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Treball final de grau

Introduction to Conformal Geometry and Penrose Diagrams

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Abstract

Conformal geometry is the branch of mathematics that studies the transformations on manifolds that preserve the angles. It has a myriad of applications, both in mathematics and in physics. In this work we present an introduction to conformal geometry and describe its relation to Penrose diagrams, which are representations of spacetimes that preserve their causal structure. To this end, we start by providing the necessary tools for doing this work from semi-Riemannian geometry and conclude by giving examples of these diagrams.

Resum

La geometria conforme és la branca de les matemàtiques que estudia les transformacions sobre varietats que conserven els angles. Té una infinitat d'aplicacions, tant en matemàtiques com en física. En aquest treball presentem una introducció a la geometria conforme i descrivim la seva relació amb els diagrames de Penrose, que són representacions d'espaitemps que conserven l'estructura causal d'aquests. Per a això, comencem proporcionant-nos les eines necessàries per fer aquest treball a partir de la geometria semi-Riemanniana i acabem posant exemples d'aquests diagrames.

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1 Introduction

Both mathematicians and physicists use pictures and diagrams as a way of representing ideas. These are powerful tools that facilitate their conceptual work. Not only are these representations useful for them to clarify their research but also to help them disseminate their ideas to the community.

Among all these ideas to represent, in this work we focus on the transformations that preserve angles, the conformal transformations. By way of illustration, the Mercator projection, which represents the nearly spherical surface of the Earth on a perfectly cylindrical surface is a conformal transformation. In the Mercator projection, angles between pieces of land are an accurate representation of the real surface of Earth. Nevertheless, the closer you get to the poles, the more magnified everything looks, which distorts reality. The distortion is so important in the Mercator projection that poles, which are points on the globe, turn into lines as big as the Equator.

The branch of mathematics which studies this kind of transformations is conformal geometry. Conformal geometry can be thought as a generalisation of semi-Riemannian geometry. That is in the sense that semi-Riemannian geometry studies the properties of semi-Riemannian manifolds (M, g), which consist of a differentiable manifold with a non-degenerate metric tensor, whereas conformal geometry studies conformal manifolds $(M, [g])$, which consist of a differentiable manifold with an equivalence class of metric tensors. This equivalence class of metric tensors is called conformal class or conformal metric.

One feature that differentiates conformal geometry from semi-Riemannian geometry is that for the latter, at every point of a manifold one has a well defined metric with which angles and lengths of vectors in the respective tangent space can be calculated. On the other hand, given a point in a conformal manifold, one only has a class of metrics defined at that point. Although this class of metrics does not allow the calculation of distances, it is still possible to compute angles between vectors. Physicists have something to say here. Albert Einstein's theories of relativity, which are well described in the books [Wal84, HE73, Sch80] use Lorentzian manifolds to describe spacetime. Lorentzian manifolds are a specific case of semi-Riemannian manifold. This, together with the fact that conformal geometry is a generalisation of semi-Riemannian geometry, was what motivated our study of semi-Riemannian geometry. Moreover, it turns out that conformal transformations of a Lorentzian manifold preserve the causal structure of such manifold.

The causal structure of a Lorentzian manifold classifies, for every point of the manifold, all the other points in three categories: the points that can be influenced by a this given point, the ones that could have been influenced by this given point and those points that can not have any causal relation with the given point. The latter group exists because any kind of information can travel faster than the speed of light. After applying a conformal transformation to a Lorentzian manifold this classification does not change. If one point can influence another, the image of the first point through a conformal transformation can influence the image of the second through the same transformation.

The power that physicist could extract from conformal transformations was very significant. Concerning problems such as the ones in electromagnetism, they need to do assumptions about how would the electromagnetic field magnitude surrounding a charge be at infinity, for example. Since it decreases inversely proportional to the square of the distance, they normally assume it is zero at infinity. However, considering "the infinity" normally has little physical meaning. As conformal transformations allow the distortion of distances without changing the angles, Roger Penrose stated that considering the conformal structure of spacetime, "points at infinity can be treated at the same basis as finite points" [Pen64b]. His work lead to what we know today as Penrose diagrams.

Penrose diagrams were developed by Roger Penrose between 1962 and 1966, shortly after he finished his PhD at Cambridge University. Some of the earliest hand-drawn representations of Penrose diagrams can be found in [HP70]. These diagrams rapidly spread from the most sophisticated scientific congresses to summer school for students and pedagogical articles. For instance, less than two years after the first apparition of Penrose diagrams, they were presented to advanced graduate students and researchers at the 1963 Les Houches Summer School in theoretical physics by Roger Penrose himself [Pen64a]. However, it was not only Roger Penrose who worked on these conformal manifolds. The same year, at the same summer school, physicist Rainer K. Sachs also presented his work with which he also brought all Minkowski spacetime to a finite sheet of paper through a conformal transformation [Sac64]. Brandon Carter, a physicist at Cambridge University, also did a lot of work with Penrose diagrams. He used them to represent spacetimes much more complex than the Schwarzschild metric case [Car66]. That is why they are also known as Penrose-Carter diagrams.

These new diagrams were useful to represent the Universe and black holes, concepts which were difficult to imagine. An important and useful feature of Penrose diagrams is that on them light rays always follow paths of 45° from vertical. Thus, Penrose diagrams helped to understand general relativity and cosmology and were used to apply topology and conformal transformatins to general relativity. More specifically, the idea behind Penrose diagrams is to identify, by means of a conformal transformation, the open Lorentzian manifold representing spacetime with the interior of a manifold with boundary. This boundary is called the conformal infinity. The manifold with boundary is said to be unphysical in the sense that it is not the actual spacetime. We have never seen the boundary of our spacetime. However, these unphysical manifolds can be very convenient for solving physical problems. The idea is to translate the problem to the unphysical compact manifold, solve it there using the conformal infinity and then translating the solution back to the physical spacetime.

Everything discussed above is what motivated the structure of this work. In Section 2 give an introduction to semi-Riemannian geometry with special mention to Lorentzian manifolds. This section also made us lay the foundations of differential geometry, which have been necessary for the whole work. There, we also had the objective of defining smooth manifolds in an accurate way, closing the section with this discussion. Analogously, in Section 3 our main goal is to give a well-formulated description of conformal manifolds. To do so, we start by presenting the conformal transformations and their classification, for what we needed to introduce the conformal Killing fields, which are a generalisation of the Killing fields in semi-Riemannian geometry. For these two sections our main sources of information were [O'N83] and [Sch08], respectively.

For the classification of the conformal transformations we have considered manifolds embedded in the n-dimensional semi-Euclidean space $\mathbb{R}^{p,q}$. We distinguished three cases: manifolds of dimension $n > 2$, the Euclidean plane and the Minkowski plane. For the first case, we saw that any conformal transformation can be described as a composition of a translation, an orthogonal transformation, a dilatation and a special conformal transformation (see Case 3.3.1). Orthogonal transformations are those that preserve the inner product. For instance, rotations, reflections and their combinations are orthogonal transformations. By preserving the inner product, these transformations would be an example of length-preserving conformal transformations.

For Case 3.3.2, the Euclidean plane $\mathbb{R}^{2,0}$, it is useful to identify it with the complex plane $\mathbb{C} \cong \mathbb{R}^{2,0}$. With that, we saw that its conformal transformations are holomorphic or antiholomorphic functions. Dilatations and rotations would also be examples of conformal transformations of the Euclidean plane. We also give special mention to a group of conformal transformations named after the mathematician August F. Möbius. Möbius transformations are well-known for their applications in mathematics [VB21] and physics [Oli02, Appendix B]. A nice and intuitive way to think of Möbius transformations is by doing an inverse steregraphic projection from the plane to the two-sphere, rotating this two-sphere through the plane and finally stereographically projecting it to the plane. We refer the reader to [AR08] for a visual and more detailed explanation of this procedure.

Finally, for Case 3.3.3, we studied the conformal transformations one can apply on the Minkowski plane, which is the semi-Euclidean 2-dimensional space named after the mathematician Hermann Minkowski. The Minkowski plane is an example of Lorentzian manifold. This case has a relevant importance for physicists because the theory of special relativity uses it to describe spacetime. We saw that for the Minkowski plane the orientation-preserving conformal transformations are composed by dilatations and boosts or Lorentz transformations, which are analogue to rotations in Euclidean space and can be thought as hyperbolic rotations preserving the relativistic interval ds^2.

We conclude in Section 4 by presenting conformal compactifications to achieve Penrose diagrams (Figure 1), as an application of everything studied before. As an illustration of them, we present here the most widely seen Penrose diagram, which is the maximally extended Penrose diagram for the 4-dimensions Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_{II}^2 \, ,$$

where $d\Omega_{II}^2 = r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2$, in spherical coordinates. For the sake of simplicity, from now on we consider the Schwarzschild radius $r_s := 2M$, which we will take it to be $r_s := 1$ when representing the maximally extended Penrose diagram. Moreover, we can forget about the angular part to represent it in a plane by considering that every point in the plane represents a two-sphere of radius r. Now, following [LV19], we proceed to do the following conformal transformations

$$(t,r) \xrightarrow{\text{Kruskal}} (T,X) \xrightarrow{\text{Rotation}} (u,v) \xrightarrow{\text{Rescaling}} (U,V) \xrightarrow{\text{Rotation}} (\tau,\rho) \, ,$$

where Kruskal coordinates are the ones that allow us to define four regions: region I ($r > r_s$, exterior of the black hole), region II ($0 < r < r_s$, interior of the black hole), region III ($r > r_s$, parallel universe) and region IV ($0 < r < r_s$, interior of the white hole). Kruskal coordinates are,

$$\text{I:} \qquad X = \left|\frac{r}{r_s} - 1\right|^{1/2} \exp\left(\frac{r}{2r_s}\right) \cosh\frac{t}{2r_s} \, , \quad T = \left|\frac{r}{r_s} - 1\right|^{1/2} \exp\left(\frac{r}{2r_s}\right) \sinh\frac{t}{2r_s} \, ,$$

$$\text{II:} \qquad X = \left|\frac{r}{r_s} - 1\right|^{1/2} \exp\left(\frac{r}{2r_s}\right) \sinh\frac{t}{2r_s} \, , \quad T = \left|\frac{r}{r_s} - 1\right|^{1/2} \exp\left(\frac{r}{2r_s}\right) \cosh\frac{t}{2r_s} \, .$$

For region III (resp. IV) we can take the Kruskal coordinates as in region I (resp. II) but changing the sign of X (resp. T) coordinate. Now we can rotate the axis applying the change of coordinates

$$u := T - X \, , \quad v := T + X \, .$$

Now we rescale the metric to bring the infinity into a finite place with

$$U := \arctan u\,, \quad V := \arctan v\,.$$

Finally, we rotate the axis back with

$$\rho := V - U\,, \quad \tau := V + U\,,$$

obtaining Figure 1.

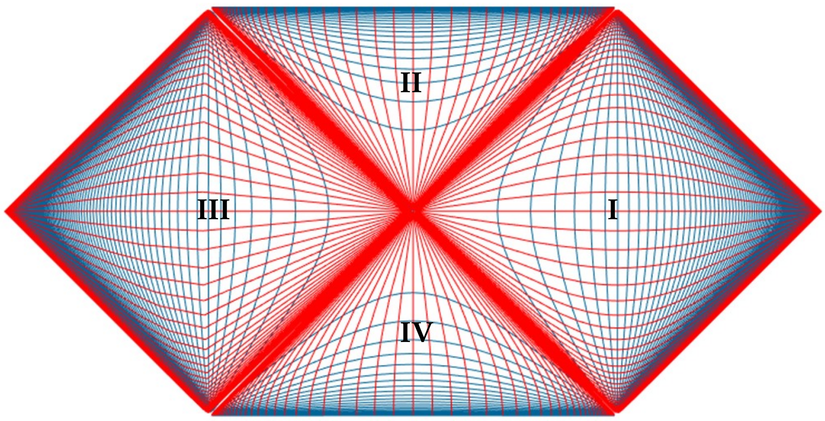

Figure 1: Extended Penrose diagram for the Schwarzschild metric in the (ρ, τ) plane. Red (resp. blue) lines represent hypersurfaces of constant t (resp. r).

With that, since light beams follow 45° trajectories in this diagram, and special relativity tells us that no particle can travel through spacetime faster than speed of light we can see that a particle crossing from region I to region II will never be able to return to region I. This censorship also applies to photons, the quantum of the electromagnetic field, which constitutes light. That is what drove its name of black hole.

For the mathematical part we refer to [CG18] and for the Penrose diagrams to [HE73], were a detailed and thorough explanation of different Penrose diagrams can be found, as well as an accurate explanation of Albert Einstein's theories of relativity. Finally, we present two cases as exemplifications. We begin with the simplest case which is the Minkowski spacetime and finally we present a not so usual case obtained when modifying a two dimensional model, developed by Roman Jackiw [Jac85] and Claudio Bunster (whose name was Claudio Teitelboim until 2005) [Tei83], and thus known as JT gravity [Ai21].

2 Semi-Riemannian Geometry

Semi-Riemannian geometry is the branch of differential geometry which studies semi-Riemannian manifolds. In turn, a semi-Riemannian manifold is a smooth manifold equipped with a metric tensor. As its name suggests, semi-Riemannian geometry is a generalisation of Riemannian geometry. For the latter, the metrics considered must be positive-definite. The former arises as a generalisation of Riemannian geometry after relaxing the condition of positive-definiteness on the metric. Thus, the only requirement for semi-Riemannian manifold's metrics is to be symmetric and non-degenerate.

The tangent space of a semi-Riemannian manifold is a semi-Euclidean vector space. Therefore, in broad terms, one can think of semi-Riemannian manifolds as being locally a semi-Euclidean space. Semi-Riemannian geometry thereby allows to generalise well-defined concepts of semi-Euclidean spaces, such as the angle between two vectors or the length of a curve, to semi-Riemannian manifolds, in the same way that differential geometry does to smooth manifolds.

As mentioned in the introduction, a special case of semi-Riemannian manifolds are Lorentzian manifolds for their applicability to describe spacetimes within A. Einstein's theories of relativity. We thus make special mention of them in this section, as well as providing the necessary differential geometry and tensor calculations tools to work in a mathematically rigorous way. For this section, we have mainly followed [O'N83] and [Sch08].

2.1 Smooth Manifolds

Definition 2.1. Let $V \subseteq \mathbb{R}^m$ be an open set. A function $f : V \to \mathbb{R}^n$ is said to be *smooth* if for every $p \in V$ its partial derivatives for all $k \geq 0$

$$\frac{\partial^k f}{\partial (x^1)^{\alpha_1} \dots \partial (x^m)^{\alpha_m}}(p),$$

where α_i are non-negative integrers such that $\alpha_1 + \cdots + \alpha_m = k$, exist and are continuous at p. Smooth functions are also called $\mathcal{C}^\infty$ functions.

Definition 2.2. An *n-dimensional chart* (U, φ) on a topological space M is an open set $U \subseteq M$ together with a homeomorphism $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$, where $\varphi(U)$ is an open set. If we write

$$\varphi(p) = (\varphi^1(p), \varphi^2(p), \dots, \varphi^n(p)) \subseteq \mathbb{R}^n \quad \text{for every } p \in M\,,$$

the functions $\varphi^1, \varphi^2, \dots, \varphi^n$ are called the *coordinate functions* of φ.

Definition 2.3. Two n-dimensional coordinate systems (U, φ) and (V, ψ) are said to *overlap smoothly* if the function $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$, and its inverse $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ are both smooth.

Definition 2.4. An *atlas* $\mathscr{A} = \{(U_i, \varphi_i)\}_{i \in I}$ of dimension n on a topological space M is a collection of $k = card(I)$ n-dimensional charts in M such that $M = \bigcup_{i \in I} U_i$ and any two given charts in $\mathscr{A}$ overlap smoothly.

An atlas $\mathscr{A}$ is said to be *maximal* if it contains every chart on M that overlaps smoothly with every chart in $\mathscr{A}$.

Note that equivalently to the definition of maximal atlas, we have that a maximal atlas on M is one which is not contained in any other different atlas on M. This will be useful for the proof of Lemma 2.6.

Definition 2.5. A *smooth manifold* or $\mathcal{C}^\infty$ *manifold* of dimension n is a topological space M (Hausdorff and second countable) of dimension n together with a maximal atlas.

From now on, when talking about a *manifold*, we will assume it is in fact a *smooth manifold*, unless noted otherwise. The *dimension* $n := \dim M$ of a manifold M is the dimension of its atlas and it is usually written as M^n.

Lemma 2.6. *Let M be a manifold. For every atlas $\mathscr{A} = \{(U_i, \varphi_i)\}_{i \in I}$ of M there exists a unique maximal atlas of M containing $\mathscr{A}$.*

Proof. Let $\mathscr{A} = \{(U_i, \varphi_i)\}_{i \in I}$ be an atlas of M. To prove the existence start by considering the set $\mathcal{A}$ of all charts in M that overlap smoothly with every chart in $\mathscr{A}$. The set $\mathcal{A}$ exists by its definition. Consider $(U, \varphi), (V, \psi) \in \mathcal{A}$. We want to see that $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ and its inverse $\psi \circ \varphi^{-1}$ are smooth. Take an arbitrary $p \in (U \cap V)$. Since $\mathscr{A}$ is an atlas, there exists a chart $(W, \phi) \in \mathscr{A}$ such that $p \in W$. The charts $(U, \varphi), (V, \psi)$ overlap smoothly with (W, ϕ), by the definition of $\mathcal{A}$. Therefore, we have in particular that $\varphi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are smooth in a neighborhood of $\phi(p)$ and $\psi(p)$, respectively. Now, since the composition of smooth maps is smooth we have that $\varphi \circ \phi^{-1} \circ \phi \circ \psi^{-1} = \varphi \circ \psi^{-1}$ is smooth in a neighborhood of $\psi(p)$. Analogously, we get that $\psi \circ \varphi^{-1}$ is also smooth in a neighborhood of $\varphi(p)$. Since we have chosen p arbitrarily, we have that both compositions are smooth and therefore, that $\mathcal{A}$ is an atlas.

Now we want to see that $\mathcal{A}$ is a maximal atlas. Suppose that $\mathcal{A}$ is contained in another atlas $\mathcal{B}$. Then $\mathscr{A}$ is also contained in $\mathcal{B}$ and therefore all charts in $\mathcal{B}$ overlap smoothly with all charts in $\mathscr{A}$. Thus, by definition of $\mathcal{A}$, we have $\mathcal{A} = \mathcal{B}$.

Finally, to see that $\mathcal{A}$ is unique, suppose that there exists another maximal atlas $\mathcal{C}$ containing $\mathscr{A}$. Then every chart in $\mathcal{C}$ overlaps smoothly with every chart in $\mathscr{A}$ and therefore $\mathcal{C} \subseteq \mathcal{A}$. Now since $\mathcal{C}$ is maximal we have that $\mathcal{C} = \mathcal{A}$. $\square$

Thanks to Lemma 2.6 a manifold can be properly determined by giving a topological space and an atlas which may not be necessarily maximal.

Examples 2.7. Let $n \geq 1$ be an natural number.

1. The space $\mathbb{R}^n$ with its identity map $id_{\mathbb{R}^n}$ as atlas is a manifold. The manifold $M = (\mathbb{R}^n, id_{\mathbb{R}^n})$ is also called the *Euclidean n-space*.

2. If M is a manifold, then any open $V \subseteq M$ is a manifold. If $\{(U_i, \varphi_i)\}_{i \in I}$ is an atlas for M, we just need to consider $\{(U_i \cap V, \varphi_i|_{U_i \cap V})\}_{i \in I}$ as an atlas for V.

3. The *graph of a smooth function* $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, defined as

$$G(f) = \{(x, f(x)) \in U \times \mathbb{R}^m\},$$

   is a manifold together with the one-chart atlas $(G(f), \varphi)$, where

$$\begin{aligned} \varphi : G(f) &\longrightarrow U \\ (x, f(x)) &\mapsto x\,. \end{aligned}$$

4. Let us consider the *n-sphere*

$$S^n = \{x = (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} x_i^2 = 1\}.$$

   Define charts

$$\begin{array}{cc} U_N = \{x \in S^n | x_{n+1} \geq 0\} & U_S = \{x \in S^n | x_{n+1} \leq 0\} \\ \varphi_N : U_N \longrightarrow \mathbb{R}^n & \varphi_S : U_S \longrightarrow \mathbb{R}^n \\ (x_1, x_2, \ldots, x_{n+1}) \mapsto (x_1, x_2, \ldots, x_n) & (x_1, x_2, \ldots, x_{n+1}) \mapsto (x_1, x_2, \ldots, x_n). \end{array}$$

   S^n is an n-dimensional manifold with the atlas $\mathscr{A} = \{(U_N, \varphi_N), (U_S, \varphi_S)\}$.

5. If M and N are manifolds with $\{U_i, \varphi_i\}_{i \in I}$ and $\{V_j, \psi_j\}_{j \in J}$ as their respective atlases, then the product $M \times N$ is a manifold with the atlas

$$\{(U_i \times V_j, \varphi_i \times \psi_j : (U_i \times V_j) \to \mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m})\}_{(i,j) \in I \times J}\,.$$

Definition 2.8. We say that a *function* $f : M^n \to \mathbb{R}$ is *smooth* if for every chart $\varphi : U \subseteq M^n \to \mathbb{R}^n$ the function $f \circ \varphi^{-}1 : \varphi(U) \to \mathbb{R}$ is smooth.

Definition 2.9. Let M and N be two manifolds. A *map* $g : M \to N$ is said to be *smooth* if and only if for every pair of charts (U, φ) of M and (V, ψ) of N, the function $\psi \circ g \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth.

Remark 2.10. Note that all smooth maps are continuous but not all continuous maps are smooth. For instance, consider the map

$$\begin{aligned} g : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto g(x) = \begin{cases} 3x & \text{if } x \leq 0 \\ 2x & \text{if } x > 0\,, \end{cases} \end{aligned}$$

which is continuous at all $\mathbb{R}$ but is not smooth at $x = 0$.

Since smoothness is a local property, it will be useful to note that for any two smooth charts of a manifold M

$$\varphi : U \subseteq M \to \mathbb{R}^n \quad \text{and} \quad \psi : V \subseteq M \to \mathbb{R}^n\,,$$

if $\varphi(p) = \psi(p)$ for every $p \in U \cap V$, then $\phi : U \cup V \subseteq M \to \mathbb{R}^n$ is smooth.

Definition 2.11. A smooth map $g : M \to N$ is a *diffeomorphism* if it has a smooth inverse map $g^{-1} : N \to M$. If such g exists, M and N are said to be *diffeomorphic*.

Remark 2.12. Note that all diffeomorphisms are homeomorphisms but not all homeomorphisms are diffeomorphisms. For instance, $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is a smooth homeomorphism, yet it is not a diffeomorphism, since it's inverse $f^{-1}(x) = x^{1/3}$ is not smooth.

2.2 Calculus on Manifolds

Definition 2.13. Given a manifold M, denote by $\mathcal{F}(M)$ the *set of all smooth real-valued functions on* M:

$$\mathcal{F}(M) := \{f : M \to \mathbb{R} \mid f \in \mathcal{C}^\infty\}.$$

Remark 2.14. The set $\mathcal{F}(M)$ has the structure of a real vector space with the operations

$$(f+g)(p) := f(p) + g(p) \quad \text{and} \quad (\lambda f)(p) := \lambda \cdot f(p)\,,$$

for $f, g \in \mathcal{F}(M)$ and $\lambda \in \mathbb{R}$. It is also a commutative ring with the multiplication

$$(fg)(p) := f(p) \cdot g(p).$$

Definition 2.15. Let M be a manifold and let $p \in M$. A *tangent vector to* M *at* p is a real-valued function $v : \mathcal{F}(M) \to \mathbb{R}$ such that for all $a, b \in \mathbb{R}$ and all $f, g \in \mathcal{F}(M)$:

(i) ($\mathbb{R}$-linear) $v(af + bg) = av(f) + bv(g)$,

(ii) (Leibniz rule) $v(fg) = v(f)g(p) + f(p)v(g)$.

The set of all tangent vectors of M at p is called the *tangent space* of M at p and it is denoted by T_pM.

Remark 2.16. Note that T_pM also has the structure of a vector space over $\mathbb{R}$, with the operations

$$(v+w)(f) := v(f) + w(f) \quad \text{and} \quad (av)(f) := a \cdot v(f)$$

for every $v, w \in T_pM$, every $f \in \mathcal{F}(M)$ and every $a \in \mathcal{R}$.

Let $\varphi = (x^1, x^2, \ldots, x^n) : U \subseteq M^n \to \mathbb{R}^n$ be a chart of M^n such that $p \in U$. Let us consider the function $\partial_i|_p : \mathcal{F}(M) \to \mathbb{R}$ defined for every $f \in \mathcal{F}(M)$ by

$$\partial_i|_p(f) \equiv \frac{\partial f}{\partial x^i}(p) := \frac{\partial(f \circ \varphi^{-1})}{\partial u^i}(\varphi(p)), \quad i \in \{1, 2, \ldots, n\},$$

where $\{u^i\}_{i=1}^n$ are the natural coordinate functions of $\mathbb{R}^n$. The function $\partial_i|_p$, which sends each function $f \in \mathcal{F}(M)$ to $\frac{\partial f}{\partial x^i}(p)$, verifies Definition 2.15 and is therefore a tangent vector of M at p. In fact, we have:

Theorem 2.17 (Basis Theorem, see Theorem 1.12 of [O'N83]). *Let (U, φ) be a chart on a manifold M of dimension n and $p \in U$. Then, it's respective tangent vectors $\{\partial_i|_p\}_{i=1}^n$ form a basis of T_pM and $dim T_pM = dim M$.*

Once we have taken a chart at a point p, the above theorem allows to write every vector $v \in T_pM$ as

$$v = \sum_i v(x^i)\partial_i|_p = v(x^i)\partial_i|_p,$$

where for the second identity we introduce the A. Einstein summation convention.

The same way we can approximate a smooth manifold M near each point $p \in M$ by its tangent vector space T_pM, it will be useful too to approximate smooth maps like $\phi : M \to N$ to linear transformations from M to N near each point $p \in M$. To do so, we will begin defining the differential map of ϕ at p.

Definition 2.18. Let $\phi : M \to N$ be a smooth map between two manifolds. For every $p \in M$ we define the *differential map of ϕ at p* by

$$\begin{aligned} d\phi_p : T_pM &\longrightarrow T_{\phi(p)}N \\ v &\longmapsto d\phi_p(v) := v_\phi\,, \end{aligned}$$

where $v_\phi(f) = v(f \circ \phi)$, for every $v \in T_pM$ and every $f \in \mathcal{F}(N)$.

Remark 2.19. It follows from Definition 2.15 and Remark 2.16 that the differential map $d\phi_p$ of a smooth map ϕ at a point $p \in M$ is a linear map between the vector spaces T_pM and $T_{\phi(p)}N$.

Definition 2.20. The *rank* of a smooth map $\phi : M \to N$ at a point $p \in M$ is the rank of the differential map $d\phi_p$. Since $d\phi_p$ is a linear map between vector spaces, it has a well defined rank which is the dimension of its image, and then

$$\mathrm{rank}_p(\phi) := \mathrm{rank}(d\phi_p) = \dim(\mathrm{Im}(d\phi_p).$$

A smooth map of *maximal rank* ϕ is a smooth map with $\mathrm{rank}_p(\phi) = \dim(T_{\phi(p)}N)$ for every $p \in M$.

Lemma 2.21. *Let $\phi : M^m \to N^n$ be a smooth map. Take charts (U, φ), (V, ψ) of M^m and N^n, respectively, such that $p \in U \subseteq M^m$ and $\phi(p) \in V \subseteq N^n$, then*

$$d\phi_p(\partial_j|_p) = \sum_{i=1}^{n} \frac{\partial(y^i \circ \phi)}{\partial x^j} \partial_i|_{\phi(p)}, \quad \text{for every } j \in \{1, 2, \ldots, m\},$$

where $\varphi = (x^1, x^2, \ldots, x^m)$, $\psi = (y^1, y^2, \ldots, y^n)$, and

$$\partial_j|_p = \left(\frac{\partial}{\partial x^j}\right)\bigg|_p, \qquad \partial_i|_{\phi(p)} = \left(\frac{\partial}{\partial y^i}\right)\bigg|_{\phi(p)}.$$

Proof. Let $v := d\phi_p(\partial_j|_p) \in M_{\phi(p)}N^n$. By Theorem 2.17 we can write

$$v = \sum_{i=1}^{n} v(y^i)\partial_i|_{\phi(p)} = \sum_{i=1}^{n} d\phi_p(\partial_j|_p)(y^i)\partial_i|_{\phi(p)} = \sum_{i=1}^{n} \frac{\partial(y^i \circ \phi)}{\partial x^j}\partial_i|_{\phi(p)},$$

where in the last equality we have used the definition of differential map (Definition 2.18). $\square$

Definition 2.22. Let $\{\frac{\partial}{\partial x^j}\}_{j=1}^m$ and $\{\frac{\partial}{\partial y^i}\}_{i=1}^n$ be the coordinate basis for T_pM^m and $T_{\phi(p)}N^n$, respectively. The matrix of $d\phi_p$ with respect to these coordinate basis,

$$\left(\frac{\partial(y^i \circ \phi)}{\partial x^j}(p)\right)_{1 \le i \le n, 1 \le j \le m},$$

is called the *Jacobian matrix* of ϕ at p relative to φ and ψ in Lemma 2.21.

Lemma 2.23. *Let $\phi : M \to N$ and $\psi : N \to P$ be smooth maps. Then,*

$$d(\psi \circ \phi)|_p = d\psi_{\phi(p)} \circ d\phi_p, \quad \text{for every } p \in M.$$

Proof. Given $u \in T_pM$ and $f \in \mathcal{F}(P)$, we have

$$d(\psi \circ \phi)|_p(u)(f) = u(f \circ \psi \circ \phi) = d\phi|_p(u)(f \circ \psi) = (d\psi|_{\phi(p)} d\phi|_p(u))(f). \qquad \square$$

From now on, we may omit the subscript p whenever it is clear from the context.

Definition 2.24. A manifold S is a *submanifold* of a manifold M if S is a topological subspace of M, the inclusion map $i : S \to M$ is smooth and at each point $p \in S$ its differential map di is injective.

Example 2.25. Every open subset $N \subseteq M$ is a submanifold of M.

Theorem 2.26. *Let $\phi : M \to N$ be a smooth map between manifolds. Then, the differential map $d\phi_p : T_pM \to T_{\phi(p)}N$ is a linear isomorphism if and only if there is a neighborhood U of p such that $\phi|_U : U \to \phi(U)$ is a diffeomorphism.*

Proof. Let (U, φ) be a chart on M with $p \in U$ and (V, ψ) a chart on N with $\phi(p) \in V$. We assume $\varphi : U \to \mathbb{R}^n$ and $\psi : V \to \mathbb{R}^n$. Then, this implication is hold straightforward by applying the Inverse Function Theorem to the function $f := \psi \circ \phi \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$.

The converse is straightforward. Indeed, since a diffeomorphism is a smooth map with smooth inverse, if $\phi|_U$ is a diffeomorphism, its differential at any $p \in U$ will be in particular a linear isomorphism. $\square$

Definition 2.27. A *local diffeomorphism* is a smooth map $\phi : M \to N$ such that $d\phi_p$ is a linear isomorphism for all $p \in M$.

Note that if a local diffeomorphism is also bijective, then it is a diffeomorphism.

Having now defined the basics of manifolds we can consider some specific maps called curves. These maps play a key role in physics because they can represent trajectories of motion.

Remark 2.28. The open interval I is also a submanifold of $\mathbb{R}$ with the chart (I, id_I).

Definition 2.29. A *curve* on a manifold M is a smooth map $\gamma : I \to M$ where $I \subseteq \mathbb{R}$ is an open interval.

Definition 2.30. Let $\gamma : I \to M$ be a curve. The *velocity vector* $\gamma'(t)$ of γ at $t \in I$ is defined as

$$\gamma'(t) := d\gamma_t \left(\left. \frac{d}{du} \right|_t \right) \in T_{\gamma(t)}M,$$

where u is the coordinate function of the manifold I which, in this case, is the identity.

Remark 2.31. Using Definition 2.18 we have that the tangent vector $\gamma'(t)$ applied to a function $f \in \mathcal{F}(M)$ can be written as

$$\gamma'(t)f = \left(\frac{d}{du}\bigg|_t \right)(f \circ \gamma) = \frac{d(f \circ \gamma)}{du}(t).$$

Letting $x^1, \dots, x^n$ be a chart on M and using Theorem 2.17, we get

$$\gamma'(t) = \sum_{i=1}^{n} \frac{d(x^i \circ \gamma)}{du}(t)\, \partial_i|_{\alpha(t)} \,.$$

Definition 2.32. Let $\gamma : I \to M$ be a curve and $s : J \to I$ be a smooth function, where $J \subseteq \mathbb{R}$ is an open interval. Then $\delta := \gamma(s) : J \to M$ is a curve and it is called the *reparametrisation* of γ with respect to s.

Remark 2.33. Using the chain rule we get that the *velocity vector* of δ is:

$$\delta' = \frac{ds}{du}(r) \cdot \gamma'(s(r))\,, \quad \text{for every } r \in J.$$

Then, considering now the map $\phi : M \to N$ and applying again the chain rule we get

$$d\phi(\gamma'(t)) = (\phi \circ \gamma)'(t)\,, \quad \text{for every } t \in I.$$

Definition 2.34. A curve $\gamma : I \to M$ is said to be *regular* if and only if $\gamma'(t) \neq 0$ for all $t \in I$.

From now on, since we can always consider the parametrisation $s(t) = t + c$ for all $c \in \mathbb{R}$, we will assume that $\gamma(0) \in M$ exists and is well defined. Such parametrisation does not change the velocity vector of the curve.

Definition 2.35. Let M be a manifold of dimension n. Define a map

$$TM := \{(p, v) | p \in M, v \in T_pM\} \xrightarrow{\pi} M$$

by letting $\pi(p, v) = p$. Let us define a smooth manifold structure on TM as follows: given a chart (U, φ) of M with coordinates $\varphi = (x^1, \dots, x^n)$, let

$$\begin{aligned} \psi_U : \pi^{-1}(U) &\longrightarrow \mathbb{R}^{2n} \\ (p, v) &\longmapsto (p^1, \dots, p^n, v^1, \dots, v^n), \end{aligned}$$

where $\varphi(p) = (p_1, \dots, p_n)$ and $v = \sum_{i=1}^n v_i \left(\frac{\partial}{\partial x^i} \right)$. The topology of TM is generated by the preimages of ψ_U for all open sets of $\mathbb{R}^{2n}$ and all charts of M. The collection $\mathscr{A} = \{(\pi^{-1}(U_i), \psi_{U_i}\}_{i \in I}$ is an atlas of TM. With these definitions, the map π is smooth and is called the *tangent bundle of M*.

Definition 2.36. A *vector field* X on a manifold M is a smooth map

$$\begin{aligned} X : M &\longrightarrow TM \\ p &\longmapsto (p, X_p). \end{aligned}$$

The set $\mathcal{X}(M)$ of all smooth vector fields on M is a module over the ring $\mathcal{F}(M)$ by considering the operations, for every $V, W \in \mathcal{X}(M)$ and every $f \in \mathcal{F}(M)$,

$$\begin{aligned} \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M) \quad &; \quad (V+W)_p := V_p + W_p \quad \text{and} \\ \mathcal{F}(M) \times \mathcal{X}(M) \to \mathcal{X}(M) \quad &; \quad (fV)_p := f(p) V_p \,. \end{aligned}$$

Definition 2.37. Let $\varphi = (x^1, x^2, \dots, x^n)$ be a chart on $U \subseteq M$. Then, for every $i \in \{1, 2, \dots, n\}$, the vector field $\partial_i : U \to TM$ sending each $p \in M$ to $\partial_i|_p$ is called the *coordinate vector field* of φ. Since $\partial_i(f) = \frac{\partial f}{\partial x^i}$ and f is smooth, we see that these coordinate vector fields are smooth.

From Theorem 2.17 we directly obtain that every vector field $V \in \mathcal{X}(M)$ can be written as

$$V = \sum_i V(x^i) \partial_i.$$

Example 2.38. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued smooth function. Consider $\mathbb{R}^n$ as a manifold. The *gradient* of f at a point $p \in \mathbb{R}^n$

$$\begin{aligned} \nabla f : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ p &\longmapsto \left(\frac{\partial f}{\partial x^1}(p), \dots, \frac{\partial f}{\partial x^n}(p) \right) \end{aligned}$$

is a vector field on $\mathbb{R}^n$.

Definition 2.39. The *Lie bracket* $[X, Y]$ of two vector fields $X, Y : M \to TM$ is the only vector field such that

$$[X, Y](f) = X(Yf) - Y(Xf).$$

If we take for instance a chart (U, φ) with $\varphi = (x^1, \dots, x^n)$ so that, using again the A. Einstein summation convention,

$$X = X^i \partial_i \,, \quad Y = Y^i \partial_i \,,$$

where X^i and Y^i are both smooth functions, we finally get

$$[X, Y] = \left(X^j \partial_j Y^i - Y^j \partial_j X^i \right) \partial_i.$$

2.3 Tensor Fields

To explain the basic concepts of semi-Riemannian manifolds, we need to introduce first the concept of *tensor*.

Definition 2.40. Let V be a module over a ring K. For integers $r, s \geq 0$. A *tensor of type* (r, s) over V is a K-multilinear function

$$A : \overbrace{V^* \times \ldots \times V^*}^{r)} \times \overbrace{V \times \ldots \times V}^{s)} \longrightarrow K,$$

where V^* is the dual module of V.

Remark 2.41. As the elements of V give an element of K when applied to elements of V^*, a tensor of type (r, s) can also be thought as a K-multilinear function

$$A : \overbrace{V \times \ldots \times V}^{s)} \longrightarrow \overbrace{V \times \ldots \times V}^{r)}.$$

From now on we will denote by $\mathcal{I}^r_s(M)$ the set of all tensor fields of type (r, s) on M. For instance, when $r = s = 0$, we have $\mathcal{I}^0_0(M) = \mathcal{F}(M)$.

Example 2.42. The differential df of a smooth function $f : \mathbb{R} \to \mathbb{R}$ is a one-form , and therefore is a $(0, 1)$ tensor too.

Definition 2.43. A *tensor field* A on a manifold M is a tensor over the $\mathcal{F}(M)$-module $\mathcal{X}(M)$. If A is of type (r, s), it is an $\mathcal{F}(M)$-multilinear function

$$A : \mathcal{X}^*(M)^r \times \mathcal{X}(M)^s \longrightarrow \mathcal{F}(M).$$

Then, A is a mathematical object that after being given r one-forms and s vector fields returns a real-valued function $f \in \mathcal{F}(M)$. We say that A has r *contravariant* slots and s *covariant* slots.

Remark 2.44. A tensor $A \in \mathcal{I}^r_s(M)$ is said to transform as a tensor if its components satisfy

$$A^{i'_1,\ldots,i'_r}_{j'_1,\ldots,j'_s} = \Lambda^{i'_1}_{i_1} \ldots \Lambda^{i'_r}_{i_r} \Lambda^{j_1}_{j'_1} \ldots \Lambda^{j_s}_{j'_s} A^{i_1,\ldots,i_r}_{j_1,\ldots,j_s},$$

where $\Lambda^i_{i'}$ are the change of basis matrices and $\Lambda^{i'}_i$ are the inverse change of basis matrices, and therefore, the transposed ones.

Considering the basis

$$\{\partial_i\}^n_{i=1} := \left\{\frac{\partial}{\partial_i}\right\}^n_{i=1}$$

as a set of vectors, one can define its dual basis $\{dx^i\}_{i=1}^n$ as a set of one-forms such that $dx^i(\partial_j) = \frac{\partial x^i}{\partial x^j} = \delta^i_j$. With this definition, every one-form ω can be written as $\omega = \omega_i dx^i$ and every vector v can be written as $v = v^i \partial_i$. In particular, the differential of a smooth function is $df = \frac{\partial f}{\partial x^i} dx^i$, hence $(df)_i = \frac{\partial f}{\partial x^i}$.

Definition 2.45. Let a, b, r, s be integers such that $1 \leq a \leq r$ and $1 \leq b \leq s$. The *contraction* $\mathcal{C}^a_b : \mathcal{I}^r_s(M) \to \mathcal{I}^{r-1}_{s-1}(M)$ is the operation defined as follows: given

$$A = A^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s} \in \mathcal{I}^r_s(M)\,,$$

where each index is summed from 1 to n and with components

$$A^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} = A(dx^{i_1}, \ldots, dx^{i_r}, \partial_{j_1}, \ldots, \partial_{j_s})\,,$$

then

$$\begin{aligned} \mathcal{C}^a_b A := A^{i_1,\ldots,i_{a-1},k,i_{a+1},\ldots,i_r}_{j_1,\ldots,j_{b-1},k,j_{b+1},\ldots,j_s} & \partial_{i_1} \otimes \cdots \otimes \partial_{i_{a-1}} \otimes \partial_{i_{a+1}} \otimes \cdots \otimes \partial_{i_r} \\ & \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_{b-1}} \otimes dx^{j_{b+1}} \otimes \cdots \otimes dx^{j_s} \in \mathcal{I}^{r-1}_{s-1}(M)\,. \end{aligned}$$

Example 2.46. Consider the tensor $A \in \mathcal{I}^3_2(M)$, where M is an n-dimensional manifold. Its components would be $A^{ijk}{}_{lm}$ with i, j, k, l, m ranging from 1 to n. Then the contraction $\mathcal{C}^2_1$ of A is given by

$$\mathcal{C}^2_1 A = A^{iak}{}_{am} \partial_i \otimes \partial_k \otimes dx^m \in \mathcal{I}^2_1(M)\,.$$

Definition 2.47. Let r, s be integers. A *tensor derivation* $\mathcal{D}$ on a smooth manifold M is a set of $\mathbb{R}$-linear functions

$$\mathcal{D} := \{\mathcal{D}^r_s : \mathcal{I}^r_s \to \mathcal{I}^r_s\}_{r,s \geq 0}$$

such that, for every pair of tensors A and B:

i) $\mathcal{D}(A \otimes B) = (\mathcal{D}A) \otimes B + A \otimes (\mathcal{D}B)$,

ii) $\mathcal{D}(\mathcal{C}A) = \mathcal{C}(\mathcal{D}A)$, for every contraction $\mathcal{C}$.

Definition 2.48. Let $X \in \mathcal{X}(M)$. The *Lie derivative* L_X relative to X is the tensor derivation such that

$$\begin{aligned} L_X(f) &= Xf && \text{for all } f \in \mathcal{F}(M), \\ L_X(Y) &= [X, Y] && \text{for all } Y \in \mathcal{X}(M), \end{aligned}$$

where $[X, Y]$ denotes the Lie bracket of the vector fields X and Y.

The Lie derivative can be also visualised as a directional derivative in the direction of the vector field X. Actually, the Lie derivative can be generalised so that it can be applied to any type of tensor, $L_X : \mathcal{I}^r_s \to \mathcal{I}^r_s$ (see, for instance, [Wal84] or [Fec06]). It is thus extremely important in theoretical physics.

Finally, let us introduce the concept of connection on M, which will be of relevant importance in Section 2.4. The connection also allows to define the covariant derivative of a vector field for this connection with respect to another vector field.

Definition 2.49. A *connection* D on a smooth manifold M is a function

$$\begin{aligned} D : \mathcal{X}(M) \times \mathcal{X}(M) &\longrightarrow \mathcal{X}(M) \\ (X, Y) &\longmapsto D_X Y \end{aligned}$$

such that, for every $X, Y, Z \in \mathcal{X}(M)$ and every $f, g \in \mathcal{F}(M)$,

i) ($\mathcal{F}(M)$-linear in the first argument) $D_{fX+gY}Z = fD_X Z + gD_Y Z$,

ii) ($\mathbb{R}$-linear in the second argument) $D_X(Y + Z) = D_X Y + D_X Z$,

iii) (Leibniz Rule) $D_X(fY) = (Xf)Y + fD_X Y$.

$D_X Y$ is called the *covariant derivative* of Y with respect to X for the connection D.

2.4 Semi-Riemannian Manifolds

Definition 2.50. The *index* ν of a symmetric bilinear form g on a real vector space V is the largest integer which coincides with the dimension of a subspace $W \subseteq V$ on which $g|_W$ is negative definite.

Remark 2.51. The index ν is always $0 \leq \nu \leq \dim V$ and g is non-negative definite if and only if $\nu = 0$.

Definition 2.52. A *metric tensor* g on a smooth manifold M is a symmetric non-degenerate $(0, 2)$ tensor field on M of constant index ν.

The metric tensor g assigns to every $p \in M$ a non-degenerate and symmetric bilinear form on the tangent space T_pM:

$$g_p : T_pM \times T_pM \to \mathbb{R}.$$

Definition 2.53. A *semi-Riemannian manifold* is a pair (M, g) where M is a smooth manifold of dimension n and g is a metric tensor.

Consider (U, φ) as a chart on $U \subseteq M$ with $\varphi = (x^1, \dots, x^n) : U \to \varphi(U) \subseteq \mathcal{R}^n$. For every $p \in U$ the bilinear form g_p can be written as

$$g_p(X, Y) = g_{ij}(p) X^i Y^j.$$

where $X = X^i \partial_i$, $Y = Y^j \partial_j \in T_p M$ and $\partial_i := \frac{\partial}{\partial x^i}$ for every $i \in \{1, \dots, n\}$. Now, for simplicity, as it happened with manifolds, we shall denote a semi-Riemannian manifold just by M, even though different smooth tensor fields g on the same manifold define different semi-Riemannian manifolds.

In Definition 2.52, g_p being non-degenerate means that if $g_p(X, Y) = 0$ for every $Y \in T_p M$ then $X = 0$, or equivalently, that the kernel of g_p is $0 \in T_p M$. Moreover, the matrix

$$(g_{ij}(p))$$

is, by assumption, non-degenerate and symmetric, for every $p \in U$, which means that

$$det(g_{ij}(p)) \neq 0 \quad \text{and} \quad (g_{ij}(p))^T = (g_{ij}(p)).$$

Even more, since g is smooth, the matrix $(g_{ij}(p))$ depends smoothly on p. This means that the coefficients $g_{ij} = g_{ij}(x)$ are smooth functions.

For *Riemannian manifolds* the matrix $(g_{ij}(p))$ is required to be positive definite, thus $g_{ij}(p) X^i Y^j \geq 0$, whereas for the more general *semi-Riemannian manifolds* this condition is not required to be fulfilled. This feature is useful to distinguish them.

Example 2.54. We denote $\mathbb{R}^{p,q} = (\mathbb{R}^{p,q}, g^{p,q})$, for $p, q \in \mathbb{N}$, the semi-Riemannian manifold of dimension $p + q$, with

$$g^{p,q}(X, Y) := \sum_{i=1}^{p} X^i Y^i - \sum_{i=p+1}^{p+q} X^i Y^i.$$

Therefore,

$$(g_{ij}) = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} = \text{diag}(\overbrace{1, \dots, 1}^{p)}, \overbrace{-1, \dots, -1}^{q)}).$$

With this notation, all $\mathbb{R}^{p,q}$ with p, q natural numbers are semi-Riemannian manifolds. For instance, we have that $\mathbb{R}^{2,0}$ is the *Euclidean plane* and $\mathbb{R}^{3,1}$ (or $\mathbb{R}^{1,3}$) is the *Minkowski space*.

An also relevant example of semi-Riemannian manifolds are the Lorentzian manifolds. They are interesting for they are used to describe spacetime within A. Einstein's theories of relativity.

Definition 2.55. *Lorentzian manifolds* are semi-Riemannian manifolds with $\nu = 1$ and dimension $n \geq 2$.

Let us now introduce the Killing vector fields, whose flows are continuous isometries of the manifold. In particular, moving each point of an object in the direction of the Killing vector will not change the distances of the object. Moreover, in general relativity, Killing vector fields allow to define invariants and conservation laws.

Definition 2.56. A vector field $X \in \mathcal{X}(M)$ on a semi-Riemannian manifold (M, g) is a *Killing vector field* if the Lie derivative with respect to X of the metric g vanishes,

$$\mathcal{L}_X g = 0.$$

This equation is called the *Killing equation*.

The following theorem states in particular the existence of a connection for every semi-Riemannian manifold. Specifically, it guarantees the existence and uniqueness, for semi-Riemannian manifolds, of a specific type of connections, which we will denote as *Levi-Civita connection* in Definition 2.66.

Theorem 2.57 (see Theorem 3.11 of [O'N83])**.** *On a semi-Riemannian manifold M there exists a unique connection D such that, for every $X, Y, Z \in \mathcal{X}(M)$,*

i) $[X, Y] = D_X Y - D_Y X,$

ii) $Xg(Y, Z) = g(D_X Y, Z) + g(Y, D_X Z),$

with the corresponding metric g of M. This connection is characterised by the Koszul formula

$$\begin{aligned} 2g(D_Y Z, X) = {} & Yg(Z, X) + Zg(X, Y) - Xg(Y, Z) - g(Y, [Z, X]) \\ & + g(Z, [X, Y]) + g(X, [Y, Z]). \end{aligned}$$

Proof. We will give a proof for Koszul formula and for the uniqueness of such a connection. For the existence of this connection we will give a sketch of the proof.

A direct way to prove the Koszul formula from the existence of such a connection D is to apply the condition $ii)$ of the theorem to the first three terms of the formula and the condition $i)$ to the last three terms. Ten of the resulting twelve terms cancel in pairs and the only term which is left is $2g(D_Y Z, X)$, using that the scalar product is commutative.

To prove the uniqueness of this connection we will use that for a scalar product if $g(X, Z) = g(Y, Z)$ for every $Z \in \mathcal{X}(M)$, then $X = Y$. Considering now both this property and the proven Koszul formula we directly get that D is unique. Any other $\bar{D}$ satisfying $i)$ and $ii)$ would also hold Koszul formula and therefore, by the mentioned property of the scalar product, we would have that $D = \bar{D}$.

The idea to prove the existence of this connection is to prove first that the Koszul formula is well defined as a one-form and then check that this one-form satisfies the three conditions of Definition 2.49 and the two conditions of Theorem 2.57. Given $X, Y \in \mathcal{X}(M)$, consider the one form

$$\begin{aligned} F(X,Y,\cdot) : \mathcal{X}(M) &\to \mathbb{R} \\ Z &\mapsto Yg(Z,X) + Zg(X,Y) - Xg(Y,Z) \\ &\quad - g(Y,[Z,X]) + g(Z,[X,Y]) + g(X,[Y,Z]). \end{aligned}$$

Since $F(X,Y,\cdot)$ is $\mathcal{F}(M)$-linear, it is a one-form. It then follows that there exists a unique vector field, namely $D_Y Z$, such that $2g(D_Y Z, X) = F(X,Y,Z)$ for every $X \in \mathcal{X}(M)$ (see for instance Proposition 10 of [O'N83], which is used to prove it). Having now seen that the Koszul formula holds, it can be applied, together with the property $[fZ,X] = -(Xf)Z + f[Z,X]$ [O'N83, pp. 45–46], to proove $i), ii)$ and $iii)$ of Definition 2.49 and $i)$ and $ii)$ of Theorem 2.57. □

From now on, we will let M be a semi-Riemannian manifold.

Definition 2.58. Given a coordinate basis of an open $U \subseteq M$, the *Christoffel symbols* Γ^i_{jl} are real-valued functions on U such that

$$D_{\partial_i}(\partial_j) = \sum_{k=1}^{n} \Gamma^k_{ji}\partial_K, \quad 1 \leq i,j \leq n,$$

where the order of the indices on Γ is not arbitrary [Sch80, p. 205].

Remark 2.59. Naming for simplicity $\tilde{\omega}^j := dx^j$ the elements of the dual basis such that $\tilde{\omega}^j(\partial_i) = \delta^j_i$, we get

$$\Gamma^k_{ji} = \tilde{\omega}^k D_{\partial_i}(\partial_j).$$

Remark 2.60. Since D is not a tensor, Γ^k_{ji} do not need to obey the usual tensor transformation rule under change of coordinates.

Even though the Christoffel symbols Γ^i_{jk} of M are not the components of any tensor, we have the following property:

Proposition 2.61. *The combination $T^i_{jk} := 2\Gamma^i_{[jk]} = \Gamma^i_{jk} - \Gamma^i_{kj}$ is a tensor.*

Proof. Let $\Lambda^i_{i'}$ be the change of basis matrices and $\Lambda^{i'}_i$ be the inverse change of basis matrices, and therefore, the transposed ones, so that $\partial'_j = \Lambda^j_{j'}\partial_j$, for example. Using Remark 2.59 one can write

$$\begin{aligned} \Gamma^{i'}_{j'k'} &= \tilde{\omega}^{i'}\left(D_{\partial_{k'}}\partial_{j'}\right) = \Lambda^{i'}_i\tilde{\omega}^i\left(D_{\partial_{k'}}\left(\Lambda^j_{j'}\partial_j\right)\right) \\ &= \Lambda^{i'}_i\Lambda^j_{j'}\tilde{\omega}^i\left(D_{\partial_{k'}}\left(\partial_j\right)\right) + \Lambda^{i'}_i\tilde{\omega}^i\left(\left(D_{\partial_{k'}}\Lambda^j_{j'}\right)\partial_j\right) = \Lambda^{i'}_i\Lambda^j_{j'}\Lambda^k_{k'}\Gamma^i_{jk} + \Lambda^{i'}_i\left(D_{\partial_{k'}}\Lambda^i_{j'}\right). \end{aligned}$$

Since the second term does not always vanish we conclude that the Christoffel symbols do not transform as tensors in general. However, if we let $\{\partial_i\}$ to be a coordinate basis, which are the ones where all the mutual Lie derivatives of its elements vanish and are also called holonomic basis, $\Lambda^i_{j'} = \frac{\partial x^i}{\partial x^{j'}}$ and the last term

$$\Lambda^{i'}_i \left(D_{\partial_{k'}} \Lambda^i_{j'} \right) = \Lambda^{i'}_i \frac{\partial^2 x^i}{\partial x^{k'} \partial x^{j'}}$$

is symmetric with respect to k', j' due to Schwarz Theorem on the commutativity of second partial derivatives. Therefore, we obtain that $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$ transforms as a tensor because the last term above is cancelled and does not contribute. □

Having defined the Christoffel symbols of M and a connection on M (cf. Definition 2.49), it is straightforward and useful to define the torsion tensor of a manifold (see, for instance, [KF18, Sch80]).

Definition 2.62. The *torsion tensor* of a connection D on M is the tensor T of type $(1,2)$ defined as

$$\begin{aligned} T : \mathcal{X}(M) \times \mathcal{X}(M) &\longrightarrow \mathcal{X}(M) \\ (X,Y) &\longmapsto T(X,Y) = D_X Y - D_Y X - [X,Y]. \end{aligned}$$

Remark 2.63. When considering a coordinate basis the components of the torsion tensor coincide with the tensor introduced in Proposition 2.61. This can be found in [HE73].

Definition 2.64. A *symmetric connection* D on M is a connection where the torsion tensor $T(X,Y)$ equals $0 \in \mathcal{X}(M)$ for every $X, Y \in \mathcal{X}(M)$.

For the sake of convenience, we now introduce the notation $D_{\partial_i} := D_i$.

Proposition 2.65. *In a coordinate basis a connection is symmetric if and only if* $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Proof. In a coordinate basis $[\partial_i, \partial_j] = 0$. If the connection is symmetric, in particular we will have

$$T(\partial_i, \partial_j) = D_i \partial_j - D_j \partial_i - [\partial_i, \partial_j] = D_i \partial_j - D_j \partial_i = 0 \Rightarrow D_i \partial_j = D_j \partial_i$$

and, using Remark 2.59, $\Gamma^k_{ij} = \Gamma^k_{ji}$. Conversely, if $\Gamma^k_{ij} = \Gamma^k_{ji}$, using the same remark $D_i \partial_j = D_j \partial_i$. Let $X = X^i \partial_i$, $Y = Y^i \partial_i$ be to arbitrary vector fields on M. We have

$$\begin{aligned} T(X,Y) = T(X^i \partial_i, Y^i \partial_i) &= D_X Y - D_Y X - [X^i \partial_i, Y^j \partial_j] \\ &= X^j D_j (Y^i \partial_i) - Y^j D_j (X^i \partial_i) - \left(X^j \partial_j Y^i - Y^j \partial_j X^i \right) \partial_i \\ &= X^j \partial_j Y^i \partial_i + X^j Y^i D_j \partial_i - Y^j \partial_j X^i \partial_i - Y^j X^i D_j \partial_i - X^j \partial_j Y^i \partial_i + Y^j \partial_j X^i \partial_i \\ &= X^j Y^i \left(D_j \partial_i - D_i \partial_j \right) = 0, \end{aligned}$$

and therefore the connection is symmetric. □

From now on we are going to consider always a coordinate basis and a symmetric connection.

Definition 2.66. The *Levi-Civita connection* on M is the only connection D on M satisfying $i)$ and $ii)$ of Theorem 2.57.

Remark 2.67. The covariant derivative with respect to a vector field X can also be denoted by ∇_X. Then, Theorem 2.57 can also be interpreted as follows: for a given semi-Riemannian manifold M, there exists a unique connection ∇ which is torsion free, $T(X,Y) = 0$ for every $X, Y \in \mathcal{X}(M)$, and compatible with the metric g in the sense that $\nabla_X g = 0$ for every $X \in \mathcal{X}(M)$. This conection ∇ is the Levi-Civita connection on M.

Let us introduce now the notation

$$f_{,i} := \partial_i f = \frac{\partial f}{\partial x^i}\,, \quad \dot{f} := \frac{df}{dt}\,, \quad X_i := g_{ij}X^j\,,$$

where f is an arbitrary function, x^i are the coordinate system, and X^i are the components with respect to a coordinate system of a vector field on a manifold with metric g. The notation "," changes to ";" when referring to the covariant derivative. This notation is commonly used in theoretical physics.

Proposition 2.68. *Let $\varphi = (x^1, \dots, x^n)$ be a chart on $U \subseteq M$ and $W \in \mathcal{X}(M)$. Let us consider the Levi-Civita connection. Then*

$$D_{\partial_i} W = \left(W^k_{\ ,i} + \Gamma^k_{ij} W^j \right) \partial_k\,,$$

where

$$\Gamma^k_{ij} = \frac{1}{2} g^{km} \left(g_{jm,i} + g_{im,j} - g_{ij,m} \right).$$

Proof. Consider $iii)$ in Definition 2.49 with $X = \partial_i$, $f = W^k$ and $Y = \partial_k$, both for every $1 \leq k \leq n$. We get

$$\begin{aligned} D_{\partial_i} W &= D_{\partial_i}\left(\sum_k W^k \partial_k \right) = \sum_k D_{\partial_i}\left(W^k \partial_k \right) = \sum_k \left((\partial_i W^k)\partial_k + W^k D_{\partial_i}\partial_k \right) \\ &= \sum_k (\partial_i W^k)\partial_k + \sum_j W^j D_{\partial_i}\partial_j = \left(\partial_i W^k + W^j \Gamma^k_{ij} \right)\partial_k, \end{aligned}$$

where in the second equality $ii)$ in Definition 2.49 has been used and in the last equality we have used A. Einstein summation convention and Definition 2.58.

To derive the expression of the Christoffel symbols as a function of the metric, consider the Koszul formula taking $X = \partial_m$, $Y = \partial_i$ and $Z = \partial_j$. Since $[\partial_i, \partial_j] = 0$ for every $1 \leq i, j \leq n$ due to Schwarz Theorem, we get

$$\begin{aligned} 2g\left(D_{\partial_i}\partial_j, \partial_m\right) = 2g\left(\Gamma_{ij}^k \partial_k, \partial_m\right) = 2\Gamma_{ij}^k g_{km} &= \partial_i g(\partial_j, \partial_m) + \partial_j g(\partial_m, \partial_i) - \partial_m g(\partial_i, \partial_j) \\ &= g_{jm,i} + g_{mi,j} - g_{ij,m}. \end{aligned}$$

Dividing by 2, using that g is symmetric and applying g^{km} on both sides we obtain

$$\Gamma_{ij}^k = \frac{1}{2} g^{km}\left(g_{jm,i} + g_{im,j} - g_{ij,m}\right).$$

□

Examples 2.69. We will compute the Christoffel symbols for two metrics with importance in general relativity.

1. Consider the $2D$ metric in the conformal gauge $g_{ij} = e^{2\rho}\eta_{ij}$, which is widely used for studying black holes in 2 dimensional gravity models [Str94]. We will compute its Christoffel symbols in the so called light-cone coordinates $x^{\pm} = x^0 \pm x^1$.

   Using the invariance of the scalar product, we get

$$\begin{aligned} ds^2 := g_{ij}dx^i dx^j = e^{2\rho}\left(-d(x^0)^2 + d(x^1)^2\right) &= e^{2\rho}(-dx^+ dx^-) \\ &= g_{+-}dx^+ dx^- + g_{+-}dx^- dx^+. \end{aligned}$$

   From that we get $g_{++} = g_{--} = 0$ and $g_{+-} = -\frac{1}{2}e^{2\rho}$. Therefore,

$$g_{+-,+} = -e^{2\rho}\partial_+\rho \quad \text{and} \quad g_{+-,-} = -e^{2\rho}\partial_-\rho\,.$$

   From this, using equation Proposition 2.68, we get:

$$\begin{aligned} \Gamma_{++}^+ &= \frac{1}{2}g^{+l}\left(2g_{l+,+}\right) = g^{+-}g_{-+,+} = (-2e^{-2\rho})(-e^{2\rho}\partial_+\rho) = 2\partial_+\rho\,, \\ \Gamma_{--}^- &= g^{-+}g_{+-,-} = 2\partial_-\rho\,, \\ \Gamma_{+-}^+ &= \frac{1}{2}g^{+l}\left(g_{l+,-} + g_{l-,+} - g_{+-,l}\right) \\ &= \frac{1}{2}g^{+-}\left(g_{-+,-} + g_{--,+} - g_{+-,-}\right) = 0\,, \\ \Gamma_{-+}^+ &= \Gamma_{+-}^- = \Gamma_{-+}^- = 0\,, \\ \Gamma_{--}^+ &= \Gamma_{++}^- = \frac{1}{2}g^{+l}\left(2g_{l-,-} - g_{--,l}\right) = 0\,. \end{aligned}$$

2. To study a $2D$ gravity model such as *JT gravity* [Wit20], consider the metric

$$ds^2 := g_{ij}dx^i dx^j = -A(x)dt^2 + \frac{1}{G(x)}dx^2.$$

Let us compute its Christoffel symbols, using $g_{00} = g_{00}(x) = -\frac{1}{\mathrm{A(x)}}$ and $g_{11} = g_{11}(x) = G(x)$:

$$\begin{aligned}
\Gamma^0_{00} &= \Gamma^0_{11} = \Gamma^1_{01} = \Gamma^1_{10} = 0\,, \\
\Gamma^0_{01} &= -\frac{1}{2A(x)}(-A'(x)) = \frac{A'(x)}{2A(x)} = \Gamma^0_{10}\,, \\
\Gamma^1_{00} &= \frac{1}{2}G(x)(A'(x)) = \frac{G(x)A'(x)}{2}\,, \\
\Gamma^1_{11} &= \frac{1}{2}G(x)\left(\frac{1}{G(x)}\right)' = -\frac{G'(x)}{2G(x)}\,.
\end{aligned}$$

We have now all the necessary tools to define the Riemann or curvature tensor and the Weyl or conformal tensor. Weyl tensor will be useful for next chapter, since it will help us identify, for instance, if a manifold is conformally flat or not.

Definition 2.70. Consider the Levi-Civita connection on M. The *Riemann curvature tensor* on M is a tensor of type $(1,3)$ defined as

$$\begin{aligned}
R : \mathcal{X}(M)^3 &\longrightarrow \mathcal{X}(M) \\
(X,Y,Z) &\longmapsto R_{XY}(Z) := D_{[X,Y]}Z - [D_X, D_Y]Z,
\end{aligned}$$

for every $X, Y, Z \in \mathcal{X}(M)$.

We refer to [O'N83] to see a proof that it is indeed a tensor.

Proposition 2.71. *On a coordinate basis, the Riemann curvature tensor satisfies*

$$R_{\partial_k \partial_l}\left(\partial_j\right) = R^i_{jkl}\partial_i,$$

where the components of the Riemann curvature tensor have the form

$$R^i_{jkl} = \Gamma^i_{kj,l} - \Gamma^i_{lj,k} + \Gamma^i_{lm}\Gamma^m_{kj} - \Gamma^i_{km}\Gamma^m_{lj}.$$

Proof. On a coordinate basis $[\partial_j, \partial_k] = 0$, then

$$\begin{aligned}
R_{\partial_k \partial_l}\left(\partial_j\right) &= -[D_{\partial_k}, D_{\partial_l}]\partial_j = D_{\partial_l}(D_{\partial_k}\partial_j) - D_{\partial_k}(D_{\partial_l}\partial_j) \\
&= D_{\partial_l}\left(\Gamma^m_{jk}\partial_m\right) - D_{\partial_j}\left(\Gamma^m_{jl}\partial_m\right) \\
&= \Gamma^m_{jk,l}\partial_m + \Gamma^m_{jk}\Gamma^r_{ml}\partial_j - \Gamma^m_{jl,k}\partial_m - \Gamma^m_{jl}\Gamma^r_{mk}\partial_r \\
&= \left(\Gamma^i_{jk,l} + \Gamma^m_{jk}\Gamma^i_{ml} - \Gamma^i_{jl,k} - \Gamma^m_{jl}\Gamma^i_{mk}\right)\partial_i \\
&= \left(\Gamma^i_{kj,l} - \Gamma^i_{lj,k} + \Gamma^i_{lm}\Gamma^m_{kj} - \Gamma^i_{km}\Gamma^m_{lj}\right)\partial_i,
\end{aligned}$$

where in the last equality we have used that we are assuming that we have a coordinate basis and a symmetric connection. $\square$

Now we can define another tensor and a scalar which are relevant for general relativity. Furthermore, we will need them to define the Weyl curvature tensor.

Definition 2.72. The *Ricci curvature tensor* Ric is a type $(0,2)$ tensor given by the contraction

$$\mathrm{Ric} = \mathcal{C}^1_2 R,$$

therefore, its components in a coordinate basis are $R_{ij} = R^m_{imj}$, where

$$\mathrm{Ric} = R_{ij} dx^i \otimes dx^j.$$

Definition 2.73. The *scalar curvature* S (commonly also R or Sc) is defined as the trace of the Ricci curvature tensor with respect to the metric

$$S = \mathrm{Tr}_g \mathrm{Ric}.$$

It can computed by $S = g^{ij} R_{ij}$.

Remark 2.74. The scalar curvature S is the simplest curvature invariant of a Riemannian manifold.

Definition 2.75. An *Einstein manifold* is a semi-Riemannian manifold (M, g) whose Ricci curvature tensor is porportional to its metric

$$Ric = kg\,, \quad \text{for some constant } k \in \mathbb{R}\,.$$

Remark 2.76. In local coordinates, (M, g) is an Einstein manifold if $R_{ij} = kg_{ij}$ for some $k \in \mathbb{R}$. Taking the trace of both sides we have that the constant k is related with the scalar curvature S by $S = nk$, where n is the dimension of M.

Let us now introduce the notation

$$A^{n_1 \dots n_p}_{[m_1 \dots m_k] m_{k+1} \dots m_q} := \frac{1}{k!} \sum_{\sigma(m_1 \dots m_k)} (-1)^{\mathrm{sgn}\,\sigma} A^{n_1 \dots n_p}_{\sigma(m_1) \dots \sigma(m_k) m_{k+1} \dots m_q},$$

where the sum runs over all possible permutations of the indexes between the square brackets. It is also applicable to different tensors as in

$$A_{i[j} B_{k]l} := \frac{1}{2} \left(A_{ij} B_{kl} - A_{ik} B_{jl} \right),$$

for example. Finally, we conclude this section with the following definition:

Definition 2.77. The *Weyl curvature tensor* W is a type $(1,3)$ tensor defined for manifolds of dimension $n \geq 3$ by the equations

$$W^i{}_{jkl} = g^{im} W_{mjkl} \quad \text{and}$$

$$W_{ijkl} = g_{im} R^m{}_{jkl} + \frac{2}{n-2}\left(g_{j[k}R_{l]i} - g_{i[k}R_{l]j}\right) + \frac{2}{(n-1)(n-2)} g_{a[c}g_{d]b}R.$$

Remark 2.78. The Weyl curvature tensor's behaviour under conformal transformations of the metric [Wal84, Appendix D] is the reason why it is sometimes named *conformal tensor*. Actually, the Weyl curvature tensor is invariant under conformal transformations of the metric [Wal84].

Remark 2.79. The Weyl curvature tensor is the traceless component of the Riemann curvature tensor [d'I92]. Thus, any contraction applied on W make it vanish.

3 Conformal Geometry

As mentioned in the introduction, conformal geometry studies the conformal transformation on manifolds. Those are transformations on manifolds that locally preserve angles but not necessarily lengths.

Our objective here is to study conformal transformations in order to describe conformal manifolds. Since any semi-Riemannian manifold (M, g) of dimension $n \geq 3$ can be associated to a conformal manifold $(M, [g])$, the latter can be thought as a generalisation of the former. Going from a semi-Riemannian manifold to its respective conformal manifold means losing the notions of lengths but retaining the notion of angles. Analogously, for a Lorentzian manifold, in physics terminology, passing to its respective conformal manifolds means dropping the spacetime intervals, which play the role of distances between events, but keeping the light-cone structure, and thus its causal structure.

To give a classification of conformal transformations we had to describe the conformal Killing fields. Those are a special kind of vector fields whose flow defines a conformal transformation. Conformal Killing fields are a generalisation of Killing vector fields in the sense that the latter are said to preserve the metric, satisfying the Killing equation $\mathcal{L}_X g = 0$, whereas the former satisfy $\mathcal{L}_X g = \kappa g$, where κ is a smooth real-valued function. More details will be provided later on. For this chapter we have mainly used [Sch08] and [CG18].

From now on, when talking about the image of a vector field, we will consider we are only refering to the vector of the corresponding tangent space, and not the pair (point, vector). We do this because it is a common convention.

3.1 Conformal Transformations

In this subsection, we aim to study the transformations we can apply to a manifold in such a way that angles are preserved even though distances can change. To do so, we must be aware that by considering the metric tensor as a smooth scalar product in all the tangent spaces of a manifold, the metric can be used to generalise concepts frequently used in $\mathbb{R}^n$ like the distance between points or the angle between vectors.

Definition 3.1. Let (M, g) and (M', g') be two semi-Riemannian manifolds of the same dimension n, and $U \subseteq M, U' \subseteq M'$ be two open subsets. A *conformal transformation* from U to U' is a smooth map $\varphi : U \to U'$ of maximal rank, in the sense of Definition 2.20, such that

$$\varphi^* g' = \Omega^2 g \quad \text{where } \Omega : U \to \mathbb{R}^+ \text{ is a smooth function.}$$

When applied to $X, Y \in T_pU$, the left-hand term in the expression $\varphi^* g' = \Omega^2 g$ means

$$\varphi^* g'(X, Y) := g'(d\varphi_p(X), d\varphi_p(Y)),$$

being $d\varphi_p : T_pU \to T_pV$ the differential map of φ at p. The function Ω is called the *conformal factor* of φ.

Consider local coordinates for M and M'. Then, for every $p \in U \subseteq M$

$$(\varphi^* g')_{ab}(p) = g'_{ij}(\varphi(p)) \partial_a \varphi^i \partial_b \varphi^j.$$

Thus, φ is conformal if and only if, in a coordinate neighborhood of each point,

$$\Omega^2 g_{ab} = (g'_{ij} \circ \varphi) \partial_a \varphi^i \partial_b \varphi^j$$

holds.

Remark 3.2. For a conformal transformation φ, tangent maps $T_p\varphi : T_pM \to T_{\varphi(p)}M'$ are bijective. Therefore, by the Inverse Mapping Theorem, conformal transformations are always locally invertible.

Examples 3.3.

1. Local isometries are smooth maps φ with $\varphi^* g' = g$. Then, they are conformal transformations with conformal factor $\Omega = 1$.

2. Consider $\mathbb{R}^2$ with the metric g given by the bilinear form

$$\langle (x, y), (x', y') \rangle := \frac{1}{2} \left(xy' + yx' \right).$$

The light-cone, which for a given $p = (p_1, p_2) \in \mathbb{R}^2$ is

$$\mathcal{L}_C(p) := \{(x,y) \in \mathbb{R}^2 : \langle (x-p_1, y-p_2), (x-p_1, y-p_2) \rangle = 0\},$$

coincides with the set of coordinate axes for $p = 0 \in \mathbb{R}^2$

$$\mathcal{L}_C(0) = \{(a,0), (0,b) : a, b \in \mathbf{R}\}.$$

There is an isometric isomorphism between $(\mathbb{R}^2, g)$ and $\mathbb{R}^{1,1}$, given by

$$\begin{aligned} \psi : \mathbb{R}^{1,1} &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (x+y, x-y). \end{aligned}$$

3. The *stereographic projection*

$$\begin{aligned} \pi : \mathbb{S}^2 \backslash \{(0,0,1)\} &\longrightarrow \mathbb{R}^{2,0} \\ (x,y,z) &\longmapsto \frac{1}{1-z}(x,y) \end{aligned}$$

is conformal with conformal factor $\Omega = \frac{1}{1-z}$. Indeed, the inverse map

$$\begin{aligned} \varphi := \pi^{-1} : \mathbb{R}^{2,0} &\longrightarrow \mathbb{S}^2 \subseteq \mathbb{R}^{3,0} \\ (x,y) &\longmapsto \frac{1}{1+x^2+y^2}(2x, 2y, x^2+y^2-1) \end{aligned}$$

is a conformal transformation with conformal factor $\Omega^{-1} = \frac{2}{1+x^2+y^2}$.

4. Identify $\mathbb{R}(2,0) \cong \mathbb{C}$ and consider, for every $z \in \mathbb{C}$, $z = x + iy$ with $x, y \in \mathbb{R}$. A smooth map $\varphi = (u,v) : M \subseteq \mathbb{C} \to \mathbb{C}$, where M is connected and open and $u = \mathrm{Re}(\varphi)$, $v = \mathrm{Im}(\varphi)$ is a conformal transformation with conformal factor $\Omega \neq 0$ if an only if

$$u_x^2 + v_x^2 = u_y^2 + v_y^2 = \Omega^2, \tag{1}$$

$$u_x u_y + v_x v_y = 0. \tag{2}$$

Holomorphic ($u_x = v_y, u_y = -v_x$) and antiholomorphic ($u_x = -v_y, u_y = v_x$) functions φ from M to $\mathbb{C}$ satisfy (1) and (2) with conformal factor $|\det(D\varphi)| = |\varphi'| =^2 u_x^2 + v_x^2 = \Omega^2$, where $D\varphi$ is the respective Jacobi matrix. Conversely, for every conformal transformation φ, (1) and (2) imply that φ is necessarily holomorphic or antiholomorphic.

3.2 Conformal Killing Fields

Consider the open subsets U, U' of the manifold $R^{p,q}$, with $p+q=n\geq 1$. Let $\varphi : U \to U'$ be a conformal transformation and $X : U \subseteq \mathbb{R}^{p,q} \to T\mathbb{R}^{p,q}$ be a smooth vector field. For smooth curves $\gamma = \gamma(t)$ on M, consider the autonomous differential equation

$$\dot{\gamma} = X(\gamma).$$

For this differential equation, for an arbitrary $p \in U$, consider its unique maximal solution $\varphi^X(\cdot, p) : I \subseteq \mathbb{R} \to M$, defined on the maximal interval $\left(t_p^-, t_p^+\right)$, which satisfies

$$\frac{d}{dt}\left(\varphi^X(t,p)\right) = X\left(\varphi^X(t,a)\right), \quad \varphi^X(0,p) = p\,.$$

Let $M_t := \{p \in M | t_p^- < t < t_p^+\}$ and $\varphi_t^X(p) := \varphi^X(t,p)$. Then $M_t \subseteq M$ and $\varphi_t^X : M_t \to M_{-t}$ is a diffeomorphism, which can be visualised as a displacement of "duration" t along the curves $\varphi^X(\cdot, p)$ defined by the vector field X. With this notation, if $p \in M_{t+s} \cap M_s = \{b \in M | t_b^- < t+s, s < t_b^+\}$ then $\varphi_t^X \circ \varphi_s^X(p) = \varphi_{t+s}^X(p)$. Moreover, $\varphi_0^X = id_M$ and $M_0 = M$.

Definition 3.4. Within the notation introduced above we have that the local one-parameter group $(\varphi_t^X)_{t\in\mathbb{R}}$ satisfies

$$\frac{d}{dt}\left(\varphi_t^X\right)_{t=0} = X,$$

which is called the *flow equation*.

With this, one can understand the so called *geodesic flow* as the action on the tangent bundle TM of M defined by the flow equation restricted to the unit tangent bundle $SM := \{(p,v) \in TM \mid g(v,v) = 1\}$ [Pat99]. Since throughout a geodesic the speed remains constant, the diffeomorphisms φ_t leave SM invariant.

Definition 3.5. A vector field X on $M \subseteq \mathbb{R}^{p,q}$ is a *conformal Killing field* if φ_t^X is conformal for every t in a neighborhood of 0.

Theorem 3.6. *Let $M \subseteq \mathbb{R}^{p,q}$ be an open manifold, $g = g^{p,q}$ be a constant metric on M and $X = (X^1, \dots, X^n) = X^i\partial_i$ be a conformal Killing field where X^i are its coordinates with respect to the canonical Cartesian coordinates on $\mathbb{R}^n$. There exists a smooth function $\kappa : M \to \mathbb{R}$ such that*

$$X_{i,j} + X_{j,i} = \kappa g_{ij}.$$

Proof. Let (φ_t) be the local one-parameter group associated to a conformal Killing field X, and $\Omega_t : M_t \to \mathbb{R}^+$ its conformal factor. Consider an arbitrary $p \in M$. Then, by Definition 3.5 and Definition 3.1,

$$(\varphi_t^* g)_{ij}(p) = g_{ab}(\varphi_t(p))\partial_i \varphi_t^a \partial_j \varphi_t^b = (\Omega_t(p))^2 g_{ij}(p).$$

Let us now differentiate with respect to t at $t = 0$, assuming g_{ab} do not depend on t, obtaining

$$\begin{aligned}
\frac{d}{dt}\left(\Omega_t^2(p) g_{ij}(p)\right)\Big|_{t=0} &= \frac{d}{dt}\left(g_{ab}(\varphi_t(p))\partial_i \varphi_t^a \partial_j \varphi_t^b\right)\Big|_{t=0} \\
&= g_{ab}(\varphi_0(p))\partial_i \dot{\varphi}_0^a \partial_j \varphi_0^b + g_{ab}(\varphi_0(p))\partial_i \varphi_0^a \partial_j \dot{\varphi}_0^b \\
&= g_{ab}(\varphi_0(p))\partial_i X^a(p)\delta_j^b + g_{ab}(\varphi_0(p))\delta_i^a \partial_j X^b(p) \\
&= \partial_i X_j(p) + \partial_j X_i(p) = \frac{d}{dt}\left(\Omega_t^2(p)\right)\Big|_{t=0} g_{ij}(p),
\end{aligned}$$

where $\partial_a \varphi_0^b = \delta_a^b$. Therefore, we have proved the theorem taking as conformal factor $\kappa(p) = \frac{d}{dt}\left(\Omega_t^2(p)\right)\big|_{t=0}$. □

Within semi-Riemannian geometry, Killing vector fields are those satisfying $\mathcal{L}_X g = 0$. Now, conformal Killing fields satisfy $\mathcal{L}_X g = \kappa g$. Therefore, conformal Killing fields can be considered as a natural generalisation of Killing vector fields.

For the case where g_{ij} is not constant, an then it varies smoothly across the points $p \in M$, the *Lie derivative* L_X of g with respect to the vector field X locally gets the form

$$(L_X g)_{ij} = X_{i;j} + X_{j;i} = \kappa g_{ij},$$

where the semicolon in the index denotes the covariant derivative corresponding to the Levi-Civita connection for g.

Definition 3.7. A *conformal Killing factor* κ is a smooth real-valued function on a semi-Riemannian manifold M such that, on coordinate neighborhoods,

$$(L_X g)_{ij} = X_{i;j} + X_{j;i} = \kappa g_{ij}$$

(for the case $M \subseteq \mathbb{R}^{p,q}$ the previous condition becomes $X_{i,j} + X_{j,i} = \kappa g_{ij}$).

Theorem 3.8. *A smooth function $\kappa : M \to \mathbb{R}$ is a conformal Killing factor if and only if*

$$(n-2)\kappa_{,ij} + g_{ij}\Delta_g \kappa = 0,$$

where $\Delta_g = g^{ab}\partial_a \partial_b$ is the Laplace-Beltrami operator *for $g = g^{p,q}$ and $dimM = n$.*

Proof. We will give a proof for the case $M \subseteq \mathbb{R}^{(p,q)}$. Let $\kappa : M \to \mathbb{R}$ be a conformal Killing factor. Then we have $X_{i,j} + X_{j,i} = \kappa g_{ij}$. From

$$\partial_a \partial_b (X_{i,j}) = \partial_j \partial_a (X_{i,b}) \,,$$

one can write

$$0 = \partial_a \partial_b (X_{i,j} + X_{j,i}) - \partial_b \partial_i (X_{a,j} + X_{j,a}) + \partial_i \partial_j (X_{a,b} + X_{b,a}) - \partial_j \partial_a (X_{i,b} + X_{b,i}).$$

Now, since κ is conformal, from Definition 3.7,

$$X_{i,j} + X_{j,i} = \kappa g_{ij},$$

and therefore

$$\begin{aligned}
0 &= \partial_a \partial_b (\kappa g_{ij}) - \partial_b \partial_i (\kappa g_{aj}) + \partial_i \partial_j (\kappa g_{ab}) - \partial_j \partial_a (\kappa g_{ib}) \\
&= g_{ij} \kappa_{,ab} - g_{aj} \kappa_{,bi} + g_{ab} \kappa_{,ij} - g_{ib} \kappa_{,ja} = g^{ab} g_{ij} \kappa_{,ab} - g^{ab} g_{aj} \kappa_{,bi} + g^{ab} g_{ab} \kappa_{,ij} - g^{ab} g_{ib} \kappa_{,ja} \\
&= g^{ab} g_{ij} \kappa_{,ab} - \delta^b_j \kappa_{,bi} + \delta^a_a \kappa_{,ij} - \delta^a_i \kappa_{,ja} = g_{ij} g^{ab} \partial_a \partial_b \kappa - \kappa_{,ji} + n \kappa_{,ij} - \kappa_{,ji} \\
&= (n-2) \kappa_{,ij} g_{ij} \Delta_g \kappa \,,
\end{aligned}$$

where in the second equality we have used that the derivatives of the metric vanish and in the last equality we have used the Schwarz Theorem to say $\kappa_{,ij} = \kappa_{,ji}$, since κ is smooth by definition. We have also used that $g^{ia} g_{aj} = \delta^i_j$ and that g is symmetric. For the converse we refer to [Sch08, Chapter 1], where it is explained throughout some cases. $\square$

This theorem also holds for a general open semi-Riemannian manifold M changing the partial derivative ”," for the covariant derivative ";".

Remark 3.9. Consider the case of $M \subseteq \mathbb{R}^{p,q}$, $p + q = n$. If $n = 2$, κ is conformal if and only if $\Delta_g \kappa = 0$. If $n \geq 2$, κ is conformal if and only if

$$\begin{cases} \kappa_{,ij} = 0, & \text{if} \quad i \neq j \\ \kappa_{,ij} = \pm (n-2)^{-1} \Delta_g \kappa \,, & \text{if} \quad i = j \,. \end{cases}$$

3.3 Classification of Conformal Transformations

The main goal of this section is to determine all conformal transformations on connected open $M \subseteq \mathbb{R}^{p,q}$. To do so, we will first determine all conformal Killing fields using Theorem 3.8. In all this section $\dim M := n = p + q$. We will also need the following definition:

Definition 3.10. Given a vector field $X \in \mathcal{X}(M)$, its *respective conformal transformation* $\varphi(x)$, $x \in M$, is given by the solution $\tilde{\varphi}(x,t)$, $t \in \mathbb{R}$, of the flow equation

$$\frac{d\tilde{\varphi}(x,t)}{dt} = X(\tilde{\varphi}(x,t)) \,, \quad \tilde{\varphi}(x,0) = x \,,$$

after taking $\varphi(x) := \tilde{\varphi}(x,1)$.

3.3.1 Case $p+q=n>2$

Proposition 3.11. *Let $n > 1$. Then, for a conformal Killing factor κ we have $\Delta_g \kappa = 0$.*

Proof. Applying g^{ij} in both sides of Theorem 3.8 we get

$$\begin{aligned} 0 &= g^{ij}(n-2)\kappa_{,ij} + g^{ij} g_{ij} \Delta_g \kappa = (n-2) g^{ij} \partial_i \partial_j \kappa + n \Delta_g \kappa \\ &= (n-2)\Delta_g \kappa + n\Delta_g \kappa = 2(n-1)\Delta_g \kappa. \end{aligned}$$

Therefore, for $n > 1$ we obtain that $\Delta_g \kappa = 0$. $\square$

Remark 3.12. The previous result for $n = 2$ can be obtained straightforward from Theorem 3.8.

Let us now introduce a theorem from [Sch08, Section 1.4.1]. After stating it, we will proceed to justify some of its results.

Theorem 3.13. *For a connected and open $M \subseteq \mathbb{R}^{(p,q)}$ with $n = p + q > 2$, every conformal transformation $\varphi : M \to \mathbb{R}^{(p,q)}$ is a composition of*

- *a translation $p \mapsto p + c$, $c \in \mathbb{R}^{(p,q)}$,*
- *an orthogonal transformation $p \mapsto \Lambda p$, $p \in O(p, q)$,*
- *a dilatation $p \mapsto e^{\lambda} p$, $\lambda \in \mathbb{R}$, and*
- *a special conformal transformation*

$$p \mapsto \frac{p - g(p,p)b}{1 - 2g(p,b) + g(p,p)g(b,b)}, \; b \in \mathbb{R}^{(p,q)}.$$

Combining Theorem 3.8 and Proposition 3.11 we have that $\kappa_{,ij} = 0$ holds for $n > 2$ and for every $1 \leq i, j \leq n$. Thus, there exist n constants $\alpha_i \in \mathbb{R}$ such that for every $p = (p^1, \ldots, p^n) \in M$

$$\kappa_{,i}(p) = \alpha_i, \quad i = 1, \ldots, n.$$

It follows that for $n > 2$ the conformal Killing factors κ are affine linear maps $\kappa : M \to \mathbb{R}$ such that

$$\kappa(p) = \lambda + \alpha_i p^i, \quad \lambda \in \mathbb{R}.$$

Let us begin justifying the translation. To do so, consider the case $\kappa = 0$. The corresponding conformal Killing field X then satisfies $X_{i,j} + X_{j,i} = 0$ because we have $M \subseteq \mathbb{R}^{p,q}$, and in particular $X_{i,i} + X_{i,i} = 0$, thus $X_{i,i} = 0$ and therefore X^i

does not depend on p^i. Now, from $X_{i,j} + X_{j,i} = 0$ and the fact that the derivatives of the metric vanish, we have

$$g_{ij} X^i{}_{,j} = -g_{ij} X^j{}_{,j} = 0 \Rightarrow X^i{}_{,j} = 0.$$

Combining both results, $X^i{}_{,i} = X^i{}_{,j} = 0$, we have that X^i can be written as $X^i(p) = c^i + \omega^i{}_j p^j$, with $c^i, \omega^i{}_j \in \mathbb{R}$. We can write $c \in \mathbb{R}^{(p,q)}$ and $\omega \in \mathbb{R}^{n\times n}$.

Consider the case $\omega = 0$, $X^i(p) = c^i$, which can be written as $X(p) = c$. To find its respective conformal transformation φ_c we compute

$$\frac{\tilde{\varphi}_c(p,t)}{dt} = c \Rightarrow \tilde{\varphi}_c(p,t) = ct + p_0.$$

Since $\tilde{\varphi}_c(p,0) = p$, we have $p_0 = p$ and therefore its respective conformal transformations are the translations $\varphi_c(p) = \tilde{\varphi}_c(p,1) = p + c$.

Consider now the case $X^i(p) = \omega^i{}_j p^j$, which can be written as $X(p) = \omega p$. As above, to find its respective conformal transformation we compute

$$\frac{\tilde{\varphi}_\Lambda(p,t)}{dt} = \omega p \Rightarrow \tilde{\varphi}_\Lambda(p,t) = p_0 e^{\omega t}, \quad p_0 \in \mathbb{R}^{(p,q)}.$$

Since $\tilde{\varphi}_\Lambda(p,0) = p$, we have $p_0 = p$ and therefore its respective conformal transformations are $\varphi_\Lambda(p) = \tilde{\varphi}_\Lambda(p,1) = pe^{\omega}$. From Theorem 3.6 we get

$$\partial_j \left(g_{ir}\omega^r{}_s p^s\right) + \partial_i \left(g_{jr}\omega^r{}_u p^u\right) = g_{ir}\omega^r{}_j + g_{jr}\omega^r{}_i = \omega^T g + g\omega = 0,$$

which in particular means that ω is antisymmetric. By the antisymmetry of ω we have, by a linear algebra result, the orthogonality of $\Lambda := e^{\omega}$. Therefore, its respective conformal transformations can be written as $\varphi_\Lambda(p) = \Lambda p$, where

$$\Lambda = e^{\omega} \in O(p,q) := \{\Lambda \in GL(n,\mathbb{R}) | \Lambda^T g^{p,q} \Lambda = g^{p,q}\}$$

and $GL(n,\mathbb{R})$ is the general linear group of degree n over $\mathbb{R}$, which is the group of the $n \times n$ invertible matrices together with the multiplication of matrices.

Finally, assuming the most general form for a conformal Killing field for $\kappa = 0$, $X^i(p) = c^i + \omega^i{}_j p^j$, we have seen that its respective conformal transformations are of the form

$$\varphi(p) = c + \varphi_\Lambda(p), \quad c \in \mathbb{R}^{(p,q)},\ \Lambda \in O(p,q).$$

As we will see in more detail in the next case, for constant conformal Killing factors $\kappa = \lambda \in \mathbb{R} \setminus \{0\}$ we have $X(p) = \lambda p$ as conformal Killing fields. Therefore, its respective conformal transformations are the *dilatations* $\varphi(p) = e^{\lambda} p$.

Finally, one could also consider the case $\kappa = \kappa(p)$, but we will omit it for the sake of brevity. From this latter case, one obtains the special conformal transformations of Theorem 3.13. To see this discussion we refer to [Sch08, Section 1.4].

3.3.2 Case *p=2*, *q=0*: the Euclidean plane

Theorem 3.14. *Let $f = u + iv : M \to \mathbb{R}^{2,0} \cong \mathbb{C}$ be an holomorphic function on an open $M \subseteq \mathbb{R}^{2,0}$ such that $f'(p) \neq 0$ for every $p \in M$. Then, f is an orientation-preserving conformal transformation with conformal Killing factor $\kappa = u_x^2 + u_y^2 = \det Df = |f'|^2$. Conversely, every orientation-preserving and conformal transformation $f : M \to \mathbb{R}^{2,0} \cong \mathbb{C}$ is an holomorphic function.*

Proof. We will provide an idea of the proof, which is related to the last example of Examples 3.3. For a transformation $f = u + iv$, being conformal (Definition 3.1) implies it must satisfy (1) and (2) in Section 3.1, which at the same time imply that f is either a holomorphic or antiholomorphic function. If we also ask f to be orientation-preserving, then it can only be holomorphic. □

The objective is the same as above. We want to analyze conformal Killing fields and conformal Killing factors in order to describe the conformal transformations in the Euclidean plane.

Consider a conformal Killing field $X = (u, v) : M \to \mathbb{C}$ such that for every $z = x + iy \in M$, $X(x, y) = u(x, y) + iv(x, z)$, where $u, v \in \mathcal{F}(M)$. Since $n = 2$, due to Theorem 3.8, $\Delta_g \kappa = 0$. Furthermore, by Definition 3.7, we have

$$\begin{aligned} X_{x,x} + X_{x,x} = 2X_{x,x} = \kappa g_{xx} = \kappa \quad &\Rightarrow \quad X_{x,x} = u_x = \kappa/2, \\ X_{y,y} + X_{y,y} = 2X_{y,y} = \kappa g_{y,y} = \kappa \quad &\Rightarrow \quad X_{y,y} = v_y = \kappa/2, \\ X_{x,y} + X_{y,x} = \kappa g_{x,y} = 0 \quad &\Rightarrow \quad X_{x,y} + X_{y,x} = u_y + v_x = 0. \end{aligned}$$

What we have seen is that X satisfies the Cauchy-Riemann equations, $u_x = v_y$ and $u_y = -v_x$, and thus X is holomorphic.

For the case with $\kappa = 0$, using now, and from now on, the usual notation $z = x + iy \in \mathbb{C} \cong \mathbb{R}^{2,0}$, we obtain $u(x, y) = u(y)$ and $v(x, y) = v(x)$ such that $u_y = -v_x$. Its general solution can be expressed as

$$X(z) = z_0 + i\theta z, \quad z \in \mathbb{C},$$

where $\theta \in \mathbb{R}$ and $z_0 \in \mathbb{C}$. We can now compute

$$\begin{aligned} \frac{d\tilde{\varphi}(z,t)}{dt} = X(\tilde{\varphi}(z,t)) \Rightarrow \frac{d\tilde{\varphi}}{dt} = z_0 + i\theta\tilde{\varphi} \Rightarrow \int \frac{d\tilde{\varphi}}{z_0 + i\theta\tilde{\varphi}} = \int dt \\ \Rightarrow \frac{\ln(z_0 + i\theta\tilde{\varphi})}{i\theta} = t + C \Rightarrow z_0 + i\theta\tilde{\varphi} = De^{i\theta t}, \end{aligned}$$

where $D = e^{i\theta C} \in \mathbb{C}$ and $C \in \mathbb{R}$ is a constant of integration even though it depends on the point $z \in \mathbb{C}$ in which we are computing the respective conformal transformation. We must apply now the initial condition

$$\tilde{\varphi}(z,0) = \frac{D}{i\theta}e^{i\theta 0} - \frac{z_0}{i\theta} = z \Rightarrow D = z_0 + i\theta z.$$

Therefore, its respective conformal transformations are

$$\varphi(z) = \tilde{\varphi}(z,1) = \frac{z_0}{i\theta}\left(e^{i\theta} - 1\right) + ze^{i\theta} \Rightarrow \varphi(z) = z_0' + ze^{i\theta},$$

where we have redefined $z_0' := \frac{z_0}{i\theta}\left(e^{i\theta} - 1\right)$.

For the case $\kappa = \lambda \in \mathbb{R} \setminus \{0\}$, we have $u_x = v_y = \lambda/2$ and $u_y = -v_x$, one gets

$$X(z) = z_0 + \left(\frac{\lambda}{2} + i\theta\right)z, \quad z \in \mathbb{C},$$

where $\theta \in \mathbb{R}$. In particular, if $\kappa = 2\lambda$, $z_0 = 0$, and considering $\theta = 0$, i.e. $u_y = v_x = 0$, one gets the dilatations

$$X(z) = \lambda z, \quad z \in \mathbb{C},$$

and, as before,

$$\frac{d\tilde{\varphi}(z,t)}{dt} = \lambda\tilde{\varphi}(z,t) \Rightarrow \frac{d\tilde{\varphi}(z,t)}{\tilde{\varphi}(z,t)} = \lambda dt \Rightarrow \tilde{\varphi}(z,t) = Ce^{\lambda t}, \quad C \in \mathbb{C}.$$

Applying the initial condition

$$\tilde{\varphi}(z,0) = C = z_0 \Rightarrow \tilde{\varphi}(z,t) = ze^{\lambda t}.$$

Therefore, its respective conformal transformation is $\varphi(z) = \tilde{\varphi}(z,1) = e^{\lambda}z$.

Let us now introduce one widely studied example of conformal transformation. Specifically, Möbius transformations have been studied within geometry and complex analysis.

Definition 3.15. Let $\varphi : \mathbb{C} \to \mathbb{C}$ be an holomorphic function. The function φ is called a *Möbius transformation* if there exists a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) \quad \text{such that} \quad \varphi(z) = \frac{az+b}{cz+d} \text{ with } cz+d \neq 0,$$

where $SL(2,\mathbb{C})$ is the special linear group of degree 2 over $\mathbb{C}$, which is the group of the 2×2 matrices with determinant $+1$ and coefficients in $\mathbb{C}$ together with the multiplication of matrices.

It is worth noting that, as it is a conformal transformation, the Möbius transformation preserves the angles (see Figure 2). These transformations can be also obtained by applying an inverse stereographic projection from the plane to the

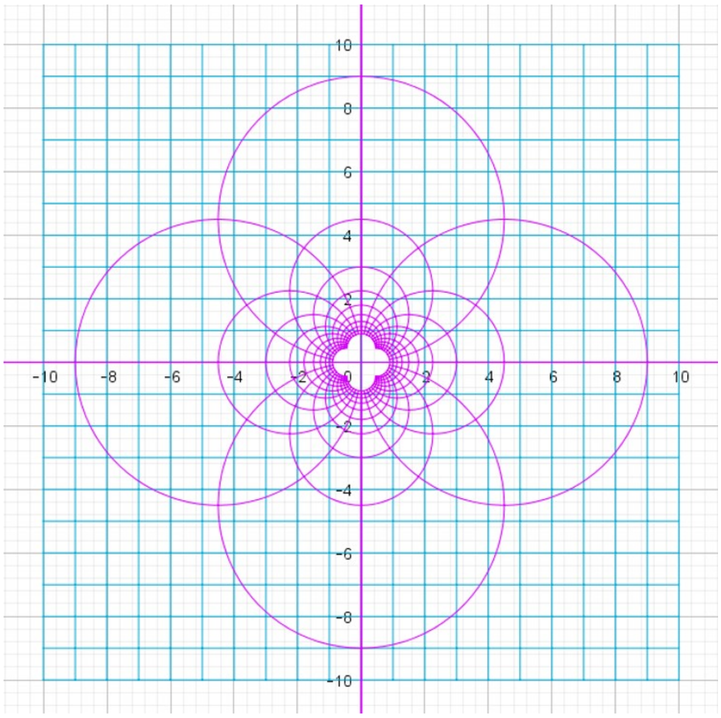

Figure 2: Representation of the Möbius transformation (purple) $\varphi(z) = \frac{9}{z}$, (corresponding to $a = d = 0$, $b = 3i$, $c = i/3$ in Definition 3.15), of the grid (blue) $\{\mathrm{Re}(z) = -10, -9, \ldots, 10\} \cup \{\mathrm{Im}(z) = -10, -9, \ldots, 10\}$.

unit two-sphere S^2, rotating the sphere through the plane to a new position and orientation and finally stereographically projecting it back to the plane. See [AR08] for a more detailed explanation of this procedure.

Let us now refer the interested reader to [Sch08, Chapter 2], where a more detailed explanation of the Möbius transformations can be found. Together with that, we also want to note the result found in Chapter 1 of the same reference which states that the linear conformal Killing factors κ describe the Möbius transformations.

Generally, for $\kappa \neq 0$, Theorem 3.6 implies that on a connected open subset $M \subseteq \mathbb{C}$, there exists a vector field $X = (u, v)$ with $u_y + v_x = 0$ and $u_x = v_y = \kappa/2$. Therefore, $u_x = v_y$ and $u_y = -v_x$, and X is holomorphic.

3.3.3 Case *p=q=1*: the Minkowski plane

This case is very important for physicists because it is used to describe spacetime in special theory of relativity. Similarly to Theorem 3.14 here we have Theorem 3.16.

Theorem 3.16. *A smooth map $\varphi = (u, v) : M \to \mathbb{R}^{1,1}$, where $M \subseteq \mathbb{R}^{1,1}$ is connected and open, is conformal if and only if*

$$u_x^2 > v_x^2 \text{ and } u_x = v_y, u_y = v_x \text{ or } u_x = -v_y, u_y = -v_x.$$

Proof. Consider first the next relation. Applying Definition 3.1, to our case, $\varphi^* g = \Omega g$ with $g = g^{1,1} = \mathrm{diag}(1, -1)$, we obtain

$$\begin{aligned}
\Omega^2 g_{00} &= \Omega^2 = (g_{ij} \circ \varphi)\partial_0 \varphi^i \partial_0 \varphi^j = u_x^2 - v_x^2 \Rightarrow \Omega^2 = u_x^2 - v_x^2\,, \\
\Omega^2 g_{11} &= -\Omega^2 = (g_{ij} \circ \varphi)\partial_1 \varphi^i \partial_1 \varphi^j = u_y^2 - v_y^2 \Rightarrow \Omega^2 = v_y^2 - u_y^2\,, \\
0 = \Omega^2 g_{01} &= (g_{ij} \circ \varphi)\partial_0 \varphi^i \partial_1 \varphi^j = u_x u_y - v_x v_y \Rightarrow 0 = u_x u_y - v_x v_y\,.
\end{aligned}$$

Therefore, we have that the condition $\varphi^* g = \Omega g$, with $g = g^{1,1} = \mathrm{diag}(1, -1)$, is equivalent to the three equations above.

Assume now that φ is conformal. From $\Omega^2 = u_x^2 - v_x^2$, since φ is conformal and therefore $\Omega^2 > 0$, we have $u_x^2 > v_x^2$. Moreover, since $v_x^2 \neq u_x^2$ because $u_x^2 > v_x^2$,

$$\begin{aligned}
&\begin{cases} \Omega^2 = u_x^2 - v_x^2 \\ \Omega^2 = v_y^2 - u_y^2 \end{cases} \Rightarrow u_x^2 - v_x^2 = v_y^2 - u_y^2 \Rightarrow u_x^2 - v_x^2 = v_y^2 - \frac{v_x^2 v_y^2}{u_x^2} \\
&\qquad \Rightarrow u_x^2 - v_x^2 = v_y^2 \left(1 - \frac{v_x^2}{u_x^2}\right) \Rightarrow u_x^2 - v_x^2 = v_y^2 \left(\frac{u_x^2 - v_x^2}{u_x^2}\right) \\
&\qquad \Rightarrow v_y^2 = u_x^2 \Rightarrow v_y = \pm u_x.
\end{aligned}$$

Now, if $v_y = u_x$, $0 = u_x(u_y - v_x)$ and $u_y = v_x$. Analogously, if $v_y = -u_x$, $0 = u_x(u_y + v_x)$ and $u_y = -v_x$. We have used that $u_x \neq 0$ because otherwise $\Omega = u_x^2 - v_x^2 = -v_x^2 \leq 0$ and we have $\Omega > 0$.

Conversely, assume that $u_x^2 > v_x^2$. Then

$$0 = 2u_x u_y - 2v_x v_y = (u_x + u_y)^2 - (v_x + v_y)^2$$

and therefore $u_x + u_y = \pm(v_x + v_y)$. For $u_x + u_y = v_x + v_y$,

$$\begin{aligned}
0 &= u_x^2 - u_x^2 + v_x v_y - u_x u_y = u_x^2 - u_x(u_x + u_y) + v_x v_y \\
&= u_x^2 - u_x(v_x + v_y) + v_x v_y = (u_x - v_x)(u_x - v_y),
\end{aligned}$$

from what we get $u_x = v_y$, since $u_x = v_x$ would contradict $u_x^2 > v_x^2$. Therefore, $u_x = v_y$ and $u_y = v_x$ for the assumption made $u_x + u_y = v_x + v_y$. Similarly, for $u_x + u_y = -(v_x + v_y)$,

$$\begin{aligned}
0 &= u_x^2 - u_x^2 + v_x v_y - u_x u_y = u_x^2 - u_x(u_x + u_y) + v_x v_y \\
&= u_x^2 + u_x(v_x + v_y) + v_x v_y = (u_x + v_x)(u_x + v_y),
\end{aligned}$$

from what we get $u_x = -v_y$, since $u_x = -v_x$ would contradict $u_x^2 > v_x^2$. Therefore, $u_x = -v_y$ and $u_y = -v_x$ for the assumption made $u_x + u_y = -(v_x + v_y)$. Since u and v are smooth by hypothesis, defining $\Omega^2 := u_x^2 - v_x^2 = v_y^2 - u_y^2 > 0$, we have that it is also smooth. Therefore, by the equivalence stated at the beginning, we have that φ is conformal since we also have $u_x u_y - v_x v_y = 0$ in both cases. $\square$

Lemma 3.17. *In $1+1$ dimensions, the solutions of the wave equation $\Delta\kappa = \kappa_{xx} - \kappa_{yy} = 0$ can be written as*

$$\kappa(x,y) = f(x+y) + g(x-y), \quad \textit{with } f, g \textit{ smooth real functions.}$$

Proof. Let's define the light-cone coordinates as $x_+ = x + y$, $x_- = x - y$. Applying the chain rule on the wave equation we obtain

$$\Delta\kappa = 0 \Rightarrow \frac{\partial^2\kappa}{\partial x^2} = \frac{\partial^2\kappa}{\partial y^2} \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial\kappa}{\partial x_-}\frac{\partial x_-}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial\kappa}{\partial x_-}\frac{\partial x_-}{\partial y}\right) \Rightarrow \frac{\partial^2\kappa}{\partial x\partial x_-} = -\frac{\partial^2\kappa}{\partial y\partial x_-}$$
$$\Rightarrow \frac{\partial^2\kappa}{\partial x_+\partial x_-}\frac{\partial x_+}{\partial x} = -\frac{\partial^2\kappa}{\partial x_+\partial x_-}\frac{\partial x_+}{\partial y} \Rightarrow \frac{\partial^2\kappa}{\partial x_+\partial x_-} = 0,$$

which has the solution

$$\kappa(x_+, x_-) = f(x_+) + g(x_-), \quad \text{with } f, g \text{ smooth real functions.}$$

Finally, undoing the change of coordinates we get what we wanted to prove. $\square$

Thus, using Proposition 3.11, any conformal factor κ in the Minkowski plane has the form in Lemma 3.17.

Corollary 3.18. *The orientation-preserving linear and conformal maps $\varphi : \mathbb{R}^{1,1} \to \mathbb{R}^{1,1}$ have a matrix representation of the form*

$$A_\varphi = e^t A_+(s) \quad \textit{or} \quad A_\varphi = e^t A_-(s),$$

with $(s,t) \in \mathbb{R}^2$ and where

$$A_+(s) = \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix} \quad \textit{and} \quad A_-(s) = \begin{pmatrix} -\cosh s & \sinh s \\ \sinh s & -\cosh s \end{pmatrix}$$

Proof. Consider the standard basis of $\mathbb{R}^2$ and A_φ the matrix representing $\varphi = (u,v)$ with respect to this basis,

$$A_\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Hence, $u(x,y) = ax + by$, $v(x,y) = cx + dy$ and $u_x = a, u_y = b, v_x = c, v_y = d$. Applying now Theorem 3.16, since φ is conformal, $a^2 > c^2$, and since

$$\det A_\varphi = ad - cb > 0, \quad \text{i.e. } \varphi \text{ is orientation-preserving,}$$

we are in the case $a = d, b = c$, because otherwise, in the case $a = -d, b = -c$, the orientation is not preserved by φ. Since $\det A_\varphi > 0$, it can be expressed as $\det A_\varphi = e^{2t}$, for some $t \in \mathbb{R}$. The solution $t \in \mathbb{R}$ of the equation $e^{2t} = a^2 - c^2 = \det A_\varphi$ is unique. For a given $t \in \mathbb{R}$, the solution $s \in \mathbb{R}$ of $\sinh s = ce^{-t}$ is also unique. We have $c^2 = e^{2t}\sinh^2 s$, from what $a^2 = c^2 + e^{2t} = e^{2t}(\sinh^2 s + 1) = (e^t \cosh s)^2$. Therefore, $a = e^t \cosh s = d$ or $a = -e^t \cosh s$ and $c = e^t \sinh s = b$. □

Having seen that, one can now interpret the action of t as a dilatation and the action of s as a boost or Lorentz transformation. Using the identities

$$\sinh x + y = \sinh x \cosh y + \sinh y \cosh x \quad \text{and}$$
$$\cosh x + y = \sinh x \sinh y + \cosh y \cosh x\,,$$

it can be seen that

$$A_\varphi(t,s) \cdot A_\varphi(t',s') = e^t A_\pm(s) e^{t'} A_\pm(s') = e^{t+t'} A_\pm(s+s') = A_\varphi(t+t', s+s').$$

Now, in order to introduce the final result of this subsection, let us introduce some concepts within group theory.

Definition 3.19. Let $(G,*), (H,\odot)$ be two groups and $\varphi : (G,*) \to (AutH, \circ)$ a group homomorphism. Consider the action of G on H, denoted by ϕ, defined by $\phi(g,h) := \varphi(g)(h)$, for every $g \in G$ and every $h \in H$. The *semidirect product of* $(H,\odot)$ *and* $(G,*)$ *with respect to* φ, denoted by $(H \rtimes_\varphi G, \times)$, is

$$H \rtimes_\varphi G := \{(h,g) | h \in H, g \in G\}, \text{ together with the operation}$$

$$(h,g) \times (h',g') := (h \odot \phi(g,h'), g * g').$$

Let us now define two groups which are named after the distinguished Dutch physicist H. Lorentz and French mathematician H. Poincaré. It can be seen that Lorentz group is a subgruop of Poincaré group. A further discussion can be found in [Pen74].

Definition 3.20. The *Lorentz group* can be mathematically described as the *indefinite orthogonal group* $O(1,3)$ and is defined as the set of all linear transformations of an 4-dimensional real vector space that leave invariant a non-degenerate, symmetric bilinear form of index $\nu = 1$. It can be represented as

$$O(1,3) = \{\Lambda \in GL(4,\mathbb{R}) \mid \Lambda^T \eta \Lambda = \eta\}\,, \quad \text{where } \eta = \mathrm{diag}(-1,1,1,1)\,.$$

Definition 3.21. The *Poincaré group* is the group of all the isometries of Minkowski spacetime, which are the transformations that preserve the relativistic interval between the points of Minkowski spacetime, named *events*. Mathematically, the Poincaré group is a semidirect product of the translations and the Lorentz group,

$$\mathcal{P}(1,3) = \mathbb{R}^{(1,3)} \rtimes_{\varphi} O(1,3),$$

where φ is the homomorphism defined by $\varphi(\alpha)(\Lambda) := \Lambda \cdot \alpha$, for every $\alpha \in \mathbb{R}^{(1,3)}$ and every $\Lambda \in O(1,3)$, as the usual multiplication of a matrix by a scalar. Therefore, the product in the Poincaré group is:

$$(\alpha, \Lambda) *_{\mathcal{P}} (\alpha', \Lambda') = (\alpha + \Lambda \cdot \alpha', \Lambda\Lambda').$$

We can now conclude with (see [Sch08, Remark 1.15]):

Proposition 3.22. *The Lorentz group $O(1,1)$ is isomorphic to $\mathbb{R}$. The corresponding Poincaré group $\mathcal{P}(1,1)$ is the semidirect product $\mathbb{R}^{(1,1)} \rtimes_{\varphi} O(1,1) \cong \mathbb{R}^{(1,1)} \rtimes_{\varphi} \mathbb{R}$ with respect to the group homomorphism*

$$\begin{aligned} \varphi : \mathbb{R} &\longrightarrow GL(2, \mathbb{R}) \\ s &\longmapsto A_+(s) := \begin{pmatrix} \cosh(s) & \sinh(s) \\ \sinh(s) & \cosh(s) \end{pmatrix}. \end{aligned}$$

3.4 Conformal Manifolds

We will introduce now the concept of conformal manifolds in the context of conformal geometry, which is analogous to the concept of semi-Riemannian manifolds within semi-Riemannian geometry. For further, though still introductory, information on conformal manifolds see [Eas96, CG18, FH20].

Definition 3.23. Consider two metrics g, g' on a manifold M. These metrics g, g' are *conformally equivalent*, $g \sim g'$, if there exists a smooth function $\Omega : M \to \mathbb{R}$ such that $\Omega(p) \neq 0$ for every $p \in M$ and for every $X, Y \in T_pM$

$$g'(X, Y) = \Omega^2(p) g(X, Y).$$

Then, for every M one can define the *conformal metric* as the equivalence class

$$[g] := \{\text{metrics } g' \text{ on } M \mid g \sim g'\}.$$

Example 3.24. Let M be the Riemannian manifold $\mathbb{R}^{2,0}$. The positive-definite metrics g, g' defined as

$$(g_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \text{ and } (g'_{ij}) = \begin{pmatrix} 2e^{2(x+y)} & -e^{2(x+y)} \\ -e^{2(x+y)} & 3^{2(x+y)} \end{pmatrix}$$

for every $(x, y) \in \mathbb{R}^{2,0}$ are conformally equivalent, since $g'(X, Y) = e^{2(x+y)}g(X, Y)$ for every $X, Y \in T_{(x,y)}M$.

Definition 3.25. Let g be a metric on M. Then the pair $(M, [g])$ is a *conformal manifold*.

Example 3.26. For the Lorentzian ($\nu = 1$) flat manifold, the metric can be expressed as $\eta = \text{diag}(-1, 1, \ldots, 1)$. Therefore, $(\mathbb{R}^{1,1}, [g])$ with

$$[g] = \{g = \omega^2(p)\text{diag}(-1, 1) | \omega : \mathbb{R}^{-1,1} \to \mathbb{R} \text{ is smooth}\}$$

is a conformal manifold.

Definition 3.27. Let g be a metric on M. Then the pair $(M, [g])$ is a *conformally flat manifold* if g is conformally equivalent to the flat metric η.

Example 3.28. Let $\left(\mathbb{R}^{1,1}, [g]\right)$ be a conformal manifold, considering the coordinates $\{t, x\}$, with $g = e^{-t^2+x^2}\text{diag}(-1, 1)$. Then, $\left(\mathbb{R}^{1,1}, [g]\right)$ is conformally flat.

On conformally flat manifolds, every point has a neighbourhood which can be mapped by a conformal transformation to a flat manifold, whose Riemann curvature tensor (Definition 2.70) is everywhere zero.

Considering these definitions, a conformal manifold can be also understood as the equivalence class of a smooth manifold, with respect to the conformal equivalence of the metrics on it. Therefore, on a conformal manifold lengths can not be considered. However, one can still define and measure angles, between, for instance, tangent vectors or curves, and ratios of lengths at a point $p \in M$.

An interesting discussion can be found in [d'I92, Section 6.13] or [FZ11, Section 3.6]. Let us present an interesting result in [d'I92]:

Proposition 3.29 (see Section 6.13 of [d'I92])**.** *Every 2-dimensional semi-Riemannian manifold is conformally flat.*

Remark 3.30. For a semi-Riemannian manifolds of dimension $n \geq 4$, it can be seen [FZ11] that it is conformally flat if its Weyl tensor vanishes. For the case $n = 3$ the semi-Riemannian manifold is conformally flat if and only if the so called *Cotton tensor* vanishes. The components of the Cotton tensor are

$$C_{ijk} = 2R_{i[j;k]} - \frac{1}{n-1}g_{i[j}R_{;k]},$$

which for $n = 3$ get the form

$$C_{ijk} = 2R_{i[j;k]} - \frac{1}{2}g_{i[j}R_{;k]}\,.$$

Conformal Lorentzian manifolds have a notorious relevance for theoretical physics. For these manifolds, studied within A. Einstein's theories of relativity, the spacetime interval, which is analogue to the concept of distance, can no longer be considered. Yet the light-cone structure, which is closely related to the causal structure, is preserved by conformal transformations and can still be considered.

4 Conformal Compactifications and Penrose Diagrams

Penrose diagrams have been widely used both in scientific literature and in pedagogy of mathematical physics [HP70, Wri14]. They consist of a conformal transformation of the physical Lorentzian manifold together with the conformal infinity, which is the boundary of this new unphysical compact manifold. That is why they are a conformal compactification of spacetimes.

Light beams always travel at 45° within a Penrose diagram and causality is preserved, so they allow to study the causal structure of spacetimes. The boundaries of these diagrams correspond either to infinity or singularities. Light must end at some point the boundary. With that, Penrose diagrams are also useful to study asymptotic properties of spacetimes, as well as singularities.

In this section, we provide a brief description of Penrose diagrams following [Wri14], after defining the so called conformal compactifications [CG18, FH20]. We also provide two detailed examples.

4.1 Conformal Compactifications

Definition 4.1. Let Σ be a submanifold of M. A *defining function* for Σ is a smooth real-valued function on M, r, such that $r^{-1}(0) = \Sigma$ and dr is non-vanishing at every point of Σ.

Definition 4.2. A compact semi-Riemannian manifold $(\bar{M}, \bar{g})$ with boundary Σ is a *conformal compactification* of (M, g) if M is the interior of $\bar{M}$ and there is a defining function r of ∂M such that

$$\bar{g}|_M = r^2 g.$$

The boundary (Σ, γ), where $\gamma = \bar{g}|_\Sigma$ is the boundary metric associated to the compactification $\bar{g}$, is called the *conformal infinity*.

For a given (M, g), there are many possible conformal compatifications. That is why only the conformal class $[\gamma]$ is uniquely determined by (M, g).

These concepts give physicists a powerful tool for solving problems related to the asymptotics of quantities in spacetimes. They rescale their original problem from the original physical semi-Riemannian manifold to the unphysical compactified manifold where they solve the problem on conformal infinity to finally undo the rescaling back to the physical manifold.

For a further discussion on conformal compactifications and their relation to physics we refer to [And06, And05]. There, you may find studies regarding the behavior of gravitational fields at infinity introduced by R. Penrose [Pen65] or the AdS/CFT correspondence introduced by J. M. Maldacena [Mal98], respectively.

4.2 Penrose Diagrams

Representations of spacetime diagrams begun with Minkowski's contribution to Special Relativity, considering the real plane and representing time on the vertical axis and space, as the radius from a center for example, on the horizontal axis. For this diagram to represent the "fabric" of spacetime, both axis should have the same units. That is why the vertical axis represents in fact the speed of light times the time, ct, even though physicists usually take $c = 1$ and omit it. It explains why light travels at 45º in this diagram, drawing the so called light-cones within which every observer is forced to move: the fact that nothing can travel faster than the speed of light c forces all observer's possible world-lines to have a slope greater than 45°.

It can be thought that every point in this diagram represents a spatial two-sphere, S^2. The angular dimensions are not represented in the diagram because the paper has only two dimensions and we use one to represent time, so the representation of only one spatial dimension is the clearest option.

Penrose diagrams allow to represent the infinite spacetime within a finite paper sheet and which also preserved the feature that photons followed straight trajectories at 45°. One of the most widely recognised Penrose diagram is the one for the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_{II}^2 .$$

As explained in the introduction, the procedure to obtain the Penrose diagram for the Schwarzschild metric is essentially a conformal transformation and can be also found in [LV19], where it also explains that there is an essential singularity at $r = 0$. The scalar of curvature diverges at this point, where the manifold is not smooth. It is represented in Figure 3.

We now continue with two examples of Penrose diagrams together with the explanation of how we obtain them.

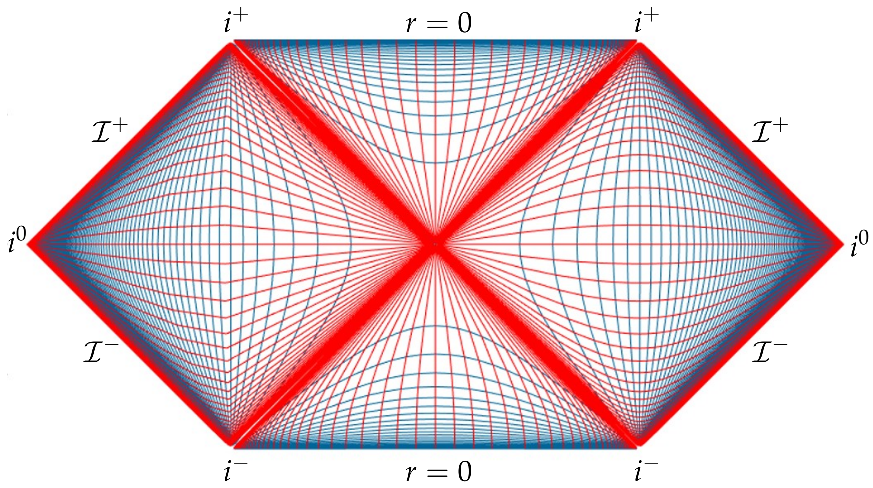

Figure 3: Extended Penrose diagram for the Schwarzschild metric. Red (resp. blue) lines represent hypersurfaces of constant t (resp. r). The labbels correspond to those explained in Case 4.2.1.

4.2.1 Minkowski spacetime

As it can be seen both in [Wri14] or [Str94] or many other papers, we start with the Minkowski spacetime metric

$$ds^2 := g_{ij}dx^i dx^j = -dt^2 + dx^2 + dy^2 + d^2 = -dt^2 + dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$$
$$= -dt^2 + dr^2 + r^2 d\Omega^2_{II}\,. \tag{3}$$

From now on, we will follow the procedure used in [Str94]. We will focus on the (r,t) plane. The basic idea is to start by rotating the spacetime (r,t) diagram so that the axis represent the light trajectories, then bringing the infinity to our paper sheet and finally rotating the diagram again so that light follows 45° straight lines again. We begin changing (3) to light-cone coordinates (u,v)

$$\begin{cases} u = t - r \\ v = t + r \end{cases} \Rightarrow -dudv = -(dt^2 - dr^2) = -dt^2 + dr^2 \Rightarrow ds^2 = -dudv + r^2 d\Omega^2_{II}\,, \tag{4}$$

where $-\infty < u,v < \infty$. The conformal infinity is a disjoint union of timelike, lightlike and spacelike infinity. These concepts are defined as follows:

- *Future timelike infinity* (i^+): where we would get if we fix r and take t to ∞.
- *Past timelike infinity* (i^-): where we would get if we fix r and take t to $-\infty$.
- *Spacelike infinity* (i^o): where we would get if we fix t and take r to ∞.

- *Future null infinity* ($\mathcal{I}^+$): where we would get if we fix u and take v to ∞.
- *Past null infinity* ($\mathcal{I}^-$): where we would get if we fix v and take u to $-\infty$.

Note that, for Minkowski space, every photon would end at $\mathcal{I}^+$ and could come from $\mathcal{I}^-$. Now we have settled these new light-cone coordinates expressing the metric with them (4) and defined the conformal infinity, it is time to bring the infinity closer. Here we have many possibilities, like the ones used by R. Sachs in [Sac64]. Among all the possibilities we are going to use the change of coordinates used in [Str94]

$$v = t + r = \tan\left(\frac{1}{2}(\psi+\zeta)\right), \quad u = t - r = \tan\left(\frac{1}{2}(\psi-\zeta)\right), \tag{5}$$

with $\zeta \pm \psi < \pi$, $\zeta > 0$, since $r \geq 0$, forming a half diamond (Figure 4). From (5),

$$dv = \cos^{-2}\left(\frac{1}{2}(\psi+\zeta)\right)\frac{1}{2}(d\psi + d\zeta), \quad du = \cos^{-2}\left(\frac{1}{2}(\psi-\zeta)\right)\frac{1}{2}(d\psi - d\zeta),$$

and therefore

$$dudv = \frac{1}{4}\cos^{-2}\left(\frac{1}{2}(\psi+\zeta)\right)\cos^{-2}\left(\frac{1}{2}(\psi-\zeta)\right)(d\psi^2 - d\zeta^2). \tag{6}$$

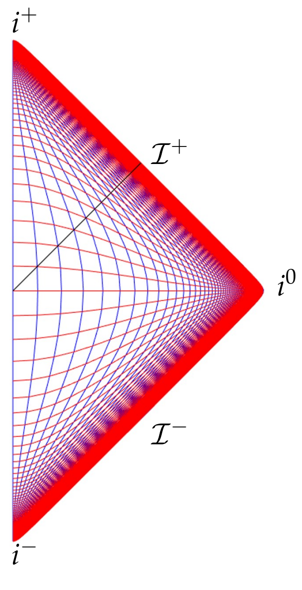

Figure 4: Penrose diagram for the Minkowski spacetime in the (ζ, ψ) plane. The black line represents a light beam from the origin of coordinates to future null infinity $\mathcal{I}^+$.

Using (6), we can rewrite metric (4) as

$$\begin{aligned}
ds^2 &= -dudv + r^2 d\Omega_{II}^2 \\
&= -\frac{1}{4}\cos^{-2}\left(\frac{1}{2}(\psi+\zeta)\right)\cos^{-2}\left(\frac{1}{2}(\psi-\zeta)\right)(d\psi^2 - d\zeta^2) + r^2 d\Omega_{II}^2 \\
&= \Omega^{-2}(\psi,\zeta)(-d\psi^2 + d\zeta^2) + r^2 d\Omega_{II}^2,
\end{aligned}$$

where $\Omega(\psi,\zeta) = 2\cos\left(\frac{1}{2}(\psi+\zeta)\right)\cos\left(\frac{1}{2}(\psi-\zeta)\right)$ is the conformal factor, introducing a new unphysical metric $\bar{g}_{ij}$ which is conformal to the physical metric,

$$\bar{g}_{ij} = \Omega^2 g_{ij}\,.$$

Equivalently we have $\Omega(r,t) = 2\cos(\arctan(t+r))\cos(\arctan(t-r))$.

Since we have the combinations $\psi+\zeta$ and $\psi-\zeta$ in the new redefinition of u and v (5), we have already done the second rotation so that the light particles follow a straight line of 45° from vertical. At the same time, we brought the infinities to the paper sheet by applying the tangent to $\frac{1}{2}(\psi+\zeta)$ and $\frac{1}{2}(\psi-\zeta)$, respectively.

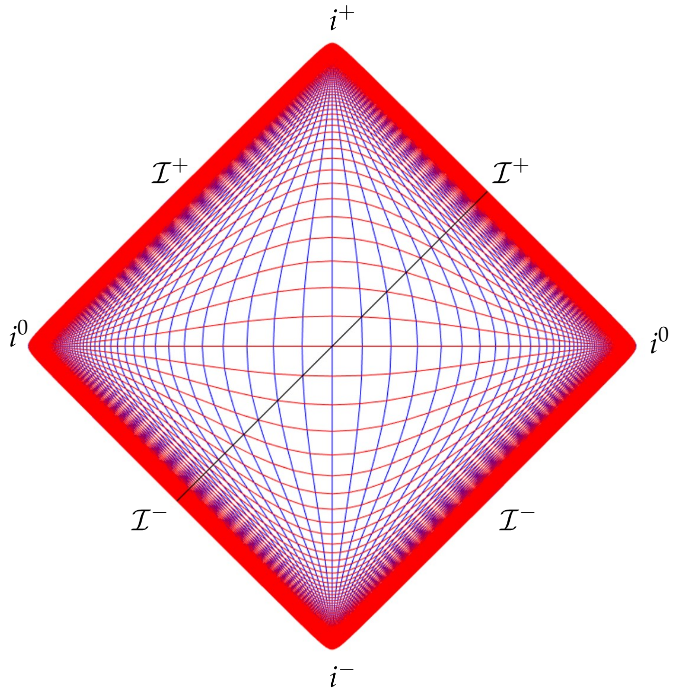

Figure 5: Extended Penrose diagram for the Minkowski spacetime in the (ζ,ψ) plane. The black line represents a light beam from the past null infinity (left) $\mathcal{I}^-$ to the future null infinity (right) $\mathcal{I}^+$.

Now, to obtain the full diamond as a representation of the $1+1$ dimensional Minkowski spacetime we can represent the following metric

$$ds^2 = -dt^2 + dx^2 = -dx^+ dx^- \,,$$

where we have used $x^\pm \equiv t \pm x$ and $x^\pm = \tan\left(\frac{1}{2}(\psi \pm \zeta)\right)$ as before, but with the difference that now, since $x \in (-\infty, +\infty)$, $|\zeta \pm \psi| < \pi$ and we therefore get the full diamond (Figure 5). To show that light particles follow indeed a 45° straight line we have also plotted a light beam in Figure 4 and in Figure 5.

4.2.2 Modified JT gravity

Following [Ai21, Wit20], we present here a model which has recently returned to the attention of physicists. The modified Jackiw-Teitelboim (JT) gravity model bulk action is

$$I_{\mathrm{JT}} = \frac{1}{16\pi G} \int d^2x \sqrt{-g}\, [\phi S + W(\phi)] \,,$$

where G is the gravitational Newton constant in $2D$, ϕ is a scalar field, called dilaton, S is the scalar of curvature of the 2-dimensional spacetime and $W(\phi)$ is a potential. The original JT model corresponds to $W(\phi) = 2\Lambda\phi$, where Λ is the cosmological constant. We shall consider the modified JT model with the potential $W(\phi) = \mathrm{sech}^2\phi$. Consider the action I_{JT} together with the metric

$$ds^2 = -A(x)dt^2 + \frac{1}{G(x)}dx^2.$$

Applying Hamilton's principle, which states that the action of the system must satisfy $\frac{\delta I_{\mathrm{JT}}}{\delta q(x)} = 0$ for $q(x) \in \{A(x), G(x)\}$, and considering the Schwarzschild-like case $A(x) = G(x)$, one can obtain [Ai21] the equations

$$\phi' A' - W(\phi) = 0\,, \quad \phi'' = 0\,,$$

where the prime symbol denotes the derivative with respect to the coordinate x. Solving these equations (after an appropriate transformation so that $\phi(x) = x$), one can rewrite the metric as

$$ds^2 = -(\tanh x + C)dt^2 + \frac{1}{\tanh x + C}dx^2, \tag{7}$$

where $C \in \mathbb{R}$ is a constant of integration. We consider $C \in (-1, 1)$ in order to have an event horizon, which requires the function $A(x) = \tanh x + C$ to vanish at some $x \in \mathbb{R}$. We can now compute the null geodesics, for which $ds^2 = 0$, obtaining

$$t = \pm \frac{Cx - \log\left(C\cosh x + \sinh x\right)}{C^2 - 1} + t_0\,. \tag{8}$$

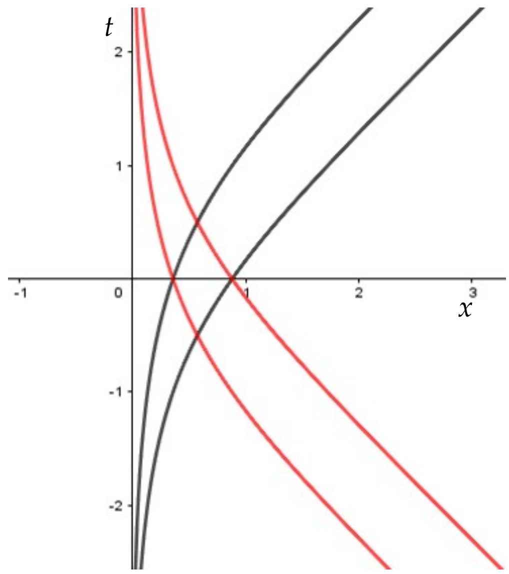

Figure 6: Plot of the outgoing (black) and ingoing (red) null geodesics (8) for $C = 0$.

The $+$ (resp. $-$) sign of $\pm$ in (8) corresponds to outgoing (resp. ingoing) light beams and t_0 is a constant (Figure 6).

To obtain a maximally extended Penrose diagram we will consider the following coordinate changes:

$$\begin{tikzcd}
(t,x) \arrow[r, "\text{Tortoise}"] & (t,x_*) \arrow[r, "\text{Retarded}", "\text{Advanced}"'] & (u,v) \arrow[d, "\text{Kruskal-like}"] \\
(T,X) & (\bar{U},\bar{V}) \arrow[l, "\text{Rotation}"'] & (U,V) \arrow[l, "\text{Rescaling}"']
\end{tikzcd} .$$

Consider tortoise coordinates (t, x_*), which must satisfy

$$\frac{dx_*}{dx} = A^{-1}(x) = \frac{1}{C + \tanh x},$$

so then, after integrating,

$$x_* = \frac{Cx - \log(C * \cosh x + \sinh x)}{C^2 - 1}\,, \quad x_* \in (-\infty, +\infty)\,. \tag{9}$$

Introducing retarded and advanced coordinates (u, v),

$$u = t - x_*\,, \quad v = t + x_*\,, \quad u, v \in (-\infty, +\infty)\,,$$

one can define the Kruskal-like coordinates

$$U = -e^{-\left(1-C^2\right)u/2}, \quad V = e^{\left(1-C^2\right)v/2}.$$

Here, $U < 0$ and $V > 0$. Consider how we can rewrite the metric (7). From (9),

$$dx_* = \frac{dx}{\tanh x + C} \Rightarrow ds^2 = -(\tanh x + C)dt^2 + \frac{1}{\tanh x + C}dx^2$$
$$= -(\tanh x + C)(dt^2 - dx_*^2) = -(\tanh x + C)dudv\,.$$

From the Kruskal-like coordinates, using that $v - u = 2x_*$,

$$\begin{cases} dU = -\frac{(-1+C^2)}{2}e^{(-1+C^2)\frac{u}{2}}du \\ dV = -\frac{(-1+C^2)}{2}e^{-(-1+C^2)\frac{v}{2}}dv \end{cases} \Rightarrow dUdV = \frac{\left(1-C^2\right)^2}{4}e^{(-1+C^2)\frac{u-v}{2}}dudv$$
$$\Rightarrow dudv = \frac{4}{\left(1-C^2\right)^2}e^{(-1+C^2)x_*}dUdV,$$

so we can rewrite the metric (7) as

$$ds^2 = -(C + tanhx)dudv = -(C + tanhx)\frac{4}{\left(1-C^2\right)^2}e^{Cx-\log(C\cdot\cosh x+\sinh x)}dUdV$$
$$= -\frac{C\cdot\cosh x + \sinh x}{\cosh x}\frac{4}{\left(1-C^2\right)^2}e^{Cx}\frac{1}{(C\cdot\cosh x + \sinh x)}dUdV$$
$$= -\frac{4e^{Cx}}{\left(1-C^2\right)^2\cosh x}dVdU.$$

Now, in order to bring the infinity into the sheet of paper, we define

$$\bar{U} = \arctan U\,, \quad \bar{V} = \arctan V\,, \quad \bar{U}, \bar{V} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

with $dUdV = (1 + U^2)(1 + V^2)d\bar{U}d\bar{V}$. After rotating the axis by changing to

$$T = \frac{1}{2}(\bar{V} + \bar{U})\,, \quad X = \frac{1}{2}(\bar{V} - \bar{U})\,, \quad T, X \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

with $-dT^2 + dX^2 = -d\bar{U}d\bar{V}$, one obtains the Penrose diagram in Figure 7.

Let us now compute the final metric:

$$ds^2 = -\frac{4e^{Cx}}{\left(1-C^2\right)^2\cosh x}dVdU = -\frac{4e^{Cx}}{\left(1-C^2\right)^2\cosh x}(1 + U^2)(1 + V^2)d\bar{U}d\bar{V}$$
$$= -\frac{4e^{Cx}}{\left(1-C^2\right)^2\cosh x}\left(1 + e^{-\left(1-C^2\right)(t-x_*)}\right)\left(1 + e^{\left(1-C^2\right)(t+x_*)}\right)d\bar{U}d\bar{V}$$
$$= \frac{4e^{Cx}}{\left(1-C^2\right)^2\cosh x}\Big[1 + 2\cosh\left(\left(1-C^2\right)t - Cx + \log\left(C\cdot\cosh x + \sinh x\right)\right)$$
$$+ e^{-2Cx}(C\cdot\cosh x + \sinh x)\Big]\left(-dT^2 + dX^2\right).$$

If we consider the case $C = 0$, the metric gets the form

$$ds^2 = \frac{4}{\cosh x} \left(1 + 2\cosh t + \log \sinh x + \sinh x\right) \left(-dT^2 + dX^2\right),$$

from which we obtain the conformal factor

$$\Omega(t,x) = \frac{1}{2} \left(\frac{\cosh x}{1 + 2\cosh t + \log \sinh x + \sinh x} \right)^{1/2}$$

of a new unphysical metric $\bar{g}_{ij}$ which is conformal to the physical metric g_{ij},

$$\bar{g}_{ij} = \Omega^2 g_{ij}\,.$$

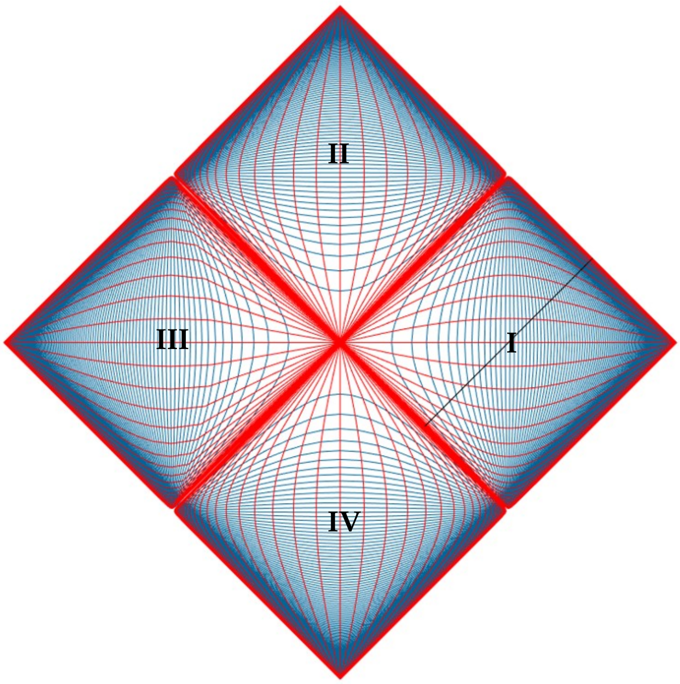

Figure 7: Extended Penrose diagram for the metric (7) with $C = 0$ in the (X,T) plane. Red (resp. blue) lines represent hypersurfaces of constant t (resp. x). Region I corresponds to $x > 0$ and region III is a copy of region I. Region II (resp. IV) corresponds to $x < 0$ for the ingoing (resp. outgoing) case in (8). The black line in region I is a null geodesic.

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