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Undergraduate Thesis

MAJOR IN MATHEMATICS

Study of price and assortment competition in retail

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Abstract

In this work we study the competition between two retailers that have to choose which of the available products to sell and at which price, knowing that each customer relates a product and its price with a value and will only acquire the product with highest value.

In the first part of this project we present our framework by defining some of the basic concepts of Game Theory and introducing new terminology. Moreover, we also review two of the pioneering models in duopolistic competition: the Cournot and Bertrand model.

The second part of the project formally defines and models the competitive situation. We formally establish the actions available for both retailers and determine how each customer ranks all products for sale. Moreover, in this chapter we also study the effects that this competition has, the possible equilibria of this competitive market and under which circumstances both retailers can reach such equilibria.

The third part of the project further studies the relationship between the price a retailer sets for his products and whether the product is common or exclusive. We attempt to model this relationship with a linear regression model and we study the results with data from clothing retailers Gucci and Farfetch.

Resum

En aquest treball estudiem la competència entre dos minoristes que han d'escollir quins dels productes disponibles posar a la venda i a quin preu, sabent que cada client relaciona un producte i el seu preu amb un valor i només adquirirà el product de major valor.

En la primera part del projecte presentem el marc de treball definint alguns conceptes bàsics de la Teoria de Jocs i introduïm terminologia nova. A més, revisem dos dels models pioners en la competència duopolística: els models de Cournot i Bertrand.

La segona part del projecte defineix i modela formalment la situació competitiva. Establim les accions disponibles per cada minorista i determinem com cada client classifica els productes a la venda. A més, també estudiem els efectes que té aquesta competència així com els possibles equilibris d'aquest mercat competitiu i en quines circumstàncies es poden assolir.

La tercera part del projecte aprofundeix en la relació entre els preus fixats per cada minorista i si el producte és exclusiu o comú. Intentem modelar aquesta relació amb un model de regressió lineal i estudiem els resultats amb les dades dels minoristes Gucci i Farfetch.

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¿Qué es la vida? Un frenesí. ¿Qué es la vida? Una ilusión, una sombra, una ficción; y el mayor bien es pequeño; que toda la vida es sueño, y los sueños, sueños son. («La vida es sueño»).

Calderón de la Barca (1600-1681)

Contents

Introduction

How many times have we heard that a company has decided to stop selling or lower the price of a product that we like or we regularly buy? Sometimes companies withdraw a product from the market because of sanitary reasons. We have heard many times that a product does not meet minimum sanitary requirements and authorities remove it from shops.

However, in other occasions products are removed from the market because they are no longer interesting to the seller or their price is lowered as a last resource in order to stimulate customers to buy them. For example, a very recent commotion in retailers' coexistence caused by the descend of 2000 Aldi products [8]. This can happen due to various reasons such as a change in customers' preferences, an increase in the cost of producing the products or the appearance of better products offered by the competitors, among others.

Therefore, it is clear that retailers face difficult planning problems when selecting which of the available products to offer and at which prices. They have to take into consideration different factors such as how customers select products or how they distinguish two similar products offered by different retailers in order to capture as many customers as possible. Consequently, retailers have to be very informed and up to date on recent tendencies such as products and prices offered by their competitors, new interesting products on trend or, in general, any new socioeconomic trend.

In this project, we focus on the competition between two retailers that have to decide, simultaneously, which of the available products to offer to the public and at which price, taking into account that some of the available products are only available to one of the retailers, while other products are available to both retailers. We deeply study the case where there are no display limitations, in which retailers can offer all of the available products at once and we also briefly comment what difficulties arise when retailers face display limitations.

As we commented above, a key concept for this kind of competition, that retailers have to take into consideration and try to predict trends, is the behaviour of customers. For this study, we consider that customers' behaviour follows a multinomial Logit model: customers relate each product with a value, rank the values of all offered products and only buy the product with the highest ranking. This model is widely used in Operation Research when modelling discrete choice situations because of its simplicity.

Project structure

The first chapter of this study is dedicated to recalling some of the basic concepts of Game Theory and introducing new terminology. We formally define the concept of a game in strategic form and best responses, as well as two important strategic situations for the competitors, the Nash equilibrium and the Pareto optimal. Moreover, we review two of the pioneering models in duopolistic competition: the Cournot and Bertrand models. Finally, we briefly introduce lattice theory and comment the importance of Fixed Point theorems when applied to Game Theory.

In the second chapter of this study we focus on defining and modelling the competition. On the one hand, we describe the available actions of each retailer, while on the other hand we introduce the multinomial Logit model applied to the behaviour of the consumers. Moreover, we study what consequences the model has when applied to competition in products only available to one of the retailers, obtaining interesting results such as the existence of Nash equilibria or the existence of an equilibrium that both retailers prefer.

The third chapter of this work is dedicated to studying the consequences of the model when applied to competition in both exclusive and common products. Similarly to the case of competition only in exclusive products, we obtain interesting results such as the existence of Nash equilibria when retailers compete and face no capacity limitations. However, in this case, we see that we cannot guarantee that there is an equilibrium that both retailers prefer nor the existence of Nash equilibria when retailers face capacity limitations.

In the fourth chapter of this study we focus on studying the relationship between the price a retailer sets for a product and whether a product is exclusive or common. We study the particular case of competition between clothing retailers Farfetch and Gucci during the winter season of 2020, and attempt to model the relation between the price and the exclusiveness with a linear regression model.

Related literature

Although the multinomial Logit model is relatively recent, in the past years it has been widely used in Operations or Marketing research when modelling customers' choice. As we comment later, one of the pioneers of developing the multinomial Logit model theory was Daniel McFadden in 1974 [16].

Furthermore, the theory of this model was further developed by Guadagni and Little in 1983 [11] where they applied the multinomial Logit model for coffee brand choice and studied the fidelity of clients; and Ben-Akiva and Lerman in 1985 [2] where they applied this model to the choice of travel companies.

More theory particularly related to competition between retailers under the multinomial Logit model of choice has been developed by Misra in 2008 [17] where he studies retailers' best responses when competing in both assortment selection with exclusive products and prices selection. This topic was further studied by Besbes and Sauré in 2016 [4] where they study the more general case of competition in assortment selection with common and exclusive products and prices selection and find equilibrium existence and uniqueness. This project follows the model presented by Besbes and Sauré.

Furthermore, the oligopolistic competition in assortment selection and prices selection has also been studied from an empirical point of view with interesting papers such as

Draganska and Jain in 2006 [6] or Draganska et al. in 2009. [7]

Finally but not less important, the theory of monopolistic optimization has also been thoroughly studied. For example, Mahajan and van Ryzin [14] in 2001 study this topic when the retailer also faces inventory decisions and customers' choice is modelled by a multinomial Logit model. Furthermore, Maddah and Bish [13] in 2007 continue this work and also study the case where the retailer also selects prices.

Chapter 1

Preliminaries

In this chapter we recall some of the basic concepts of Game Theory, which are needed for the work in the next pages. The first section of this chapter introduces basic terminology, following the notation used by Pérez, Jimeno and Cerdá in *Teoría de Juegos* (2004) [21]. The second part of this chapter studies the best individual decisions of each player and introduces the notion of equilibrium of a game and the Pareto efficiency. The third part of this chapter is dedicated to the Cournot and Bertrand models, two of the first models of duopolistic competition. Another interesting reading that served as inspiration for this chapter is *Game theory for applied economists* (1992) [10] by Robert S. Gibbons.

1.1 The game

Informally speaking, we say that a game is every situation controlled by some rules and with a well-defined result characterized by strategic interdependence, that is, the outcome depends on the actions taken by the players. Those rules dictate the possible actions each player has at disposition, the order in which every action takes place and how every action affects the outcome of the situation. Furthermore, every player is assumed to be rational, or in other words, players select actions that bring the best possible outcome, knowing that the result is affected by the actions of all players.

The grounds of Game Theory were set by John von Neumann and Oskar Morgenstern in *Theory of Games and Economic Behavior* (1944) [19], where they studied the zero-sum games and applied their results to war strategy. In that work, they introduced the axioms of the expected utility theory and the notion of game was first formalized.

With that said, we can now formally define a game.

Definition 1.1. *A* game in strategic form *is a triplet* $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ *where* N *is a finite set (the players),* S_i *is a countable or uncountable set (the strategy space of player* i*) and* $u_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ *is a function (the utility function of player* i*).*

The combination of strategies $s = (s_1, s_2, \ldots, s_n) \in S_1 \times S_2 \times \cdots \times S_n$, where component i designates a strategy for player i, is also called a *strategy profile*. In order to simplify the notation, we say that S_{-i} is the strategy space of all players but player i, hence $S_{-i} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$. The elements of S_{-i} are the strategy profiles $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \in S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$.

However, we notice that not all strategies are equal. Rational players select strategies to maximize their utility, and therefore we may think that there are strategies that are better than others.

Let Γ be a game. Consider $s_i, s_i' \in S_i$ two strategies of player i. We say that s_i *strictly dominates* s_i' if $\forall i \in N$ and $\forall s_{-i} \in S_{-i}$

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}).$$

We say that s_i *weakly dominates* s_i' if $\forall i \in N$ and $\forall s_{-i} \in S_{-i}$

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}).$$

Definition 1.2. *Let Γ be a game. Let $i \in N$, then the strategy $s_i^* \in S_i$ is a* best response *to a strategy profile $s_{-i} \in S_{-i}$ if*

$$u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \textit{for all} \quad s_i \in S_i.$$

It is worth mentioning that best responses are not necessarily unique.

1.2 The Nash equilibrium

Years later John Forbes Nash[1] obtained his doctorate in Mathematics with the article *Non-cooperative games* (1950) [18], where he defined the equilibrium: a particular solution of a game, a more general solution than the one previously given by Von Neumann and Morgenstern for zero-sum games. This concept revolutionized the theory developed until the date and would become one of the fundamental notions of Game Theory. Throughout the history, many crucial decisions related to economics or politics were taken having in consideration the Nash equilibrium, and that shows the importance of John Nash's work. In 1994 John Forbes Nash, Reinhard Selten and John Harsanyi were awarded the Economics Nobel Prize *"for their analysis of equilibria in non-cooperative games theory"*.

Definition 1.3. *Let Γ be a game. A* Nash equilibrium *is a strategy profile $s^* = (s_1^*, \ldots, s_n^*) \in S_1 \times \ldots \times S_n$ such that $\forall i \in N$ and $\forall s_i \in S_i$*

$$u_i(s^*) \geq u_i(s_i, s_{-i}^*).$$

We observe that a Nash equilibrium is a combination of strategies where every strategy is a best response to the others' strategies.

In order to better illustrate and exemplify the terminology we introduced in these past sections, we study the well-known *Prisoner's dilemma*.

Example 1.1. *The police investigate a robbery and caught two suspects. The suspects are in prison and the police want them to confess their crime because they have not enough proofs. In order to do so, each suspect is individually interrogated and is given two options: to confess or to remain silent.*

[1] John Forbes Nash (1928-2015) was an American mathematician, specialized in Game Theory, Differential Geometry and Partial Derivative Equations. He won the Economics Nobel Prize in 1994 and the Abel Prize in 2015.

If a suspects confesses and the other remains silent, the confessor is freed and the other suspect is imprisoned for 4 years. If both suspects confess, they will both be imprisoned for 3 years. Finally, if both suspects remain silent, they will both be imprisoned for 1 year.

Therefore, we have a 2-player game, $N = \{1, 2\}$*, where the strategy space of each player is* $S_1 = \{C, S\}$ *and* $S_2 = \{C, S\}$*, where* C *denotes the strategy of confessing and* S *denotes the strategy of remaining silent.*

We can now the payoff matrix of this game.

	C	S
C	($\underline{-3}$, $\underline{-3}$)	($\underline{0}$, -4)
S	(-4, $\underline{0}$)	(-1, -1)

Table 1.1: Payoff matrix of the game

In this table, the best responses are underlined. For example, we can see that for player 2 to confess is a best response *to player 1's confession because* $u_2(C, S) = -4 < -3 = u_2(C, C)$.

Moreover, we have that for both players to confess dominates *remaining silent because* $u_1(C, C) = -3 > -4 = u_1(S, C)$ *and* $u_1(C, S) = 0 > -1 = u_1(S, S)$*, and similarly for player 2.*

Finally, we notice that there exists one Nash equilibrium that is (C, C) *because in this situation both players play a best response. Therefore, in equilibrium, both players confess and are imprisoned during 3 years. It is clear that this outcome is not "efficient", since both players would have a better outcome if they both remained silent. We see a formal definition of efficiency in the next section.*

1.3 Pareto efficiency

Many times we are interested in studying if an economic situation's outcome is "optimal". An interesting measure of this optimality was introduced by Italian economist Vilfredo Pareto in *Manual of Political Economy* (1906) [20]. In this section we introduce the terminology used by Pareto following modern-day notation.

Informally speaking, we say that an economic situation can be *Pareto improved* when it is possible to improve someone's outcome. If an economic situation cannot be Pareto improved, we say that it is *Pareto efficient*.

Definition 1.4. *Let* Γ *be a game. We say that a strategic profile* $s \in S_1 \times \ldots \times S_n$ *can be* Pareto improved *if there exists another strategic profile* $s' \in S_1 \times \ldots \times S_n$ *such that*

$$u_i(s') \geq u_i(s) \text{ for all } i \in N \text{ and } u_i(s') > u_i(s) \text{ for some } i \in N.$$

Moreover, we say that a strategic profile $s \in S_1 \times \ldots \times S_n$ *is* Pareto efficient *if there are no possible Pareto improvements.*

An interesting way to visually represent strategic profiles that are Pareto efficient is through a particular curve called *Pareto frontier*. This representation is used in economics and engineering. If we consider Γ to be a game, we formally define the Pareto frontier as the set

$$P(\Gamma) = \{s \in S_1 \times \ldots \times S_n : s \text{ cannot be Pareto improved}\}$$

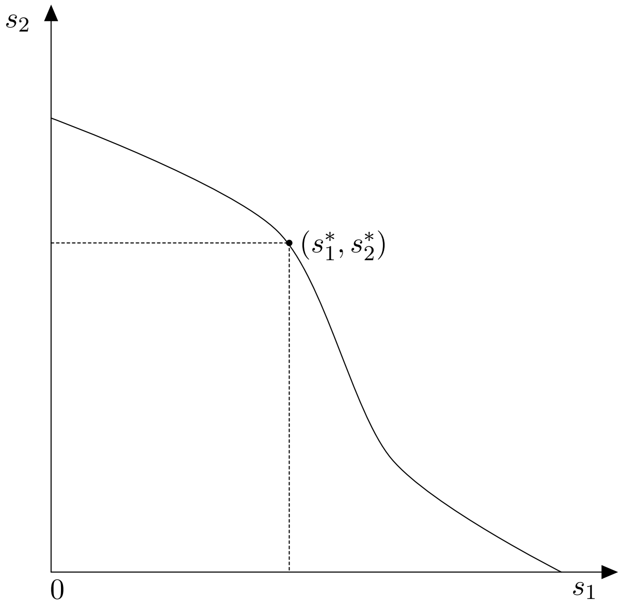

Figure 1.1: Example of a Pareto frontier with two players

1.4 The Cournot competition

One of the main applications of Game Theory is the study of competition between various companies that dominate a certain market (oligopolies). As commented by Mas-Colell et al. (1995) [15] or Jehle et al. (2011) [12], it is natural to study this type of competition as a game because decisions and strategic interactions are crucial. This topic has been widely studied until this date, particularly the distinction between competition in prices and competition in quantities.

Antoine Augustin Cournot is considered to be the pioneer of quantity competition, as he first defined and studied this model in *Recherches sur les principes mathématiques de la théorie des richesses* (1838) [5]. In order to define the model, we use the approach and notation of modern-day microeconomics. Moreover, we study this model as a non-cooperative game where strategic variables are the quantities a company is willing to produce.

The simplified Cournot model considers two companies A and B that compete in quantities of a homogeneous product, that is, both companies have the same product and they choose how many units they produce. We assume that companies simultaneously choose to produce quantities q_A and q_B, respectively. We also assume that the inverse demand function (the price as a function of quantity) is decreasing and linear, that is,

$$P(Q) = \begin{cases} a - bQ & \text{if } bQ < a, \\ 0 & \text{if } bQ \geq a, \end{cases}$$

where $a > b > 0$ are constant values and $Q = q_A + q_B$.

Furthermore, we consider marginal costs for both companies to be constant, equal and less than a, and therefore the cost functions are $C_A(q_A) = cq_A$ and $C_B(q_B) = cq_B$, for some constant $c < a$, and that all products will be sold.

Therefore, the profit functions for $q_A, q_B \in [0, \frac{a}{b}]$ that each company will try to maximize, given their competitor quantity selection, are

$$u_A(q_A, q_B) = q_A\,(a - bq_A - bq_B) - cq_A,$$
$$u_B(q_A, q_B) = q_B\,(a - bq_A - bq_B) - cq_B.$$

Using partial differentiation, we can find the best response functions: a function that assigns the company's best response for a given quantity of the other company. Finally, intersecting both best response functions we find the equilibrium

$$(q_A^*, q_B^*) = \left(\frac{a-c}{3b}, \frac{a-c}{3b}\right),$$

which is known as the Cournot equilibrium. These quantities are the Nash equilibrium of this game.

1.5 The Bertrand competition

In 1883, Joseph Bertrand published a book review [3] where he criticized Cournot's book and proposed his own model, where companies compete in prices. As in the previous case, we define this model using the modern-day notation of microeconomics. Furthermore, we study this model as a non-cooperative game where strategic variables is the price at which a company is willing to sell a product.

The simplified Bertrand model considers two companies A and B that compete in prices of a homogeneous product, that is, both companies have the same product and they choose at which price they sell the product. Once the price is fixed, the demand function determines the quantity that is sold. For this simplified model, we consider that the demand function is $q(p) = a - p$ for some constant $a > 0$.

We assume that companies simultaneously choose a price p_A and p_B, respectively, and buyers only buy from the company with the lowest price or in equal quantities if the price is equal. Therefore, the demand function (the quantity as a function of price) of company A is

$$q_A(p_A, p_B) = \begin{cases} 0 & \text{if } p_A > p_B, \\ \frac{q(p_A)}{2} & \text{if } p_A = p_B, \\ q(p_A) & \text{if } p_A < p_B, \end{cases}$$

and similarly for company B.

Moreover, we consider marginal costs for both companies to be constant and equal, therefore the cost functions are $C_A(q_A) = cq_A$ and $C_B(q_B) = cq_B$, for some constant $c > 0$.

Therefore, company A's profit function for $p_A, p_B \in [0, \infty]$ is

$$u_A(p_A, p_B) = \begin{cases} 0 & \text{if } p_A > p_B, \\ \frac{(p_A - c)(a - p_A)}{2} & \text{if } p_A = p_B, \\ (p_A - c)(a - p_A) & \text{if } p_A < p_B, \end{cases}$$

and similarly for company B.

In this situation, both companies tend to lower prices and therefore leading to the equilibrium

$$(p_A^*, p_B^*) = (c, c),$$

which is known as the Bertrand equilibrium. This equilibrium is a Nash equilibrium of this game. We observe that, in this case, the profit of both companies is 0, and therefore the model considered by Bertrand is ruinous for the competitors and more favourable to the buyers than the model considered by Cournot.

1.6 Fixed point theorems

In this section we are particularly interested in presenting a fixed point theorem proven by Topkis (1998) [23]. We recall that fixed point theory focuses on finding under which circumstances a given function $f : X \to X$, where X is a set, has a fixed point, that is, an $x_0 \in X$ such that $f(x_0) = x_0$.

Fixed point theory is originated by Henri Poincaré in 1895 with the paper *Analysis situs* where he studies manifolds and introduces the homology theory while studying differential equations. However, it was during the 1900s when this theory greatly developed. Mathematicians such as Brouwer or Kakutani proved very important theorems that years later would lead to results that revolutionized mathematics, such as Nash's equilibrium existence.

For example, Brouwer's fixed point theorem states that every continuous function defined on a nonempty convex compact of an Euclidean space and with values in this compact has a fixed point, while Kakutani's fixed point theorem is a generalization of Brouwer's fixed point theorem and states that every set-valued function defined on a nonempty convex compact of an Euclidean space and whose graph is closed has a fixed point.

As we see in the next pages, the problem of finding a best response for a particular strategy of the competitor is not always an easy problem to solve. However, we can find a more manageable transformation of the problem where we can apply this fixed point theorem.

In order to do so, we first introduce some terminology, following the notation used by Topkis (1998) [23] or Vives (1999) [24].

Let $(X, \geq)$ be a pair, where X is a nonempty set and $\geq$ is a binary relation on X. We say $(X, \geq)$ is a *partially ordered set* if $\geq$ is reflexive, transitive and antisymmetric. For example, the plane $\mathbb{R}^2$ with the ordering $(x, y) \geq (x', y')$ if $x \geq x'$ and $y \geq y'$ is a partially ordered set.

We say that a partially ordered set $(X, \geq)$ is a *lattice* if for every two elements $x \in X$ and $y \in X$ we have a supremum, that is, $\sup_X(x, y) = \inf\{z \in X : z \geq x, z \geq y\}$; and an infimum, that is, $\inf_X(x, y) = \sup\{z \in X : x \geq z, y \geq z\}$, in X. Again, for example, the plane $\mathbb{R}^2$ with the ordering $\geq$ we defined above is a lattice.

We say that a lattice $(X, \geq)$ is *complete* if for every nonempty subset $A \subset X$ we have that $\sup_X(A) \in X$ and $\inf_X(A) \in X$.

We say that a nonempty subset $A \subset X$ is a *sublattice of* X if for every $x \in A$ and $y \in A$ we have that $\sup_X(x, y) \in A$ and $\inf_X(x, y) \in A$. Moreover, we say that a sublattice Y of lattice X is *subcomplete* if for every nonempty subset $A \subseteq Y$ we have that $\sup_X(A) \in Y$ and $\inf_X(A) \in Y$. We denote by $\mathcal{L}(X)$ the collection of all sublattices of a lattice X.

Given a lattice $(X, \geq)$, the *induced set order* $\sqsupseteq$ defined on all nonempty elements of the power set $\mathcal{P}(X) \setminus \{\emptyset\}$ is defined as: for $Y \in \mathcal{P}(X) \setminus \{\emptyset\}$ and $Z \in \mathcal{P}(X) \setminus \{\emptyset\}$, we say that $Y \sqsupseteq Z$ if for every $y \in Y$ and $z \in Z$ we have that $\sup_X(y, z) \in Y$ and $\inf_X(y, z) \in Z$.

Finally, a correspondence on X is a function defined on X whose range is included in $\mathcal{P}(X)$. We now assume that X is a lattice and therefore we consider the correspondence $f : X \to \mathcal{L}(X)$, where we consider the induced set order $\sqsupseteq$ in $\mathcal{L}(X)$. We say that this correspondence f is *increasing in* X if for every $x \in X$ and $y \in X$ we have that if $x \geq y$ then $f(x) \sqsupseteq f(y)$.

We now present the fixed point theorem for lattices of $\mathbb{R}$ proven by Topkis [23] in 1998.

Theorem 1.1. *Let $X \subseteq \mathbb{R}$ be a nonempty complete lattice, and consider an increasing correspondence $Y : X \to \mathcal{L}(X)$, where we consider the induced set order $\sqsupseteq$ in $\mathcal{L}(X)$. We assume that for every $x \in X$, $Y(x)$ is subcomplete. Then,*

1. *There exists at least one fixed point of Y(x). Moreover,*

   $$sup_X\left(\{x \in X : Y(x) \cap [x, \infty] \text{ is nonempty}\}\right)$$

   is the greatest fixed point and

   $$inf_X(\{x \in X : Y(x) \cap [-\infty, x] \text{ is nonempty}\})$$

   is the smallest fixed point.

2. *The set of fixed points of Y(x) is a complete lattice.*

Chapter 2

Competition: the model I

In this chapter we present the competitive model analyzed by Besbes and Sauré [4] and study its consequences. We have in mind the competition of retailers such as Farfetch or Gucci, and the applied case will be studied in the last chapter. In the first section we describe how the retailers compete, that is, what are the possible decisions of every player. In the second section we characterize the model that describes clients' choice process. In the third section we study the best responses of the players and the existence of Nash equilibria when retailers compete only with exclusive products. The case of competition with common products is studied in the third chapter.

2.1 Competition between two retailers

We consider a duopolistic market where retailers A and B compete in product assortment and in products' price. We note by P_A and P_B the finite set of products that retailer A and B, respectively, can offer. In order to compete, each retailer will select a subset of this set and a price for every product. Due to limited space in shops, we introduce constant values C_A and C_B that designate the maximum number of products retailers A and B can offer at once.

We denote by $\{1, 2, \ldots, |P|\}$ the elements of $P = P_A \cup P_B$ and for every product $i \in P$ and we denote by $c_{A,i} \geq 0$ the constant marginal cost that retailer A has to pay to produce or to buy from a manufacturer the product i, and similarly for retailer B. Moreover, the price that retailer A charges for product $i \in P_A$ is denoted by $p_{A,i}$, and similarly for retailer B.

We assume that $p_{A,i} > c_{A,i}$ for all $i \in S_A$, and similarly for retailer B. We observe that this assumption is consistent to the fact that competitors are seeking for utility maximization and therefore they will choose higher prices than the cost of producing or buying the product from a manufacturer. This is somewhat different to the model described by Bertrand, where competitors reached equilibrium when the prices were set equal to the marginal cost of the product. However, in practice, it is more logical to think that retailers would prefer not to sell a product instead of selling it at production cost, due to other factors such as display expenses (which we neglect in our model).

Moreover, we say that product $i \in P$ is *exclusive* to retailer A if $i \in P_A \setminus P_B$, that is, product i can only be offered by retailer A; and similarly for retailer B. We say that product $i \in P$ is *common* if $i \in P_A \cap P_B$, that is, product i can be offered by both retailers. For example, Hacendado and Carrefour products are exclusive to Mercadona or Carrefour, respectively, while Campofrio products are common.

Finally, we designate by $A_A \subset \mathcal{P}(P_A)$ and $A_B \subset \mathcal{P}(P_B)$ the set of all possible assortment selection, where $\mathcal{P}(C)$ is the power set of C. In other words,

$$A_A = \{C \subseteq P_A;\ card(C) \leq C_A\}$$

and

$$A_B = \{C \subseteq P_B;\ card(C) \leq C_B\}.$$

With that being said, we can now characterize the possible strategies for the retailers. A strategy of retailer A is a pair (S_A, p_A) where $S_A \in A_A$ is the assortment selection of retailer A and $p_A = (p_{A,1}, p_{A,2}, \ldots, p_{A,|S_A|})$ is the vector of prices, where for all $i \in S_A$ we have $p_{A,i} > c_{A,i}$. And similarly for retailer B.

2.2 Behaviour of consumers

Discrete choice models explain the behavior of decision-making individuals, like people or firms, that choose between various alternatives. In our case, decision makers are clients that search a certain product of their interest. Furthermore, the alternatives' set, also called the *choice set*, has to fulfill three conditions:

- The choice set is finite.
- Alternatives are *mutually exclusive*, that is, clients select one and only one product, discarding all other alternatives.
- The choice set is *exhaustive*. All alternatives are available to the client.

The most used discrete choice model is the logit model. This model was popularized, due to its simplicity and the fact that the choice probability can be expressed as a closed formula, by the American economist Daniel McFadden in his work *Conditional Logit Analysis of Qualitative Choice Behavior* (1974) [16] where he developed the theory of this choice model and applied his results to empirical problems. In 2000, Daniel McFadden was awarded, together with James Heckman, the Economics Nobel Prize *"for his development of theory and methods for analyzing discrete choice"*.

In that paper, McFadden studies the case where decision makers have to select between two options: option 1 and option 2. The most important idea of the paper is that it assumes each decision maker to have an utility function that assigns a real value to the two options, U_1 and U_2. The paper considers that this utility function has two parts, one that models the observable information that is available to the researcher, V_1 and V_2, and the other part that models the information unknown by the researcher, ε_1 and ε_2, which we suppose independent and identically distributed random variables following a standard Gumbel distribution.

The Gumbel distribution is useful for representing the maximum or the minimum of a number of samples from various distributions, and it can be obtained from the well-known exponential distribution. If the random variable X follows an exponential distribution with mean 1, the random variable $-log(X)$ follows a standard Gumbel distribution. In other words, for $i \in \{1,2\}$ the random variable of errors ε_i has cumulative distribution function

$$F(\varepsilon_i) = e^{-e^{-\varepsilon_i}},$$

and probability density function

$$f(\varepsilon_i) = e^{-\varepsilon_i} e^{-e^{-\varepsilon_i}}.$$

We follow the notation used by McFadden (1974) and Anderson et al. in *Discrete Choice Theory of Product Differentiation* (1992) [1] and adapt it to our model.

We consider that every client x assigns a real value $U_{A,i}(x)$ to acquiring product $i \in P_A$ from retailer A, and a real value $U_0(x)$ to not acquiring any product, where the utility function is defined as

$$\begin{aligned} U_{A,i}(x) = &\ \mu_{A,i} - \alpha_{A,i} p_{A,i} + \varepsilon_i^x, \\ U_0(x) = &\ \varepsilon_0^x, \end{aligned}$$

and similarly for retailer B.

In the above expressions, we consider that $\mu_{A,i}$ is a constant value related to acquiring product i from retailer A, and similarly for B. Moreover, we consider $\alpha_{A,i} > 0$ to be the importance that clients give to the price of product i from retailer A, and similarly for B. And finally, we consider that ε_i^x for $i \in P \cup \{0\}$ are independent and identically distributed random variables following the standard Gumbel distribution that represent the errors. These errors are independent of the retailer the client acquires product i from.

2.3 Competition in exclusive products

In this section we are particularly interested in studying if a Nash equilibrium can be achieved by the competitors and under which circumstances. We study this model as a game between two players (retailers A and B) whose strategies are pairs $(S_A, p_A) \in A_A \times \mathbb{R}_+^{|S_A|}$, and similarly for retailer B. However, we are still missing to express the utility function of each retailer, which we will assume is its expected profit.

As stated before, one of the main reasons to use this model is that the choice probability can be expressed as a closed form. We recall that clients will choose a product that maximizes their utility. From now on, we suppose that retailers compete in exclusive products, that is $S_A \cap S_B = \emptyset$. The general case will be studied in the next chapter.

In the next lemma we compute the probability that a client chooses product i from a retailer. The first step of the proof is to find an expression of the probability of a customer acquiring a certain product i conditioned to the values of ε_i^x, where we strongly use that the random variables ε_i^x for $i \in S_A \cup S_B \cup \{0\}$ are independent and identically distributed. The second step of the proof is to integrate over all possible values of ε_i^x in order to find this probability.

However, we need some notation in order to simplify the expressions. The *attraction factor of product i offered by retailer A* is defined as

$$v_{A,i} = e^{\mu_{A,i} - \alpha_{A,i} p_{A,i}},$$

and similarly for retailer B. The attraction factor of not acquiring any product is $v_0 = 1$.

Lemma 2.1. *Let (S_A, p_A) and (S_B, p_B) be two strategies of retailer A and B, respectively. The probability that a client acquires product $i \in S_A$ from retailer A can be expressed as*

$$q_{A,i} = \frac{v_{A,i}}{1 + \sum_{j \in S_A} v_{A,j} + \sum_{j \in S_B} v_{B,j}},$$

and similarly for retailer B. On the other hand, the probability of not acquiring any product is

$$q_0 = \frac{1}{1 + \sum_{j \in S_A} v_{A,j} + \sum_{j \in S_B} v_{B,j}}.$$

Proof. We first study a particular case. We observe that the probability of a client x choosing product $i \in S_A$ over product $j \in S_A$, $j \neq i$, is

$$\begin{aligned} P(U_{A,i}(x) > U_{A,j}(x)) &= P(\mu_{A,i} - \alpha_{A,i} p_{A,i} + \varepsilon_i^x > \mu_{A,j} - \alpha_{A,j} p_{A,j} + \varepsilon_j^x) \\ &= P(\varepsilon_j^x < \mu_{A,i} - \mu_{A,j} - \alpha_{A,i} p_{A,i} + \alpha_{A,j} p_{A,j} + \varepsilon_i^x) \\ &= F(\mu_{A,i} - \mu_{A,j} - \alpha_{A,i} p_{A,i} + \alpha_{A,j} p_{A,j} + \varepsilon_i^x) \\ &= e^{-\frac{v_{A,j}}{v_{A,i}} e^{-\varepsilon_i^x}}, \end{aligned}$$

where in the third step we assume ε_i^x to be known, and F is the cumulative distribution function of ε_j^x, that is, $F(s) = e^{-e^{-s}}$.

Therefore, the probability of a client x choosing product $i \in S_A$ over all other offered products is the probability that the utility of product $i \in S_A$ is greater than the utility of all other offered products, that is,

$$q_{A,i} = P(U_{A,i}(x) > U_0(x), U_{A,i}(x) > U_{A,j}(x) \ \forall j \in S_A \ \ j \neq i, U_{A,i}(x) > U_{B,j}(x) \ \forall j \in S_B).$$

Moreover, since ε_j^x for $j \in S_A \cup S_B \cup \{0\}$ are independent and identically distributed, we have that the probability of client x to choose product $i \in S_A$ over all other products conditioned to ε_i^x is

$$P(\text{choose } i \text{ over all products} | \varepsilon_i^x) = e^{-\frac{1}{v_{A,i}} e^{-\varepsilon_i^x}} \prod_{j \in S_A \setminus \{i\}} e^{-\frac{v_{A,j}}{v_{A,i}} e^{-\varepsilon_i^x}} \prod_{j \in S_B} e^{-\frac{v_{B,j}}{v_{A,i}} e^{-\varepsilon_i^x}}.$$

Therefore, $q_{A,i}$ is the integral of $P(\text{choose } i \text{ over all products} | \varepsilon_i^x)$ over $\mathbb{R}$ weighted by the density of ε_i^x:

$$\begin{aligned} q_{A,i} &= \int_{-\infty}^{\infty} \left(e^{-\frac{1}{v_{A,i}} e^{-s}} \prod_{j \in S_A \setminus \{i\}} e^{-\frac{v_{A,j}}{v_{A,i}} e^{-s}} \prod_{j \in S_B} e^{-\frac{v_{B,j}}{v_{A,i}} e^{-\varepsilon_i^x}} \right) e^{-s} e^{-e^{-s}} ds \\ &= \int_{-\infty}^{\infty} \exp\left(-\frac{1}{v_{A,i}} e^{-s} - \sum_{j \in S_A} \frac{v_{A,j}}{v_{A,i}} e^{-s} - \sum_{j \in S_B} \frac{v_{B,j}}{v_{A,i}} e^{-s} \right) e^{-s} ds \\ &= \int_{-\infty}^{\infty} \exp\left(-e^{-s} \left(\frac{1}{v_{A,i}} + \sum_{j \in S_A} \frac{v_{A,j}}{v_{A,i}} + \sum_{j \in S_B} \frac{v_{B,j}}{v_{A,i}} \right) \right) e^{-s} ds. \end{aligned}$$

Doing the change of variables $t = e^{-s}$ we obtain that

$$\begin{aligned} q_{A,i} &= \int_0^\infty \exp\left(-t\left(\frac{1}{v_{A,i}} + \sum_{j\in S_A}\frac{v_{A,j}}{v_{A,i}} + \sum_{j\in S_B}\frac{v_{B,j}}{v_{A,i}}\right)\right) dt \\ &= \left.\frac{\exp\left(-t\left(\frac{1}{v_{A,i}} + \sum_{j\in S_A}\frac{v_{A,j}}{v_{A,i}} + \sum_{j\in S_B}\frac{v_{B,j}}{v_{A,i}}\right)\right)}{-\frac{1}{v_{A,i}} - \sum_{j\in S_A}\frac{v_{A,j}}{v_{A,i}} - \sum_{j\in S_B}\frac{v_{B,j}}{v_{A,i}}}\right|_0^\infty \\ &= \frac{v_{A,i}}{1 + \sum_{j\in S_A} v_{A,j} + \sum_{j\in S_B} v_{B,j}}. \end{aligned}$$

The proof for q_0 is analogous.

□

Similarly with the case of Cournot and Bertrand model, we can now find the expression of the utility function of both competitors. The utility function in this case will be the expected profit of each retailer, that is,

$$u_A(S_A, p_A, S_B, p_B) = \sum_{i\in S_A} (p_{A,i} - c_{A,i}) q_{A,i},$$

for retailer A, and similarly for retailer B.

These are the functions that each retailer will try to maximize, given their competitor assortment and price selection, that is, given a strategy (S_B, p_B) of retailer B, retailer A will try to find a strategy (S_A^*, p_A^*) such that

$$u_A(S_A^*, p_A^*, S_B, p_B) = \max_{(S_A, p_A)} u_A(S_A, p_A, S_B, p_B),$$

or in other words, A will try to find a best response to the strategy (S_B, p_B), and similarly for retailer B.

However, this may seem as a problem that is not easy to solve, because it is a combinatorial optimization problem. In order to solve this problem, firstly we restrict ourselves to the case with fixed prices, where retailers compete only in assortment selection. In that case, we find a problem transformation that transforms the problem of finding a best response to a more manageable optimization problem.

Competition with fixed prices

We now suppose that vectors p_A and p_B are fixed. We observe that strategic variables are only assortment selections $S_A \in A_A$ and $S_B \in A_B$.

As stated before, given a strategy $S_B \in A_B$ of retailer B, retailer A will try to find a strategy S_A^* such that

$$u_A(S_A^*, S_B) = \max_{S_A\in A_A} u_A(S_A, S_B),$$

and similarly for retailer B.

We notice that, since S_A and S_B are finite, A_A and A_B are also finite. Therefore, given a strategy of a retailer, there always exists at least one best response for the competitor.

We now present a lemma that transforms the problem of finding a best response of a retailer, given a strategy of the competitor, into a different optimization problem. Similar problem transformations can be found in Gallego et al. (2004) [9] or in Besbes and Sauré (2016) [4].

Lemma 2.2. *An assortment selection $S_A^* \in A_A$ for retailer A maximizes $u_A(S_A, S_B)$ for $S_A \in A_A$ and for a given strategy $S_B \in A_B$ of retailer B if and only if S_A^* maximizes*

$$y(S_A, \lambda) = \sum_{i \in S_A} (p_{A,i} - c_{A,i} - \lambda) v_{A,i} - \lambda \sum_{i \in S_B} v_{B,i},$$

for $S_A \in A_A$ where λ is the solution to the problem $\max \lambda \in (0, \infty)$ *such that*

$$\max_{S_A \in A_A} y(S_A, \lambda) \geq \lambda,$$

and similarly for retailer B.

Proof. We observe that, for a given strategy S_B, retailer A will maximize the function

$$u_A(S_A, S_B) = \sum_{i \in S_A} (p_{A,i} - c_{A,i}) q_{A,i} = \frac{\sum_{i \in S_A} (p_{A,i} - c_{A,i}) v_{A,i}}{1 + \sum_{j \in S_A} v_{A,j} + \sum_{j \in S_B} v_{B,j}},$$

for $S_A \in A_A$, or in other words, will try to find the maximum $\lambda \in (0, \infty)$ such that

$$\max_{S_A \in A_A} \left(\frac{\sum_{i \in S_A} (p_{A,i} - c_{A,i}) v_{A,i}}{1 + \sum_{j \in S_A} v_{A,j} + \sum_{j \in S_B} v_{B,j}} \right) \geq \lambda.$$

We can now cross multiply and group similar terms, and get that the problem of maximizing $u_A(S_A, S_B)$ for a given S_B is equivalent to maximizing $\lambda \in (0, \infty)$ such that

$$\max_{S_A \in A_A} y(S_A, \lambda) \geq \lambda.$$

□

In order to further simplify the notation, we can express the problem of retailer A finding a best response for the strategy $S_B \in A_B$ as the optimization problem,

$$\max \lambda \in (0, \infty) \text{ such that } \max_{S_A \in A_A} \left(\sum_{i \in S_A} \theta_{A,i}(\lambda) \right) \geq \lambda \left(1 + \sum_{i \in S_B} v_{B,i} \right), \quad (2.1)$$

where we denote $\theta_{A,i}(\lambda) = (p_{A,i} - c_{A,i} - \lambda) v_{A,i}$.

Notice that the term $\sum_{i \in S_B} v_{B,i}$ is independent of the assortment selection of retailer A, and we can think of this term as the *total attraction* of competitor's assortment selection. In order to solve

$$\max_{S_A \in A_A} \left(\sum_{i \in S_A} \theta_{A,i}(\lambda) \right) \geq \lambda (1 + \sum_{i \in S_B} v_{B,i}),$$

retailer A can rank all his available products computing $\theta_{A,i}(\lambda)$ for $i \in P_A$ and then selecting the products with highest, and positive, value $\theta_{A,i}(\lambda)$ until the limit of display C_A is reached. This process is called $\theta_{A,i}(\lambda)$ *product ranking.*

We now study two crucial characteristics of retailer A's best response for a strategy $S_B \in A_B$ for retailer B with total attraction e. We denote by

$$\lambda_A : (0, \infty) \to (0, \infty)$$

the solution to (2.1), that is, $\lambda_A(e)$ is the expected profit per customer of retailer A when facing an assortment selection of total attraction e from retailer B. Moreover, following the notation used by Sundaram [22] for correspondences, we denote by $a_A(e)$ the correspondence

$$a_A : (0, \infty) \to A_A$$

that associates a total attraction e with a retailer A's best response, that is, $a_A(e)$ is an assortment selection that maximizes $\sum_{i \in S_A} \theta_{A,i}(\lambda_A(e))$. We notice that, in general, the image of a_A is not unique.

Proposition 2.1. *With the notation defined above, we have that*

1. *$\lambda_A(\cdot)$ is a strictly decreasing function.*
2. *Let us consider two strategies $S_B, S'_B \in A_B$ for retailer B with total attractions e and e', respectively. If $e > e'$, for all $a \in a_A(e)$ and for all $a' \in a_A(e')$, the total attraction of a is greater or equal than the total attraction of a'.*

Proof. We first observe that for any strategy $S_B \in A_B$ with total attraction $e > 0$, and for all $a \in a_A(e)$, we have that

$$\sum_{i \in a} \theta_{A,i}(\lambda_A(e)) = \lambda_A(e)(1 + e)$$

1. We assume that $e > e' > 0$, and we want to prove that $\lambda_A(e') > \lambda_A(e)$.

We have that

$$\max_{S_A \in A_A} \left(\sum_{i \in S_A} \theta_{A,i}(\lambda_A(e)) \right) = \lambda_A(e)(1 + e) > \lambda_A(e)(1 + e').$$

We now consider functions $f : (0, \infty) \to (0, \infty)$ defined by

$$f(\lambda) = \max_{S_A \in A_A} \left(\sum_{i \in S_A} \theta_{A,i}(\lambda) \right)$$

and $g : (0, \infty) \to (0, \infty)$ defined by

$$g(\lambda) = \lambda(1 + e').$$

We have that both functions f and g are continuous. Therefore, there exists an $\varepsilon > 0$ such that for all $\lambda \in (\lambda_A(e) - \varepsilon, \lambda_A(e) + \varepsilon)$ we have that

$$\max_{S_A \in A_A} \left(\sum_{i \in S_A} \theta_{A,i}(\lambda) \right) = f(\lambda) > g(\lambda) = \lambda(1 + e').$$

Since $f(\lambda)$ is decreasing in λ and $g(\lambda)$ is increasing in λ, we have that $\lambda_A(e') > \lambda_A(e')$.

2. We assume that $e > e' > 0$. We consider a retailer A's best response to an assortment selection with total attraction e', $a' \in a_A(e')$. As we commented before this preposition, retailer A solves

$$\max_{S_A \in A_A} \left(\sum_{i \in S_A} \theta_{A,i}(\lambda) \right) \geq \lambda(1 + \sum_{i \in S_B} v_{B,i})$$

by ranking the $\theta_{A,i}(\lambda)$ value for every product available to them and then selecting the products with highest and positive $\theta_{A,i}(\lambda)$. Therefore, if a product is selected, it has a higher or equal (because the display limit could be reached) value than any other not selected product, that is, for any $i \in a'$ and $j \in P_A \setminus a'$ we have that $\theta_{A,i}(\lambda_A(e')) \geq \theta_{A,j}(\lambda_A(e'))$.

Moreover, we consider a retailer A's best response to an assortment selection with total attraction e, $a \in a_A(e)$, and for any $i \in a$ and $j \in P_A \setminus a$ we have that

$$\begin{aligned}
\theta_{A,i}(\lambda_A(e')) + v_{A,i}(\lambda_A(e') - \lambda_A(e)) &= v_{A,i}(p_{A,i} - c_{A,i} - \lambda_A(e')) + v_{A,i}(\lambda_A(e') - \lambda_A(e)) \\
&= \theta_{A,i}(\lambda_A(e)) \geq \theta_{A,j}(\lambda_A(e)) \\
&= v_{A,j}(p_{A,j} - c_{A,j} - \lambda_A(e')) + v_{A,j}(\lambda_A(e') - \lambda_A(e)) \\
&= \theta_{A,j}(\lambda_A(e')) + v_{A,j}(\lambda_A(e') - \lambda_A(e)).
\end{aligned}$$

Therefore, for any $a \in a_A(e)$ and $a' \in a_A(e')$, if we consider products $i \in a \setminus a'$ and $j \in a' \setminus a$, we have that $\theta_{A,j}(\lambda_A(e')) \geq \theta_{A,i}(\lambda_A(e'))$ because, in particular, $j \in a'$. Moreover, from the inequality above we deduce that

$$\begin{aligned}
v_{A,i}(\lambda_A(e') - \lambda_A(e)) &= \theta_{A,i}(\lambda_A(e')) - \theta_{A,i}(\lambda_A(e')) + v_{A,i}(\lambda_A(e') - \lambda_A(e)) \\
&\geq \theta_{A,j}(\lambda_A(e')) - \theta_{A,i}(\lambda_A(e')) + v_{A,j}(\lambda_A(e') - \lambda_A(e)) \\
&= v_{A,j}(\lambda_A(e') - \lambda_A(e)).
\end{aligned}$$

We know from the first part of this proposition that $\lambda_A(e') - \lambda_A(e) > 0$, therefore we obtain that $v_{A,i} \geq v_{A,j}$ for all $i \in a \setminus a'$ and for all $j \in a' \setminus a$.

We now observe that $\theta_{A,i}(\lambda)$ is a decreasing function in λ. Therefore, as λ increases, there will be less products with positive $\theta_{A,i}(\lambda)$ value and therefore there will be less selected products. As $\lambda_A(e') > \lambda_A(e)$, for all $a \in a_A(e)$ and $a' \in a_A(e')$ we have that $|a'| \leq |a|$. Therefore,

$$|a' \setminus a| = |a'| - |a' \cap a| \leq |a| - |a \cap a'| = |a \setminus a'|.$$

We recall that retailer A is selling product $i \in S_A$ at positive profit for any strategy S_A, that is, $p_{A,i} > c_{A,i}$ for all $i \in S_A$, and thus $v_{A,i} > 0$ for any $i \in S_A$. Therefore,

$$\sum_{i \in a \setminus a'} v_{A,i} \geq |a \setminus a'| \min_{i \in a \setminus a'}(v_{A,i}) \geq |a' \setminus a| \max_{i \in a' \setminus a}(v_{A,i}) \geq \sum_{i \in a' \setminus a} v_{A,i}.$$

To conclude, for any $a \in a_A(e)$ and $a' \in a_A(e')$,

$$\sum_{i \in a} v_{A,i} = \sum_{i \in a \cap a'} v_{A,i} + \sum_{i \in a \setminus a'} v_{A,i} \geq \sum_{i \in a \cap a'} v_{A,i} + \sum_{i \in a' \setminus a} v_{A,i} = \sum_{i \in a'} v_{A,i}.$$

□

This proposition gives us some better view on the behaviour of retailers' assortment selection. The first part of the proposition expresses that, as the competitor's assortment selection total attraction increases, the expected profit of the retailer will decrease. This characteristic is what we expect from the competition between two retailers: consumers will tend to buy from the retailer that offers a more attractive assortment, either because the products' quality is higher or because more products are offered.

The second part of the proposition states that if the competitor increases the total attraction of his assortment selection, the retailer will also increase the total attraction of his assortment selection. Again, this is what we expect from this competition: if the competitor's quality or quantity of products increases, the retailer will tend to keep up with his competitor by adding new products or better ones.

In the next theorem we prove that there exists at least one Nash equilibrium of the game. Also, in the case of multiple equilibria, there exists an equilibrium that is preferred by both retailers.

Theorem 2.1. *Let us suppose that retailers A and B compete in assortment selection only with only exclusive products. Then there always exists at least one equilibrium.*

Moreover, in the case of multiple equilibria, there exists an equilibrium that Pareto-dominates all other equilibria.

Proof. For retailer A and B we consider the sets of all total attraction values

$$Z_A = \{e_1^A, e_2^A, \dots, e_{k_A}^A\}$$

and

$$Z_B = \{e_1^B, e_2^B, \dots, e_{k_B}^B\},$$

where for $j \in \{1, 2, \dots, k_A\}$ each $e_j^A = \sum_{i \in S_A} v_{A,i}$ for some $S_A \in A_A$, and similarly for retailer B. Moreover, we suppose that the total attraction values are ordered, that is, $e_1^A < e_2^A < \dots < e_{k_A}^A$ and $e_1^B < e_2^B < \dots < e_{k_B}^B$.

We also consider functions $Y_A : Z_B \to \mathcal{P}(Z_A)$ defined by

$$Y_A(e^B) = \left\{ \sum_{i \in a} v_{A,i} : a \in a_A(e^B) \right\}$$

and similarly $Y_B : Z_A \to \mathcal{P}(Z_B)$ defined by

$$Y_B(e^A) = \left\{ \sum_{i \in a} v_{A,i} : a \in a_B(e^A) \right\}.$$

Finally we consider the correspondence $Y : Z_A \times Z_B \to \mathcal{P}(Z_A) \times \mathcal{P}(Z_B)$ defined by $Y(e^A, e^B) = (Y_A(e^B), Y_B(e^A))$.

We first observe that $Z_A \times Z_B$ is a nonempty complete lattice.

Indeed, $Z_A \times Z_B$ with the ordering relation $\geq$ where $(e^A, e^B) \geq (f^A, f^B)$ if $e^A \geq f^A$ and $e^B \geq f^B$ is a partially ordered set.

Moreover, we have that $Z_A \times Z_B$ is a lattice with

$$\sup_{Z_A \times Z_B} ((e^A, e^B), (f^A, f^B)) = (\max(e^A, f^A), \max(e^B, f^B))$$

and

$$\inf_{Z_A \times Z_B}((e^A, e^B), (f^A, f^B)) = (\min(e^A, f^A), \min(e^B, f^B)).$$

With this characterization of supremum and infimum, $Z_A \times Z_B$ is a complete lattice.

In Proposition 2.1 we have seen that if $e > e'$, we have that for all $a \in a_A(e)$ and $a' \in a_A(e')$, the total attraction of a is greater or equal than the total attraction of a'. Therefore, if $(e^A, e^B) \geq (f^A, f^B)$ we have that on one hand,

$$\max_{a \in a_B(e^A) \cup a_B(f^A)} \sum_{i \in a} v_{A,i} \in Y_A(e^B)$$

and

$$\max_{a \in a_A(e^B) \cup a_A(f^B)} \sum_{i \in a} v_{B,i} \in Y_B(e^A).$$

While in the other hand,

$$\min_{a \in a_B(e^A) \cup a_B(f^A)} \sum_{i \in a} v_{A,i} \in Y_A(f^B)$$

and

$$\min_{a \in a_A(e^B) \cup a_A(f^B)} \sum_{i \in a} v_{B,i} \in Y_B(f^A).$$

Therefore, Y is an increasing correspondence.

By the fixed point theorem on lattices, Theorem 1.1, we have that Y has at least one fixed point.

We observe that if retailers select assortments with total attraction equal to the total attraction in the fixed point, they would reach an equilibrium.

In order to find an equilibrium that Pareto-dominates the others, we observe that one additional consequence of the fixed point Theorem 1.1 is that the set of fixed points of Y is a complete lattice. Therefore, there exists a fixed point (e_*^A, e_*^B) that satisfies $(f^A, f^B) \geq (e_*^A, e_*^B)$ for every other fixed point (f^A, f^B) of Y.

We recall, from Proposition 2.1, that the expected profit of retailer A is strictly decreasing in retailer B's assortment selection total attraction. Therefore, retailer A prefers an equilibrium that minimizes the total attraction of retailer B's assortment selection, and similarly for retailer B.

In conclusion, both retailers prefer the equilibrium associated with total attraction e_*^A and e_*^B, respectively. □

Competition in prices and assortment

We now study the case where retailers compete in both assortment selection and products' prices. In order to study this case, we follow a similar structure as in the case of competition in assortment selection: we first obtain monotonicity results of the best responses and then we utilize those results to prove the existence of Nash equilibria of the game.

As we stated in the previous section where we defined the model, strategies are now pairs (S_A, p_A) where $S_A \in A_A$ is the assortment selection of retailer A and $p_A = (p_{A,1}, p_{A,2}, \ldots, p_{A,|S_A|})$ is the vector of prices, where for all $i \in S_A$ we have that $p_{A,i}$ is the price set for product i and $p_{A,i} > c_{A,i}$. And similarly for retailer B. Moreover, we consider that retailers select these strategies simultaneously, without knowing their competitor's strategy.

Similarly to the case of competition in assortment selection, given a strategy (S_B, p_B) for retailer B, retailer A's best response will be the solution to the problem of maximizing λ such that

$$\max_{(S_A,p_A)} \left(\sum_{i \in S_A} (p_{A,i} - c_{A,i} - \lambda) v_{A,i}(p_{A,i}) \right) \geq \lambda \left(1 + \sum_{i \in S_B} v_{B,i}(p_{B,i}) \right).$$

We notice that we can calculate best responses by first fixing an assortment selection and computing the best prices for those products and then fixing the prices and calculating the best assortment selection, a problem that we have already solved.

For a given assortment and price selection of retailer B, and for a given $\lambda \in (0, \infty)$, if the assortment selection of retailer A is fixed, retailer A faces the optimization problem

$$\max_{p_A} \left(\sum_{i \in S_A} (p_{A,i} - c_{A,i} - \lambda) v_{A,i}(p_{A,i}) \right)$$

whose solution, using partial differentiation analysis, is

$$p^*_{A,i}(\lambda) = \frac{1}{\alpha_{A,i}} + c_{A,i} + \lambda,$$

for each product i in the assortment selection of retailer A.

Similarly to the case of competition in assortment selection only, we introduce

$$\lambda_A : (0, \infty) \to (0, \infty)$$

the solution to the maximization problem above, that is, $\lambda_A(e)$ is the expected profit per customer of retailer A when facing an assortment and price selection of total attraction e from retailer B.

Moreover, following the notation used by Sundaram [22] for correspondences, we denote by $a_A(e)$ the correspondence

$$a_A : (0, \infty) \to A_A$$

that associates a total attraction e with a retailer A's best assortment selection, that is, $a_A(e)$ is an assortment selection that maximizes $\sum_{i \in S_A} \theta_{A,i}(\lambda_A(e))$. We notice that, in general, the image of a_A is not unique.

Proposition 2.2. *With the notation defined above, we have that*

1. $\lambda_A(\cdot)$ *is a strictly decreasing function.*
2. *Given two strategies* $(S_B, p_B), (S'_B, p'_B)$ *for retailer* B *with total attraction* e *and* e'*, respectively. If* $e > e'$ *and for all* $a \in a_A(e)$ *and for all* $a' \in a_A(e')$*, the total attraction of* a *is greater or equal than the total attraction of* a'*.*

Proof. We first observe that for any strategy (S_B, p_B) with total attraction $e > 0$, and for all $a \in a_A(e)$, we have that

$$\sum_{i\in a} \theta_{A,i}(\lambda_A(e)) = \lambda_A(e)(1+e),$$

where now $\theta_{A,i}(\lambda_A(e)) = v_{A,i}(p^*_{A,i}(\lambda_A(e)))(p^*_{A,i}(\lambda_A(e)) - c_{A,i} - \lambda_A(e))$.

1. We assume that $e > e' > 0$, and we want to prove that $\lambda_A(e') > \lambda_A(e)$.

We have that

$$\max_{S_A \in A_A} \left(\sum_{i\in S_A} \theta_{A,i}(\lambda_A(e)) \right) = \lambda_A(e)(1+e) > \lambda_A(e)(1+e').$$

We now consider functions $f : (0,\infty) \to (0,\infty)$ defined by

$$f(\lambda) = \max_{S_A \in A_A} \left(\sum_{i\in S_A} \theta_{A,i}(\lambda) \right)$$

and $g : (0,\infty) \to (0,\infty)$ defined by

$$g(\lambda) = \lambda(1+e').$$

We have that both functions f and g are continuous. Therefore, there exists an $\varepsilon > 0$ such that for all $\lambda \in (\lambda_A(e) - \varepsilon, \lambda_A(e) + \varepsilon)$ we have that

$$\max_{S_A \in A_A} \left(\sum_{i\in S_A} \theta_{A,i}(\lambda) \right) = f(\lambda) > g(\lambda) = \lambda(1+e').$$

Since $f(\lambda)$ is decreasing in λ and $g(\lambda)$ is increasing in λ, we have that $\lambda_A(e') > \lambda_A(e')$.

2. We assume that $e > e' > 0$. We consider retailer A's best responses $a \in a_A(e)$ and $a' \in a_A(e')$. By the product ranking commented in the previous section, we have that for all $i \in a' \setminus a$ and $j \in a \setminus a'$

$$\theta_{A,i}(\lambda_A(e')) \geq \theta_{A,j}(\lambda_A(e')),$$

and for all $i \in a \setminus a'$ and $j \in a' \setminus a$ we have that

$$\theta_{A,i}(\lambda_A(e)) \geq \theta_{A,j}(\lambda_A(e)).$$

Furthermore, we notice that if $\lambda > 0$, $\theta_{A,i}(\lambda) > 0$ for every product i. We recall that retailers will choose products with highest, and positive, $\theta_{A,i}(\lambda)$ value. As every product has a positive value, we conclude that best responses have as many products as possible, that is, $|a| = C_A$. Therefore, for all $i \in a \setminus a'$ and $j \in a' \setminus a$ we have that

$$\theta_{A,i}(\lambda_A(e)) - \theta_{A,i}(\lambda_A(e')) \geq \theta_{A,j}(\lambda_A(e)) - \theta_{A,j}(\lambda_A(e')).$$

Moreover, we have that $\theta_{A,i} : (0,\infty) \to (0,\infty)$ is a continuous and differentiable function that satisfies for all $\lambda > 0$

$$\frac{d\theta_{A,i}}{d\lambda}(\lambda) = -e^{\mu_{A,i}-\alpha_{A,i}(c_{A,i}+\lambda)-1} = -v_{A,i}(p^*_{A,i}(\lambda)) < 0.$$

Moreover, we observe that for every product i, $v_{A,i}(p^*_{A,i}(\cdot))$ is a decreasing function.

Therefore, by the mean value theorem, we have that there is a $\tilde{\lambda} \in (\lambda_A(e), \lambda_A(e'))$ such that

$$\frac{\theta_{A,i}(\lambda_A(e) - \theta_{A,i}(\lambda_A(e'))}{\lambda_A(e') - \lambda_A(e)} = -\frac{d\theta_{A,i}}{d\lambda}(\tilde{\lambda}) = v_{A,i}(p^*_{A,i}(\tilde{\lambda})).$$

Utilizing that $v_{A,i}(p^*_{A,i}(\lambda))$ is a decreasing function and multiplying in cross, we obtain that for every $i \in a \setminus a'$ and $j \in a' \setminus a$,

$$\begin{aligned} v_{A,i}(p^*_{A,i}(\lambda_A(e)))(\lambda_A(e') - \lambda_A(e)) &\geq \theta_{A,i}(\lambda_A(e)) - \theta_{A,i}(\lambda_A(e')) \\ &\geq \theta_{A,j}(\lambda_A(e)) - \theta_{A,j}(\lambda_A(e')) \\ &\geq v_{A,j}(p^*_{A,j}(\lambda_A(e')))(\lambda_A(e') - \lambda_A(e')), \end{aligned}$$

where in the last inequality we used the mean value theorem again. Therefore, as $\lambda_A(e') - \lambda_A(e) > 0$ we have that for all $i \in a \setminus a'$ and $j \in a' \setminus a$, $v_{A,i}(p^*_{A,i}(\lambda_A(e))) \geq v_{A,j}(p^*_{A,j}(\lambda_A(e')))$. Finally, we have that for all $a \in a_A(e)$ and $a' \in a_A(e')$,

$$\begin{aligned} \sum_{i\in a} v_{A,i}(p^*_{A,i}(\lambda_A(e))) &= \sum_{i\in a\setminus a'} v_{A,i}(p^*_{A,i}(\lambda_A(e))) + \sum_{i\in a\cap a'} v_{A,i}(p^*_{A,i}(\lambda_A(e))) \\ &\geq \sum_{i\in a'\setminus a} v_{A,i}(p^*_{A,i}(\lambda_A(e'))) + \sum_{i\in a\cap a'} v_{A,i}(p^*_{A,i}(\lambda_A(e))) \\ &\geq \sum_{i\in a'} v_{A,i}(p^*_{A,i}(\lambda_A(e'))), \end{aligned}$$

□

With this proposition, we have obtained similar information about the competition between the two retailers as in the case of competition in only assortment selection. The first part of the proposition states that, as the competitor's assortment selection total attraction increases, the expected profit of the retailer will decrease. This is what we expect from the competition between two retailers: consumers will tend to buy from the retailer that offers a more attractive assortment, either because the products' quality is higher or are cheaper, or because more products are offered.

The second part of the proposition states that if the competitor increases the total attraction of his assortment selection, the retailer will also increase the total attraction of his assortment selection. Again, this is what we expect from this competition: if the competitor's quality increases, the products are cheaper or he offers a wider selection of products, the retailer will tend to keep up with his competitor by adding new products, better ones or cheaper ones.

In the next theorem we prove that there exists at least one Nash equilibrium of the game. Also, in the case of multiple equilibria, there exists an equilibrium that is preferred by both retailers.

Theorem 2.2. *If retailer A and B compete in assortment with exclusive products and price selection, there always exists at least one equilibrium. Moreover, in the case of multiple equilibria, there exists an equilibrium that Pareto-dominates all other equilibria.*

Proof. The proof of this theorem is very similar to the analogous result of competition in assortment selection only.

For retailer A and B we consider the sets of all total attraction values

$$Z_A = \{e_1^A, e_2^A, \dots, e_{k_A}^A\}$$

and

$$Z_B = \{e_1^B, e_2^B, \dots, e_{k_B}^B\},$$

where for $j \in \{1, 2, \dots, k_A\}$ each $e_j^A = \sum_{i \in S_A} v_{A,i}(p^*_{A,i}(\lambda)))$ for some $S_A \in A_A$, and similarly for retailer B. Moreover, we suppose that the total attraction values are ordered, that is, $e_1^A < e_2^A < \dots < e_{k_A}^A$ and $e_1^B < e_2^B < \dots < e_{k_B}^B$.

We also consider functions $Y_A : Z_B \to \mathcal{P}(Z_A)$ defined by

$$Y_A(e^B) = \left\{ \sum_{i \in a} v_{A,i}(p^*_{A,i}(\lambda_A(e^B))) : a \in a_A(e^B) \right\}$$

and similarly $Y_B : Z_A \to \mathcal{P}(Z_B)$ defined by

$$Y_B(e^A) = \left\{ \sum_{i \in a} v_{A,i}(p^*_{B,i}(\lambda_B(e^A))) : a \in a_B(e^A) \right\}.$$

Finally we consider the correspondence $Y : Z_A \times Z_B \to \mathcal{P}(Z_A) \times \mathcal{P}(Z_A)$ defined by $Y(e^A, e^B) = (Y_A(e^B), Y_B(e^A))$.

We first observe that $Z_A \times Z_B$ is a nonempty complete lattice. Indeed, $Z_A \times Z_B$ with the ordering relation $\geq$ where $(e^A, e^B) \geq (f^A, f^B)$ if $e^A \geq f^A$ and $e^B \geq f^B$ is a partially ordered set.

Moreover, we have that $Z_A \times Z_B$ is a lattice with

$$\sup_{Z_A \times Z_B} ((e^A, e^B), (f^A, f^B)) = (\max(e^A, f^A), \max(e^B, f^B))$$

and

$$\inf_{Z_A \times Z_B} ((e^A, e^B), (f^A, f^B)) = (\min(e^A, f^A), \min(e^B, f^B)).$$

With this characterization of supremum and infimum, $Z_A \times Z_B$ is a complete lattice.

In Proposition 2.2 we have seen that if $e > e'$, we have that for all $a \in a_A(e)$ and $a' \in a_A(e')$, the total attraction of a is greater or equal than the total attraction of a'. Therefore, if $(e^A, e^B) \geq (f^A, f^B)$ we have that on one hand,

$$\max_{a \in a_B(e^A) \cup a_B(f^A)} \sum_{i \in a} v_{A,i}(p^*_{A,i}(\lambda_A(e^B))) \in Y_A(e^B)$$

and

$$\max_{a \in a_A(e^B) \cup a_A(f^B)} \sum_{i \in a} v_{B,i}(p^*_{B,i}(\lambda_B(e^A))) \in Y_B(e^A).$$

While in the other hand,

$$\min_{a \in a_B(e^A) \cup a_B(f^A)} \sum_{i \in a} v_{A,i}(p^*_{A,i}(\lambda_A(e^B))) \in Y_A(f^B)$$

and

$$\min_{a \in a_A(e^B) \cup a_A(f^B)} \sum_{i \in a} v_{B,i}(p^*_{B,i}(\lambda_B(e^A))) \in Y_B(f^A).$$

Therefore, Y is an increasing correspondence.

By the fixed point theorem on lattices, Theorem 1.1, we have that Y has at least one fixed point.

We observe that if retailers select assortments with total attraction equal to the total attraction in the fixed point, they would reach an equilibrium.

In order to find an equilibrium that Pareto-dominates the others, we observe that one additional consequence of the fixed point Theorem 1.1 is that the set of fixed points of Y is a complete lattice. Therefore, there exists a fixed point (e^A_*, e^B_*) that satisfies $(f^A, f^B) \geq (e^A_*, e^B_*)$ for every other fixed point (f^A, f^B) of Y.

We recall, from Proposition 2.2, that the expected profit of retailer A is strictly decreasing in retailer B's assortment and price selection total attraction. Therefore, retailer A prefers an equilibrium that minimizes the total attraction of retailer B's assortment and price selection, and similarly for retailer B.

In conclusion, both retailers prefer the equilibrium associated with total attraction e^A_* and e^B_*, respectively. □

With this result we proved a very interesting result that shows the existence of Nash equilibria when retailers compete in assortment with only exclusive products and in price selection. Moreover, we proved that in the case of multiple equilibria, there is a particular equilibrium that Pareto-dominates all other and thus both retailers prefer. In the next chapter we see what happens when retailers compete with common products and if similar results can be found.

Chapter 3

Competition: the model II

In this chapter we study the case of competition in both exclusive and common products, that is, given two assortment selections S_A and S_B for retailer A and B, respectively, we have that in general $S_A \cap S_B \neq \emptyset$.

Similarly to the case of competition in exclusive products, we study if competitors can achieve a Nash equilibrium and under which circumstances. In order to do so, we follow a similar structure to the previous case. We firstly restrict ourselves in the case where retailers compete only in assortment selection, this time with both exclusive and common products. Later, we study the case where retailers compete both in assortment selection and prices selection.

3.1 Competition between two retailers

In the next lemma we compute the probability that a client chooses product i from a retailer. However, we notice that in this case one common product i might have the highest attraction factor but this value is equal for both retailers. In this case, the customer is interested in acquiring this product but has to choose from which retailer to buy it. If this is the case, we suppose that the customer chooses the retailer with the same probability.

Lemma 3.1. *Let (S_A, p_A) and (S_B, p_B) be two strategies of retailer A and B, respectively. The probability that a client acquires product $i \in S_A$ from retailer A can be expressed as*

$$q_{A,i} = \frac{v_{A,i}\left(\mathbb{1}_{S_A \setminus S_B}(i) + \delta_{A,i}\mathbb{1}_{S_B}(i)\right)}{1 + \sum_{j \in S_A \setminus S_B} v_{A,j} + \sum_{j \in S_A \cap S_B} \max(v_{A,j}, v_{B,j}) + \sum_{j \in S_B \setminus S_A} v_{B,j}},$$

and similarly for retailer B. On the other hand, the probability of not acquiring any product is

$$q_0 = \frac{1}{1 + \sum_{j \in S_A \setminus S_B} v_{A,j} + \sum_{j \in S_A \cap S_B} \max(v_{A,j}, v_{B,j}) + \sum_{j \in S_B \setminus S_A} v_{B,j}},$$

where $\mathbb{1}_A$ is the indicator function of set A, [1] *and we denoted by*

$$\delta_{A,j} = \begin{cases} 1 & \text{if } v_{A,j} > v_{B,j} \\ \frac{1}{2} & \text{if } v_{A,j} = v_{B,j}. \end{cases}$$

[1] Let X be a set. The indicator function of a subset $A \subset X$ is a function $\mathbb{1}_A : X \to \{0,1\}$ defined as $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ if $x \notin A$.

Proof. Similarly to the proof of Lemma 2.1, we have to find the probability of a client x choosing product $i \in S_A$, that is,

$$q_{A,i} = P(U_{A,i}(x) \geq U_0(x), U_{A,i}(x) \geq U_{A,j}(x)\ \ \forall j \in S_A\ \ j \neq i, U_{A,i}(x) \geq U_{B,j}(x)\ \ \forall j \in S_B).$$

Unlike the case of Lemma 2.1, we now have common products. In order to solve this problem, we look at three groups of products: the products in $S_A \setminus S_B$, the products in $S_A \cap S_B$ and the products in $S_B \setminus S_A$.

Similarly to Lemma 2.1, the probability of a customer choosing product i over all $j \in S_A \setminus S_B$ conditioned to ε_i^x is

$$P(U_{A,i}(x) \geq U_{A,j}(x)\ \ \forall j \neq i \mid \varepsilon_i^x) = \prod_{j \in (S_A \setminus S_B) \setminus \{i\}} e^{-\frac{v_{A,j}}{v_{A,i}} e^{-\varepsilon_i^x}},$$

and similarly for products in $S_B \setminus S_A$.

In the case of the products in $S_A \cap S_B$ we have that the probability of a customer choosing product i over all $j \in S_A \cap S_B$ conditioned to ε_i^x is

$$P(U_{A,i}(x) \geq U_{A,j}(x), U_{A,i}(x) \geq U_{B,j}(x)\ \ \forall j \neq i \mid \varepsilon_i^x).$$

We notice that for every product $j \in S_A \cap S_B$, $j \neq i$ we have that

$$\begin{aligned}
&P(U_{A,i}(x) \geq U_{A,j}(x), U_{A,i}(x) \geq U_{B,j}(x) \mid \varepsilon_i^x) \\
&= P(\mu_{A,i} - \alpha_{A,i} p_{A,i} + \varepsilon_i^x \geq \mu_{A,j} - \alpha_{A,j} p_{A,j} + \varepsilon_j^x, \mu_{A,i} - \alpha_{A,i} p_{A,i} \varepsilon_i^x \geq \mu_{B,j} - \alpha_{B,j} p_{B,j} \varepsilon_j^x \mid \varepsilon_i^x) \\
&= P(\varepsilon_j^x \leq -\mu_{A,i} + \alpha_{A,i} p_{A,i} + \max(\mu_{A,j} - \alpha_{A,j} p_{A,j}, \mu_{B,j} - \alpha_{B,j} p_{B,j}) - \varepsilon_i^x \mid \varepsilon_i^x) \\
&= e^{-\frac{\max(v_{A,j}, v_{B,j})}{v_{A,i}} e^{-\varepsilon_i^x}}.
\end{aligned}$$

Therefore, for the group of items in $S_A \cap S_B$ we have that

$$P(U_{A,i}(x) \geq U_{A,j}(x), U_{A,i}(x) \geq U_{B,j}(x)\ \ \forall j \neq i \mid \varepsilon_i^x) = \prod_{j \in (S_A \cap S_B) \setminus \{i\}} e^{-\frac{\max(v_{A,j}, v_{B,j})}{v_{A,i}} e^{-\varepsilon_i^x}}.$$

Finally, we have to distinguish whether i is a common or exclusive product.

If i is an exclusive product, then

$$\begin{aligned}
&P(\text{choose } i \text{ over all products}|\varepsilon_i^x) \\
&= \exp\left(-e^{-\varepsilon_i^x}\left(\frac{1}{v_{A,i}} + \sum_{j \in (S_A \setminus S_B) \setminus \{i\}} \frac{v_{A,j}}{v_{A,i}} + \sum_{j \in S_B \cap S_A} \frac{\max(v_{A,j}, v_{B,j})}{v_{A,i}} + \sum_{j \in S_B \setminus S_A} \frac{v_{B,j}}{v_{A,i}}\right)\right).
\end{aligned}$$

On the other hand, if i is a common product we have that

$$\begin{aligned}
&P(\text{choose } i \text{ over all products}|\varepsilon_i^x) \\
&= P(\text{choose } i \text{ from retailer } A, \text{ choose } i \text{ over all products different of } i|\varepsilon_i^x) \\
&= \delta_{A,i} \exp\left(-e^{-\varepsilon_i^x}\left(\frac{1}{v_{A,i}} + \sum_{j \in S_A \setminus S_B} \frac{v_{A,j}}{v_{A,i}} + \sum_{j \in (S_B \cap S_A) \setminus \{i\}} \frac{\max(v_{A,j}, v_{B,j})}{v_{A,i}} + \sum_{j \in S_B \setminus S_A} \frac{v_{B,j}}{v_{A,i}}\right)\right).
\end{aligned}$$

In both cases we obtain the result integrating $P(\text{choose } i \text{ over all products}|\varepsilon_i^x)$ over $\mathbb{R}$ weighted by the density of ε_i^x and applying similar algebraic manipulations to the ones previously done in Lemma 2.1. □

Similarly to the previous case, we first study the case of competition with fixed prices. In order to study this case, we find a more general problem transformation than the one presented in Lemma 2.2.

Competition with fixed prices

We suppose that the price vectors for both retailers p_A and p_B are fixed. Given a strategy $S_B \in A_B$ of retailer B, retailer A will try to find a strategy S_A^* such that

$$u_A(S_A^*, S_B) = \max_{S_A \in A_A} u_A(S_A, S_B),$$

and similarly for retailer B.

We now present a problem transformation, similar to the one presented in Lemma 2.2 that transforms our problem of finding a best response into another optimization problem. The proof is the same as the proof of Lemma 2.2.

Lemma 3.2. *An assortment selection $S_A^* \in A_A$ for retailer A maximizes $u_A(S_A, S_B)$ for $S_A \in A_A$ and for a given strategy $S_B \in A_B$ of retailer B if and only if S_A^* maximizes*

$$y(S_A, \lambda) = \sum_{i \in S_A} (p_{A,i} - c_{A,i} - \lambda) v_{A,i} + \sum_{j \in S_A \cap S_B} (\delta_{A,i}(p_{A,i} - c_{A,i}) - \lambda) \max(v_{A,i}, v_{B,i}) - \lambda \sum_{i \in S_B} v_{B,i},$$

for $S_A \in A_A$ where λ is the solution to the problem $\max \lambda \in (0, \infty)$ *such that*

$$\max_{S_A \in A_A} y(S_A, \lambda) \geq \lambda,$$

and similarly for retailer B.

We can simplify the expression above by noticing that we can split the term $\max(v_{A,i}, v_{B,i}) = \delta_{A,i} v_{A,i} + \delta_{B,i} v_{B,i}$ for every common product i and by introducing the following notation

$$\theta_{A,i}(\lambda) = \begin{cases} (p_{A,i} - c_{A,i} - \lambda) v_{A,i} & \text{if } i \text{ is an exclusive product} \\ \delta_{A,i}((p_{A,i} - c_{A,i} - \lambda) v_{A,i} + \lambda v_{B,i}) & \text{if } i \text{ is a common product.} \end{cases}$$

Therefore, for a given strategy $S_B \in A_B$ for retailer B, retailer A has to solve the optimization problem

$$\max \lambda \in (0, \infty) \;\; \text{such that} \;\; \max_{S_A \in A_A} \left(\sum_{i \in S_A} \theta_{A,i}(\lambda) \geq \lambda (1 + \sum_{i \in S_B} v_{B,i}) \right).$$

The problem of finding a Nash equilibrium in this case is more complex. In the next result we show that if retailers face no capacity limitation, that is, if they can offer all available products at once, a Nash equilibrium can be found. The proof of this result can be found in Besbes and Sauré [4]. However, if they face capacity limitation, in general, no Nash equilibrium can be found. We will see an example next.

Theorem 3.1. *We suppose that retailers A and B compete in assortment selection only with both exclusive and common products and face no capacity limitations, that is, $C_A = |P_A|$ and $C_A = |P_B|$. Then there always exists at least one Nash equilibrium.*

Proof. In order to prove this theorem, we consider a sequence of assortment selection pairs. We assume that each retailer starts with a certain assortment selection and each turn each retailer selects an assortment that is a best response to the competitor's previous assortment selection. This procedure is known in the literature as a *tatônnement* process, where both competitors reach an equilibrium by trial and error (*tatônnement* in French).

We consider the sequence of assortment selection pairs $\{(S_A^n, S_B^n)\}_{n\geq 1}$ defined by

$$S_A^1 = S_B^1 = \{i \in P_A \cap P_B : v_{A,i} = v_{B,i}\}$$

and for $n > 1$, for a given pair of assortment selection (S_A^{n-1}, S_B^{n-1}), we consider

$$\lambda_A^n = \max\left\{\lambda \in (0,\infty) : \max_{S_A \in P_A} \sum_{i\in S_A} \theta_{A,i}(\lambda, S_B^{k-1}) \geq \lambda(1 + \sum_{i\in S_B^{n-1}} v_{B,i})\right\},$$

$$S_A^n \in \left\{a \in A_A : \sum_{i\in a} \theta_{A,i}(\lambda_A^n, S_B^{k-1}) = \max_{S_A\in A_A} \sum_{i\in S_A} \theta_{A,i}(\lambda_A^n, S_B^{k-1}) \text{ and } |a| \text{ is maximal}\right\},$$

and similarly for retailer B.

Therefore, we have a sequence of assortment selection pairs (S_A^n, S_B^n) for $n \geq 1$ where retailer's A assortment selection S_A^n is a best response to retailer's B assortment selection S_B^{n-1}, and similarly for retailer B. Moreover, each assortment selection in the sequence has the maximum number of products possible (among other best responses).

We now show that this sequence is increasing, that is, for all $n > 1$, $S_A^{n-1} \subseteq S_A^n$ and $S_B^{n-1} \subseteq S_B^n$. We prove this result by induction.

For $n = 1$, we consider the pair (S_A^1, S_B^1). We have that for every product $i \in S_A^1$

$$\theta_{A,i}(\lambda) = \frac{1}{2}((p_{A,i} - c_{A,i} - \lambda)v_{A,i} + \lambda v_{B,i}) = \frac{1}{2}(p_{A,i} - c_{A,i})v_{A,i} > 0,$$

where we used that if $i \in S_A^1$ then $v_{A,i} = v_{B,i}$. Therefore, the products in S_A^1 are added in every assortment selection for retailer A. Since there are no capacity limitations we have that $S_A^1 \subseteq S_A^2$, and similarly for retailer B.

We assume that $S_A^{k-1} \subseteq S_A^k$ for every $2 \leq k \leq n$, and similarly for retailer B. We want to prove that $S_A^n \subseteq S_A^{n+1}$ and $S_B^n \subseteq S_B^{n+1}$.

We first notice that for all $k > 1$ and for a fixed assortment selection S_B^k of retailer B, we have that for all $S_A \in A_A$

$$\sum_{i\in S_A} \theta_{A,i}(\lambda_A^k, S_B^k) \leq \lambda_A^k \left(1 + \sum_{i\in S_B^k} v_{B,i}\right). \tag{3.1}$$

Indeed, in order to prove this inequality we can group the common products in three groups $P_A \cap P_B = X \cup Y \cup Z$ where $X = \{i \in P_A \cap P_B : v_{A,i} > v_{B,i}\}$, $Y = \{i \in P_A \cap P_B : v_{A,i} = v_{B,i}\}$ and $Z = \{i \in P_A \cap P_B : v_{A,i} < v_{B,i}\}$.

As we already stated before, the products in Y are added in every assortment selection for each retailer. Therefore, for a fixed assortment selection S_B^k and for every $\lambda \geq 0$ we can write

$$\begin{aligned}
\sum_{i\in S_A^k} \theta_{A,i}(\lambda, S_B^k) &= \sum_{i\in S_A^k\cap(P_A\setminus P_B)} \theta_{A,i}(\lambda, S_B^k) + \sum_{i\in S_A^k\cap Z\setminus S_B^k} \theta_{A,i}(\lambda, S_B^k) + \sum_{i\in S_A^k\cap X\cap S_B^k} \theta_{A,i}(\lambda, S_B^k)\\
&+ \sum_{i\in S_A^k\cap X\setminus S_B^k} \theta_{A,i}(\lambda, S_B^k) + \sum_{i\in Y} \theta_{A,i}(\lambda, S_B^k)\\
&= \sum_{i\in S_A^k\cap(P_A\setminus P_B)} v_{A,i}(p_{A,i} - c_{A,i} - \lambda) + \sum_{i\in S_A^k\cap Z\setminus S_B^k} v_{A,i}(p_{A,i} - c_{A,i} - \lambda)\\
&+ \sum_{i\in S_A^k\cap X\cap S_B^k} v_{A,i}\left(p_{A,i} - c_{A,i} - \lambda\left(\frac{v_{A,i} - v_{B,i}}{v_{A,i}}\right)\right)\\
&+ \sum_{i\in S_A^k\cap X\setminus S_B^k} v_{A,i}(p_{A,i} - c_{A,i} - \lambda) + \frac{1}{2}\sum_{i\in Y} v_{A,i}(p_{A,i} - c_{A,i} - \lambda)\\
&= \sum_{i\in S_A^k\cap(P_A\setminus P_B)} v_{A,i}(p_{A,i} - c_{A,i} - \lambda) + \sum_{i\in S_A^k\cap Z\setminus S_B^{k-1}} v_{A,i}(p_{A,i} - c_{A,i} - \lambda)\\
&- \sum_{i\in S_A^k\cap Z\cap S_B^k\setminus S_B^{k-1}} v_{A,i}(p_{A,i} - c_{A,i} - \lambda)\\
&+ \sum_{i\in S_A^k\cap X\cap S_B^{k-1}} v_{A,i}\left(p_{A,i} - c_{A,i} - \lambda\left(\frac{v_{A,i} - v_{B,i}}{v_{A,i}}\right)\right)\\
&+ \sum_{i\in S_A^k\cap X\setminus S_B^{k-1}} v_{A,i}(p_{A,i} - c_{A,i} - \lambda) + \sum_{i\in S_A^k\cap X\cap S_B^k\setminus S_B^{k-1}} v_{B,i}\lambda\\
&+ \frac{1}{2}\sum_{i\in Y} v_{A,i}(p_{A,i} - c_{A,i} - \lambda).
\end{aligned}$$

In particular, since for a fixed strategy S_B^{k-1} for retailer B and for every $i \in S_A^k \setminus S_B^{k-1}$ we have that $\theta_{A,i}(\lambda_A^k, S_B^{k-1}) = v_{A,i}(p_{A,i} - c_{A,i} - \lambda_A^k) \geq 0$, we deduce the following inequality for $\lambda = \lambda_A^k$

$$\begin{aligned}
\sum_{i\in S_A^k} \theta_{A,i}(\lambda_A^k, S_B^k) &= \lambda_A^k\left(1 + \sum_{i\in S_B^{k-1}} v_{B,i} + \sum_{i\in S_A^k\cap X\cap S_B^k\setminus S_B^{k-1}} v_{B,i}\right)\\
&- \sum_{i\in S_A^k\cap Z\cap S_B^k\setminus S_B^{k-1}} v_{A,i}(p_{A,i} - c_{A,i} - \lambda_A^k)\\
&\leq \lambda_A^k\left(1 + \sum_{i\in S_B^{k-1}} v_{B,i} + \sum_{i\in S_A^k\cap X\cap S_B^k\setminus S_B^{k-1}} v_{B,i}\right)
\end{aligned} \tag{3.2}$$

In conclusion, for every assortment selection $S_A \in A_A$ and for a fixed assortment selection S_B^k of retailer B we have that

$$\begin{aligned}
\sum_{i \in S_A} \theta_{A,i}(\lambda_A^k, S_B^k) &= \sum_{i \in S_A \cap S_B^k} \theta_{A,i}(\lambda_A^k, S_B^k) + \sum_{i \in S_A \setminus S_B^k} \theta_{A,i}(\lambda_A^k, S_B^k) \\
&\leq \sum_{i \in S_A^k} \theta_{A,i}(\lambda_A^k, S_B^k) + \sum_{i \in (S_A \setminus S_A^k) \cap (S_B^k \setminus S_B^{k-1}) \cap X} (\theta_{A,i}(\lambda_A^k, S_B^{k-1}) + \lambda_A^k v_{B,i}) \\
&+ \sum_{i \in (S_A \setminus S_A^k) \cap S_B^{k-1}} \theta_{A,i}(\lambda_A^k) + \sum_{i \in (S_A \setminus S_A^k) \cap (X \setminus S_B^k)} \theta_{A,i}(\lambda_A^k) \\
&\leq \lambda_A^k \left(1 + \sum_{i \in S_B^{k-1}} v_{B,i} + \sum_{i \in S_A^k \cap X \cap S_B^k \setminus S_B^{k-1}} v_{B,i} + \sum_{i \in (S_A \cap S_A^k) \cap (S_B^k \setminus S_B^{k-1}) \cap X} v_{B,i} \right) \\
&\leq \lambda_A^k \left(1 + \sum_{i \in S_B^k} v_{B,i} \right).
\end{aligned}$$

In the first inequality above we used that $\theta_{A,i}(\lambda_A^k, S_B^k) = \theta_{A,i}(\lambda_A^k, S_B^{k-1}) \geq 0$ if $i \in S_A^k \cap S_B^{k-1}$, $\theta_{A,i}(\lambda_A^k, S_B^k) = \theta_{A,i}(\lambda_A^k, S_B^{k-1}) + \lambda_A^k v_{B,i} \geq 0$ if $i \in S_A^k \cap (S_B^k \setminus S_B^{k-1})$ and $\theta_{A,i}(\lambda_A^k, S_B^k) = 0$ if $i \in S_A^k \cap (S_B^k \setminus S_B^{k-1}) \cap Z$. With these characteristics in mind, we deduce that $\theta_{A,i}(\lambda_A^k, S_B^k) \geq 0$ for every $i \in S_A^k$. Moreover, we have that $\theta_{A,i}(\lambda_A^k, S_B^k) = 0$ for every $i \in (S_A \setminus S_A^k) \cap (S_B^k \setminus S_B^{k-1}) \cap Z$.

In the second inequality above we used the inequality previously found in (3.2) and the fact that $\theta_{A,i}(\lambda_A^k, S_B^{k-1}) < 0$ if $i \in P_A \setminus S_A^k$, because S_A^k is a best response to S_B^{k-1}.

To conclude the induction, we have that the inequality in (3.1) is true for every assortment selection $S_A \in A_A$. In particular it is also true when applied to a best response to S_B^k. Therefore we have that $\lambda_A^{k+1} \leq \lambda_A^k$. Moreover, since $\theta_{A,i}(\lambda_A^k, S_B^k) \geq 0$ for every $i \in S_A^k$ and $\theta_{A,i}(\cdot, S_B^k)$ is a decreasing function, we deduce that $\theta_{A,i}(\lambda_A^{k+1}, S_B^k) \geq 0$. Therefore, we have that $S_A^k \subseteq S_A^{k+1}$.

To conclude the proof, we notice that we constructed a succession of assortment selection pairs $\{(S_A^n, S_B^n)\}_{n \geq 1}$ such that $S_A^k \subseteq S_A^{k+1}$ and $S_B^k \subseteq S_B^{k+1}$ for every $k \geq 1$. Since the sets of all available products P_A and P_B are finite, we have that there exists $n_1 \geq 1$ and $n_2 \geq 1$ such that $S_A^n = S_A^{n_1}$ for all $n \geq n_1$ and $S_B^n = S_B^{n_2}$ for all $n \geq n_2$. Therefore we have that $(S_A^{n_0}, S_B^{n_0})$ is a Nash equilibrium, where $n_0 = \max(n_1, n_2)$.

□

In the next example we present a competition between two retailers where they have limited space in shops, and we show that no Nash equilibrium can be found.

Example 3.1. *Let A and B be to retailers that compete over a market. We consider that they have access to common products $P_A = \{1, 2, 3\}$ and $P_B = \{1, 2, 3\}$, respectively. Moreover, we assume that they face capacity limitations in shops such that retailer A can only display two products at once and retailer B can only display one product at once. In other words, $C_A = 2$ and $C_B = 1$.*

Therefore, their strategy space is

$$A_A = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\},$$

and

$$A_B = \{\{1\}, \{2\}, \{3\}\},$$

respectively.

Furthermore, we assume that all products have a marginal cost of production equal to 1 and a selling price of 2, therefore the profit of each retailer per sale is 1. Also, we assume that all products have similar attraction factors such that $v_{A,1} = v_{B,1} = 4.9$, $v_{A,2} = v_{B,2} = 5$ *and* $v_{A,3} = v_{B,3} = 5.1$.

With that being said, we can compute the probabilities that a customer selects a product i *for two given strategies* $S_A \in A_A$ *and* $S_B \in A_B$ *following the formula given in Lemma 2.3.*

In order to show an example of how these probabilities are computed, we suppose that $S_A = \{1, 2\}$ *and* $S_B = \{1\}$. *Then, we have that the probability of a customer buying product 1 from retailer A is*

$$q_{A,1} = \frac{1}{2} \cdot \frac{v_{A,1}}{1 + v_{A,1} + v_{A,2}} = \frac{49}{218}.$$

Finally, we can write the matrix of outcomes where every entry is computed as

$$u_A(S_A, S_B) = \sum_{i \in S_A} (p_{A,i} - c_{A,i}) q_{A,i}(S_A, S_B) = \sum_{i \in S_A} q_{A,i}(S_A, S_B),$$

and similarly for retailer B.

Therefore, we have that the matrix of outcomes is:

	$\{1\}$	$\{2\}$	$\{3\}$
$\{1\}$	$(\frac{49}{118}, \frac{49}{118})$	$(\frac{49}{109}, \frac{50}{109})$	$(\frac{49}{110}, \underline{\frac{51}{110}})$
$\{2\}$	$(\frac{50}{109}, \frac{49}{109})$	$(\frac{5}{12}, \frac{5}{12})$	$(\frac{50}{111}, \underline{\frac{17}{37}})$
$\{3\}$	$(\frac{51}{110}, \frac{49}{110})$	$(\frac{17}{37}, \underline{\frac{50}{111}})$	$(\frac{51}{122}, \frac{51}{122})$
$\{1, 2\}$	$(\frac{149}{218}, \frac{49}{218})$	$(\frac{74}{109}, \frac{25}{109})$	$(\frac{99}{160}, \underline{\frac{51}{160}})$
$\{1, 3\}$	$(\underline{\frac{151}{220}}, \frac{49}{220})$	$(\frac{5}{8}, \underline{\frac{5}{16}})$	$(\frac{149}{220}, \frac{51}{220})$
$\{2, 3\}$	$(\frac{101}{160}, \underline{\frac{5}{16}})$	$(\underline{\frac{76}{111}}, \frac{25}{111})$	$(\underline{\frac{151}{222}}, \frac{17}{74})$

Table 3.1: Payoff matrix of the game of Example 3.1

In this table, the best responses are underlined. As we can see, there are no Nash equilibria of the game.

Competition in prices and assortment

We now study the case where retailer compete in both assortment selection and price fixing. In other words, strategies are now pairs (S_A, p_A) with $S_A \in A_A$ is the assortment selection of retailer A and $p_A = (p_{A,1}, p_{A,2}, \ldots, p_{A,|S_A|})$ is the vector of prices, where for all $i \in S_A$ we have that $p_{A,i}$ is the price at which retailer A is willing to sell product i and $p_{A,i} > c_{A,i}$. And similarly for retailer B. Moreover, we consider that retailers select these strategies simultaneously, without knowing their competitor's strategy.

Similarly to the case of competition in assortment selection, given a strategy (S_B, p_B) for retailer B, retailer A's best response will be the solution to the problem of maximizing $\lambda \geq 0$ such that

$$\max_{(S_A,p_A)} \left(\sum_{i \in S_A} \theta_{A,i}(\lambda, p_A, S_B, p_B) \right) \geq \lambda \left(1 + \sum_{i \in S_B} v_{B,i}(p_{B,i}) \right),$$

where we recall that

$$\theta_{A,i}(\lambda, p_A, S_B, p_B) = \begin{cases} (p_{A,i} - c_{A,i} - \lambda) v_{A,i} & \text{if } i \notin S_B \\ \delta_{A,i}((p_{A,i} - c_{A,i} - \lambda) v_{A,i} + \lambda v_{B,i}) & \text{if } i \in S_B. \end{cases}$$

Notice that we can calculate best responses by first fixing an assortment selection and calculating the best prices for those products and then fixing the prices and calculating the best assortment selection, a problem that we have already solved.

Let us suppose that retailer B offers assortment selection S_B and price selection p_B. Moreover, we suppose that retailer A's assortment selection S_A is fixed. For a given $\lambda \geq 0$ we have that the price at which retailer A will sell exclusive product i is

$$p^*_{A,i}(\lambda) = \frac{1}{\alpha_{A,i}} + c_{A,i} + \lambda,$$

which we already computed before. However, the study of prices for common products is more complex, as retailers will tend to lower the price of a product in order to capture the whole market. In order to study this case, for every common product i we introduce $p^{min}_{A,i}$ the minimum price at which retailer A is willing to sell common product i, and therefore will not further lower the price of product i past $p^{min}_{A,i}$. And similarly for retailer B.

We now consider the set of common products whose attraction level with the minimum price is higher for retailer A, that is,

$$\overline{S_A} = \{ i \in S_A \cap S_B : v_{A,i}(p^{min}_{A,i}) > v_{B,i}(p^{min}_{B,i}) \}.$$

For the products in this subset $\overline{S_A}$ retailer A is willing to compete with retailer B and lower product i's price in order to overtake the whole market, and therefore retailer A will fix prices

$$p^*_{A,i} = \min \left(\frac{\mu_{A,i} - \mu_{B,i} + \alpha_{B,i} p^{min}_{B,i}}{\alpha_{A,i}}, \frac{1}{\alpha_{A,i}} + c_{A,i} + \lambda \right)$$

while for the common products i not in this subset, retailer A is not able to overtake the whole market, and therefore will fix prices $p^*_{A,i} = p^{min}_{A,i}$.

With the next result we show that if retailers can offer all available products at once (no capacity limitations are present), then a Nash equilibrium can be found.

Theorem 3.2. *We suppose that retailers A and B compete in assortment selection and price selection with both exclusive and common products and face no capacity limitations, that is, $C_A = |P_A|$ and $C_A = |P_B|$. Then there always exists at least one Nash equilibrium.*

Proof. We suppose retailer B offers a fixed strategy (S_B, p_B). We define

$$\theta^*_{A,i}(\lambda, S_B, p_B) = \max_{p_A} \ \theta_{A,i}(\lambda, p_A, S_B, p_B).$$

We observe that by the definition of $\theta_{A,i}$ we have that $\theta^*_{A,i}(\lambda, S_B, p_B) \geq 0$, since for every λ we can find a feasible price such that $\theta_{A,i}(\lambda, p_A, S_B, p_B) \geq 0$. Therefore, all products in P_A will be added to the assortment selection of retailer A, since no capacity limitations are present. Therefore, we can suppose that both retailers will offer all available products. Furthermore, the maximization problem that retailer A has to solve can be re-expressed as

$$\max \left\{ \lambda \geq 0 : \sum_{i \in P_A} \theta^*_{A,i}(\lambda, P_B, p_B) \geq \lambda(1 + \sum_{i \in P_B} v_{B,i}(p_{B,i}) \right\}.$$

Consequently, any pair (λ_A, λ_B) leads to an equilibrium, where λ_A denotes the expected profit of retailer A in equilibrium and similarly for retailer B, if they are solution to equations

$$\begin{cases} \sum_{i \in P_A} \theta^*_{A,i}(\lambda_A, P_B, p^*_B(\lambda_B)) = \lambda_A(1 + \sum_{i \in P_B} v_{B,i}(p^*_{B,i}(\lambda_B)) \\ \sum_{i \in P_B} \theta^*_{B,i}(\lambda_B, P_A, p^*_A(\lambda_A)) = \lambda_B(1 + \sum_{i \in P_A} v_{A,i}(p^*_{A,i}(\lambda_A)). \end{cases}$$

In order to solve each equation we observe that by the definition of $\theta^*_{A,i}(\lambda_A, P_B, p^*_B(\lambda_B))$, there exists a positive constant $M > 0$ such that $\theta^*_{A,i}(\lambda_A, P_B, p^*_B(\lambda_B))$ is positive and decreasing for $\lambda \in [0, M]$. Moreover, in each equation we have that $\lambda_A(1+\sum_{i \in P_B} v_{B,i}(p^*_{B,i}(\lambda_B))$ is increasing in λ and for $\lambda = 0$, it equals 0. Therefore, there exists an unique solution to the first equation which we denote $Y_A(\lambda_B)$, and similarly for the second equation.

Furthermore, we notice that in each equation, the term $\theta^*_{A,i}(\lambda_A, P_B, p^*_B(\lambda_B))$ is independent of λ_B while the term $\lambda_A(1 + \sum_{i \in P_B} v_{B,i}(p^*_{B,i}(\lambda_B))$ is decreasing in λ_B; and similarly for the second equation. Therefore, the solutions $Y_A(\cdot)$ and $Y_B(\cdot)$ is increasing.

Consequently, we have that

$$0 \leq Y_A(Y_B(0)) \leq Y_A(Y_B(M)) \leq M.$$

Since prices $p^*_A(\cdot)$ and $p^*_B(\cdot)$ are continuous by definition, we have that $Y_A(\cdot)$ and $Y_B(\cdot)$ are also continuous. Therefore, by Bolzano's Theorem applied to continuous function $Y_A(Y_B(\lambda)) - \lambda$ in $[0, M]$ and utilizing the inequalities above, we have there exists $\lambda^* \in [0, M]$ such that $Y_A(Y_B(\lambda^*)) = \lambda^*$. Finally, we notice that the pair (λ^*, Y^*_B) solves the first equation in the system above. Doing a similar reasoning for the second equation, we find and equilibrium. □

With this result we proved a very interesting result that shows the existence of Nash equilibria when retailers compete in assortment with both common and exclusive products and in price selection and face no capacity limitation. However, contrary to the case of competition only in exclusive products, in the case of multiple equilibria we cannot guarantee the existence of an equilibrium that Pareto-dominates all other equilibria.

Chapter 4

Application: competition between Gucci and Farfetch

In the previous chapter we studied the consequences of the competition between two retailers in assortment selection and price selection, considering that customers select products according to an utility ranking like we described before. We came to the conclusion that retailers set optimal prices for exclusive products and tend to lower the prices for common products in order to take over the whole market, similarly to what happens in the Bertrand model.

In this chapter we further study the relationship between the price that a retailer sets for a product and whether a product is exclusive or not. In order to do so, we consider the competition between clothing retailers Farfetch and Gucci during the winter season of 2020.

The data used for this chapter was collected by IESE professor Victor Martínez de Albéniz during December 2020 from the retailers' website and was classified in two *.RData* files, one for each retailer, following the *schema.org* [1] data structure. Due to limited space, the data is available on demand. The crucial information for each product we can find in those files is the name of the product or a brief description, the seller, the price at which the product was put on sale and the time each product was added to the website.

There is a total of $52,426$ Gucci products that were collected during the whole month of December, and a total of $968,690$ Farfetch products that correspond to the Farfetch products added to their website in the date of 13/12/2020.

In order to clear the data, we firstly filter the Gucci products by the date they were added to their website, and only select the products added to the website during the date of 13/12/2020. Moreover, we remove the data for both retailers that does not have a proper name or price by searching the rows of the data frame that have only a space, no data (*NA*) or not a valid number (*NaN*) under the name or price column and eliminating them.

Furthermore, we remove the products that correspond to brands other than Gucci from the data collected from Farfetch, since we are particularly interested in studying which products are only offered by Gucci and which products are offered by both Gucci and Farfetch.

[1] The schema.org data structure is a series of characteristics in which data is classified. It was created in order to homogenize the structure of the data on the Internet.

In order to select which products offered by Gucci are exclusive and which are both offered by Gucci and Farfetch, we use the *agrep()* function in R, which is used to find similar groups of characters or *strings*. This function is the implementation in R of the Levenshtein[2] distance.

The Levenshtein distance is a string metric that measures the minimum number of insertions, elimination or substitution of characters required to transform one string into another. For example, the Levenshtein distance between the two strings "rice" and "race" is 1, since only one letter substitution is required to change one string into the other.

However, we can not do a proper product matching only using this distance. For example, if we consider two of the products in the list such as "Sudadera con estampado de Mad Cookies" and "sudadera bordado mad cookies", we observe that these two products are equal or very similar (one is printed and the other is embroidered), and we assume they are common products. Nevertheless, the Levenshtein distance between the two string is 15.

We notice that this comparison is not good enough, since we can find two very different products such as "camiseta mad cookies" and "anillo de oro" whose Levenshtein distance is also equal to 15. Therefore, we have to further refine our product matching.

In order to do so, we have to remove words that do not contain crucial information from the strings such as "para", "de", "con" or "Gucci" among others. With these changes done, we can finally search for products that match in both lists. As we commented before, we use the *agrep()* function of R an consider that two products are the same if the Levenshtein distance between the two strings is less or equal than 3. This is for anticipating possible typos when introducing the name of the product.

Once we have cleared and the available data and searched for common products, we can now study the relationship between the price retailer Gucci sets for its products and whether the product is exclusive to Gucci or it is also sold by Farfetch.

In the first place, we attempt to model the relationship between the price and the exclusiveness of a product by a linear relation and an error term, which is also known as the simple linear regression model.

In other words, we assume that the available sample of n observations comes from a simple linear regression model where the price p_i of a product i is given by the expression

$$p_i = \beta_0 + \beta_1 X(i) + \varepsilon_i \ \text{ for all } \ 1 \leq i \leq n,$$

where X is the explanatory variable that takes value 1 if the product is exclusive and 0 if the product is common, also known as a *dummy variable* that models the qualitative feature of exclusiveness. Moreover, β_0 and β_1 are the regression coefficients, and ε_i is the error term, where ε_i for $1 \leq i \leq n$ are independent and identically distributed random variables that are centered ($E(\varepsilon_i) = 0$ for all $1 \leq i \leq n$) and with finite and constant variance $Var(\varepsilon_i) = \sigma^2$ for all $1 \leq i \leq n$.

With that being said, we can interpret β_0 as the average price of a common product and β_1 as the average difference in prices between exclusive and common products.

We have that the results for this linear regression model are:

[2]Vladimir Levenshtein (1935-2017) was a Russian mathematician specialized in information theory.

```
Call:
lm(formula = Price ~ Exclusive, data = dataframe)

Residuals:
    Min      1Q  Median      3Q     Max 
-1253.9  -575.1  -365.1   514.9  6434.9 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  1465.11      27.95  52.425   <2e-16 ***
Exclusive     108.81      91.38   1.191    0.234    
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 899.9 on 1142 degrees of freedom
Multiple R-squared:  0.00124,	Adjusted R-squared:  0.0003656 
F-statistic: 1.418 on 1 and 1142 DF,  p-value: 0.234
```

Figure 4.1: Summary of linear regression with exclusiveness as only explanatory variable.

As we can observe, we obtain a higher p-value than the significance level 0.05 and therefore we fail to reject the null hypothesis. We recall that the null hypothesis is that the coefficients of regression are zero, that is, there is no relationship between the price Gucci sets for a product and whether the product is exclusive or not. Moreover, we notice the presence of outliers situated at the higher end of the available prices, that is, some products are way overpriced in comparison with the available data.

This, in particular, shows that the model we used to relate the price and the exclusiveness of each product is not good enough or that there is no relationship between the two.

In order to further improve the study, we remove the outliers that we detected and we introduce an additional explanatory variable that models the type of each product. We consider three groups of products: the group A where we place products such as shirts, T-shirts, sweatshirts or sweaters; the group B where we place jackets, coats or dresses; and group C where we place trousers, skirts and others.

Therefore, we now consider that the available sample of n observations comes from a simple linear regression model where the price p_i of a product i is given by the expression

$$p_i = \beta_0 + \beta_1 X(i) + \beta_2 Y(i) + \varepsilon_i \quad \text{for all} \quad 1 \leq i \leq n,$$

where X, β_0, β_1 and ε_i are the same as in the previous model, β_2 is a regression coefficient and Y is the explanatory variable that takes value 1 if the product is in group A, 2 if the product is in group B and 3 if the product is in group C.

We have that the results for this linear regression model are:

```
Call:
lm(formula = Price ~ Exclusive + Type, data = data)

Residuals:
    Min      1Q  Median      3Q     Max
-1254.4  -518.5  -331.5   553.0  2073.0

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  1284.07      62.83  20.438   <2e-16 ***
Exclusive      75.92      78.86   0.963   0.3359
Type           71.48      29.86   2.394   0.0168 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 769.4 on 1130 degrees of freedom
Multiple R-squared:  0.005744,  Adjusted R-squared:  0.003984
F-statistic: 3.264 on 2 and 1130 DF,  p-value: 0.0386
```

Figure 4.2: Summary of linear regression with exclusiveness and product type as explanatory variables.

As we can observe, this time we obtain a lower p-value than the significance level 0.05 and therefore we reject the null hypothesis. Therefore, we can now conclude that our model is statistically significant and thus we observe a relationship between the price retailer Gucci sets for his products and whether the products are common or exclusive and which is the type of each product.

However, we notice that the model is still not good enough: we notice that the residuals are still very big and they are not randomly distributed around 0. Moreover, we observe that the multiple and the adjusted R-squared are very insignificant, and therefore our model does not explain well our data.

Therefore, we conclude that the linear regression model we used does not fit well enough the available data and we cannot affirm that there is a linear relation between the price retailer Gucci sets for his products and whether the product is exclusive or common. Further improvements could be the introduction of new explanatory variables or the application of another model such as the logistic regression.

Bibliography

[1] Anderson, Simon P., De Palma, André and Thisse, Jacques-François (1992). *Discrete Choice Theory of Product Differentiation.* The MIT Press.

[2] Ben-Akiva, M. and Lerman, S. R. (1985). *Discrete Choice Analysis: Theory and Application to Travel Demand.* The MIT Press.

[3] Bertrand, J. (1883). Book review of "Théorie mathématique de la richesse sociale" and of "Recherches sur les principes mathématiques de la théorie des richesses". *Journal de Savants*, 67, 499-508.

[4] Besbes, O. and Sauré, D. (2016). Product Assortment and Price Competition under Multinomial Logit Demand. *Production and Operations Management Society*, Vol. 25, No. 1, January 2016, 114-127.

[5] Cournot, A. (1838). *Recherches sur les principes mathématiques de la théorie des richesses.* Paris: Hachette. Œuvres complètes de A.-A. Cournot, 8.

[6] Draganska, M. and Jain, D. C. (2006). Consumer preferences and product-line pricing strategies: An empirical analysis. *Marketing Science*, Vol. 25, No. 2, 164–174.

[7] Draganska, M., Mazzeo, M. and Seim, K. (2009). Beyond plain vanilla: Modeling pricing and product assortment choices. *Quantitative Marketing and Economics.* Vol. 7, 105–146.

[8] Food Retail & Shoppers. (2021). *Aldi agita la guerra de precios con una bajada del 1% en 2000 productos.* Accessed June 2021. `https://www.foodretail.es/retailers/Aldi-guerra-precios-bajada-productos-surtido_0_1550844923.html`

[9] Gallego, G., Iyengar, R., Phillips, R. and Dubey, A. (2004). Managing Flexible Products on a Network. *CORC Technical Report*, TR 2004-01.

[10] Gibbons, R. S. (1992). *Game theory for applied economists.* Princeton University Press.

[11] Guadagni, P. M. and Little, J.D. (1983). A logit model of brand choice calibrated on scanner data. *Marketing Science*, Vol. 2, Issue 3, 203-238.

[12] Jehle, G. A., and Reny, P. J. (2011). *Advanced microeconomic theory.* Pearson Education.

[13] Maddah, B. and Bish, E. K. (2007). Joint pricing, assortment, and inventory decisions for a retailers product line. *Naval Research Logistics.* Vol. 54, 315–330.

[14] Mahajan, S. and van Ryzin, G. (2001). Stocking retail assortments under dynamic consumer substitution. *Operations Research*. Vol. 49, 334–351.

[15] Mas-Colell, A., Whinston, M. D. and Green, J. R. (1995). *Microeconomic Theory*. Oxford University Press.

[16] McFadden, D. (1974). *Conditional Logit Analysis of Qualitative Choice Behavior*, in P. Zarembka (ed.) Frontiers in Econometrics (Economic theory and mathematical economics), 105–142, Academic Press.

[17] Misra, K. (2008). Understanding retail assortments in competitive markets. *London Business School*.

[18] Nash, J. (1951). Non-cooperative games. *The Annals of Mathematics*, 54(2), second series, 286–295.

[19] Von Neumann, J., and Morgenstern, O. (1944). *Theory of games and economic behavior*. Princeton University Press.

[20] Pareto, V. (1906). *Manual of Political Economy*. (originally Manuale di economia politica – con una introduzione alla Scienza Sociale. Milano, 1906). Translation of 1927 French edition. Augustus M. Kelley Publishing, 1971

[21] Pérez, J., Jimeno, J. L. and Cerdá, E. (2004). *Teoría de Juegos*. Pearson Educación.

[22] Sundaram, Rangarajan K. (1996) *A First course in optimization theory.* Cambridge University Press.

[23] Topkis, D. (1998). *Supermodularity and Complementarity*. Princeton University Press.

[24] Vives, X. (1999). *Oligopoly Pricing: Old Ideas and New Tools*. The MIT Press.

Appendix A

Used code

In this Appendix we can see the code used for filtrating the data in Chapter 4.

```
PFar <- PFar[(PFar$brand.id == 25354),]
PFar <- PFar[!(PFar$name == ""),]
PFar <- PFar[!(PFar$name == " "),]
PFar <- PFar[!is.na(PFar$name),]
PGuc <- PGuc[!(PGuc$name == " "),]
PGuc <- PGuc[!(PGuc$name == ""),]
PGuc <- PGuc[!is.na(PGuc$name),]
PFar[[2]] <- tolower(PFar[[2]]
PGuc[[6]] <- tolower(PGuc[[6]])

PFar[[2]] <- gsub(pattern= " a ", replacement=" ", PFar[[2]])
PFar[[2]] <- gsub(pattern= " para ", replacement=" ", PFar[[2]])
PFar[[2]] <- gsub(pattern= " hombre ", replacement=" ", PFar[[2]])
PFar[[2]] <- gsub(pattern= " mujer ", replacement=" ", PFar[[2]])
PGuc[[6]] <- gsub(pattern= " mujer ", replacement=" ", PGuc[[6]])
PGuc[[6]] <- gsub(pattern= " hombre ", replacement=" ", PGuc[[6]])
PGuc[[6]] <- gsub(pattern= " a ", replacement=" ", PGuc[[6]])
PGuc[[6]] <- gsub(pattern= " para ", replacement=" ", PGuc[[6]])
PFar[[2]] <- gsub(pattern= " de ", replacement=" ", PFar[[2]])
PFar[[2]] <- gsub(pattern= " con ", replacement=" ", PFar[[2]])
PFar[[2]] <- gsub(pattern= " y ", replacement=" ", PFar[[2]])
PFar[[2]] <- gsub(pattern= " en ", replacement=" ", PFar[[2]])
PGuc[[6]] <- gsub(pattern= " de ", replacement=" ", PGuc[[6]])
PGuc[[6]] <- gsub(pattern= " con ", replacement=" ", PGuc[[6]])
PGuc[[6]] <- gsub(pattern= " y ", replacement=" ", PGuc[[6]])
PGuc[[6]] <- gsub(pattern= " en ", replacement=" ", PGuc[[6]])
PGuc[[6]] <- gsub(pattern= " estampado ", replacement=" ", PGuc[[6]])
PGuc[[6]] <- gsub(pattern= " Gucci ", replacement=" ", PGuc[[6]])
PFar[[2]] <- gsub(pattern= " Gucci ", replacement=" ", PFar[[2]])
PFar[[2]] <- gsub(pattern= " estampado ", replacement=" ", PFar[[2]])
```

```
for(i in 1:1144){
  c <- agrep(PGuc[[6]][[i]], PFar[[2]], max.distance=3)
  if(length(c) != 0){
        PGuc[[16]][[i]] <- 0
        PFar <- PFar[-c,]
  }
}

 for(i in 1:1144){
     c <- agrep(PGuc[[6]][[i]], PGuc[[6]], max.distance=3)
     if(length(c) != 0){
         PGuc[[16]][[i]] <- PGuc[[16]][[c[1]]]
     }
 }
```