant γ_i's, but not all of them are centered at the origin. Assume that there exists $k \in \{1, \ldots, n\}$ such that Γ_k is a circle with radius r_k centered at $x_k \neq 0$. Since γ_k is a constant, D_k^ϵ is an annulus given by $B(x_k, r_k + \epsilon\gamma_k) \setminus B(x_k, r_k)$. The symmetry of D_k^ϵ about x_k immediately leads to $p^\epsilon|_{D_k^\epsilon} = -\frac{1}{2}|x - x_k|^2 + \frac{1}{2}(r_k + \epsilon\gamma_k)^2$. An elementary computation gives

$$\begin{aligned}\epsilon^{-1}\int_{D_k^\epsilon}(|x|^2 + \nabla p^\epsilon \cdot x)dx &= \epsilon^{-1}\int_{D_k^\epsilon}|x|^2 - (x - x_k)\cdot x dx = \epsilon^{-1}|x_k|^2|D_k^\epsilon| \\ &\geq 2\pi r_k\gamma_k|x_k|^2 > 0,\end{aligned}$$

where in the second-to-last step we used that $|D_k^\epsilon| = 2\pi\epsilon r_k\gamma_k + \pi\epsilon^2\gamma_k^2$. Setting $c_0 := 2\pi r_k\gamma_k|x_k|^2$ gives $I^\epsilon \geq c_0 > 0$, thus we can conclude. □

Now we are ready to prove Theorem 1.1. Note that for $\Omega < 0$, the symmetry result immediately follows from Proposition 5.1 and Corollary 5.3. For $\Omega = 0$, Proposition 5.1–5.2 already imply that a stationary vortex sheet with positive strength must be a union of nested circles with constant strength on each of them. To finish the proof, we only need to show that these nested circles must be concentric.

Proof of Theorem 1.1. For a uniformly-rotating vortex sheet with $\Omega < 0$, the symmetry result for $\Omega < 0$ is a direct consequence of Proposition 5.1 and Corollary 5.3. Next we focus on the stationary (i.e. $\Omega = 0$) case.

Combining Propsitions 5.1–5.2, we obtain that Γ is a union of nested circles, and γ_i is constant on Γ_i for all $i = 1 \ldots, n$. It remains to show that all Γ_i's are concentric. Let us denote by $\mathbf{v}_i$ the contribution to the velocity field by Γ_i. Since Γ_i is a circle with constant strength γ_i, a quick application of the divergence theorem yields that $\mathbf{v}_i \equiv 0$ in the open disk enclosed by Γ_i, whereas $\mathbf{v}_i(x) = \dfrac{\gamma_i L_i (x - x_i^0)^\perp}{2\pi|x - x_i^0|^2}$ in the open set outside Γ_i, where x_i^0 is the center of the circle Γ_i.

Without loss of generality, let us reorder the indices such that Γ_i is nested inside Γ_j for $i < j$. Towards a contradiction, let $k > 1$ be such that Γ_k is the first circle that is not concentric with Γ_1. From the above discussion, we know that $\mathbf{v}_i = 0$ on Γ_k for $i = k+1, \ldots, n$ (since Γ_k is nested inside Γ_i), whereas for $i = 1, \ldots, k-1$ we have $\mathbf{v}_i = \dfrac{\gamma_i L_i (x - x_1^0)^\perp}{2\pi|x - x_1^0|^2}$ on Γ_k, since all these Γ_i's have the same center x_1^0 and are nested inside Γ_k. Summing them up (and also using the fact that Γ_k contributes zero normal velocity on itself, since it is a circle with constant strength), we have

$$BR(x) \cdot \mathbf{n} = \sum_{i=1}^{n} \mathbf{v}_i(x) \cdot \mathbf{n} = \left(\sum_{i=1}^{k-1} \gamma_i L_i \right) \frac{(x - x_1^0)^\perp \cdot \mathbf{n}}{2\pi |x - x_1^0|^2} \quad \text{on } \Gamma_k,$$

where the right hand side is not a zero function since Γ_k has a different center from x_1^0. This causes a contradiction with the fact that $\omega = \omega_0$ is stationary. As a result, all $\Gamma_1, \ldots, \Gamma_n$ must be concentric circles, finishing the proof. $\square$

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