Universitat de Barcelona
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GRAU DE MATEMÀTIQUES

Treball final de grau

CATEGORY THEORY

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Abstract

The main goal of this project is the investigation of the mathematical structures called *categories*, looking at their most important features and applications. It will also be see the concept of *functors*, how they make sense when working with categories and two of the most relevant type of functors, *representable and adjoint functors*. The development of the study is based on the book *Categories* by *T.S. Blyth*.

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Acknowledgements

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Finally, I would like to recognize my greatest support, my family, especially my parents. I would not be here without them.

Introduction

Motivation

Category theory, developed in the middle of the last century, is a relevant theory in mathematics which tries to *organize* different mathematical *objects* and structures. This report introduces the most important aspects of *categories*, how these frameworks allow us to work different aspects of mathematics like *sets*, *groups*, *vector spaces*, *topological spaces*, etc., and look at some similarities between these mathematical structures. To do it we need to manage categories, *functors*, the construction of *universal objects* like *products* or *pullbacks*. We will relate two types of functors, *adjoint*, and *representable* functors.

Structure of the project

In the first chapter we look at *classes* and how important they are to have an appropriate definition for categories. The second chapter is when we identify some of the most important examples of categories, and some of the most important elements in categories. This chapter will be the basics for the development of the next chapters.

In the third chapter, we will talk about new particular elements in categories, these are products, pullbacks, intersections, and equalisers, all these are known as *universal objects*. To find them we need to construct new categories where they have the sense to exist, and then translate any result to our original category.

The last chapter is the most relevant in our project, this one talks about functors and natural transformations. The reason to study them is that functors relate categories, and natural transformations relate functors. At this point, we will be ready to announce Yoneda's Theorem and then define representable functors. Finally, we define adjoint functors, adjoint pairs, and show a theorem that relates adjoint pair with representable functor.

Chapter 1

Classes

1.1 Context

Set theory is the base of mathematics as we know nowadays, it would be impossible to work out of these fundamentals. Nevertheless, to define the notion of a *category* we should start with something more than *sets*, we need the notion of *classes*. This necessity arises from the theoretical limitations of sets. Let see a historical example of this:

Example 1.1. *Russell's Paradox*: Consider the "set" of all sets that does not belong to themselves. This would be:

$$A = \{x | x \notin x\}$$

We arrive at a contradiction with this "set", that is:

$$A \in A \iff A \notin A$$

Here the problem that we have is to work with A as a set. According to Zermelo-Fraenkel's axiomatic there is no set that belongs to itself, so A would be the "set" of all sets, but there is no such "set".

The point with this example is the limitations of how sets are defined. Well, with *classes* we won't have this problem.

Definition 1.2. *We define a class as a free reunion of objects.*

To see the difference between sets and classes, we figured out that every set is a class, but not every class is a set, this is why sets are also called small classes. The classes which are not sets are named proper classes. The point is that we can define a class with a property that its objects satisfy, without the condition to be first in another class. Recalling *Russell's Paradox*, from example 1.1 we can not have the set of all sets, this arrives to a contradiction, but with classes is different. For example, we can work with the class of all classes or classes that belong to themselves (assuming the belonging of classes the same way that is in sets).

We need classes to define concepts in *category theory* in a general vision, but mostly we will be working with the *set category*. Nevertheless, we will show how a class is really a set in a particular case of this report.

Chapter 2

Categories

2.1 Notion of category

Definition 2.1. *We define a category as a class* $\mathbf{C}$ *of objects such that:*

(1) for every pair of objects $\boldsymbol{X}$*,* $\boldsymbol{Y}$ *of* $\mathbf{C}$ *there is a set* $Mor_{\mathbf{C}}(\boldsymbol{X},\boldsymbol{Y})$*, called the the set of morphisms from* $\boldsymbol{X}$ *to* $\boldsymbol{Y}$*, with* $Mor_{\mathbf{C}}(\boldsymbol{X},\boldsymbol{Y})$ *and* $Mor_{\mathbf{C}}(\boldsymbol{X'},\boldsymbol{Y'})$ *disjoint unless* $\boldsymbol{X} = \boldsymbol{Y}$ *and* $\boldsymbol{Y} = \boldsymbol{Y'}$ *in which case they coincide;*

(2) for any three objects $\boldsymbol{X}$*,* $\boldsymbol{Y}$*,* $\boldsymbol{Z}$ *of* $\mathbf{C}$ *there is a mapping*

$$Mor_{\mathbf{C}}(\boldsymbol{X},\boldsymbol{Y}) \times Mor_{\mathbf{C}}(\boldsymbol{Y},\boldsymbol{Z}) \longrightarrow Mor_{\mathbf{C}}(\boldsymbol{X},\boldsymbol{Z})$$

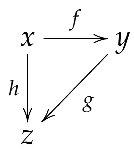
described by $(g, f) \longmapsto f \circ g$*, with the following properties:*

(α) for every object $\boldsymbol{X}$ *there is a morphism* $id_X \in Mor_{\mathbf{C}}(\boldsymbol{X},\boldsymbol{X})$ *which is a right identity under* $\circ$ *for the elements of* $Mor_{\mathbf{C}}(\boldsymbol{X},\boldsymbol{Y})$ *and a left identity under* $\circ$ *for the elements of* $Mor_{\mathbf{C}}(\boldsymbol{Y},\boldsymbol{X})$*;*

(β) $\circ$ *is 'associative' in the sense that when the composites* $h \circ (g \circ f)$ *and* $(h \circ g) \circ f$ *are defined they are equal.*

Notice that in Definition 2.1 we do not require the morphisms to be maps, we take abuse of notation but we could also name the morphisms *arrows*. To have a better notion of *categories* we will see some examples:

Example 2.2. A graph:



Here we have that objects of C constitute the *set* $\{x, y, z\}$ and morphisms the *set* $\{id_x, id_y, id_z, f, g, h\}$, where $g \circ f = h$

Example 2.3. The second category we are going to show is the **Set** category. This category has as objects all the sets; as morphisms (or arrows) the maps from one set to another. The composition between the arrows is the common composition of functions, and the identity morphism of an object in **Set** is the identity map. As we are used to work within the *set theory* it is clear that we obtain a category with this class.

When objects are sets we say the category is a *small category*. In the next examples, as they are small categories, we will not mention the composition between them, so composition is the same as in Example 2.3.

Example 2.4. The next example is another mathematical structure, the *groups*. As we know well the groups, it is natural to take as arrows the *group homomorphisms*. This is because, as we will see in the next examples, we maintain the mathematical structure. So we have the category **Grp.**

Example 2.5. $\mathbf{Vec}_{\mathbb{K}}$ is the category with objects vector spaces over a field $\mathbb{K}$ with finite dimension, and arrows linear transformations between them.

Example 2.6. Ab the category based on abelian groups with the group morphisms between abelian groups.

Example 2.7. Topological spaces generate another category with the continuous maps. This category is known as **Top**.

Example 2.8. The category **Ring** based on rings with ring morphisms.

Observation: when working with categories it is common to show just the objects, the morphisms tend to be the most obvious to compose between them, the ones that preserve the structure.

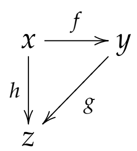
Another interesting and important concept we will be working with during the project is the *dual category*.

Definition 2.9. *Given a category* **C** *we denote* **C*** *as the dual category generated with the same objects as* **C** *and morphisms are defined:*

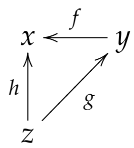
$$Mor_{\mathbf{C}^*}(A, B) := Mor_{\mathbf{C}}(B, A)$$

This category will be useful to get more important results and properties of categories. Let see an example:

Example 2.10. Consider the category given by the directed graph from Example 2.2:



In this case its associated dual category is:



We see that objects do not change, the difference with the category **C** is the direction of the edges. Of course, the identity morphisms are still the same.

2.2 Particular objects

We are going to talk now about particular objects in a category that may exist but it is not a necessity. These are *universal objects*.

Definition 2.11. *Let **C** be a category, we say that U is an initial object if it is an object of C such that* $Mor_{\mathbf{C}}(U, A)$ *is a singleton for every object A of **C**. Dually, U is said to be a terminal object if U is an initial object in **C***, this is* $Mor_{\mathbf{C}}(A, U)$ *is singleton for every object A in **C**.*

Example 2.12. $\emptyset$ is an initial object in **Set** due to there is just one map from $\emptyset$ to any set, the empty map.
The set $\{x\}$, with x a set, is a terminal object. Given any set Y, there is just one map f_Y from Y to $\{x\}$ defined by $y \longmapsto f_Y(y) = x$ for all $y \in Y$.

Notice that we require every object of **C** being *connected* to the universal object, in the sense that there has to be a morphism between them, and of course to be unique. The sense of being *connected* in a category goes this way (clearly without the singleton condition).

Definition 2.13. *Let **C** be a category. We say that **C** is a connected category if and only if, for every two pair of objects X, Y of **C**, the set* $Mor_{\mathbf{C}}(X, Y) \neq \emptyset$.

To introduce a very useful theorem that relates these universal objects, we need first the concept of isomorphic objects.

Definition 2.14. *We say* $f : \boldsymbol{A} \longrightarrow \boldsymbol{B}$ *is an isomorphism if and only if there exist (a necessarily unique)* $g : \boldsymbol{B} \longrightarrow \boldsymbol{A}$ *such that* $f \circ g = id_B$ *and* $g \circ f = id_A$. *If we have* $h_1, h_2 : \boldsymbol{B} \longrightarrow A$ *such that* $f \circ h_1 = id_B$ *and* $h_2 \circ f = id_A$ *it gives:*

$$h_1 = id_A \circ h_1 = (h_2 \circ f) \circ h_1 = h_2 \circ (f \circ h_1) = h_2 \circ id_B = h_2$$

We write g as f^{-1}.

Definition 2.15. *We say objects X, Y of **C** are isomorphic if $Mor_{\mathbf{C}}(X, Y)$ contains an isomorphism. In this case we often write $X \simeq Y$.*

Theorem 2.16. *Universal objects of the same type are isomorphic.*

Proof. Let U_1 and U_2 be initial objects in a category **C** (the same proof would work for terminal objects). We have to proof that $Mor_{\mathbf{C}}(U_1, U_2)$ is a singleton that consists of an isomorphism. As U_1, U_2 are initial object for every object X of **C** $Mor_{\mathbf{C}}(U_1, X($ and $Mor_{\mathbf{C}}(U_2.X)$ are singleton. In particular, for X = U_2, $Mor\mathbf{C}(U_1, U_2) = \{\alpha\}$, $Mor_{\mathbf{C}}(U_2, U_2) = \{id_{U_2}\}$; for X = U_1, $Mor_{\mathbf{C}}(U_1, U_1) = \{id_{U_1}\}$, $Mor\mathbf{C}(U_2, U_1) = \{\beta\}$. So we have that $\alpha \circ \beta = id_{U_2}$. Similarly $\beta \circ \alpha = id_{U_1}$. Consequently α is isomorphic to β with $\beta = \alpha^{-1}$. □

This is an important result, for example recalling the Example 2.12 we can say that just singleton sets are terminal objects, and ∅ is the only initial object.
So it would be a reasonable question if there exists an object that satisfies both properties, being an initial and terminal object.

Definition 2.17. *Let **C** be a category, we say U is a zero object if it is both terminal and initial object.*

Example 2.18. In **Grp** category we have that trivial group {*} is a zero object. The proof of being a terminal object goes the same way we saw in Example 2.12. The point is seeing it as an initial object. Looking to group homomorphisms, we have that *identity element goes to identity element*; in our case, given a group G there is just one group homomorphism $f : \{*\} \longrightarrow G$ defined by $f(*) = 1_G$ with 1_G identity element of G.

Theorem 2.19. *Let **C** be a category with initial and terminal objects. Then, **C** is connected if and only if **C** has a zero object.*

Proof. $\Leftarrow$
Assume 0 is a zero object of **C**. Let A and B be two objects of **C**. We want to show $Mor_{\mathbf{C}}(A, B) \neq \emptyset$. We will use the zero object as a *"factorizator"*; this is, as 0 is a terminal object there is a unique $\alpha \in Mor_{\mathbf{C}}(A, 0)$, and as 0 is an initial object there is a unique $\beta \in Mor_{\mathbf{C}}(0, B)$. So now we have $\beta \circ \alpha : A \longrightarrow 0 \longrightarrow B$, we have just found an element $\gamma = \beta \circ \alpha \in Mor_{\mathbf{C}}\{A, B\}$. As we choose A and B arbitrarily we have C a connected category.
$\Rightarrow$
We will not show this implication because we need some concepts not explained in this work. □

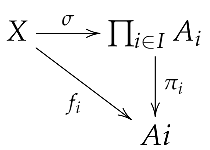
Chapter 3

Universal Constructions

In the first chapters, we defined categories and looked at the most important and basic concepts. In this new chapter, we will see some applications of categories. In a general vision, our goal is to generate categories from given categories. This will be a way to define new concepts in categories thanks to the constructed ones, this process is known as "turning theorems into definitions".

3.1 Products and coproducts

We begin considering a family of sets $\{A_i\}_{i\in I}$ indexed by a set I. Let $\prod_{i\in I} A_i$ be the cartesian product of this family, and $\pi_j : \prod_{i\in I} A_j \longrightarrow A_i$ the projections for each set A_j defined by $\pi_j((a_i)_{i\in I}) = a_j$. Suppose now that X is a set and a family of mappings $f_i : X \longrightarrow A_i$. Our goal is to have a unique function σ such that the diagram



is commutative for every $i \in I$.

Choosing the mapping $\sigma : X \longrightarrow \prod_{i\in I} A_j$ defined by $\sigma(x) = (f_i(x))_{i\in I}$ it would be a mapping that makes our diagram a commutative diagram. This is because

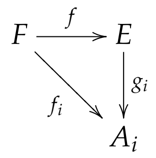
$$\forall x \in X \quad (\pi_i \circ \sigma)(x) = \pi_i(\sigma(x)) = \pi_i((f_j(x))_{j\in I}) = f_i(x)$$

So $\pi_i \circ \sigma = f_i$. To proof the uniqueness let suppose another mapping $\psi : X \longrightarrow \prod_{i\in I} A_j$ such that $\pi_i \circ \psi = f_i$. Given $x \in X$ let $\psi(x) = (e_i)_{i\in I}$. Then

$$f_j(x) = \pi_j(\psi(x)) = \pi_j((e_i)_{i\in I}) = e_j$$

hence $\psi(x) = (e_i)_{i\in I} = (f_i(x))_{i\in I} = \sigma(x)$ and so $\psi = \sigma$.

Let turn this "*theorem*" into a definition. Let **C** be a category and let $(A_i)_{i\in I}$ be a family of objects of **C**. We define a new category **K** in the following way: the objects of **K** are pairs $(E, \{g_i\}_{i\in I})$ with E an object of **C** and $g_i : E \longrightarrow A_i$ morphisms of C. A morphism f is in the set $Mor_{\mathbf{K}}((F, \{f_i\}), (E, \{g_i\}))$ if it is in $Mor_{\mathbf{C}}(F, E)$ such that the diagram



is commutative for every $i \in I$.

Definition 3.1. *Let $(P, \{p_i\}_{i\in I})$ be a terminal object in the previous category, then we say that $(P, \{p_i\}_{i\in I})$ is a product of the family $(A_i)_{i\in I}$.*

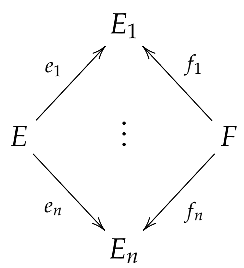
Dual notion of the product would be an object $(Q, \{q_i\}_{i\in I})$ such that it is an initial object, we say it is a coproduct.

We will say that **C** has (finite) products if every (finite) family of objects of **C** has a product in the category **K**.

Notice that the sense of products (coproducts) are unique up to isomorphism, given that every terminal (initial) object is unique up to isomorphism as we saw in Theorem 2.16. Let see an example of a product to visualize it better.

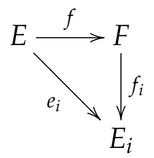
Example 3.2. Let $\mathbf{Vec}_{\mathbb{K}}$ be the category of vector spaces over a field $\mathbb{K}$ with finite dimension as we saw in Example 2.5. Consider a finite interval $I = \{1, ..., n\}$ and a family of vector spaces $\{E_i\}_{i\in I}$ in $\mathbf{Vec}_{\mathbb{K}}$.

Our new category **K** related with $\mathbf{Vec}_{\mathbb{K}}$ is the one with pairs $(E, \{e_i\}_{i\in I})$ with E a vector space and each $e_i : E \longrightarrow E_i$ a linear transformation between vector spaces. To visualize the objects of **K** let see the diagram



here we focus on the family $\{E_i\}_{i\in I}$ and take as objects of **K** the objects of $\mathbf{Vec}_{\mathbb{K}}$ with their respective family of linear transformations.

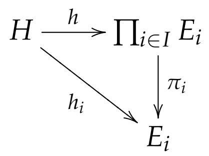
Now if there is a morphism $f : E \longrightarrow F$ such that the diagram



is commutative for every $i \in I$ then it would be on the set

$$Mor_{\mathbf{K}}((E, \{e_i\}_{i\in I}), (F, \{f_i\}_{i\in I}))$$

Let us have a look at a terminal object in this category. A *product* in this category $\mathbf{K}$ is the pair $(\prod_{i\in I} E_i, \{\pi_i\}_{i\in I})$ with $\prod_{i\in I} E_i$ the vector space generated by the cartesian product of the family $\{E_i\}_{i\in I}$, and the family $\{\pi_i\}_{i\in I}$ are the projecctions $\pi_j : \prod_{i\in I} E_i \longrightarrow E_j$ of each E_j.
Let us see this is a terminal object. Given $(H, \{h_i\}_{i\in I})$ we have to proof the existence of $h : (H, \{h_i\}_{i\in I}) \longrightarrow (\prod_{i\in I} E_i, \{\pi_i\}_{i\in I})$ the unique morphisms between this two objects such that the diagram

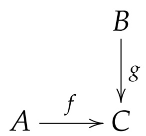


is commutative for every $i \in I$. Looking at first part of this section we can say that there is a unique h defined by $h(x) = (h_1(x), ..., h_n(x))$ with $x \in H$. As we choose $(H, \{h_i\}_{i\in I})$ arbitrarily we can say $(\prod_{i\in I} E_i, \{\pi_i\}_{i\in I})$ is a terminal object.

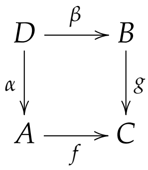
In this last example, we can see how is the translation from a "theorem" to a definition thanks to categories. Notice that we need a terminal object $(P, \{p_i\}_{i\in I})$ in the category $\mathbf{K}$. Working with products we will say that P is a product of $\mathbf{C}$ without mention the morphisms p_i because these tend to be the ones that commute the diagram.

3.2 Pullbacks and pushouts

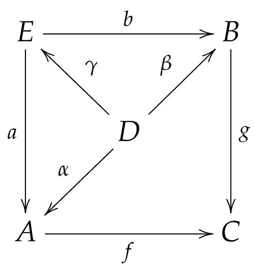
Let us have a look at another important universal object, *pullbacks* and their dual version *pushouts*. Let $\mathbf{C}$ be a category and two morphisms $f : A \longrightarrow C$ and $g : B \longrightarrow C$.



Generate the category **K** in the following way: take as objects the commutative diagrams of the form



with D object of **C**, we will name this object as $[A, B, C; D]$. Given 2 objects $[A, B, C; D]$ and $[A, B, C; E]$ we have the set $Mor_{\mathbf{K}}([A, B, C; D], [A, B, C; E])$ is constitute by morphisms $\gamma : D \longrightarrow E$ such that the diagram

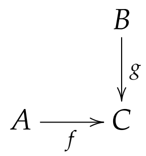


is commutative.

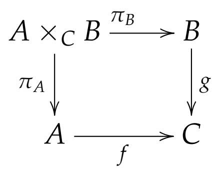
Definition 3.3. *Let* **K** *be the category defined above. We say that* $[A, B, C; P]$ *is a pullback if it is a terminal object in* **K**. *The dual notion is known as pushout. We will say that* P *is a pullback*

We say **C** has *pullbacks* if for every pair of morphisms, $f : A \longrightarrow C$ and $g : B \longrightarrow C$, there exists a pullback. Let us have a look at an example.

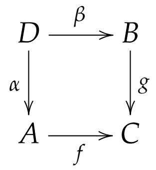
Example 3.4. Recalling the category **Set** the category of all sets. Let $f : A \longrightarrow C$ and $g : B \longrightarrow C$ with A, B, C three sets, and f, g two functions with the same codomain



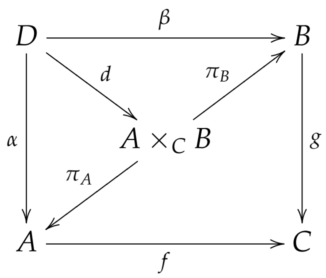
We will prove the set $A \times_C B := \{(a, b) \in A \times B \mid f(a) = g(b)\}$ is a terminal object with the projections to sets A, B.



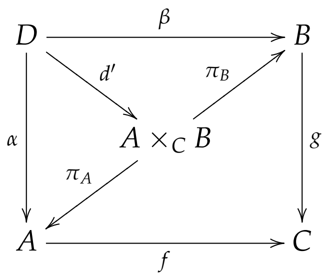
Let D a set and two functions $\alpha : D \longrightarrow A$ and $\beta : D \longrightarrow B$ such that the diagram



is commutative. Let see that there is a unique function $d : D \longrightarrow A \times_C B$ such that the diagram



is commutative. We define $d : D \longrightarrow A \times_C B$ as $d(x) = (\alpha(x), \beta(x))$ with $x \in D$. Let see this is a well defined function. As we have our diagram a commutative diagram, we have $f(\alpha(x)) = g(\beta(x))$ for every $x \in D$, this means $(\alpha(x), \beta(x)) \in A \times_C B$ for every $x \in D$. To prove the uniqueness we assume there is another function $d' : D \longrightarrow A \times_C B$ such that the diagram



is commutative. Looking at the diagram we figured out that for every $x \in D$ the equalities $\pi_B(d'(x)) = \beta(x)$ and $\pi_A(d'(x)) = \alpha(x)$ hold. This means the projections of d' are $d'(x) = (\alpha(x), \beta(x))$, and this is how we defined d, then $d' = d$.

This last example shows us that **Set** has pullbacks.

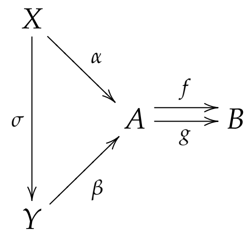
3.3 Other universal objects

We just saw 2 universal objects in categories. Now we will mention another 2 different universal objects that will help us to characterize categories.

- **Equalisers and coequialisers**
  Let **C** be a category and two morphisms $f, g \in Mor_{\mathbf{C}}(A, B)$. Consider the category **K** with objects the commutative diagrams

$$X \xrightarrow{\alpha} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

  this is $f \circ \alpha = g \circ \alpha$. For the morphisms of **K** between two objects we take the morphisms $\sigma : X \longrightarrow Y$ such that the diagram



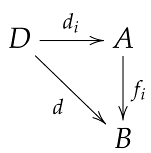
  is commutative.
  If there exists a terminal object T in the category **K**, then we will say that T is an *equaliser* in the category **C**. Its dual notion is known as *coequaliser*. We will say that category **C** has equalisers if, for every pair of morphisms $f, g : A \longrightarrow B$ there exists an equaliser.

- **Intersections**
  To understand this universal object we need a previous concept.

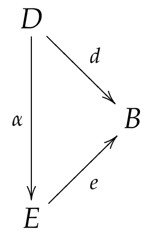
  Definition 3.5. *Let B be an object in the category **C**. We say that (A,f) is a subobject of B if A is an object of **C** and $f : A \longrightarrow B$ is a left cancellable morphism, this is:*

$$f \circ g = f \circ h \quad \Rightarrow \quad g = h$$

  Now, let $(A_i, f_i)_{i \in I}$ be a family of subobjects of an object B, indexed by the set I. We consider the category **K** choosing as objects the subobjects (D, d) of B such that the triangle



  commutes for every $i \in I$. The morphisms of this category **K** are the morphisms $\alpha : D \longrightarrow E$ such that the diagram



is commutative. If there is a terminal object T in the category **K** we say that there is an intersection of the family $(A_i, f_i)_{i \in I}$ of subobjects of B. We say **C** has finite intersections if for every finite family of subobjects of **C** there is an intersection.

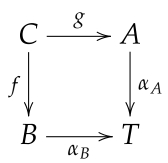
After looking at the most important universal objects of categories, we can characterize categories with the next theorem.

Theorem 3.6. *Let **C** be a category. The following statements are equivalent:*

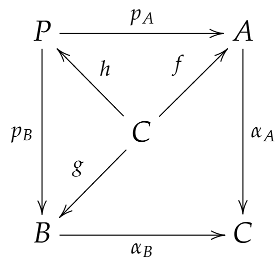
- *1) **C** has finite products and equalisers;*
- *2) **C** has finite products and finite intersections;*
- *3) **C** has pullbacks and a terminal object.*

Proof. 2) $\Rightarrow$ 1) and 1) $\Rightarrow$ 3) implications are proved with concepts out of this project, so we will just show the implication 3) $\Rightarrow$ 2).

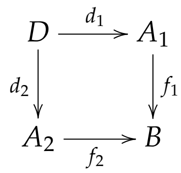
3) $\Rightarrow$ 2) Suppose **C** has a terminal object T, this is for all X in **C** the set $Mor_{\mathbf{C}}(X, T) = \{\alpha_X\}$, a singleton set. Let $f : C \longrightarrow A$ and $g : C \longrightarrow A$ be two morphisms of **C**. Since T is terminal, $Mor_{\mathbf{C}}(C, T)$ is also singleton, and therefore the following diagram commutes:



As **C** has pullbacks, we can considerate a pullback P with morphisms $p_A : P \longrightarrow A$ and $p_B : P \longrightarrow B$, of the pair (α_A, α_B). Then there is a unique morphisms $h : C \longrightarrow P$ such that $p_b \circ h = g$ and $p_A \circ h = f$, resulting the diagram



Then we have that (P, p_A, p_B) is a product of $\{A, B\}$.
Let us see **C** has finite intersections. We can have a look at the definitions and observe that if (A_1, f_1) and (A_2, f_2) are subobjects of B then an subobject (D, d) is an intersection of (A_1, f_1) and (A_2, f_2), if and only if $[A_1, A_2, B; D]$:



is a pullback. So from the fact **C** has pullbacks, we can say **C** has finite intersections. □

Chapter 4

Functors

In the previous chapters we look at different concepts of categories, we realized that objects do not have the same relevance as morphisms. Moreover, we could think of a category **C** as the morphisms it is made of, this would be the category that has as objects the identity morphisms; and morphisms the same from category **C**. The point is, that most important in categories are the relations, the morphisms. This allows us to think of how categories relate to them. We are going to define the concept of *functor* which, in informal words, means a morphism between categories. Also, we will look at particular collection of functors, and even how they relate between them.

4.1 Notion of functor

Definition 4.1. *A covariant functor from the category **C** to category **D** is a prescription that assigns to every objet A of **C** an object FA of **D**, and to every morphisms $\alpha : A \longrightarrow B$ of **C** a morphisms $F\alpha : FA \longrightarrow FB$ of **D**, such that*

- $Fid_A = id_{FA}$ *for every object A of **C***
- *if $\beta \circ \alpha$ is defined in **C** then $F\beta \circ F\alpha$ is defined in **D** and $F\beta \circ F\alpha = F(\beta \circ \alpha)$.*

We have a similar definition for a *contravariant functor*, the difference from the previous one is that every morphism $\alpha : A \longrightarrow B$ is assigned to a morphism $F\alpha : FB \longrightarrow FA$. We could understand a contravariant functor as a covariant functor from category **C** to the dual category of **D**, $\mathbf{D}^d$.

A representation of these concepts could be:

$$\begin{array}{ccc} A & \xrightarrow{F} & FA \\ \alpha\Big\downarrow & \xrightarrow{F} & \Big\downarrow F\alpha \\ B & \xrightarrow[F]{} & FB \end{array}$$

is a diagram of a covariant functor. The next one is a diagram of a contravariant functor:

$$\begin{array}{ccc} A & \xrightarrow{F} & FA \\ \alpha\Big\downarrow & \xrightarrow{F} & \Big\uparrow F\alpha \\ B & \xrightarrow[F]{} & FB \end{array}$$

Nevertheless, when we talk of a *functor* we will refer to a covariant functor. Let us have a look at some examples.

Example 4.2. A topological example of a functor would be the fundamental group Π, the first group of homotopy. This is the functor that assigns a group to each topological space, and each continuous map to a morphism of groups. This is $\Pi : \mathbf{Top} \longrightarrow \mathbf{Grp}$ goes from the category **Top** to the category **Grp**. Really, every homotopy and homology group is a functor from **Top** to **Grp**.

Example 4.3. The *forgetful functor* is the functor that goes from **Top** to **Set**. Every topological space (U, T_U) is assigned to a set, the set of points U; and every continuous application $f : (U, T_U) \longrightarrow (V, T_V)$ is assigned to a function between sets $Ff : U \longrightarrow V$ defined as $Ff(u) = f(u) \in V$. What the forgetful functor does is avoiding the topological structure and look at the topological space without its topology, a set. Notice that forgetful functor could be defined over every mathematical structure based on sets.

Example 4.4. Let us work now an example that is relevant for the next sections. Consider the category **C** and a object A of **C**. We will define a functor of the form

$$Mor_{\mathbf{C}}(A, \_) : \mathbf{C} \longrightarrow \mathbf{Set}$$

This is a set valued functor, meaning that our functor goes from the category **C** to the category **Set**. This functor assigns to each object B of **C** the set:

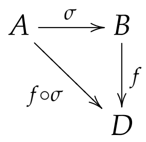
$$Mor_{\mathbf{C}}(A, \_) := Mor_{\mathbf{C}}(A, B)$$

Given two objects B and D from **C** we assign to each morphism $f : B \longrightarrow D$ the function $Mor_{\mathbf{C}}(A, \_)(f) := Mor_{\mathbf{C}}(id_A, f)$ defined by:

$$Mor_{\mathbf{C}}(id_A, f) : Mor_{\mathbf{C}}(A, B) \longrightarrow Mor_{\mathbf{C}}(A, D)$$

$$\sigma \longmapsto Mor_{\mathbf{C}}(id_A, f)(\sigma) := f \circ \sigma$$

This function could be visualized in the next diagram



We will denote this functor as the *right morphism functor*, and to simplify notation we denote $h_A = Mor_{\mathbf{C}}(A, \_)$. Similary we have the *left morphism functor*:

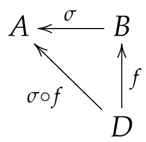
$$Mor_{\mathbf{C}}(\_, A) : \mathbf{C}^d \longrightarrow \mathbf{Set}$$

This functor assigns to each object B of $\mathbf{C}^d$ the set $Mor_{\mathbf{C}}(B, A)$; and every morphism $f : D \longrightarrow B$ of $\mathbf{C}^d$ is associated to the function:

$$Mor_{\mathbf{C}}(f, id_A) : Mor_{\mathbf{C}}(B, A) \longrightarrow Mor_{\mathbf{C}}(D, A)$$

$$\sigma \longmapsto Mor_{\mathbf{C}}(f, id_A)(\sigma) := \sigma \circ f$$

This function is represented in the diagram



We denote the *left morphism functor* $h^A = Mor_{\mathbf{C}}(\_, A)$. Notice this last functor is an example of a contravariant functor.

The advantage of using functors is that we can work in a particular category **C**, if you find any result in **C** then you can translate this one thanks to a functor $F : C \longrightarrow D$, whenever this makes sense. Another interesting curiosity of functors is that they allow us to create the *category of all categories*; this is the category that has as objects all categories, and as morphisms the functors between categories. This last sentence allows us to think about composition between functors.

Definition 4.5. *Let $F : \boldsymbol{C} \longrightarrow \boldsymbol{D}$ and $G : \boldsymbol{D} \longrightarrow \boldsymbol{E}$ be two functors. We denote the functor $G \circ F : \boldsymbol{C} \longrightarrow \boldsymbol{E}$ in the following way: if A is an object of C, then $G \circ F(A) = G(FA)$; also, if α is a morphism in $\boldsymbol{C}$ then $G \circ F(\alpha) = G(F\alpha)$.*

4.2 Yoneda's Theorem and representable functors

We just saw how to relate two different categories, it would a reasonable question to ask for relations between functors. This question leads us to *natural transformations*.

Definition 4.6. *Let F, G be two functors from the category $\boldsymbol{C}$ to the category $\boldsymbol{D}$. We say a natural transformation η is a rule that assigns to each object A of $\boldsymbol{C}$ a morphism $\eta_A : FA \longrightarrow GA$ of $\boldsymbol{D}$ in such a wat that associated with every morphism $f : A \longrightarrow B$ in $\boldsymbol{C}$ there is a commutative diagram:*

$$\begin{array}{ccccc} A & & FA & \xrightarrow{\eta_A} & GA \\ {\scriptstyle f}\downarrow & & {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ B & & FB & \xrightarrow[\eta_B]{} & GB \end{array}$$

Let have a look at an example:

Example 4.7. Given a category **C**, we consider a morphism $\sigma : X \longrightarrow Y$. Take functors $h^X = Mor_{\mathbf{C}}(\_, X)$ and $h^Y : Mor_{\mathbf{C}}(\_, Y)$, these are the corresponding left morphism functors from Example 4.4. Let us see how we can generate a natural transformation η between these two functors, this means we would have a commmutative diagram:

$$\begin{array}{ccccc} A & & h^X A = Mor_{\mathbf{C}}(A, X) & \xrightarrow{\eta_A} & h^Y A = Mor_{\mathbf{C}}(A, Y) \\ {\scriptstyle f}\downarrow & & {\scriptstyle h^X f}\downarrow & & \downarrow{\scriptstyle h^Y f} \\ B & & h^X B = Mor_{\mathbf{C}}(B, X) & \xrightarrow[\eta_B]{} & h^Y B = Mor_{\mathbf{C}}(B, Y) \end{array}$$

We can define the natural transformation η in the following way:

$$\eta_A : h^X A \longrightarrow h^Y A$$

$$\alpha \longmapsto \eta_A(\alpha) = \sigma \circ \alpha$$

Resulting the triangle:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ & {\scriptstyle \sigma\circ\alpha}\searrow & \downarrow{\scriptstyle \sigma} \\ & & Y \end{array}$$

What really does this natural transformation is to compose the given morphism $\sigma : X \longrightarrow Y$ with any morphism $\alpha : A \longrightarrow X$ resulting a morphism $\beta = \sigma \circ \alpha : A \longrightarrow Y$, looking at them in the category **Set**.

Now we could consider the class of natural transformations from functor F to functor G, the class $Nat(F, G)$. An interesting question would be if this class constitutes a set because if it does we could, for example, consider the category of functors and the natural transformations as morphisms. Well, in general, this is no longer true. Nevertheless, if we are working with a small category (i.e. its objects are sets) then we can have an interesting result, Yoneda's theorem.

Theorem 4.8. *(Yoneda) Let* $\mathbf{C}$ *be a category, an object* A *of* $\mathbf{C}$*, and a set valued functor* $F : \mathbf{C} \longrightarrow \mathbf{Set}$*. Then the class* $Nat(h_A, F)$ *of natural transformations from* h_A *to* F *constitutes an equipotent set to* FA*.*

Remember that functor h_A refers to the right morphism functor $Mor_{\mathbf{C}}(A, \_)$.

Proof. Let $\eta : h_A \longrightarrow F$ be a natural transformation, this is for every objects X and Y of $\mathbf{C}$ and every morphism $f : X \longrightarrow Y$ we have the commutative diagram

$$\begin{array}{ccccc} Y & & h_A Y = Mor_{\mathbf{C}}(A, Y) & \xrightarrow{\eta_Y} & FY \\ {\scriptstyle f}\downarrow & & {\scriptstyle h_A f}\downarrow & & \downarrow{\scriptstyle Ff} \\ X & & h_A X = Mor_{\mathbf{C}}(A, X) & \xrightarrow[\eta_X]{} & FX \end{array}$$

In particular taking $Y = A$ and a morphism $f : A \longrightarrow X$ we have:

$$\begin{array}{ccccc} A & & h_A A & \xrightarrow{\eta_A} & FA \\ {\scriptstyle f}\downarrow & & {\scriptstyle h_A f}\downarrow & & \downarrow{\scriptstyle Ff} \\ X & & h_A X & \xrightarrow[\eta_X]{} & FX \end{array}$$

Notice f is an object of $h_A X$, so we can see from the diagram that

$$Ff(\eta_A(id_A)) = \eta_X(h_A f(id_A)) = \eta_X(f \circ id_A) = \eta_X(f)$$

So, we figured out that every morphism η_X is completely determined by the element $\eta_X(id_A)$ of FA thanks to the morphism Ff in **Set**. Since FA is a set, it follows that so also is $Nat(h_A, f)$.
We need to proof now $Nat(h_A, f)$ being an equipotent set to FA. To see this we consider the mapping

$$\sigma : Nat(h_A, f) \longrightarrow FA$$
$$\eta \longmapsto \sigma(\eta) = \eta_A(id_A)$$

So, our objective now is to produce an inverse mapping to σ, this is a mapping $\lambda : FA \longrightarrow Nat(h_A, F)$.

Suppose an element $a \in FA$. Given an $f : A \longrightarrow X$ we have $Ff : FA \longrightarrow FX$ and so we have $Ff(a) \in FX$. We can therefore define a mapping

$$\lambda_X(a) : h_A X \longrightarrow FX$$
$$f \longmapsto \lambda_X(a)(f) = Ff(a)$$

This mapping gives rise to a natural transformation $\lambda(a) : h_A \longrightarrow F$, as can be seen in the commutative diagram

$$\begin{array}{ccccc} X & & h_A X & \xrightarrow{\lambda_X(a)} & FX \\ {\scriptstyle g}\downarrow & & {\scriptstyle h_A g}\downarrow & & \downarrow{\scriptstyle Fg} \\ Y & & h_A Y & \xrightarrow[\lambda_Y(a)]{} & FY \end{array}$$

This is

$$Fg(\lambda_X(a)(f)) = Fg(Ff(a))$$
$$\lambda_Y(a)(h_A g(f)) = \lambda_Y(a)(g \circ f) = F(g \circ f)(a) = Fg(Ff(a))$$

So we will define our mapping $\lambda : Nat(h_A, f) \longrightarrow FA$ by the prescription $a \mapsto \lambda(a)$.
Let us show this is an inverse of σ previously defined. Consider η in $Nat(h_A, F)$ then

$$\lambda_X(\eta_A(id_A))(f) = Ff(\eta_A(id_A)) = \eta_X(f)$$

and so $\lambda(\eta_A(id_A)) = \eta$ and $\lambda(\sigma(\eta)) = \eta$. This shows that $\lambda \circ \sigma = id_{Nat(h_A,F)}$. Now for every $a \in FA$ we have

$$\sigma(\lambda(a)) = (\lambda_A(id_A)) = Fid_A(a) = id_{FA}(a)$$

and hence $\sigma \circ \lambda = id_{FA}$. □

This last result leads us to the study of the isomorphism λ generated in the previous demonstration $\lambda : FA \longrightarrow Nat(h_A, F)$. This morphism assigns a natural transformation for each set $a \in FA$, the natural transformation $\lambda(a)$. The question here is when this natural transformation $\lambda(a)$ becomes a natural isomorphism. Whenever this situation happens we will say the functor F is a *representable functor* by the pair (A, a).

Example 4.9. Let us consider the forgetful functor from Example 4.3 U : **Grp** $\longrightarrow$ **Set**. This is the functor that assigns to each group its set, and morphisms of groups become functions between sets. We will show that U is represented by the pair $(\mathbb{Z}, 1)$. Notice that 1 is being seen as an element of the set $\mathbb{Z}$, not as the neutral

element of the group $\mathbb{Z}$.
We need $\lambda(1) : h_{\mathbb{Z}} = Mor_{\mathbf{Grp}}(\mathbb{Z}, \_) \longrightarrow U$ to be a natural isomorphism. To check it, we suppose a morphism of groups $\sigma : \mathbb{Z} \longrightarrow G$ and we see that

$$\lambda_G(1) : h_{\mathbb{Z}}G \longrightarrow UG$$
$$\sigma \longmapsto \lambda_G(1)(\sigma)$$

is an isomorphism. This is because given an $x \in UG = G$ (G as set) we have a unique group morphism $\sigma : \mathbb{Z} \longrightarrow G$ such that $U\sigma(1) = x$, given that every group morphisms is completely determined by the image of the neutral element. Then to assign every group morphism an element of G, we first calculate $\sigma(1)$. So we have an unique antiimage for every $x \in UG = G$, then we have that our function $\lambda_G(1) : h_{\mathbb{Z}}G \longrightarrow UG$ is an isomorphism, and so is $\lambda(1)$.

4.3 Adjoint functors

In this section, we will define a new relation between functors and we will show the relationship between the previous concepts.

Definition 4.10. *Given categories **C**, **D**, and functors $F : \mathbf{C} \longrightarrow \mathbf{D}$, $G : \mathbf{D} \longrightarrow \mathbf{C}$ such that*

$$Mor_{\mathbf{C}}(G\_, \_) \approx Mor_{\mathbf{D}}(\_, F\_)$$

This is $Mor_{\mathbf{C}}(GX, Y) \approx Mor_{\mathbf{D}}(X, FY)$ for every $X \in \mathbf{D}$ and every $Y \in \mathbf{C}$. We say F is a right adjoint of G, G is a left adjoint of F and (G,F) is an adjoint pair.

Let us see a topological example.

Example 4.11. Let $U : \mathbf{Top} \longrightarrow \mathbf{Set}$ be the forgetful functor.

First, let $D : \mathbf{Set} \longrightarrow \mathbf{Top}$ be the functor that assigns to each set A the discrete topology $D(A) = (A, \mathcal{P}(A))$; and each function $f : A \longrightarrow B$ is assign to the continuous map $Df : (A, \mathcal{P}(A)) \longrightarrow (B, \mathcal{P}(B))$ defined by $Df(a) = f(a) \in (B, \mathcal{P}(B))$ for $a \in A$.
We will show that D is a left adjoint of U; this is for every $X \in \mathbf{Set}$ and every $Y \in \mathbf{Top}$:

$$Mor_{\mathbf{Top}}(DX, Y) \approx Mor_{\mathbf{Set}}(X, UY)$$

Let $f : X \longrightarrow UY$ be a function, we have to assign f an unique continuous map $g : DX \longrightarrow Y$. We define g in the following way:

$$g : DX \longrightarrow Y$$
$$x \longmapsto g(x) = f(x)$$

To show g is a continuous map we suppose an open $V \subset Y$, and we have to see that $g^{-1}(V) = \{p \in DX : \quad g(p) \in V\}$ is an open on X. Well this is true because every subset of X is an open in $DX = (X, \mathcal{P}(X))$, in particular for $g^{-1}(V)$.

On the other side, let $T : \mathbf{Set} \longrightarrow \mathbf{Top}$ be the functor that assigns to each set A the trivial topology $TA = (A, \{\emptyset, A\})$; and every function $f : A \longrightarrow B$ is assign to a continuous map $Tf : (A, \{\emptyset, A\}) \longrightarrow (B, \{\emptyset, B\})$ defined by $Tf(a) = f(a) \in (B, \{\emptyset, B\})$ for $a \in A$.
We will show that T is a right adjoint of U; this is for every $X \in \mathbf{Top}$ and every $Y \in \mathbf{Set}$:

$$Mor_{\mathbf{Set}}(UX, Y) \approx Mor_{\mathbf{Top}}(X, TY)$$

Let $f : UX \longrightarrow Y$ be a function, we have to assign f an unique continuous map $g : TY \longrightarrow X$. We define g in the following way:

$$g : X \longrightarrow TY$$

$$x \longmapsto g(x) = f(x)$$

To show g is a continuous map we suppose an open $V \subset TY$, and we have to see that $g^{-1}(V) = \{p \in DX : \quad g(p) \in V\}$ is an open on X. In this case $V \in \{\emptyset, Y\}$: if $V = \emptyset$ then $g^{-1}(V) = \emptyset \subset X$ is an open; and if $V = Y$ then $g^{-1}(V) = X$, and so it is an open.

To end, we will relate these concepts of adjoint pair and representable functors as it can be seen in the following theorem:

Theorem 4.12. *Let $F : \boldsymbol{C} \longrightarrow \boldsymbol{D}$ be a functor between categories $\boldsymbol{C}$ and $\boldsymbol{D}$. F has a left adjoint if and only if, for every object D of $\boldsymbol{D}$, the functor $F_D = h_D \circ F : \boldsymbol{C} \longrightarrow \boldsymbol{Set}$ is representable. The dual notion of this is: F has a right adjoint if and only if, for every D object of $\boldsymbol{D}$ the functor $F^D = h^D \circ F : \boldsymbol{C} \longrightarrow \boldsymbol{Set}$ is representable.*

Example 4.13. Remember the forgeful functor $U : \mathbf{Top} \longrightarrow \mathbf{Set}$ we saw in Example 4.11 had a left adjoint functor $D : \mathbf{Set} \longrightarrow \mathbf{Top}$, the discrete topology functor. Let us use the theorem 4.12, this means the functor $U_B = h_B \circ U : \mathbf{Top} \longrightarrow \mathbf{Set}$ is a representable functor for every set B.
We have to find a pair (A,a) with A a topological space, and $a \in U_B A = (h_B \circ U)A = h_B(UA) = Mor_{\mathbf{Set}}(B, UA)$, so a is of the form $a : B \longrightarrow UA$. Let A be DB, the discrete topological space of B; and $a = id_B$. So we have to prove that functor U_B is represented by the pair (DB, id_B). This allow us to check if the natural

transformation $\lambda(id_B) : h_{DB} \longrightarrow U_B$

$$\begin{array}{ccccc} X & & h_{DB}X = Mor_{\mathbf{Top}}(DB, X) & \xrightarrow{\lambda_X(id_B)} & U_BX = Mor_{\mathbf{Set}}(B, UX) \\ \Big\downarrow f & & \Big\downarrow h_{DB}f & & \Big\downarrow U_Bf \\ Y & & h_{DB}Y = Mor_{\mathbf{Top}}(DB, Y) & \xrightarrow[\lambda_Y(id_B)]{} & U_BY = Mor_{\mathbf{Set}}(B, UY) \end{array}$$

is an natural isomorphism, with

$$\lambda_X(id_B) : h_{DB}X \longrightarrow U_BX$$

defined by $g \longmapsto \lambda_X(id_B)(g) = U_Bg(id_B) = (h_B \circ U)g(id_B) = h_B(U_g(id_B)) = Ug$. Then we can say (DB, id_B) represents the functor U_B.

Notice that as we already knew that D was a left adjoint of U, we had the isomorphic sets $Mor_{\mathbf{Top}}(DB, X) \approx Mor_{\mathbf{Set}}(B, UX)$ for every X topological space. There is an isomorphism, the point is that we had to prove that $\lambda(id_B)$ was a natural isomorphism.

Chapter 5

Conclusions

To summarize, at first, we started this project without the knowledge of what a category is. We defined them and realized that it really makes sense to *categorize* mathematical structures. Moreover, as categories have their own mathematical structure, we can have the category of categories, possible because we introduce the notion of classes to define a category.

Moving forward, in the last section of chapter 2 and during all chapter 3 we look at how can we take advantage of categories. This work was done when we study particular elements in categories, and also try to use as much as we can the notion of "turning theorems into definitions". To do it we made new categories to find these *universal objects* called products, pullbacks, equalisers, and intersections.

In chapter 4 the most relevant work with categories was done. We investigate over relations between categories, these are the functors, and how important they are to categories, they are literally the morphisms of categories. Moreover, we focused on relations between functors called natural transformation, and when the class of natural transformations between two functors constitutes a set. In Yoneda's Theorem we realized there is one situation where the class of natural transformations between two functors become a set. This allowed us to define representable and adjoint functors, which are very related concepts.

We conclude the project saying that this research of categories was focused on the relations between objects, we tried to *define* mathematical objects not by a proper definition but by how it is related to other objects.

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