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A consistency result on long cardinal sequences ☆

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ABSTRACT

For any regular cardinal κ and ordinal $\eta < \kappa^{++}$ it is consistent that 2^κ is as large as you wish, and every function $f : \eta \longrightarrow [\kappa, 2^\kappa] \cap Card$ with $f(\alpha) = \kappa$ for $cf(\alpha) < \kappa$ is the cardinal sequence of some locally compact scattered space.

1. Introduction

If X is a locally compact, scattered Hausdorff (in short: LCS) space and α is an ordinal, we let $\mathrm{I}_\alpha(X)$ denote the αth Cantor-Bendixson level of X. The cardinal sequence of X, $CS(X)$, is the sequence of the cardinalities of the infinite Cantor-Bendixson levels of X, i.e.

$$\mathrm{CS}(X) = \langle |I_\alpha(X)| : \alpha < \mathrm{ht}^-(X)\rangle,$$

where $\mathrm{ht}^-(X)$, the *reduced height* of X, is the minimal ordinal β such that $\mathrm{I}_\beta(X)$ is finite. The *height* of X, denoted by $\mathrm{ht}(X)$, is defined as the minimal ordinal β such that $\mathrm{I}_\beta(X) = \emptyset$. Clearly $\mathrm{ht}^-(X) \leq \mathrm{ht}(X) \leq \mathrm{ht}^-(X) + 1$.

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If α is an ordinal, let $\mathcal{C}(\alpha)$ denote the class of all cardinal sequences of LCS spaces of reduced height α and put

$$\mathcal{C}_\lambda(\alpha) = \{s \in \mathcal{C}(\alpha) : s(0) = \lambda \wedge \forall \beta < \alpha \; s(\beta) \geq \lambda\}.$$

Let $\langle\kappa\rangle_\alpha$ denote the constant κ-valued sequence of length α.

In [4] it was shown that the class $\mathcal{C}(\alpha)$ is described if the classes $\mathcal{C}_\kappa(\beta)$ are characterized for every infinite cardinal κ and ordinal $\beta \leq \alpha$. Then, under GCH, a full description of the classes $\mathcal{C}_\kappa(\alpha)$ for infinite cardinals κ and ordinals $\alpha < \omega_2$ was given.

The situation becomes, however, more complicated for $\alpha \geq \omega_2$. In [9] we gave a consistent full characterization of $\mathcal{C}_\kappa(\alpha)$ for any uncountable regular cardinals κ and ordinals $\alpha < \kappa^{++}$ under GCH.

If GCH fails, much less is known on $\mathcal{C}_\kappa(\alpha)$ even for $\alpha < \kappa^{++}$.

In [11] it was proved that $\langle\omega\rangle_{\omega_1} {}^\frown \langle\omega_2\rangle \in \mathcal{C}_\omega(\omega_1 + 1)$ is consistent.

In [5] a similar result was proved for uncountable cardinals instead of ω: if κ is a regular cardinal with $\kappa^{<\kappa} = \kappa > \omega$ and $2^\kappa = \kappa^+$, then in some cardinality preserving generic extension of the ground model we have

$$\langle\kappa\rangle_{\kappa^+} {}^\frown \langle\kappa^{++}\rangle \in \mathcal{C}(\kappa^+ + 1).$$

In [10] we proved that if κ and λ are regular cardinals with $\kappa \leq \lambda$, $\kappa^{<\kappa} = \kappa$, $2^\kappa = \kappa^+$, and $\delta < \kappa^{++}$ with $cf(\delta) = \kappa^+$, then in some cardinality preserving generic extension of the ground model we have

$$\langle\kappa\rangle_\delta {}^\frown \langle\lambda\rangle \in \mathcal{C}(\delta + 1).$$

In this paper we will prove a much stronger result than the above mentioned one.

Theorem 1.1. *Assume that κ and λ are regular cardinals, $\kappa^{++} \leq \lambda$, $\kappa^{<\kappa} = \kappa$, $2^\kappa = \kappa^+$, $\lambda^{\kappa^+} = \lambda$ and $\delta < \kappa^{++}$. Then, in some cardinality preserving generic extension of the ground model, we have $2^\kappa = \lambda$ and*

$$\{f \in {}^\delta([\kappa, \lambda] \cap Card) : f(\alpha) = \kappa \textit{ whenever } cf(\alpha) < \kappa\} \subset \mathcal{C}_\kappa(\delta).$$

Definition 1.2. Let $\mathcal{C}$ be a family of sequences of cardinals. We say that an LCS space X is *universal for* $\mathcal{C}$ iff $CS(X) \in \mathcal{C}$ and for each $s \in \mathcal{C}$ there is an open subspace $Z \subset X$ with $CS(Z) = s$.

Remark. The assumption $\delta < \kappa^{++}$ is essential in the construction as we will explain in a Remark on page 8.

So, we do not know whether Theorem 1.1 can be generalized to $\delta = \kappa^{++}$. In fact, if κ is a specific uncountable cardinal, the problem whether it is relatively consistent with ZFC that $\langle\kappa\rangle_{\kappa^{++}} \in C(\kappa^{++})$ is a long-standing open question. Nevertheless, by a well-known result of Baumgartner and Shelah, it is known that it is relatively consistent with ZFC that $\langle\omega\rangle_{\omega_2} \in C(\omega_2)$ (see [2]).

Instead of Theorem 1.1 we prove the following stronger result:

Theorem 1.3. *Assume that κ and λ are regular cardinals, $\kappa^{++} \leq \lambda$, $\kappa^{<\kappa} = \kappa$, $2^\kappa = \kappa^+$, $\lambda^{\kappa^+} = \lambda$ and $\delta < \kappa^{++}$. Then, in some cardinal preserving generic extension, we have $2^\kappa = \lambda$ and there is an LCS space X which is universal for*

$$\mathcal{C} = \{f \in {}^\delta\big([\kappa, \lambda] \cap Card\big) : f(\alpha) = \kappa \textit{ whenever } cf(\alpha) < \kappa\}.$$

Definition 1.4. Let $\kappa < \lambda$ be cardinals, δ be an ordinal, and $A \subset \delta$. An LCS space X of height δ is called $(\kappa, \lambda, \delta, A)$-*good* iff there is an open subspace $Y \subset X$ such that

(1) $CS(Y) = \langle \kappa \rangle_\delta$,
(2) $\mathrm{I}_\zeta(Y) = \mathrm{I}_\zeta(X)$, and so $|\,\mathrm{I}_\zeta(X)| = \kappa$, for $\zeta \in \delta \setminus A$,
(3) $|\,\mathrm{I}_\zeta(X)| = \lambda$ for $\zeta \in A$,
(4) for $\zeta \in A$ the set $Z_\zeta = \mathrm{I}_{<\zeta}(Y) \cup \mathrm{I}_\zeta(X)$ is an open subspace of X such that
   (a) $\mathrm{I}_\xi(Z_\zeta) = \mathrm{I}_\xi(Y)$ for $\xi < \zeta$,
   (b) $\mathrm{I}_\zeta(Z_\zeta) = \mathrm{I}_\zeta(X)$.

Theorem 1.3 follows immediately from Koszmider's Theorem, Theorem 1.6 and Proposition 1.7 below.

The following result of Koszmider can be obtained by putting together [7, Fact 32 and Theorem 33]:

Definition 1.5 *(See [6,7]).* Assume that $\kappa < \lambda$ are infinite cardinals. We say that a function $\mathcal{F} : [\lambda]^2 \longrightarrow \kappa^+$ is a κ^+-*strongly unbounded function on* λ iff for every ordinal $\vartheta < \kappa^+$ and for every family $\mathcal{A} \subset [\lambda]^{<\kappa}$ of pairwise disjoint sets with $|\mathcal{A}| = \kappa^+$, there are different $a, b \in \mathcal{A}$ such that $\mathcal{F}\{\alpha, \beta\} > \vartheta$ for every $\alpha \in a$ and $\beta \in b$.

Koszmider's Theorem. *If κ, λ are infinite cardinals such that $\kappa^{++} \leq \lambda$, $\kappa^{<\kappa} = \kappa$, $2^\kappa = \kappa^+$ and $\lambda^{\kappa^+} = \lambda$, then in some cardinal preserving generic extension $\kappa^{<\kappa} = \kappa$, $\lambda^\kappa = \lambda$ and there is a κ^+-strongly unbounded function on λ.*

For an ordinal $\delta < \kappa^{++}$ let

$$\mathcal{L}^\delta_\kappa = \big\{\alpha < \delta : \mathrm{cf}(\alpha) \in \{\kappa, \kappa^+\}\big\}.$$

Theorem 1.6. *If $\kappa < \lambda$ are regular cardinals with $\kappa^{<\kappa} = \kappa$, $\lambda^\kappa = \lambda$, and there is a κ^+-strongly unbounded function on λ, then for each $\delta < \kappa^{++}$ there is a κ-complete κ^+-c.c poset $\mathcal{P}$ of cardinality λ such that in $V^{\mathcal{P}}$ we have $2^\kappa = \lambda$ and there is a $(\kappa, \lambda, \delta, \mathcal{L}^\delta_\kappa)$-good space.*

We will prove Theorem 1.6 in Section 2.

Proposition 1.7. *If $\kappa < \lambda$ are regular cardinals and $\delta < \kappa^{++}$, then a $(\kappa, \lambda, \delta, \mathcal{L}^\delta_\kappa)$-good space is universal for*

$$\mathcal{C} = \{f \in {}^\delta\big([\kappa, \lambda] \cap Card\big) : f(\alpha) = \kappa \text{ whenever } cf(\alpha) < \kappa\}.$$

Proof. Let X be a $(\kappa, \lambda, \delta, \mathcal{L}^\delta_\kappa)$-good space. Fix $f \in \mathcal{C}$. For $\zeta \in \mathcal{L}^\delta_\kappa$ pick $T_\zeta \in \big[\mathrm{I}_\zeta(X)\big]^{f(\zeta)}$, and let

$$Z = Y \cup \bigcup\{T_\zeta : \zeta \in \mathcal{L}^\delta_\kappa\}.$$

Since $\mathrm{I}_{<\zeta}(Y) \cup T_\zeta$ is an open subspace of X for $\zeta \in \mathcal{L}^\delta_\kappa$, for every $\alpha < \delta$ we have

$$\mathrm{I}_\alpha(Z) = \mathrm{I}_\alpha(Y) \cup \bigcup\{\mathrm{I}_\alpha(\mathrm{I}_{<\zeta}(Y) \cup T_\zeta) : \zeta \in \mathcal{L}^\delta_\kappa\}.$$

Since

$$\mathrm{I}_\alpha(\mathrm{I}_{<\zeta}(Y) \cup T_\zeta) = \begin{cases} \mathrm{I}_\alpha(Y) & \text{if } \alpha < \zeta, \\ T_\zeta & \text{if } \alpha = \zeta, \\ \emptyset & \text{if } \zeta < \alpha, \end{cases}$$

we have

$$\mathrm{I}_\alpha(Z) = \begin{cases} \mathrm{I}_\alpha(Y) & \text{if } \alpha \notin \mathcal{L}^\delta_\kappa, \\ \mathrm{I}_\alpha(Y) \cup T_\alpha & \text{if } \alpha \in \mathcal{L}^\delta_\kappa. \end{cases}$$

Since $|\,\mathrm{I}_\alpha(Y)| = \kappa$ and $|\,\mathrm{I}_\alpha(Y) \cup T_\alpha| = \kappa + f(\alpha) = f(\alpha)$, we have $CS(Z) = f$, which was to be proved. □

2. Proof of Theorem 1.6

2.1. Graded posets

In [5], [8], [11] and in many other papers, the existence of an LCS space is proved in such a way that instead of constructing the space directly, a certain "graded poset" is produced which guaranteed the existence of the wanted LCS-space. From these results, Bagaria, [1], extracted the notion of s-posets and established the formal connection between graded posets and LCS-spaces. For technical reasons, we will use a reformulation of Bagaria's result introduced in [12].

If $\preceq$ is an arbitrary partial order on a set X then define the topology $\tau_\preceq$ on X generated by the family $\{\mathrm{U}_\preceq(x), X \setminus \mathrm{U}_\preceq(x) : x \in X\}$ as a subbase, where $\mathrm{U}_\preceq(x) = \{y \in X : y \preceq x\}$.

In what follows, if i is a partial function from $[X]^2$ to X where X is the domain of some poset, for every $\{s,t\} \in [X]^2 \setminus \mathrm{dom}(i)$ we will write $i\{s,t\} = undef$. So, we will write $i : \left[X\right]^2 \longrightarrow X \cup \{undef\}$ in order to represent a partial function i from $[X]^2$ to X.

Proposition 2.1 *([12, Proposition 2.1]). Assume that $\langle X, \preceq\rangle$ is a poset, $\{X_\alpha : \alpha < \delta\}$ is a partition of X and $i : \left[X\right]^2 \longrightarrow X \cup \{undef\}$ is a function satisfying (a)–(c) below:*

(a) *if $x \in X_\alpha$, $y \in X_\beta$ and $x \preceq y$ then either $x = y$ or $\alpha < \beta$,*
(b) *$\forall \{x,y\} \in \left[X\right]^2$ $\big(\, \forall z \in X\ (z \preceq x \wedge z \preceq y)$ iff $z \preceq i\{x,y\}\, \big)$,*
(c) *if $x \in X_\alpha$ and $\beta < \alpha$ then the set $\{y \in X_\beta : y \preceq x\}$ is infinite.*

Then $\mathcal{X} = \langle X, \tau_\preceq\rangle$ is an LCS space with $\mathrm{I}_\alpha(\mathcal{X}) = X_\alpha$ for $\alpha < \delta$.

Definition 2.2. Let $\kappa < \lambda$ be cardinals, δ be an ordinal, and $A \subset \delta$. Assume that $\langle X, \preceq\rangle$ is a poset, $\{X_\alpha : \alpha < \delta\}$ is a partition of X and $i : \left[X\right]^2 \longrightarrow X \cup \{undef\}$ is a function satisfying conditions (a)–(c) from Proposition 2.1.

We say that poset $\langle X, \preceq\rangle$ is $(\kappa, \lambda, \delta, A)$-*good* iff there is a set $Y \subset X$ such that:

(d) if $x_0 \preceq x_1$, then either $x_0 = x_1$ or $x_0 \in Y$;
(e) $X_\zeta \in \left[Y\right]^\kappa$ for $\zeta \in \delta \setminus A$;
(f) $|X_\zeta| = \lambda$ and $|X_\zeta \cap Y| = \kappa$ for $\zeta \in A$.

Proposition 2.3. *Let $\kappa < \lambda$ be cardinals, δ be an ordinal, and $A \subset \delta$. If $\langle X, \preceq\rangle$ is a $(\kappa, \lambda, \delta, A)$-good poset, then $\mathcal{X} = \langle X, \tau_\preceq\rangle$ is a $(\kappa, \lambda, \delta, A)$-good space.*

Proof. By Proposition 2.1, $\mathcal{X} = \langle X, \tau_\preceq\rangle$ is an LCS space with $\mathrm{I}_\alpha(\mathcal{X}) = X_\alpha$ for $\alpha < \delta$.

By (d), the subspace Y is open, and so $\mathrm{I}_\zeta(Y) = \mathrm{I}_\zeta(X) \cap Y$. Thus $|\,\mathrm{I}_\zeta(Y)| = \kappa$ by (e) and (f). So $CS(Y) = \langle\kappa\rangle_\delta$, i.e. 1.4(1) holds.

If $\zeta \in \delta \setminus A$, then $\mathrm{I}_\zeta(X) \subset Y$ by (e), so $\mathrm{I}_\zeta(X) = \mathrm{I}_\zeta(Y)$. Thus 1.4(2) holds. Moreover $\mathrm{I}_\zeta(Y) = \mathrm{I}_\zeta(X) \cap Y$. 1.4(3) follows from (f).

Also, for $\zeta \in A$ (a) and (d) imply that $\mathrm{U}_{\preceq}(s) \subset Z_\zeta$ for $s \in Z_\zeta$, and so Z_ζ is an open subspace of $\mathcal{X}$. Hence $\mathrm{I}_\xi(Z_\zeta) = \mathrm{I}_\xi(X) \cap Z_\zeta = X_\xi \cap Z_\zeta$.

Thus $\mathrm{I}_\xi(Z_\zeta) = \mathrm{I}_\xi(Y)$ for $\xi < \zeta$, and $\mathrm{I}_\zeta(Z_\zeta) = X_\zeta$. So 1.4(4) also holds.

Thus $\mathcal{X}$ is a $(\kappa, \lambda, \delta, A)$-good space. □

So, instead of Theorem 1.6, it is enough to prove Theorem 2.4 below.

Theorem 2.4. *If $\kappa < \lambda$ are regular cardinals with $\kappa^{<\kappa} = \kappa$, $\lambda^\kappa = \lambda$, and there is a κ^+-strongly unbounded function on λ, then for each $\delta < \kappa^{++}$ there is a κ-complete κ^+-c.c poset $\mathcal{P}$ of cardinality λ such that in $V^{\mathcal{P}}$ we have $2^\kappa = \lambda$ and there is a $(\kappa, \lambda, \delta, \mathcal{L}^\delta_\kappa)$-good poset.*

So, assume that κ, λ and δ satisfy the hypothesis of Theorem 2.4. In order to construct the required poset $\mathcal{P}$, first we need to recall some notion from [8, Section 1].

2.2. Orbits

If $\alpha \le \beta$ are ordinals let

$$[\alpha, \beta) = \{\gamma : \alpha \le \gamma < \beta\}.$$

We say that I is an *ordinal interval* iff there are ordinals α and β with $I = [\alpha, \beta)$. Write $I^- = \alpha$ and $I^+ = \beta$.

If $I = [\alpha, \beta)$ is an ordinal interval let $\mathrm{E}(I) = \{\varepsilon^I_\nu : \nu < \mathrm{cf}(\beta)\}$ be a cofinal closed subset of I having order type $\mathrm{cf}(\beta)$ with $\alpha = \varepsilon^I_0$ and put

$$\mathcal{E}(I) = \{[\varepsilon^I_\nu, \varepsilon^I_{\nu+1}) : \nu < \mathrm{cf}\,\beta\}$$

provided β is a limit ordinal, and let $\mathrm{E}(I) = \{\alpha, \beta'\}$ and put

$$\mathcal{E}(I) = \{[\alpha, \beta'), \{\beta'\}\}$$

provided $\beta = \beta' + 1$ is a successor ordinal.

Define $\{\mathcal{I}_n : n < \omega\}$ as follows:

$$\mathcal{I}_0 = \{[0, \delta)\} \text{ and } \mathcal{I}_{n+1} = \bigcup\{\mathcal{E}(I) : I \in \mathcal{I}_n\}.$$

Put $\mathbb{I} = \bigcup\{\mathcal{I}_n : n < \omega\}$.

Note that $\mathbb{I}$ is a *cofinal tree of intervals* in the sense defined in [8]. So, the following conditions are satisfied:

(i) For every $I, J \in \mathbb{I}$, $I \subset J$ or $J \subset I$ or $I \cap J = \emptyset$.
(ii) If I, J are different elements of $\mathbb{I}$ with $I \subset J$ and J^+ is a limit ordinal, then $I^+ < J^+$.
(iii) $\mathcal{I}_n$ partitions $[0, \delta)$ for each $n < \omega$.
(iv) $\mathcal{I}_{n+1}$ refines $\mathcal{I}_n$ for each $n < \omega$.
(v) For every $\alpha < \delta$ there is an $I \in \mathbb{I}$ such that $I^- = \alpha$.

Then, for each $\alpha < \delta$ we define

$$\mathrm{n}(\alpha) = \min\{n : \exists I \in \mathcal{I}_n \text{ with } I^- = \alpha\},$$

and for each $\alpha < \delta$ and $n < \omega$ we pick

$$\mathrm{I}(\alpha, n) \in \mathcal{I}_n \text{ such that } \alpha \in \mathrm{I}(\alpha, n).$$

Proposition 2.5. *Assume that $\zeta < \delta$ is a limit ordinal. Then, there is an interval*

$$J(\zeta) \in \mathcal{I}_{n(\zeta)-1} \cup \mathcal{I}_{n(\zeta)}$$

such that ζ is a limit point of $E(J(\zeta))$.

If $cf(\zeta) = \kappa^+$, then $J(\zeta) \in \mathcal{I}_{n(\zeta)}$ and $J(\zeta)^+ = \zeta$.

Proof. If there is an $I \in \mathcal{I}_{\mathrm{n}(\zeta)}$ with $I^+ = \zeta$ then $J(\zeta) = I$. If there is no such I, then ζ is a limit point of $\mathrm{E}(\mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1))$, so $J(\zeta) = \mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1)$.

Assume now that $\mathrm{cf}(\zeta) = \kappa^+$. Then $\zeta \in \mathrm{E}(\mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1))$, but $|\,\mathrm{E}(\mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1)) \cap \zeta| \leq \kappa$, so ζ can not be a limit point of $\mathrm{E}(\mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1))$. Therefore, it has a predecessor ξ in $\mathrm{E}(\mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1))$, i.e. $[\xi, \zeta) \in \mathcal{I}_{\mathrm{n}(\zeta)}$, and so $J(\zeta) = [\xi, \zeta)$ and $J(\zeta) \in \mathcal{I}_{n(\zeta)}$. □

If $\mathrm{cf}(J(\zeta)^+) \in \{\kappa, \kappa^+\}$, we denote by $\{\epsilon^\zeta_\nu : \nu < \mathrm{cf}(J(\zeta)^+)\}$ the increasing enumeration of $\mathrm{E}(J(\zeta))$, i.e. $\epsilon^\zeta_\nu = \varepsilon^{J(\zeta)}_\nu$ for $\nu < \mathrm{cf}(J(\zeta)^+)$.

Now if $\zeta < \delta$, we define the *basic orbit* of ζ (with respect to $\mathbb{I}$) as

$$\mathrm{o}(\zeta) = \bigcup\{(\mathrm{E}(\mathrm{I}(\zeta, m)) \cap \zeta) : m < \mathrm{n}(\zeta)\}.$$

We refer the reader to [8, Section 1] for some fundamental facts and examples on basic orbits. In particular, we have that $\alpha \in o(\beta)$ implies $o(\alpha) \subset o(\beta)$.

If $\zeta \in \mathcal{L}^\delta_\kappa$, we define the *extended orbit* of ζ by

$$\overline{\mathrm{o}}(\zeta) = \mathrm{o}(\zeta) \cup (\mathrm{E}(J(\zeta)) \cap \zeta).$$

Observe that if $J(\zeta) \in \mathcal{I}_{n(\zeta)-1}$ then $\overline{o}(\zeta) = o(\zeta)$.

The underlying set of our poset will consist of blocks. The following set $\mathbb{B}$ below serves as the index set of our blocks:

$$\mathbb{B} = \{S\} \cup \mathcal{L}^\delta_\kappa.$$

Let

$$B_S = \delta \times \kappa$$

and

$$B_\zeta = \{\zeta\} \times [\kappa, \lambda)$$

for $\zeta \in \mathcal{L}^\delta_\kappa$.

The underlying set of our poset will be

$$X = \bigcup\{B_T : T \in \mathbb{B}\}.$$

To obtain a $(\kappa, \lambda, \delta, \mathcal{L}^\delta_\kappa)$-good poset we take $Y = B_S$ and

$$X_\zeta = \begin{cases} \{\zeta\} \times \kappa & \text{if } \zeta \in \delta \setminus \mathcal{L}^\delta_\kappa, \\ \{\zeta\} \times \lambda & \text{if } \zeta \in \mathcal{L}^\delta_\kappa. \end{cases}$$

Define the functions $\pi : X \longrightarrow \delta$ and $\rho : X \longrightarrow \lambda$ by the formulas

$$\pi(\langle\alpha,\nu\rangle) = \alpha \text{ and } \rho(\langle\alpha,\nu\rangle) = \nu.$$

Define

$$\pi_B : X \longrightarrow \mathbb{B} \text{ by the formula } x \in B_{\pi_B(x)}.$$

Finally we define the *orbits* of the elements of X as follows:

$$\mathrm{o}^*(x) = \begin{cases} \mathrm{o}(\pi(x)) & \text{for } x \in B_S, \\ \overline{\mathrm{o}}(\pi(x)) & \text{for } x \in X \setminus B_S. \end{cases}$$

Observe that $\mathrm{o}^*(x) \in \left[\pi(x)\right]^{\leq\kappa^+}$ and

$$|\,\mathrm{o}^*(x)| \leq \kappa \text{ unless } x \in B_\xi \text{ with } cf(\xi) = \kappa^+.$$

To simplify our notation, we will write $\mathrm{o}(x) = \mathrm{o}(\pi(x))$ and $\overline{\mathrm{o}}(x) = \overline{\mathrm{o}}(\pi(x))$.

2.3. Forcing construction

Let $\Lambda \in \mathbb{I}$ and $\{x,y\} \in \left[X\right]^2$. We say that Λ *separates* x *from* y if

$$\Lambda^- < \pi(x) < \Lambda^+ < \pi(y).$$

Let $\mathcal{F} : \left[\lambda\right]^2 \longrightarrow \kappa^+$ be a κ^+-strongly unbounded function.
Define

$$\mathrm{f} : \left[X\right]^2 \longrightarrow \left[\delta\right]^{\leq\kappa}$$

as follows:

$$\mathrm{f}\{x,y\} = \begin{cases} o(x) \cup \{\epsilon_\zeta^{\pi(x)} : \zeta < \mathcal{F}\{\rho(x),\rho(y)\}\} & \text{if } \pi_B(x) = \pi_B(y) \neq S, \\ & \text{and } cf(\pi(x)) = \kappa^+, \\ \mathrm{o}^*(x) \cap \mathrm{o}^*(y) & \text{otherwise.} \end{cases}$$

Observe that

$$|\,\mathrm{f}\{x,y\}| \leq \kappa$$

for all $\{x,y\} \in \left[X\right]^2$.

Definition 2.6. We define the poset $\mathcal{P} = \langle P, \leq\rangle$ as follows: $\langle A, \preceq, i\rangle \in P$ iff the following conditions hold:

(P1) $A \in \left[X\right]^{<\kappa}$;
(P2) $\preceq$ is a partial order on A such that $x \preceq y$ implies $x = y$ or $\pi(x) < \pi(y)$;
(P3) if $x \preceq y$ and $\pi_B(x) \neq S$, then $x = y$;
(P4) $\mathrm{i} : \left[A\right]^2 \longrightarrow A \cup \{\mathrm{undef}\}$ such that for each $\{x,y\} \in \left[A\right]^2$ we have

$$\forall a \in A([a \preceq x \wedge a \preceq y] \text{ iff } a \preceq \mathrm{i}\{x,y\});$$

(P5) for each $\{x,y\} \in [A]^2$ if x and y are $\preceq$-incomparable but $\preceq$-compatible, then

$$\pi(\mathrm{i}\{x,y\}) \in \mathrm{f}\{x,y\};$$

(P6) If $\{x,y\} \in [A]^2$ with $x \prec y$, and $\Lambda \in \mathbb{I}$ separates x from y, then there is $z \in A$ such that $x \prec z \prec y$ and $\pi(z) = \Lambda^+$.

The ordering on P is the extension: $\langle A, \preceq, \mathrm{i}\rangle \leq \langle A', \preceq', \mathrm{i}'\rangle$ iff $A' \subset A$, $\preceq' = \preceq \cap (A' \times A')$, and $\mathrm{i}' \subset \mathrm{i}$.

Remark. Property (P5) will be used to prove that $\mathcal{P}$ satisfies the κ^+-chain condition. For this, we will use in an essential way that $\delta < \kappa^{++}$ and $f : [X]^2 \to [\delta]^{\leq\kappa}$. Then, if $R = \langle r_\nu : \nu < \kappa^+\rangle$ is a subset of P of size κ^+ with $r_\nu = \langle A_\nu, \preceq_\nu, \mathrm{i}_\nu\rangle$ for $\nu < \kappa^+$, by using the assumption that $\kappa^{<\kappa} = \kappa$, we can assume that $\{A_\nu : \nu < \kappa^+\}$ forms a Δ-system with kernel A_Δ and that the conditions r_ν $(\nu < \kappa^+)$ are pairwise isomorphic. Note that if $\kappa^+ < \delta < \kappa^{++}$, we can not assume that A_Δ is an initial segment of each A_ν for $\nu < \kappa^+$. However, since $|f\{x,y\}| \leq \kappa$ for all $\{x,y\} \in [X]^2$, we can assume by (P5) that if $x, y \in A_\Delta$ with $x \neq y$ and $\nu < \mu < \kappa^+$, we have that $i_\nu\{x,y\} = i_\mu\{x,y\}$. Then, by using the fact that $\mathcal{F}$ is a κ^+-strongly unbounded function, we will be able to find two different conditions r_ν and r_μ in R that are compatible in $\mathcal{P}$. To show that r_ν and r_μ are compatible, we will be able to define the infimum of pairs of elements $\{x,y\}$ where $x \in A_\nu \setminus A_\mu$ and $y \in A_\mu \setminus A_\nu$ by using the properties of trees of intervals and orbits (specially Proposition 2.5). Note that if $\delta = \kappa^{++}$, we can not define the notion of a basic orbit of an element $\zeta < \delta$ on a tree of intervals $\{\mathcal{I}_n : n < \omega\}$ where $\mathcal{I}_0 = \{[0,\delta)\}$ in such a way that $|o(\zeta)| \leq \kappa$.

For $p \in P$ write $p = \langle A_p, \preceq_p, \mathrm{i}_p\rangle$.

To complete the proof of Theorem 2.4 we will use the following lemmas which will be proved later:

Lemma 2.7. *$\mathcal{P}$ is κ-complete.*

Lemma 2.8. *$\mathcal{P}$ satisfies the κ^+-c.c.*

Lemma 2.9.

(a) *For all $x \in X$, the set*

$$D_x = \{q \in P : x \in A_q\}$$

is dense in $\mathcal{P}$.

(b) *If $x \in X$, $\alpha < \pi(x)$ and $\zeta < \kappa$, then the set*

$$E_{x,\alpha,\zeta} = \{q \in P : x \in A_q \wedge \exists b \in A_q \cap (\{\alpha\} \times (\kappa \setminus \zeta))\ b \preceq_q x\}$$

is dense in $\mathcal{P}$

Since $\lambda^{<\kappa} = \lambda$, the cardinality of P is λ. Thus, Lemma 2.7 and Lemma 2.8 above guarantee that forcing with P preserves cardinals and $2^\kappa = \lambda$ in the generic extension.

Let $G \subset P$ be a generic filter. Put $A = \bigcup\{A_p : p \in G\}$, $i = \bigcup\{\mathrm{i}_p : p \in G\}$ and $\preceq = \bigcup\{\preceq_p : p \in G\}$. Then $A = X$ by Lemma 2.9(a).

We claim that $\langle X, \preceq\rangle$ is a $(\kappa, \lambda, \delta, \mathcal{L}^\delta_\kappa)$-poset.

Recall that we put $X_\zeta = \{\zeta\} \times \kappa$ for $\zeta \in \delta \setminus \mathcal{L}^\delta_\kappa$ and $X_\zeta = \{\zeta\} \times \lambda$ for $\zeta \in \mathcal{L}^\delta_\kappa$. Then the poset $\langle X, \preceq\rangle$, the partition $\{X_\zeta : \zeta < \delta\}$, the function i and $Y = \delta \times \kappa$ clearly satisfy conditions 2.1(a,b) and 2.2(d,e,f) by the definition of the poset $\mathcal{P}$.

Finally condition 2.1(c) holds by Lemma 2.9(b).

So to complete the proof of Theorem 2.4 we need to prove Lemmas 2.7, 2.8 and 2.9.

Since κ is regular, Lemma 2.7 clearly holds.

Proof of Lemma 2.9. (a) Let $p \in P$ be arbitrary. We can assume that $x \notin A_p$.

Let $A_q = A_p \cup \{x\}$, $\preceq_q = \preceq_p \cup \{\langle x, x\rangle\}$, and define $i' \supset i$ such that $i'\{a, x\} = undef$ for $a \in A_p$. Then $q = \langle A_q, \preceq_q, \mathrm{i}_q\rangle \in D_x$ and $q \le p$.

(b) Let $p \in P$ be arbitrary. By (a) we can assume that $x \in A_p$. Write $\beta = \pi(x)$.

Let m be the natural number such that $I(\alpha, m) = I(\beta, m)$ and $I(\alpha, m + 1) \ne I(\beta, m + 1)$. We put $I_k = I(\alpha, k)$ for $k \ge m + 1$. Let $K = \{\alpha\} \cup \{I_k^+ : m + 1 \le k < n(\alpha)\}$.

For each $\gamma \in K$ pick $b_\gamma \in (\{\gamma\} \times (\kappa \setminus \zeta)) \setminus A_p$. So $\pi(b_\gamma) = \gamma$.

Let $A_q = A_p \cup \{b_\gamma : \gamma \in K\}$,

$$\preceq_q = \preceq_p \cup \{\langle b_\gamma, b_{\gamma'}\rangle : \gamma, \gamma' \in K, \gamma \le \gamma'\} \cup \{\langle b_\gamma, z\rangle : \gamma \in K, z \in A_p, x \preceq_p z\}.$$

We let $\mathrm{i}_q\{y, z\} = \mathrm{i}_p\{y, z\}$ if $\{y, z\} \in \big[A_p\big]^2$, $\mathrm{i}_q\{b_\gamma, b_{\gamma'}\} = b_\gamma$ if $\gamma, \gamma' \in K$ with $\gamma < \gamma'$, $\mathrm{i}_q\{b_\gamma, z\} = b_\gamma$ if $\gamma \in K$ and $x \preceq_p z$, and $\mathrm{i}_q\{b_\gamma, z\} = undef$ otherwise.

Let $q = \langle A_q, \preceq_q, \mathrm{i}_q\rangle$. Next we check that $q \in P$. Clearly $(P1), (P2), (P3)$ and $(P5)$ hold for q. (P4) also holds because if $y \in A_p$ and $\gamma \in K$ then either $b_\gamma \preceq_q y$ or they are $\preceq_q$-incompatible.

To check (P6) assume that $b_\gamma \prec_q y$ and Λ separates b_γ from y. If $\Lambda^+ < \beta$, then $z = b_{\Lambda^+}$ meets the requirements of (P6). If $\Lambda^+ = \beta$, we have $b_\gamma \prec_q x \prec_q y$ and $\pi(x) = \beta$, and so we are done. And if $\Lambda^+ > \beta$, we apply condition (P6) for p, and so there is $z \in A_p$ such that $x \prec_p z \prec_p y$ and $\pi(z) = \Lambda^+$, and hence $b_\gamma \prec_q z \prec_q y$.

By the construction, $q \le p$.

Finally $q \in E_{x,\alpha,\zeta}$ because $b_\alpha \in A_q \cap \big(\{\alpha\} \times (\kappa \setminus \zeta)\big)$ and $b_\alpha \preceq_q x$. □

The rest of the paper is devoted to the proof of Lemma 2.8.

Proof of Lemma 2.8. Assume that $\langle r_\nu : \nu < \kappa^+\rangle \subset P$ with $r_\nu \ne r_\mu$ for $\nu < \mu < \kappa^+$.

In the first part of the proof, till Claim 2.16, we will find $\nu < \mu < \kappa^+$ such that r_ν and r_μ are twins in a strong sense, and r_ν and r_μ form a *good pair* (see Definition 2.15). Then, in the second part of the proof, we will show that if $\{r_\nu, r_\mu\}$ is a good pair, then r_ν and r_μ are compatible in $\mathcal{P}$.

Write $r_\nu = \langle A_\nu, \preceq_\nu, \mathrm{i}_\nu\rangle$ and $A_\nu = \{x_{\nu,i} : i < \sigma_\nu\}$.

Since we are assuming that $\kappa^{<\kappa} = \kappa$, by thinning out $\langle r_\nu : \nu < \kappa^+\rangle$ by means of standard combinatorial arguments, we can assume the following:

(A) $\sigma_\nu = \sigma$ for each $\nu < \kappa^+$.

(B) $\{A_\nu : \nu < \kappa^+\}$ forms a Δ-system with kernel A_Δ.

(C) For each $\nu < \mu < \kappa^+$ there is an isomorphism $h_{\nu,\mu} : \langle A_\nu, \preceq_\nu, \mathrm{i}_\nu\rangle \longrightarrow \langle A_\mu, \preceq_\mu, \mathrm{i}_\mu\rangle$ such that for every $i, j < \sigma$ the following holds:
   - (a) $h_{\nu,\mu} \restriction A_\Delta = \mathrm{id}$,
   - (b) $h_{\nu,\mu}(x_{\nu,i}) = x_{\mu,i}$,
   - (c) $\pi_B(x_{\nu,i}) = \pi_B(x_{\nu,j})$ iff $\pi_B(x_{\mu,i}) = \pi_B(x_{\mu,j})$,
   - (d) $\pi_B(x_{\nu,i}) = S$ iff $\pi_B(x_{\mu,i}) = S$,
   - (e) if $\{x_{\nu,i}, x_{\nu,j}\} \in \big[A_\Delta\big]^2$ then $x_{\nu,i} = x_{\mu,i}$, $x_{\nu,j} = x_{\mu,j}$ and $\mathrm{i}_\nu\{x_{\nu,i}, x_{\nu,j}\} = \mathrm{i}_\mu\{x_{\mu,i}, x_{\mu,j}\}$,
   - (f) $\pi(x_{\nu,i}) \in \mathrm{o}(x_{\nu,j})$ iff $\pi(x_{\mu,i}) \in \mathrm{o}(x_{\mu,j})$,
   - (g) $\pi(x_{\nu,i}) \in \overline{\mathrm{o}}(x_{\nu,j})$ iff $\pi(x_{\mu,i}) \in \overline{\mathrm{o}}(x_{\mu,j})$,
   - (h) $\pi(x_{\nu,i}) \in \mathrm{o}^*(x_{\nu,j})$ iff $\pi(x_{\mu,i}) \in \mathrm{o}^*(x_{\mu,j})$,

(i) $\pi(x_{\nu,k}) \in \mathrm{f}\{x_{\nu,i}, x_{\nu,j}\}$ iff $\pi(x_{\mu,k}) \in \mathrm{f}\{x_{\mu,i}, x_{\mu,j}\}$.
(j) $\mathrm{cf}(\pi(x_{\nu,i})) = \kappa^+$ iff $\mathrm{cf}(\pi(x_{\mu,i})) = \kappa^+$.

Note that in order to obtain (C)(e) we use condition (P5) and the fact that $|\mathrm{f}\{x, y\}| \leq \kappa$ for all $x \neq y$.

Also, we may assume the following:

(D) There is a partition $\sigma = K \cup^* F \cup^* D \cup^* M$ such that for each $\nu < \mu < \kappa^+$:
(a) $\forall i \in K\ x_{\nu,i} \in A_\Delta$ and so $x_{\nu,i} = x_{\mu,i}$. $A_\Delta = \{x_{\nu,i} : i \in K\}$.
(b) $\forall i \in F\ x_{\nu,i} \neq x_{\mu,i}$ but $\pi_B(x_{\nu,i}) = \pi_B(x_{\mu,i}) \neq S$.
(c) $\forall i \in D\ x_{\nu,i} \notin A_\Delta$, $\pi_B(x_{\nu,i}) = S$ and $\pi(x_{\nu,i}) \neq \pi(x_{\mu,i})$.
(d) $\forall i \in M\ \pi_B(x_{\nu,i}) \neq S$ and $\pi(x_{\nu,i}) \neq \pi(x_{\mu,i})$.
(E) If $\pi(x_{\nu,i}) = \pi(x_{\nu,j})$ then $\{i, j\} \in \left[K \cup F\right]^2 \cup \left[D \cup M\right]^2$.

By [3, Corollary 17.5], if $\sigma < \kappa = \kappa^{<\kappa}$ then the following partition relation holds:

$$\kappa^+ \longrightarrow (\kappa^+, (\omega)_\sigma)^2.$$

(i.e. given any function $c : [\kappa^+]^2 \longrightarrow 1 + \rho$ either there is a set $A \in [\kappa^+]^{\kappa^+}$ such that $c''[A]^2 = \{0\}$, or for some $\xi < \sigma$ there is a set $B \in [\kappa^+]^\omega$ such that $c''[B]^2 = \{1 + \xi\}$.)

Hence we can assume:

(F) $\pi(x_{\nu,i}) \leq \pi(x_{\mu,i})$ for each $i \in \sigma$ and $\nu < \mu < \kappa^+$.

For $i \in \sigma$ let

$$\delta_i = \begin{cases} \pi(x_{\nu,i}) & \text{if } i \in K \cup F, \\ \sup\{\pi(x_{\nu,i}) : \nu < \kappa^+\} & \text{if } i \in D \cup M. \end{cases}$$

Claim 2.10. *(a) If $i \in D \cup M$, then the sequence $\langle \pi(x_{\nu,i}) : \nu < \kappa^+ \rangle$ is strictly increasing, $cf(\delta_i) = \kappa^+$ and $sup(J(\delta_i)) = \delta_i$. Moreover for every $\nu < \kappa^+$ we have $\pi(x_{\nu,i}) < \delta_i$.*
(b) If $\{i, j\} \in [M]^2$ and $x_{\nu,i} \preceq_\nu x_{\nu,j}$, then $x_{\nu,i} = x_{\nu,j}$.

Proof. If $i \in D \cup M$, then (F) and (D)(c-d) imply that the sequence $\{\pi(x_{\nu,i}) : \nu < \kappa^+\}$ is strictly increasing. Hence $cf(\delta_i) = \kappa^+$ and $\pi(x_{\nu,i}) < \delta_i$ for $i \in D \cup M$.

Thus Proposition 2.5 implies $\sup(J(\delta_i)) = \delta_i$. So (a) holds.

(D)(d) and condition (P3) imply (b). □

We put

$$Z_0 = \{\delta_i : i \in \sigma\}.$$

Since $\pi'' A_\Delta = \{\delta_i : i \in K\}$ we have $\pi'' A_\Delta \subset Z_0$. Then, we define Z as the closure of Z_0 with respect to $\mathbb{I}$:

$$Z = Z_0 \cup \{I^+ : I \in \mathbb{I}, I \cap Z_0 \neq \emptyset\}.$$

Observe that

$$|Z| < \kappa.$$

By Claim 2.10(a), the sequence $\langle \pi(x_{\nu,i}) : \nu < \kappa^+\rangle$ is strictly increasing for $i \in D \cup M$. Since $|Z| < \kappa$, and $|\mathrm{o}^*(x_{\nu,k})| \leq \kappa$ for $x_{\nu,k} \in B_S \cap A_\vartriangle$, we can assume that

(G) $\pi(x_{\nu,i}) \notin \mathrm{o}^*(x_{\nu,k})$ for $x_{\nu,k} \in B_S \cap A_\vartriangle$ and $i \in D \cup M$.

Our aim is to prove that there are $\nu < \mu < \kappa^+$ such that the forcing conditions r_ν and r_μ are compatible. However, since we are dealing with infinite forcing conditions, we will need to add new elements to $A_\nu \cup A_\mu$ in order to be able to define the infimum of pairs of elements $\{x, y\}$ where $x \in A_\nu \setminus A_\mu$ and $y \in A_\mu \setminus A_\nu$. The following definitions will be useful to provide the room we need to insert the required new elements.

Let

$$\sigma_1 = \{i \in \sigma \setminus K : \mathrm{cf}(\delta_i) = \kappa\}$$

and

$$\sigma_2 = \{i \in \sigma \setminus K : \mathrm{cf}(\delta_i) = \kappa^+\}.$$

Assume that $i \in \sigma_1 \cup \sigma_2$. Let

$$\xi_i = \min\{\zeta \in \mathrm{cf}(\delta_i) : \epsilon_\zeta^{J(\delta_i)} > \sup(\delta_i \cap Z)\}.$$

Since $|Z| < \kappa \leq cf(\delta_i)$, the ordinal ξ_i is defined and $\delta_i > \epsilon_{\xi_i}^{J(\delta_i)}$.

Then, if $i \in \sigma_1$ we put

$$\underline{\gamma}(\delta_i) = \epsilon_{\xi_i}^{J(\delta_i)} \text{ and } \gamma(\delta_i) = \delta_i,$$

and if $i \in \sigma_2$ we put

$$\underline{\gamma}(\delta_i) = \epsilon_{\xi_i}^{J(\delta_i)} \text{ and } \gamma(\delta_i) = \epsilon_{\xi_i+\kappa}^{J(\delta_i)}.$$

For $i \in \sigma_2$, since $\gamma(\delta_i) < \delta_i$ and $\delta_i = \lim\{\pi(x_{\nu,i}) : \nu < \kappa^+\}$ by Claim 2.10(a) for all $i \in D \cup M$, we can assume that

(H) $\pi(x_{\nu,i}) \in J(\delta_i) \setminus \gamma(\delta_i)$, and so $\pi(x_{\nu,i}) \notin Z$, for all $i \in D \cup M$.

We will use the following fundamental facts.

Claim 2.11. *If $x_{\nu,i} \preceq_\nu x_{\nu,j}$ then $\delta_i \leq \delta_j$.*

Proof. $x_{\nu,i} \preceq_\nu x_{\nu,j}$ implies $\pi(x_{\nu,i}) \leq \pi(x_{\nu,j})$ by (P2). □

Claim 2.12. *Assume $i, j \in \sigma$. If $x_{\nu,i} \preceq_\nu x_{\nu,j}$ then either $\delta_i = \delta_j$ or there is $a \in A_\vartriangle \cap B_S$ with $x_{\nu,i} \preceq_\nu a \preceq_\nu x_{\nu,j}$.*

Proof. Assume that $i, j \notin K$ and $\delta_i \neq \delta_j$. By Claim 2.11, we have $\delta_i < \delta_j$. Since $i \in F \cup M$ and $x_{\nu,i} \preceq_\nu x_{\nu,j}$ imply $x_{\nu,i} = x_{\nu,j}$ and so $\delta_i = \delta_j$, we have that $i \in D$, and so $\pi(x_{\nu,i}) < \delta_i$, $\mathrm{cf}(\delta_i) = \kappa^+$ and $J(\delta_i)^+ = \delta_i$ by Proposition 2.5.

Since $\delta_i < \delta_j$, we have $\delta_i < \gamma(\delta_j) < \pi(x_{\nu,j})$ by (H), and so $J(\delta_i)$ separates $x_{\nu,i}$ from $x_{\nu,j}$. By (P6), we infer that there is an $a = x_{\nu,k} \in A_\nu$ such that $\pi(a) = \delta_i$ and $x_{\nu,i} \preceq_\nu a \preceq_\nu x_{\nu,j}$.

Since $x_{\nu,k} \neq x_{\nu,j}$, we have $x_{\nu,k} \in B_S$, and so $k \in K \cup D$. But as $\pi(x_{\nu,k}) = \delta_i \in Z$ we obtain $k \notin D$ by (H), and so $k \in K$, which implies $a = x_{\nu,k} \in A_\vartriangle \cap B_S$. □

Claim 2.13. *If* $x_{\nu,i} \in A_\vartriangle \cap B_S$ *and* $x_{\nu,j} \in A_\nu$ *are compatible but incomparable in* r_ν*, then* $x_{\nu,k} = \mathrm{i}_\nu\{x_{\nu,i}, x_{\nu,j}\} \in A_\vartriangle \cap B_S$.

Proof. First, (P2) implies $x_{\nu,k} \in B_S$.

Since $\pi(x_{\nu,k}) = \pi(\mathrm{i}_\nu\{x_{\nu,i}, x_{\nu,j}\}) \in \mathrm{f}\{x_{\nu,i}, x_{\nu,j}\} = \mathrm{o}^*(x_{\nu,i}) \cap \mathrm{o}^*(x_{\nu,j}) \subset \mathrm{o}^*(x_{\nu,i})$ by (P5), and $x_{\nu,i} \in A_\vartriangle \cap B_S$, we have $k \notin D \cup M$ by (G). Thus $k \in K$, and so $x_{\nu,k} \in A_\vartriangle$.

Hence $x_{\nu,k} = \mathrm{i}_\nu\{x_{\nu,i}, x_{\nu,j}\} \in A_\vartriangle \cap B_S$. □

Claim 2.14. *Assume that* $x_{\nu,i}$ *and* $x_{\nu,j}$ *are compatible but incomparable in* r_ν*. Let* $x_{\nu,k} = \mathrm{i}_\nu\{x_{\nu,i}, x_{\nu,j}\}$*. Then either* $x_{\nu,k} \in A_\vartriangle$ *or* $\delta_i = \delta_j = \delta_k$.

Proof. If $\delta_k \neq \delta_i$, we infer that there is $b \in A_\vartriangle \cap B_S$ with $x_{\nu,k} \preceq_\nu b \preceq_\nu x_{\nu,i}$ by Claim 2.12. So $x_{\nu,k} = \mathrm{i}_\nu\{b, x_{\nu,j}\}$ and thus $x_{\nu,k} \in A_\vartriangle$ by using Claim 2.13.

Similarly, $\delta_k \neq \delta_j$ implies $x_{\nu,k} \in A_\vartriangle$. □

Definition 2.15. $\{r_\nu, r_\mu\}$ is a *good pair*
iff the following holds:

(a) for all $i \in F$ with $cf(\delta_i) = \kappa^+$ we have

$$\mathrm{f}\{x_{\nu,i}, x_{\mu,i}\} \supset \overline{o}(\delta_i) \cap \gamma(\delta_i), \tag{$\blacktriangledown$}$$

(b) for all $\{i, j\} \in [F]^2$ with $\delta_i = \delta_j$ and $\mathrm{cf}(\delta_i) = \kappa^+$ we have

$$\mathrm{f}\{x_{\nu,i}, x_{\mu,j}\} \supset \overline{o}(\delta_i) \cap \gamma(\delta_i). \tag{$\blacktriangle$}$$

Claim 2.16. *There are* $\nu < \mu < \kappa^+$ *such that the pair* $\{r_\nu, r_\mu\}$ *is good.*

Proof. Let

$$\vartheta = \sup\{\xi_\ell + \kappa : \ell \in \sigma_2 \cap F\}.$$

Since $\mathcal{F}$ is a κ^+-strongly unbounded function on λ we can find $\nu < \mu < \kappa^+$ such that
for all $i \in F$ we have

$$\mathcal{F}\{\rho(x_{\nu,i}), \rho(x_{\mu,i})\} \geq \vartheta$$

and for all $\{i, j\} \in [F]^2$ with $\delta_i = \delta_j$ and $\mathrm{cf}\,\delta_i = \kappa^+$ we have

$$\mathcal{F}\{\rho(x_{\nu,i}), \rho(x_{\mu,j})\} \geq \vartheta.$$

Hence, $\{r_\nu, r_\mu\}$ is good. □

To finish the proof of Lemma 2.8 we will show that

$$\text{If } \{r_\nu, r_\mu\} \text{ is a good pair, then } r_\nu \text{ and } r_\mu \text{ are compatible.} \tag{$\dagger$}$$

So, assume that $\{r_\nu, r_\mu\}$ is a good pair.

Write $\delta_{x_{\nu,i}} = \delta_{x_{\mu,i}} = \delta_i$.

If $s = x_{\nu,i}$ write $s \in K$ iff $i \in K$. Define $s \in F$, $s \in M$, $s \in D$ similarly.

In order to amalgamate conditions r_ν and r_μ, we will use a refinement of the notion of amalgamation given in [8, Definition 2.4].

Let $A' = \{x_{\nu,i} : i \in F \cup D \cup M\}$. For $x \in (A_\nu \setminus A_\mu) \cup (A_\mu \setminus A_\nu)$ define the *twin* x' of x in a natural way: $x' = h_{\nu,\mu}(x)$ for $x \in A_\nu \setminus A_\mu$, and $x' = h^{-1}_{\nu,\mu}(x)$ for $x \in A_\mu \setminus A_\nu$.

Let $\text{rk} : \langle A', \preceq_\nu \restriction A' \rangle \longrightarrow \theta$ be an order-preserving injective function for some ordinal $\theta < \kappa$, and for $x \in A'$ let

$$\beta_x = \epsilon^{\delta_x}_{\underline{\gamma}(\delta_x)+\text{rk}(x)}.$$

Since $\text{cf}(\gamma(\delta_x)) = \kappa$ and $|A'| < \kappa$ we have

$$\beta_x \in \left(\overline{\text{o}}(\delta_x) \cap [\underline{\gamma}(\delta_x), \gamma(\delta_x))\right) \setminus \sup\{\beta_z : \text{rk}(z) < \text{rk}(x)\}.$$

For $x \in A'$ let

$$y_x = \langle \beta_x, 0 \rangle\,,$$

and put

$$Y = \{y_x : x \in A'\}.$$

So, for every $x \in A'$, $y_x \in B_S$ with $\pi(y_x) < \pi(x)$.

Define the functions $g : Y \longrightarrow A_\nu$ and $\bar{g} : Y \longrightarrow A_\mu$ as follows:

$$g(y_x) = x \text{ and } \bar{g}(y_x) = x',$$

where x' is the "twin" of x in A_μ.

Now, we are ready to start to define the common extension $r = \langle A, \preceq, \text{i} \rangle$ of r_ν and r_μ. First, we define the universe A as

$$A = A_\nu \cup A_\mu \cup Y.$$

Clearly, A satisfies (P1). Now, our purpose is to define $\preceq$.

Extend the definition of g as follows: $g : A \longrightarrow A_\nu$ is a function,

$$g(x) = \begin{cases} x & \text{if } x \in A_\nu, \\ x' & \text{if } x \in A_\mu \setminus A_\nu, \\ s & \text{if } x = y_s \text{ for some } s \in A'. \end{cases}$$

We introduce two relations on $A_p \cup A_q \cup Y$ as follows:

$$\begin{aligned} \preceq^{R1} &= \{\langle y, x \rangle \in Y \times A : g(y) \preceq_\nu g(x)\}, \\ \preceq^{R2} &= \{\langle x, z \rangle \in A \times A : \exists a \in A_\triangle \; g(x) \preceq_\nu a \preceq_\nu g(z)\}. \end{aligned}$$

Then, we put

$$\preceq = \preceq_\nu \cup \preceq_\mu \cup \preceq^{R1} \cup \preceq^{R2}. \tag{$\bigstar$}$$

The following claim is well-known and straightforward.

Claim 2.17. *$\preceq_{\nu,\mu} = \preceq \restriction (A_\nu \cup A_\mu)$ is the partial order on $A_\nu \cup A_\mu$ generated by $\preceq_\nu \cup \preceq_\mu$.*

The following straightforward claim will be used several times in our arguments.

Claim 2.18. *If $x \preceq z$ then $g(x) \preceq_\nu g(z)$.*

Sublemma 2.19. *$\preceq$ is a partial order on $A_\nu \cup A_\mu \cup Y$.*

Proof. We should check that $\preceq_\nu$ is transitive, because it is trivially reflexive and antisymmetric.

So let $s \preceq t \preceq u$. We should show that $s \preceq u$.

Since $x \preceq z$ implies $g(x) \preceq_\nu g(z)$, we have $g(s) \preceq_\nu g(t) \preceq_\nu g(u)$ and so

$$g(s) \preceq_\nu g(u). \tag{$\star$}$$

If $\langle s, u\rangle \in (Y \times A) \cup (A_\nu \times A_\nu) \cup (A_\mu \times A_\mu)$, then ($\star$) implies $s \preceq^{R1} u$ or $s \preceq_\nu u$ or $s \preceq_\mu u$, which implies $s \preceq u$ by ($\bigstar$).

So we can assume that $s \in A_\nu$ (the case $s \in A_\mu$ is similar), and so $u \in Y$ or $u \in A_\mu$.

Case 1. $u \in A_\mu$.

If $t \in A_\nu \cup A_\mu$, then $s \preceq_{\nu,\mu} t \preceq_{\nu,\mu} u$, and so $s \preceq_{\nu,\mu} u$ by Claim 2.17. So $s \preceq u$.

Assume that $t \in Y$. Then $s \preceq^{R2} t$, and so there is $a \in A_\triangle$ such that $g(s) \preceq_\nu a \preceq_\nu g(t)$. Since $t \preceq u$ implies $g(t) \preceq_\nu g(u)$, we have $g(s) \preceq_\nu a \preceq_\nu g(u)$, and so $s \preceq^{R2} u$. Thus $s \preceq u$.

Case 2. $u \in Y$.

If $t \in Y$, then $s \preceq^{R2} t$, and so there is $a \in A_\triangle$ such that $g(s) \preceq_\nu a \preceq_\nu g(t)$. Since $t \preceq u$ implies $g(t) \preceq_\nu g(u)$, we have $g(s) \preceq_\nu a \preceq_\nu g(u)$, and so $s \preceq^{R2} u$. Thus $s \preceq u$.

Assume that $t \in A_\nu \cup A_\mu$. Then $t \preceq^{R2} u$, and so there is $a \in A_\triangle$ such that $g(t) \preceq_\nu a \preceq_\nu g(u)$. Then $g(s) \preceq_\nu a \preceq_\nu g(u)$, and so $s \preceq^{R2} u$. Thus $s \preceq u$. □

So, by the previous Sublemma 2.19 and by the construction, (P2) and (P3) hold for $\preceq$.

Next define the function $\mathrm{i} : [A]^2 \longrightarrow A \cup \{\mathrm{undef}\}$ as follows:

$$\mathrm{i} \supset \mathrm{i}_\nu \cup \mathrm{i}_\mu,$$

and for $\{s, t\} \in [A]^2 \setminus ([A_\nu]^2 \cup [A_\mu]^2)$ such that s and t are $\preceq$-compatible, put $\mathrm{i}\{s, t\} = \mathrm{i}\{s, y_s\} = \mathrm{i}\{t, y_s\} = y_s$ if $s \in A'$ and $t = s'$, and otherwise consider the element

$$v = \mathrm{i}_\nu\{g(s), g(t)\},$$

and let

$$\mathrm{i}\{s,t\} = \begin{cases} v & \text{if } v \in A_\triangle, \\ y_v & \text{if } v \notin A_\triangle. \end{cases}$$

Let

$$\mathrm{i}\{s, t\} = undef$$

if s and t are not $\preceq$-compatible.

If s and t are compatible, then so are $g(s)$ and $g(t)$ because $x \preceq y$ implies $g(x) \preceq_\nu g(y)$ by Claim 2.18. Moreover $\mathrm{i}_\nu\{s,t\} = \mathrm{i}_\mu\{s,t\}$ for $\{s,t\} \in \left[A_\Delta\right]^2$ by condition (C)(e), so the definition above is meaningful, and gives a function i.

Claim 2.20. *If $v \in A_\Delta$ and $s \in A$, then $\pi(v) \in \mathrm{o}^*(g(s))$ iff $\pi(v) \in \mathrm{o}^*(s)$.*

Proof. If $s \in A_\nu \cup A_\mu$ then $g(s) = s$ or $g(s) = s'$, and so $\pi(v) \in \mathrm{o}^*(g(s))$ iff $\pi(v) \in \mathrm{o}^*(s)$ by (C)(b) and (C)(h).

Consider now the case $s = y_x \in Y$. Then $\pi(s) \in E(J(\delta_x)) \cap [\underline{\gamma}(\delta_x), \gamma(\delta_x))$, and so

$$\mathrm{o}^*(s) = \mathrm{o}(\pi(s)) = \bigcup\{E(I) : I \in \mathbb{I}, I^- < \pi(s) < I^+\} \cap \pi(s) = \bigcup\{E(I) : I \in \mathbb{I}, J(\delta_x) \subset I\} \cap \pi(s).$$

We distinguish the following two cases.

Case 1. $\pi(x) < \delta_x$.

If $x \in B_S$ then $\gamma(\delta_x) < \pi(x) < \delta_x$ by (H), and so

$$\mathrm{o}^*(x) \cap \pi(s) = \mathrm{o}(\pi(x)) \cap \pi(s) = \bigcup\{E(I) : J(\delta_x) \subset I\} \cap \pi(s) = \mathrm{o}^*(s).$$

If $x \notin B_S$ then $x \in M$ and $\gamma(\delta_x) < \pi(x) < \delta_x$ by (H), and so

$$\mathrm{o}^*(x) \cap \pi(s) = \overline{\mathrm{o}}(\pi(x)) \cap \pi(s) = (\bigcup\{E(I) : J(\delta_x) \subset I\} \cup E(J(\pi(x)))) \cap \pi(s) = \bigcup\{E(I) : J(\delta_x) \subset I\} \cap \pi(s) = \mathrm{o}^*(s).$$

Case 2. $\pi(x) = \delta_x$.

Then $x \in F$ and so

$$\mathrm{o}^*(x) = \overline{\mathrm{o}}(\pi(x)) = o(x) \cup (E(J(\delta_x)) \cap \delta_x) = \big(\bigcup\{E(I) : I^- < \pi(x) < I^+\} \cup E(J(\delta_x))\big) \cap \pi(x) = \bigcup\{E(I) : J(\delta_x) \subset I\} \cap \pi(x),$$

so $\mathrm{o}^*(s) = \mathrm{o}^*(x) \cap \pi(s)$.

So in both cases $\mathrm{o}^*(s) = \mathrm{o}^*(x) \cap \pi(s)$. Also, note that as $v \in A_\Delta$, we have that $\pi(v) \notin (\underline{\gamma}(\delta_x), \delta_x)$, and hence if $v \in \mathrm{o}^*(g(s))$ then $\pi(v) < \pi(s)$. So, $\pi(v) \in \mathrm{o}^*(x) = \mathrm{o}^*(g(s))$ iff $\pi(v) \in \mathrm{o}^*(s)$. $\square$

Claim 2.21. *If $\{s,t\} \in \left[A\right]^2$, $v \in A_\Delta$ and $\pi(v) \in \mathrm{f}\{g(s), g(t)\}$ then $\pi(v) \in \mathrm{f}\{s,t\}$.*

Proof. We should distinguish two cases.

Case 1. $\mathrm{f}\{g(s), g(t)\} = o^*(g(s)) \cap o^*(g(t))$.

As $\pi(v) \in \mathrm{f}\{g(s), g(t)\}$, we have $\pi(v) \in o^*(g(s)) \cap o^*(g(t))$. Since $\pi(v) \in \mathrm{o}^*(g(s))$ implies $\pi(v) \in \mathrm{o}^*(s)$ and $\pi(v) \in \mathrm{o}^*(g(t))$ implies $\pi(v) \in \mathrm{o}^*(t)$ by Claim 2.20, we have $\pi(v) \in o^*(s) \cap o^*(t) = \mathrm{f}\{s,t\}$.

Case 2. $\mathrm{f}\{g(s), g(t)\} = o(g(s)) \cup \big\{\epsilon_\zeta^{\pi(g(s))} : \zeta < F\{\rho(g(s)), \rho(g(t))\}\big\}$.

So $\pi_B(g(s)) = \pi_B(g(t)) \neq S$ and $cf(\pi(g(s))) = \kappa^+$. We can assume that $s \in A_\nu \setminus A_\mu$ and $t \in A_\mu \setminus A_\nu$. If $g(s) \in M$, then $g(t) \in M$ by (E). Then as $[\gamma(\delta_{g(s)}), J(\delta_{g(s)})^+) \cap \pi'' A_\Delta = \emptyset$, we infer that $\pi(v) \in o(g(s)) = o(g(t))$, and thus $\pi(v) \in o(s) \cap o(t) \subset f\{s,t\}$. Now assume that $g(s), g(t) \in F$. So $s, t \in F$, and $\delta' = \delta_{g(s)} = \delta_{g(t)}$ has cofinality κ^+. So,

$$\pi(v) \in \mathrm{f}\{g(s), g(t)\} = \mathrm{o}(\delta') \cup \{\epsilon_\zeta^{\delta'} : \zeta < F\{\rho(g(s)), \rho(g(t))\}\}. \qquad (\triangle)$$

Since $\pi'' A_\Delta \cap (\gamma(\delta'), \delta') = \emptyset$, $(\triangle)$ implies

$$\pi(v) \in \overline{\mathrm{o}}(\delta') \cap \gamma(\delta').$$

But, by ($\blacktriangle$)

$$\overline{\mathrm{o}}(\delta') \cap \gamma(\delta') \subset \mathrm{f}\{s,t\},$$

and so $\pi(v) \in \mathrm{f}\{s,t\}$. □

Sublemma 2.22. $\langle A, \preceq, \mathrm{i}\rangle$ *satisfies (P4) and (P5).*

Proof. Let $\{s,t\} \in [A]^2$ be a pair of $\preceq$-incomparable and $\preceq$-compatible elements. We distinguish the following cases.

Case 1. $\{s,t\} \in [A_\nu]^2$. (The case $\{s,t\} \in [A_\mu]^2$ is similar)

Since $\preceq_\nu \subset \preceq$, we have $\mathrm{i}_\nu\{s,t\} \preceq s,t$, so to check (P4) we should show that $x \preceq s,t$ implies $x \preceq \mathrm{i}_\nu\{s,t\}$. We can assume that $x \notin A_\nu$.

If $x \in Y$, then $x \preceq^{R1} s$ and $x \preceq^{R1} t$, i.e. $g(x) \preceq_\nu g(s), g(t)$ and so $g(x) \preceq_\nu \mathrm{i}_\nu\{g(s), g(t)\} = \mathrm{i}_\nu\{s,t\} = g(\mathrm{i}_\nu\{s,t\})$, and so $x \preceq^{R1} \mathrm{i}_\nu\{s,t\}$. Thus $x \preceq \mathrm{i}_\nu\{s,t\}$.

If $x \in A_\mu \setminus A_\nu$, then $x \preceq^{R2} s$ and $x \preceq^{R2} t$, i.e. $g(x) \preceq_\nu a \preceq_\nu g(s)$ and $g(x) \preceq_\nu b \preceq_\nu g(t)$ for some $a, b \in A_\Delta$. Then $c = \mathrm{i}_\nu\{a,b\} \in A_\Delta$, and so $g(x) \preceq_\nu c \preceq_\nu \mathrm{i}_\nu\{g(s), g(t)\} = \mathrm{i}_\nu\{s,t\} = g(\mathrm{i}_\nu\{s,t\})$, and so $x \preceq^{R2} \mathrm{i}_\nu\{s,t\}$. Thus $x \preceq \mathrm{i}_\nu\{s,t\}$.

Finally (P5) holds in Case 1 because r_ν satisfies (P5).

Case 2. $\{s,t\} \notin [A_\nu]^2 \cup [A_\mu]^2$.

To check (P4) we should prove that $\mathrm{i}\{s,t\}$ is the greatest common lower bound of s and t in $\langle A, \preceq\rangle$.

Assume first that s and t are not twins. Note that by Claim 2.18, $g(s)$ and $g(t)$ are $\preceq_\nu$-compatible. Write $v = \mathrm{i}_\nu\{g(s), g(t)\}$.

Case 2.1. $v \in A_\Delta$, and so $\mathrm{i}\{s,t\} = v$.

Since $v = g(v) \preceq_\nu g(s)$ and $v \in A_\Delta$, we have $v \preceq^{R2} s$. Similarly $v \preceq^{R2} t$. Thus v is a common lower bound of s and t.

To check that v is the greatest lower bound of s,t in $\langle A, \preceq\rangle$ let $w \in A$, $w \preceq s,t$. Then $g(w) \preceq_\nu g(s), g(t)$. Thus $g(w) \preceq_\nu \mathrm{i}_\nu\{g(s), g(t)\} = v$.

Since $v \in A_\Delta$, $g(w) \preceq_\nu v$ implies $w \preceq^{R2} v$. Thus $w \preceq v$. Thus (P4) holds.

To check (P5) observe that $g(s)$ and $g(t)$ are incomparable in A_ν. Indeed, $g(s) \preceq_\nu g(t)$ implies $v = g(s) \in A_\Delta$ and so $g(s) \preceq_\nu g(t)$ implies $s \preceq^{R2} t$, which contradicts our assumption that s and t are $\preceq$-incomparable.

Thus, by applying (P5) in r_ν,

$$\pi(v) \in \mathrm{f}\{g(s), g(t)\}.$$

Thus $\pi(v) \in \mathrm{f}\{s,t\}$ by Claim 2.21, and so (P5) holds.

Case 2.2. $v \notin A_\vartriangle$, and so $\mathrm{i}\{s,t\} = y_v$.

First, we show that $\delta_v = \delta_{g(s)} = \delta_{g(t)}$. Note that if $g(s)$ and $g(t)$ are $\preceq_\nu$-comparable, then $v = g(s)$ or $v = g(t)$, and we have that $\delta_{g(s)} = \delta_{g(t)}$, because otherwise we would infer from Claim 2.12 that s,t are $\preceq$-comparable, which is impossible.

Now assume that $g(s)$ and $g(t)$ are $\preceq_\nu$-incomparable.

If $\delta_v < \delta_{g(s)}$, then there is $a \in A_\vartriangle \cap B_S$ with $v \preceq_\nu a \preceq_\nu g(s)$ by Claim 2.12. Thus $v = \mathrm{i}_\nu\{a, g(t)\}$ and so $v \in A_\vartriangle$ by Claim 2.13, which is impossible. Thus $\delta_v = \delta_{g(s)}$, and similarly $\delta_v = \delta_{g(t)}$. Hence

$$\delta_{g(s)} = \delta_{g(t)} = \delta_v.$$

And we have

$$\pi(y_v) \in E(J(\delta_v)) \cap [\underline{\gamma}(\delta_v), \gamma(\delta_v)).$$

Then, if $s,t \in F$ and $\mathrm{cf}(\delta_v) = \kappa^+$, by condition (▲), we deduce that $E(J(\delta_v)) \cap \gamma(\delta_v) \subset f\{s,t\}$, and so as $\pi(y_v) < \gamma(\delta_v)$, we have $\pi(y_v) \in f\{s,t\}$. Otherwise,

$$E(J(\delta_v)) \cap \min(\pi(s), \pi(t)) \subset \mathrm{f}\{s,t\}.$$

Then as $v = \mathrm{i}_\nu\{g(s), g(t)\}$, we have $\pi(v) < \pi(g(s)), \pi(g(t))$, hence $\pi(y_v) < \pi(s), \pi(t)$ and thus $\pi(y_v) \in \mathrm{f}\{s,t\}$.

Thus (P5) holds.

To check (P4) first we show that $y_v \preceq s,t$. Indeed $g(v) \preceq_\nu g(s)$ implies $y_v \preceq^{R1} s$. We obtain $y_v \preceq^{R1} t$ similarly.

Let $w \preceq s,t$.

Assume first that $\delta_{g(w)} < \delta_v$. Since $w \preceq s,t$ we have $g(w) \preceq_\nu g(s), g(t)$ by Claim 2.18 and hence $g(w) \preceq_\nu \mathrm{i}_\nu\{g(s), g(t)\} = v$. By Claim 2.12 there is $a \in A_\vartriangle$ such that $g(w) \preceq_\nu a \preceq_\nu v$. Thus $w \preceq^{R2} y_v$.

Assume now that $\delta_{g(w)} = \delta_v$.

Then, we have that $w \in Y$. To check this fact, assume on the contrary that $w \in A_\nu \cup A_\mu$. So, we have $\delta_w = \delta_{g(w)} = \delta_v = \delta_{g(s)} = \delta_{g(t)}$. Note that if $s \in Y$, then $\pi(s) \in [\underline{\gamma}(\delta_w), \gamma(\delta_w))$, which contradicts the assumption that $w \preceq s$. So $s \notin Y$, and analogously $t \notin Y$.

Assume that $w \in A_\nu$. If $s \in A_\mu$, as $w \preceq s$ there is $b \in A_\vartriangle$ such that $w \preceq b \preceq s$, which is impossible because $\pi(w) > \gamma(\delta_w) = \gamma(\delta_s)$ and $[\gamma(\delta_s), J(\delta_s)^+) \cap \pi'' A_\vartriangle = \emptyset$. Thus $s \notin A_\mu$. And by means of a parallel argument, we can show that $t \notin A_\mu$. So $s,t \in A_\nu$, which was excluded. Analogously, $w \in A_\mu$ implies $s,t \in A_\mu$.

Therefore, $w = y_z$ for some $z \in A'$. Then $z \preceq_\nu g(s)$ and $z \preceq_\nu g(t)$, and so $z \preceq_\nu \mathrm{i}_\nu\{g(s), g(t)\} = v$. Thus $y_z \preceq^{R1} y_v$.

Now, assume that s and t are twins. So $t = s'$ and $i\{s,s'\} = y_s$. If $s \in F$ and $cf(\pi(s)) = \kappa^+$, we have that $\pi(y_s) \in \overline{o}(\delta_s) \cap \gamma(\delta_s) \subset f\{s,s'\}$ by (▼). Otherwise, $\pi(y_s) \in o^*(\pi(s)) \cap o^*(\pi(s')) = f\{s,s'\}$. Thus (P5) holds. To check (P4), it is clear that $y_s \prec s,s'$. So, assume that $w \prec s,s'$. If $w = y_u \in Y$, then as $w \prec s$ we infer that $u \preceq s$, and thus $w \preceq y_s$. Now, suppose that $w \in A_\nu \cup A_\mu$. Then, there is $b \in A_\vartriangle$ such that either $w \preceq b \preceq s$ or $w \preceq b \preceq s'$. In both cases, we have $w \preceq y_s$.

So we proved Sublemma 2.22. □

Sublemma 2.23. $\langle A, \preceq, \mathrm{i}\rangle$ *satisfies* $(P6)$.

Proof. Assume that $\{s,t\} \in [A]^2$, $s \preceq t$ and Λ separates s from t, i.e.,

$$\Lambda^- < \pi(s) < \Lambda^+ < \pi(t).$$

We should find $v \in A$ such that $s \preceq v \preceq t$ and $\pi(v) = \Lambda^+$.

Note that since $s \preceq t$, we have $\delta_{g(s)} \leq \delta_{g(t)}$ by Claim 2.11.

We can assume that $\{s,t\} \notin [A_\nu]^2 \cup [A_\mu]^2$ because r_ν and r_μ satisfy (P6).

We distinguish the following cases.

Case 1. $\delta_{g(s)} < \delta_{g(t)}$.

As $g(s) \preceq_\nu g(t)$, there is $a \in A_\vartriangle \cap B_S$ with $g(s) \preceq_\nu a \preceq_\nu g(t)$ by Claim 2.12.

Case 1.1. $\pi(a) \in \Lambda$.

Thus Λ separates a from $g(t)$.

Applying (P6) in r_ν for a and $g(t)$ and Λ we obtain $b \in A_\nu$ such that $a \preceq_\nu b \preceq_\nu g(t)$ and $\pi(b) = \Lambda^+$.

Note that as $\pi(a) \in \Lambda, a \in A_\vartriangle$ and $\pi(b) = \Lambda^+$, we have that $\pi(b) \in Z$. Thus $b \in A_\vartriangle$ by (H).

Thus $g(s) \preceq_\nu b \preceq_\nu g(t)$ implies $s \preceq^{R2} b \preceq^{R2} t$, and so $s \preceq b \preceq t$.

Case 1.2. $\pi(a) \notin \Lambda$.

If $\Lambda^+ = \pi(a)$, then we are done because $g(s) \preceq_\nu a \preceq_\nu g(t)$ implies $s \preceq a \preceq t$.

So we can assume that $\Lambda^+ < \pi(a)$.

Since r_ν and r_μ satisfy $(P6)$ and Λ separates s from a, we can assume that $s \notin A_\nu \cup A_\mu$.

Hence $s = y_{g(s)}$ and Λ separates $g(s)$ from a because $\pi(s) \in J(\delta_{g(s)}) \subset \Lambda$. (If $\Lambda \subsetneq J(\delta_{g(s)})$, then $\Lambda^- < \pi(s) < \Lambda^+$ is not possible.)

Thus there is $b \in A_\nu$ such that $g(s) \preceq_\nu b \preceq_\nu a$ and $\pi(b) = \Lambda^+$.

Since $\delta_{g(s)} \in Z_0$, we have $\pi(b) \in Z$, and so $b \in A_\vartriangle$ by (H).

Thus $s = y_{g(s)} \preceq^{R1} b \preceq^{R2} t$, and so $s \preceq b \preceq t$.

Case 2. $\delta_{g(s)} = \delta_{g(t)}$.

We will see that this case is not possible.

Case 2.1. $s \in A_\nu$.

Note that if $t \in A_\mu$, then since $s \preceq t$ there is $b \in A_\vartriangle$ such that $s \preceq b \preceq t$, which is impossible because $\pi(s) > \gamma(\delta_s)$ and $[\gamma(\delta_s), J(\delta_s)^+) \cap \pi'' A_\vartriangle = \emptyset$. Thus $t \notin A_\mu$.

Since $s \in A_\nu$, $s \preceq t$ and $\delta_s = \delta_{g(t)}$ we have $t \notin Y$, and so $t \in A_\nu$, which was excluded.

By means of a similar argument, we can show that $s \in A_\mu$ is also impossible.

Case 2.2. $s = y_{g(s)}$.

Then $\pi(s) \in E(J(\delta_{g(s)}))$ and so $\Lambda^- < \pi(s) < \Lambda^+$ implies $J(\delta_{g(s)}) \subset \Lambda$. But then $\pi(t) \leq \Lambda^+$, so Λ can not separate s from t.

Thus (P6) holds.

So we proved Sublemma 2.23. □

Thus we proved that r is a common extension of r_ν and r_μ.

This completes the proof of Lemma 2.8, i.e. $\mathcal{P}$ satisfies κ^+-c.c. □

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