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LOCAL CONNECTIVITY OF JULIA SETS

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Abstract

Complex dynamics is a part of Mathematics that did not start to shine until the arrival of prominent figures like Koenigs, Fatou and Julia. In particular, one of the most innovative ideas was the Julia set of a given function f. The particular shape and characteristics of these sets do not leave any mathematician indifferent, and a useful way to try to understand them is to study their topology. We aim to determine which Julia sets are locally connected, considering the relations that this topological property has with major questions of complex dynamics, as for example the MLC conjecture. In this thesis we will focus on hyperbolic rational maps, the maps that have the simplest dynamics, and which are conjectured to be dense among all rational maps (HD conjecture). The goal is to prove the following theorem: hyperbolic rational maps of degree larger than 1 with a connected Julia set have a locally connected Julia set.

To do so, we first present the preliminaries on different aspects of Mathematics, such as hyperbolic geometry, Montel's theory and, of course, complex dynamics. It is followed by the proof of Carathéodory's theorem, which gives a crucial criterion about which sets are locally connected. Finally, the last chapter is dedicated to the proof of the main theorem.

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Contents

Introduction

The goal of this thesis is to prove the following statement: Let R be a **hyperbolic** rational map of degree greater than or equal to 2. Then, if its **Julia set** $\mathcal{J}(R)$ is **connected**, it must be **locally connected**. In order to understand the significance and importance of the previous theorem, I will first introduce some background in complex dynamics.
Holomoprhic dynamics is an area of Mathematics whose development started with the arrival of mathematicians like Koenigs, Julia or Fatou. The foundation is considered to have been brought by **Ernst Schröder** (1841-1902) [AL]. Although his main fields of study were set theory and logic, he began to be interested in complex iteration when studying Newton's method, an algorithm to find the roots of a given function f using iteration, given by

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}$$

The German mathematician gave his particular point of view focusing on neighbourhoods in the complex plane instead of isolated points in $\mathbb{R}$. He managed to arrive at the following conclusion:

Theorem 0.1 (Scröder's Fixed Point Theorem)**.** *Let $f \in \mathcal{H}(D)$ be an analytic function in a neighbourhood D of a fixed point w (that is, a point such that $f(w) = w$) with $|f'(w)| < 1$. Then, for all points $z \in D' \subseteq D$ of a possibly smaller neighbourhood of w, D',*

$$lim_{n\to\infty} f^{\circ n}(z) = w$$

In other words, he discovered what we call now an **attracting fixed point** of f.
Schröder tried to explicitly calculate forward iterates of complex functions so that he could understand better their behaviour. In the process, he realized the usefulness of **conjugacies**. Two functions f, g are said to be (conformally) conjugate on a domain D if there exists a (conformal) homeomorphism $\varphi \in \mathcal{H}(\mathcal{D})$ such that:

$$\varphi \circ f(z) = g \circ \varphi(z)$$

Notice that φ is a conjugacy also between the iterates of f since

$$f^{\circ n}(z) = \varphi^{-1} \circ g^{\circ n} \circ \varphi(z)$$

This implies, for example, that the orbits of f map to orbits of g under φ. Hence, if g is "much simpler" than f, the relation can provide a lot of new information on the dynamics of f. He was particularly interested in conjugacies to linear or affine maps like $g(z) = \lambda z$ or $g(z) = z + \lambda$, giving us the Schröder and Abel functional equations:

$$\varphi(f(z)) = \lambda\varphi(z), \text{ or} \quad (1)$$
$$\varphi(f(z)) = \varphi(z) + \lambda. \quad (2)$$

If φ is a diffeomorphism we can differentiate both sides of the Schröder functional equation (1) obtaining

$$\varphi'(w)f'(w) = \lambda\varphi'(w) \to f'(w) = \lambda$$

At this point, complex dynamics moved from Germany to France with **Gabriel Koenigs** (1858-1931). He was considered the most important mathematician related to complex iteration in the XIX century. Collaborating with **Gaston Darboux** (1842-1917), Koenigs used Darboux's results

of uniform convergence of real functions with complex ones. With his tools and resources, he studied the forward orbit of an initial condition z_0, that is, the set of points $\{z_0, f(z_0), f^{\circ 2}(z_0), ...\}$ and he noticed that certain subsequences could converge to a periodic fixed point, that is, a point w such that $f^p(w) = w$ for a minimal integer $p \geq 2$. Moreover, he proved the converse of the fixed point theorem, stating that if the sequence $\{f^{\circ n}(z)\}_{n \geq 0}$ remained in a domain D, converging to a point w, then $|f'(w)| \leq 1$.
Koenigs' influence was remarkable, as two of his pupils studied the cases of $|f'(w)| = 0$ (Grévy) and $|f'(w)| = e^{2\pi i p/q}$ (Leau and the Parabolic Flower Theorem).
Despite his success, Koenigs had some questions that he was not able to solve, as he had not the necessary tools (set theory, topology and Montel's theory) to do that. He suspected that there would exist a partition of the Riemann Sphere (appendix A.2) depending on the behaviour of the iterates. The problem arose when he thought there could be an infinite number of components of the partition, as the equation $f^n(z) = z$ has infinite solutions when n goes to infinity. With the new techniques of the forecoming years, this problem would disappear.

Fortunately, a new generation of brilliant French mathematicians, inspired by the ideas of **Camille Jordan** (1838-1922), appeared. Some of them were **Émile Borel** (1871-1965), **René-Louis Baire** (1874-1932) and **Henri Lebesgue** (1875-1941). This last one guided one of the pioneers of the complex dynamics: **Pierre Fatou**. Fatou (1878-1929) benefited from Lebesgue's ideas and he studied the set that separated the different components (which we will denote $\mathcal{J}$) that Koenigs predicted. He observed that it could come in very different ways. For example, a set is called totally disconnected if its only connected components are trivial, and this set is, moreover, perfect if every limit point belongs to the set. Then he proved the next result.

Theorem 0.2. *Let $R(z) = P(z)/Q(z)$ be a rational map of degree $max(deg(P(z), deg(Q(z)) > 1$ with only an attracting fixed point. Besides, suppose that the critical values (the points c with $R(w) = c$ and $R'(w) = 0$) converge to the fixed point. Then, $\mathcal{J}$ must be totally disconnected and perfect.*

He also saw that $\mathcal{J}$ could be a continuous curve that is non-differentiable almost everywhere.
Nevertheless, the formal definition and the exceptional properties of $\mathcal{J}$ would not have been possible without the contribution of **Paul Montel** (1876-1975). He was Lebesgue's pupil and he brought along a theory of normal families.

A family of holomorphic functions $\mathcal{F}$ is said to be normal in a domain D if every sequence of maps in $\mathcal{F}$ contains a subsequence which converges locally uniformly on D to an analytic function or diverges locally uniformly. Montel's best known contribution was to prove that every family of rational functions in a domain D that omits three different values is normal (see Chapter 2). Although being the creator of this theory, Fatou and our final main character, **Gaston Julia** (1893, 1978) would be the ones that actually used it in complex dynamics.

Julia was a prominent genius. The Algerian published his first paper in 1913, whilst in 1915 he faced a German attack, proving his bravery and gaining the respect of selected mathematicians as Picard or Humbert. His popularity arose in 1918, with **The Grand Prix des Sciences mathmétmatiques**, a prize awarded by the French Academy. That year, following the tendencies of the moment, the winner would be the one with the best results focused on **global** theory of complex iteration, instead of the local point of view. Julia won that prize, followed by **Samuel Lattès** (1873-1918).

Fatou did not win the prize but he did get some recognition. As it happened before in the history of science, he reached lots of results that Julia proved too at the same time. A huge difference between both works is that Fatou's investigation was based on functional equations whilst Julia's research (more rigorous and extensive) was focused purely on iteration.
Now we can formally define what this set $\mathcal{J}$ is. The Julia set, $\mathcal{J}$, of a rational map R is the set of points such that the family of iterates $\{R^{\circ n}\}_{n\geq 0}$ is not normal in any neighbourhood of the points. The Fatou set is its complement. Further properties will be explained in Chapter 3 but the definition and properties of the Julia set have always attracted the interest of many mathematicians, and in particular their intricate fractal nature.

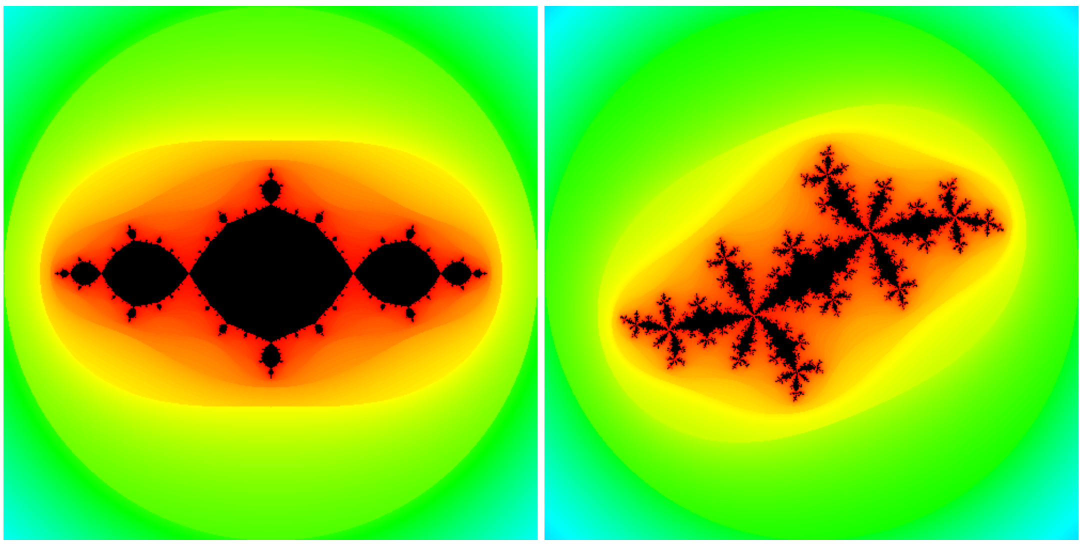

Figure 1: The Julia set (boundary of the black part) of two quadratic polynomials

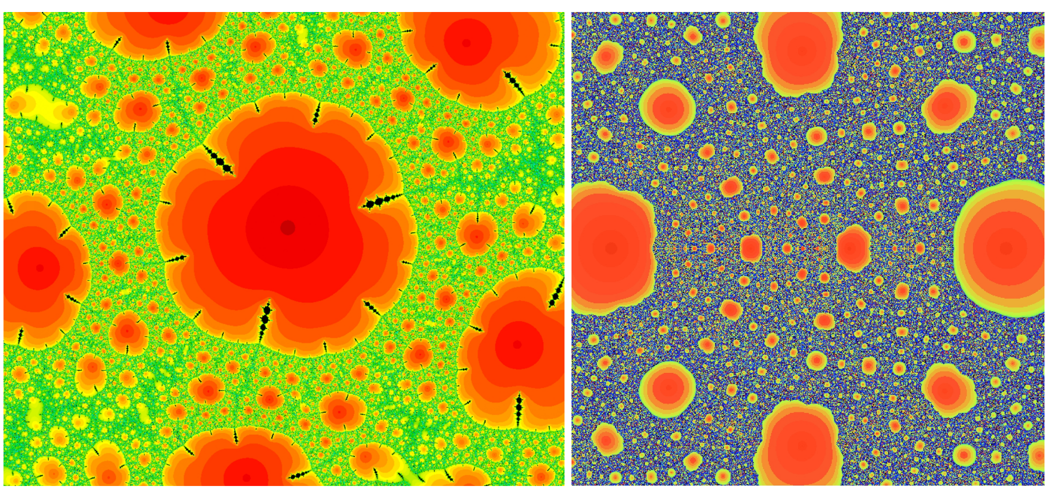

Figure 2: Julia sets homeomoprhic to Sierpinski Carpets

The topological properties of Julia sets are diverse and interesting. Because of the dynamical nature of the definition of these sets, we find outstanding connections between the dynamics of the map and the topology of its Julia set. One of the topological properties most discussed and still not completely understood is local connectivity. A set X is connected if there does not exist any two disjoint open subsets U, V such $X = U \cup V$. Thus, a set X is locally connected if for any point $z \in X$ and for any neighbourhood U_z of the point, there exists another neighbourhood N_z with $z \in N_z \subset U_z$ such that $N_z \cap X$ is connected.

Local connectivity is particularly important in complex dynamics. For example, we define the **Mandelbrot set** $\mathcal{M} = [c \in \mathbb{C},\ \mathcal{J}(z^2 + c)\ \textit{is connected}]$. One of the most important conjectures in complex dynamics states that the Mandelbrot set is locally connected (MLC conjecture). The reason of its relevance is the connection it has to many other aspects of dynamics, as we explain below.

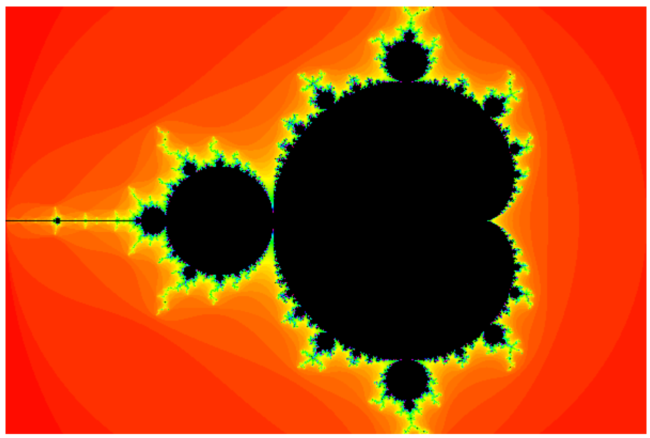

Figure 3: The Mandelbrot set (in black)

To understand the power of local connectivity, we will prove in Chapter 4 the well-known Carathéodory's theorem. Using it, if $\mathcal{M}$ was locally connected, then there would exist a continuous map $\phi : \partial\mathbb{D} \to \partial M$, showing that, despite the fractality of its boundary (having Hausdorff dimension 2) it is nevertheless a continuous curve.

Julia sets, on the other hand, may or may not be locally connected. It would be interesting to have a criterion to determine whether a given Julia set is locally connected or not, but this question is too general to be answered. Even so, there are some classes of maps for which local connectivity is known to hold. The largest such set of functions is the class of **hyperbolic maps**. Generally, in dynamical systems theory, hyperbolic dynamics are important because of the presence of what we call stable and unstable subspaces, whose points have either a contracting or an expanding behaviour. An introductory example in linear maps could be a 2×2 diagonalizable matrix with eigenvalues $0 < a_1 < 1 < a_2$. Hyperbolic rational maps represent the (most) well-behaved rational maps in complex iteration. A rational map R is hyperbolic (in the sense of complex dynamics) if R is expanding on $\mathcal{J}$, that is to say, for any $z \in \mathcal{J}$, there exists a conformal metric such that for any vector v in the tangent space at z,

$$||DR_z(v)|| > ||v||.$$

As it turns out, this expansive property is equivalent to having very simple dynamics (see Chapter 5): a rational map is hyperbolic if and only if its critical points tend to attracting periodic

orbits. Using some results proved in Chapter 3, all this means that any orbit in the Fatou set must converge towards an attracting periodic orbit.
Attracting periodic orbits are the stable equilibria of the system, and they stay stable under light perturbations. Therefore, hyperbolic maps are robust, which makes them desirable in modelling.
Another conjecture in the field is the HD conjecture which states that the set of hyperbolic rational maps is dense among rational maps (that is, every rational map can be approximated by hyperbolic rational ones). Moreover, there is also the HD2 conjecture, which says that every quadratic polynomial can be approximated by hyperbolic quadratic polynomials. In fact, if the MLC conjecture turns out to be true, the HD2 conjecture will be instantly proved [MCM].

The main goal of this thesis is to prove the following theorem.

Theorem 0.3 (Main theorem)**.** *Let $R(z) = P(z)/Q(z)$ with $P(z), Q(z)$ polynomials be a hyperbolic rational map (of degree greater than 1). Then, if its Julia set is connected, it must be locally connected.*

To reach this goal, the first chapter introduces hyperbolic geometry and the Schwarz-Pick lemma, an important result that relates hyperbolic metric and holomorphic functions. The second chapter is a quick summary of Riemann surfaces to understand Montel's theorem, previously cited. The third one is the longest. It gives a general view of dynamics, going through the different types of fixed points and orbits, and characterising properties of Julia sets and Fatou sets. The fourth chapter is focused on proving Carathéodory's theorem, which gives crucial information about maps that can be extended onto the boundary of the domains. The final chapter contains the proof of the main theorem. This thesis also includes two appendixes for readers without any background in complex analysis, starting with the definition of a holomorphic function and ending with the Riemann Mapping Theorem. Some background of topology and manifolds is also given in a short appendix.

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Chapter 1

Hyperbolic geometry and Schwarz-Pick lemma

1.1 Basics of metric spaces

The goal of this chapter is to introduce the hyperbolic metric and to prove the Schwarz-Pick lemma, which states that all holomorphic maps are contractions with respect to this metric. First we start with some preliminaries on metric spaces, and before applying these concepts to the hyperbolic metric, we will give examples in the Euclidean metric. The main references for this chapter are [KL] and [BM].

Definition 1.1 (Metric)**.** *Let X be a topological space. A metric in X is a function $d : X \times X \to \mathbb{R}$ such that for $x, y, z \in X$ we have*

$$\begin{aligned} d(x,y) &> 0 \; if \; x \neq y \\ d(x,x) &= 0 \\ d(x,y) &= d(y,x) \\ d(x,z) &\leq d(x,y) + d(y,z) \end{aligned}$$

The Euclidean metric in the complex plane is the distance we are used to and we denote it by for every pair of points $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, we have

$$d(z_1, z_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In geometry, maps that preserve distances are called **isometries**.

Definition 1.2 (Isometry)**.** *Let X, Y be topological spaces and d_X, d_Y the corresponding metrics. An isometry is a map $f : X \to Y$ such that for any $x_1, x_2 \in X$*

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$$

Intuitively, it is easy to see that rotations or translations are isometries with respect to the Euclidean metric, but a scaling map (a function of the form $f(z) = cz$ for $|c| \neq 1$) is not.
We are also interested in maps that preserve angles.

Definition 1.3 (Angle between curves)**.** *Let α, β be two differentiable curves on complex plane that intersect at a point z. The angle between α and β at z is the angle between the two tangent lines at this curves.*

This is expressed by the formula

$$cos(\theta) = \frac{\alpha'(z) \cdot \beta'(z)}{||\alpha'(z)|| \cdot ||\beta'(z)||}$$

where $|| \cdot ||$ denotes the Euclidean norm

Definition 1.4 (Conformal map)**.** *A (real) differentiable map $f : D \to \mathbb{C}$ that preserves angles at a point z where D is a neighbourhood of z is called **conformal** (at z). A diffeomorphism f from D to $\mathbb{C}$ is called conformal if it is conformal at every point of D.*

With these concepts in mind we are going to give a little twist to measuring curves and distances. It can happen that our domains are not what we call **convex**, which means that there are two points that cannot be joined by a segment inside the domain. In that case, what would the distance be? This is when we investigate on the possible paths to join one point with the other.

Imagine our domain as a grid-like set. The objective is to choose the set of points that, joining them together, form a curve that connects our two initial points, with minimal length. At this point, the infinitesimal idea arises. We extend the concept of distance between two points to neighbourhoods that are extremely small (infinitesimal) so, in the limit, the metric d induces what we call a **density** $d(t)|dt|$ defined at each point of the set. Now our new concept of the length of a curve is, simply, the integral of the density along the curve.

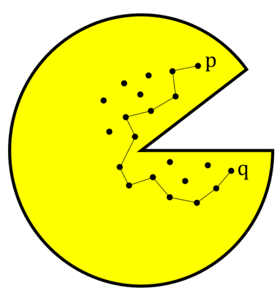


Figure 1.1: Distances in a non-convex space

Definition 1.5 (Length of a path)**.** *Set ψ a path from p to q in U. Then we define the length of the path as*

$$d(\psi) = \int_{\psi} d(t)|dt|$$

where $d(t)$ is the density of the corresponding metric.

Under these conditions, $d(p, q) = inf_{\psi} d(\psi)$, the infimimum of the $d(\psi)$ where ψ is every possible path from p to q.
In the case of the Euclidean distance d, we want it to be invariant under translations, so $d(t) = d(t + t_0)$ for any t, t_0, which induces $d(t)$ to be constant.

Another important concept to finally construct hyperbolic metric is the metrics that are inherited by the functions. Set $f : U \to V$ a conformal isomorphism, $\alpha(t)$ a curve in U and $\widetilde{\alpha}(t) := f(\alpha(t))$ a curve in V. Then, if $d(t)$ is the density of the corresponding metric in U,

$$\int_{\widetilde{\alpha}} |dt| = \int_{\alpha} d(f(t))|f'(t)||dt| \tag{1.1}$$

1.2 Constructing the hyperbolic metric

Now we are in a position to begin to talk about hyperbolic geometry. First, some historical background. When the well-known Euclid released his work: **The Elements of Euclid**, he built the whole geometry from five postulates:

1. Two points are joined by a unique segment
2. A segment can be extended up to infinity
3. Any circumference is defined by its center and its radius
4. All right angles are equal among them
5. Given a line and an exterior point, we can draw a parallel line through the aforementioned point

The last postulate has intrigued a lot of mathematicians deeply, as it could not be proved using the other 4 postulates. What would happen if we consider it to be false? With this question in mind, **Riemann** gave birth to the non-Euclidean geometry.

We are going to focus on the unit disk $\mathbb{D}$ of the complex plane. We have seen that the distance can be given by the density function $\rho(t)|dt|$ in the unit disk with respect to this metric.

Definition 1.6. *As in the case of Euclidean geometry, we define the ρ-length of ψ as*

$$\rho(\psi) = \int_{\psi} \rho(t)|dt|$$

In the Euclidean metric, straight lines are the shortest paths from one point to another. Formally, we say that the shortest path between two points is a **geodesic**. We are going to adapt the definition to the hyperbolic metric.

Definition 1.7 (Geodesic)**.** *A hyperbolic **geodesic** ψ is a curve such that for every tuple $t_1 < t_2 < t_3$ with $\psi(t_i) = p_i$ for $i \in 1 \leq i \leq 3$, we have*

$$\rho(p_1, p_3) = \rho(p_1, p_2) + \rho(p_2, p_3)$$

In this case, we want the distance to be invariant under **conformal homeomorphisms**, that is to say, under all Möbius transformations from $\mathbb{D}$ to itself

$$\varphi_{-a}(z) := \frac{z + a}{1 + \bar{a}z}$$

Recalling (1.1), we want

$$\rho(\varphi_{-a}(t))|\varphi'_{-a}(t)| = \rho(t)$$

Applying $\varphi_{-a}(0) = a$ and setting $\rho(0) = 1$ (it depends on the author, sometimes this value is 2, it is strongly related to the geometry on the upper-half plane), we obtain

$$\rho(a) = \frac{1}{1 - |a|^2}$$

Now let us suppose we want to find the distance between 0 and p. Let ψ be the (straight) segment that joins them, $\psi(t) = tp,\ t \in [0,1]$. Then,

$$\rho(\psi) = \int_\psi \rho(t)|dt| = \int_0^1 \frac{|p|dt}{1 - (t|p|)^2}$$

which gives us

$$\rho(\psi) = \frac{1}{2}log(\frac{1 + |p|}{1 - |p|})$$

Now we are going to check if this is a geodesic. First, we will see that it minimizes distances. Set α another path. A typical tool to compare distances is to split the segment into a partition. Set $0 = t_1 < t_2 < ... < t_{n-1} < t_n = 1$. So, estimating the integral,

$$Est_{(\alpha,n)} = \sum_{i=1}^{n-1} \rho(\alpha(t_i))|\alpha(t_{i+1}) - \alpha(t_i)|$$

Therefore, in order to compare it with our initial guess, we are going to project each $\alpha(t_i)$ onto the straight line ψ. Our first try would be to proceed with an orthogonal projection, but we are going to take advantage of the invariance under rotations of ρ, and we are going to project them radially. This is well defined: draw the circle of center 0 and radius $|\alpha(t_i)|$. There is only one intersection between the circumference and the segment ψ.

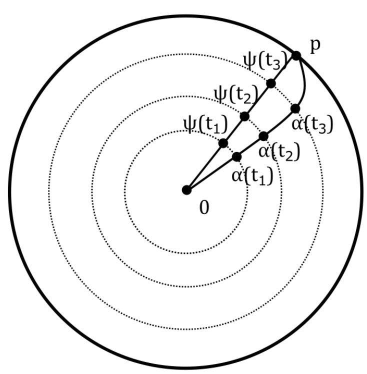


Figure 1.2: Representation of radial projections

Now we have another estimation,

$$Est_{(\psi,n)} = \sum_{i=1}^{n-1} \rho(\psi(t_i))|\psi(t_{i+1}) - \psi(t_i)| = \sum_{i=1}^{n-1} \rho(\alpha(t_i))|\psi(t_{i+1}) - \psi(t_i)|$$

Following Euclidean geometry,

$$|\psi(t_{i+1}) - \psi(t_i)| \leq |\alpha(t_{i+1}) - \alpha(t_i)|$$

So,

$$\rho(\psi) = lim_{n\to\infty} Est_{(\psi,n)} \leq lim_{n\to\infty} Est_{(\alpha,n)} = \rho(\alpha)$$

We have just proved our statement, but only if one of the two points is the origin. What would happen if we had two arbitrary points? Indeed, let $p, q \in \mathbb{D}$. Consider

$$\varphi_p(z) = \frac{z-p}{1-\overline{p}z}$$

We have $\varphi_p(p) = 0$ and $\varphi_p(q) = w$. So,

$$\rho(p,q) = \rho(0,w) = \frac{1}{2}log(\frac{1+|w|}{1-|w|}) = \frac{1}{2}log(\frac{1+|\varphi_p(q)|}{1-|\varphi_p(q)|})$$

And the previous equality gives us

$$\rho(p,q) = \frac{1}{2}log(\frac{|1-\overline{p}q|+|q-p|}{|1-\overline{p}q|-|q-p|})$$

Proposition 1.8. *The aforementioned curve ψ for the origin case and the image under $\varphi_p(z)$ for the two arbitrary points case are geodesics.*

Proof. Set $r \in \mathbb{C}$ such that there exists $t' \in (0,1)$ with $\psi(t') = r$. Denote $\psi_1 = \psi_{[0,t']}$ and $\psi_2 = \psi_{[t',1]}$. By the definition 1.6, we already know that $\rho(\psi) = \rho(\psi_1) + \rho(\psi_2)$

We have to check if $\rho(p,q) = \rho(p,r) + \rho(r,q)$. We are going to argue by contradiction, knowing that $\rho(\psi)$ is minimal, which leads to $\rho(p,q) = \rho(\psi)$. Suppose that $\rho(\psi_1) \neq \rho(p,r)$. Then, there would be a shorter path that would join p and r, and concatenating it with ψ_2, we would obtain a shorter curve than ψ. Similarly, $\rho(\psi_2) = \rho(r,q)$, and we obtain

$$\rho(p,q) = \rho(p,r) + \rho(r,q)$$

□

A natural question that can emerge is, are there more than one geodesic given two points? Are there infinitely many? Well, the answer will be given in the forecoming theorem.

Theorem 1.9. *There exists one and only one shortest path that joins two given points p and q.*

The proof follows from the next lemma.

Lemma 1.10. *The only geodesic joining 0 to p is the aforementioned $\psi(t) = tp$.*

Proof. Let's denote ψ as in the statement. Set γ another curve that is also a geodesic. There exists a point q such that there is a time $t \in \mathbb{R}$ with $\gamma(t) = q$ and there is no time that holds that $\psi(t) = q$. We are going to project this point radially onto ψ, with $q_0 = |\frac{q}{p}|p$. Note that $|q| \leq |p|$ and $|q_0| = |q|$. Also, note that both q and q_0 lie in the circle of radius $|q|$ and center 0, which we are going to call C.
Because of the geodesic condition,

$$\rho(0,p) = \rho(0,q) + \rho(q,p)$$

$$\rho(0,p) = \rho(0,q_0) + \rho(q_0,p)$$

As $p(0,z)$ depends only on $|z|$, we deduce that $\rho(q,p) = \rho(q_0,p)$. Turning back to our main tools, we can apply the map

$$\varphi_p(z) = \frac{z-p}{1-\overline{p}z}$$

Set $\widetilde{q} = \varphi_p(q)$ and $\widetilde{q_0} = \varphi_p(q_0)$. Thus, by invariance under conformal mappings,

$$\rho(0,\widetilde{q}) = \rho(p,q) = \rho(p,q_0) = \rho(0,\widetilde{q_0})$$

which again, only holds if $|\widetilde{q}| = |\widetilde{q_0}|$. But now focus on the transformation of C under φ_p. The center goes to $-p$, and p goes to the origin. Now we have that $\widetilde{q_0}$ and $\widetilde{q}$ lie in the image of the circle, which does not include the origin. Finally, for any point of the segment line between 0 and p (let us say, tp),

$$\varphi_p(tp) = \frac{t-1}{1-\widetilde{p}tp}p = k(t)p$$

so the segment line is still a straight line under φ_p and, by the definition of $\widetilde{q_0}$, it is closer to the origin than $\widetilde{q}$, which contradicts $|\widetilde{q}| = |\widetilde{q_0}|$. □

Then, we can prove theorem 1.9.

Proof. It is immediate by applying the map φ_p, its inverse φ_{-p} and using last lemma. □

Now we are ready to prove the **Schwarz-Pick lemma**. For the proof we are going to recall similar techniques we used in the proof of **the Riemann Mapping Theorem** (appendix B).

Theorem 1.11 (Schwarz-Pick Lemma)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Then f is both an infinitesimal and a global contraction with respect to the hyperbolic metric on $\mathbb{D}$. In other words, for every $w, z \in \mathbb{D}$, $\rho(z,w) \geq \rho(f(z), f(w))$*

Proof. Given f, we are going to apply a composition of functions so that zero maps to itself, in order to apply **Schwarz lemma**. Indeed, consider the function $g = \varphi_{f(t)} \circ f \circ \varphi_{-t}$.
A simple calculation gives us $g(0) = 0$, and then, by the Schwarz lemma, $|g'(0)| \leq 1$. But if we compute that derivative

$$g'(0) = (\varphi_{f(t)}(f(t)) \circ f(t) \circ \varphi_{-t}(0))'$$

The derivative of $\varphi_{f(t)}$ is

$$\varphi'_{f(t)}(z) = \frac{1-|f(t)|^2}{(1-\overline{f(t)}z)^2}$$

Then,

$$g'(0) = \frac{1-|f(t)|^2}{(1-|f(t)|^2)^2} \cdot f'(t) \cdot (1-|t|^2) = \frac{f'(t)(1-|t|^2)}{1-|f(t)|^2}$$

So,

$$|\frac{f'(t)(1-|t|^2)}{1-|f(t)|^2}| \leq 1$$

which gives us $|f'(t)|\rho(f(t)) \leq \rho(t)$, the infinitesimal contraction. In order to obtain the global contraction set $p, q \in \mathbb{D}$. We want to see if $\rho(f(p), f(q)) \leq \rho(p,q)$. Recalling our previous lemma, we know there exists a geodesic ψ that joins p with q. By definition of the ρ-length,

$$\rho(f(p), f(q)) \leq \rho(f(\psi)) = \int_{f(\psi)} \rho(f(t))|f'(t)||dt| \leq \int_{\psi} \rho(t)|dt| = \rho(\psi) = \rho(p,q)$$

We obtain, hereby, the global contraction. □

We will stop the section here as we do not need any more results of this topic in this thesis. However, I would like to give a short explanation of the fact that this geometry contradicts the fifth postulate.
First, there is the fact that, due to the invariance under conformal mappings, we can transport the hyperbolic metric from the disk to the **Upper Half-plane** $\mathbb{H}_+ = \{(x, y) \in \mathbb{C}, x > 0\}$ (also called "Le demi plan de Poincaré"), so we can work in that space. In this case, the conformal mapping has a name, the **Cayley transformation**

$$z \to \frac{z - i}{z + i}$$

Secondly, it is not hard to prove that vertical lines are geodesics in this plane. Finally, we can notice that the conformal map $z \to -\frac{1}{z}$ is a biholomorphism in $\mathbb{H}_+$. These lines have semi-circles as images (and note these semi-circles have center in $\partial\mathbb{H}_+ = \{(x, y) \in \mathbb{C}, x = 0\}$. Again, by the invariance under conformal mappings, these semi-circles can be seen as geodesics between two points that lie on the same circle.
Hence, choose your favorite semicircle L and your favorite point $p \notin L$. There exist infinitely many semi-circles that contain p and do not intersect with L. So, here, the fifth postulate does not hold.

Chapter 2

Normal families and Montel's Theorem

2.1 Riemann surfaces

As explained in the introduction, the basic definition and results in complex dynamics are based on the theory of normal families. Our goal in this chapter is to develop some background in this field, with some preliminaries on Riemann surfaces. I would like to remark that I could have shortened this section a lot, as we will focus only on subsets of the Riemann Sphere $\overline{\mathbb{C}}$ in the forecoming chapters, but I wanted to fill the reader in about the generality of this theory. The references of this chapter are [CG], [MIL] and [KL]

Definition 2.1 (Manifold)**.** *A (n-)**manifold** M is a topological space that has three properties:*

- *It is locally Euclidean: For every point of M, p, there exists a neighbourhood of p, U, which is homeomorphic to an open set of* $\mathbb{R}^n$
- *It is Hausdorff: For every two points of M, p, q, there exist two neighbourhoods U, V of p and q respectively, such that* $U \cap V = \emptyset$
- *Second-countability: Its topology has a countable base*

Definition 2.2 (Riemann surface)**.** *A **Riemann surface** is a one-dimensional complex and connected manifold.*

We are going to classify the types of Riemann surfaces we can find.

Definition 2.3 (Covering map)**.** *Let M and N be two connected manifolds. A **covering map** is a map* $\phi : M \to N$ *such that every point* $p \in N$ *has a neighbourhood U homeomorphic to* $\phi^{-1}(U)$*. For every aforementioned N there exists a simply connected manifold* $\widetilde{N}$ *such that* $\phi : \widetilde{N} \to N$ *is a covering map. In this case, we denote that map as a universal covering map, and* $\widetilde{N}$ *is the universal covering.*

Definition 2.4 (Fundamental group)**.** *Let M be a topological space. A loop is a closed curve in M. Define as* $\wr$ *the equivalency class of the loops of M that are homotopically equivalent (C.12). Then, the **fundamental group** of M,* $\pi_1(M)$ *is the group of equivalence classes of* $\wr$*.*

There is an equivalent construction of the fundamental group, which will be useful later. A **deck transformation** associated with a covering map $\phi : S \to T$ is a continuous map $\alpha : S \to S$ which holds that $\phi \circ \alpha = \phi$. Then, the fundamental group can also be defined as all the deck transformations for the universal covering $\phi : \widetilde{T} \to T$.

For the classification of the surfaces, we are going to use the following remarkable theorem. To see a complete proof, consult [SG]

Theorem 2.5 (Uniformization Theorem)**.** *Every Riemann surface S is (conformally) isomorphic to $\widetilde{S}/\pi_1(S)$, with $\widetilde{S}$ simply connected, and then isomorphic to either $\mathbb{D}$, $\mathbb{C}$ or $\overline{\mathbb{C}}$.*

Remark 2.6. Then we have the following classification:

- Spherical Case: S is conformally isomorphic to $\overline{\mathbb{C}}$
- Euclidean Case: In this case, $\pi_1(S)$ can have zero, one or two generators. Respectively, we have that S is conformally isomorphic to $\mathbb{C}$, a cylinder or a torus.
- Hyperbolic case: $\widetilde{S}$ is conformally isomorphic to the unit disk $\mathbb{D}$.

In the previous chapter we built the hyperbolic metric by imposing it to be invariant under conformal isomorphisms. Given a hyperbolic surface S, we have the universal covering map $\pi : \widetilde{S} \cong \mathbb{D} \to S$. Using the fact that $\widetilde{S}$ is conformally isomorphic to $\mathbb{D}$, we can choose the inherited metric. This way, $\widetilde{S}$ will be invariant under conformal isomorphisms (and particularly, under deck transformations) and therefore, the universal covering map will map small enough open sets of $\widetilde{S}$ to the corresponding open sets of S, but preserving the distances. This is what we call a **local isometry**.
For example, the upper-half plane $\mathbb{H}_+$ is also hyperbolic and conformally isomorphic to the unit disk, by applying the Cayley transformation $z \to \frac{z-i}{z+i}$.

Corollary 2.7. *Riemann surfaces can be expressed as a (countable) union of compact sets. This is what we call $\sigma-$compactness.*

Proof. This follows from theorem 2.5, or it can be found in the appendix (C.16). □

Lemma 2.8. *Holomorphic maps from spherical surfaces to either hyperbolic or Euclidean ones must be constant. It also holds for maps from Euclidean surfaces to hyperbolic ones.*

Proof. First note that we can extend the holomorphic map onto the universal coverings by definition. Therefore, in the first case we have a holomorphic map going from $\overline{\mathbb{C}}$ to either $\mathbb{C}$ or $\mathbb{D}$ and, by theorem A.22 in the first possibility, and by Liouville theorem and maximum modulus principle in the second one, we obtain the result. The second case also uses Liouville theorem in a similar way. □

Corollary 2.9. *The Triply Punctured Sphere, which is $\overline{\mathbb{C}}$ without three points, is a hyperbolic surface. In fact, $\overline{\mathbb{C}}$ without more than three points is also a hyperbolic surface.*

Proof. It comes from the classification theorem, discarding the Euclidean and spherical cases. □

2.2 Montel's theorem

Montel's theorem shows a relation between maps in the Riemann Sphere without three points, and the desired concept of the chapter: normal families. In order to be able to state and understand Montel's theorem we have to state some theorems related to these hyperbolic surfaces. It is extremely useful to understand better the space of the maps going from a Riemann surface, S, to another one, T, and later on we will focus on the maps that are holomorphic. We denote the first space as $Map(S,T)$, and the second as $Hol(S,T)$. We shall construct an appropriate topology, in this case, the **topology of uniform convergence**. Some of the theorems below will be stated without a proof, since they fall out of the scope of this thesis.

Definition 2.10. *Let S be locally compact (C.15) (indeed, all manifolds are locally compact), and T be a metric space, with distance d. Then, for any $f \in Map(S,T)$, we set $N_{K,\epsilon}(f)$ as the elements of $Map(S,T)$ holding the following property: for a compact set $K \subset S$ and $\epsilon > 0$, $N_{K,\epsilon}(f)$ is composed by all $g \in Map(S,T)$ such that*

$$d(f(p), g(p)) < \epsilon \ \ \forall p \in K$$

These sets form a basis and therefore they generate the topology.

With this definition we can extract some properties.

Proposition 2.11. *The topology on $Map(S,T)$ does not depend on the choice of the metric of T. Also, $Map(S,T)$ admits a metric (it is metrizable).*

Proof. Let us prove the first statement. The definition of basic neighbourhoods is apparently dependent on the metric on T. The idea is to find an equivalent definition without using a particular choice of metric d. Indeed, consider the product space $T \times T$ and let U be a neighbourhood of the diagonal $\Gamma = \{(q,q), q \in T\}$. Choosing a compact set in S, K and $f \in Map(S,T)$, we define $N_{K,U}(f)$ as the set of maps of $Map(S,T)$ such that $(f(p), g(p)) \in U$ for all $p \in K$.
Now we are going to link these sets with the previous topology depending on the distance to conclude that, in fact, both basis form the same topology. Choosing a small enough $\epsilon > 0$ such that for any $p \in K$ with $d(f(p), q) < \epsilon$, it holds that $(f(p), q) \in U$, so $N_{K,\epsilon}(f) \subset N_{K,U}(f)$. Also, we can choose a small enough neighbourhood of Γ, U', such that for every (q_1, q_2) at a distance strictly less than ϵ, they are included in U'. Hence, we obtain the other inclusion.

Next, we are going to move onto the next statement. First, let's change the current distance we defined before by the bounded distance $d'(q_1, q_2) = Min(1, d(q_1, q_2))$. Both metrics generate the same topology in T. Indeed, set $q \in U \subset T$ with U an open set with respect to the metric d'. Then, by definition of open set there exists an open ball $B_{r,d'}(q) \subset U$ with $r > 0$. Now setting $\epsilon \in (0,1)$ and $r' = min(r, \epsilon) < 1$ we have that $B_{r',d}(q) = B_{r',d'}(q) \subset B_{r,d'}(q)$. The converse inclusion is trivial.
As S is a manifold, it admits an exhaustion by compact sets (C.16), that is, S can be written as the union of a nested sequence of compact sets K_i with $K_i \subset int(K_{i+1})$. Thus, we can define a new distance in $Map(S,T)$:

$$d'_{S,T}(f,g) = \sum_{n \geq 1} Max\{d'(f(p), g(p)),\ p \in K_n\}/2^n$$

Define the $\epsilon-$neighbourhoods, $N'_\epsilon(f)$, as the maps $g \in Map(S,T)$ such that $d'_{S,T}(f,g) < \epsilon$. We are going to check that these neighbourhoods and $N_{K_m,\epsilon}(f)$ define the same topology.

Set $\epsilon > 0$ and choose a big enough m with $2^{-m} < \epsilon/2$. Then, we are going to check that $N_{K_m,\epsilon/2}(f) \subset N'_\epsilon(f)$. Suppose that $g \in N_{K_m,\epsilon/2}(f)$, then $d(f(p),g(p)) < \epsilon/2$ for every $p \in K_m$. Hence,

$$g \in N'_\epsilon(f) \iff d'_{S,T}(f,g) < \epsilon \iff \sum_{n\geq 1} Max\{d'(f(p),g(p)),\ p \in K_n\}/2^n < \epsilon \iff$$

$$\sum_{m\geq n\geq 1} Max\{d'(f(p),g(p)),\ p \in K_n\}/2^n + \sum_{n>m} Max\{d'(f(p),g(p)),\ p \in K_n\}/2^n < \epsilon$$

Using that $(1/2^n) < \epsilon/2$ for $n > m$, and by definition of d',

$$\sum_{n>m} Max\{d'(f(p),g(p)),\ x \in K_n\}/2^n < (\epsilon/2)(\sum_{j\geq 1} Max\{d'(f(p),g(p)),\ x \in K_{m+j}\}/2^j) < \epsilon/2$$

With the hypothesis that $g \in N_{K_m,\epsilon/2}(f)$ and that for every $j < k$, $K_j \subset K_k$,

$$\sum_{m\geq n\geq 1} Max\{d'(f(p),g(p)),\ p \in K_n\}/2^n < \epsilon/2$$

The other inclusion is easier. Given a compact set K and $\epsilon > 0$ define m as the minimum integer such that $K \subset K_m$ (by the exhaustion by compact sets). We can see that $N'_{\epsilon/2^m}(f) \subset N_{K,\epsilon}(f)$. In fact, if we suppose that $g \in N'_{\epsilon/2^m}(f)$, then, $d'_{S,T}(f,g) < \epsilon/2^m$ which implies that the summation of non-negative terms is strictly less than $\epsilon/2^m$. The result arrives when we look at the $m^{th}-$term. □

It is immediate to see that $Hol(S,T)$ is a closed set of $Map(S,T)$, and therefore, it is another metric space. To understand better the latter, we can state the following theorem.

Theorem 2.12. *Let S, T be hyperbolic surfaces. Then $Hol(S,T)$ is locally compact. Moreover, if $K \subset S$ and $K' \subset T$ are compact sets, the set of holomorphic functions f that hold that the image $f(K) \subset K'$ is a compact set of $Hol(S,T)$. Also note that if T is compact, $Hol(S,T)$ is as well.*

We finally arrive at the concept of normal families of maps between Riemann surfaces.

Definition 2.13 (Uniform convergence)**.** *Let S, T be two Riemann surfaces. A family of holomorphic functions $\{g_n : S \to T\}$ is said to* ***converge uniformly*** *to a function g if for $\epsilon > 0$ there exists a natural number N such that for every $n \geq N$,*

$$|g_n(x) - g(x)| \leq \epsilon,\ x \in S$$

Definition 2.14 (Normal families)**.** *Denote A an arbitrary index set. A family of holomorphic functions $\mathcal{F} = \{f_t : S \to T,\ t \in A\}$ is called* ***normal*** *if every infinite sequence of functions of $\mathcal{F}$ contains either a subsequence that either converges locally uniformly or diverges locally uniformly.*

This definition is valid for any pair of arbitrary Riemann surfaces, but the case of the sphere is the one we will be paying attention to, after stating Montel's theorem. Then, as $\overline{\mathbb{C}}$ is compact, we can immediately discard the divergence statement:

Definition 2.15 (Normal families in the Riemann Sphere)**.** *A family of holomorphic functions $\mathcal{F} = \{f_t : \overline{\mathbb{C}} \to \overline{\mathbb{C}},\ t \in A\}$ is called* ***normal*** *if every infinite sequence of functions of $\mathcal{F}$ contains a subsequence that converges locally uniformly.*

This concept is related to the type of surface we are working with. An equivalent definition can be presented.

Definition 2.16. *Let S be a Riemann surface and T be a compact Riemann surface. A family of functions* $\mathcal{F} = \{f_n : S \to T\}$ *is normal if* $\overline{\mathcal{F}}$ *is a compact in* $Hol(S, T)$.

For hyperbolic surfaces we have a strong criterion to determine whether a given family of holomorphic functions is normal or not. The result is amazing.

Theorem 2.17. *If S and T are hyperbolic, any family of holomorphic maps* $\mathcal{F} = \{f_t : S \to T,\ t \in A\}$ *is normal.*

Proof. We have to check the convergence and divergence cases. For any base point $s_0 \in S$, if $\{f(s_0); f \in \mathcal{F}\} \subset K \subset T$ with K compact, then using the theorem 2.12, $\overline{\mathcal{F}}$ is compact. If not, for a base point $t_0 \in T$, there is a sequence $f_n \subset \mathcal{F}$ that holds $dist_T(t_0, f_n(s_0)) \to \infty$. As f_n are holomorphic, we saw in the Schwarz-Pick lemma that the sequence f_n diverges locally uniformly from T (because of the global contraction feature). □

Almost at the end of the chapter, we see what happens if we compare normal families $\mathcal{F} = \{f_t : S \to T,\ t \in A\}$ when they are looked into their original "destination" T, or if they arrive at a larger or smaller Riemann surface. This will be useful, for example, for maps in the Riemann Sphere which omit three different points, for which we would like to be able to transmit our knowledge of normal families in hyperbolic surfaces (as the sphere without three points) to a spherical one (the Riemann Sphere).

Lemma 2.18. *Let* S, T, U *be three Riemann surfaces,* $U \subset T$ *and* $\mathcal{F}$ *a family of holomorphic maps going from S to U. If they diverge locally uniformly from U, but not from T, then there exists a subsequence that converges locally uniformly to a point in* ∂U *(as a constant map).*

Corollary 2.19. *A family is normal if and only if it is normal when maps are looked into a larger surface.*

With all these tools, we can state Montel's theorem.

Theorem 2.20 (Montel's theorem)**.** *Let S be a Riemann surface,* $\mathcal{F} = \{f_t : S \to \overline{\mathbb{C}} \setminus \{a, b, c\}\}_t$ *a family of holomorphic functions and* $a, b, c \in \overline{\mathbb{C}}$ *three distinct points. Then,* $\mathcal{F}$ *is normal.*

Chapter 3

Complex dynamics

After the preliminaries introduced in the previous sections we are now ready to start discussing about dynamics. Our aim in this chapter is to explain the basic background in holomorphic dynamics, leading to the statement and proof of the central result of the thesis.
To do so, the first section introduces basic properties of Julia and Fatou sets. The second, third and fourth sections study the cases of the different types of orbits, to conclude with statements involving the presence of critical points. The last parts of the chapter describe the structure of Fatou set.

3.1 Fatou and Julia sets

Let S be a compact Riemann surface (e.g. the Riemann Sphere), $f : S \to S$ a holomorphic map, and $f^{\circ n}$ denote the n^{th}-iterate. We classify the points $z \in S$ depending on whether z is in the domain of normality of $\mathcal{F} = \{f^{\circ n} : S \to S\}_{n \in \mathbb{N}}$ or not.

Definition 3.1 (Fatou set and Julia set)**.** *The* ***Fatou set*** $\mathcal{F}$ *of a given map* $f : S \to S$ *is the set of points* $z \in S$ *such that* $\{f^{\circ n}\}_{n \in \mathbb{N}}$ *is a normal family in a neighbourhood of* z*. The* ***Julia set*** $\mathcal{J}$ *is the complement of the Fatou set.*

Both Fatou and Julia sets can be denoted as $\mathcal{F}(f)$ and $\mathcal{J}(f)$ or $\mathcal{F}$ and $\mathcal{J}$, when the map is clear from the context.

Example 3.2. Consider the map $z \to z^2$. As $f^{\circ n}(z) = z^{2^n}$, if $|z| < 1$, then $z \to 0$ and if $|z| > 1$, $z \to \infty$. Note that if we are working in the Riemann Sphere, this second condition means that the iterates converge locally uniformly to the constant function equal to ∞. If we are in the usual complex plane, the iterates diverge locally uniformly. If $|z| = 1$, no matter which neighbourhood we choose, there will be points that converge to 0, others to ∞, and the rest will be bouncing in the unit circumference. Hence, z belongs to the Julia set.

For functions as simple as $Q(z) = z^2 + c,\ c \in \mathbb{C}$, Julia sets can be both complicated and beautiful. Some examples are shown

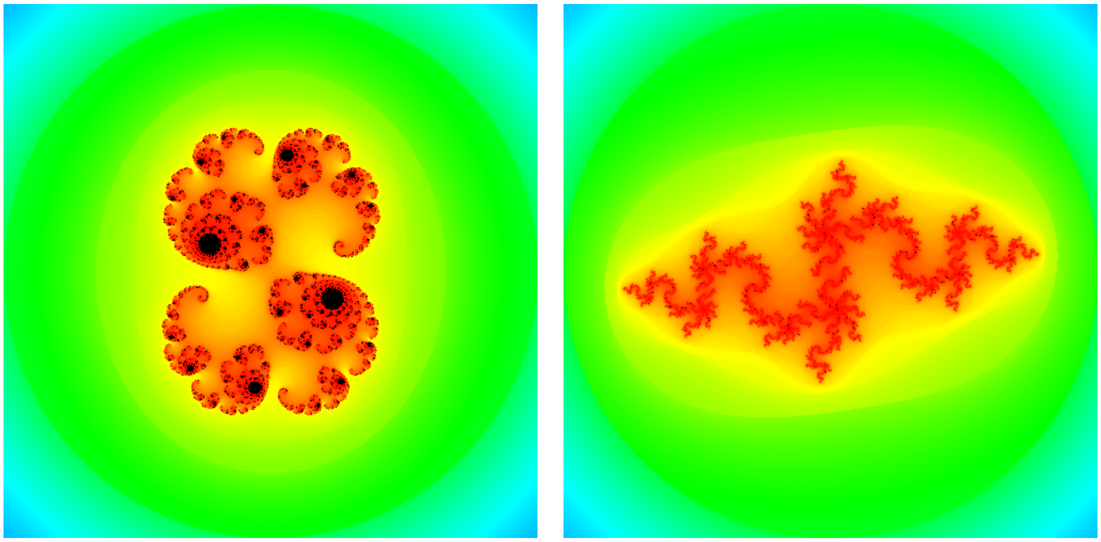


Figure 3.1: Julia sets of $Q(z) = z^2 + 0.285 + 0.01i$ and $Q(z) = z^2 - 0.8696 + 0.26i$

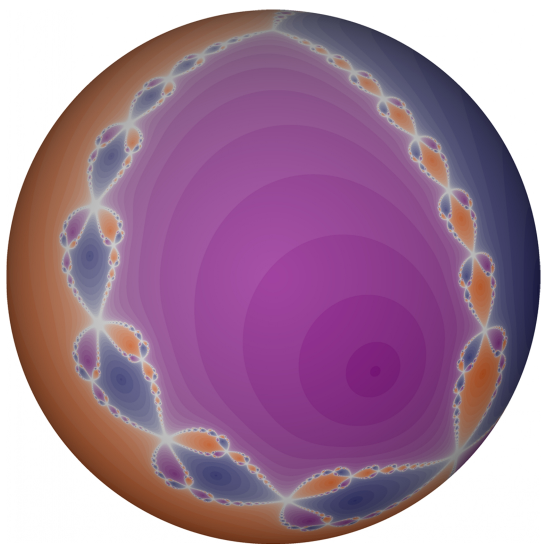


Figure 3.2: Julia set of $f(z) = z(z-1)(z-i)$, courtesy of Jordi Taixés

We are going to focus on rational maps $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, $R(z) = P(z)/Q(z)$ where P, Q are complex polynomials with no common factors. The degree, d, is the maximum between the degrees of the polynomials (and we are going to assume it is strictly larger than 1). Most of the results will be proved on $R(z)$ being a rational map, but, in fact, some of them still hold for an arbitrary map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$.

As Julia sets are the Philosopher's stone of this thesis, we will state some of the many properties of these sets.

Lemma 3.3 (Complete invariance). *$\mathcal{J}(R) = \mathcal{J}$ and $\mathcal{F}(R) = \mathcal{F}$ are completely invariant, that is to say, $R(\mathcal{J}) = R^{-1}(\mathcal{J}) = \mathcal{J}$ and the same holds for $\mathcal{F}$.*

$R^{-1}(U)$ is understood as all points $z \in \overline{\mathbb{C}}$ such that $R(z) \in A$ since R does not have a well-defined global inverse branch (as its degree is larger than 1).

Proof. Suppose $z \in \mathcal{F}(R)$, then, by definition of normality, $R^{-1}(z) \in \mathcal{F}(R)$. Next we will see $R(z) \in \mathcal{F}(R)$. As $z \in \mathcal{F}(R)$, there is a neighbourhood U of z such that the iterates form a normal family, and therefore they converge locally uniformly. As we can see in the appendix A, rational functions are holomorphic, and the Open mapping theorem holds in the sphere, so $R(U)$ is an open neighbourhood of $R(z)$ where normality still holds. Therefore, $R(z) \in \mathcal{F}(R)$. Then $\mathcal{F}(R)$ is completely invariant. As the Julia set is the complement of the Fatou set, it is completely invariant too. □

Lemma 3.4 (Iteration lemma). *For any $k \in \mathbb{N}$, $\mathcal{J}(R) = \mathcal{J}(R^{\circ k})$ and $\mathcal{F}(R) = \mathcal{F}(R^{\circ k})$*

Proof. The inclusion $\mathcal{F}(R^{\circ k}) \subset \mathcal{F}(R)$ is a consequence of the fact that if $\{R^{\circ kn}\}_{n\in\mathbb{N}}$ is normal in an open set U, then it means that there exists a subsequence which converges locally uniformly, and at the same time, it is a subsequence of the family $\{R^{\circ n}\}_{n\in N}$.
Now set $z \in \mathcal{F}(R^k)$ and we will see that $z \in \mathcal{F}(R)$, concluding the proof. The family $\{R^{\circ kn}\}_{n\in\mathbb{N}}$ is normal in a neighbourhood of z, denoted by U_z. As R is a rational map, it is uniformly continuous on compact sets, so we can conveniently choose different families of functions of the form $\{R^{\circ kn+i}\}_{n\in\mathbb{N}}$ for $0 \leq i \leq k-1$ that will indeed be normal in U. Then, z must belong to the Fatou set of R as every subsequence of the forward iterates of R will have infinite iterates of the form $nk + i_0$ for a given choice of $i_0 \in \{0, ..., k-1\}$. □

Lemma 3.5. *Let $z \in \mathcal{J}(R)$ and U be a neighbourhood of z. Then $\bigcup_{n\in\mathbb{N}} R^{\circ n}(U)$ takes every value of $\overline{\mathbb{C}}$ except, at most, two points.*

Proof. We are going to argue by contradiction. Suppose three values are omitted. As the family of maps is not normal (due to the fact that z belongs to the Julia set), we use the Montel's theorem, and we reach the contradiction. □

Denote the two omitted points in lemma 3.5 by E_z.

Theorem 3.6. *E_z is independent of z. Moreover, if E_z has a single point, $R(z)$ is conformally conjugated to a polynomial. If E_z has 2 points, $R(z)$ is conformally conjugated to a polynomial of the form $\pm z^d$.*

Due to the independence with respect to z, we will denote E_z simply by E, the **exceptional set** of R.

Proof. By definition, $R^{-1}(E_z) \subset E_z$. In the case of one point, we can apply a suitable transformation so that $E_z = \{\infty\}$, and $R^{-1}(\infty) = \infty$, so we obtain that there are not other poles and hereby R is a polynomial. Moving to the other case, we can again apply another transformation setting $E_z = \{0, \infty\}$. Then, we have two cases. If $R(0) = 0$ and $R(\infty) = \infty$ we are still in the case of the polynomial and, locally, it can be written as $R(z) = Cz^d$, as 0 is the only root of the polynomial. Otherwise, if $R(0) = \infty$ and $R(\infty) = 0$, we obtain $R(z) = Cz^{-d}$. □

We will later see some other properties of the exceptional set E. As $E \cap \mathcal{J}(R) = \emptyset$, we have the following corollary.

Corollary 3.7. *Let R be a rational map. Then,*

1. *The backward iterates of $z \in \mathcal{J}(R)$ are dense in $\mathcal{J}(R)$*
2. *If $D \subset \mathcal{F}$ is a completely invariant open set, then $\mathcal{J}(R) = \partial D$.*
3. *If the Julia set has an interior point, it must be equal to the whole $\overline{\mathbb{C}}$.*

Proof. The first and second statements are a direct consequence of lemma 3.5, the previous theorem and the invariance of the Julia set. For the last statement, choose a neighbourhood of a point of the Julia set (inside of this set). By lemma 3.5, and by the invariance of the Julia set, the union of the images by R is dense in $\overline{\mathbb{C}}$. As $\mathcal{J}(R)$ is closed, we obtain $\mathcal{J}(R) = \overline{\mathbb{C}}$. □

Now we focus our attention on the fixed points and periodic fixed points of R. In what follows we shall use the notation f for the function to be iterated since results hold not only for rational maps. But first, an introductory definition.

Definition 3.8 (Orbit)**.** *The (forward)* ***orbit*** *of a point z_0 under a map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is the set of forward iterates $\{f^n(z_0)\}_{n \geq 0}$.*

Definition 3.9 (Fixed points and periodic fixed points)**.** *A* ***fixed point*** *is a point $z_0 \in \overline{\mathbb{C}}$ such that $f(z_0) = z_0$. A* ***periodic fixed point*** *is a point $z_0 \in \overline{\mathbb{C}}$ such that there exists a minimal integer k (its* ***period****) such that*

$$f^{\circ k}(z_0) = z_0, \text{ but } f^{\circ j}(z_0) \neq z_0, \text{ for } j < k$$

As we have seen that $\mathcal{J}(f^k) = \mathcal{J}(f)$, we can adapt the definitions of periodic fixed points onto fixed points.

Definition 3.10 (Cycle and multiplier)**.** *Let z_0 be a periodic fixed point of period k. The set of points that compose its orbit is denoted by* ***cycle****:*

$$z_0 \to f(z_0) = z_1 \to f(f(z_0)) = z_2 \to ... \to z_m = z_0$$

We define the ***multiplier*** *of the orbit as the number $\lambda = (f^{\circ k}(z_0))' = f'(z_0)f'(z_1)...f'(z_{k-1})$*

We may note that $\lambda = 0$ if and only if one of the points of the orbit has a vanishing derivative.

Definition 3.11 (Critical points)**.** *A point c that holds that $f'(c) = 0$ is a* ***critical point****, and its orbit is a* ***critical orbit****. The set of all critical points and their critical orbits is the* ***postcritical set of*** *f, $P(f)$.*

We can classify the different periodic orbits by the values of λ. A periodic orbit is:

- Attracting if $|\lambda| < 1$. More specifically, it is geometrically attracting if $0 < |\lambda| < 1$ and superattracting if $|\lambda| = 0$
- Repelling if $|\lambda| > 1$
- Rationally neutral if $|\lambda| = 1$ and there exists an integer n such that $\lambda^n = 1$
- Irrationally neutral otherwise.

Focusing on fixed points, note that we have used the term "attracting" fixed point if $|\lambda| < 1$. This, of course, has a reason. For a small enough neighbourhood of z_0, we have

$$|f(z) - f(z_0)| = |f(z) - z_0| \leq \rho|z - z_0|$$

for $|\lambda| < \rho < 1$. Iterating, we deduce that the points of our initial neighbourhood get closer and closer to z_0. For the repelling case it is immediate to check that the repelling fixed point is an attracting fixed point of f^{-1}. In the case of periodic fixed points, it follows similarly.

Definition 3.12 (Basin of attraction)**.** *We define the* ***basin of attraction*** *of an attracting $k-$periodic orbit $z_0 \to z_1 \to ... \to z_{k-1} \to z_0$ as the set of points that tend to the orbit under iteration of f. By definition, this set is open, and it can have different connected components.*
In the case of a fixed point z_0, we denote its basin of attraction as $A(z_0)$, and we moreover define the immediate basin of attraction $A_0(z_0)$ as the connected component of the basin of attraction of z_0 where this point belongs.

Then we have this powerful lemma.

Lemma 3.13. *Attracting orbits belong to the Fatou set $\mathcal{F}(f)$, whilst repelling orbits belong to the Julia set $\mathcal{J}(f)$.*

By definition, if this holds, the basin of attraction of the orbit belongs to the Fatou set, too.

Proof. First we are going to see the problem only for fixed points, and later we will solve the orbit case. Suppose $f(z_0) = z_0$ with multiplier λ. Let's check the two cases. If $|\lambda|$ is greater than 1, we have to recall that if we have a sequence of holomorphic functions f_n that converge locally uniformly to a certain f, its derivatives converge, too. As the multiplier is greater than 1, its derivatives do not converge to 0, which denies the possibility of convergence. As a result, it belongs to the Julia set $\mathcal{J}(f)$. Now suppose $|\lambda| < 1$. Then we may recall a technique we used before. In a enough small neighbourhood of z_0, there exists a $\rho \in \mathbb{R}$ with $|\lambda| < \rho < 1$ such that $|f(z) - z_0| < \rho|z - z_0|$ and, therefore, it converges locally uniformly to the constant function with value z_0. As the Julia set of f is equal to the Julia set of $f^{\circ m}$, the result for periodic orbits is immediate. □

Now we define a special of periodic fixed points.

Definition 3.14 (Parabolic periodic fixed points)**.** *A periodic fixed point $z_0 = f^{\circ n}(z_0)$ is said to be parabolic if the corresponding multiplier is 1, but $f^{\circ n}$ is not the identity map.*

These points are extremely related to Julia sets.

Lemma 3.15. *If a periodic point z_0 is parabolic, it belongs to the Julia set.*

Proof. The proof is quite short, we have to use again the convergence of the derivatives of a sequence of holomorphic functions. By a suitable change of coordinates we may assume the periodic point is 0, and, therefore the function $f^{\circ n}$ will be written as $z + a_m z^m + H.O.T$, with $m \geq 2$ and $a_m \neq 0$ with m-th derivative $(m!)a_m$. Looking at the family $\{f^{\circ nk}\}$, the derivative for an arbitrary $k \in \mathbb{N}$ at the origin is $(m!)a_m k$, which tends to infinity as $k \to \infty$. Using the convergence criteria that we recalled, we obtain the non-normality. □

Coming back to the rational maps, we have a demolishing statement about Julia sets.

Theorem 3.16. *The Julia set of a rational map (of degree $d \geq 2$) is never empty.*

Proof. Arguing by contradiction, suppose $\mathcal{J}(R)$ is empty. Then, $\{R^n\}$ is a normal family on all $\overline{\mathbb{C}}$, so there is a subsequence n_k such that $R^{n_k} \to f$ for $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. As R^n are holomorphic, their limit is holomorphic too. Therefore, f is a rational function. The degree of functions $R^{\circ n}$ is d^n (R has degree d, $R \circ R$ has degree d^2...), but, as it tends to infinity, we have a rational function with not finite degree (which is a contradiction). □

Now we introduce an important definition. Let R be a rational map of degree $d \geq 2$ and set $z_0 \in \overline{\mathbb{C}}$. We denote by **grand orbit** $GO(z_0, R)$, the set of points whose orbits eventually intersect the orbit of z_0. Another way to write $GO(z_0, R)$ is $\{z' \in \mathbb{C},\ R^{\circ m}(z') = R^{\circ n}(z_0)\ \ n, m \in \mathbb{N}\}$. A point $\tilde{z}$ is called grand orbit finite ($\tilde{z} \in \mathcal{E}(R)$) if its grand orbit is a finite set.

Theorem 3.17. *There are, at most, two points that are grand orbit finite and these must be superattracting periodic points ($E = \mathcal{E}(R)$).*

Proof. Suppose $\tilde{z}$ is a grand orbit finite point. So, there are only finitely many points of $\overline{\mathbb{C}}$ whose orbits intersect the orbit of $\tilde{z}$. Hence, R maps $GO(\tilde{z}, R)$ with itself, forming a periodic orbit $z_0 \to z_1 \to ... \to z_m = z_0$.
This means $R(z_0) = z_1$, $R(z_1) = z_2$... A natural question is, can there be any other point w such that $R(w) = z_k$, for a given k? The answer is negative, as if it was any, it would belong to $GO(\tilde{z}, R)$. So, $R(z_j) = z_{j+1}$ for j modulus m and for an only z_j. As the degree of $d \geq 2$, z_j is a zero of the equation $R(z) - z_{j+1}$ with multiplicity d. But now notice that, if it is a zero, then $R(z) - z_j$ is of the form $(z - z_{j+1})^d F(z)$ for a given holomorphic function $F(z)$ that does not vanish. If we compute the derivative, $(R(z))'_{|z=z_j} = (R(z) - z_{j+1})'_{|z=z_j} = (d(z-z_j)^{d-1}F(z) + (z-z_j)^d F'(z))_{|z=z_j} = 0$, so each point is a critical one, and hence, we are in the case of a superattracting orbit.

Now, suppose there are 3 or more grand orbit finite points. Then $\overline{\mathbb{C}} \setminus \mathcal{E}(R)$ is as a sphere without three points, which is an hyperbolic surface. Also note that, $R(\overline{\mathbb{C}} \setminus \mathcal{E}(R)) = \overline{\mathbb{C}} \setminus \mathcal{E}(R)$, so $\{R^{\circ n}\}_{|\overline{\mathbb{C}} \setminus \mathcal{E}(R)}$ is a normal family by Montel's theorem. But then, both $\mathcal{E}(R)$ and $\overline{\mathbb{C}} \setminus \mathcal{E}(R)$ belong to the Fatou set, contradicting the fact that the Julia set is not empty. □

This theorem gives us a remarkable consequence, which allows us to see lemma 3.5 in a different way.

Corollary 3.18. *Let $z \in \mathcal{J}(R)$ and N be a neighbourhood of z. Then, if we denote $U = \bigcup_{n \geq 0}(R^{\circ n}(N))$,*

- *$U \cap \mathcal{J}(R) = \mathcal{J}(R)$*
- *U is the whole $\overline{\mathbb{C}}$ except, at most, 2 points, which are the grand orbit finite points, if N does not contain any of those.*

Proof. We split the proof in two parts:

- U does not reach, at most, 2 points: Arguing by contradiction, let's suppose that U does not reach 3 points. Note that U is invariant under R by definition (and therefore $R(\overline{\mathbb{C}} \setminus U)$ is also invariant). Then, we are in a hyperbolic surface and we can apply Montel's theorem in order to state that U belongs to the Fatou set, which is impossible as it contains points of the Julia set.

- Those 2 points are grand orbit finite points: If they were not, their backward iterates would eventually reach U, contradicting that they do not belong to this set. As $\mathcal{J}(R) \cap \mathcal{E}(R) = \emptyset$ (due to the fact that superattracting points belong to the Fatou set), $\mathcal{J}(R) \subset U$, so $U \cap \mathcal{J}(R) = \mathcal{J}(R)$.

□

Corollary 3.19. *Let $\mathcal{A}$ be a basin of attraction of an attracting periodic orbit. Then $\partial\mathcal{A} = \mathcal{J}(R)$.*

Remark 3.20. Basins of attraction are open sets, so $\mathcal{A} \cap \partial\mathcal{A} = \emptyset$. Therefore, we do not contradict the fact that $\mathcal{J}(R) \cap \mathcal{F} = \emptyset$.

Proof. We will prove both inclusions $\partial\mathcal{A} \subset \mathcal{J}(R)$ and $\mathcal{J}(R) \subset \partial\mathcal{A}$.

- $\partial\mathcal{A} \subset \mathcal{J}(R)$: Set $\tilde{z} \in \partial\mathcal{A}$, and let $N_{\tilde{z}}$ be an arbitrary neighbourhood of $\tilde{z}$. Note that as R is holomorphic, the Open mapping theorem holds, so the (holomorphic) iterates $\{R^{\circ n}\}_{|N_{\tilde{z}}}$ hold it, too. Note that every point of $\mathcal{A}$ eventually converges to an attracting periodic orbit, but this is not the case of the boundary points, so this cannot be a normal family, and hence, $\partial\mathcal{A} \subset \mathcal{J}(R)$.

- $\mathcal{J}(R) \subset \partial\mathcal{A}$: Now set $\tilde{w} \in \mathcal{J}(R)$ and let $N_{\tilde{w}}$ be an arbitrary neighbourhood of $\tilde{w}$. Using corollary 3.18, there is an n_0 such that $R^{\circ n_0}(N_{\tilde{w}})$ intersects $\mathcal{A}$. By the definition of basin of attraction, $N_{\tilde{w}}$ intersects $\mathcal{A}$, so $\mathcal{J}(R) \subset \partial\mathcal{A}$, as $\mathcal{A} \cap \mathcal{J}(R) = \emptyset$.

□

In the following sections we will see how critical points interact with the different type of orbits, an information that will be of great utility at the time of proving remarkable statements in Chapter 5.

3.2 Attracting and repelling fixed points

In order to understand better the behaviour of a given map f, we shall use the power of congucacies. Recall from the introduction that a function $f : D \to D$ is conformally conjugate to $g : D' \to D'$ if there exists a conformal map $\varphi : D \to D'$ such that $\varphi(f(z)) = g(\varphi(z))$. Note that $g^{\circ n} = \varphi \circ f^{\circ n} \circ \varphi^{-1}$, and besides, if z_0 is a (perodic) fixed point of f, then $\varphi(z_0)$ is a (periodic) fixed point of g. Furthermore, the basin of attraction for f is the basin of attraction for g under the conjugacy φ. We are going to study maps as

$$f(z) = z_0 + \lambda(z - z_0) + a(z - z_0)^p + \dots,\ p \geq 2$$

With a suitable change of coordinates we can suppose $f(0) = 0$. Our mission is to find best possible conjugacies to our functions, and we are going to see that they depend, basically, on λ.

Theorem 3.21 (Kœnigs Linearization). *If $0 < |\lambda| < 1$ there exists a unique (up to multiplication by scalar) local holomorphic change of coordinate $w = \phi(z)$ such that $\phi \circ f \circ \phi^{-1} = g$, where $g(w) = \lambda w$. This solves the so-called Schröder functional equation*

$$\phi \circ f \circ \phi^{-1}(w) = \lambda w$$

Proof. First we prove the uniqueness. Suppose there exist two changes of coordinates ϕ, φ that are solutions to the equation. Then, if $g(w) = \lambda w$, $\phi \circ f \circ \phi^{-1}(w) = \lambda w$ and $\varphi \circ f \circ \varphi^{-1}(w) = \lambda w$. But now, look at the map $\varphi \circ \phi^{-1}$. If we try to apply this map as a conjugacy to g,

$$(\varphi \circ \phi^{-1}) \circ g \circ (\varphi \circ \phi^{-1})^{-1} = \varphi \circ \phi^{-1} \circ g \circ \phi \circ \varphi^{-1}$$

And now applying $g(w) = \phi \circ f \circ \phi^{-1}(w) = \lambda w$,

$$\varphi \circ \phi^{-1} \circ g \circ \phi \circ \varphi^{-1} = \varphi \circ f \circ \varphi^{-1} = g$$

So, $\varphi \circ \phi^{-1}$ is a conjugacy of g. Therefore, as $g(w) = \lambda w$,

$$\varphi \circ \phi^{-1}(\lambda w) = \lambda(\varphi \circ \phi^{-1})(w) \tag{3.1}$$

Applying the hypothesis, we know that $\varphi \circ \phi^{-1}$ can be expressed as $a_1 w + a_2 w^2 + H.O.T$. Then, looking at the equality (3.1), we obtain

$$\lambda a_1 w + \lambda a_2 w^2 + ... = a_1(\lambda w) + a_2(\lambda w)^2 + ...$$

Looking at term by term, as λ is not either zero or a root of unity, we obtain that

$$\lambda a_n w^n = a_n(\lambda w)^n \iff a_n = 0 \; \forall n \geq 2$$

Hence, $\varphi \circ \phi^{-1} = a_1 w$, or, similarly, using $w = \phi(z)$, $\varphi(z) = a_1 \phi(z)$
Now we shall prove the existence. As $0 < |\lambda| < 1$ we can find a constant c such that $c^2 < |\lambda| < c < 1$. Then, in a small enough neighbourhood of 0, we can find a disk inside, namely, $B_r(0)$ such that for any $z \in B_r(0)$ and $r \leq 1$, $|f(z)| < c|z|$. We are going to see what happens to an orbit of an arbitrary point of $B_r(0)$. Set $z_0 \in B_r(0)$, and suppose the orbit is $z_0, z_1, z_2, ...$ Then $|z_1| \leq c|z_0|$ and, iteratively, $|z_n| \leq c^n|z_0| < rc^n$. Looking at the Taylor's expansion of f, it exists a constant K such that $|f(z) - \lambda z| \leq K|z|^2$. Hence, $|f(z_n) - z_n| = |z_{n+1} - z_n| \leq K|z_n|^2 \leq K(c^n r)^2$
Now we are going to determine which is the map ϕ. Set $k = Kr^2/|\lambda|$ and $w_n = z_n/\lambda^n$. This sequence $\{w_n\}$ will converge to ϕ. Indeed,

$$|w_{n+1} - w_n| = (|\lambda|)^{-n}|z_{n+1} - \lambda z_n| \leq Kr^2 c^{2n}/|\lambda|^n = k(c^2/|\lambda|)^n \to 0$$

as $n \to \infty$ Hence, the $\{w_n\}$ converge locally uniformly to ϕ, with $\phi(z_0) = lim_{n\to\infty} z_n/\lambda^n$Also note that, due to the convergence of derivatives, it is a local isomorphism. □

Corollary 3.22. *There is also a Kœnigs linearization for $|\lambda| > 1$, the repelling case.*

Proof. It is immediate by using the last theorem with f^{-1} and $|\lambda^{-1}| = 1/|\lambda| \in (0, 1)$. □

Corollary 3.23. *Suppose $\tilde{z}$ is a geometrically attracting fixed point, with basin of attraction $\mathcal{A} = \mathcal{A}(\tilde{z})$. Then, we can extend the previous ϕ to $\mathcal{A}$, so that*

$$\phi \circ f \circ \phi^{-1}(w) = \lambda w, \; w \in \mathcal{A}$$

Proof. The idea is quite simple. For any $p_0 \in \mathcal{A}$ we can choose the first element of its orbit p_k where we can apply Koenigs linearization. Therefore, we can define $\phi(p_0) := \phi(p_k)/\lambda^k$. □

Theorem 3.24 (Böttcher coordinates)**.** *Using the same method, it is easy to see that, in the superattracting case ($\lambda = 0$), we have a unique conjugation to $\phi(z) = z^p$ for $p \geq 2$.*

Now we can focus on rational maps, since in this case we can provide more information about the linearization.
Set $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ a rational map of degree 2 and suppose $\tilde{z}$ is an attracting fixed point. Set $\mathcal{A}_0$ the immediate basin of attraction and $\mathcal{A}$ the whole basin of attraction. We can find a local inverse of the previous $\phi : \mathcal{A} \to \overline{\mathbb{C}}$ such that its image is inside $\mathcal{A}_0$. Indeed, we can choose a small enough disk $B_\epsilon(0)$ such that $\psi_\epsilon : B_\epsilon(0) \to \mathcal{A}_0$ holds that $\phi(\psi_\epsilon(w)) = w$.

Lemma 3.25 (Extension of linearizing coordinates)**.** *There is a unique extension $\psi : B_r(0) \to \mathcal{A}_0$ of ψ_ϵ in a maximal disk $B_r(0)$ ($r < \infty$) such that $\phi(\psi(w)) = w$. Moreover, we can extend it homeomorphically to the boundary with $\psi(\partial B_r(0)) \subset \mathcal{A}_0$. The image of the boundary contains a critical point of R.*

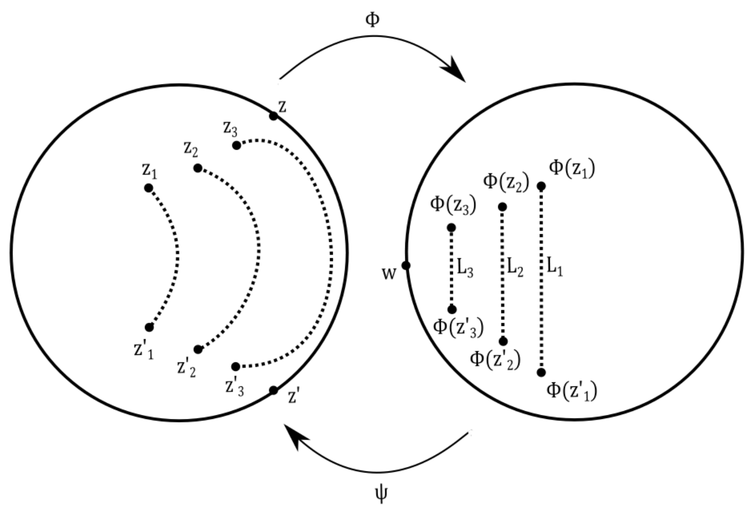
Proof. First we will find the maximal disk $B_r(0)$. We should start by checking that we cannot extend the map to the whole $\mathbb{C}$. Suppose this was the case. Then $\psi(\mathbb{C}) \subset \mathcal{A}_0$ and $\phi(\psi(w)) = w$. If $\infty \in \psi(\mathbb{C})$ then $\phi(\overline{\mathbb{C}}) \subset \mathbb{C}$. As $\overline{\mathbb{C}}$ is compact, its image is also compact in $\mathbb{C}$, so $\phi_{|\mathbb{C}}$ is constant by Liouville's Theorem. The only continuous extension to $\overline{\mathbb{C}}$ is setting ϕ to be a constant map, which is a contradiction. Then, we can only have the case where $\overline{\mathbb{C}} \setminus \psi(\mathbb{C}) = \{\infty\}$. By the commutativity hypothesis $R_{|\psi(\mathbb{C})}$ is bijective, and as the complement of $\psi(\mathbb{C})$ has only one element, the map is bijective everywhere, which contradicts the fact that R has degree 2.
Hence there must exist a maximal disk so that the map extends homeomorphically under that disk, but we have to make sure that we can study its boundary to check the critical points feature. In fact, denote U as the image $\psi(B_r(0))$. Let's look at the commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{R} & R(U) \\ \phi\downarrow & & \downarrow\phi \\ B_r(0) & \xrightarrow{f_\lambda} & \lambda B_r(0) \end{array}$$

where f_λ denotes the map $z \to \lambda z$. As $\lambda < 1$ it follows that $f_\lambda((B_r(0)))$ is contained in a compact set $K_\lambda \subset B_r(0)$. Therefore, using the commutativity, $R(U)$ is contained in a compact set of U, K. Hence, $R(\overline{U}) \subset U \subset \mathcal{A}$, and by definition of basin of attraction, $\overline{U} \subset \mathcal{A}$ so ϕ is defined on the boundary.

The next step is proving that this boundary contains a critical point. Conversely, suppose that it does not contain any critical points of R. Set $\tilde{w} \in \partial B_r(0)$ and then choose an accumulation point $z_{\tilde{w}}$ of $\psi(t\tilde{w})$, for t tending to 1. As it is not a critical point we can choose a well defined branch of the local inverse of R in a neighbourhood of $z_{\tilde{w}}$, which we will call $R^{-1}_{\tilde{w}}$. Note that $\lambda\tilde{w} \in B_r(0)$, so $\psi(w)$ is defined in a neighbourhood of $\tilde{w}$, and hence we can apply $R^{-1}_{\tilde{w}}$ to that result. This composition coincides with ϕ in an open neighbourhood so there must be the same map by the uniqueness of the analytic extension. Hence, if we suppose there is no critical point in the boundary, we can repeat the process for every point of $\partial B_r(0)$ so we can define the map ϕ on an open set bigger than $B_r(0)$, contradicting the maximality of this one.

The last step is to check that ψ extends homeomorphically on the boundary $\partial B_r(0)$. As we have seen that ϕ is defined on $\overline{U}$, it is equivalent to see that ϕ maps $\overline{U}$ onto $\overline{B_r(0)}$. We only need to prove the injectivity, as the construction of ψ gives as the remaining properties. Arguing by contradiction, suppose there exist two points $z,\ z' \in \partial U$ such that $\phi(z) = \phi(z') = w \in \partial B_r(0)$. Then, we can find two sequences $\{z_j\}_{j\geq 1},\ \{z'_j\}_{j\geq 1}$ in U that converge to z and z' respectively. We can therefore consider the two sequences $\{\phi(z_j)\}_{j\geq 1}, \{\phi(z'_j)\}_{j\geq 1}$ that converge towards w but $\phi(z_j) \neq \phi(z'_j)$. Hence we can consider the lines that join them, L_j. Finally, let us define X in ∂U as the limit of the images $\psi(L_j)$ which is compact and connected by the structure of the lines in $B_r(0)$. Just in order to imagine what is actually happening, whilst L_j are becoming smaller when j tends to infinity, their images become bigger, eventually tending to a piece of the boundary, connecting z and z'.



But now, using $R = \psi \circ f_\lambda \circ \phi$ and that $\lambda w \in B_r(0)$, $R(X)$ has only one element whilst X has (at least) both $z,\ z'$. Recalling that X is connected, it contradicts the fact that R is holomorphic. □

This result has an immediate consequence which differentiates holomorphic dynamics from real ones.

Theorem 3.26. *Set R to be a rational map of degree $d \geq 2$. Then, the immediate basin of every attracting periodic orbit contains at least one critical point.*

Proof. We have to split it in three cases.

- Geometrically attracting fixed point case: It follows from the previous lemma.
- Superattracting fixed point case: The fixed point is, indeed, the critical point.
- An attracting m-periodic orbit z_n. We have that $R(\mathcal{A}_0(z_j)) \subset \mathcal{A}_0(z_{j+1})$. Arguing by contradiction, suppose there are not any critical points in any of $\mathcal{A}_0(z_j)$. Then, if we composed the map R m times, we would have that $R^m : \mathcal{A}_0(z_j) \to \mathcal{A}_0(z_j)$ would not have any critical points, which is not true, as it is a rational function.

□

In particular, this result implies that maps can have at most as many attracting equilibria as critical points. It is straightforward to see that the number of critical points of a rational function of degree d is $2d-2$ (counting multiplicity).

Corollary 3.27 (Bound on attracting periodic points). *Let R be a rational map of degree 2, then the number of periodic orbits is at most equal to the number of critical points, which is finite (at most, $2d-2$).*

There is a similar extension result for the case of superattracting points. We are going to reuse the aforementioned ψ, and now $\widetilde{z}$ is an superattracting fixed point.

Theorem 3.28 (Extension of Böttcher coordinates). *In the similar case of lemma 3.25, ψ can be analytically extended to an open ball $B_r(0)$. If $r=1$, ψ maps $B_1(0)$ onto $\mathcal{A}_0$ biholomorphically, and then $\widetilde{z}$ is the only critical point. If not, there is a critical point in the boundary of $\psi(B_r(0))$.*

We have a final theorem that will help us to understand the structure of immediate basins of attraction.

Theorem 3.29 (Connectivity of immediate attracting basins). *Let A_0 be the immediate basin of attraction for either a geometrically attracting or superattracting fixed point $\widetilde{w}$. Then its complement is either connected or has uncountably many connected components. Hence, A_0 is either connected or infinitely connected. Note that the simply connected condition is due to normality of A_0.*

Proof. First we are going to construct a structure of A_0 based on open disks. Let B_0 be an open disk that contains $\widetilde{w}$ and such that $f(\overline{B_0}) \subset B_0$ and $\partial B_0 \cap P = \emptyset$ where P is the postcritical set of R. Now denote B_k the k-th preimage of B_0 that still contains $\widetilde{w}$.
So, we have obtained a nested chain

$$B_0 \subset B_1 \subset ... \subset B_k \subset ...$$

Now, for any point $w \in A_0$, there exists a path α that connects w to $\widetilde{w}$. By the compactness of the path, there exists a number n such that $f^k(\alpha)$ is inside B_0, which implies that α is in B_k (by connectivity condition). Hence, $w \in B_k$, and as it was an arbitrary point of A_0, it turns out that $A_0 = \bigcup_{j\geq 0} B_j$. Using this structure we can analyze what happens at the complementary. Each component B_k can have diverse boundaries, but essentially there are two cases. If every B_k has for boundary a simple closed curve, then the complement of each B_k is connected, so $\overline{\mathbb{C}} \setminus A_0 = \overline{\mathbb{C}} \setminus \bigcup_{j\geq 0} B_j = \overline{\mathbb{C}} \setminus lim_{j\to\infty} B_j$ is connected. Contrary to this case, it can exist a minimal k such that B_k has more than one simple closed curve as boundary, but this is not the case for B_{k-1}. For simplicity sake we are going to relabel the indexes as $B_{k-1} \to B_0$ and $B_k \to B_1$ (and so on). Denote this set of curves as $\alpha_1, ..., \alpha_n$, that are also the boundaries of connected sets $D_1, ..., D_n$ that belong to $\overline{\mathbb{C}} \setminus B_1$.
Set $m \geq 1$. In order to show the existence of an infinite number of components, we are going to prove that no matter which (finite) sequence of numbers $(i_1, ..., i_m)$, between 1 and n, we choose that there exists a component that has a boundary labelled as $\alpha_{i_1,...,i_m}$ such that:

- It is the boundary of a connected component of $\overline{\mathbb{C}} \setminus B_m$
- $f(\alpha_{i_1,...,i_m}) = \alpha_{i_1,i_2,...,i_{m-1}}$

Inductively, we will have proved the existence of an infinite number of components.
First, it is immediate to see that compactness is maintained under $f : \overline{B_m} \to \overline{B_{m-1}}$ (for every B_m)

and, hence, this property holds for $f : \overline{B_m} \setminus f^{-1}(B_0) \to \overline{B_{m-1}} \setminus B_0$.
Second, notice that each of the components of $\overline{B_m} \setminus f^{-1}(B_0)$ (for example, labelled as B_{m_i}) belongs to a complementary set labelled as D_{m_i}, each one with boundary $\alpha_{i_1,i_2,..,i_{m-1},i}$.
Hence, the boundaries $\alpha_{i_1,i_2,...,i_{m-1}}$ in ∂B_{m-1} are the image of some $\alpha_{i_1,i_2,...,i_{m-1},i}$ of $\partial B_m \cap D_{m_i}$, giving us the desired result. □

3.3 Parabolic fixed points

Next, we are going to treat the case of multiplier $\lambda = 1$, that is

$$f(z) = z + az^{n+1} + ... \, , \; n \geq 1 \textit{ and } a \neq 0$$

This is the case where dynamics "combine" in a beautiful way with botany. First, let's define the preliminaries. A sequence of numbers $\{z_j\}_{j=1}^{\infty}$ is said to converge nontrivially to zero if $|z_j| > 0$ for any j and $lim_{j\to\infty} z_j = 0$.
In the forecoming definition we are going to treat numbers as vectors (just for the interpretation), so we will use the typical letter v.

Definition 3.30. *Set $v \in \mathbb{C}$. We say that v is an **attraction vector** if $nav^n = 1$, and a **repulsion vector** if $nav^n = 1$. Then, we denote $v_0, ..., v_{2n-1}$ these vectors, with $v_j = e^{\pi ij/n} v_0$ so that $nav_j^n = (-1)^j$.*

We shall study an orbit that converges to the fixed point 0. We will see it is extremely related to the repulsion or attraction vectors.

Lemma 3.31. *Suppose an orbit under f, $z_0 \to z_1 \to ...$ converges nontrivially to zero. Then, it converges to $v_j / \sqrt[n]{k}$ for a given attraction vector v_j. It happens the same with an orbit under f^{-1}, in this case, with a repulsion vector.*

Before starting the proof, we are going to define some useful concepts. Let $\mathbb{R}_+$ be the positive (half) real line, $\mathbb{R}_+ = \{z \in \mathbb{C},\ z = Re(z) \geq 0\}$. Then $\mathbb{R}_+ v_j$ is called an attracting or repelling ray (if v_j is either an attraction or repulsion vector). Note that, as we have $2n$ vectors of this kind, we can split the complex plane in a star-shaped way, with $2n$ sectors, delimited by the rays. I would also like to recall a notion that I was introduced to in my analytical number theory course, the "little o". Let f and g be some analytic functions. We say that $f = o(g)$ if $lim_{z\to\infty} \frac{f(z)}{g(z)} = 0$. As we will focus on points near zero, we can write our f in a more formal way (rather than using the ...): $f(z) = z(1 + az^n + o(z^n))$ when $z \to 0$.

Proof. All we know about repulsion and attraction vectors is that $nav_j^n = (-1)^j$, so it would be a good idea to get rid of the na part. Also, we want to see the behaviour of an orbit really close to zero, so another way to study it is using a change of coordinates so that we study the infinity. Therefore, we denote $w = \varphi(z) = c/z^n$ for $c = -1/na$. Then, $\varphi(v_j) = (-1)^{j+1}$.
The inverse function of φ is $\varphi^{-1}(z) = \sqrt[n]{c/w}$, but it is not uniquely defined in the whole complex plane, as there are n branches. This is when we recall the construction of the rays. Denote by $\triangle_j$ the two sectors that are delimited by $\mathbb{R}_+ v_j$. This new space has for boundary $\mathbb{R}_+ v_{j+1} \cup \mathbb{R}_+ v_{j-1}$, which are attraction vectors if v_j is a repulsion vector, and the opposite, if v_j is an attraction vector. Note that $z \in \triangle_j$ if $z = Ce^{i\theta} v_j$, for $-\pi/n < \theta < \pi/n$. Note that $\triangle_j \cap \triangle_{j+1}$ is one of the sectors we defined before the proof.

Suppose that v_j is a repulsion vector (j is even). Then both $\mathbb{R}_+ v_{j+1}$ and $\mathbb{R}_+ v_{j-1}$ are attracting rays, and therefore

$$\varphi(Cv_{j+1}) = \varphi(Cv_{j-1}) = \frac{c}{(Cv_{j-1})^n} = \frac{-1}{na(Cv_{j-1})^n} = \frac{(-1)^{1+j-1}}{C^n} = \frac{1}{C^n}$$

for every $C \in R_+$, hence

$$\varphi(\mathbb{R}_+ v_{j-1}) = \mathbb{R}_+$$

Then,

$$\varphi(\triangle_j) = \mathbb{C} \setminus \mathbb{R}_+$$

We can apply the same process if v_j is an attracting vector and we obtain that

$$\varphi(\triangle_j) = \mathbb{C} \setminus (-\mathbb{R}_+)$$

where $-\mathbb{R}_+ = \{-z \in \mathbb{C},\ z \in \mathbb{R}_+\}$. So, we can define a unique branch of the inverse function φ^{-1} (which will be labelled as ψ_j) so that $\psi_j : \mathbb{C} \setminus ((-1)^j \mathbb{R}_+) \to \triangle_j$ and $\varphi \circ \psi_j = id$.
Now that we understand our sectors $\triangle_j$ better, we can choose one and study the dynamics there. Let $0 \leq j \leq 2n-1$ and be $\triangle_j$ the corresponding sector. Instead of looking at the raw f, consider the j-dependent transformation

$$w \to \varphi \circ f \circ \psi_j(w) = F_j(w)$$

So, $f(\psi_j(w)) = \sqrt[n]{\frac{c}{w}}(1 + a\frac{c}{w} + o(\frac{c}{w})) = \sqrt[n]{\frac{c}{w}}(1 + a\frac{c}{w} + o(\frac{1}{w}))$ as $|w| \to \infty$.
Then,

$$F_j(w) = \frac{w}{(1 + a\frac{c}{w} + o(\frac{1}{w}))^n}$$

Using the Negative Binomial Theorem, and recalling the definition of c:

$$F_j(w) = w(1 - \frac{nac}{w} + o(\frac{1}{w})) = w + 1 + w(o(\frac{1}{w})) = w + 1 + o(1) \tag{3.2}$$

This last equation can be translated as there exists an $R > 0$ such that for $|w| > R$,

$$|F_j(w) - w - 1| < 1/2$$

Then, as $Re(z) < |z|\ \forall z \in \mathbb{C}$ and $|z| = |-z|$, we obtain that $|F_j(w) - w - 1| = |w + 1 - F_j(w)| > Re(w + 1 - F_j(w)) = Re(w) + 1 - Re(F_j(w))$, which leads to the inequality

$$Re(F_j(w)) > Re(w) + 1/2 \tag{3.3}$$

Substituting $w = \varphi(z) = c/z^n$ we obtain that $F_j(\varphi(z)) = (\varphi \circ f \circ \psi_j) \circ (\varphi(z)) = \varphi(f(z))$ for a small enough z. So,

$$Re(\varphi(f(z)) > Re(\varphi(z)) + 1/2 \tag{3.4}$$

Let's choose a large enough R so that previous equalities hold and denote $\mathbb{H}_R = \{z \in \mathbb{C},\ Re(z) > R\}$. Then, we can define the image $\psi_j(\mathbb{H}_R)$, which we will conveniently denote $\mathcal{P}_j(R)$. Note that $F_j(\mathbb{H}_R) \subset \mathbb{H}_R$ by equation (3.3). So,

$$F_j(\mathbb{H}_R) = (\varphi \circ f \circ \psi_j)(\mathbb{H}_R) = \varphi \circ f(\mathcal{P}_j(R)) \subset \mathbb{H}_R$$

Applying ψ_j to both sides of the last equality,

$$f(\mathcal{P}_j(R)) \subset \mathcal{P}_j(R)$$

Again, by the inequality, iterating $\mathcal{P}_j(R)$ under f we obtain that the images are getting smaller, converging uniformly to 0. Therefore, we denote $\mathcal{P}_j(R)$ as an attracting petal. Let's move back to the statement of the lemma. Suppose there exists an orbit $z_0 \to z_1 \to ...$ that converges to zero nontrivially. By the inequality (3.4), there exists an n_0 such that for all $m \geq n_0$, $Re(\varphi(z_m)) > R$, so $z_m \in \mathcal{P}_j(R)$ for some odd j (here we have used the invariance of the petal under f).

Now let's work with the w's. We may consider $w_0 = \varphi(z_0) \to w_1 = \varphi(z_1) \to ....$ So, for every $m \geq n_0$, $w_m \in \mathbb{H}_R$. Looking at the inequalities we obtain that $Re(w_m) \to \infty$, so $|w_m| \to \infty$. But here it is when we really use the f with the little o. As $F(w) = w + 1 + o(1)$, $F(w_m) - w_m \to 1$ as $n \to \infty$. Hence,

$$\frac{w_m - w_0}{m} = \sum_{j=0}^{m-1} \frac{w_{j+1} - wj}{m} \to 1$$

and, hereby, $w_m \to m$. Finally, as $w_m = \frac{-1}{naz_m}$, $naz_m \to \frac{-1}{m} = \frac{v_j^n}{m}$, which leads to $z_m \to \frac{v_j}{\sqrt[n]{m}}$, as z_m belongs to a petal $\mathcal{P}_j(R)$. □

The inequalities give us an immediate consequence.

Corollary 3.32. *Recalling equation (3.4), we can see there are not any cycles close to a parabolic fixed point.*

With this lemma, we can extend this result when λ is a root of unity. Suppose then $\lambda = e^{2\pi i p/q}$ for $p, q \in \mathbb{N}$, $gcd(p, q) = 1$. Then, $f(z) = \lambda z + az^{n+1} + ...$
Set $\tilde{z}$ a fixed point under f. In order to apply the previous lemma, we shall consider $f^{\circ q}$ (note that if we compose f q times we obtain a new function that is of the form $1 + bz + ...$). Indeed, $f^{\circ 2}(z) = \lambda(\lambda z + az^{n+1} + ...) + H.O.T = \lambda^2 z + \lambda az^{n+1} + H.O.T$, so $f^{\circ q}(z) = \lambda^q z + bz^{n+1} + ... = z + bz^{n+1} + ...$ As $f^{\circ q}$ has n attraction vectors, we can choose one, v, and therefore, there is an orbit $z_0 \to z_q \to z_{2q}...$ that converges to $\tilde{z}$ under $f^{\circ q}$ in the direction of v.
The image under f gives us another orbit $z_1 \to z_{q+1} \to z_{2q+1} \to ...$ that converges to $f(\tilde{z}) = \tilde{z}$ in the direction of the attracting vector λv. Look at what is happening here: every time we apply f, we are rotating the attraction vector, but when we do it q times we obtain the same orbit as the very first one. But due to the fact we have n attraction vectors, then $n = kq$ for an integer k.
In order to link this back to the Julia sets, we can propose some definitions:

Definition 3.33. *Set $f : S \to S$ with S a Riemann surface and let $\tilde{z}$ be a parabolic fixed point with multiplier* 1*. Then, given an attraction vector v_j we can define the parabolic basin of attraction $\mathcal{A}_j$ associated to v_j as the set of points whose orbit tend to $\tilde{z}$ following the direction of v_j. As in previous definitions, the* ***immediate basin of attraction*** *$\mathcal{A}_j^0$ is the connected component of $\mathcal{A}_j$ that has the fixed point $\tilde{z}$.*

Remark 3.34. The case of $\lambda = e^{\pi i p/q}$ is equivalent with using $f^{\circ q}$.

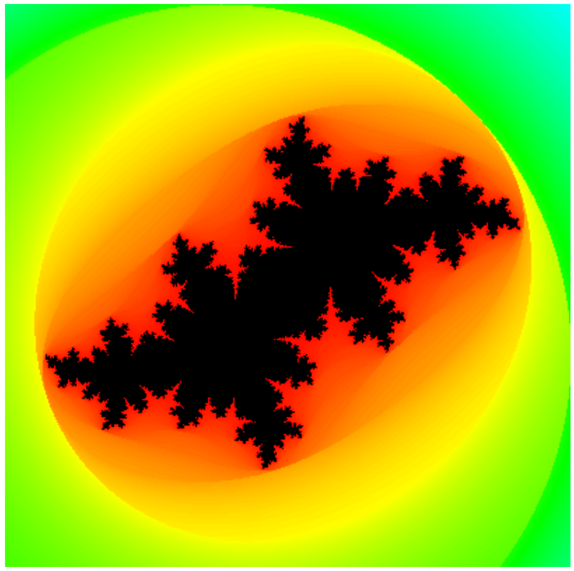


Figure 3.3: Julia set of $f(z) = z^2 + e^{\pi i 3/4} z$

Theorem 3.35 (Parabolic basins belong to the Fatou set)**.** *$\mathcal{A}_j$ belongs to the Fatou set $\mathcal{F}(f)$ and $\partial\mathcal{A}_j$ belongs to the Julia set $\mathcal{J}(f)$.*

Proof. Due to the remark, we can assume $\lambda = 1$. By lemma 3.15, the fixed point belongs to the Julia set. Moreover, by definition of $\mathcal{A}_j$, it belongs to the Fatou set. Take a point p whose orbit converges trivially to $\tilde{z}$, then, by the invariance of the Julia set, its orbit belongs to the Julia set. The interesting case arrives. Suppose that there exists a point $z_0 \in \partial\mathcal{A}_j$ such that $(\bigcup_{j=0}^{\infty}\{z_j\}) \cap \{\tilde{z}\} = \emptyset$. We have seen that the orbit of z_0 does not converge to $\tilde{z}$ (either trivially, by assumption, or nontrivially, as it belongs to the boundary of $\mathcal{A}_j$), therefore, we can find a subsequence $\{z_{j_k}\}$ such that $d(z_{j_k}, \tilde{z}) > K$ for $K > 0$, but, as $\mathcal{A}_j$ converges to z_0, the family of iterates cannot be normal in a neighbourhood of z_0, as any neighbourhood intersects $\mathcal{A}_j$. Hence, $\partial\mathcal{A}_j \subset \mathcal{J}(f)$ ☐

Now we can define the particular basins of attraction.

Definition 3.36 (Petals)**.** *More generally, suppose that $f : S \to S$ has a parabolic fixed point $\tilde{z}$ such that its multiplicity is $n+1$ for $n \geq 1$. Suppose f is injective in a neighbourhood of S of $\tilde{z}$, N. We say that $\mathcal{P} \subset N$ is an attracting petal for an attraction vector v_j if $\mathcal{P}$ is invariant under f and an orbit converges to $\tilde{z}$ following the direction v_j if and only if the orbit, eventually, gets inside $\mathcal{P}$. We can rewrite the definition in the case of a repelling petal, by substituting f for f^{-1}.*

Theorem 3.37 (Parabolic Flower Theorem)**.** *Suppose $\tilde{z}$ is a parabolic fixed point of multiplicity $n+1$. Then there exist $2n$ attracting and repelling petals $\mathcal{P}_j$ (depending whether j is odd or even) such the intersection of two consecutive petals is another petal, which does not intersect with the others.*

Proof. It follows from its definition and the lemmas we have proved. ☐

And now, I will state two powerful additional theorems that I won't prove (and some consequences). Further details can be found in [MIL], [CG] and [BEA]

Theorem 3.38 (Parabolic Linearization Theorem, Abel functional equation)**.** *Suppose that $f : S \to S$ is globally holomorphic and it has a parabolic fixed point $\tilde{z}$. Then, for any petal $\mathcal{P}$, there is, up to* *a composition with translation, one and only one conformal embedding $\alpha : \mathcal{P} \to \mathbb{C}$ that satisfies Abel functional equation:*

$$\alpha(f(z)) = 1 + \alpha(z),\ z \in \mathcal{P} \cap f^{-1}(\mathcal{P})$$

α is called the Fatou map.

Corollary 3.39. *If $\mathcal{P}$ is attracting, it can be extended to the basin of attraction of $\mathcal{P}$. Furthermore, this extension still satisfies the Abel functional equation.*

Corollary 3.40. *If $\mathcal{P}$ is a repelling petal, then $\alpha^{-1} : \alpha(\mathcal{P}) \to \mathcal{P}$ can be extended to a unique holomorphic map $\widetilde{\alpha} : \mathbb{C} \to S$ that satisfies*

$$f(\widetilde{\alpha}(w)) = \widetilde{\alpha}(1 + w)$$

In analogy to the attracting case, there is a strong relationship between the basins of attraction of parabolic fixed points and the critical points of R.

Theorem 3.41. *Let R be a rational map. Suppose $\widetilde{z}$ is a parabolic fixed point of multiplicity $n+1$ and multiplier $\lambda = 1$. Then, each immediate basin of attraction contains at least one critical point of R.*

Recalling that we also had the presence of critical points of attracting basins of attraction, we obtain a remarkable consequence.

Corollary 3.42. *A rational map with degree d has, at most $2d - 2$ attracting and parabolic cycles.*

3.4 Cremer and Siegel points

The last case to consider is when the multiplier λ has modulus 1 but it is not a root of the unity, so $\lambda = e^{2\pi i \xi}$ for $\xi \in \mathbb{R} \setminus \mathbb{Q}$. Again, by a suitable change of coordinates, we can consider the function

$$f(z) = \lambda z + a_n z^n + ... \ for\ n \geq 2$$

Then we say that the origin is an **irrationally indifferent fixed point** and ξ is the **rotation number**. As the other sections, we want to find out whether f admits a local linearization φ: $\varphi(f(z)) = \lambda\varphi(z)$, and $\varphi'(0) = 1$. Setting $z = h(w)$, $h = \varphi^{-1}$, we are looking for an h such that:

$$f(h(w)) = h(\lambda w)$$

If this linearization is possible, we will say the fixed point is a **Cremer point**. Otherwise, it will be tagged as a **Siegel point**. A connected component of the Fatou set where f is conformally conjugate to a rotation of the unit disk is a **Siegel disk**.

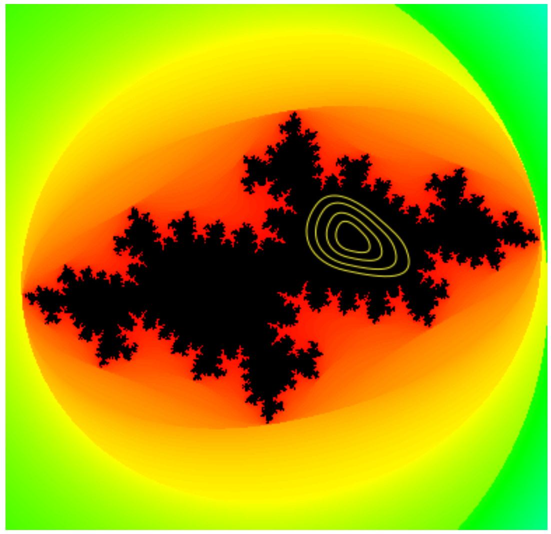


Figure 3.4: In $f(z) = z^2 + e^{2\pi i\sqrt{2}}z$ there is a Siegel disk

We can give an immediate result:

Lemma 3.43. *h is univalent (injective and holomorphic).*

Proof. The holomorphic part is already done by assumption. Let's see what happens with the injectivity. Suppose there are w_1, w_2 such that $h(w_1) = h(w_2)$. Then, the Schröder equation implies that:

$$h(\lambda w_1) = f(h(w_1)) = f(h(w_2)) = h(\lambda w_2)$$

Applying the same procedure,

$$h(\lambda^2 w_1) = f(h(\lambda w_1)) = f(f(h(w_1))) = f(f(h(w_2))) = f(h(\lambda w_2)) = h(\lambda^2 w_2)$$

Then, $h(\lambda^n w_1) = h(\lambda^n w_2)$ for every integer $n \geq 0$. But note that, as λ has an irrational rotation number, the family $\{\lambda^n\}_{n\geq 1}$ is dense in the circumference. Then, as h is holomorphic, $h(e^{i\theta}w_1) = h(e^{i\theta}w_2)$ for every $\theta \in \mathbb{R}$. Again, as h is holomorphic, $h(w_1 z) = h(w_2 z)$ for every $z \in \mathbb{D}$. Using the fact that $h'(0) = 1$, we can calculate the derivative at zero:

$$(h(w_1 z))'|_{z=0} = (h(w_2))'|_{z=0} \iff w_1 h'(w_1 z)|_{z=0} = w_2 h'(w_2 z)|_{z=0} \iff w_1 = w_2$$

□

We can prove a relation between the existence of h and the family of iterates of f.

Lemma 3.44. *Such h exists if and only if $\{f^{\circ n}\}_{n\geq 1}$ is uniformly bounded in a neighbourhood of the origin.*

Proof. First suppose the function h exists. Then,

$$f(h(w)) = h(\lambda w)$$

But this implies that $f(z) = h(\lambda h^{-1}(z))$. Iteratively,

$$f^{\circ 2}(z) = f(f(z)) = f(h(\lambda h^{-1}(z))) = h(\lambda(\lambda h^{-1}(z))) = h(\lambda^2 h^{-1}(z)))$$

So, $f^{\circ n}(z) = h(\lambda^n h^{-1}(z))$, which is bounded.
Now let's assume $|f^{\circ n}| < K$. We shall define the functions

$$\varphi_n(z) = \frac{1}{n}\sum_{j=0}^{n-1} \lambda^{-j} f^{\circ j}(z)$$

As $\{f^{\circ n}\}_{n\geq 1}$ is uniformly bounded, so $\{\varphi_n\}_{n\geq 1}$ is also uniformly bounded. Using **Montel's theorem for sequences of holomorphic functions** (B.6), there exists a subsequence that converges locally uniformly to some holomorphic function φ.
But now consider $\varphi_n \circ f$.

$$\varphi_n(f(z)) = \frac{1}{n}\sum_{j=0}^{n-1} \lambda^{-j} f^{\circ j}(f(z)) = \frac{1}{n}\sum_{j=0}^{n-1} \lambda^{-j} f^{\circ j+1}(z) = \lambda \frac{n}{n+1} \varphi_{n+1}(z)$$

So, in the limit, $\varphi \circ f = \lambda f$.
Furthermore, using the fact that $\lambda = f'(0)$,

$$\varphi_n'(z)|_{z=0} = \frac{1}{n}\sum_{j=0}^{n-1} \lambda^{-j} (f^j)'(z)|_{z=0} \iff \varphi_n(0) = \frac{1}{n}\sum_{j=0}^{n-1} (\frac{f'(0)}{\lambda})^j = \frac{1}{n} n = 1$$

so we conclude that $\varphi'(0) = 1$ and hereby we can define the inverse function h in a neighbourhood of 0.

□

We are going to give a method to find Siegel points. Further information can be found in [CG] and [MIL]. An irrational number ξ is called **diophantine** if there exist $c > 0$ and $\mu < \infty$ such that for all p, q with $gcd(p, q) = 1$ we have

$$|\xi - \frac{p}{q}| \geq c/q^\mu$$

Theorem 3.45. *Suppose that f has a irrationally indifferent fixed point $\widetilde{z}$ with multiplier $\lambda = e^{2\pi\xi i}$. If ξ is diophantine, then $\widetilde{z}$ is a Siegel point.*

Now we will define a new type of sets that are strongly related to Siegel disks.

Definition 3.46 (Herman ring)**.** *A periodic component U of the Fatou set with period n is a **Herman ring** if, essentially, it is conformally isomorphic to an annulus $\mathcal{A}_{r,R} = \{z \in \overline{\mathbb{C}},\ r < |z| < R\}$ and $R^{\circ n}$ is conjugate to either a rotation on the annulus or a composition of a rotation and an inversion. Siegel disks and Herman rings are **rotation domains**.*

To sum up, Siegel disks are simply connected components, whilst Herman rings are doubly connected. Following the steps of the previous sections, rotation domains are related to the presence of critical points.

Theorem 3.47. *Boundaries of Siegel disks and Herman rings are contained in the closure of the post-critical set of R. Moreover, if H is a Herman ring, then every boundary point belongs to the closure of a critical orbit.*

3.5 Study of cycles in rational maps

Now that we have seen all possible types of fixed points, we will study the periodic orbits of a rational map, which we recall they are called **cycles**. As in the case of periodic orbit, a cycle will be attracting, repelling or indifferent depending on the multiplier λ. We have already seen that we can only have $2d-2$ attracting cycles and parabolic ones (corollary 3.42).

We would like to see that if $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a rational map of degree $d \geq 2$ has a finite number of indifferent cycles. Indeed,

Theorem 3.48. *The number of indifferent cycles with multiplier $\lambda \neq 1$ is finite.*

The proof of this theorem is quite hard as it involves quasiconformal surgery, which consists of perturbing our initial function without altering its dynamical behaviour. In this case, it can be proved that the bound is $6d-6$, by perturbing the function transforming its indifferent periodic points into attracting fixed points. The best bound ever found, in this case by Shishikura, is $2d-2$ [SHI].

With that case covered, let us study the repelling cycles. But before, a necessary lemma.

Lemma 3.49. *Let $\mathcal{J}(R)$ be the Julia set of a rational map of degree 2. Then, it does not contain any isolated points.*

Proof. Set $z_0 \in \mathcal{J}(R)$ and U an arbitrary neighbourhood. We must check there exists a point z such that $\mathcal{J}(R) \cap U$. We are going to split it in three cases:

- Suppose z_0 is not periodic. Then, by the invariance of the Julia set, there exists a $z_1 \in \mathcal{J}(R)$ that is a preimage of z_0, which means, $R(z_1) = z_0$. For all n, $R^{\circ n}(z_0) \neq z_0$, so $R^{\circ n}(z_0) \neq z_1$. As the backward iterates of a point of the Julia set are dense, there exists an $n_0 > 0$ and a $z \in U$ such that $R^{\circ n_0}(z) = z_1$. Hence, $z \in \mathcal{J}(R) \cap U$.
- Now suppose z_0 is periodic (of period n) and it is the only solution to the equation $R^{\circ n}(z_0) = z_0$. This means that $R^{\circ n}(z)$ is of the form $(z - z_0)^m + z_0$, and it is hereby a superattracting point, so it does not belong to $\mathcal{J}(R^{\circ n})$, which leads to a contradiction as $\mathcal{J}(R^{\circ n}) = \mathcal{J}(R)$.
- Finally, suppose there is an alternative solution z_1 to $R^{\circ n}(z_1) = z_0$. Also, $R^{\circ j}(z_0) \neq z_1$ since if this happened, $z_0 \neq z_1 = R^{\circ j}(z_0) = R^{\circ (n+j)}(z_0) = R^{\circ n}(z_1) = z_0$. Then, we would be back to the first case.

□

With this lemma in mind, we can prove the following theorem, which states that repelling periodic points are dense in the Julia set

Theorem 3.50. *$\mathcal{J}(R)$ is the closure of the set of repelling periodic points.*

Proof. We are going to use Fatou's proof although there is an alternative proof using Julia's method. The idea is to show that every point in the Julia set can be approximated by repelling periodic points.

First recall that, as $\mathcal{J}(R)$ does not have isolated points, we can remove finitely many elements of

the sets so the statement remains true. Therefore, it is convenient to remove critical values and fixed points. The idea behind this is to apply the **Inverse Function Theorem**.
So suppose z_0 is neither a critical value nor a fixed point. Hence, we can denote z_j as the $j^{th}-$ preimage $((R^{-1})^{\circ j}(z_0))$, for $1 \leq j \leq d$.
As we advanced before (and noticing that $R'(z_0) \neq 0$), we can use IFT to find a (holomorphic) function φ_1 defined in some neighbourhood of z_0, U, such that $R \circ \varphi_1(z) = z$ for $z \in U$ and $\varphi_1(z_0) = z_1$. But there is no reason why we should stop here. We can repeat the process finding holomorphic functions φ_j such that $R(\varphi_j(z)) = z$ and $\varphi_j(z_0) = z_j$, $j \leq d$.
Recall that our objective is to approximate points of the Julia set by repelling periodic orbits. Our claim is that there exists an n and a $\tilde{z} \in U$ such that $R^{\circ n}(\tilde{z})$ is either $\tilde{z}$, $\varphi_l(\tilde{z})$ or $\varphi_k(\tilde{z})$ for $1 \leq l < k \leq d$. Arguing by contradiction, let's suppose this is not the case.
The (complex) **cross-ratio** of four points of $\overline{\mathbb{C}}$ is defined to be the number

$$\mathcal{X}(a,b,c,d) = \frac{(c-a)(d-b)}{(b-a)(d-c)}$$

This number is very important when we study automorphisms on $\overline{\mathbb{C}}$, but now we will treat it as a function. Consider,

$$\mathcal{X}_n = \frac{(R^{\circ n}(\tilde{z}) - \varphi_l(\tilde{z}))(\tilde{z} - \varphi_k(\tilde{z}))}{(R^{\circ n}(\tilde{z}) - \varphi_k(\tilde{z}))(\tilde{z} - \varphi_l(\tilde{z}))}$$

$\mathcal{X}_n$ clearly avoids 0 and ∞ but also 1, so we can apply **Montel's theorem** to conclude that $\{\mathcal{X}_n\}$ is a normal family on U, contradicting that U is a neighbourhood of a point of the Julia set. Therefore, we can be sure that either $R^{\circ n}(\tilde{z}) = \tilde{z}$, $R^{\circ n}(\tilde{z}) = R^{-l}(\tilde{z})$ (which means that $R^{\circ(n+l)}(\tilde{z}) = \tilde{z}$) or finally $R^{\circ n}(\tilde{z}) = R^{-k}(\tilde{z})$ (so, again, $R^{\circ(n+k)}(\tilde{z}) = \tilde{z}$)
Then, every point of the Julia set has an arbitrarily close point that is periodic. Using both corollary 3.42 and theorem 3.48 (the finiteness of non-repelling orbits), the only possible option is that $\tilde{z}$ belongs to a repelling periodic orbit. □

With this theorem we can prove the following statement.

Corollary 3.51. *Let U be an open set whose intersection with the Julia set is not empty. Then there exists an* $n \geq 1$ *such that* $\mathcal{J}(R) = R^{\circ n}(U \cap \mathcal{J}(R))$

Remark 3.52. Note that this corollary is much stronger than corollary 3.18, as here n is finite.

3.6 The Classification of Fatou components

We have arrived at the main theorem of the chapter. We will not prove the theorem but we have learnt and used all the tools we need to, indeed, prove it. First, an introductory definition.

Definition 3.53 (Fatou component). *A **Fatou component** is a connected component of* $\mathcal{F}$*, the Fatou set.*

We can classify the Fatou components that map onto themselves.

Theorem 3.54 (Classification of periodic Fatou components). *Let U be a Fatou component of a rational map R and suppose that there exists an integer* $k \geq 1$ *such that* $R^{\circ k}(U) = U$*. Then, U only can be:*

- *the immediate basin of attraction of an attracting fixed point*
- *the immediate basin of attraction of a petal of a parabolic fixed point (with multiplier* $|\lambda| = 1$*)*
- *a Siegel disk*
- *a Herman ring*

Components that eventually fall into a cycle are not very relevant. Finally, it could happen that there existed components that are neither periodic or preperiodic (which are called **wandering domains**), and neither Julia nor Fatou could solve this problem. It was in fact, Sullivan, 65 years later, who proved the following crucial result.

Theorem 3.55 (No wandering domains)**.** *Let R be a rational map of degree at least 2. Then, there does not exist any wandering domain.*

His proof uses sophisticated techniques involving quasiconformal surgery, prime ends (Chapter 4)... A complete proof can be found in [SUL].

Chapter 4

Carathéodory's theorem

We recall at this point that our goal in this work was to connect dynamics and topology, and more precisely, hyperbolicity and local connectivity. In this chapter we deal with one of the main tools to prove that a certain set is locally connected. Recall that in the Riemann Mapping Theorem, we had a biholomorphic map from a simply connected set to the unit disk. Observe that if we can extend the inverse of this map to the boundaries, it means that the boundary of the set is, essentially, a circumference (as boundaries map to other boundaries, using the Open mapping theorem). We will study the cases when we can extend the map.

The chapter is distributed as follows: first, an introductory topological part that introduces the concept of prime ends. Next, we must prove some analytical tools to finally arrive at the third part whose goal is to prove the Carathéodory's theorem.

Most of the results can be found in [MIL] and in [CG].

4.1 Topological part: Prime ends

Let U be a simply connected set such that it does not include the point at infinity ∞ (as stated in Riemann Mapping Theorem). We will try to understand better its boundary in a purely topological way.

The idea will be drawing lines that split our set U in two parts, making them smaller and smaller. Let's formalize this idea.

Definition 4.1 (Crosscut)**.** *We say that a **crosscut** is a subset $A \subset U$ such that:*

- $\partial A \subset \partial U$
- *A is homeomorphic to an open interval*

So, a trivial example could be drawing a line joining two points of ∂U that is completely inside U. Notice that this set divides U in two connected components (this follows clearly from Jordan curve theorem C.11). These components will be called **crosscut neighbourhoods** of U. Usually, we will choose one of them.

Definition 4.2 (Fundamental chain). *A **fundamental chain** $\mathcal{N} = \{N_i\}_{i\geq 1} \subset U$ is a sequence of crosscut neighbourhoods such that*

- $N_{i+1} \subsetneq N_i$
- *The closures are* $A_i = U \cap \partial N_i$
- $A_j \cap A_k$ *for* $j \neq k$
- *diameter*$(A_i) \to 0$ *as* $i \to \infty$

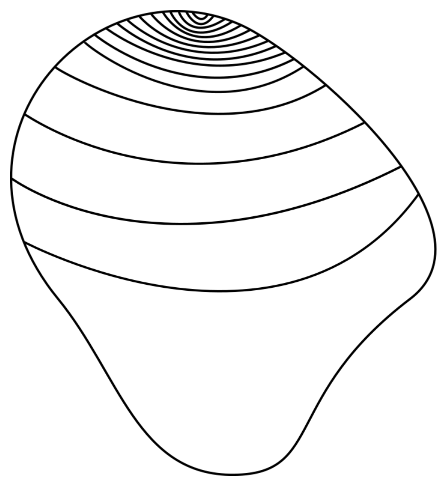


Figure 4.1: A visual example of a fundamental chain

We will omit the sub-indexes of the fundamental chains for the rest of the chapter. We can track the behaviour of these chains. It may happen that two chains $\{N_i\}$, $\{N'_j\}$ behave pairwise. This means that no matter which choice of i, there will exist some indices j_i such that $N'_{j_i} \subset N_i$, and conversely swapping their roles. In this case we say they are **equivalent**. This tag is not chosen randomly:

- $\{N_i\}$ is equivalent to itself
- If $\{N_i\}$ is equivalent to $\{N'_j\}$ then $\{N'_j\}$ is equivalent to $\{N_i\}$
- If $\{N_i\}$ is equivalent to $\{N'_j\}$ and $\{N'_j\}$ is equivalent to $\{N''_k\}$, then $\{N_i\}$ is equivalent to $\{N''_k\}$, by definition.

So, we can define an equivalence class $\mathcal{E}$, denoted as **prime end**.

Example 4.3. Suppose that $U = \mathbb{D}$. Denote $\{N_i\}$ the fundamental chain that we build as follows: take the crosscuts $A_i = \{(x, y) \in \mathbb{D},\ y = \sum_{j\geq 1}^{i} (1/2)^j\}$. Similarly let $\{N'_k\}$ be another fundamental chain with $A'_k = \{(x, y) \in \mathbb{D},\ y = \sum_{j\geq 1}^{k} (1/3)^j\}$.

Focusing on the behaviour of the fundamental chain, we denote its **impression** as $\bigcap_{i\geq 1} \overline{N_i}$.

Lemma 4.4. *The impression is a non-empty, compact and connected subset of* ∂U *whilst* $\bigcap_{i\geq 1} N_i$ *is always empty.*

Proof. First we will prove that $\bigcap_{i\geq 1} N_i$ has not any elements. Let $z \in U$ arbitrary. Notice that $N_1 \subset U$ is not equal to U as the crosscut splits U in two parts. So choose $z_1 \in U \setminus N_1$. Denote P a path inside of U that joins z_1 and z. As P is compact, the distance from P to ∂U is strictly bigger than zero, let us say ϵ. Hence, there exists a big enough j such that $diameter(N_j) < \epsilon$. So, $P \cap N_j = \emptyset$, therefore z does not belong to any N_j. As it was arbitrary, $\bigcap_{i\geq 1} N_i = \emptyset$. By construction, the impression is not empty, and we have seen that it belongs to ∂U (if this was not the case, $\emptyset \neq \bigcap_{i\geq 1} \overline{N_i} \cap U = \bigcap_{i\geq 1} N_i = \emptyset$). The compactness is immediately given as it is a closed subset of ∂U.

Finally, we shall prove the connectivity feature. Contrarily, suppose that the impression is not connected. So, there exist two disjoint closed sets R and T such that $\bigcap_{i\geq 1} \overline{N_i} = R \cup T$. By the T_4 property (inherited of metric spaces) there exist some disjoint open sets $R \subset V$ and $T \subset W$ such that $\bigcap_{i\geq 1} \overline{N_i} \subset V \cup W$. Now consider $O_i := \overline{N_i} \setminus V \cap W$. Note that O_i is compact, $O_{i+1} \subset O_i$ and $\bigcap_{i\geq 1} O_i = \bigcap_{i\geq 1}(\overline{N_i} \setminus V \cup W) = \emptyset$. As we have seen before, this cannot be true unless one of the sets is empty, for example O_j (and therefore, all the following O_{j+k}). Then $\overline{N_j} \subset V \cap W$. As $\bigcap_{i\geq 1} \overline{N_i} = R \cap T$, it follows that $\overline{N_j}$ has to be necessarily included in both of the sets, which is impossible since the $\overline{N_i}$ are connected. □

The simplest case is when the impression is a single point. In this case we say the fundamental chain converges to the given point. On the other side of the coin, it could happen that two fundamental chains $\{N_i\}$ and $\{N'_j\}$ had totally different behaviours. That is, there exist some i_0, j_0 such that $N_{i_0} \cap N_{j_0} = \emptyset$. In this case, they are said to be **eventually disjoint**

Lemma 4.5. *The only possible scenarios are the equivalence and the eventually disjoint cases.*

Proof. If there existed some i,j such that $N_i \cap N'_j = \emptyset$, they would be eventually disjoint. Therefore, suppose that for all i, j, $N_i \cap N'_j \neq \emptyset$, and we will prove that for each i there exists a j_0 such that $N'_{j_0} \subset N_i$ (and hence, for every $j > j_0$, $N'_j \subset N_i$). Fix an i. For every j, $N_{i+1} \cap N'_j \neq \emptyset$, and recall that $N'_j \subset N'_{j+k}$ for every $k > 0$. By the previous lemma, we have seen that $\bigcap N'_j = \emptyset$, so at some point A'_j, the $j^{th}-$crosscut intersects N_{i+1}. Now we are going to give it a twist. Arguing by contradiction, suppose there exists no j_0 such that $N'_{j_0} \subset N_i$. Then, as none of the N'_j is empty, we would have that $(U \setminus N_i) \cap N'_j \neq \emptyset$. Then (again), at some point A'_j would intersect with $U \setminus N_i$, and using what we have seen, it would also intersect with N_{i+1}. Hence, $diameter(A'_j) \geq distance(A_i, A_{i+1}) > 0$, which contradicts the fact that $diameter(A'_j) \to 0$ □

4.2 Analytical part

In order to support the previous lemmas, and to use the biholomorphic map from U to $\mathbb{D}$, we will need some analytical tools. Suppose ρ is a conformal metric on the square $I^2 = (0, \epsilon) \times (0, \epsilon)$. From previous courses of analysis it is well-known that the area $\mathcal{A}$ of I^2 is given by

$$\mathcal{A} = \int\int_{I^2} \rho(x + iy)^2 dxdy$$

and the length of a segment with $y = c$ is

$$L(c) = \int_I \rho(x + ic)dx$$

Lemma 4.6. *If $\mathcal{A}$ is finite, then the length of a horizontal segment is finite for almost every choice of $y = c$ and*

$$\int_I L(y)^2 dy \leq \epsilon \mathcal{A}$$

Proof. As well-reputed analysts, we are going to use the Cauchy-Schwarz Inequality. Recall that given two measurable functions $f, g : I \to \mathbb{R}$ we have:

$$(\int_I f(x)g(x)dx)^2 \leq \int_I f^2(x)dx \int_I g^2(x)dx$$

Remark 4.7. We have proved this inequality thousands of times in the degree, so we will assume it holds.

Choosing the convenient functions $f(x) = \rho(x + iy)$ and $g(x) = 1$. Then,

$$L(y)^2 = (\int_I \rho(x+iy)dx)^2 = (\int_I 1\rho(x+iy)dx)^2 \leq (\int_I dx)(\int_I \rho(x+iy)^2 dx) = \epsilon(\int_I \rho(x+iy)^2 dx)$$

Integrating both sides with respect to y, we obtain the desired result. □

Now we will prove a property that will be useful to prove further results.

Corollary 4.8. *Set $S = \{y \in I,\ L(y) \leq \sqrt{2\mathcal{A}}\}$. Then if we denote the Lebesgue measure as $leb()$, $leb(S) \geq leb(I)/2$*

Proof. First recall that if U, V are a pair of sets such that $leb(U \cap V) = 0$, then $leb(U \cup V) = leb(U) + leb(V)$. So in this case, as $S \subset I$, $\epsilon = leb(I) = leb(I \setminus S \cup S) = leb(I \setminus S) + leb(S)$. Using the previous lemma,

$$\epsilon \mathcal{A} \geq \int_I L(y)^2 dy = \int_{I \setminus S} L(y)^2 dy + \int_S L(y)^2 \geq \int_{I \setminus S} (\sqrt{2\mathcal{A}})^2 = 2\mathcal{A} leb(I \setminus S)$$

Hence, $leb(S) = leb(I) - leb(I \setminus S) \geq \epsilon - \epsilon/2 = \epsilon/2$ □

In order to use these results in $\overline{\mathbb{C}}$, it seems a good idea to consider a map from $I^2 \to \overline{\mathbb{C}}$. If we restrict it to be a conformal isomorphism, we can use the metric of $\overline{\mathbb{C}}$ to define a metric $\rho(z)|dz|$ in I^2. Using the corollaries of the section we can see that almost all horizontal (and vertical) line segments are mapped to finite curves and the majority of these ones have spherical length at most $\sqrt{2\mathcal{A}}$.
Now, sharpening the conditions a little bit, we can consider the Riemann map $\phi : \mathbb{D} \to U$ for a simply connected set U with $\infty \notin U$.

Lemma 4.9. *For almost every $\theta \in \mathbb{R}$ the curve $\alpha(r) = re^{i\theta}$ is mapped to a curve of finite length in U.*

Proof. If $\mathbb{D}$ was some set of the kind of I^2 we would already obtain the result. In order to follow this approach, consider the map $(x, y) \to e^{x+iy}$. Note that the modulus of the image is e^x, so it is interesting to consider $x < 0$, so $|e^{x+iy}| \leq 1$. Moreover, in order to be injective, we should only consider $y \in [0, 2\pi)$ (actually, every interval of length 2π would have a great fit, but just for simplicity we consider the most common one). Hence, applying what we have seen, the result is straightforward. □

A final result will be used in an essential lemma of the next session. We will not show the proof as it has some concepts that are tedious to explain and prove, but the proof essentially uses those concepts and what we have seen so far.

Theorem 4.10. *In the same case as last theorem, the radial limit*

$$limit_{r\to 1}\phi(\alpha(r)) = \phi(re^{i\theta}) \in \partial U$$

exists for almost every $\theta \in \mathbb{R}$. Conversely, for a given $w \in \partial U$, for almost every value $\theta \in \mathbb{R}$,

$$limit_{r\to 1}\phi(re^{i\theta}) \neq w$$

4.3 Proof of the Carathéodory's Theorem

With these two points of view we can prove the Carathéodory's Theorem, a way to connect local connectivity with extension maps. First, let's formally define what local connectivity means.

Definition 4.11 (Local connectivity)**.** *A set X is* ***locally connected*** *if for every point $x \in X$ there exist arbitrarily small connected neighbourhoods of x. There are other possible definitions:*

- *Imposing those neighbourhoods to be open*
- *If every open subset of X can be written as the union of connected open sets of X*

If X is a metric space, we can also give an alternative:

Definition 4.12 ($\epsilon - \delta$ definition)**.** *X is locally connected if for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any two points of X with $d(x, y) < \delta$, they are contained in a small connected subset of diameter ϵ.*

Now we are a few steps away of proving the Carathéodory's theorem, but for now we will have to prove some lemmas. Again, set U a simply connected set (with $\infty \notin U$) and let $\phi : \mathbb{D} \to U$ the corresponding biholomorphic map.

Lemma 4.13. *If there exist two values r_1, r_2 such that the curves $\phi_1(r) = \phi(re^{i\alpha})$, $\phi_2(r) = \phi(re^{i\beta})$ hold that $\phi_1(r_1) = \phi_2(r_2) = z_0 \in \partial U$ then z_0 disconnects ∂U.*

Proof. It is an application of Jordan curve theorem. □

Lemma 4.14. *Set $e^{i\theta}$ for $\theta \in \mathbb{R}$. Then, there exists a fundamental chain $\{N_i\}$ that converges to $e^{i\theta}$ and, moreover, $\{\phi(N_i)\}$ is a fundamental chain in U.*

Proof. First, we shall recall the strong similarity of the unit disk with the half-planes. In this case, we will use the following set: $\mathbb{H}^- = \{(x, y) \in \mathbb{C},\ x < 0\}$ and the biholomorphic map $f : \mathbb{H}^- \to \mathbb{D}$ such that $f(x + iy) = e^x e^{iy}$ (note that, indeed, f is the exponential map $f(z) = e^z$). As $\partial\mathbb{H}^-$ is a straight line, it can be easier (at least, visually) to build the fundamental chain. Indeed, we are going to take some appropriate rectangles. First, suppose we have already built the first $i - 1$ rectangles. Now let $K_i^- = \{x + iy \in \mathbb{H}^-,\ x \in (-\epsilon, 0), y \in (y_{1,i}, y_{2,i})\}$. In this case, we can choose $\epsilon, y_{1,i}, y_{2,i}$ in function of $\delta < 1/i$ such that:

- $\epsilon = -2\delta$
- $y_{1,i} = \theta - \delta$
- $y_{2,i} = \theta + \delta$
- $K_i^- \subset N_{i-1}^-$

The area of each K_i^- tends to zero as i tends to infinity (the area is, at most, $\frac{4}{i^2}$). Now, mapping K_i^- through f, we obtain a region in $\mathbb{D}$. Therefore, we can map it through ϕ so that we obtain another region on U, whose area vanishes when i tends to infinity.
Note that $\sqrt{2\mathcal{A}} = 2\sqrt{2}\delta$. By corollary 4.8, the set of points y such that $L(y) \leq \sqrt{2\mathcal{A}}$ is not empty (as it has a measure strictly bigger than zero). Hence, we can choose two points c_1, c_2 such that $y_{1,i} < c_1 < \theta < c_2 < y_{2,i}$ and the length of both horizontal segments $L(c_1)$, $L(c_2)$ tend to zero (as they are at most $\sqrt{2\mathcal{A}}$, and the area tends to zero). Mapping to $\mathbb{D}$ under f, we find two segments that its length also tend to zero. We may also choose an appropriate δ so that the length of the vertical arc is also smaller than $\sqrt{2\mathcal{A}}$.
The only problem could arise if, somehow, the images of the boundaries of the segments achieved endpoints, crossed with one another... If this was not the case, we would choose the rectangle $N_i^- = \{x + iy, \ -\epsilon < x < 0 \ \textit{and} \ c_1 < y < c_2\} \subset K_i^-$, and we would prove the lemma by using the images under f and $\phi \circ f$. Recalling the theorem 4.10 and the lemma 4.14, we have wiped out any possible problem. □

We can extend this result to the map ϕ^{-1}, making it sharper.

Lemma 4.15. *Paths of U that land into a point on ∂U map to paths of $\mathbb{D}$ that land into a point on $\partial\mathbb{D}$. Moreover, two different paths of U that land into distinct boundary points behave similarly in $\mathbb{D}$.*

Proof. Set the biholomorphic map $\phi : \mathbb{D} \to U$. Suppose that $\alpha : [0,1) \to U$ is a curve in U that tends to a well defined point in ∂U. Now we can consider the curve $\phi^{-1}(\alpha(t))$ and we can choose an accumulation point $e^{i\theta}$. We will prove that this point is actually a limit. Using the techniques we have learnt, let $\{N_i\}$ be a fundamental chain converging to $e^{i\theta}$. Recall that $\{\phi(N_i)\}$ is also a fundamental chain by lemma 4.14. We will prove that $\phi^{-1}(\alpha(t))$ always stays in the fundamental chain for t close enough to 1, imposing that it is, indeed, the limit.
Suppose that this is not the case, so there exists a sequence of values t_j and an index k such that $\phi^{-1}(\alpha(t_j))$ does not belong to N_k. Hence, the curve will cross infinitely many times the region delimited by A_k and A_{k+1}. Therefore, the image under ϕ will cross infinitely many times the corresponding region in U. This is not possible as we supposed that there the curves tend to a point p when $t \to 1$ but we have seen that for any t close enough to 1 there will be a $1 > t_j > t$ such that $d(\alpha(t_j), p) > d(\alpha(t_j), A_{k+1}) > 0$ (due to the fact that consecutive crosscuts do not intersect).

The second part is easier. Suppose that $\alpha, \ \beta : [0,1) \to U$ converging to different points but both $\phi^{-1}(\alpha(t))$, $\phi^{-1}(\beta(t))$ converge to the same point when $t \to 1$. Choosing a similar fundamental chain in $\mathbb{D}$ we have that any crosscut will cut both curves, but taking images under ϕ we would obtain that the different limit points would be contained in the limit, but it is impossible as diameter of the crosscuts tends to zero.

□

Corollary 4.16. *Fundamental chains on U map on fundamental chains on $\mathbb{D}$ under ϕ^{-1}.*

Proof. This result is immediate using last lemma and lemma 4.14.
Suppose that $\{N_i\}$ is a fundamental chain on U and let A_i be the respective crosscuts. We have seen in the previous lemma that the images under ϕ^{-1} of paths that land on well defined points on ∂U land on well defined points on $\partial\mathbb{D}$, so $\phi^{-1}(A_i)$ are crosscuts and therefore $\phi^{-1}(N_i)$ are crosscut neighbourhoods. By the second part of previous lemma, the new crosscuts $\phi^{-1}(A_i)$ are disjoint (paths that land on different boundary points are mapped to the same kind of paths). Hence, we have all the properties except that the diameter of the crosscuts must tend to zero. To prove this, we simply have to choose an accumulation point $e^{i\theta} \in \partial\mathbb{D}$ of the crosscuts $\phi^{-1}(A_i)$. Then we can pick a fundamental chain $\{\widetilde{N_j}\}$ converging to this point. By lemma 4.14, $\{\phi(\widetilde{N_j})\}$ is a fundamental chain in U. As $e^{i\theta}$ is both an accumulation point of $\phi^{-1}(N_i)$ and the limit of $\{\widetilde{N_j}\}$, $\phi^{-1}(N_i) \cap \widetilde{N_j} \neq \emptyset$ for every i, j, so, as ϕ is homeomorphic in $\mathbb{D}$, $N_i \cap \phi(\widetilde{N_j}) \neq \emptyset$. As both $\{N_i\}$, $\{\phi(\widetilde{N_j})\}$ are fundamental chains in U, they are equivalent. Hence, $N_i \subset \phi(\widetilde{N_j})$ for some i, j and vice-versa, and taking images under ϕ^{-1}, we obtain that the diameter of $\phi^{-1}(N_i)$ will tend to zero. □

With these results we have stated the relation between prime ends of $\mathbb{D}$ and U. We define the **Carathéodory compactification** $\hat{U}$ as the union of U with all its prime ends. This definition brings along a new topology, whose basis is defined as follows: for any crosscut neighbourhood N, group N itself and all possible prime ends coming from further fundamental chains contained in N.

Using the previous lemmas we have seen that any prime end of $\mathbb{D}$ is a well defined point on $\partial\mathbb{D}$ and any point of $\partial\mathbb{D}$ is the impression of only one prime end. Therefore $\widehat{\mathbb{D}} \cong \overline{\mathbb{D}}$. By the previous relations of the prime ends of $\mathbb{D}$ and the prime ends of U, we can extend the Riemann map ϕ onto the Carathéodory compactifications, $\phi : \widehat{\mathbb{D}} \to \widehat{U}$.
Following the same idea we can state the main theorem of the chapter, but just before that, a purely topological lemma.

Lemma 4.17. *Let $f : X \to Y$ be a continuous function, X be compact and locally connected and Y be the image ($f(X) = Y$) and Hausdorff. Then Y is also compact and locally connected.*

Proof. Let $\{U_i\}_{i=1}^{\infty}$ be an open cover of Y, then $\{f^{-1}(U_i)\}_{i=1}^{\infty}$ is an open cover of X, so there exists a finite subcover $\{f^{-1}(U_i)\}_{i=1}^{k}$. Taking images of f and using that $f(X) = Y$, we obtain the compactness.

Using a similar technique, let $y \in Y$ and let $N(y)$ be an open (not necessarily connected) neighbourhood of y. Then consider the compact set $f^{-1}(y)$ and its neighbourhood $f^{-1}(N(y))$. As X is locally connected, this neighbourhood can be written as the union of connected open sets $\bigcup_{i=1}^{\infty} U_i$ that intersect $f^{-1}(y)$. Taking images under f we obtain a union of connected sets, being a connected subset of N. We require it to be a neighbourhood of y but this already holds as it contains the set $Y \setminus f(X \setminus \bigcup_{i=1}^{\infty} U_i)$, due to the fact that $\bigcup_{i=1}^{\infty} f(U_i) = Y \setminus (Y \setminus \bigcup_{i=1}^{\infty} f(U_i)) \supset Y \setminus f(X \setminus \bigcup_{i=1}^{\infty} U_i)$. □

Finally, we can prove the main theorem of the chapter.

Theorem 4.18 (Carathéodory)**.** *Let $\phi : \mathbb{D} \to U$ be a biholomorphic map. Then ϕ can be extended analytically onto the boundaries if and only if ∂U is locally connected (or equivalently, if the complementary of U is locally connected).*

Proof. First we are going to suppose that either $\overline{\mathbb{C}} \setminus U$ or ∂U are locally connected. The idea is to choose appropriate ϵ and δ of the $\epsilon - \delta$ definition 4.12. Consider a fundamental chain in U, $\{N_j\}$. Our goal is to see that the impression is a single point, and then, using our previous arguments, it is equivalent to say that ϕ extends onto the boundary. Set ϵ and δ such that $diameter(A_j) < \delta$ for a given j so that the two endpoints are in a compact connected set of diameter less than ϵ. Denote this set as $Y \subset \overline{\mathbb{C}} \setminus U$. Now consider the new (compact) set $Y \cup A_j$, which separates N_j and $U \setminus \overline{N_j}$. If this was not the case we could find a curve (that would not intersect $Y \cup A_j$) that would join a point in N_j and a point in $U \setminus \overline{N_j}$, and we could find another curve joining these two points but crossing A_j. The union of the curves would be a Jordan curve, contradicting the connectivity of Y.
Note that as $diameter(Y) < \epsilon$ and $diameter(A_j) < \delta$, the diameter of the union cannot exceed $\epsilon + \delta$. The trick is to choose arbitrarily small values of ϵ. As we have advanced, let $\epsilon > 0$ such that:

- $\overline{\mathbb{C}} \setminus Y \cup A_j$ has a major connected component whilst the others have diameter less than $\epsilon + \delta$
- $diameter(U \setminus \overline{N_1}) > \epsilon + \delta$

With these conditions, $U \setminus \overline{N_1}$ must be contained in the major connected component, so N_k has a diameter at most $\epsilon + \delta$ for every $k \geq 1$. As we can always reduce ϵ, the diameter of N_j tends to zero for further indexes, which implies that the impression has only a point.

Now we will prove the other part. Suppose that ϕ extends analytically onto the boundary. So, as ϕ was a biholomoprhic map from $\mathbb{D}$ to U, boundaries map onto themselves. As $\partial\mathbb{D}$ is locally connected and $\overline{\mathbb{C}}$ is a metric space (and therefore, a Hausdorff space), ∂U is locally connected by previous lemma. Now it only remains to look at $\overline{\mathbb{C}} \setminus U$. As points bounded away of ∂U are not an inconvenient, we shall see the case for $w \in \partial U$. Using that ∂U is locally connected, we can choose a connected neighbourhood $N_w \subset \partial U$. Then set a small enough (connected) ball B that intersects N_w. Hence, the set $(B \cap U) \cup N_w$ is a connected neighbourhood of w. □

Corollary 4.19. *The same ϕ can be extended onto the boundaries if ∂U is a Jordan curve.*

Chapter 5

Main theorem

After setting up all preliminary background in the preceding sections, we are now ready to state and prove the main theorem of this thesis. The main references of this last chapter are [BEA] and [MIL], with some complementary results of [FB] and [WD].

We start first with the concept of hyperbolicity of a rational map.

Definition 5.1 (Change of scale)**.** *Let R be a rational map of degree at least 2 and let $\omega(z)|dz|$ be a well defined density of a metric in the Julia set $\mathcal{J}(R)$ (that is to say, defined in an open set W containing $\mathcal{J}(R)$). Given any point $z \in W$ such that $z \in R^{-1}(W)$ we can define the* ***change of scale*** *of R at z as*

$$||R'(z)||_\omega = \frac{\omega(R(z))|R'(z)|}{\omega(z)}$$

Notice it is well defined, as ω is a Riemannian metric. Now, we introduce one of the most important definitions:

Definition 5.2 (Hyperbolic map)**.** *The same map R is said to be* ***hyperbolic*** *if it is expanding on $\mathcal{J}$, which means that there exist $c > 0$ and $\lambda > 1$ such that*

$$||(R^n)'(z)||_\omega \geq c\lambda^n$$

Remark 5.3. The first thing to notice is that we can choose an arbitrary metric. Indeed, suppose that $\omega_1(z)|dz|$ and $\omega_2(z)|dz|$ are two metrics and R is expanding with respect to ω_1. Looking at the map $z \to \frac{\omega_1(z)}{\omega_2(z)}$, we see it is continuous and well defined. Furthermore, as $\mathcal{J}$ is compact, for any $z \in \mathcal{J}$ there exist two constants m, M such that $m < \frac{\omega_1(z)}{\omega_2(z)} < M$. Using that

$$||R'(z)||_{\omega_1} = \frac{\omega_1(R(z))|R'(z)|}{\omega_1(z)} \text{ and } ||R'(z)||_{\omega_2} = \frac{\omega_2(R(z))|R'(z)|}{\omega_2(z)}$$

We conclude:

$$\frac{||R'(z)||_{\omega_1}}{||R'(z)||_{\omega_2}} = \frac{\omega_1(R(z))\omega_2(z)}{\omega_1(z)\omega_2(R(z))} \leq \frac{M}{m}$$

And hence,

$$||R'(z)||_{\omega_1} \leq (M/m)||R'(z)||_{\omega_2}$$

Repeating the process with the map $z \to \frac{\omega_2(z)}{\omega_1(z)}$ we end up with two inequalities of the form

$$||R'(z)||_{\omega_1} \leq A||R'(z)||_{\omega_2} \leq B||R'(z)||_{\omega_1}$$

for $A, B > 0$, which gives us the result using the expanding hypothesis.

The second thing that we can deduce from the definition is that if $||R'(z)||_\omega \geq k > 1$, then R maps paths of length $d(\psi)$ to paths of length larger than or equal to $kd(\psi)$, which implies by definition that distances are multiplied by a factor of k, too. This value k is said to be the **expansion constant**.

Now we will give a strong criterion to find hyperbolic maps.

Theorem 5.4. *Let R be a rational map of degree $d \geq 2$. Each critical point of R accumulates at a (super)attracting orbit of R (or equivalently, $\overline{P} \cap \mathcal{J} = \emptyset$, where $\overline{P}$ is the closure of the postcritical set of R) if and only if R is hyperbolic.*

Proof. Let us prove the forward implication first. Let P be the postcritical set of R and let V be the union of disjointed discs around attracting cycles of R, with $R(V) \subset V$. If we suppose these discs to be small enough, we can assume $W = \overline{\mathbb{C}} \setminus (P \cup V)$ to be connected.
Moreover, as $R(P) \subset P$ and $R(V) \subset V$ by definition, we have that $R^{-1}(W) \subset W$. Denote $\rho(z)|dz|$ the hyperbolic metric on W and $\pi : \mathbb{D} \to W$ the universal covering map. Let S be a well defined branch of R^{-1} in W (notice that $P \cap W = \emptyset$) with a point w such that $\pi(0) = w$ and $R^{-1}(w) \neq w$. With this definition there exists a point $\theta \in \mathbb{D}$ such that $\pi(\theta) = S(w) \neq w$.
In order to apply the Schwarz-Pick lemma, we must build a map going from $\mathbb{D}$ to $\mathbb{D}$. Therefore we may look at the composition $\varphi = \pi^{-1} \circ S \circ \pi : \mathbb{D} \to \mathbb{D}$. Looking at the value at zero, $\varphi(0) = \pi^{-1} \circ S(w) = \pi^{-1}(\pi(\theta)) = \theta$ we can extend it analytically in $\mathbb{D}$ to obtain a holomorphic map $\varphi : \mathbb{D} \to \mathbb{D}$.
Hence, by the Schwarz-Pick lemma, φ is a contraction. As π is a local isometry, R^{-1} is also a contraction, which means R is expanding.

Now let's prove the other implication.
Suppose R is hyperbolic and $\rho(z)|dz|$ a given metric on a neighbourhood of $\mathcal{J}$, V, with the condition that $||R'(z)||_\rho \geq k > 1$. It is crucial to notice that there cannot be any critical point c in V, as if this was the case,

$$1 < ||R'(c)||_\rho = \frac{\omega(R(c))|R'(c)|}{\omega(c)} = 0$$

Let's define the $\epsilon-$neighbourhood of $\mathcal{J}$, $N_\epsilon(\mathcal{J})$, as the set of points of $\overline{\mathbb{C}}$ that are at a distance strictly less than ϵ from the Julia set. Then, we can choose this value ϵ if this set follows two conditions:

1. For every point $z \in N_\epsilon(\mathcal{J})$ there exists a geodesic ψ_z in V that joins z and $\mathcal{J}$
2. $R^{-1}(N_\epsilon(\mathcal{J})) \subset V$

Now using the second condition, the definition of the change of scale and the invariance of the Julia set, $d_\rho(R(z), \mathcal{J}) \geq kd_\rho(z, \mathcal{J})$, for $z \in R^{-1}(N_\epsilon(\mathcal{J}))$
Again, for $z \in N_\epsilon(\mathcal{J})$, a minimal geodesic between z and $\mathcal{J}$ will be included in $N_\epsilon(\mathcal{J})$, so taking

a branch of the preimages, we will have a path of length not longer than $d_\rho(R(z), \mathcal{J})/k$.
Therefore, every orbit starting at $N_\epsilon(\mathcal{J}) \setminus \mathcal{J}$ will eventually leave this set as distances keep growing. As $\mathcal{J} \subset N_\epsilon(\mathcal{J})$, the limit belongs to the Fatou set where we can choose an accumulation point w. If U is the connected component, then by the accumulation feature, there exists an integer $n \geq 1$ such that $R^n(U) \subset U$. By the Sullivan Classification of Fatou components, we only have four possibilities, but the non-presence of any critical points in V rules out a lot of possibilities. Not only it cannot be a parabolic basin (by definition) but also it cannot be a rotation domain, due to theorem 3.47. So, the orbit of a critical point will converge to an attracting periodic orbit, and hence, $\overline{P} \cap \mathcal{J} = \emptyset$. □

Note that this theorem implies that, for hyperbolic maps, every orbit of the Fatou set converges to an attracting periodic orbit. We can now state the main theorem of the thesis.

Theorem 5.5 (Main theorem)**.** *Let R be a hyperbolic rational map of degree $d \geq 2$ with a connected Julia set. Then the Julia set is locally connected.*

To prove the theorem we will need both Whyburn's theorem and two forecoming lemmas.

Theorem 5.6 (Whyburn's theorem)**.** *Let $X \subset \overline{\mathbb{C}}$ be a compact set. Then X is locally connected if and only if the following conditions are satisfied:*

- *For every $\epsilon > 0$, the number of connected components of $\overline{\mathbb{C}} \setminus X$ with diameter greater than ϵ is finite.*
- *The boundary of each connected component of $\overline{\mathbb{C}} \setminus X$ is locally connected*

Proof. We will only prove one of the implications, as it is the only one we will use. Also, we will consider the spherical metric $\rho(z)|dz|$. Set $x \in X$ and define $N_\epsilon(x) = \{z \in \overline{\mathbb{C}},\ d_\rho(x,z) < \epsilon\}$. Take into account that $N_\epsilon(x)$ could not be included in X. Following the equivalent definitions of local connectivity, we would like to find another neighbourhood $N_\delta(x)$ with $\delta < \epsilon$ such that every point of $X \cap N_\delta(x)$ can be connected to x by a connected subset of $N_\epsilon(x) \cap X$, giving us the result we want.
We are going to denote U_j the complementary connected components, as we will use this theorem with $\overline{\mathbb{C}} \setminus \mathcal{J}$ (so they will be Fatou components). Notice that ∂U_j will intersect X as X is closed and U_j is a complementary connected component. Set $\delta < \epsilon/2$ with the following condition: for every $x_\delta, y_\delta \in \partial U_j$ (for a given j) with $d_\rho(x_\delta, y_\delta) < \delta$, they can be connected through a connected set of ∂U_j with diameter strictly less than $\epsilon/2$. Note that for the connected components of diameter strictly less than $\epsilon/2$, we do not have any problem. With the other ones, we must remember that there is a finite number of them, by assumption
Proceed as follows: set $y \in X \cap N_\delta(x)$ and consider the geodesic $I_{x,y}$. Now, replace every connected component of $I_{x,y} \setminus X$ by boundaries of a suitable U_j, which will be of diameter strictly less than $\epsilon/2$. After this construction, x and y are in a subset with the following properties:

- It is connected due to the local connectivity of complementary components
- It is contained in $N_\epsilon(x)$ by the sum of the diameters.
- It is in X as ∂U_j lies there

So, X is locally connected.

□

We will use Whyburn's theorem to prove the main result. Hence, we need to prove the two equivalent conditions of the previous theorem. The first condition will be much easier to prove, whilst the second one needs a little more detail.

Theorem 5.7 (First condition)**.** *Consider the spherical metric ρ on $\overline{\mathbb{C}}$. Suppose R is a hyperbolic rational map of degree at least* 2*. Let $\mathcal{U}$ be the set formed by all Fatou components. Then for every $\epsilon > 0$, all but finitely many elements of $\mathcal{U}$ have diameter smaller than ϵ.*

Proof. We are going to use similar techniques to those in theorem 5.4. Indeed, suppose $k > 1$ is the expansion constant in a neighbourhood V of $N_\epsilon(\mathcal{J})$ with ϵ such that $R^{-1}(N_\epsilon(\mathcal{J})) \subset N_\epsilon(\mathcal{J})$ and $\overline{N_\epsilon(\mathcal{J})}$ is contained in V. It would be a mistake to directly consider the spherical metric ρ, as the conditions of the theorem 5.4 ensure that there exists a choice of the metric that holds the aforementioned conditions, but it does not ensure if it is, indeed, the spherical one.

Using the distance feature, for every $U \in \mathcal{U}$ in $N_\epsilon(\mathcal{J})$, it holds that $diam_\omega(R^{-1}(U)) \leq diam_\omega(U)/k$, being $R^{-1}(U)$ a component.
Now consider the open cover of $\overline{\mathbb{C}}$, $\bigcup_{j=1} U_j \cup N_\epsilon(\mathcal{J})$ for $U_j \in \mathcal{U}$. By the compactness of Riemann Sphere, there exists an open subcover $U_1 \cup U_2 \cup \ \ldots\ \cup U_i \cup N_\epsilon(\mathcal{J})$ (with relabelled indexes). Therefore, it does not matter whether there are finitely or infinitely many Fatou components, but we have seen that there are only finitely many ones that land outside $N_\epsilon(\mathcal{J})$.
Using the theorem 5.4, after iteration by R, these components eventually escape from $N_\epsilon(\mathcal{J})$. There exist some (minimal) values n_j, what we call levels, such that each $R^{\circ n_j}(U_j)$ escapes from $N_\epsilon(\mathcal{J})$ (following attracting orbits).
Some of them may be already immediate basins of attraction, so in that case that value will be zero (in this case, these Fatou components are $U_1, ..., U_i$). There are only finitely many components such that their level is equal to 1, as if this was not the case, we would obtain infinitely many components of level 0, contradicting the last statement. Inductively, fixing a level, there are finitely many components with that level.
Set $M = max\{diam_\omega(U),\ U \in \mathcal{U}$ and its level is $1\}$. Then, if U has level m,

$$diam_\omega(U) \leq M/k^{m-1}$$

Taking into account that $\overline{N_\epsilon(\mathcal{J})}$ is compact, we use the remark 5.3 to obtain the same result using the spherical metric. □

Theorem 5.8 (Second condition)**.** *Let R be a hyperbolic rational map of degree strictly greater than* 1 *with a connected Julia set. Then every Fatou component U has a locally connected boundary.*

Proof. We will focus on the case that $R(U) \subset U$ as the case of a periodic Fatou component will be covered applying the result for $R^{\circ k}$. Using theorem 5.4, set z_0 as an attracting fixed point and $U = A_0(z_0)$ its immediate basin of attraction. As $\mathcal{J}$ is connected, using theorem 3.29 and the definition of Julia set, U must also be simply connected.
Applying what we saw in lemma 3.25 and using that U is simply connected, we can find a conformal isomorphism $\psi : \mathbb{D} \to U$ such that $\psi(0) = z_0$. As the Julia set maps to itself, we can consider the proper map (A.23, A.24 and A.25) $B = \psi^{-1} \circ R \circ \psi : \mathbb{D} \to \mathbb{D}$ with $\psi(0) = 0$ (using that $R(z_0) = z_0$). Hence, B is a Blaschke product. Recalling lemma 3.25, there is at least a point $\widetilde{c} \in \mathbb{D}$ such that $\psi(\widetilde{c}) = c$ is a critical point of R. Then $B'(\widetilde{c}) = \left(\psi^{-1}\right)'(R(c))R'(c)\psi'(\widetilde{c}) = 0$ as c is a critical point of R, so B has as many critical points as R has in U (recall that ψ is a conformal isomorphism, so it does not have any critical points in $\mathbb{D}$). Using again that ψ is a conformal

isomorphism, the degree of B, d, will be the same as the degree of $R_{|U}$, which is greater than 1, because the presence of critical points in $\mathbb{D}$ implies, by Schwarz lemma, that $B(z) \neq e^{i\theta}z$.
From now on, we try to extend ψ onto the boundaries so that we can apply Carathéodory's theorem to confirm that ∂U is locally connected. To do this, we are going to search an appropriate sequence of curves starting at $\mathbb{D}$ and then going to U, converging uniformly to the corresponding boundaries.

First, note that, as the Julia set does not contain any critical points (theorem 5.4), the corresponding critical points of B (denoted by $crit(B)$) are contained in a disk centered at 0 with radius $0 < r' < 1$, $D(0, r')$ (as there are only finitely many critical points of R). Here I used the $D(0, r')$ notation instead of the $B_{r'}(0)$ one to avoid confusions with de Blaschke product. Considering a possibly larger $r < 1$, choose another disk $D(0, r)$ such that:

1. $\overline{B(D(0,r))} \subset D(0,r)$: This can be held applying the Schwarz lemma, which states that holomorphic maps in the disk are contractions or rotations (and we have wiped out the rotation possibility).

2. $crit(B) \subset D(0,r)$

Denote $\widetilde{\gamma}_0$ the boundary curve of $D(0,r)$, which is a Jordan curve (C.10). Now we shall study the components of $B^{-1}(\widetilde{\gamma}_0)$. First of all, by the first condition, there is one of the components, denoted as $\widetilde{\gamma}_1$, such that $\widetilde{\gamma}_0 \subset int(\widetilde{\gamma}_1)$. We will see that $\widetilde{\gamma}_1$ is, indeed, a Jordan curve. For that, note that it is a closed curve as B is continuous. Then we must check that $\widetilde{\gamma}_1$ is simple. Arguing by contradiction, suppose that $\widetilde{\gamma}_1$ is not simple. Then, there exists a point w of $\widetilde{\gamma}_1$ where the curve crosses itself. Using that $crit(B) \cap \widetilde{\gamma}_1 = \emptyset$, we obtain that the derivative of B in w does not vanish. Hence, there exists a neighbourhood of the point in the curve that is homeomorphic to the image, but it contradicts the fact that $\widetilde{\gamma}_0$ is a simple curve.

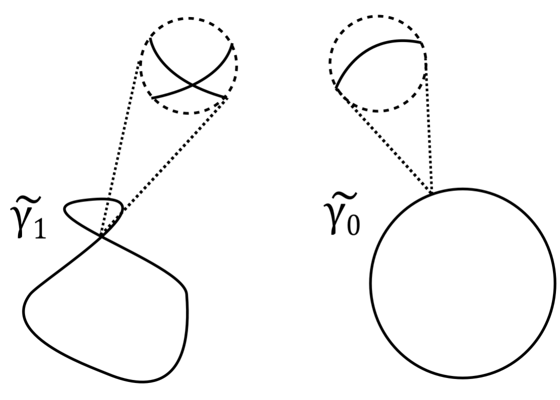


Figure 5.1: There cannot be a local homeomorphism

After that, we define γ_0 as $\psi^{-1}(\widetilde{\gamma_0}) \cap U$ and γ_1 also as $\psi^{-1}(\widetilde{\gamma}_1) \cap U$. In order to parameterize both curves, first we start with an arbitrary parameterization of γ_0, setting $\gamma_0 : [0,1] \to U$. Next, we are going to parameterize γ_1 in a very careful way. Set $\gamma_1 : [0,1] \to U$ with $R(\gamma_1(t)) = \gamma_0(dt \ mod\ 1)$ (taking into account that d is the degree of B).

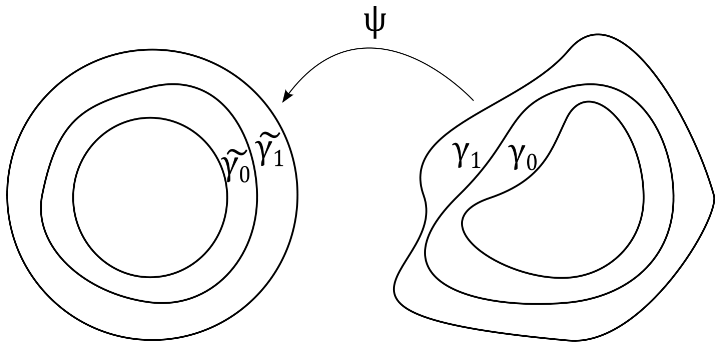


Figure 5.2: Representation of the curves

Following the same procedure we obtain two sequences of nested curves $\{\gamma_k\}_{k\geq 0} \subset U$ and $\{\widetilde{\gamma}_k\}_{k\geq 0} \subset \mathbb{D}$, with $R(\gamma_k(t)) = \gamma_{k-1}(dt \ mod \ 1)$. Arguing likewise, these curves are Jordan curves. Furthermore, the sequence γ_k tends to ∂U as k tends to ∞. The reason is quite simple: γ_k is a curve in the immediate basin of attraction of z_0 that holds that, after applying R k times, it maps to γ_0 (and consequently, the interior of γ_k will map to the interior of γ_0). If these curves did not tend to the boundary, there would be an accumulation point $w \in U$. Thus, the points in a small enough neighbourhood of w (contained in U) would require an infinite number of iterates to converge to the neighbourhood of z_0 bounded by γ_0, contradicting the fact that these curves belong to $U = A_0(z_0)$. In addition to that result, we should highlight that, if we denote the interior component of the curves γ_k as D_k, then $U = \bigcup_{k\geq 0} D_k$, with $D_k \subset D_{k+1}$. Using that ψ is a conformal isomorphism, the sequence of curves $\{\widetilde{\gamma}_n\}_{n\geq 0}$ tends to the boundary of the unit disk.
The idea is to see that $\{\gamma_n\}_{n\geq 0}$ is a Cauchy sequence such that $|\gamma_n(t) - \gamma_m(t)| \to 0$ uniformly on t.
But first, the reader may have noticed that we have not taken into account a (crucial) hypothesis yet. Exactly, we have not used the hyperbolic hypothesis! Then, set V the neighbourhood of $\mathcal{J}$ such that $k > 1$ is an expansion constant. Denote the compact set $K = \partial V \cap U$ and using that $U = \bigcup_{k\geq 0} D_k$ (where $D_k \subset D_{k+1}$), there exists a number n_0 such that for every $n \geq n_0$, $K \subset D_n$. As we are going to focus on (only) the corresponding curves, we can relabel them as follows:

$$\gamma_{n_0} \to \gamma_0,\ \gamma_{n_0+1} \to \gamma_1 ... \ and \ \widetilde{\gamma}_{n_0} \to \widetilde{\gamma}_0, \widetilde{\gamma}_{n_0+1} \to \widetilde{\gamma}_1$$

Next, it will be useful to estimate the distance between two consecutive curves $|\gamma_{n+1}(t) - \gamma_n(t)|$. Recall that, from the definition of the length of a curve (A.8), given $f \in \mathcal{H}(\Omega)$ and $\alpha : [a, b] \to \Omega$ a curve with endpoints z_1 and z_2,

$$length(f(\alpha)) = \int_\alpha |f'(z)|dz = \int_a^b |f'(\alpha(t))||\alpha'(t)|dt \leq max_{w\in[z_1,z_2]}(|f'(w)|)length(\alpha)$$

As the Euclidean distance between two points is defined to be the minimum of the lengths of the paths that joins them, we obtain a trivial consequence of the last inequality:

$$|f(z_2) - f(z_1)| \leq max_{w\in[z_1,z_2]}|f'(w)||z_2 - z_1|$$

where $[z_1, z_2]$ denotes the straight segment between z_1 and z_2. Let g be the well defined branch of R^{-1} such that $\gamma_n(t) = g(\gamma_{n-1}(dt \ mod \ 1))$. Then,

$$|\gamma_{n+1}(t) - \gamma_n(t)| = |g(\gamma_n(dt \ mod \ 1)) - g(\gamma_{n-1}(dt \ mod \ 1))| \leq$$

$$\leq max_{z\in[\gamma_n(dt\ mod\ 1),\gamma_{n-1}(dt\ mod\ 1)]}(|g'(z)|)|\gamma_n(dt\ mod\ 1)-\gamma_{n-1}(dt\ mod\ 1)|$$

But before using this inequality in our problem we have to pay attention to the segment, as it could not be in U. To solve this issue, we are going to take advantage of the parameterization we have defined before. The idea will be that for every $t\in[0,1]$ and $n\geq 0$ we will define a curve of finite length $\alpha_n^t:[0,1]\to U$ such that

- $\alpha_n^t(0)=\gamma_n(t)$
- $\alpha_n^t(1)=\gamma_{n+1}(t)$
- $\alpha_n^t(s)\in int(\gamma_{n+1})\setminus\overline{int(\gamma_n)}$ for $s\in(0,1)$ (what we call the annulus bounded by γ_n and γ_{n+1}, A_n)

Moreover, we would like these curves to be related to g, so that they conserve the structure that the γ_k curves have. Thus, we will start with the curves α_0^t and we will define the corresponding α_k^t iteratively.

Lemma 5.9. *Set A_0, γ_0 and γ_1 as before. Then, there exists a bound M such that any pair of points $w_0\in\gamma_0$ and $w_1\in\gamma_1$ can be joined together through a curve within A_0 of length at most M.*

Proof. Recalling B.17, there exists a value $0<r<1$ such that $\varphi:A_r\to A_0$ is a conformal isomorphism, where $A_r=\{z\in\mathbb{C},\ r<|z|<1\}$. We are going to use the following result, which can be found as the theorem 2.9 of [FB]:

Theorem 5.10. *Let G be a Jordan domain (a bounded domain whose boundary is composed by Jordan curves) and $\varphi:G\to\mathbb{C}$ be a conformal isomorphism. Then if ∂G is $\mathcal{C}^k$, we can extend φ onto the boundary of G such that the resultant map is $\mathcal{C}^{k-1}$.*

As both γ_0 and γ_1 are $\mathcal{C}^2$, we can extend φ onto the boundaries and the result is $\mathcal{C}^1$. Now, note that $\varphi^{-1}(w_0)$ and $\varphi^{-1}(w_1)$ can be joined by a curve $\widetilde{\alpha}$ within A_r of length at most 2π. Furthermore, as $\overline{A_r}$ is compact and φ is $\mathcal{C}^1$, $|\varphi'|$ is bounded by a constant $\widetilde{M}$. Hence, if we define $\alpha=\varphi(\widetilde{\alpha})$, α connects w_0 with w_1 and

$$length(\alpha)\leq max_{z\in\overline{A_r}}(|\varphi'(z)|)length(\alpha)\leq\widetilde{M}2\pi:=M$$

□

With this lemma we can choose a family of curves $\{\alpha_0^t\}$ holding the aforementioned conditions and denote M the bound of the lengths (whose finiteness has been proved in the previous lemma). Then, we can define the next families of curves in the corresponding annulus with the condition that $R(\alpha_k^t)=\alpha_{k-1}^{dt\ (mod\ 1)}$.

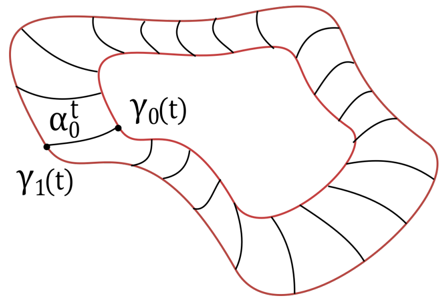


Figure 5.3: An example of the $\{\alpha_0^t\}$

So, for $n \geq 0$ we have that

$$|\gamma_{n+1}(t) - \gamma_n(t)| \leq length(\alpha_n^t)$$

as the distance is defined to be the infimum of the lengths of the curves joining $\gamma_{n+1}(t)$ and $\gamma_n(t)$. Thus, using A.8 (definition of the length of a curve) and the construction of α_n^t,

$$length(\alpha_n^t) = length(g(\alpha_{n-1}^{dt})) = \int |g'(\alpha_{n-1}^{dt})| = \int_0^1 |g'(\alpha_{n-1}^{dt}(s))||\alpha_{n-1}^{dt}{}'(s)|ds$$

where dt is considered to be mod 1. That last integral is bounded by

$$C\int_0^1 |\alpha_{n-1}^{dt}{}'(s)|ds = C \cdot length(\alpha_{n-1}^{dt})$$

where $C = max_{z\in\alpha_{n-1}^{dt}}(|g'(z)|)$. By the definition of α_{n-1}^{dt}, the derivative is computed in points of U, so we can apply the inequalities. As

$$1 = id'(z) = (R \circ g)'(z) = R'(g(z))g'(z)$$

and using the expansion constant, we obtain that $C \leq \frac{1}{k}$ Then, we can repeat the process until we arrive at the first $\alpha_0^{t'}$:

$$|\gamma_{n+1}(t) - \gamma_n(t)| \leq length(\alpha_n^t) \leq \left(\frac{1}{k}\right) length(\alpha_{n-1}^{dt(mod\ 1)}) \leq ... \leq \left(\frac{1}{k}\right)^n length(\alpha_0^{d^n t\ (mod\ 1)}) \leq \left(\frac{1}{k}\right)^n M$$

So finally, for every $m > n$,

$$|\gamma_m(t) - \gamma_n(t)| \leq |\gamma_{n+1}(t) - \gamma_n(t)| + |\gamma_{n+2}(t) - \gamma_{n+1}(t)| + ... + |\gamma_m(t) - \gamma_{m-1}(t)|$$

and using the previous result,

$$|\gamma_m(t) - \gamma_n(t)| \leq M\left(\frac{1}{k^n} + \frac{1}{k^{n+1}} + ... + \frac{1}{k^{m-1}}\right)$$

This bound shows two major consequences:

- Tending m to infinity, we can see that the bound represents a convergent series (as $k > 1$), so tending n to infinity we obtain the tail of a convergent series. Then, the bound must tend to zero

- The convergence is uniform on t

To sum up, we have proved that $\gamma_n(t) \rightrightarrows \gamma(t)$ for a certain curve $\gamma \subset \partial U$. But now, looking at the curves in $\mathbb{D}$, $\widetilde{\gamma}_n$, we have seen that they converge to the boundary of the disk, but also, $\psi(\widetilde{\gamma}_n(t)) = \gamma_n(t) \rightrightarrows \gamma(t)$, giving the extension of the map ψ onto the boundaries. Applying Carathéodory's theorem, we obtain that ∂U is locally connected.

□

Appendix A

Elementary results of Complex Analysis and the Riemann Sphere

A.1 The complex plane and holomorphic functions

In this first appendix we will introduce basic theorems of complex analysis that can be found in any introductory course. Different notations I am going to use:
D: a domain of the complex plane.
U: an open set of the complex plane.
$\mathbb{D}$: the unit disk (other authors use $\triangle$).

Definition A.1. *A function $f : U \to \mathbb{C}$ is $\mathbb{C}-differentiable$ in a point a of U if the following limit exists:*

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

A function that is $\mathbb{C}-$differentiable in every point of U is holomorphic.

Theorem A.2. *Set $f : U \to \mathbb{C}$ holomorphic. If $z = x + iy$, $Re(f) = u(x, y)$ and $Im(f) = v(x, y)$ then*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These are the ***Cauchy-Riemann equations***

Theorem A.3. *If f is holomorphic, f is analytic, which means that given a point of U, p, there exists a small enough neighbourhood of p, U' such that*

$$f(z) = \sum_{n \geq 0} a_n \frac{(z - p)^n}{n!}$$

for $a_n \in \mathbb{C}$ and $z \in U'$.

Theorem A.4 (Liouville)**.** *A holomorhpic function (in all $\mathbb{C}$) $f : \mathbb{C} \to \mathbb{C}$ (entire) that is bounded is, necessarily bounded.*

Theorem A.5 (Open mapping theorem)**.** *Holomorphic functions are open.*

Theorem A.6 (Maximum modulus principle)**.** *Set a holomorphic function $f : U \to \mathbb{C}$. For any $D \subset U$, if $sup_{z \in D}|f(z)| = max_{z \in D}|f(z)|$ then f is constant.*

Theorem A.7 (Uniqueness of analytic continuation)**.** *Let f be a holomorphic function defined in an open set U. Set W an open set such that $W \cap U \neq \emptyset$ and there exists an analytic extension on W of f, then it must be unique.*

Next we are going to see some theorems related to the integrals in the complex plane. First, this introductory definition.

Definition A.8. *Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function on a (smooth) curve $\alpha : [a, b] \to \mathbb{C}$. Hence, the integral of the function through the curve α is*

$$\int_\alpha f(z)dz = \int_a^b f(\alpha(t))\alpha'(t)dt$$

Hence, the length of the curve is

$$\int_a^b |\alpha'(t)|dt$$

Theorem A.9 (Cauchy)**.** *If $f : U \to \mathbb{C}$ is a holomorphic function, then for every closed curve in U, α,*

$$\int_\alpha f(z)dz = 0$$

Theorem A.10 (Morera's theorem)**.** *Set $f : U \to \mathbb{C}$. If for every triangle $\triangle \subset U$ we have*

$$\int_\triangle f(z)dz = 0$$

then f is holomorphic.

Now let's recall one of the most beautiful formulas I have been shown throughout my degree.

Theorem A.11 (Cauchy's integral formula)**.** *Let $f : U \to \mathbb{C}$ be a holomorphic function. Set $w \in U$. Denote $B_r(w) \subset U$ the disk of radius r and center w. Denote γ the simple closed piecewise smooth and positively oriented curve surrounding the disk. Then, for every $z_0 \in B_r(w)$,*

$$\int_\gamma \frac{f(z)}{z - z_0} = 2\pi i f(z_0)$$

And there is also a formula for the derivatives.

Theorem A.12 (Cauchy's integral formula for the derivatives)**.** *Denote $f^{(n)}$ the $n^{th}-$derivative of f. Under the same conditions of the function of the last theorem,*

$$\int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} = \frac{2\pi i}{n!} f(z_0)$$

Now we will prove a crucial theorem in this field.

Lemma A.13 (Schwarz lemma)**.** *Let $f : B_1(0) \to B_1(0)$ a holomorphic function such that$f(0) = 0$, then $f(z) \leq |z|$ and $f'(0) \leq 1$. If one of the inequalities is, indeed, an equality, the other one is also an equality and the function is necessarily of the kind $f(z) = e^{i\theta}z$.*

Proof. As $f(0) = 0$, we can factorize the function as follows:
$f(z) = zg(z)$, for $g : B_1(0) \to \mathbb{C}$. For every $r < 1$, with $z \in B_r(0)$, we obtain

$$|g(z)| \leq |f(z)|/|z| \leq 1/r$$

Letting r approximate to 1, we obtain the first inequality. Due to the fact that $g(0) = f'(0)$, we immediately have the second one.
Applying **maximum modulus principle**, if $|g|$ contains the value 1 in the unity disk, then $|f| = |z|$ and hence $f(z) = e^{i\theta}z$. □

Finally, we will study some important functions that will stand out in the main chapters.

Definition A.14. *A function is called biholomorphic if it is holomorphic, bijective and its inverse is holomorphic.*

We are going to focus on the bilohomorphic functions in the unit disk $\mathbb{D}$.

Theorem A.15. *Set $z_0 \in \mathbb{D}$ and define $\varphi_a : \mathbb{D} \to \mathbb{C}$, as:*

$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$$

Then, φ_a is biholomorphic in $\mathbb{D}$, and if f is another biholomorphic map in $\mathbb{D}$, it is necessarily the product of a rotation and a function of the kind φ_a.

Proof. First we are going to see that $\varphi_a(\mathbb{D}) \subset \mathbb{D}$. Notice that $|\bar{a}z| < |z|$, so the denominator cannot vanish (this ensures holomorphic property). Now, suppose that $|\varphi_a(z)| < 1$. That happens if and only if $|\frac{z-a}{1-\bar{a}}| < 1$, so then $(|\frac{z-a}{1-\bar{a}}|)^2 < 1$. Using the property that $|z|^2 = z\bar{z}$,

$$\frac{(z-a)(\bar{z}-\bar{a})}{(1-\bar{a}z)(1-a\bar{z})} = \frac{|z|^2 - \bar{z}a - \bar{a}z + |a|^2}{(1-\bar{a}z)(1-a\bar{z})} = \frac{|z|^2 - \bar{z}a - \bar{a}z + |a|^2}{1 - \bar{z}a - \bar{a}z + |az|^2} < 1$$

So,

$$|z|^2 - \bar{z}a - \bar{a}z + |a|^2 < 1 - \bar{z}a - \bar{a}z + |az|^2 \iff |z|^2 + |a|^2 < 1 + |az|^2 \iff |a|^2 - |a|^2|z|^2 < 1 - |z|^2$$

As $|a| < 1$, this last condition always holds. Next we shall prove bijectivity. This is extremely simple, as note that if we consider $\varphi(-a)$, we obtain that

$$\varphi_a(\varphi_{-a}(z)) = \frac{\frac{z-a}{1-\bar{a}z} + a}{1 + \bar{a}\frac{z-a}{1-\bar{a}z}} = \frac{\frac{z-a+a-|a|^2z}{1-\bar{a}z}}{\frac{1-|a|^2-\bar{a}z+\bar{a}z}{1-\bar{a}z}} = z$$

Hence, we have obtained the inverse, carrying with it the bijectivity. Now set $f : \mathbb{D} \to \mathbb{D}$ another biholomorphic map. Then, for no matter which function φ_a, $f \circ \varphi_a : \mathbb{D} \to \mathbb{D}$. Therefore, choose $a = -f^{-1}(0)$. Hence,

$$f \circ \varphi_{-f^{-1}(0)}(0) = f(f^{-1}(0)) = 0$$

Hence, we can apply Schwarz lemma to deduce that $|f(\varphi_{-f^{-1}(0)}(z))| \leq |z|$. A typical use for the Schwarz lemma is to apply it, not one, but two times, in this case to the inverse function $(\varphi_{-f^{-1}(0)} \circ f)^{-1}$, so $|\varphi_{f^{-1}(0)}(f^{-1}(w))| \leq |w|$. With $w = f(\varphi(z)) \in \mathbb{D}$, we conclude that $|f(\varphi(z))| = |z|$, so there exists a $\theta \in \mathbb{R}$ with $f(\varphi_{-f^{-1}(0)}(z)) = e^{i\theta}$, or better written, $f(z) = e^{i\theta}\varphi_{f^{-1}(0)}(z)$ □

A.2 The Riemann Sphere and further results

Curiously, we will not often use the complex plane in the main chapters, as the dynamics we want to study involve rational functions, and the zero does not behave very well in fractions. This is why I will introduce a new whole space. I am talking about the **Riemann Sphere**.
There are several ways to come up with the concept of this sphere; for example, because it is the compactification of Alexandroff of the complex plane. But, according to what we have already seen, I prefer to introduce it as follows. Look at the map $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad f(z) = \frac{1}{z}$. It is a biholomorphic map from $\mathbb{C} \setminus \{0\}$ to itself (the inverse is given by the proper f). So it would be extremely appropriate to extend this map to the whole complex plane. Here is why we add the point ∞. Intuitively, $f(\infty) = 0$ and $f(0) = \infty$. Therefore we denote $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ as the new space, the Riemann Sphere. But, this is just a small grasp. To understand better how it works, we should define which are the open sets. A common sense is that an open set of $\mathbb{C}$ should be an open set of $\overline{\mathbb{C}}$, and, indeed, we are not wrong. But then, what happens with the infinity point?

Definition A.16. *An open set $\widetilde{U}$ in $\overline{\mathbb{C}}$ is of the form:*

- *If $\infty \notin \widetilde{U}$, then $\widetilde{U}$ is an open set of $\mathbb{C}$*
- *If $\infty \in \widetilde{U}$, then $\overline{\mathbb{C}} \setminus \widetilde{U}$ has to be equal to $\mathbb{C} \setminus K$ for a K compact set of $\mathbb{C}$*

Once we have defined the topology there, we should focus on the functions. What is the notion of (complex) differentiability? Here is why the map $z \to \frac{1}{z}$ is so important.

Definition A.17. *As before, denote $\widetilde{U} \subset \overline{\mathbb{C}}$ an open set of the Riemann Sphere. A continuous function $f : \widetilde{U} \to \overline{\mathbb{C}}$ is complex-differentiable at a given point z if:*

- *If $f(z) = z = \infty$, then $\frac{1}{f(\frac{1}{z})}$ is complex differentiable in $\mathbb{C}$*
- *If only $f(z) = \infty$, then $\frac{1}{f(z)}$ is complex differentiable in $\mathbb{C}$*
- *If only $z = \infty$, then $f(\frac{1}{z})$ is complex differentiable in $\mathbb{C}$*
- *Finally, if neither $f(z)$ nor z are ∞, then $f(z)$ is complex differentiable in $\mathbb{C}$*

I will not go any further proving some characteristics of holomorphic functions on the Riemann Sphere, but I would like to state some theorems so that we can check their similarities with the complex plane.

Theorem A.18. *Open mapping theorem holds in the Riemann Sphere. Concretely, if D is a domain in $\overline{\mathbb{C}}$, then for an holomorphic function f, $f(D)$ is, again, a domain.*

Theorem A.19. *Any polynomial P defined on the complex plane can be extended to the Riemann Sphere by setting $P(\infty) = \infty$. Moreover, if $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a holomorphic function such that $f(\mathbb{C}) \subset \mathbb{C}$, then f is a polynomial in $\mathbb{C}$.*

Theorem A.20. *Every holomorphic function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a quotient of two polynomials of $\overline{\mathbb{C}}$. More explicitly, that f can be written as*

$$f(z) = \frac{P(z)}{Q(z)}$$

for two given polynomials $P(z), Q(z) : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$.

Corollary A.21. *The only biholomorhpic functions of $\overline{\mathbb{C}}$ are the Möbius transformations,*

$$f(z) = \frac{az+b}{cz+d},\ ad-cb \neq 0$$

Theorem A.22. *Every holomorphic map from $\overline{\mathbb{C}}$ to $\mathbb{C}$ must be constant.*

Proof. As $\overline{\mathbb{C}}$ is compact, its image under f is compact, so we have $f(\mathbb{C}) \subset f(\overline{\mathbb{C}}) \subset K$ for K a compact set. Hence $f_{|\mathbb{C}}$ is constant and the only possible continuous extension is setting f to be constant. □

Almost at the end of this chapter I will state some proprieties of extremely useful maps, which will be used in Chapter 5.

Theorem A.23 (Blaschke product)**.** *A rational map R of degree d that leaves $\mathbb{D}$ invariant can only be of the form:*

$$R(z) = e^{i\theta}\beta_{a_1}(z)...\beta_{a_d}(z)$$

where $\beta_{a_i}(z) = \frac{1-\bar{a}_i}{1-a_i}\frac{z-a_i}{1-\bar{a}_i z}$ *and* $a_i \in \overline{\mathbb{C}} \setminus \partial\mathbb{D}$*, for* $1 \leq i \leq d$

Proof. Although the result is fundamental, the proof is quite simple. First of all, if a couple of coefficients a_j, a_k held that $a_j\bar{a}_k = 1$, then

$$\beta_{a_j}(z)\beta_{a_k}(z) = \frac{(1-\bar{a}_j)(1-\bar{a}_k)(z-a_j)(z-a_k)}{(1-a_j)(1-a_k)(1-\bar{a}_j z)(1-\bar{a}_k z)} = \frac{(1-\bar{a}_j-\bar{a}_k+\bar{a}_j\bar{a}_k)(z^2-za_j-za_k+a_ja_k)}{(1-a_j-a_k+a_ja_k)(1-\bar{a}_j z-\bar{a}_k z+\bar{a}_j\bar{a}_k z^2)} =$$

$$\frac{z^2-za_j-za_k+a_ja_k-\bar{a}_jz^2+|a_j|^2z+z-|a_j|^2a_k-\bar{a}_kz^2+z+|a_k|^2z-a_j|a_k|^2+\bar{a}_j\bar{a}_kz^2-\bar{a}_jz-\bar{a}_kz+1}{z^2-za_j-za_k+a_ja_k-\bar{a}_jz^2+|a_j|^2z+z-|a_j|^2a_k-\bar{a}_kz^2+z+|a_k|^2z-a_j|a_k|^2+\bar{a}_j\bar{a}_kz^2-\bar{a}_jz-\bar{a}_kz+1}$$

which is 1, reducing the degree of R.
The construction is straightforward: choose any solution a_i such that $R(a_i) = 0$ and then divide $R(z)$ by $\beta_{a_i}(z)$. We obtain a rational map with strictly less degree.
Finally, the uniqueness is automatically given as $R(1) = e^{i\theta}$ and the coefficients $\{a_1, ..., a_d\}$ are given by the equation $R(z) = 0$ □

Definition A.24. *A proper map is a map $f : U \to V$ such that for any compact set $K \subset V$, $f^{-1}(K)$ is a compact set in X. Equivalently, proper maps are the maps that map to .*

Theorem A.25. *Proper maps f in the unit disk are Blaschke products.*

Proof. By the definition of proper map, for every $r < 1$, $f^{-1}(\overline{B_r(0)})$ is a compact set contained in a open disk of $\mathbb{D}$, so

$$lim_{|z|\to 1^-}|f(z)| = 1$$

Now we consider a Blaschke product $B(z) = \beta_{a_1}(z)...\beta_{a_d}(z)$ where $a_1, ..., a_d$ are the zeros of f (with multiplicity).
Finally, we can consider the map $g(z) = f(z)/B(z)$. Note that we can extend it to a holomorphic map as the zeros of $B(z)$ are the zeros of f, so there are only removable singularities. It can be seen that the modulus of each of the β_{a_i} is 1 (as they are, essentially, conformal isomorphisms in the unit disk). Then,

$$lim_{|z|\to 1^-}|g(z)| = 1$$

Using the **minimum modulus principle**, g is constant, so $f(z) = aB(z)$ with $|a| = 1$. □

Appendix B

The Riemann Mapping Theorem

B.1 Sequences of holomorphic functions

In this appendix we are about to prove one of the most useful theorems in complex analysis, which allows us to characterize certain open sets, the **Riemann Mapping Theorem**. But first, we shall define some notions about sequences of holomorphic functions.

Definition B.1. *A sequence of holomorphic functions $f_n : U \to \mathbb{C}$ is said to converge locally uniformly to a function $f : U \to \mathbb{C}$ if for every $w \in U$ exists a $r > 0$ such that*

$$sup_{z \in B_r(0)} |f_n(z) - f(z)| \to 0, \ n \to \infty$$

This definition is equivalent to say that f_n converge on compact sets.

Theorem B.2. *Let f_n be as the previous definition. Then f is holomorphic.*

Proof. Let $\triangle$ be an arbitrary triangle in U. As local convergence implies convergence under compact sets, we have

$$\lim_{n \to \infty} \int_{\triangle} f_n(z) dz = \int_{\triangle} f(z) dz$$

Using **Cauchy's theorem**, the first integral vanishes and then, every integral of f under a triangle vanishes too, which by **Morera's theorem** confirms that f is holomorphic. □

Proposition B.3. *Using again the same f_n, the derivatives are also locally uniformly convergent to the derivatives of f.*

Observation B.4. *We shall notice this does not happen in the real case, for example with functions as $f_n(x) = \frac{x}{1+(nx)^2}$. The f_n converge locally uniformly to 0, the first derivative converge to 1.*

Proof. Let $r > 0$ such that $B_{3r}(w) \subset U$. Using the **Cauchy's integral formula for the derivatives**, for any point $w' \in B_r(0)$

$$f^{(k)}(z) - f_n^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial B_{2r}(w)} \frac{f(z) - f_n(z)}{(z - w')^{k+1}} dz$$

But then, we can bound the fraction:

$$sup_{w' \in B_r(w)} |f^{(k)} - f_n^{(k)}| \leq \frac{2k!}{r^k} sup_{z \in B_{2r}(w)} |f(z) - f_n(z)| \to 0, \ n \to \infty$$

□

Corollary B.5. *Let D be a domain and $f_n : D \to \mathbb{C}$ a sequence of holomorphic functions that converge locally uniformly to some $f : D \to \mathbb{C}$. If for every $n \in \mathbb{N}$, f_n has at most k zeros, then either f has at most k zeros or f is identically zero.*

Theorem B.6 (Montel's theorem for sequences of holomorphic functions)**.** *Let $f_n : U \to \mathbb{C}$ be a sequence of holomorphic functions that is locally bounded , which means that for every $w \in U$ there exist $r > 0, K < \infty$ such that*

$$sup_{n\in\mathbb{N}} sup_{z\in B_r(w)} |f_n(z)| < K$$

Then, there exists a subsequence f_{n_k} of f_n that converges locally uniformly to some holomorphic function $f : u \to \mathbb{C}$.

Corollary B.7 (Vitali)**.** *Let $f_n : D \to \mathbb{C}$ be a sequence of holomorphic functions that is locally uniformly bounded. Suppose the set $\{z \in D : lim_{n\to\infty} f_n(z) \ exists\}$ has an accumulation point in D, then f_n converges locally uniformly to a holomorphic function $f : D \to \mathbb{C}$.*

Proof. First, let's prove pointwise convergence. We are going to argue by contradiction. Assume there exists a point $w \in D$ such that $f_n(w)$ does not converge. Using the bound hypothesis, we can apply Montel's theorem for sequences of holomorphic functions and therefore, we find subsequences $f_{n_{j,1}}, f_{n_{j,2}}$ that converge locally uniformly to g and h respectively, but $g(w) \neq h(w)$. By assumption, we know that both functions agree in $\{z \in D : lim_{n\to\infty} f_n(z) \ exists\}$, which is a contradiction by the identity theorem. Now suppose f_n does not converge locally uniformly to a function f. By definition, there exist a subsequence (not relabelled), a point z and $\epsilon > 0$ such that $lim_{n\to\infty}|f_n(z) - f(z)| > \epsilon$. Using again the previous theorem, there must exist a subsequence that converges locally uniformly to some g, but by pointwise convergence $f = g$.

□

B.2 The Riemann Mapping Theorem

Once we have learnt these concepts let us talk about simply connected sets and some of the properties of functions that are defined under these sets.

Definition B.8. *Let $G \subset \mathbb{C}$ be an open set. G is simply connected if it is path-connected and if every closed curve of G can be contracted to a point, which means that there exists a continuous function $F : [0,1] \times [0,1] \to G$ that for a closed curve $\alpha(t), t \in [0,1]$ of G it holds:*

$$\begin{aligned} F(0,t) &= \alpha(t) && \forall t \in [0,1] \\ F(1,t) &= p, && p \in G, \forall t \in [0,1] \\ F(s,0) &= F(s,1) && \forall s \in [0,1] \end{aligned}$$

Before moving forward we are going to recall a basic theorem of complex analysis.

Theorem B.9. *Let $G \subset \mathbb{C}$ be a simply connected domain and $f : G \to \mathbb{C}$ a holomorphic function. Then there exists another (holomorphic) function $F : G \to C$ that $F'(z) = f(z) \ \forall z \in \mathbb{C}$*

Corollary B.10. *Let $G \subset \mathbb{C}$ be a simply connected domain and let $f : G \to \mathbb{C} \setminus \{0\}$ be a holomorphic function.Then, there exists a holomorphic function $log(f) : G \to \mathbb{C}$ which is the inverse of the complex exponential, $exp(log(f)) = f$.*

Proof. As f does not have any zeros we can consider the logarithmic derivative, $g = \frac{f'}{f}$ defined on G. Thus, we can choose a primitive, $h : G \to \mathbb{C}$, as G is simply connected, which holds that $h(z_0) = log(f(z_0))$.
Deriving, we obtain the following equality:

$$\frac{d}{dz}(f(z)exp(-h(z))) = f'(z)exp(-h(z)) - f(z)exp(-h(z))g(z)$$

Using that $g(z) = \frac{f'(z)}{f(z)}$, the last expression vanishes for every $z \in G$. Therefore, $f(z)exp(-h(z)) = c(z)$ for a constant $c(z)$ that depends on the connected components of G. As G is connected, $c(z) = c$ for every $z \in G$, and then, $f(z)exp(-h(z)) = c$.
Finally, as $h(z_0) = log(f(z_0))$, we obtain $f(z_0) = exp(-h(z_0))$. Using the connectivity feature, $log(f) = h(z)$. □

Corollary B.11. *Using a function f with the same conditions as the previous corollary, it exists the $n^{th} -$ root of f, namely, a function $\sqrt[n]{}:G \to \mathbb{C}$ with $(\sqrt[n]{f})^n = f$.*

Proof. Using the aforementioned corollary, $\sqrt[n]{(f)} = exp(h(z)/n)$ □

In the next proposition we'll give a fundamental property to find this kind of functions.

Proposition B.12. *Let f be holomorphic and injective. Then f' does not vanish anywhere and it has a well defined holomorphic inverse.*

Proof. As f is holomorphic, f can be expressed as an analytic series in a neighbourhood of every point of the domain. Then, suppose there exists a point z_0 such that $f'(z_0) = 0$. Making a quick change of variables $z \to z + z_0 \to f(z + z_0) \to f(z + z_0) - f(z_0)$, we can therefore assume that z_0, $f(z_0)$ i $f'(z_0)$ vanish. Then, there exist some coefficients $a_k \in \mathbb{C}$ for $k \geq 2$ that

$$f(z) = z_0 + f'(z_0)z + \sum_{k=2} a_k z^k \;=\; \sum_{k=2} a_k z^k$$

Thus, there is a k that $a_k \neq 0$ and then,

$$f(z) = z^k g(z)$$

for a certain function g that does not vanish at $z = 0$. From here, we are going to find a contradiction with the injectivity of f. There exists a $r > 0$ such that $B_r(0) \subset U$. As g is never zero, we can find a function $h : B_r(0) \to \mathbb{C}$ that $h^k = g_{|B_r(0)}$. That would mean that $f(z) = (zh(z))^k$. Using the **Open mapping theorem**, we can find a small enough radius, r' and two points z_1, z_2 that

$$z_1 h(z_1) = r'$$

$$z_2 h(z_2) = r' exp(2\pi i/k)$$

But in this case, raising to k, we would contradict the injectivity of f.
Now let us prove the second part.

Applying again the **Open mapping theorem**, $f(U)$ is an open set.
As f' does not vanish in U, we can use the **inverse function theorem** to confirm that the inverse is well defined as a $\mathbb{R}-$differentiable function. In order to check the holomorphic feature, we will focus on the structure of df. df is a rotation due to **Cauchy-Riemann equations**, so the inverse of the differential is another rotation, and hence, the inverse is holomorphic. □

Finally, we will study the solution of an extremal problem

Problem B.13. *Let $D \subset \mathbb{C}$ be a simply connected domain that contains the origin. Set*

$$M = \{f : D \to B_1(0), f \text{ holomorphic and injective}, \ f(0) = 0\}$$

Suppose that M is not empty. Them, prove that there exists a function of M that solves the extremal problem

$$\rho = \sup\{|f'(0)|, \ f \in M\}$$

Solution 1. *Let f_n be a sequence of holomorphic functions of M. Suppose also that*

$$lim_{n\to\infty}|f_n'(0)| = \rho$$

*Notice that the functions $f_n(D) \subset B_1(0)$are uniformly bounded by 1, so we can use the **Montel theorem for sequences of functions** and we therefore obtain a subsequence of functions f_{n_k} that converge locally uniformly in D. We may remark that for a fixed z,*

$$f(z) = lim_{n\to\infty} f_n(z)$$

*Then, $f : D \to \overline{B_1(0)}$. If at a given point of D $|f|$ reaches the unity, due to **maximum modulus principle**, f would be constant, and it would contradict that $f(0) = 0$ ($1 = |f| = |f(0) = 0$). Using the convergence of the derivatives, $|f'(0)| = lim_{n\to\infty}|f_n'(0)| = \rho$. In order to finish the exercise we have to see that $f \in M$, so we have to check whether f is injective or not. Arguing by contradiction, if f was not injective, there would exist two points $v, w \in D$ with $f(v) = f(w)$. As $f_n \in M$ are injective functions, their derivatives do not vanish. In this case, f' does not vanish either, because of corollary B.5. f cannot be constant, so the function $g(z) := f(z) - f(v)$ is not zero everywhere, but it has a zero in w and in v. But looking at the corresponding $g_n := f_n(z) - f_n(v)$, $g_n \to g(z)$ locally uniformly, and they only have a zero in D (due to injectivity). Using again corollary B.5, we reach the wished contradiction.*

The moment has arrived, we can finally prove the main theorem. The goal of this chapter is to characterize the open sets that are biholomorphic to the unit disk. We can notice that the whole complex plane is not one of them, as if would contradict **Liouville's theorem**.

Theorem B.14 (Riemann Mapping Theorem)**.** *Let $G \subsetneq \mathbb{C}$ a simply connected domain. Then there exists a biholomorphic function $\phi : G \to \mathbb{D}$.*

Remark B.15. This theorem guarantees its existence, but it does not give us any clue of how to find the biholomorphism.

Remark B.16. It is curious that the searched sets are, precisely, all the simply connected sets (except the whole complex plane).

Proof. Using the proposition B.12, we shall find a function $\phi : G \to \mathbb{D}$ bijective and holomorphic. We divide the proof in 3 steps.

1. Firstly, we will find an injective and holomorphic map from G to $\mathbb{D}$, with $0 \in g(G)$.
2. Secondly, we will prove that surjectivity is given by the maximality of $|f'(0)|$.
3. We will use the problem we solved to conclude the proof.

1. This is maybe the trickiest part. We will focus on the complementary of G. The easiest case is when we assume there is an open ball outside G. We shall notice this is not always the case, as, for example, the domain of the logarithm is the complex plane without a semi-line (for example, $\mathbb{C} \setminus \{(x, 0),\ x < 0\}$). So, suppose that there exist a $w \in G$ and a $r > 0$ that $B_{2r}(w) \subset \mathbb{C} \setminus G$. Let $h : G \to \mathbb{D}$, $h(z) = \frac{r}{z-w}$. Then $h(G) \subset B_{\frac{1}{2}}(0)$. Moreover, h is well defined, holomorphic and injective. Let $w' \in h(G)$ be a point, and we will study the map $\widetilde{h}(z) = \frac{1}{2}(h(z) - w')$. It is, again, well defined, injective and holomorphic. Also, the zero is the image of w', so we obtain the desired set.
   Now let us consider the tricky case. The idea is to consider logarithms and $n^{th}-$roots. Suppose that there exists an $w \in \mathbb{C} \setminus G$. Now we can introduce the transformations $z \to z - w \to \sqrt{z - w}$. In this way, this function does not vanish. I affirm it is injective. If it was not the case, there would exist two points z_1, z_2 such that $\sqrt{z_1 - w} = \sqrt{z_2 - w}$, but raising to the square we obtain $z_1 - w = z_2 - w$.
   In order to apply what we have previously proved, we shall check $\mathbb{C} \setminus \sqrt{G - w}$ contains an open ball. Using the **Open mapping theorem**, there exist a point $\widetilde{z} \neq 0$ and a radius $r > 0$ with $B_{2r}(\widetilde{z}) \subset \sqrt{G - w}$. We will prove that $-B_{2r}(\widetilde{z}) \subset \mathbb{C} \setminus \sqrt[2]{G - w}$, intuitively denoting $-B_{2r}(\widetilde{z})$ as $\{z \in \mathbb{C} : -z \in B_{2r}(\widetilde{z})\}$. If this was not the case, there would exist two points $z_1, z_2 \in G$ that hold $\sqrt[2]{z_1 - w} = -\sqrt[2]{z_1 - w}$. Similarly, raising to the square, we obtain $z_1 = z_2$, and therefore $z_1 = w$, which is impossible because $w \in \mathbb{C} \setminus G$.
   To sum up, we have proved this part in order to consider that G includes the origin, and taking a product by scalar, we may consider $G \subset \mathbb{D}$.

2. Suppose we have a function $f : G \to \mathbb{D}$ that is holomorphic, injective, and $f(0) = 0$, but it is not surjective. Then, we will find another function $\phi : G \to \mathbb{D}$ that is holomorphic and injective, it holds $\phi(0) = 0$ and $|\phi'(0)| > |f'(0)|$.
   First, we are going to use the remarkable function we studied in the first appendix. For $a \in \mathbb{D}$ consider
$$\varphi_a : \mathbb{D} \to \mathbb{C},\ \varphi_a = \frac{z - a}{1 - \bar{a}z}$$
   Suppose f is not surjective. Then, let $a \in \mathbb{D} \setminus f(G)$. Consider $\varphi_a \circ f : G \to \mathbb{D}$ and $\varphi_a(f(z)) \neq 0$. In order to save some ink, I will denote the last composition g. By corollary B.11, it exists a holomorphic square root (we are using again the same trick), $\sqrt{g}$ which is injective. Set $b = \sqrt{g(0)} \in \mathbb{D}$. We can therefore define
$$\phi = \varphi_b \circ \sqrt{g(z)}$$
   Notice that the function $\psi := \varphi_a^{-1} \circ \varphi_b^{-2}$ holds the property $\psi \circ \phi = f$, and then $\psi(0) = 0$. Moreover, by Schwarz lemma $\psi'(0) < 1$. So,
$$|f'(0)| = |(\psi \circ \phi)'(0)| = |(\psi'(0))(\phi'(0))| < |\phi'(0)|$$

3. We are almost crossing the finishing line. Consider the identity function $id : G \to \mathbb{C},\ id(z) = z$. Recalling the problem we solved, $id \in M$. So, M is not empty, and then, we can find a function that maximizes ρ, which finishes the proof.

□

Finally, I will state a more complex result whose proof can be found in the fourth chapter of [KR].

Theorem B.17. *Let Ω be a doubly connected domain such that any of the components is the complement of a point. Then Ω is conformally equivalent to an annulus* $\{z \in \mathbb{C},\ r < |z| < R\}$

Appendix C

Topology and Manifolds

The aim of this chapter is to provide information about some famous results related to basic topology and manifolds.

C.1 Elementary results in topology

Definition C.1. *Let X be a set of points. A topology in X, $\mathcal{T}$ is a collection of subsets of X such that:*

- *$\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$*
- *If $X_1, ..., X_k ... \in \mathcal{T}$ then $\bigcup_{j \geq 1} X_j \in \mathcal{T}$*
- *If $X_1, ..., X_k \in \mathcal{T}$ then $X_1 \cap ... \cap X_k \in \mathcal{T}$*

A topological space is a pair $(X, \mathcal{T})$ where $\mathcal{T}$ is a topology in X. The elements of the topology are called open sets.
A set A is said to be closed if $X \setminus A$ is open (or equivalently, $X \setminus A \in \mathcal{T}$).

Example C.2. For example consider the real line $\mathbb{R}$. A topology could be the set of open intervals.

Definition C.3. *Let $(X, \mathcal{T})$ be a topological space. A basis $\mathcal{B}$ of this space is a collection of open sets B_i such that for every point $p \in X$ with $p \in U \in \mathcal{T}$ there exists a $B \in \mathcal{B}$ such that $p \in B \subset U$*

We have seen that a topology gives birth to a basis but, can we swap their roles?

Proposition C.4. *Let X be a set and let $\mathcal{P}(X)$ be the power set of X (equivalently, $\mathcal{P}(X) = \{P$, P is a subset of $X\}$. Then $\mathcal{B} \subset \mathcal{P}(X)$ is the basis of (a) topology if the following conditions are held:*

- *If $B_1, B_2 \in \mathcal{B}$ then $B_1 \cap B_2 \in X$*
- *$\bigcup_{B \in \mathcal{B}} B = X$*

We can classify the different types of topological spaces. Here we recall some important ones.

Definition C.5. *A topological space $(X, \mathcal{T})$ is:*

- *Frechet or T_1 if for every pair of points $x, y \in X$ there exists an open set $U(x)$ such that $x \in U(x)$ and $y \notin U(x)$*

- *Hausdorff or T_2 if for every pair of points $x, y \in X$ there exists a couple of open sets U, V with $U \cap V = \emptyset$ such that $x \in U$ and $y \in V$*
- *T_3 if for every point p and every closed set A with $p \notin A$, there exist two disjointed open sets U, V with $A \subset U$ and $p \in V$. We say $(X, \mathcal{T})$ is regular if both T_1 and T_3 hold*
- *T_4 if for every disjointed closed sets A, B there exist two disjointed open sets with $A \subset U$ and $B \subset V$. When $(X, \mathcal{T})$ hold both properties T_1 and T_4 we say it is normal*

Remark C.6. If it does not lead to a confusion, writers usually avoid to keep writing $(X, \mathcal{T})$ and, instead, they only denote the topological space as, simply, X.

Definition C.7. *Let X be a topological space. A curve is a continuous function $\alpha : I \to X$ with I an interval of the real numbers.*

Remark C.8. We usually denote curve as the image of the function.

Definition C.9. *A curve $\alpha : [a, b] \to X$ is said to be closed if $\alpha(a) = \alpha(b)$.*

Definition C.10. *A curve $\alpha : [a, b] \to X$ is said to be simple if its image is injective, or equivalently, the image does not have any auto-intersections. A simple closed curve with $X = \mathbb{R}^2$ is a Jordan curve.*

Theorem C.11 (Jordan curve Theorem)**.** *A Jordan curve disconnects the plane in two connected components, and one of them is not bounded.*

Definition C.12. *A pair of closed curves $\alpha(t), \beta(t) : [a, b] \to \mathbb{R}^n$ is said to be homotopically equivalent if there exists a continuous map $F : [a, b] \times [0, 1] \to \mathbb{R}^n$ such that:*

- $F(t, 0) = \alpha(t)$
- $F(t, 1) = \beta(t)$
- $F(0, s) = F(1, s)$

C.2 Manifolds

For further properties or proofs, please see [LE]

Definition C.13. *A topological space is called an n-manifold if it satisfies:*

- *Hausdorff property*
- *Second countability (countable basis)*
- *Locally Euclidean property: Every point has a neighbourhood homeomorphic to an open set of $\mathbb{R}^n$*

Definition C.14. *Let M be an $n-$manifold. A coordinate chart consists of:*

- *A subset U of M*
- *A subset V of $\mathbb{R}^n$*
- *A homeomorphism $\phi : U \to V$*

Theorem C.15. *Let M be a manifold. Then every point has a neighbourhood which is contained in a compact subset of M (this is what we call local compactness).*

Proof. It is easy to see that every topological manifold admits a countable basis of precompact balls, which means that their closures are compact. □

Theorem C.16. *A manifold M admits an exhaustion by compact sets. That is, there exists a nested sequence of compact sets $\{K_i\}_{i\geq 1}$ such that $M = \bigcup_{i\geq 1} K_i$ and $K_i \subset int(K_{i+1})$.*

Proof. We have seen that M admits a basis made out of precompact balls, and by second countability, it is countable. Then we can denote this set as $\{B_i\}_{i\geq 1}$. We are going to construct the compact sets as follows. First, take $K_1 = \overline{B_1}$. Now suppose that, by induction, we have already built $K_1, .., K_n$ such that $B_j \subset K_j$ and $K_j \subset int(K_{j+1})$. Using that K_n is compact, there exists an index n' such that $K_n \subset B_1 \cup B_2 \cup ... \cup B_{n'}$. What has to be done finally is to choose K_{n+1} as the closure of the previous set. □

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