UNIVERSITAT DE BARCELONA

DOUBLE DEGREE IN MATHEMATICS AND BUSINESS ADMINISTRATION

Final degree project

LOCAL RISK MINIMIZATION STRATEGIES FOR OPTION PRICING

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Abstract

The main goal of this work is to introduce the local risk-minimizing strategy that can be applied in incomplete financial markets to hedge options, together with some applications. An option is a financial asset mainly used as an insurance product to protect the investor from the different market risks. The major risk for an option seller is not to be able to cover his future payments obligations with the option price received. This is what option hedging tries to solve, finding the ideal strategy and option price to cover the risk.

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I would also like to acknowledge all the professors who introduced me to the captivating and interesting topic of financial markets.

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Introduction

This work is the result after six years studying the double degree in Mathematics and Business Administration in the Universitat de Barcelona.

Business Administration covers a large range of areas, and two years ago I decided that I wanted to focus my career in finance. This is why in the penultimate semester I enrolled for a practical course on stock exchange and financial analysis sponsored by the university. In this course I greatly increased my knowledge in financial markets, and I came to the conclusion that I wanted my final project to be related to this area.

However, the financial market also covers a lot of topics and I was hesitant about which to choose. Dr. Oriol Roch suggested me to collect the different pricing models that can be found in incomplete markets. But for this I needed the mathematical base of financial markets, which I had not yet seen. Dr. Josep Vives lectures a basic course in this subject and he generously borrowed me his lecture notes so I could start my work.

During the research I realised that there was a strategy that was highly interesting, and that I had enough information to just focus on it. Therefore, I decided to just concentrate on risk-minimizing strategies, specifically in the local risk-minimizing method. In addition, I also believed that it would be great to add some real examples and not just focus on the theory part.

The work is divided into four parts. The first two chapters introduce the financial and mathematical theory needed to understand the main chapter of the work, which is chapter three. In this chapter I begin introducing some strategies that can be applied in incomplete markets, and later I concentrate on risk minimization strategies, in particular in a variant called local risk-minimizing. Finally, in the last chapter some examples of application of this strategy are given, together with a couple of numerical examples.

Chapter 1

Fundamentals

Basic financial concepts needed to understand the forthcoming chapters will be introduced in this first chapter.

1.1 Options

Every financial transaction comes with a risk, and since the formation of the financial market, players have always tried to find ways to mitigate it. One of the most common methods nowadays is with the use of derivatives.

Definition 1.1. A **contingent claim or derivative** is a financial product whose value is reliant on the performance of an underlying asset. It is created by means of a legal contract, in which the conditions are previously agreed between the seller and the buyer.

The typical underlying assets are bonds, commodities, currencies, interest rates and stocks. Futures and options are the most common traded derivatives in regulated financial markets, and are mainly used to mitigate risk and speculate thanks to their financial properties.

A *future* is a derivative contract in which the buyer has both the right and the obligation to buy a fixed amount of the underlying asset at the established price and date. In contrast, the seller has the obligation to sell that underlying asset.

Definition 1.2. An **option** is a derivative contract in which the buyer has the right, but not the obligation, to either buy or sell an arranged amount of the underlying asset for a fixed price, also called the *strike price*, at a specific date called *expiration date*. The execution of the right will be called *exercising the option*.

The choice opportunity is one of the main advantages that options offer in comparison with futures, together with an easiness to create market strategies[1]. As a consequence options have been object of multiple mathematical and financial studies.

Definition 1.3. We can distinguish between two types of options:

- **Call options**, which give the buyer the right to buy the underlying asset.
- **Put options**, which give the buyer the right to sell the underlying asset.

An agent in the financial market can either buy or sell a call or a put, giving place to four basic strategies.

We will work from the point of view of the option seller, who has the obligation to sell or to buy the underlying asset. Therefore he demands a *premium* from the buyer at the formalisation of the contract. This premium will be also interpreted as the *market option price*.

Definition 1.4. Additionally, options can be divided depending on the allowed execution time:

- **American options** can be exercised any time before the end of the contract, and even on that date.
- **European options** can only be exercised at the expiration date.

Deciding when is the best time to exercise the option adds complexity to the price valuation, therefore many of the financial studies and models are centered on European options, and so is this work.

With an European option, at the expiration day the right to buy or to sell will only take place if the holder of the option can get a benefit from it. This means that the holder can either exercise the option, or just do nothing, losing the premium paid to the seller.

The options market is a *zero-sum game*. In other words, the wins and the losses of both agents, the seller and the buyer of the option, get exactly balanced. And the quantity that one agent wins is the same amount that the other part loses.

[1]More advantages of options over futures and other financial products can be found in [6]: chapter 2, page 44.

Example 1.5. Two simple examples of the utility of options in the real market are given.

- An airline expects that the jet fuel price will increase in the next year and it wants to ensure the acquisition of it at the current market price. Therefore, the airline buys today a jet fuel call option, which gives it the right to buy the fuel in the future at the current price. If the price does increase, the company is not impacted by the price rise, whereas if the price decreases, the airline doesn't have to exercise the option and it can buy the jet fuel direct on the market for a lower price, losing only the premium payed for the option.
- A farmer expects that the harvest will not be good this season and he is worried that his products will have to be sold at a lower price than the current market price. Therefore, he buys a put option for a premium amount, which gives him the right to sell his products at the current price. If the harvest is poor, he can sell his products at a higher price, whereas if the harvest goes better than expected, he doesn't have to exercise the option and he can sell the products for the market price.

Both examples show that with the use of options agents can be secured against uncertain and unwanted price changes from the financial market, only by paying the option price.

1.2 Aspects of options

From the book of J.C. Cox and M. Rubinstein in 1985 [6], it can be concluded that there are six fundamental direct determinants that can influence the option market price prior to expiration. Their effect on the option price will differ if it is a call or a put. We focus first on call options:

- **The current market price of the underlying asset:** The greater the value of the underlying asset in the market, the higher the call value.
- **The fixed or strike price of the option:** The higher the strike price at expiration, the lower the initial call price.
- **The asset volatility[2]:** The higher the volatility, the higher the possible future prices. Therefore, more risk for the seller implies a higher call price.
- **The time to expiration:** The longer it takes to expire, the higher the volatility. Hence, the higher the premium that will have to be payed.

[2]We understand asset volatility as a measure of the dispersion of possible future asset prices.

- **The interest rate:** The higher the interest rate, the lower the present value of the strike price. This implies a higher call value.
- **The cash dividends from the underlying asset:** Higher future dividends of the underlying asset will lower the current call price.

For a put option the effect of the determinants work in the opposite direction, except for the volatility, which impacts in the same way.

Remark 1.6. For this work we will consider options with underlying assets that pay no dividend, such as stocks.

Definition 1.7. At the expiration date (T) we can differentiate three different option positions regarding the market stock price (S_T) and the strike price (K):

- **In the money:** When $S_T > K$ for call options and $S_T < K$ for put options. In this position the call holder can buy the underlying asset for a lower price than the one from the market, thus he obtains a benefit from it and he will exercise the option.
- **At the money:** When $S_T = K$. The price of the underlying asset coincides with the strike price, therefore it does not matter if the call holder buys the asset in the market or exercises the option.
- **Out of the money:** When $S_T < K$ for call options and $S_T > K$ for put options. In the case of exercising the call option, the holder would pay a higher price than if he buys it directly in the market.

Remark 1.8. When the option is in the money, if the call holder just after exercising the option and buying the underlying asset for a price K immediately sells the asset back to the market for the price S_T, he can obtain a positive amount of $S_T - K$. For market and theory simplicity we will consider that when the option ends in the money, the seller pays this amount to the call buyer. This is the *payment or payoff of the option*, and it can be denoted by $(S_T - K)^+ := max\,[S_T - K,\, 0]$. On the other hand, if the option ends out of the money the call holder will buy the underlying asset in the market loosing the premium.

As a result, the *profit* that a call holder can obtain with the option is $(S_T - K)^+ -$ premium. A large option price will decrease the option profit.

A similar argument holds for the put option, where the payment is denoted by $(K - S_T)^+ := max\,[K - S_T,\, 0]$.

We can determine a price relation between a call and a put option.

Proposition 1.9. *(Call-put parity for European options)*

Given a call option with price C_t at time t, strike price K and expiration date T, and a put option P_t with the same conditions. Then it satisfies

$$C_t - P_t = S_t - \frac{K}{(1+r)^t},$$

where $r \geq 0$ is a fixed interest rate and S_t is the market stock price.

Proof. Can be found in [12]: chapter 1, page 28. □

Proposition 1.10. *The price call C_t satisfies*

$$S_t \geq C_t \geq \left(S_t - \frac{K}{(1+r)^t} \right)^+ .$$

Proof. The higher bound can easily be seen by contradiction and the lower bound using the call-put parity with a positive put price. □

Remark 1.11. Note that the call price is a convex function of the stock price.

Similarly, for a put option the boundaries will be

$$\left(\frac{K}{(1+r)^t} - S_t \right)^+ \geq P_t \geq \frac{K}{(1+r)^t}.$$

The main difficulty that a seller encounters in the financial market is to be able to find an appropriate selling price for the product. For this reason, this work provides a strategy to solve this common problem.

Chapter 2

Mathematical framework

Before introducing the central model that will help determining the option price, mathematical concepts of the financial market have to be introduced.

The first two sections are mainly based on the lecture notes [4] and [21], and the third section in [10]: chapter 10.

2.1 Martingales

Definition 2.1. A **stochastic process** $X := \{X_t;\ t \in \mathbb{T}\}$ is a sequence of random variables defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and within a certain period of time $\mathbb{T}$.

We will be working with a discrete-time set $\mathbb{T} := \{0, 1, ..., T\}$, for a certain fixed $T \in \mathbb{N}$ or $\mathbb{T} = \mathbb{N}$. We denote by $t = 0$ the *current time*.

Definition 2.2. A **filtration** $\mathbb{F}$ **associated with a probability space** $(\Omega, \mathcal{F}, \mathbb{P})$ is a sequence of σ-algebras $\mathbb{F} := \{\mathcal{F}_t;\ t \in \mathbb{T}\}$ such that

- $\mathcal{F}_t \subseteq \mathcal{F}$, for all $t \in \mathbb{T}$,
- $\mathcal{F}_{t-1} \subseteq \mathcal{F}_t$, for all $t \in \mathbb{T} - \{0\}$.

A filtration $\mathcal{F}_t$ at time $t \in \mathbb{T}$ represents the option price information available in the market at that time. A probability space with an associated filtration is called a *filtered probability space*, $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$.

Definition 2.3. The **natural filtration** $\mathbb{F}^N$ of a stochastic process X is a sequence of σ-algebras such that

$$\mathcal{F}_t := \sigma \left\{ X_j,\ j \leq t \right\}.$$

This is, the σ-algebras generated by the own variables of the process.

Definition 2.4. A stochastic process X defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is **adapted with respect to a filtration** $\mathbb{F}$ if X_t is $\mathcal{F}_t$-measurable for any $t \in \mathbb{T}$.

This means that given an information $\mathcal{F}_t$, for every instant $t \in \mathbb{T}$, we know the value of X_t. And for every time $j > t$, we can only know its probability.

Remark 2.5. Every stochastic process is adapted to its natural filtration.

The natural filtration is the information that can be deducted from the process, and it can be complemented with other information from the market. For now we will consider that $\mathcal{F} = \mathcal{F}^N$.

Definition 2.6. A stochastic process X is **predictable with respect to a filtration** $\mathbb{F}$ if X_0 is $\mathcal{F}_0$-measurable, and X_t is $\mathcal{F}_{t-1}$-measurable for any $t \in \mathbb{T} - \{0\}$.

This means that given an information $\mathcal{F}_{t-1}$ we know the value of X_t.

Definition 2.7. We say that a process $M := \{M_t;\ t \in \mathbb{T}\}$ is a **martingale with respect to a filtration** $\mathbb{F}$ in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if

i) M is a stochastic process adapted to $\mathbb{F}$,

ii) $\mathbb{E}(|M_t|) < \infty$, for every $t \in \mathbb{T}$,

iii) $\mathbb{E}[M_{t+1}|\mathcal{F}_t] = M_t$, for every $t \in \mathbb{T}$.

$\mathbb{E}[M_{t+1}|\mathcal{F}_t]$ is called the *conditional expectation of M_{t+1} based on the information at time t*, and it satisfies the linear property, i.e. for $a, b \in \mathbb{R}$

$$\mathbb{E}\left(aM + bM' \middle| \mathcal{F}_t\right) = a\mathbb{E}(M|\mathcal{F}_t) + b\mathbb{E}\left(M' \middle| \mathcal{F}_t\right).$$

Remark 2.8. The third property says that the expected value in the next point of time $t+1$ is equal to the present value M_t. It can also be rewritten in two other ways

- denoting $\Delta M_{t+1} = M_{t+1} - M_t$ and $\forall t \in \mathbb{T}$,

$$\mathbb{E}[\Delta M_{t+1}|\mathcal{F}_t] = 0. \tag{2.1}$$

- $\mathbb{E}\left[M_{t+j} \middle| \mathcal{F}_t\right] = M_t$, $\forall j \geq 1$ and $\forall t \in \mathbb{T}$.

  Proof. It can be seen by recursively applying the tower property of conditional expectation for all $j \geq 2$,

$$\mathbb{E}\left[M_{t+j}|\mathcal{F}_t\right] = \mathbb{E}\left[\mathbb{E}\left[M_{t+j}|\mathcal{F}_{t+j-1}\right]|\mathcal{F}_t\right] = \mathbb{E}\left[M_{t+j-1}|\mathcal{F}_t\right] = M_t.$$

□

The tower property satisfies for $j \leq t \leq u$

$$\mathbb{E}\left[\mathbb{E}\left[M_u|\mathcal{F}_t\right]\middle|\mathcal{F}_j\right] = \mathbb{E}\left[M_u\middle|\mathcal{F}_j\right], \forall t.$$

Remark 2.9. In a finite probability space every random variable X can be integrated. This implies that its expected value is

$$\mathbb{E}\left(X\right) := \sum_{\omega\in\Omega} X\left(\omega\right)\mathbb{P}\left(\omega\right),$$

which is a finite number.

Remark 2.10. A martingale can be seen as the mathematical formalization of a zero-sum game.

We will now present some properties of martingales.

Definition 2.11. Let M be a martingale, H an adapted and bounded process with respect to a filtration $\mathbb{F}$, and $M_0 \in \mathbb{R}$ a constant. The **martingale transformation of** X **by** H is defined as

$$X_t := H_t \cdot M_t = M_0 + \sum_{j=1}^{t} H_{j-1}\Delta M_j, \; \forall t \geq 1,$$

where $\Delta M_j = M_j - M_{j-1}$.

Proposition 2.12. *The martingale transformation is a martingale.*

Proof. For every $t \geq 0$, M_t is $\mathcal{F}_t$-measurable, thus it is an adapted process. Since H is a bounded process, M_t is integrable, and finally because H is adapted,

$$\mathbb{E}\left[\Delta X_{t+1}|\mathcal{F}_t\right] = \mathbb{E}\left[H_t\left(\Delta M_{t+1}\right)|\mathcal{F}_t\right] = H_t\mathbb{E}\left[\Delta M_{t+1}|\mathcal{F}_t\right] = 0.$$

□

Proposition 2.13. *Let $M := \{M_t,\, t \geq 0\}$ be an adapted and integrable process. M is a martingale if and only if for every adapted and bounded process H and for every $t \geq 0$, we have*

$$\mathbb{E}\left(\sum_{j=1}^{t} H_{j-1}\Delta M_j\right) = 0.$$

Proof. Can be found in [21]: chapter 2. □

Definition 2.14. Two martingales M and M' are **strongly orthogonal** if their product MM' is also a martingale. Using Formula 2.1, this is for all $t \in \mathbb{T}$,

$$\mathbb{E}\left[\Delta\left(MM'\right)_{t+1}\middle|\mathcal{F}_t\right] = 0.$$

2.2 Fundamental theorems of finance

Definition 2.15. A **discrete-time financial model** is built on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ within a discrete-time period $\mathbb{T} := \{0, 1, ...T\}$, where

- Ω is a finite set of elements and every $\omega \in \Omega$ represents every possible situation or scenario of the market,
- $\mathbb{F} := \{\mathcal{F}_t, \, t \geq 0\}$ is a filtration that represents the information available in the market at time $t \in \mathbb{T}$,
- $\mathbb{P}$ is a probability, such that $\mathbb{P}(\omega) > 0$, for all $\omega \in \Omega$. This means that every state ω is possible and none can be ruled out.

Assume that the financial market is formed by $d+1$ *financial assets*, $d \in \mathbb{N}$. They can be, for instance, derivatives, bonds, commodities, or currencies. The price of each of them at time t is $S_t^0, S_t^1, ..., S_t^d$, where S_t^0 will be the only risk-free asset. It is also called the *investor bank account or savings account*, and if we suppose an initial saving of one monetary unit, $S_0^0 = 1$, and a fixed interest of $r \geq 0$, its capitalized value will be $S_t^0 = (1+r)^t$ at time t.

The rest of the products will be the risky assets and they can take any real positive value, $S_t^k \in \mathbb{R}^+$, $k \geq 1$. $\mathbf{S}_t := \left(S_t^0, S_t^1, ..., S_t^d\right)$ will be the *price array at time t* formed by random $\mathcal{F}_t$-measurable variables.

Remark 2.16. We will denote in bold the complete array containing the risk-free and risky assets.

Definition 2.17. The **discounted price of an asset** S_t^k, $k \geq 1$, $t \geq 1$ **at time** t is

$$\tilde{S}_t^k := \frac{S_t^k}{(1+r)^t} = \frac{S_t^k}{S_t^0}.$$

$\tilde{\mathbf{S}}_t := \left(\tilde{S}_t^0, \tilde{S}_t^1, ..., \tilde{S}_t^d\right)$ will be the *discounted price array at time t* and we will denote by $\tilde{S}^k := \left\{\tilde{S}_t^k, \, t \in \mathbb{T}\right\}$ the *discounted price process of the risky asset* S^k.

Remark 2.18. Note that it satisfies $\tilde{S}_t^0 = 1$, $\forall t$, and $\tilde{S}_0^k = S_0^k$, $\forall k \geq 0$.

The idea of how the financial model will work is the following: at the initial time $t = 0$ there will be a finite number of possible values. As time evolves and as the new information reveals, certain states could be excluded as being impossible, and finally, at time $t = T$ the true state $\omega \in \Omega$ will be revealed. Hence the evolution of the information is aligned with the definition of a filtration and its sequence $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}_{T-1} \subseteq \mathcal{F}_T$ as the information available increases.

Definition 2.19. A **portfolio** is a combination of risky assets with a one risk-free asset.

Definition 2.20. A **trading or investment strategy at time** $t \in \mathbb{T}$ is a $(d+1)$-array formed by adapted stochastic processes, such that

$$\mathbf{H}_t := \left(H_t^0, H_t^1, ..., H_t^d\right).$$

H_t^k denotes the number of shares of the asset k held in the portfolio at time t. $H^k := \left\{H_t^k,\ t \in \mathbb{T}\right\}$ will be the *investment strategy process of the asset* H^k, and $\mathbf{H} := \{\mathbf{H}_t,\ t \geq 0\}$ will be the *trading or investment strategy.*

Remark 2.21. Note that H_t^k can be negative. This corresponds to borrowing money from the bank in the case of the risk-free asset, or selling short[1] for the risky assets.

Definition 2.22. The **portfolio value or value of a portfolio at time** $t \in \mathbb{T}$ following a trading strategy $\mathbf{H}_t$ is the scalar product

$$\mathrm{V}_t\left(\mathbf{H}\right) := \mathbf{H}_t \cdot \mathbf{S}_t = \sum_{k=0}^{d} H_t^k\, S_t^k.$$

$\mathrm{V}_0(\mathbf{H}) = H_0 \cdot S_0$ will be the *initial investment*, and $\tilde{\mathrm{V}}_t\left(\mathbf{H}\right) = \frac{\mathrm{V}_t(\mathbf{H})}{S_t^0}$ will be the *discounted portfolio value at time* t.

Definition 2.23. A **self-financing strategy** is an investment strategy such that

$$\mathbf{H}_t \cdot \mathbf{S}_t = \mathbf{H}_{t-1} \cdot \mathbf{S}_t,\ \forall t \in \mathbb{T} - \{0\}.$$

This means that the investment in the portfolio is the same for every point in time, and that no money is added, i.e., any change in the portfolio value must be due to a gain or a loss in the risky assets.

Proposition 2.24. *(Invariability of the discount factor)*
An investment strategy ***H*** *is self-financing with respect to the prices* $\mathbf{S}_t$ *if and only if* ***H*** *is self-financing with respect to the discounted prices* $\tilde{\mathbf{S}}_t$.

Proof. $\forall t \in \mathbb{T} - \{0\}$,

$$\mathbf{H}_t \cdot \mathbf{S}_t = \mathbf{H}_{t-1} \cdot \mathbf{S}_t \Leftrightarrow \mathbf{H}_t \cdot \frac{\mathbf{S}_t}{S_t^0} = \mathbf{H}_{t-1} \cdot \frac{\mathbf{S}_t}{S_t^0} \Leftrightarrow \mathbf{H}_t \cdot \tilde{\mathbf{S}}_t = \mathbf{H}_{t-1} \cdot \tilde{\mathbf{S}}_t.$$

□

[1] In the financial market, short selling means borrowing an asset and selling it on the market, planning to return it back later for less money.

Proposition 2.25. *The following are equivalent.*

i) A trading strategy $\boldsymbol{H}$ *is self-financing.*

ii) For any $t \in \mathbb{T}$,

$$\mathrm{V}_t(\boldsymbol{H}) = \mathrm{V}_0(\boldsymbol{H}) + \sum_{j=1}^{t} \mathbf{H}_{j-1} \cdot \Delta \mathbf{S}_j,$$

where $\Delta \mathbf{S}_j = \mathbf{S}_j - \mathbf{S}_{j-1}$.

iii) For any $t \in \mathbb{T}$,

$$\tilde{\mathrm{V}}_t(\boldsymbol{H}) = \mathrm{V}_0(\boldsymbol{H}) + \sum_{j=1}^{t} \mathbf{H}_{j-1} \cdot \Delta \tilde{\mathbf{S}}_j, \tag{2.2}$$

where $\Delta \tilde{\mathbf{S}}_j = \tilde{\mathbf{S}}_j - \tilde{\mathbf{S}}_{j-1}$.

Proof. Can be found in [12]: chapter 1, page 16. □

Proposition 2.26. *Consider* $H^k = \left\{ \left(H_t^1, H_t^2, ..., H_t^d\right), \, t \in \mathbb{T} \right\}$ *an adapted stochastic process and* $\mathrm{V}_0 \geq 0$ *a constant. Then, it exists a unique adapted process* H^0 *such that*

$$\boldsymbol{H} = \left(H^0, H^1, ..., H^d\right)$$

is a self-financing strategy with initial value $\mathrm{V}_0(\boldsymbol{H}) = \mathrm{V}_0$.

Proof. Can be found in [21]: chapter 3. □

This proposition shows that a self-financing strategy is determined only by its initial value and the amount of risky assets at a certain moment of time.

Definition 2.27. A self-financing strategy $\mathbf{H}$ is an **arbitrage strategy** if $\mathrm{V}_0(\mathbf{H}) = 0$ and $\mathrm{V}_T(\mathbf{H}) \geq 0$ with a strict positive probability.

In other words, an arbitrage strategy allows the opportunity to yield a positive amount of money without taking any risk. If arbitrage opportunities existed, everyone would enter the financial market using this strategy. Hence, the financial market would not be in equilibrium. This is why modern finance is based on the absence of arbitrage opportunities or *arbitrage-free models*. We will be assuming an arbitrage-free financial market for the rest of the work.

The *law of one price* states that in a financial market cannot exist two trading strategies $\mathbf{H}$ and $\mathbf{H}'$, such that $\mathrm{V}_T(\mathbf{H}) = \mathrm{V}_T(\mathbf{H}')$ satisfying $\mathrm{V}_0(\mathbf{H}) \neq \mathrm{V}_0(\mathbf{H}')$. This means that it is not possible to have two trading strategies with same final value, but different initial value.

Remark 2.28. If there are no arbitrage opportunities, the law of one price needs to hold.

We will now introduce the two fundamental theorems of the financial market.

Definition 2.29. A market is said to be **viable** if there are no arbitrage opportunities.

Definition 2.30. A **measure of a probability** $\mathbb{P}^*$ **is equivalent to** $\mathbb{P}$**,** $\mathbb{P}^* \sim \mathbb{P}$**,** if for any $A \in \mathcal{F}, \quad \mathbb{P}^*(A) = 0 \Leftrightarrow \mathbb{P}(A) = 0$.

This implies that they have the same sets of probability zero, and in a finite probability space this simply reduces in $\mathbb{P}^*(\omega) > 0$ for all $\omega \in \Omega$.

Theorem 2.31. *(First fundamental theorem of finance)*

A finite market is viable if and only if there exists a probability measure $\mathbb{P}^*$ *equivalent to* $\mathbb{P}$ *such that the discounted prices of the risky assets,* $\tilde{S}^k$*,* $k = 1, ..., d$ *are martingales with respect to* $\mathbb{P}^*$*.*

Remark 2.32. Every price probability in which the discounted price is a martingale is called *risk neutral probability or martingale measure*. With this definition, the theorem also says that a finite market is viable if and only if there exists a risk neutral probability. Therefore, the absence of arbitrage opportunities is characterized by the existence of martingale measures.

Definition 2.33. The **set of all risk neutral probabilities** will be denoted by $\mathcal{P} := \{\mathbb{P}^*\}$, which is non empty in a viable market.

Proof. $\Leftarrow$ Assume that there exists a risk neutral probability $\mathbb{P}^* \sim \mathbb{P}$, such that the discounted prices $\tilde{S}^k$ are martingales. Consider a self-financing strategy $\mathbf{H}$. Using Formula 2.2, we have

$$\tilde{V}_t(\mathbf{H}) = V_0(\mathbf{H}) + \sum_{j=1}^{t} \sum_{k=1}^{d} H_{j-1}^k \Delta \tilde{S}_j^k,$$

where $\Delta \tilde{S}_j^k = \tilde{S}_j^k - \tilde{S}_{j-1}^k$. This is a sum of d martingale transformations because H^k are adapted processes per definition, and are also bounded since we are in a finite probability space. We have seen that the transformation of a martingale is a martingale, thus we have that $\tilde{V}_t(\mathbf{H})$ is a martingale under $\mathbb{P}^*$. If we define $\mathbb{E}^*$ as the *expected value under* $\mathbb{P}^*$, we obtain

$$\mathbb{E}^*\left(\tilde{V}_T(\mathbf{H})\right) = \mathbb{E}^*\left(\tilde{V}_0(\mathbf{H})\right).$$

Taking an strategy with $\tilde{V}_0(\mathbf{H}) = 0$, this leads to $\mathbb{E}^*\left(\tilde{V}_N(\mathbf{H})\right) = 0$. Therefore, if the expected value is zero, having $\tilde{V}_T(\mathbf{H}) \geq 0$, it necessarily has to be zero for every $\omega \in \Omega$. Otherwise there would be an arbitrage opportunity and the market would not be complete.

$\Rightarrow$ Suppose now that the market is complete and there cannot exist any arbitrage opportunity. Using Proposition 2.13, in order to see that S^k is a martingale we have to construct $\mathbb{P}^*$ in order that with a fixed $T \geq 1$, $k \in \{1, ..., d\}$ and any adapted process H^k we have

$$\mathbb{E}^*\left(\sum_{j=1}^{T} H_{j-1}^k \Delta\tilde{S}_j^k\right) = 0.$$

Assume that there are N elements in the finite probability space. Consider Γ as the set of all random non negative variables X such that $0 \notin \Gamma$. Let $S \subset \Gamma$ be the subset of variables in such a way that $\sum_{\omega\in\Omega} X(\omega) = 1$. S is convex and compact.

Consider the set

$$L := \left\{\tilde{V}_T(\mathbf{H}) : \mathbf{H} \text{ self-financing and } V_0(\mathbf{H}) = 0\right\}.$$

Observe that $L \cap S = \emptyset$. In fact, $L \cap \Gamma = \emptyset$, otherwise there would exist a self-financing strategy H with $V_0(\mathbf{H}) = 0$ such that $\tilde{V}_T(\mathbf{H}) \in \Gamma$, and that would be an arbitrage strategy.

With the Separation theorem[2]:

Theorem 2.34. *Given $K \subseteq \mathbb{R}^n$ a convex and compact set. Let $V \subseteq \mathbb{R}^n$ be a linear space. Suppose that $K \cap V = \emptyset$. Then, there exists a linear application $\xi : \mathbb{R}^n \to \mathbb{R}$ such that $\xi(x) > 0$ for all $x \in K$ and $\xi(x) = 0$ for all $x \in V$.*

We have that there exists a linear application ξ satisfying $\xi(X) > 0$ for every $X \in S$, and $\xi(X) = 0$ for every $X \in L$. Therefore, for any $X \in S$,

$$\xi(X) = \sum_{k=1}^{N} \lambda_k X(\omega_k) > 0,\ \forall \lambda_k > 0.$$

For every $k = 1, ..., N$ we can define a new probability

$$\mathbb{P}^*(\omega_k) := \frac{\lambda_k}{\sum_{j=1}^{N} \lambda_j},$$

which is positive for every ω_k, and as a result equivalent to $\mathbb{P}$.

[2]Proof of the theorem can be found in the Annex section A.1.

With a fixed process S^k, consider an adapted process H^k and an investment strategy such that at every moment we have H_j^k units of assets of S^k. We can associate that with a unique self-financing strategy with $\mathrm{V}_0(\mathbf{H}) = 0$. Hence, we have

$$\tilde{\mathrm{V}}_T(\mathbf{H}) = \sum_{j=1}^{T} H_{j-1}^k \Delta \tilde{S}_j^k \in L.$$

In consequence

$$\mathbb{E}^* \left(\sum_{j=1}^{T} H_{j-1}^k \Delta \tilde{S}_j^k \right) = \mathbb{E}^* \left(\tilde{\mathrm{V}}_T(\mathbf{H}) \right) = \frac{\xi \left(\tilde{\mathrm{V}}_T(\mathbf{H}) \right)}{\sum_{j=1}^{N} \lambda_j} = 0.$$

And finally with Proposition 2.13 we prove that $\tilde{S}^k$ is a martingale. □

Remark 2.35. $\tilde{\mathrm{V}}_t(\mathbf{H})$ is also a martingale.

Consider a $\mathcal{F}_T$-measurable and non-negative random variable B that represents the *payoff at time T* of a financial contract.

Definition 2.36. A payoff B can be **replicated** if it exists a constant V_0 and a self-financing and replicating strategy $\mathbf{H}$ such that $\mathrm{V}_T(\mathbf{H}) = B$.

A *replicating strategy* $\boldsymbol{H}$ is a an strategy that allows to replicate a certain value.

Definition 2.37. A market is said to be **complete** if every payoff can be replicated by some self-financing and replicating trading strategy. Otherwise, the market is said to be **incomplete**.

Theorem 2.38. *(Second fundamental theorem of finance)*

A finite and viable market is complete if and only if it exists a unique risk neutral probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$, under which the discounted prices are martingales.

Proof of the theorem can be found in the Annex section A.2.

Remark 2.39. As mentioned by S.R. Pliska in [15], the second theorem implies that in a complete market any payoff can be replicated with a good choice of an initial investment and strategy. In consequence, in a complete market we can always find a way to price all type of derivatives.

In the next section we will associate the mathematical concepts with the financial market theory.

2.3 Option pricing

Let D_t denote the value of an option at time t, where D_0 represents the premium or positive initial market price. The value process D_t derives from the underlying price process $\mathbf{S}_t$. Thus, D_t is measurable with respect to the σ-algebra generated by the price process. At expiration, as seen in Remark 1.8, the payoff will be $B = (\mathbf{S}_T - K)^+ = D_T$ for a call, and $B = (K - \mathbf{S}_T)^+ = D_T$ for a put.

Together with Definition 2.36, an option final value D_T is replicable if there exists a self-financing replicating trading strategy $\mathbf{H}$ whose portfolio value at $t = T$ coincides with D_T, i.e.,

$$B = D_T = \mathrm{V}_T(\mathbf{H}) = \mathbf{H}_T \cdot \mathbf{S}_T.$$

Definition 2.40. From Proposition 2.25, an option is **replicable** if and only if the discounted payoff value satisfies

$$\tilde{B} = \tilde{\mathrm{V}}_T(\mathbf{H}) = \mathbf{H}_T \cdot \tilde{\mathbf{S}}_T = \mathrm{V}_0(\mathbf{H}) + \sum_{j=1}^{T} \mathbf{H}_{j-1} \cdot \Delta\tilde{\mathbf{S}}_j,$$

where $\Delta\tilde{\mathbf{S}}_j = \tilde{\mathbf{S}}_j - \tilde{\mathbf{S}}_{j-1}$. We will say that the discounted value is also *replicable*.

In a viable arbitrage-free market with a set of risk neutral probabilities $\mathcal{P} = \{\mathbb{P}^*\}$ under which the discounted payoff $\tilde{B}$ of an European option is a martingale, for each risk neutral probability we can obtain a unique positive amount called *arbitrage-free price*. We denote by

$$\Theta\left(\tilde{B}\right) = \left\{\mathbb{E}^*\left[\tilde{B}\right] \middle| \, \mathbb{P}^* \in \mathcal{P}\right\}$$

the *set of all expected discounted prices of a payoff* $\tilde{B}$ that can be obtained in the market.

Theorem 2.41. *The set of all expected discounted prices of a payoff in an arbitrage-free market is nonempty. The lower and upper bounds are given by*

$$\tilde{D}_{\inf}\left(\tilde{B}\right) := \inf_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^*\left[\tilde{B}\right] \quad \textit{and} \quad \tilde{D}_{\max}\left(\tilde{B}\right) := \max_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^*\left[\tilde{B}\right].$$

Proof. Can be found in [10]: chapter 5, page 311. □

Remark 2.42. In an arbitrage-free market the amount $\tilde{D}_0 = D_0$ of an option with discounted payment $\tilde{B}$ is a price at which the option can be traded at time $t = 0$ without introducing arbitrage opportunities into the market. This implies, that if the option is sold for D_0, then neither of the agents can find an investment strategy that yields an opportunity to obtain a positive gain without taking any risk.

Remark 2.43. Consider an European call option with initial price C_0 and its underlying asset S_t^1. From Remark 1.8 we know that the option payoff is $B = \left(S_T^1 - K\right)^+$. In an arbitrage-free market in which the risk-free asset satisfies $S_0^0 = 1$, applying Theorem 2.41 the initial discounted price satisfies

$$\tilde{C}_0 = \mathbb{E}^*\left[\tilde{B}\right] = \mathbb{E}^*\left[\frac{B}{S_T^0}\right] = \mathbb{E}^*\left[\left(\frac{S_T^1 - K}{S_T^0}\right)^+\right] = \mathbb{E}^*\left[\left(\tilde{S}_T^1 - \frac{K}{S_T^0}\right)^+\right].$$

As seen in Proposition 1.10 and Remark 1.11, the price of a call option is a convex function always above of $\left(S_0^1 - K\right)^+$. Therefore, we have that the initial price can be bounded by

$$C_0 \geq \left(\mathbb{E}^*\left[\tilde{S}_T^1 - \frac{K}{S_T^0}\right]\right)^+ = \left(S_0^1 - \mathbb{E}^*\left[\frac{K}{S_T^0}\right]\right)^+ \geq \left(S_0^1 - K\right)^+.$$

The next theorem states that if a payoff is replicable, we are able to find an initial value for the option, and with the options that are not replicable in an incomplete market, we can at least identify an interval in which the value for the time $t = 0$ must fall.

Theorem 2.44. *Let $\Theta\left(\tilde{B}\right)$ be the set of all expected discounted prices of a payoff $\tilde{B}$.*

i) If $\tilde{B}$ is replicable, this is $\tilde{B} = \tilde{V}_T\left(\boldsymbol{H}\right)$ for an strategy $\boldsymbol{H}$, then the set $\Theta\left(\tilde{B}\right)$ consists of the single element V_0.

ii) Otherwise, then $\tilde{D}_{\inf}\left(\tilde{B}\right) < \tilde{D}_{\sup}\left(\tilde{B}\right)$ and

$$\Theta\left(\tilde{B}\right) = \left(\tilde{D}_{\inf}\left(\tilde{B}\right), \tilde{D}_{\sup}\left(\tilde{B}\right)\right).$$

Proof. Can be found in [10]: chapter 5, page 315. □

Definition 2.45. Consider an option and its payoff B.

- **Pricing an option** is to determine the price at $t = 0$ that will allow the holder of the option obtain a payoff B at time $t = T$.
- **Hedging an option** is to establish an investment strategy for the seller to cover his obligation to give a payoff B to the buyer at time $t = T$.

Since the work is from the point of view of the seller, we will estimate the selling price that the option should have in order to hedge the future obligations.

Remark 2.46. Together with Proposition 2.26, the completeness of the market ensures the existence of a self-financing and replicating strategy **H** that generates the wanted random payoff B, such that $\tilde{\mathrm{V}}_t(\mathbf{H})$ is a martingale satisfying $\tilde{\mathrm{V}}_T(\mathbf{H}) = \tilde{B}$. Therefore, $\mathrm{V}_0(\mathbf{H}) = \mathbb{E}^*\left(\tilde{B}\right)$ is the *fair price* needed to be payed by the buyer at time $t = 0$. If **H** is the trading strategy, it satisfies

$$\mathrm{V}_0(\mathbf{H}) = \mathbb{E}^*\left(\tilde{B}\right) = \mathbb{E}^*\left(\tilde{\mathrm{V}}_T(\mathbf{H})\right).$$

With the information available in the market, we have for all $t \in \mathbb{T}$

$$\mathrm{V}_t(\mathbf{H}) = S_t^0 \mathbb{E}^*\left[\tilde{B}\middle|\mathcal{F}_t\right] = S_t^0 \mathbb{E}^*\left[\frac{B}{S_T^0}\middle|\mathcal{F}_t\right] = \frac{S_t^0}{S_T^0}\mathbb{E}^*\left[B|\mathcal{F}_t\right].$$

This is the amount needed at time t to replicate B at time T by following the strategy **H**. Thus, at time $t = 0$ the seller has to sell the option for fair a price of

$$\mathbb{E}^*\left(\frac{B}{S_T^0}\right).$$

Definition 2.47. Using Remark 1.8, if K is the strike price of an European option, the **fair price of an option at time** $t = 0$ is

- for calls
$$C_0 := \mathbb{E}^*\left(\frac{(\mathbf{S}_T - K)^+}{S_T^0}\right),$$
- and for puts
$$P_0 := \mathbb{E}^*\left(\frac{(K - \mathbf{S}_T)^+}{S_T^0}\right).$$

Below, the theory of two option pricing models that will be utilized in the last chapter is introduced.

2.4 The binomial model

The binomial model is the simplest option pricing model that helps determining the strategy that an agent has to follow in financial markets. A particular feature is that at every time $t \in \mathbb{T}$ the underlying asset can only take two possible values. We will not deeply develop this model as it is not the purpose of the work and it can be found in every book related to financial markets, for example in [3] or [19]. However, we will mention the most important aspects and results that will be needed to understand the last chapter.

Remark 2.48. A particular case of the model is the well-known Cox-Ross-Rubinstein model or CRR model, which was developed by J.C. Cox, S.A. Ross and M. Rubinstein in 1979 [5]. The equivalent model for a continuous-time framework is the Black-Scholes model, developed by F. Black and M. Scholes in 1973 [1].

Assume a market model in a discrete and finite time period $\mathbb{T} = \{0, 1, ..., T\}$ with only two assets; a risk-free asset with initial price $S_0^0 = 1$ and a risky asset $S_t \geq 0$. Assume a fixed interest rate $r \geq 0$ such that $S_t^0 = (1+r)^t$ is the price at time t of the risk-free asset.

Suppose that at every time $t \in \mathbb{T}$ the risky asset price can either increase a certain amount $u \in \mathbb{R}$ with probability $P(u) = p$, or decrease a certain amount $d \in \mathbb{R}$ with probability $P(d) = 1 - p$, where $-1 < d < u$ to avoid negative prices and $p \in [0, 1]$. This means that at every moment of time we will have

$$\begin{array}{ccl} & \nearrow & S_t(u) = S_{t-1}(1+u) \\ S_{t-1} & & \\ & \searrow & S_t(d) = S_{t-1}(1+d) \end{array}$$

Proposition 2.49. *The binomial model is viable and complete if and only if $d < r < u$.*

Proof. Can be found in [21]: chapter 4. □

Corollary 2.50. The binomial model is complete.

The general method to obtain the fair initial value of an option with value D_t, consists in a step by step backward strategy starting with the payoff $B = D_T$ and concluding at the initial option price D_0. A backward strategy is the most common and effective approach to price products in the financial market.

Assuming a self-financing investment strategy $\mathbf{H}_t = \left(H_t^0, H_t\right)$, we impose

$$B = D_T = \mathrm{V}_T(\mathbf{H}) = H_{T-1}^0 S_T^0 + H_{T-1} S_T = H_{T-1}^0 (1+r)^T + H_{T-1} S_T.$$

We have to solve the system formed by the next two equations:

$$\begin{cases} B(u) = H_{T-1}^0 (1+r)^T + H_{T-1} S_{T-1} (1+u) \\ B(d) = H_{T-1}^0 (1+r)^T + H_{T-1} S_{T-1} (1+d) \end{cases}$$

The unique solution is

$$H_{T-1} = \frac{B(u) - B(d)}{S_T(u) - S_T(d)},$$

and

$$H^0_{T-1} = \frac{B(d)(1+u) - B(u)(1+d)}{(u-d)(1+r)^T}.$$

Together with Proposition 2.25, the discounted portfolio value at $t = T-1$ is

$$\tilde{V}_{T-1}(\mathbf{H}) = H^0_{T-1} + H_{T-1}\tilde{S}_{T-1},$$

with a current value of

$$V_{T-1}(\mathbf{H}) = H^0_{T-1}S^0_{T-1} + H_{T-1}S_{T-1}.$$

For the next step, we take $D_{T-1} = V_{T-1}(\mathbf{H})$, and we repeat the process, now solving

$$V_{T-1}(\mathbf{H}) = H^0_{T-2}(1+r)^{T-1} + H_{T-2}S_{T-1}.$$

In other words, since we only know the final payoff $B = D_T$ and its states u and d, applying this strategy one time we obtain the payoff at one step ahead of time, D_{T-1}. Taking this amount as the new replicable amount and repeating the process, by iteration we will be able to obtain the initial fair price of the option, D_0.

2.5 The trinomial model

In the trinomial model there are three possible stock prices at every moment instead of two. It is an incomplete model and in the last chapter we will see how to price options with the method presented in chapter three.

With the same model formalism as in the binomial model, we will now introduce an intermediate $m \in \mathbb{R}$ amount that satisfies $u > m > d$. Therefore, we will have at each time t

$$S_t = \begin{cases} S_t(u) & \text{with probability } P(u) \\ S_t(m) & \text{with probability } P(m) \\ S_t(d) & \text{with probability } P(d), \end{cases}$$

satisfying $P(u) + P(m) + P(d) = 1$.

Similarly to the binomial model, we have the next proposition.

Proposition 2.51. *The trinomial model does not admit arbitrage opportunities, i.e. it is viable, if and only if $d < r < u$.*

Proof. Can be found in [3]: chapter 2, page 21. □

With the same financial market hypothesis as in the binomial model, we should now solve a system formed by three equations and two variables

$$\begin{cases} B(u) = H^0_{T-1}(1+r)^T + H_{T-1}S_{T-1}(1+u) \\ B(m) = H^0_{T-1}(1+r)^T + H_{T-1}S_{T-1}(1+m) \\ B(d) = H^0_{T-1}(1+r)^T + H_{T-1}S_{T-1}(1+d), \end{cases}$$

which in general has no exact solution.

Remark 2.52. In an incomplete market we cannot find a unique strategy **H** that satisfies the conditions. Instead, we obtain an infinite number of strategies. This is related with the definition of an incomplete market, where not every payoff can be replicated by some trading strategy. This implies that in incomplete markets we will have at least two risk neutral probabilities resulting in different option prices.

In the next chapter we will introduce different strategies that can be applied to solve this problem, and we will go into more detail with one of them.

Chapter 3

Incomplete markets

In a complete market there exists a unique martingale measure under which the discounted option prices are martingales. This means that for each payoff in the market we can find a replicating and self-financing strategy which replicates exactly the payment. One method to obtain the fair initial value is, for example, with the CRR model. However, in reality the financial market is not a complete market, and some payoffs cannot be replicated with a both replicating and self-financing strategy.

Incomplete markets appear when risks cannot be perfectly covered by financial products. For example, when there are more states of nature than financial products, or when we add a new state to the binomial model, converting it into the trinomial model. Also, the market is incomplete when payments are not entirely determined by the financial market, or when there exist transaction costs. Examples of financial products include weather derivatives, catastrophe bonds and insurance products[1]. As a result, the information available may not only depend on the one from the assets, and the filtration may not always be assumed to be the natural filtration.

The pricing problem of not being able to find a replicating and self-financing strategy has been studied by several researchers, who have developed different strategies. This chapter introduces some of those strategies and focus in one type: risk minimization strategies. Specifically in a variant called local risk minimization.

[1]More characteristics of incomplete markets can be found in the paper by J. Staum (2007) [20].

3.1 Strategies

The study of the pricing problem in incomplete markets has lead to different methods, which can be classified into four categories.

- **Superhedging**: In this category strategies are self-financing but not replicating. The objective of the strategy is to make sure that there is enough money to pay the payoff of the option. This is by using a minimax theorem developed by F. Delbaen and W. Schachermayer in 1994 [7]. The problem with this strategy is that it chooses its cheapest superreplicating strategy, which is not usually the more convenient one.

- **Efficient hedging**: The aim is to minimize $\mathbb{E}\left[l\left(V_T - D\right)\right]$, where l is a loss function. Recall that V_T is the portfolio value at time T and D is the derivative price. This strategy is the general case of what is known as *quantile hedging*, when l is a strictly positive function. It was developed by H. Föllmer and P. Leukert in 2000 [8].

- **Utility indifference pricing**: The objective is to maximize the expected utility of the seller's wealth. It computes the *utility indifference price*, which is the price for which the seller is indifferent between selling or not the option, and for that it usually needs a non-linear pricing rule such us a quadratic utility or exponential utility. More details of it can be found in the book by R. Cont and P. Tankov from 2004 [2].

- **Quadratic hedging**: Collection of hedging strategies, derived from the study of quadratic utility.

  – **Variance-minimizing hedging**: Developed by M. Schweizer in 2001 [18], and R. Cont and P. Tankov in 2004 [2]. Strategies are self-financing but not necessarily replicating. The objective is to minimize the expected square hedging error.

  – **Risk-minimizing hedging**: In this approach strategies are replicating but not necessarily self-financing. The aim is to minimize the expected square value of the cost originated when adapting the strategy to the market changes. M. Schweizer (1999) [16], T. Moeller (2006) [13] and H. Föllmer (1988) [9] are some of the studies of this strategy.

From those strategies, quadratic hedging is the most used and practical one with a varying of papers and studies. And since the main objective in financial market is to cover the risk and minimize the cost, it seems more interesting to focus on risk-minimizing strategies.

3.2 Risk minimization strategy

The theory of risk minimization was first developed in a continuous-time framework by H. Föllmer and D. Sondermann in 1986 [11]. And later in 1988 H. Föllmer and M. Schweizer introduced the model for discrete time [9]. This strategy is a quadratic hedging strategy, which tries to find a fair price for both the seller and the buyer with a replicating but not self-financing strategy.

In incomplete markets we are not able to find a self-financing strategy that replicates exactly any option payoff. This means that in order to find the initial price, money will be added to or withdrawn from the portfolio from time to time, and these money transfers will give rise to a cost. The risk minimization strategy aims to minimize this expected cost and helps determining a trading strategy.

3.2.1 Model framework

In a discrete-time period $\mathbb{T} = \{0, 1, ..., T\}$ we have a risk-free asset $S^0 = 1$ and d financial risky assets with prices $S_t = \left(S_t^1, ..., S_t^d\right)$ at time t. Let $\mathbf{S}_t = \left(S^0, S_t\right)$ denote the price array. Recall that the discounted price of an asset is $\tilde{S}_t^k = \frac{S_t^k}{S^0}$, hence the discounted risk-free asset will be $\tilde{S}_t^0 = 1$ for each t.

We consider $\mathbf{S}_t$ and $\tilde{\mathbf{S}}_t$ to be square integrable processes.

Definition 3.1. A stochastic process S is **square integrable** if

$$\mathbb{E}\left(S_t^2\right) < \infty, \ \forall t \in \mathbb{T}.$$

We will denote $S_t \in L^2$.

We will have a probability measure P, also called *physical measure*, which gives us the randomness of the different possible values in the model. Since the market is arbitrage-free and not complete, there exist at least two martingale measures $P^* \in \mathcal{P}$ equivalent to P, such that $\tilde{S}^k$, $1 \leq k \leq d$, is a martingale with respect to each P^*.

Consider a probability space with a filtration $\mathcal{F} = \{\mathbb{F}_t, \ t \geq 0\}$. We assume S^0 and S_t to be adapted with respect to the filtration. The seller of the option follows what is called a *dynamic trading strategy*, i.e. $\mathbf{H}_t = \left(H_t^0, \ H_t\right)$, where H_t^0 is adapted and H_t is a predictable process. This means that at each time t the portfolio is successively readjusted according to the information available at that time, which is at time $t - 1$ for the risky assets. Thus, H_0 does not exist. H_t^0 is the amount in the bank account and it can be changed at the end of the time period.

The discounted value of the portfolio at time t will be

$$\tilde{\mathrm{V}}_t(\mathbf{H}) = \mathbf{H}_t \cdot \tilde{\mathbf{S}}_t = H_t^0 + H_t \cdot \tilde{S}_t,$$

with $\mathrm{V}_0(\mathbf{H}) = V_0 = H_0^0$ the initial investment.

With new information revealed at every time t, the seller will have to adjust the strategy buying or selling risky stocks and choosing H_{t+1}. The number of additional stocks is $H_{t+1} - H_t$ with a discounted cost of $(H_{t+1} - H_t) \cdot \tilde{S}_t$. This adjustment leads to a discounted gain or loss of the portfolio value of $H_{t+1} \cdot \left(\tilde{S}_{t+1} - \tilde{S}_t\right)$. From this we can obtain the next definition.

Definition 3.2. Following an investment strategy $\mathbf{H}$, the **gains process** G is given by $G_0 = 0$, and for each t we have

$$G_t := \sum_{j=1}^{t} H_j \cdot \Delta\tilde{S}_j.$$

This is, the gains obtained by the price changes in the risky assets.

Remark 3.3. We will denote by ΔX_t the incremental of any discrete-time stochastic process X at time t, i.e., $\Delta X_t = X_t - X_{t-1}$.

Finally, the bank account can also change from H_t^0 to H_{t+1}^0. Adding up, the *variation of the portfolio value at time* $t+1$ will be

$$\begin{aligned}\Delta\tilde{\mathrm{V}}_{t+1}(\mathbf{H}) &= (H_{t+1} - H_t) \cdot \tilde{S}_t + H_{t+1} \cdot \left(\tilde{S}_{t+1} - \tilde{S}_t\right) + \left(H_{t+1}^0 - H_t^0\right) \\ &= \Delta H_{t+1} \cdot \tilde{S}_t + H_{t+1} \cdot \Delta\tilde{S}_{t+1} + \Delta H_{t+1}^0.\end{aligned}$$

Definition 3.4. The **cost process or cumulative cost associated with the strategy H at time** t is the difference between the portfolio value and the gains process, i.e.

$$\pi_t(\mathbf{H}) := \tilde{\mathrm{V}}_t(\mathbf{H}) - G_t = \mathbf{H}_t \cdot \tilde{\mathbf{S}}_t - \sum_{j=1}^{t} H_j \cdot \Delta\tilde{S}_j, \tag{3.1}$$

where $\pi_0(\mathbf{H}) = V_0$ is the cost of the initial investment.

In other words, the cost process reflects the changes on the value process that are not caused by trading gains on the risky assets at time t.

Remark 3.5. Note that in a self-financing strategy, the value process is only generated by the gains process. This means, together with Proposition 2.25, that the cost process is constant over time and equal to the initial investment.

Consider now an European option with a final payoff $B = D_T$. We assume the option payoff to be an square-integrable random variable, and we will only admit strategies where each $\mathrm{V}_t(\mathbf{H})$ is square-integrable.

We know that in an incomplete market we are not able to find a self-financing and replicating strategy to price the option at the initial time. One possible method would be to minimize the conditional expected value under some martingale measure P^* of the square of the costs, taking into account $\tilde{\mathrm{V}}_T(\mathbf{H}) = \tilde{B}$. In other words, we would minimize for each $t \in \mathbb{T}$

$$R_t(\mathbf{H}) = \mathbb{E}^*\left[\left(\pi_{t+1}(\mathbf{H}) - \pi_t(\mathbf{H})\right)^2\Big|\mathcal{F}_t\right] = \mathbb{E}^*\left[\left(\Delta\pi_{t+1}(\mathbf{H})\right)^2\Big|\mathcal{F}_t\right]. \tag{3.2}$$

M. Föllmer and M. Schweizer solved this problem in [9] via backward induction starting with R_{T-1}: the seller at the end of the process wants to have an amount $\mathrm{V}_T(\mathbf{H}) = B = D_T$. Therefore, using Formula 3.1, we want to minimize

$$R_{T-1}(\mathbf{H}) = \mathbb{E}^*\left[\left(\Delta\pi_T(\mathbf{H})\right)^2\Big|\mathcal{F}_{T-1}\right] = \mathbb{E}^*\left[\left(\tilde{B} - \tilde{\mathrm{V}}_{T-1}(\mathbf{H}) - H_T \cdot \Delta\tilde{S}_T\right)^2\Big|\mathcal{F}_{T-1}\right].$$

This is, we want to project $\tilde{B} - H_T \cdot \Delta\tilde{S}_T$ in the subset of random variables $\mathcal{F}_{T-1}$-measurables. In this case, the optimal solution is

$$\mathbb{E}^*\left[\tilde{B} - H_T \cdot \Delta\tilde{S}_T\big|\mathcal{F}_{T-1}\right] = \tilde{\mathrm{V}}_{T-1}(\mathbf{H}).$$

Using the linearity property of the conditional expectation and that H_T is predictable, the problem can be rewritten as

$$\mathbb{E}^*\left[\tilde{B}\big|\mathcal{F}_{T-1}\right] = \tilde{\mathrm{V}}_{T-1}(\mathbf{H}) + H_T \cdot \mathbb{E}^*\left[\Delta\tilde{S}_T\big|\mathcal{F}_{T-1}\right],$$

which simply reduces in determining the best linear estimate of $\tilde{B}$ based on the increment of $\Delta\tilde{S}_T$. The optimal constants H_T and $\tilde{\mathrm{V}}_{T-1}(\mathbf{H})$ are given by the least squares approximation leading to the solution

$$H_T = \frac{Cov^*\left(\tilde{B}, \Delta\tilde{S}_T\big|\mathcal{F}_{T-1}\right)}{Var^*\left[\Delta\tilde{S}_T\big|\mathcal{F}_{T-1}\right]}$$

and

$$\tilde{\mathrm{V}}_{T-1}(\mathbf{H}) = \mathbb{E}^*\left[\tilde{B}\big|\mathcal{F}_{T-1}\right] - H_T \cdot \mathbb{E}^*\left[\Delta\tilde{S}_T\big|\mathcal{F}_{T-1}\right] = \mathbb{E}^*\left[\tilde{B}\big|\mathcal{F}_{T-1}\right],$$

where we used that the price is a martingale. H_T^0 can be found using the portfolio definition

$$H_T^0 = \tilde{B} - H_T \cdot \tilde{S}_T.$$

Recursively, the solution is for all $t \in \mathbb{T}$

$$H_t = \frac{Cov^*\left(\tilde{B} - \sum_{j=t+1}^{T} H_j \Delta\tilde{S}_j, \Delta\tilde{S}_t\Big|\mathcal{F}_{t-1}\right)}{Var^*\left[\Delta\tilde{S}_t\big|\mathcal{F}_{t-1}\right]} \tag{3.3}$$

and

$$H_t^0 = \tilde{V}_t(\mathbf{H}) - H_t \cdot \tilde{S}_t = \mathbb{E}^* \left[\tilde{B} - \sum_{j=t+1}^{T} H_j \Delta \tilde{S}_j \middle| \mathcal{F}_t \right] - H_t \cdot \tilde{S}_t, \tag{3.4}$$

with

$$\tilde{V}_{t-1}(\mathbf{H}) = \mathbb{E}^* \left[\tilde{B} - \sum_{j=t}^{T} H_j \Delta \tilde{S}_j \middle| \mathcal{F}_{t-1} \right]. \tag{3.5}$$

This is the *risk-minimizing strategy* that the seller has to follow to be hedged against the future payoff B, where the fair selling option price has to be

$$\tilde{V}_0(\mathbf{H}) = \mathbb{E}^* \left[\tilde{B} - \sum_{j=1}^{T} H_j \Delta \tilde{S}_j \middle| \mathcal{F}_0 \right].$$

Remark 3.6. Note that solving this problem implies $\mathbb{E}^*[\Delta \pi_t | \mathcal{F}_t] = 0$, i.e. the cost process is a martingale.

Remark 3.7. Note also that the solution from Problem 3.2 coincides with the solution of this alternative problem, consisting in minimizing at each time t

$$R_t(\mathbf{H}) = \mathbb{E}^* \left[(\pi_T(\mathbf{H}) - \pi_t(\mathbf{H}))^2 \middle| \mathcal{F}_t \right], \tag{3.6}$$

subject to the condition $\tilde{V}_T(\mathbf{H}) = \tilde{B}$.

For the equivalence of the two problems, the expected values have to be calculated with respect to the same martingale measure P^*.

Remark 3.8. As stated in the paper from T. Moeller (2006) [13], the first Problem 3.2 can also be formulated and solved by using the physical measure, this is by replacing the expected value with respect to the martingale measure P^* by an expected value with respect to P. This will lead to a similar solution, but where all expected values and variances are calculated under the objective measure. This is known as *local risk minimization*, which will be covered in the next section.

However, this cannot be applied to solve Problem 3.6, also known as *global risk minimization*, since the solution would still result in having to choose the optimal martingale measure. This was proved by M. Schweizer in [17].

This is the reason why we will be focusing on the local risk minimization method.

3.3 Local risk minimization

This method aims to minimize the expected squared value of the incremental cost. This is, by minimizing Problem 3.2 using the physical measure P instead of the martingale measure P^*. The theory can be found in chapter 10 of the book by H. Föllmer and A. Schied (2016) [10].

We assume that the discounted payoff $\tilde{B}$ and the discounted price process $\tilde{\mathbf{S}}_t$ are square-integrable with respect to the physical measure P. This is, $\tilde{B} \in L^2(P)$ and $\tilde{\mathbf{S}}_t \in L^2(P)$, $\forall t \in \mathbb{T}$.

Definition 3.9. A **square-integrable admissible strategy for** $\tilde{B}$ is a trading strategy **H** whose value satisfies

$$\tilde{\mathrm{V}}_T(\mathbf{H}) = \tilde{B}, \text{ P-almost surely}^2$$

with $\tilde{\mathrm{V}}_t \in L^2(P)$ and $G_t \in L^2(P)$ for each t, where G is the gains process presented before.

Definition 3.10. The **local risk process of an L²-admissible strategy H** $= (H^0, H)$ is the process

$$R_t^{loc}\left(H^0, H\right) := \mathbb{E}\left[(\Delta\pi_{t+1}(\mathbf{H}))^2 \middle| \mathcal{F}_t\right],\ t = 0, 1, ..., T-1.$$

An L^2-admissible strategy $\hat{\boldsymbol{H}} = \left(\hat{H}^0, \hat{H}\right)$ is called a **locally risk-minimizing strategy** if, for all t

$$R_t^{loc}\left(\hat{H}^0, \hat{H}\right) \leq R_t^{loc}\left(H^0, H\right),\ \text{P-a.s.},$$

for each L^2-admissible strategy **H** whose value process satisfies

$$\tilde{\mathrm{V}}_{t+1}\left(\hat{\boldsymbol{H}}\right) = \hat{H}^0_{t+1} + \hat{H}_{t+1} \cdot \tilde{S}_{t+1}.$$

Remark 3.11. A local risk process of an L^2-admissible strategy **H** can also be expressed as

$$R_t^{loc}(\mathbf{H}) = Var\left(\Delta\pi_{t+1}(\mathbf{H}) | \mathcal{F}_t\right) + \mathbb{E}^2\left[\Delta\pi_{t+1}(\mathbf{H}) | \mathcal{F}_t\right].$$

Definition 3.12. An L^2-admissible strategy is called **mean self-financing** if its cost process π is a martingale measure with respect to P. This is, for all t,

$$\mathbb{E}\left[\Delta\pi_{t+1}(\mathbf{H}) | \mathcal{F}_t\right] = 0,\ \text{P-a.s.}$$

[2] P-a.s. means almost sure with respect to a probability measure P.

Remark 3.13. Locally risk-minimizing strategies are not self-financing, but they are self-financing on average. This means that each additional cost is a random variable with expectation zero.

Definition 3.14. Two adapted processes M_t and M'_t are **strongly orthogonal with respect to P** if the conditional covariances

$$Cov\left(\Delta M_{t+1},\, \Delta M'_{t+1} \middle| \mathcal{F}_t\right)$$

are well defined and vanish P-a.s. for $t = 0, 1, ..., T-1$.

Remark 3.15. Suppose we have a strong orthogonality of two adapted processes. If one of them is a martingale measure under P, then the conditional covariance reduces to

$$Cov\left(\Delta M_{t+1},\, \Delta M'_{t+1} \middle| \mathcal{F}_t\right) = \mathbb{E}\left[\Delta M_{t+1} \Delta M'_{t+1} \middle| \mathcal{F}_t\right].$$

This comes from the definition of the conditional covariance of M_t and M'_t:

$$Cov\left(M_t, M'_t \middle| \mathcal{F}_t\right) := \mathbb{E}\left[M_t M'_t \middle| \mathcal{F}_t\right] - \mathbb{E}\left[M_t \middle| \mathcal{F}_t\right] \mathbb{E}\left[M'_t \middle| \mathcal{F}_t\right].$$

The next theorem characterizes the locally risk minimizing strategies.

Theorem 3.16. *An L^2-admissible strategy $\mathbf{H}$ is locally risk-minimizing if and only if it is mean self-financing and its cost process is strongly orthogonal to $\tilde{\mathbf{S}}_t$.*

Proof. From Remark 3.11, we have that the local risk process can be decomposed as

$$R_t^{loc}(\mathbf{H}) := Var\left(\Delta\pi_{t+1}(\mathbf{H}) | \mathcal{F}_t\right) + \mathbb{E}^2\left[\Delta\pi_{t+1}(\mathbf{H}) | \mathcal{F}_t\right].$$

Since the conditional variance does not change if we add random $\mathcal{F}_t$-measurable variables, the first term turns into

$$Var\left(\Delta\pi_{t+1}(\mathbf{H}) | \mathcal{F}_t\right) = Var\left(\tilde{\mathrm{V}}_{t+1}(\mathbf{H}) - H_{t+1} \cdot \Delta\tilde{S}_{t+1} \middle| \mathcal{F}_t\right).$$

The second term can be rewritten as

$$\mathbb{E}^2\left[\Delta\pi_{t+1}(\mathbf{H}) | \mathcal{F}_t\right] = \left(\mathbb{E}\left[\tilde{\mathrm{V}}_{t+1}(\mathbf{H}) \middle| \mathcal{F}_t\right] - H_{t+1} \cdot \mathbb{E}\left[\Delta\tilde{S}_{t+1} \middle| \mathcal{F}_t\right] - \tilde{\mathrm{V}}_t(\mathbf{H})\right)^2.$$

We fix t and $\tilde{\mathrm{V}}_{t+1}(\mathbf{H})$, and consider H_{t+1} and $\tilde{\mathrm{V}}_t(\mathbf{H})$ as parameters. We want to find the necessary conditions to minimize $R_t^{loc}(\mathbf{H})$ with respect to those parameters.

Note that it is possible to change the parameters H_t^0 and H_t, leaving H_{t+1} and $\tilde{\mathrm{V}}_{t+1}(\mathbf{H})$ unchanged, such that $\tilde{\mathrm{V}}_t(\mathbf{H})$ can take any given value and the strategy continues to be L^2-admissible for $\tilde{B}$. In particular, the first term is not affected, therefore it is necessary for the optimization that $\tilde{V}_t(\mathbf{H})$ minimizes the second term. This is the case if and only if this term is zero

$$\mathbb{E}\left[\tilde{V}_{t+1}(\mathbf{H})\middle|\mathcal{F}_t\right] - H_{t+1} \cdot \mathbb{E}\left[\Delta\tilde{S}_{t+1}\middle|\mathcal{F}_t\right] - \tilde{V}_t(\mathbf{H}) = 0, \tag{3.7}$$

which proves the mean-self financing property because the second term satisfies

$$\mathbb{E}\left[\Delta\pi_{t+1}(\mathbf{H})|\mathcal{F}_t\right] = \mathbb{E}\left[\tilde{V}_{t+1}(\mathbf{H}) - H_{t+1} \cdot \Delta\tilde{S}_{t+1}\middle|\mathcal{F}_t\right] - \tilde{V}_t(\mathbf{H}) = 0.$$

Since the first term is independent of $\tilde{V}_t(\mathbf{H})$ and it is a quadratic form in terms of H_{t+1}, it can be minimized if and only if H_{t+1} solves the linear equation

$$Cov\left(\tilde{V}_{t+1}(\mathbf{H}) - H_{t+1} \cdot \Delta\tilde{S}_{t+1}, \Delta\tilde{S}_{t+1}\middle|\mathcal{F}_t\right) = 0.$$

Given Formula 3.7 and using Remark 3.15, the linear equation holds if and only if

$$\mathbb{E}\left[\Delta\pi_{t+1}(\mathbf{H})\,\Delta\tilde{S}_{t+1}\middle|\mathcal{F}_t\right] = 0.$$

Thus, we have proved that an L^2-admissible strategy $\mathbf{H}$ is locally risk-minimizing if and only if it is mean self-financing and its cost process is strongly orthogonal to the price process. Backward induction on t concludes the proof. □

Remark 3.17. Minimizing $R_t^{loc}(\mathbf{H})$ determines $\hat{H}_t^0$, $\hat{H}_t$ and $\tilde{V}_{t-1}(\hat{\boldsymbol{H}})$, but we are free to choose $\hat{H}_{t-1}^0$ and $\hat{H}_{t-1}$ among all H_{t-1}^0 and H_{t-1}^0 that satisfy $\tilde{V}_{t-1}(\mathbf{H}) = H_{t-1}^0 + H_{t-1} \cdot \tilde{S}_{t-1}$. It is then natural to minimize $R_{t-1}^{loc}(\mathbf{H})$ under the condition that $\tilde{V}_{t-1}(\mathbf{H})$ is equal to $\tilde{V}_{t-1}(\hat{\boldsymbol{H}})$ previously obtained. Now the problem will be the same type as the previous one.

The proof provides a recipe for a recursive construction of a locally risk-minimizing strategy. This is, if $\tilde{V}_{t+1}(\mathbf{H})$ is given, we have to minimize

$$\mathbb{E}\left[\left(\Delta\pi_{t+1}(\mathbf{H})\right)^2\middle|\mathcal{F}_t\right] = \mathbb{E}\left[\left(\tilde{V}_{t+1}(\mathbf{H}) - \left(\tilde{V}_t(\mathbf{H}) + H_{t+1} \cdot \Delta\tilde{S}_{t+1}\right)\right)^2\middle|\mathcal{F}_t\right]$$

with respect to $\tilde{V}_t(\mathbf{H})$ and H_{t+1}. And this is just the same Problem 3.2 of determining the linear regression of $\tilde{V}_{t+1}(\mathbf{H})$ on the increment of $\Delta\tilde{S}_{t+1}$. This yields to the same set of Formulas 3.3, 3.4 and 3.5 but using the physical measure.

A useful formula is when we only have one risky asset, $d = 1$. In that case

$$\hat{H}_{t+1} := \frac{Cov\left(\tilde{V}_{t+1}(\hat{\boldsymbol{H}}), \Delta\tilde{S}_{t+1}|\mathcal{F}_t\right)}{Var\left[\Delta\tilde{S}_{t+1}|\mathcal{F}_t\right]}$$

and

$$\tilde{V}_t(\hat{\boldsymbol{H}}) := \mathbb{E}\left[\tilde{V}_{t+1}(\hat{\boldsymbol{H}})|\mathcal{F}_t\right].$$

Defining $\hat{H}_t^0 := \tilde{V}_t\left(\hat{\boldsymbol{H}}\right) - \hat{H}_t \cdot \tilde{S}_t$, we obtain an strategy $\hat{\boldsymbol{H}} = \left(\hat{H}^0, \hat{H}\right)$ that satisfies $\tilde{V}_T\left(\hat{\boldsymbol{H}}\right) = \tilde{B}$ with an initial fair price of $\tilde{V}_0(\hat{\boldsymbol{H}}) = \mathbb{E}\left[\tilde{V}_1\left(\hat{\boldsymbol{H}}\right)\middle|\mathcal{F}_0\right]$.

To sum up, as explained in the paper from M. Schweizer (1999) [16], a locally risk-minimizing strategy can be characterized by two properties:

- The cost process π must be a martingale, as seen in Remark 3.6. Although the strategy is no longer self-financing, it has to still remain mean self-financing.
- And the cost process must also be orthogonal to the price process, as seen in Theorem 3.16.

Those two conditions translate into the next corollary.

Corollary 3.18. *There exists a locally risk-minimizing strategy if and only if the discounted payoff $\tilde{B}$ admits a decomposition*

$$\tilde{B} = c + \sum_{t=1}^{T} H_t \cdot \Delta\tilde{S}_t + Q_T, \text{ P-a.s.},$$

where $c \geq 0$ is a constant, $\boldsymbol{H}$ is a predictable process satisfying

$$H_t \cdot \Delta\tilde{S}_t \in L^2,$$

and Q_T is a square integrable P-martingale strongly orthogonal to $\tilde{S}_t$.

This decomposition is called the *Föllmer-Schweizer decomposition of $\tilde{B}$.*

Remark 3.19. Note that without the orthogonal term, the decomposition is true for replicable options.

The local risk minimization method can be also applied to complete markets, in which every option can be replicated by an obvious self-financing strategy.

This is the theory behind the local risk minimization method. For the Application chapter we will be using an adapted version of it that will simplify the process. For that, we will assume an initial portfolio value of $V_0(\mathbf{H}) = 0$, and we will denote by $V_{t+}(\mathbf{H}) = \mathbf{H}_{t+1} \cdot \mathbf{S}_t$ the portfolio just after being readjusted, including the risky asset, with the information available at time t.

Next proposition is an adaptation from Proposition 4.1 that can be found in [14]: page 26, for the case of one risky asset, $d = 1$.

Proposition 3.20. *The following are equivalent for all $t = 0, ..., T-1$.*

i) $\hat{\boldsymbol{H}}_t$ *minimizes*

$$\mathbb{E}\left[\left(\Delta\pi_t\left(\boldsymbol{H}\right)\right)^2\middle|\mathcal{F}_{t-1}\right].$$

ii) $\hat{\boldsymbol{H}}_t$ *solves*

$$\mathbf{H}_t \cdot \mathbb{E}\left[\tilde{\mathbf{S}}_t \cdot \tilde{S}_t^0 \middle| \mathcal{F}_{t-1}\right] = \mathbb{E}\left[\tilde{\mathrm{V}}_t\left(\boldsymbol{H}\right)\tilde{S}_t^0\middle|\mathcal{F}_{t-1}\right]$$
$$\mathbf{H}_t \cdot \mathbb{E}\left[\tilde{\mathbf{S}}_t \cdot \tilde{S}_t^1 \middle| \mathcal{F}_{t-1}\right] = \mathbb{E}\left[\tilde{\mathrm{V}}_t\left(\boldsymbol{H}\right)\tilde{S}_t^1\middle|\mathcal{F}_{t-1}\right].$$

iii) The price process and the cost process are orthogonal martingales, this is

$$\mathbb{E}\left[\Delta\pi_t\left(\hat{\boldsymbol{H}}\right)\middle|\mathcal{F}_{t-1}\right] = 0$$

$$\mathbb{E}\left[\tilde{\boldsymbol{S}}_t\Delta\pi_t\left(\hat{\boldsymbol{H}}\right)\middle|\mathcal{F}_{t-1}\right] = 0.$$

And the fair initial price satisfies $\tilde{\mathrm{V}}_{0+}(\hat{\boldsymbol{H}}) = \hat{\boldsymbol{H}}_1 \cdot \tilde{\boldsymbol{S}}_0$.

Proof. Can be found in [14]: appendix B, page 267. □

A strategy $\hat{\boldsymbol{H}}$ satisfying one of these conditions is called *locally risk-minimizing*. Such strategy always exists, but may not be unique.

Proposition 3.21. *Let* $\hat{\boldsymbol{H}}^X$ *and* $\hat{\boldsymbol{H}}^Y$ *be locally risk-minimizing strategies of the derivative processes* X *and* Y*, respectively. Then, for* $a, b \in \mathbb{R}$ *it satisfies that* $a\hat{\boldsymbol{H}}^X + b\hat{\boldsymbol{H}}^Y$ *is a locally risk-minimizing strategy of the derivative process* $aX + bY$*.*

Proof. Can be found in [14]: chapter 4, page 28. □

This proposition shows that an option process is actually a finite sum of options, each occurring at each time t. This means that given a portfolio we can add each of the individual risk-minimizing strategies to find the portfolio risk-minimizing strategy. And that there is no need to recalculate the locally risk-minimizing strategy on the whole. Because of this, the fair price associated comes from a linear pricing rule, implying that there exist some probability measure $\hat{P} \in \mathcal{P}$ such that the fair price of an European option is given by

$$\tilde{\mathrm{V}}_{t+}\left(\hat{\boldsymbol{H}}\right) = \mathbb{E}^{\hat{P}}\left[\tilde{D}_j|\mathcal{F}_t\right],$$

for $t < j$. And this probability measure is the so called *minimal measure*.

Remark 3.22. Recall that a local risk minimization is one method of choosing one particular measure $\hat{P}$ from all the possible risk neutral probabilities in $\mathcal{P}$.

Chapter 4

Application

In this chapter we will see how the local risk minimization strategy can be applied in binomial and trinomial models. It is based on the doctoral thesis by J. Pansera (2008) [14].

It begins first with a couple of simple cases where there is only a riskless asset. Later we introduce a risky asset and indicate how to apply the strategy in a binomial and trinomial model, adding a numerical example in the trinomial model with one period and two periods.

We will suppose that the filtration is the natural filtration, thus no external information from the financial market will affect, and we will also assume a risk-free asset with initial value $S_0^0 = 1$, which results in $\tilde{S}_t^0 = 1$, $\forall t$.

4.1 Simple cases

For the simple cases we will assume a risk-free asset, such as a bank account. However, it can also be applied to risk-free bonds.

4.1.1 A single asset and a single period

We have a riskless asset that expires in one period, $T = 1$.

From Proposition 3.20, the locally risk-minimizing strategy can be found by solving the equation

$$H_1^0 \cdot \mathbb{E}\left[\tilde{S}_1^0 \cdot \tilde{S}_1^0 \middle| \mathcal{F}_0\right] = \mathbb{E}\left[\tilde{\mathrm{V}}_1\left(\mathbf{H}\right) \tilde{S}_1^0 \middle| \mathcal{F}_0\right],$$

where $\tilde{\mathrm{V}}_1\left(\mathbf{H}\right) = \tilde{D}_1 = \tilde{B}$ because we want a replicating strategy.

Since $\tilde{S}_1^0 = 1$,

$$\hat{H}_1^0 = \mathbb{E}\left[\tilde{B}\middle|\mathcal{F}_0\right] = \frac{1}{S_1^0}\mathbb{E}\left[B|\mathcal{F}_0\right]. \tag{4.1}$$

This is the number of shares of the riskless asset that the seller should hold at time $t = 0$ in order to have at time $t = 1$ the amount B.

Thus, the fair value has to be

$$\tilde{\mathrm{V}}_{0+}(\hat{\boldsymbol{H}}) = \hat{H}_1^0.$$

Example 4.1. For a first example, assume we have a bank account with initial value S_0^0 that in one period, $T = 1$, requires us to pay an amount $B = 1$, for example a bank commission. We want to know at $t = 0$ how much do we need to have in the bank account to be able to cover that risk.

With Equation 4.1, we estimate the amount that we need to have in the account at $t = 1$:

$$\hat{H}_1^0 = \frac{1}{S_1^0}\mathbb{E}\left[B|\mathcal{F}_0\right] = \frac{1}{S_1^0}.$$

This gives us a discounted fair bank account value of

$$\tilde{\mathrm{V}}_{0+}\left(\hat{\boldsymbol{H}}\right) = \hat{H}_1^0 = \frac{1}{S_1^0}.$$

This means that in order to be able to pay an amount $B = 1$ in one period, we need to have an amount $\hat{H}_1^0$ in the bank account with a discounted account value of $\tilde{\mathrm{V}}_{0+}\left(\hat{\boldsymbol{H}}\right)$.

Note that the strategy is both replicating and self-financing, and we would obtain the same result with a complete market strategy.

4.1.2 A single asset and multiple periods

We have a riskless asset which will be kept multiple finite periods, $T > 1$.

From Proposition 3.20, since $\tilde{S}_t^0 = 1$ for every t, a locally risk-minimizing strategy can be found by solving

$$H_t^0 = \mathbb{E}\left[\tilde{\mathrm{V}}_t\left(\mathbf{H}\right)|\mathcal{F}_{t-1}\right]. \tag{4.2}$$

We have seen in Proposition 3.21 that a locally risk-minimizing strategy of a financial product process can be found by adding up the locally risk-minimizing strategies of each product. This means, that we can restrict the method to a single financial asset $\tilde{D}_t$ occurring at some time $t \leq T$.

First, for $t = T$ we obtain using Equation 4.2 and $\tilde{V}_T(\mathbf{H}) = \tilde{B}$, the risk-free amount needed at time T

$$\hat{H}^0_T = \mathbb{E}\left[\tilde{B}|\mathcal{F}_{T-1}\right].$$

With a portfolio value of

$$\tilde{V}_T\left(\hat{\boldsymbol{H}}\right) = \hat{H}^0_T.$$

Since we want the locally risk-minimizing strategy to be replicating, we take this value as the payoff value that we want to have at the prior time period. This means that we need to impose

$$\tilde{V}_{T-1}\left(\hat{\boldsymbol{H}}\right) = \hat{H}^0_T.$$

We repeat now the process for this new payoff solving

$$\begin{aligned} \hat{H}^0_{T-1} &= \mathbb{E}\left[\tilde{V}_{T-1}\left(\hat{\boldsymbol{H}}\right)\middle|\mathcal{F}_{T-2}\right] = \mathbb{E}\left[\hat{H}^0_T\middle|\mathcal{F}_{T-2}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\tilde{B}\middle|\mathcal{F}_{T-1}\right]\middle|\mathcal{F}_{T-2}\right] \\ &= \mathbb{E}\left[\tilde{B}\middle|\mathcal{F}_{T-2}\right], \end{aligned} \tag{4.3}$$

where we used the tower property of conditional expectation seen in Remark 2.8.

With this we obtain the new value $\tilde{V}_{T-2}\left(\hat{\boldsymbol{H}}\right)$, and by iteration we compute the local risk-minimizing investment strategy process

$$\hat{H}^0 = \{\hat{H}^0_1, ..., \hat{H}^0_T\}.$$

Therefore, the initial fair amount needed at $t = 0$ to obtain a final payoff B at $t = T$ will be

$$\tilde{V}_{0+}\left(\hat{\boldsymbol{H}}\right) = \hat{H}^0_1.$$

Example 4.2. We take the same example as before with $B = 1$, but with $T > 1$ periods.

The local risk minimizing strategy for $t = T$ will be

$$\hat{H}^0_T = \mathbb{E}\left[\tilde{B}\left(\hat{\boldsymbol{H}}\right)|\mathcal{F}_{T-1}\right] = \frac{1}{S^0_T},$$

with a discounted portfolio value of

$$\tilde{V}_T\left(\hat{\boldsymbol{H}}\right) = \frac{1}{S^0_T}.$$

Assuming $\tilde{V}_T\left(\hat{\boldsymbol{H}}\right)$ as the new payoff value that we want to replicate, we need to calculate

$$\hat{H}^0_{T-1} = \mathbb{E}\left[\tilde{V}_{T-1}\left(\hat{\boldsymbol{H}}\right)|\mathcal{F}_{T-2}\right] = \frac{1}{S^0_T}.$$

By iteration we obtain that the locally risk-minimizing strategy is

$$\hat{H}^0_1 = ... = \hat{H}^0_T = \frac{1}{S^0_T},$$

with an initial fair amount of

$$\tilde{V}_{0+}\left(\hat{\boldsymbol{H}}\right) = \frac{1}{S^0_T}.$$

The locally risk-minimizing strategy for this example is to have at each time t a constant amount $\hat{H}^0_T$ in the bank account. Note that this strategy is also self-financing and replicating.

4.2 Two assets and a single period

We will denote by S_t the risky asset and by H_t its strategy.

From Proposition 3.20, we must solve the system of equations

$$\begin{pmatrix} H^0_1 \\ H_1 \end{pmatrix} \cdot \mathbb{E}\left[\begin{pmatrix} \tilde{S}^0_1 \\ \tilde{S}_1 \end{pmatrix} \cdot \tilde{S}^0_1 \middle| \mathcal{F}_0\right] = \mathbb{E}\left[\tilde{V}_1\left(\mathbf{H}\right)\tilde{S}^0_1 \middle| \mathcal{F}_0\right]$$

$$\begin{pmatrix} H^0_1 \\ H_1 \end{pmatrix} \cdot \mathbb{E}\left[\begin{pmatrix} \tilde{S}^0_1 \\ \tilde{S}_1 \end{pmatrix} \cdot \tilde{S}_1 \middle| \mathcal{F}_0\right] = \mathbb{E}\left[\tilde{V}_1\left(\mathbf{H}\right)\tilde{S}_1 \middle| \mathcal{F}_0\right],$$

which simplifies, when $\tilde{S}^0_1 = 1$ and $\tilde{V}_1\left(\mathbf{H}\right) = \tilde{B}$, to

$$H^0_1 + H_1 \cdot \mathbb{E}\left[\tilde{S}_1 \middle| \mathcal{F}_0\right] = \mathbb{E}\left[\tilde{B} \middle| \mathcal{F}_0\right],$$

$$H^0_1 \cdot \mathbb{E}\left[\tilde{S}_1 \middle| \mathcal{F}_0\right] + H_1 \cdot \mathbb{E}\left[\left(\tilde{S}_1\right)^2 \middle| \mathcal{F}_0\right] = \mathbb{E}\left[\tilde{B}\tilde{S}_1 \middle| \mathcal{F}_0\right].$$

We solve the $2x2$ system of equations by isolating H^0_1

$$H^0_1 = \mathbb{E}\left[\tilde{B} \middle| \mathcal{F}_0\right] - H_1 \cdot \mathbb{E}\left[\tilde{S}_1 \middle| \mathcal{F}_0\right],$$

and multiplying it for $\mathbb{E}\left[\tilde{S}_1 \middle| \mathcal{F}_0\right]$ before replacing it in the second equation:

$$\mathbb{E}\left[\tilde{B} \middle| \mathcal{F}_0\right]\mathbb{E}\left[\tilde{S}_1 \middle| \mathcal{F}_0\right] - H_1 \cdot \mathbb{E}^2\left[\tilde{S}_1 \middle| \mathcal{F}_0\right] + H_1 \cdot \mathbb{E}\left[\left(\tilde{S}_1\right)^2 \middle| \mathcal{F}_0\right] = \mathbb{E}\left[\tilde{B}\tilde{S}_1 \middle| \mathcal{F}_0\right].$$

Which solves as

$$H_1 = \frac{\mathbb{E}\left[\tilde{B}\tilde{S}_1 \middle| \mathcal{F}_0\right] - \mathbb{E}\left[\tilde{B} \middle| \mathcal{F}_0\right] \mathbb{E}\left[\tilde{S}_1 \middle| \mathcal{F}_0\right]}{\mathbb{E}\left[\left(\tilde{S}_1\right)^2 \middle| \mathcal{F}_0\right] - \mathbb{E}^2\left[\tilde{S}_1 \middle| \mathcal{F}_0\right]}.$$

Applying the definition of conditional covariance and conditional variance we obtain the locally risk-minimizing strategy similar to the one seen in the last chapter

$$\hat{H}_1 = \frac{Cov\left[\tilde{B}, \tilde{S}_1 \middle| \mathcal{F}_0\right]}{Var\left[\tilde{S}_1 \middle| \mathcal{F}_0\right]}, \tag{4.4}$$

$$\hat{H}_1^0 = \mathbb{E}\left[\tilde{B} \middle| \mathcal{F}_0\right] - \hat{H}_1 \cdot \mathbb{E}\left[\tilde{S}_1 \middle| \mathcal{F}_0\right]. \tag{4.5}$$

The initial discounted value is computed as

$$\tilde{\mathrm{V}}_{0+}\left(\hat{\boldsymbol{H}}\right) = \hat{H}_1^0 \tilde{S}_0^0 + \hat{H}_1 \tilde{S}_0 = \hat{H}_1^0 + \hat{H}_1 \tilde{S}_0,$$

with an actual value of

$$\mathrm{V}_{0+}\left(\hat{\boldsymbol{H}}\right) = \hat{H}_1^0 S_0^0 + \hat{H}_1 S_0.$$

Next, the local risk-minimization strategy is applied to the models introduced in sections 2.4 and 2.5, where certain formulas to compute the strategy can be obtained.

4.2.1 The binomial model

Assume that we have a bank account with an amount S_0^0 and a final value of $S_1^0 = S_0^0(1+r)$, where r is a fixed interest rate. It satisfies $\tilde{S}_1^0 = 1$. There is also a stock S_0 with final value

$$S_1 = \begin{cases} S_1(u) & \text{with probability } P(u) \\ S_1(d) & \text{with probability } P(d), \end{cases}$$

and an European option with premium D_0 and payoff B

$$B = \begin{cases} B(u) & \text{with probability } P(u) \\ B(d) & \text{with probability } P(d). \end{cases}$$

We can obtain the local risk-minimizing strategy with Formulas 4.4 and 4.5:

$$\hat{H}_1 = \frac{Cov\left[\tilde{B}, \tilde{S}_1 \middle| \mathcal{F}_0\right]}{Var\left[\tilde{S}_1 \middle| \mathcal{F}_0\right]} = \frac{\mathbb{E}\left[\tilde{B}\tilde{S}_1 \middle| \mathcal{F}_0\right] - \mathbb{E}\left[\tilde{B} \middle| \mathcal{F}_0\right] \mathbb{E}\left[\tilde{S}_1 \middle| \mathcal{F}_0\right]}{\mathbb{E}\left[\left(\tilde{S}_1\right)^2 \middle| \mathcal{F}_0\right] - \mathbb{E}^2\left[\tilde{S}_1 \middle| \mathcal{F}_0\right]},$$

$$\hat{H}_1^0 = \mathbb{E}\left[\tilde{B} \middle| \mathcal{F}_0\right] - \hat{H}_1 \cdot \mathbb{E}\left[\tilde{S}_1 \middle| \mathcal{F}_0\right],$$

where:

- $\mathbb{E}\left[\tilde{B}\tilde{S}_1 \middle| \mathcal{F}_0\right] = \tilde{B}(u)\,\tilde{S}_1(u)\,P(u) + \tilde{B}(d)\,\tilde{S}_1(d)\,P(d)$,
- $\mathbb{E}\left[\tilde{B} \middle| \mathcal{F}_0\right] \mathbb{E}\left[\tilde{S}_1 \middle| \mathcal{F}_0\right] = \left[\tilde{B}(u)\,P(u) + \tilde{B}(d)\,P(d)\right]\left[\tilde{S}_1(u)\,P(u) + \tilde{S}_1(d)\,P(d)\right]$,
- $\mathbb{E}\left[\left(\tilde{S}_1\right)^2 \middle| \mathcal{F}_0\right] = \left(\tilde{S}_1(u)\right)^2 P(u) + \left(\tilde{S}_1(d)\right)^2 P(d)$,
- $\mathbb{E}^2\left[\tilde{S}_1 \middle| \mathcal{F}_0\right] = \left[\tilde{S}_1(u)\,P(u) + \tilde{S}_1(d)\,P(d)\right]^2$.

For the numerator of $\hat{H}_1$, we have

$$\tilde{B}(u)\,\tilde{S}_1(u)\,P(u) + \tilde{B}(d)\,\tilde{S}_1(d)\,P(d) - \left[\tilde{B}(u)\,P(u) + \tilde{B}(d)\,P(d)\right]\left[\tilde{S}_1(u)\,P(u)\right.$$

$$\left. + \tilde{S}_1(d)\,P(d)\right] = \tilde{B}(u)\,\tilde{S}_1(u)\,P(u)\,[1 - P(u)] + \tilde{B}(d)\,\tilde{S}_1(d)\,P(d)\,[1 - P(d)]$$

$$- \tilde{B}(d)\,\tilde{S}_1(u)\,P(u)\,P(d) - \tilde{B}(u)\,\tilde{S}_1(d)\,P(u)\,P(d)\,.$$

Taking into account $P(u) = 1 - P(d)$, it reduces to

$$P(u)\,P(d)\left[\tilde{B}(u)\,\tilde{S}_1(u) + \tilde{B}(d)\,\tilde{S}_1(d) - \tilde{B}(d)\,\tilde{S}_1(u) - \tilde{B}(u)\,\tilde{S}_1(d)\right]$$

$$= P(u)\,P(d)\left(\tilde{B}(u) - \tilde{B}(d)\right)\left(\tilde{S}_1(u) - \tilde{S}_1(d)\right).$$

And for the denominator of $\hat{H}_1$, we obtain

$$\left(\tilde{S}_1(u)\right)^2 P(u) + \left(\tilde{S}_1(d)\right)^2 P(d) - \left[\tilde{S}_1(u)\,P(u) + \tilde{S}_1(d)\,P(d)\right]^2$$

$$= P(u)\left(\tilde{S}_1(u)\right)^2 [1 - P(u)] + \left(\tilde{S}_1(d)\right)^2 P(d)\,[1 - P(d)] + 2\tilde{S}_1(u)\,\tilde{S}_1(d)\,P(u)\,P(d)$$

$$= P(u)\,P(d)\left(\tilde{S}_1(u) - \tilde{S}_1(d)\right)^2.$$

This leads to the locally risk-minimizing amounts

$$\hat{H}_1 = \frac{\tilde{B}(u) - \tilde{B}(d)}{\tilde{S}_1(u) - \tilde{S}_1(d)},$$

and

$$\hat{H}_1^0 = \frac{\tilde{B}(d)\,\tilde{S}_1(u) - \tilde{B}(u)\,\tilde{S}_1(d)}{\tilde{S}_1(u) - \tilde{S}_1(d)}.$$

The discounted portfolio value will be

$$\tilde{V}_{0+}\left(\hat{\boldsymbol{H}}\right) = \hat{H}_1^0 + \hat{H}_1 \tilde{S}_0.$$

Remark 4.3. Taking the non discounted values, the initial actual value should be

$$\begin{aligned}
\mathrm{V}_{0+}\left(\hat{\boldsymbol{H}}\right) &= \hat{H}_1^0 S_0^0 + \hat{H}_1 S_0 \\
&= \frac{B(d)\,S_1(u) - B(u)\,S_1(d)}{S_1(u) - S_1(d)} S_0^0 + \frac{B(u) - B(d)}{S_1(u) - S_1(d)} S_0 \\
&= B(u)\left(\frac{-S_1(d)\,S_0^0 + S_0}{S_1(u) - S_1(d)}\right) + B(d)\left(\frac{S_1(u)\,S_0^0 - S_0}{S_1(u) - S_1(d)}\right) \\
&= \frac{S_0^0}{S_1^0}\left[B(u)\,\hat{P}(u) + B(d)\,\hat{P}(d)\right],
\end{aligned}$$

where

$$\hat{P}(u) = \frac{S_1^0/S_0^0 - S_1(d)\,S_1^0/S_0}{S_1(u)/S_0 - S_1(d)/S_0} \quad \text{and} \quad \hat{P}(d) = \frac{S_1(u)\,S_1^0/S_0 - S_1^0/S_0^0}{S_1(u)/S_0 - S_1(d)/S_0}$$

are the minimal measures of the binomial model.

Those risk neutral probabilities are aligned with the ones in the Cox-Ross-Rubinstein that can be found in [6].

4.2.2 The trinomial model

As seen in section 2.5, the trinomial model is an incomplete model. It is also the simplest example of an incomplete financial market.

For this model we also suppose that we have a bank account S^0 and a stock S, but now the stock can take three different and positive values

$$S_1 = \begin{cases} S_1(u) & \text{with probability } P(u) \\ S_1(m) & \text{with probability } P(m) \\ S_1(d) & \text{with probability } P(d), \end{cases}$$

where $P(u) + P(m) + P(d) = 1$.

The European option can also take three different payoff values

$$B = \begin{cases} B(u) & \text{with probability } P(u) \\ B(m) & \text{with probability } P(m) \\ B(d) & \text{with probability } P(d). \end{cases}$$

Like in the the binomial model, applying the local risk-minimizing strategy with Formulas 4.4 and 4.5 we can compute

$$\hat{H}_1 = \frac{Cov\left[\tilde{B}, \tilde{S}_1 \middle| \mathcal{F}_0\right]}{Var\left[\tilde{S}_1 \middle| \mathcal{F}_0\right]} = \frac{\mathbb{E}\left[\tilde{B}\tilde{S}_1 \middle| \mathcal{F}_0\right] - \mathbb{E}\left[\tilde{B} \middle| \mathcal{F}_0\right]\mathbb{E}\left[\tilde{S}_1 \middle| \mathcal{F}_0\right]}{\mathbb{E}\left[\left(\tilde{S}_1\right)^2 \middle| \mathcal{F}_0\right] - \mathbb{E}^2\left[\tilde{S}_1 \middle| \mathcal{F}_0\right]}$$

$$\hat{H}_1^0 = \mathbb{E}\left[\tilde{B}\middle|\mathcal{F}_0\right] - \hat{H}_1 \cdot \mathbb{E}\left[\tilde{S}_1\middle|\mathcal{F}_0\right],$$

where

- $\mathbb{E}\left[\tilde{B}\tilde{S}_1\middle|\mathcal{F}_0\right] = \tilde{B}(u)\,\tilde{S}_1(u)\,P(u) + \tilde{B}(m)\,\tilde{S}_1(m)\,P(m) + \tilde{B}(d)\,\tilde{S}_1(d)\,P(d)$,
- $\mathbb{E}\left[\tilde{B}\middle|\mathcal{F}_0\right]\mathbb{E}\left[\tilde{S}_1\middle|\mathcal{F}_0\right] = \left[\tilde{B}(u)\,P(u) + \tilde{B}(m)\,P(m) + \tilde{B}(d)\,P(d)\right]\left[\tilde{S}_1(u)\,P(u)\right.$ $\left. + \tilde{S}_1(m)\,P(m) + \tilde{S}_1(d)\,P(d)\right]$,
- $\mathbb{E}\left[\left(\tilde{S}_1\right)^2\middle|\mathcal{F}_0\right] = \left(\tilde{S}_1(u)\right)^2 P(u) + \left(\tilde{S}_1(m)\right)^2 P(m) + \left(\tilde{S}_1(d)\right)^2 P(d)$,
- $\mathbb{E}^2\left[\tilde{S}_1\middle|\mathcal{F}_0\right] = \left[\tilde{S}_1(u)\,P(u) + \tilde{S}_1(m)\,P(m) + \tilde{S}_1(d)\,P(d)\right]^2$.

In a similar way to the process done in the binomial model, now considering $P(u) + P(m) + P(d) = 1$, we can also obtain the formulas for the minimal measures. The discounted value satisfies

$$\tilde{V}_{0+}\left(\hat{\boldsymbol{H}}\right) = \mathbb{E}^{\hat{P}}\left[\tilde{B}\middle|\mathcal{F}_0\right],$$

where the minimal measures are defined by

$$\hat{P}(u) = P(u)\left[1 + \left(\tilde{S}_1(u) - \mathbb{E}\left[\tilde{S}_1\middle|\mathcal{F}_0\right]\right)\kappa\right],$$

$$\hat{P}(m) = P(m)\left[1 + \left(\tilde{S}_1(m) - \mathbb{E}\left[\tilde{S}_1\middle|\mathcal{F}_0\right]\right)\kappa\right],$$

$$\hat{P}(d) = P(d)\left[1 + \left(\tilde{S}_1(d) - \mathbb{E}\left[\tilde{S}_1\middle|\mathcal{F}_0\right]\right)\kappa\right],$$

with

$$\kappa := \frac{\tilde{S}_0 - \mathbb{E}\left[\tilde{S}_1\middle|\mathcal{F}_0\right]}{Var\left[\tilde{S}_1\middle|\mathcal{F}_0\right]}.$$

Remark 4.4. Note that the three minimal measures sum up to 1.

Example 4.5. Assume a bank account and a stock with a value of one monetary unit at $t = 0$, this is $S_0^0 = S_0 = 1$. Assume also a fixed interest rate of $r = 0.1$. This is, $S_1^0 = 1\,(1 + 0.1) = 1.1$.

The possible future stock prices and its probability at $t = 1$ are

$$S_1 = \begin{cases} S_1(u) = 2.0 \text{ with probability } P(u) = 0.2 \\ S_1(m) = 1.0 \text{ with probability } P(m) = 0.5 \\ S_1(d) = 0.1 \text{ with probability } P(d) = 0.3, \end{cases}$$

where $u = 1$, $m = 0$ and $d = -0.9$ satisfying $d < r < u$ in order to be in an arbitrage-free market as seen in Proposition 2.51.

Its discounted prices are computed as:

$$\tilde{S}_1 = \frac{S_1}{1+0.1} = \begin{cases} \tilde{S}_1(u) = \frac{2.0}{1.1} \text{ with probability } P(u) = 0.2 \\ \tilde{S}_1(m) = \frac{1.0}{1.1} \text{ with probability } P(m) = 0.5 \\ \tilde{S}_1(d) = \frac{0.1}{1.1} \text{ with probability } P(d) = 0.3. \end{cases}$$

Consider an European call option with strike price of one monetary unit, $K = 1$. The payoff that the buyer can obtain with it will be

$$B = \begin{cases} B(u) = (2.0-1.0)^+ = 1.0 \text{ with probability } P(u) = 0.2 \\ B(m) = (1.0-1.0)^+ = 0.0 \text{ with probability } P(m) = 0.5 \\ B(d) = (0.1-1.0)^+ = 0.0 \text{ with probability } P(d) = 0.3, \end{cases}$$

with discounted payoffs of

$$\tilde{B}(u) = \frac{1.0}{1.1} \quad \text{and} \quad \tilde{B}(m) = \tilde{B}(d) = 0.$$

Applying the local risk-minimizing formulas

$$\hat{H}_1 = \frac{\mathbb{E}\left[\tilde{B}\tilde{S}_1 \middle| \mathcal{F}_0\right] - \mathbb{E}\left[\tilde{B} \middle| \mathcal{F}_0\right] \mathbb{E}\left[\tilde{S}_1 \middle| \mathcal{F}_0\right]}{\mathbb{E}\left[(\tilde{S}_1)^2 \middle| \mathcal{F}_0\right] - \mathbb{E}^2\left[\tilde{S}_1 \middle| \mathcal{F}_0\right]}$$

$$\hat{H}_1^0 = \mathbb{E}\left[\tilde{B} \middle| \mathcal{F}_0\right] - \hat{H}_1 \cdot \mathbb{E}\left[\tilde{S}_1 \middle| \mathcal{F}_0\right],$$

where:

- $\mathbb{E}\left[\tilde{B}\tilde{S}_1 \middle| \mathcal{F}_0\right] = \frac{1}{1.1}\frac{2}{1.1}0.2 = \frac{40}{121}$,
- $\mathbb{E}\left[\tilde{B} \middle| \mathcal{F}_0\right] \mathbb{E}\left[\tilde{S}_1 \middle| \mathcal{F}_0\right] = \frac{2}{11}\frac{93}{110} = \frac{93}{605}$,
- $\mathbb{E}\left[\left(\tilde{S}_1\right)^2 \middle| \mathcal{F}_0\right] = \frac{1303}{1210}$,
- $\mathbb{E}^2\left[\tilde{S}_1 \middle| \mathcal{F}_0\right] = \left(\frac{93}{110}\right)^2$,

we obtain the strategy values

$$\hat{H}_1^0 \approx -0.2311635 \quad \text{and} \quad \hat{H}_1 \approx 0.488473.$$

The minimal measures will be

- $\hat{P}(u) = 0.2\left[1 + \left(\frac{2}{1.1} - \frac{93}{110}\right)\kappa\right] \approx 0.28304$,
- $\hat{P}(m) = 0.5\left[1 + \left(\frac{1}{1.1} - \frac{93}{110}\right)\kappa\right] \approx 0.5135814$,

- $\hat{P}(d) = 0.3\left[1 + \left(\frac{0.1}{1.1} - \frac{93}{110}\right)\kappa\right] \approx 0.2033782,$

where

$$\kappa = \frac{1 - \frac{93}{110}}{\frac{1303}{1210} - (\frac{93}{110})^2} \approx 0.426843.$$

Finally, the discounted fair value of the option is an amount

$$\tilde{V}_{0+}\left(\hat{\boldsymbol{H}}\right) = \mathbb{E}^{\hat{P}}\left[\tilde{B}\middle|\mathcal{F}_0\right] = \tilde{B}(u)\,\hat{P}(u) \approx \frac{1}{1.1}0.28304 \approx 0.2573095.$$

Remark 4.6. The local risk-minimizing strategy will be as follows: at $t = 0$, the seller sells the option for a price $\tilde{V}_{0+}\left(\hat{\boldsymbol{H}}\right)$, borrows an amount H_1^0 from the bank and buys an amount $\hat{H}_1$ of the stock in the market. This leads to a total cost of $0.2573095 - 0.488473 + 0.2311635 = 0$. At $t = 1$, depending on the state, he will have a total cost of $B + S_1\hat{H}_1 - \hat{H}_1^0(1 + 0.1)$, satisfying $\mathbb{E}\left[\pi_1\left(\hat{\boldsymbol{H}}\right)|\mathcal{F}_0\right] = 0$.

4.3 Two assets and multiple periods

As done in section 4.1.2, we will also restrict the local risk-minimizing strategy to a single European option $\tilde{D}_t$ occurring at some time $t \leq T$.

For $t = T$, we must solve the system when $\tilde{V}_t(\mathbf{H}) = \tilde{B}$

$$\begin{pmatrix} H_T^0 \\ H_T \end{pmatrix} \cdot \mathbb{E}\left[\begin{pmatrix} \tilde{S}_T^0 \\ \tilde{S}_T \end{pmatrix} \cdot \tilde{S}_T^0 \middle| \mathcal{F}_{T-1}\right] = \mathbb{E}\left[\tilde{B}\tilde{S}_T^0\middle|\mathcal{F}_{T-1}\right],$$

$$\begin{pmatrix} H_T^0 \\ H_T \end{pmatrix} \cdot \mathbb{E}\left[\begin{pmatrix} \tilde{S}_T^0 \\ \tilde{S}_T \end{pmatrix} \cdot \tilde{S}_T \middle| \mathcal{F}_{T-1}\right] = \mathbb{E}\left[\tilde{B}\tilde{S}_T\middle|\mathcal{F}_{T-1}\right].$$

We have seen that it solves as

$$\hat{H}_T = \frac{Cov\left[\tilde{B}, \tilde{S}_T\middle|\mathcal{F}_{T-1}\right]}{Var\left[\tilde{S}_T\middle|\mathcal{F}_{T-1}\right]},$$

$$\hat{H}_T^0 = \mathbb{E}\left[\tilde{B}\middle|\mathcal{F}_{T-1}\right] - \hat{H}_T \cdot \mathbb{E}\left[\tilde{S}_T\middle|\mathcal{F}_{T-1}\right].$$

The discounted value at $t = T - 1$ will be

$$\tilde{V}_{T-1}\left(\hat{\boldsymbol{H}}\right) = \hat{H}_T^0 + \hat{H}_T\tilde{S}_{T-1}.$$

Next, we want to replicate this amount at the previous time step, thus we assume $\tilde{V}_{T-1}\left(\hat{\boldsymbol{H}}\right) = \hat{H}_T^0 + \hat{H}_T\tilde{S}_T$, and we must now solve the following system

$$\begin{pmatrix} H_{T-1}^0 \\ H_{T-1} \end{pmatrix} \cdot \mathbb{E}\left[\begin{pmatrix} \tilde{S}_{T-1}^0 \\ \tilde{S}_{T-1} \end{pmatrix} \cdot \tilde{S}_{T-1}^0 \middle| \mathcal{F}_{T-2}\right] = \mathbb{E}\left[\tilde{V}_{T-1}(\mathbf{H})\,\tilde{S}_{T-1}^0\middle|\mathcal{F}_{T-2}\right],$$

$$\begin{pmatrix} H_{T-1}^0 \\ H_{T-1} \end{pmatrix} \cdot \mathbb{E}\left[\begin{pmatrix} \tilde{S}_{T-1}^0 \\ \tilde{S}_{T-1} \end{pmatrix} \cdot \tilde{S}_{T-1} \middle| \mathcal{F}_{T-2}\right] = \mathbb{E}\left[\tilde{\mathrm{V}}_{T-1}(\mathbf{H})\,\tilde{S}_{T-1} \middle| \mathcal{F}_{T-2}\right].$$

This similarly solves as before and we obtain the new values

$$\hat{H}_{T-1}^0 \quad \text{and} \quad \hat{H}_{T-1},$$

with a discounted value of

$$\tilde{\mathrm{V}}_{T-2}\left(\hat{\boldsymbol{H}}\right) = \hat{H}_{T-1}^0 + \hat{H}_{T-1}\tilde{S}_{T-2}.$$

Recursively, we will get to the initial value of the strategy

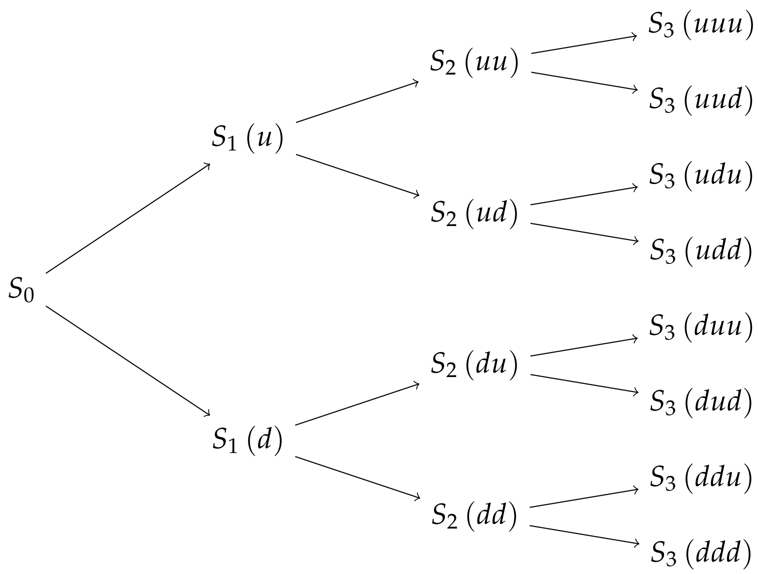
$$\tilde{\mathrm{V}}_{0+}\left(\hat{\boldsymbol{H}}\right) = \hat{H}_1^0 + \hat{H}_1\tilde{S}_0.$$

We will now apply the method again to the binomial and trinomial models.

4.3.1 The binomial model

Assume the same hypotheses as in section 4.2.1, now with $T \geq 2$.

The stock price process for $T = 3$ can be viewed as the following binomial tree:



At each period, the stock price can only move up u or down d, with $d < r < u$. This is, the state space is formed by

$$\Omega := \{u, d\}^T.$$

We will denote by E_t the *stock state process until the time t*, thus

$$S_{t+1} = \begin{cases} S_{t+1}(E_t u) & \text{with probability } P(E_t u) \\ S_{t+1}(E_t d) & \text{with probability } P(E_t d), \end{cases}$$

where $E_t u$ is the concatenation of the process E_t with the up movement, and $E_t d$ is the concatenation with the down movement.

The locally risk-minimizing strategy will be solved similarly to the one in section 4.2.1 following a backwards method. The difference when we have different states is that at every time t we will have to compute for each node its strategy and discounted value.

For each $t \leq T$, the system of equations solves as

$$\hat{H}_t(E_{t-1}) = \frac{\tilde{V}_t(E_{t-1}u) - \tilde{V}_t(E_{t-1}d)}{\tilde{S}_t(E_{t-1}u) - \tilde{S}_t(E_{t-1}d)},$$

and

$$\hat{H}_t^0(E_{t-1}) = \frac{\tilde{V}_t(E_{t-1}d)\,\tilde{S}_t(E_{t-1}u) - \tilde{V}_t(E_{t-1}u)\,\tilde{S}_t(E_{t-1}d)}{\tilde{S}_t(E_{t-1}u) - \tilde{S}_t(E_{t-1}d)}.$$

Note that $\hat{H}_t(E_{t-1})$ is the number of shares of the risky asset at the state E_{t-1}.

The portfolio value at each node E_{t-1} at time t will be

$$\tilde{V}_{(t-1)+}(\hat{\boldsymbol{H}})(E_{t-1}) = \hat{H}_t^0(E_{t-1}) + \hat{H}_t(E_{t-1})\,\tilde{S}_{t-1}(E_{t-1}).$$

Once the nodes of the tree are computed at time t, we have the next fair value for $t-1$ and we can repeat the strategy, until we arrive to the initial fair value.

Similarly to Remark 4.3, we could also obtain a explicit formula for the martingale measures.

4.3.2 The trinomial model

In the same way as in section 4.2.2, we assume three possible states; up, middle and down.

For each $t \leq T$ and at each node, the system solves as

$$\hat{H}_t(E_{t-1}) = \frac{Cov\left[\tilde{V}_t(E_{t-1}), \tilde{S}_t(E_{t-1}) \middle| \mathcal{F}_{t-1}\right]}{Var\left[\tilde{S}_t(E_{t-1}) \middle| \mathcal{F}_{t-1}\right]},$$

and

$$\hat{H}_t^0(E_{t-1}) = \mathbb{E}\left[\tilde{V}_t(E_{t-1}) \middle| \mathcal{F}_{t-1}\right] - \hat{H}_t(E_{t-1}) \cdot \mathbb{E}\left[\tilde{S}_t(E_{t-1}) \middle| \mathcal{F}_{t-1}\right].$$

The discounted value is

$$\tilde{V}_{(t-1)+}\left(\hat{\boldsymbol{H}}\right)\left(E_{t-1}\right)=\mathbb{E}^{\hat{P}}\left[\tilde{D}_t\middle|\mathcal{F}_{t-1}\right],$$

where the minimal measures are defined by

$$\hat{P}\left(E_{t-1}u|E_{t-1}\right)=P\left(E_{t-1}u|E_{t-1}\right)\left[1+(\tilde{S}_t\left(E_{t-1}u\right)-\mathbb{E}\left[\tilde{S}_t\middle|E_{t-1}\right])\kappa_t\right],$$

$$\hat{P}\left(E_{t-1}m|E_{t-1}\right)=P\left(E_{t-1}m|E_{t-1}\right)\left[1+\left(\tilde{S}_t\left(E_{t-1}m\right)-\mathbb{E}\left[\tilde{S}_t\middle|E_{t-1}\right]\right)\kappa_t\right],$$

$$\hat{P}\left(E_{t-1}d|E_{t-1}\right)=P\left(E_{t-1}d|E_{t-1}\right)\left[1+\left(\tilde{S}_t\left(E_{t-1}d\right)-\mathbb{E}\left[\tilde{S}_t\middle|E_{t-1}\right]\right)\kappa_t\right],$$

with

$$\kappa_t:=\frac{\tilde{S}_{t-1}\left(E_{t-1}\right)-\mathbb{E}\left[\tilde{S}_t\middle|E_{t-1}\right]}{Var\left[\tilde{S}_t\middle|E_{t-1}\right]}.$$

Remark 4.7. $P\left(E_{t-1}u|E_{t-1}\right)$ means the probability from state E_{t-1} to $E_{t-1}u$, and $\mathbb{E}\left[\tilde{S}_t\middle|E_{t-1}\right]$ is the expected value of $\tilde{S}_t$ at the state E_{t-1}.

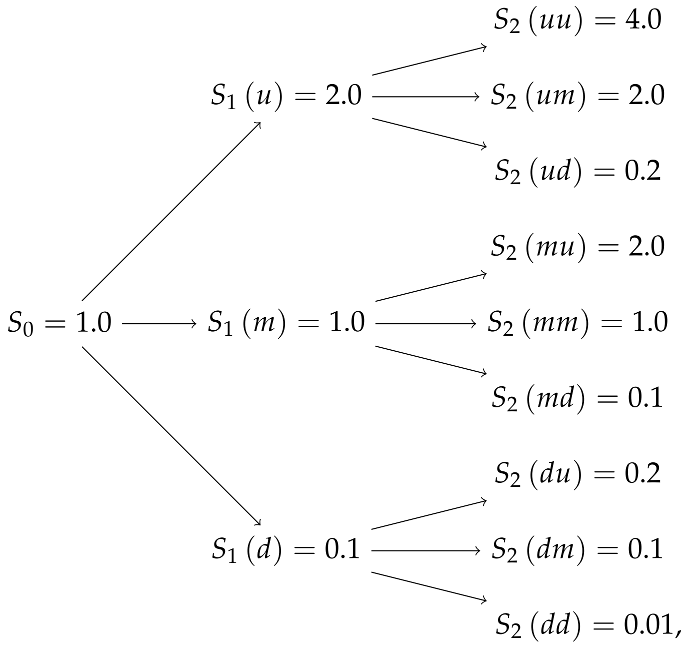
Once we have computed the locally risk-minimizing strategy and the discounted value for each of the nodes at t, we follow the same strategy by taking the discounted values as the new payoffs that we want to replicate at time $t-1$. By iteration we will obtain the initial fair value.

Example 4.8. Similar to Example 4.5, we assume a bank account and a stock with initial value of $S_0^0=S_0=1$, and a fixed interest rate of $r=0.1$. We will assume two time periods, $T=2$.

The bank account will have the next two future values:

$$S_1^0=(1+0.1)=1.1\ \text{ and }\ S_2^0=(1+0.1)^2=1.21.$$

The possible future stock prices are given by the next trinomial tree, with $u=1$, $m=0$ and $d=-0.9$, and probabilities $P\left(u\right)=0.2$, $P\left(m\right)=0.5$ and $P\left(d\right)=0.3$,



with discounted prices

$$\tilde{S}_1 = \frac{S_1}{1.1} \text{ and } \tilde{S}_2 = \frac{S_2}{1.21}.$$

We also assume an European call option with strike price $K = 1$. Therefore, the payoff will be:

- $B(uu) = 3.0$ with probability $p(uu) = 0.04$
- $B(um) = 1.0$ with probability $p(um) = 0.10$
- $B(mu) = 1.0$ with probability $p(mu) = 0.10$,

with all the other payoffs equal to 0.

We apply the local risk-minimizing strategy. First, for $t = 2$ we compute the strategy for the node $E_2 = u$:

$$\hat{H}_2(u) = \frac{Cov\left[\tilde{B}(u), \tilde{S}_2(u) \middle| \mathcal{F}_1\right]}{Var\left[\tilde{S}_2(u) \middle| \mathcal{F}_1\right]} = \frac{\mathbb{E}\left[\tilde{B}(u)\tilde{S}_2(u) \middle| \mathcal{F}_1\right] - \mathbb{E}\left[\tilde{B}(u) \middle| \mathcal{F}_1\right]\mathbb{E}\left[\tilde{S}_2(u) \middle| \mathcal{F}_1\right]}{\mathbb{E}\left[\left(\tilde{S}_2(u)\right)^2 \middle| \mathcal{F}_1\right] - \mathbb{E}^2\left[\tilde{S}_2(u) \middle| \mathcal{F}_1\right]},$$

and

$$\hat{H}_2^0(u) = \mathbb{E}\left[\tilde{B}(u) \middle| \mathcal{F}_1\right] - \hat{H}_2(u) \cdot \mathbb{E}\left[\tilde{S}_2(u) \middle| \mathcal{F}_1\right],$$

where:

- $\mathbb{E}\left[\tilde{B}(u)\,\tilde{S}_2(u)|\mathcal{F}_1\right] = \frac{3}{1.21}\frac{4}{1.21}0.2 + ... + \frac{0}{1.21}\frac{0.2}{1.21}0.3 \approx 2.322245748,$
- $\mathbb{E}\left[\tilde{B}(u)|\mathcal{F}_1\right]\mathbb{E}\left[\tilde{S}_2(u)|\mathcal{F}_1\right] \approx 1.39744553,$
- $\mathbb{E}\left[\left(\tilde{S}_2(u)\right)^2\Big|\mathcal{F}_1\right] \approx 3.559866129,$
- $\mathbb{E}^2\left[\tilde{S}_2(u)\,|\mathcal{F}_1\right] \approx (1.537190083)^2 \approx 2.362953188.$

This yields to the locally risk-minimizing strategy for $t = 2$ of

$$\hat{H}_2^0(u) \approx -0.278626149 \quad \text{and} \quad \hat{H}_2(u) \approx 0.772654645.$$

This means that if the stock prices go up at $t = 1$, the strategy to be hedged against all possible payoffs would be to have $\hat{H}_2(u)$ units of the stock and have $\hat{H}_2^0(u)$ units borrowed from the bank.

Next, we calculate $\tilde{V}_{1+}(u) = \mathbb{E}^{\hat{P}}\left[\tilde{B}|\mathcal{F}_1\right]$. The minimal measures are

$$\begin{aligned}
\hat{P}(uu|u) &= P(u)\left[1 + \left(\tilde{S}_2(uu) - \mathbb{E}\left[\tilde{S}_2|u\right]\right)\kappa_2\right] \\
&\approx 0.2\left[1 + \left(\frac{4.0}{1.21} - 1.537190083\right)\kappa_2\right] \approx 0.234763752, \\
\hat{P}(um|u) &= P(m)\left[1 + \left(\tilde{S}_2(um) - \mathbb{E}\left[\tilde{S}_2|u\right]\right)\kappa_2\right] \\
&\approx 0.5\left[1 + \left(\frac{2.0}{1.21} - 1.537190083\right)\kappa_2\right] \approx 0.283040401, \\
\hat{P}(ud|u) &= P(d)\left[1 + \left(\tilde{S}_2(ud) - \mathbb{E}\left[\tilde{S}_2|u\right]\right)\kappa_2\right] \\
&\approx 0.3\left[1 + \left(\frac{0.2}{1.21} - 1.537190083\right)\kappa_2\right] \approx 0.203378124,
\end{aligned}$$

with

$$\kappa_2 = \frac{\tilde{S}_1(u) - \mathbb{E}\left[\tilde{S}_2|u\right]}{Var\left[\tilde{S}_2|u\right]} \approx \frac{\frac{2.0}{1.1} - 1.537190083}{3.559866129 - 2.362953188} \approx 0.234763752.$$

This yields to the discounted value

$$\begin{aligned}
\tilde{V}_{1+}\left(\hat{\boldsymbol{H}}\right)(u) &= \tilde{B}(uu)\,\hat{P}(uu|u) + \tilde{B}(um)\,\hat{P}(um|u) + \tilde{B}(uu)\,\hat{P}(ud|u) \\
&\approx \frac{3.0}{1.21}0.234763752 + \frac{1.0}{1.21}0.283040401 \approx 1.126200477.
\end{aligned}$$

We repeat the process for the other two states and we obtain the following values:

- For $E_2 = m$:

$$\hat{H}_2^0\left(m\right) \approx -0.213657399 \quad \text{and} \quad \hat{H}_2\left(m\right) \approx 0.493038121,$$

  with a discounted portfolio value of $\tilde{\mathrm{V}}_{1+}\left(\hat{\boldsymbol{H}}\right)\left(m\right) \approx 0.233917687$.

- For $E_2 = d$:

$$\hat{H}_2^0\left(d\right) = 0 \quad \text{and} \quad \hat{H}_2\left(d\right) = 0,$$

  with a discounted portfolio value of $\tilde{\mathrm{V}}_{1+}\left(\hat{\boldsymbol{H}}\right)\left(m\right) = 0$.

Note that in the down state, as there is no possibility to have a strict positive payoff, it does not make sense to have any strategy.

For $t = 1$, assuming $\tilde{C}_1\left(u\right) = \tilde{V}_{1+}\left(u\right)$, $\tilde{C}_1\left(m\right) = \tilde{V}_{1+}\left(m\right)$ and $\tilde{C}_1\left(d\right) = \tilde{V}_{1+}\left(d\right)$ we have now the payoff tree for $T = 1$

$$\tilde{C}_0 \begin{array}{l} \nearrow \tilde{C}_1\left(u\right) \approx 1.1262 \\ \longrightarrow \tilde{C}_1\left(m\right) \approx 0.2339 \\ \searrow \tilde{C}_1\left(d\right) = 0. \end{array}$$

We repeat now the same process with the new three possible payoffs, obtaining the following locally risk-minimizing strategy

$$\hat{H}_1^0 \approx -0.328524148 \quad \text{and} \quad \hat{H}_1 \approx 0.736955644,$$

with an initial fair value of the European call option of

$$\tilde{\mathrm{V}}_{0+}\left(\hat{\boldsymbol{H}}\right) \approx 0.438886783.$$

Remark 4.9. To summarize the strategy:

- At $t = 0$, the seller sells the option for a fair price $\tilde{\mathrm{V}}_{0+}\left(\hat{\boldsymbol{H}}\right)$, borrows an amount $\hat{H}_1^0$ from the bank and buys another amount $\hat{H}_1$ of the stock in the market.

- At $t = 1$, the seller changes his investment strategy to $\hat{H}_2^0$ and $\hat{H}_2$ depending on the state that took place.

- At $t = 2$ the seller is hedged to every possible payment of the call.

Conclusions

The aim of this work was to introduce the local risk-minimization strategy to price and hedge options in incomplete markets.

In order to do that, we began with the required mathematical and financial theory to comprehend how the financial market worked, as well as understanding the difference between a complete and an incomplete market. In the main and third chapter we realised how common incomplete markets are, and how many different strategies exist to price and hedge financial products. We developed the theory behind the local risk-minimization strategy and, finally, in the last chapter, the application of this strategy in the binomial and trinomial models has been explained.

Although in the applications chapter we could not extend the strategy to more complicated models, the theory applies the same way. This could be a good future idea to investigate. Moreover, a comparison between the diverse strategies in incomplete markets can also be further studied.

The financial market and, in particular, the derivative market offers an immense range of financial products. It has been interesting to realise how many studies have been published related to this financial area, and how important it is to being able to price this products. At university I did not have the chance to explore in so much detail the financial market, and this work has helped me understanding the theory behind it.

After finishing this work I feel more confident about continuing my studies in this area and pursuing a master in finance.

Appendix A

Annexes

A.1 First fundamental theorem of finance

The next two theorems are needed to proof Theorem 2.31.

Theorem A.1. *(Hyperplane separation theorem)*

Let $C \subset \mathbb{R}^n$ be a convex and closed set such that $0 \notin C$. Then there exists $\alpha > 0$ and a linear application $\xi : \mathbb{R}^n \to \mathbb{R}$ such that for all $x \in C$

$$\xi(x) \geq \alpha > 0.$$

Proof. Consider $\lambda > 0$ and a closed ball $\bar{B}_\lambda(0)$ with center in the origin and radius λ. The intersection with the convex C is immediate with a sufficient big radius. We have $A = \bar{B}_\lambda(0) \cap C \neq \emptyset$. A is closed because it is an intersection of closed sets, and it is also enclosed for definition. Thus we have that it is a compact set and we can define the norm function

$$x \in \mathbb{R}^n \to \|x\| \in \mathbb{R},$$

which is continue. And every continuous function in a compact space has a maximum and a minimum, therefore

$$\exists x_0 \in A \text{ such that } \|x_0\| \leq \|x\|, \forall x \in A.$$

In addition, for every $x \in A$ we have $\|x\| > \lambda$. And $\|x_0\| \leq \|x\|, \forall x \in C$. Suppose $x \in C$, $x \neq x_0$ and a real number $t \in (0, 1]$. Given that C is convex, we have that $(1-t)x_0 + tx \in C$. Therefore the minimum x_0 satisfies

$$\|x_0\| \leq \|(1-t)x_0 + tx\| = \|x_0 + t(x - x_0)\|$$

$$\Leftrightarrow \|x_0\|^2 \leq \|x_0 + t(x - x_0)\|^2$$

$$\Leftrightarrow \|x_0\|^2 \leq \|x_0\|^2 + t^2\|x - x_0\|^2 + 2tx_0 \cdot (x - x_0)$$
$$\Leftrightarrow t^2\|x - x_0\|^2 + 2tx_0 \cdot (x - x_0) \geq 0$$
$$\Leftrightarrow t\|x - x_0\|^2 + 2x_0 \cdot (x - x_0) \geq 0$$

which is true for all $t \in (0, 1]$ and if and only if

$$x_0 \cdot (x - x_0) \geq 0 \Leftrightarrow x \cdot x_0 \geq \|x_0\|^2.$$

If we choose $\xi(x) := x \cdot x_0$ and $\alpha := \|x_0\|^2$ we finish the proof. □

Theorem A.2. *(Separation theorem)*

Given $K \subseteq \mathbb{R}^n$ a convex and compact set. Let $V \subseteq \mathbb{R}^n$ be a linear space. Suppose that $K \cap V = \emptyset$. Then, there exists a linear application $\xi : \mathbb{R}^n \to \mathbb{R}$ such that $\xi(x) > 0$ for all $x \in K$ and $\xi(x) = 0$ for all $x \in V$.

Proof. Define the set

$$C := \{x \in \mathbb{R}^n | x = y - z,\ y \in K,\ z \in V\}.$$

We will prove that it is closed, convex and does not include the 0.

- It is convex because of K.
- It does not contain the 0 because if it does, there should be an element $x = y - y$, $y \in K \cap V$. But we supposed that the intersection is void.
- To prove that it is closed, we consider a succession $x_n \in C$ convergent to $x \in \mathbb{R}^n$. For definition, $x_n = y_n - z_n$, $y_n \in K$ and $z_n \in V$, where K is compact and V is closed. Because K is compact, there exists a partial succession y_{n_k} which converges to an element $y \in K$. Therefore, $x_{n_k} = y_{n_k} - z_{n_k}$. Given that the successions x_n and y_{n_k} have a limit, we can say that $z := y - x$ is the limit of z_{n_k}, and because V is closed and $z \in V$, we have that $x = y - z \in C$. This is, the limit of the convergent succession is inside the set, therefore the set is closed.

Now we can apply the Hyperplane separation theorem and we have that there exists $\alpha > 0$ and a linear application ξ such that

$$\xi(y - z) = \xi(y) - \xi(z) \geq \alpha > 0,\ \forall y \in K,\ z \in V.$$

We can write $z = \|z\|\frac{z}{\|z\|} = \|z\|\tilde{z} = \lambda\tilde{z}$, with $\tilde{z}$ an element of V with the unit norm. Therefore,

$$\xi(y) - \xi(z) = \xi(y) - \lambda\xi(\tilde{z}) \geq \alpha > 0,\ \forall y \in K,\ \tilde{z} \in V,\ \lambda > 0.$$

If $\xi(\tilde{z}) < 0$ we can write

$$\xi(y) - \xi(z) = \xi(y) + \lambda|\xi(\tilde{z})| \geq \alpha > 0, \forall y \in K, \tilde{z} \in V, \lambda > 0.$$

In general, we have

$$\xi(y) - \xi(z) = \xi(y) - \lambda|\xi(\tilde{z})| \geq \alpha > 0, \forall y \in K, \tilde{z} \in V, \lambda \in \mathbb{R}.$$

Given that $y \in K$, K a compact set, we have that $\xi(y)$ has an upper level, which implies that it is necessary $\xi(\tilde{z}) = 0$. And we have

$$\xi(y) \geq \alpha > 0, \forall y \in K.$$

□

A.2 Second fundamental theorem of finance

Theorem A.3. *(Second fundamental theorem of finance or asset pricing)*

A finite and viable market is complete if and only if exists a unique risk neutral probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$, under which discounted prices are martingales.

Proof. $\Rightarrow$ If a market is complete we have to see that $\mathbb{P}^*$ is the unique risk neutral probability. Let $\mathbb{P}_1^*$ and $\mathbb{P}_2^*$ be two risk neutral probabilities with $\mathbb{E}_1^*$ and $\mathbb{E}_2^*$ the respectively expected values under $\mathbb{P}_1^*$ and $\mathbb{P}_2^*$. Consider B a payoff. Because the market is complete, it exists $V_0 \geq 0$ and a self-financing strategy H such that

$$B = V_0 + \sum_{j=1}^{T} \mathbf{H}_{j-1} \cdot \Delta \mathbf{S}_j = V_T\left(\mathbf{H}\right),$$

where $\Delta \mathbf{S}_j = \mathbf{S}_j - \mathbf{S}_{j-1}$. As $\mathbb{P}_1^*$ and $\mathbb{P}_2^*$ are risk neutral, the discounted value of B is the final value of a martingale with respect to this two probabilities and it satisfies

$$\mathbb{E}_1^*\left(\tilde{V}_T\left(\mathbf{H}\right)\right) = \mathbb{E}_2^*\left(\tilde{V}_T\left(\mathbf{H}\right)\right) = V_0,$$

therefore

$$\mathbb{E}_1^*\left(B\right) = \mathbb{E}_2^*\left(B\right).$$

Since B is any strict positive and $\mathcal{F}_T$-measurable value, we have that $\mathbb{P}_1^* = \mathbb{P}_2^*$ in $\mathcal{F}_T = \mathcal{F}$.

$\Leftarrow$ Consider $\mathbb{P}^*$ the unique risk neutral probability in a viable market. Assume that the market is not complete and that there exists $\mathbf{H}$ a payoff that cannot be replicated. Let Γ be the set of all discounted replicated payoffs and consider Λ

the set of all random variables in the finite probability space. In this set we can consider the scalar product $\langle X, Y\rangle = \mathbb{E}^* (X, Y)$. Observe that $\tilde{H} \in \Lambda - \Gamma$. Then we can choose a random variable $X \neq 0$, orthogonal to Γ. We define

$$\mathbb{P}^{**} (\omega) := \left(1 + \frac{X(\omega)}{2\|X\|_\infty}\right) \mathbb{P}^* (\omega),$$

where $\|X\|_\infty$ is the maximum norm. Per definition $\mathbb{P}^{**} (\omega) > 0$ for all $\omega \in \Omega$ and given that $\mathbb{E}^* (X) = \mathbb{E}^* (X \cdot 1) = 0$, we have

$$\sum_{\omega\in\Omega} \mathbb{P}^{**} (\omega) = \mathbb{E}^* \left(1 + \frac{X}{2\|X\|_\infty}\right) = 1 + \frac{\mathbb{E}^* (X)}{2\|X\|_\infty} = 1.$$

And it is clear $\mathbb{P}^{**} \sim \mathbb{P}^*$.

Finally we just have to see that $\mathbb{P}^{**}$ is risk neutral. Given a discounted process $\tilde{S}^k$, an adapted process H^k, and any T, we have

$$\mathbb{E}^{**} \left(\sum_{j=1}^{T} H^k_{j-1} \cdot \Delta\tilde{S}^k_j\right) = \mathbb{E}^* \left(\sum_{j=1}^{T} H^k_{j-1} \cdot \Delta\tilde{S}^k_j\right) + \frac{1}{2\|X\|_\infty}\mathbb{E}^* \left(X \cdot \sum_{j=1}^{T} H^k_{j-1} \cdot \Delta\tilde{S}^k_j\right),$$

which is equal to zero because $\tilde{S}$ are martingales with respect to $\mathbb{P}^*$ and

$$\sum_{j=1}^{T} H^k_{j-1} \cdot \Delta\tilde{S}^k_j \in \Gamma.$$

Thus, with Proposition 2.13 we have that $\mathbb{P}^{**}$ is also a risk neutral probability, which is a contradiction with the hypothesis of $\mathbb{P}^*$ being the only risk neutral probability. □

References

[1] Black, F. and Scholes, M. *The pricing of options and corporate liabilities* [paper]. Journal of Political Economy, Volume 81, Issue 3, 1973. Pages 637-654.

[2] Cont, R. and Tankov, P. *Financial Modelling with Jump Processes* [book]. 1st edition. Chapman and Hall, 2004.

[3] Capiński, M. and Kopp, E. *Discrete Models of Financial Markets* [book]. 1st edition. Cambridge University Press, 2012.

[4] Corcuera, J.M. *A course in Stochastic Processes* [lecture notes]. Universitat de Barcelona, 2018.

[5] Cox, J.C., Ross, S.A. and Rubinstein, M. *Option pricing: A simplified approach* [paper]. Journal of Financial Economics, Volume 7, Issue 3, 1979. Pages 229-263.

[6] Cox, J.C. and Rubinstein, M. *Options Markets* [book]. 1st edition. Prentice-Hall, 1985.

[7] Delbaen, F. and Schachermayer W. *A general version of the fundamental theorem of asset pricing* [paper]. Mathematische Annalen, Volume 300, 1994. Pages 463-520.

[8] Föllmer, H. and Leukert, P. *Efficient hedging: Cost versus short-fall risk* [paper]. Journal of Finance and Stochastics, Volume 4, 2000. Pages 117-146.

[9] Föllmer, H. and Schweizer, M. *Hedging by sequential regression: an introduction to the mathematics of option trading* [paper]. ASTIN Bulletin, Volume 18, Issue 2, 1988. Pages 147-160.

[10] Föllmer, H. and Schied, A. *Stochastic Finance: an introduction in discrete time* [book]. 4th edition. De Gruyter, 2016.

[11] Föllmer, H. and Sondermann, D. *Hedging of non-redundant contingent claims* [paper]. Contributions to Mathematical Economics, 1986.

[12] Lamberton, D. and Lapeyre, B. *Introduction to Stochastic Calculus Applied to Finance* [book]. 2nd edition. Chapman and Hall, 2008.

[13] Moeller, T. *Risk Minimization* [paper]. Encyclopedia of Actuarial Science, 2006.

[14] Pansera J. *Local risk minimization, consistent interest-rate modeling, and applications to life insurance* [thesis]. Iowa Research Online, 2008.

[15] Pliska, S.R. *Introduction to Mathematical finance: Discrete time models* [book]. 1st edition. Blackwell, 1997.

[16] Schweizer, M. *A guided tour through quadratic hedging approaches* [paper]. Technische Universität Berlin, 1999.

[17] Schweizer, M. *Hedging of options in a general semimartingale model* [paper]. ETH Zurich, 1988.

[18] Schweizer, M. *Option pricing. Interest rates and risk management. Chapter 15: A guided tour through quaratic hedging approaches* [book chapter]. Cambridge University Press, 2001.

[19] Shreve, S. *Stochastic calculus for finance I: The binomial asset pricing model* [book]. 1st edition. Springer, 2004.

[20] Staum, J. *Incomplete markets* [paper] Handbooks in Operations Research and Management Science, Volume 15, 2007. Pages 511-563.

[21] Vives, J. *Modelització* [lecture notes]. Universitat de Barcelona, 2020.