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Evolutionary Games

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Abstract

Evolutionary game theory is the application of game theory to evolving populations in different contexts as biology or economics. It defines a theoretical framework of contests and strategies into which Darwinian competition can be modelled. In this work, we will present the main concepts of this field from a mathematical perspective, being the *evolutionary stable strategies* (ESS) and the *replicator dynamics* systems the most important concepts.

Firstly, we present the basic concepts of game theory and qualitative theory of differential equations that will be used throughout the work. We also study how to represent plannar differential systems using the Poincaré compactification. Then, we develop a theoretical study of the ESS and explain the different type of games where this concept can be applied, *player versus environment* games and *player versus player* games. Finally, we explain the replicator dynamics and weaker concepts of evolution than ESS. All the sections will have different examples in order to help the reader understand and apply the theory.

Resum

La teoria de jocs evolutiva és l'aplicació de la teoria de jocs a poblacions que evolucionen en camps com la biologia o l'economia. Defineix un marc de treball teòric de competicions i estratègies on l'evolució Darwiniana pot ser estudiada. En aquest treball, presentarem els conceptes principals d'aquest camp des d'una persepectiva matemàtica, sent els conceptes d'*evolutionary stable strategy* (ESS) i *replicator dynamics* els més importants.

Primerament, presentarem els conceptes bàsics de teoria de jocs i de teoria qualitativa d'equacions diferencials que seran utilitzats en el treball. També estudiarem com representar sistemes d'equacions diferencials en el pla usant la compactificació de Poincaré. Més endavant, desenvoluparem l'estudi teòric de l'ESS i explicarem els diferents tipus de jocs on aquest concepte es pot aplicar. Finalment, explicarem els de sistems de replicator dynamics i conceptes evolutius més febles que l'ESS. Totes les seccions tindran diferents exemples per tal d'ajudar al lector o lectora a entendre i aplicar la teoria.

Agraïments

Primerament, voldria agrair als meus tutors, Josep Maria i Xavier, la seva gran ajuda per portar endavant aquest treball. Ha sigut un gran plaer treballar conjuntament amb ells perquè m'han ajudat tant a desenvolupar el projecte com a gaudir del tema gràcies al seu interés i implicació.

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Introduction

Evolution can be defined from many complex and different perspectives. However, we will use the simplest and broadest definition that can be found in a dictionary. Evolution is a process of change in a certain direction. Using this definition as the starting point of this work leaves us the opportunity to walk into many different fields. From here we could start talking about the development of societies, the creation of the universe or even about the world's most famous video game of history, Pokémon. However, almost every reader should direct his/her mind towards biology when he/she first hears about evolution. On the other hand, what are we talking about when we refer to a game? Is it the ferocious Monopoly that comes after a family lunch or maybe it's just the most important football match of the year, *el Clásico*. Once again, we will go to the dictionary and define a game as a physical or mental competition conducted according to rules with the participants in direct opposition to each other. From a biological perspective, a game could be seen as the competition for survival between different species.

Evolutionary game theory encompasses Darwinian evolution, including competition, natural selection and heredity. A big difference between classical game theory and evolutionary game theory is that players do not require to act rationally, as Maynard Smith [8] realised. The evolutionary process itself will be the one testing the adequacy of each strategy and will keep eliminating those behaviours that do not help a species to survive. That is, the survival of certain strategies or behaviours will mainly depend on the offspring left by the individuals who adopt these strategies. The most successful strategies will then be passed to the next generations and will keep living on as long as they ensure survival for those individuals who use them. Going back to Darwin, we can summarise it by using the famous phrase, "survival of the fittest", which was first used by Herbert Spencer after reading Darwin's "*Origin of Species*".

The aim of this work will be to enter the reader into the world of evolutionary game theory and approach it from a mathematical and economic point of view. The application of game theory to evolutionary problems is very common nowadays, although it was not so long ago when it all began to take shape. John Maynard Smith and George R. Price's 1973 *Nature* article, "*The Logic of Animal Conflict*", [7], is often referred as the first description of the concept of an *evolutionary stable strategy*, ESS, which will be one of the main points of this work. As we previously said, biology is the field where the concept of evolution has had a biggest importance and so it has been the field where evolutionary game theory has primarily developed. In fact, even if the paper that we just mentioned is marked as the "birth of the ESS", one of the authors, George R. Price, had already got close to this same term in a previous publication in 1968, "*Antlers, Instraspecific Combat, and Altruism*" [3]. This is how Price described this concept in his article: "A sufficient condition for a genetic strategy to be stable against evolutionary perturbation is that no better strategy exists that is possible for the species without taking a major step in intelligence or physical endowment. Hence a fighting strategy can be tested for stability by introducing perturbations in the form of animals with deviant behaviour, and determining whether selection will automatically act against such animals". As we will see, this definition is not far from the formal ESS definition that will be presented later on. In 1974, Maynard Smith, [8], also developed more theory regarding this concept and adding more complex situations.

The concept of stability in large populations had already been approached by John Forbes Nash Jr., an American mathematician who made fundamental contributions to game theory, differential geometry and the study of partial differential equations. He is the only person to be awarded both the Nobel Memorial Prize in Economic Sciences and the Abel Prize. Nash made an enormous impact in game theory, being the *Nash equilibrium* an essential concept to understand and solve games. Nash, [12], also introduced evolutionary game theory in his work with the concept of "mass-action interpretation", which refers to the situation of a population where everyone uses the same strategy. The question in this case is, will the population remain this way or can it be "invaded" by individuals using a different strategy? The answer to this question will turn around the concept of ESS, which is a refinement of the concept of Nash equilibrium, as we will see later in this work. A very good example of an ESS is the sex ratio in the majority of animal species, as Maynard Smith studied in his work *"Did Darwin get it right?"*, [9]. Why do we have a 50% ratio of males and females? The answer will be that this situation is an evolutionary stable strategy and we will develop it in the second chapter of this work.

Having said all this, the study of evolutionary situations will not conclude only with the concept of ESS. As we will see, not all games will end up having strategies that are as stable as the concept of ESS requires, the classical Rock-Paper-Scissors game will be an example. Then, to have a more complete view of the evolutionary process, we will introduce the model of *replicator dynamics*. This model explains changes in fitness that arise from changes in the population's composition. In this case we will also be working with a huge population of individuals, however, not all individuals will be using the same strategy this time, all the possible strategies will be played with a certain probability or proportion. The evolution of the population will be described by differential equations that were firstly introduced by Peter Taylor and Leo Jonker, [15], in 1978.

Once we get at this point, we will have to use the concepts related to differential equations that will have been defined in the first chapter of this work. The objective will be the representation of the phase portrait that corresponds to the equations of the replicator dynamics, that way we will be able to graphically see how the population evolves. In order to provide this representation we will use,s the Poincaré compactification, which will also allow us to study the trajectories of a planar differential linear system near infinity. Using as a reference the book "Qualitative Theory of Plannar Differential Systems" by Freddy Dumortier, Jaume Llibre and Joan C. Artés, [2], we will explain in broad terms how this process is done.

This work is organized as follows. In the first and second sections,s of chapter 1 we will present the basic concepts of Game Theory that will be needed throughout the entire work. The next two sections of the first chapter will be the most mathematical part as they introduce the necessary concepts of differential equations as well as the phase portrait representation using the Poincaré compactification. Then, chapter 2 introduces the concept of ESS and the different type of games that can be studied in evolutionary game theory. Finally, the third and last chapter presents the replicator dynamics equations and more concepts of stability that are weaker than the ESS and the relationships between them. Several examples of games will be explained with their correspondent phase portrait representations.

Chapter 1

Background

The structure and organization of Section 1.1 and 1.2 of this chapter have been inspired by the Part I of the book *"Game theory: decision, interaction, and evolution"*, [17]. However, many concepts have been changed and explained in a different way using the content provided in the Game Theory course of the Business Administration degreee who was taught by one of the advisors of this work, Josep Maria Izquierdo Aznar.

Then, in sections 1.3 and 1.4 we have used as guidance the books *"Liçóes de equaçóes diferenciais ordinárias"* by Sotomayor, [14], and *"Qualitative theory of planar differential systems"*, [2]. In both cases, we restricted our study to the topics that were necessary and more related to the purpose of this work, which is evolutionary game theory.

1.1 Games and Strategies

Game theory studies the interaction between decision-makers whose decisions affect each other. The biggest distinction that can be made in tame theory involves the existence or non-existence of cooperation between players. A game is *cooperative* if the players are able to form binding commitments externally enforced, e.g., through contract law. A game is *non-cooperative* if players cannot form alliances or if all agreements need to be self-enforcing, e.g., through credible threats.

Non-cooperative game theory is the most traditional and general of both types, as cooperative games can also be viewed from a non-cooperative perspective. While cooperative game theory describes only the structure, strategies and payoffs of coalitions that may arise, non-cooperative game theory also looks at how bargaining procedures will affect the distribution of payoffs within each coalition.

A *static game* is one in which a single decision is made simultaneously by each player, and none of them has any knowledge of the decision made by the other players before making their own decision.

We will essentially work with non-cooperative static games in this work. However, there is one part of evolutionary game theory where we will present a different kind of situation where a single player competes against a bigger opponent, the environment. But this point will be addressed later on in this document, for now we have to start with the proper definition of the games that we will be working with.

Definition 1.1. An *action* is a choice of behaviour in a single-decision problem. The set of alternative actions will be denoted A and it can be a discrete or continuous set.

Definition 1.2. A *payoff* is a function $\pi : A \to \mathbb{R}$ that associates a numerical value with every action $a \in A$.

Observation 1.1. *In this definition of payoff function, the reader could think that the payoff obtained only depends on the action chosen by the player. However, the situation is more complex since the outcomes do not only depend on what the individual is doing but also on what the rest of the players decide to do. That is, an action chosen by a player can leave a different payoff depending on what behaviour he/she is facing. In each situation that we expose during this work we will clarify which are the factors that affect the payoff and the notation will take that aspect into account. That is, in the case of having two players, the payoff function should be described as* $\pi : X_1 \times X_2 \to \mathbb{R}$ *where* X_1 *and* X_2 *are the sets of available actions for player 1 and player 2 respectively.*

Definition 1.3. An action a^* is an *optimal action* if

$$\pi(a^*) \geq \pi(a) \ \forall a \in \boldsymbol{A}.$$

That is, optimal actions are the ones that maximise the payoff $\pi(a)$. There can be more than one optimal action for each payoff function and set of actions. All the optimal actions will lead to the same payoff.

Theorem 1.1. *The optimal action is unchanged if payoffs are altered by an affine transformation*

$$\pi'(a) = \alpha\pi(a) + \beta,$$

where α *and* β *are constants and* $\alpha > 0$.

Proof. Suppose that a^* is an optimal action for the payoff function π and π' is an affine transformation of π. Then, let's check that a^* is an optimal action for the payoff function π' too.

$$\pi'(a^*) \geq \pi'(a) \iff \alpha\pi(a^*) + \beta \geq \alpha\pi(a) + \beta \iff \pi(a^*) \geq \pi(a).$$

□

Definition 1.4. A *strategy* is a plan of action, i.e., it is a rule for choosing an action at every point where a decision has to be made.

Observation 1.2. *It is important to clarify that even if we are only dealing with one-decision games, i.e., there is only one action to be taken, we will refer to this action as a strategy. So, strategies are the main concept that will be present in every part of this work. The following definitions that will refer to the action concept will naturally be extended to strategies too.*

Definition 1.5. A *pure strategy* is one in which there is no randomisation. The set of pure strategies will be denoted S. When there is only a single decision to be made, the sets of actions and pure strategies are identical.

A *mixed strategy* σ specifies the probability $p(s)$ with which each one of the pure strategies $s \in S$ is used.

Suppose that the set of strategies is $S = \{s_1, s_2, \ldots\}$, then a mixed strategy can be represented as the vector of probabilities:

$$\sigma = (p(s_1), p(s_2), \ldots),$$

where

$$\sum_{s \in S} p(s) = 1.$$

Mixed strategies can be represented as linear combinations of pure strategies. Then, a pure strategy can be represented as a vector where all the components are zero except for one that is one, for example,

$$s_1 = (1, 0, \ldots).$$

Definition 1.6. Assuming that we are in the case of a 2-player game, we will usually denote $p(s)$ as the probability of using the pure strategy s by player 1 and $q(s)$ as the probability of using the pure strategy s by player 2. Then, the payoff corresponding to a pair of mixed strategies (σ_1, σ_2) is given by

$$\pi_i(\sigma_1, \sigma_2) = \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} p(s_1) q(s_2) \pi_i(s_1, s_2).$$

As we said before in the observation, the payoff in this case depends on the strategies chosen by both players and the notation clearly identifies it. The definition of the payoff function can clearly be extended to games that involve more than two players by just adding the probabilities of the rest of the players and the correspondent payoff for every combination of strategies chosen by them.

Observation 1.3. *In some cases, authors make the distinction between "mixed strategies" and "behavioural strategies" when there is potentially more than one action to be made. However, in the situations that we will be treating in this work, that will not be necessary. Even we did, it can be proved that choosing any of the concepts is equivalent, so we will stick to the one that we have already defined.*

Now that we have already defined some important concepts that will be present throughout the work, we can give a proper definition of what a game is.

Definition 1.7. A *non-cooperative static game* is represented by the following elements:

1) *Players.* The number of individuals that take part of the game, which we will denote as N, and their notion of rationality.

2) *Strategies.* We will define them by assigning a pure strategy set, S_i, for each player that participates in the game.

3) *Preferences over outcomes*, i.e., the valuation of what each player is going to receive as an outcome of the game. We will formalize this with the already defined concept of payoff.

A tabular description of a game, using pure strategies, is called the *normal form* or *strategic form* of a game.

Example 1.1. *Strategic Innovation*
This example is inspired by the article *"Application of Evolutionary Game Theory to Strategic Innovation"*, [13], where the authors study the generation and evolution of strategic innovation from the perspective of evolutionary game theory.

Let's assume that we are in a situation where two different companies, company A and company B, are competing in the same market. Both firms have three different alternatives when competing against each other, these are, not innovate(NI), low innovation(I) and high innovation(HI), i.e., $S = \{NI, LI, HI\}$ is the set of pure strategies.

Let's also assume that the total market value is 1000M and these are the only two firms competing in the market, so they share all the benefits. We will also assume that innovating has no cost for the firms and that the firm with the higher innovation gets a higher share of the market. Now we can represent the normal form of the game with the associated payoffs matrix where company A acts is the row player and company B is the column player.

	NI	LI	HI
NI	(500,500)	(400,600)	(250,750)
LI	(600,400)	(500,500)	(300,700)
HI	(750,250)	(700,300)	(500,500)

This game has two important characteristics to highlight. The first one is that this game is clearly symmetric on the payoffs. Then, it is special type of game called *zero-sum game*, in which the total payoff for both players is always the same, in this case, 1000M and they share all of it. A decrease in a company's payoff is totally absorbed by the other company and the sum of both is always the same.

Observation 1.4. *As we already mentioned before, the definitions that we established about actions earlier in this section can be naturally extended to strategies. Therefore, the reader should expect no difference when we start working with the same terms and definitions either on actions or strategies.*

Definition 1.8. The *support* of a mixed strategy σ is the set $S(\sigma) \subseteq S$ of all the pure strategies for which σ specifies $p(s) > 0$.

1.2 Solving Games

Across this section we will dig deeper into the concept of a game and how to solve it. That is, taking into account all the possible strategies for each player and the payoffs obtained when those strategies interact, we will have to establish criteria that helps us find the "best outcome". Two related concepts that we will later define will allow us to solve the games, these are domination between strategies and Nash Equilibrium.

All the results that will be obtained in this section only make sense assuming that the individuals that take part of the game are "rational". That is, players always want to obtain the best outcome of the game, i.e., the highest possible payoff taking into account their possible actions and the counterparts'.

Definition 1.9. The *solution of a game* (not necessarily unique) is the set of strategies that each participant in the game will use as a result of the rational decision process. We will write these strategies between parenthesis in ascendant player order. For example, for a 2-player game, the solution will be (σ_1, σ_2) where the first strategy corresponds to player one and the second one to player two. Then, for an n-player game, it will naturally be $(\sigma_1, \sigma_2, \ldots, \sigma_n)$.

Definition 1.10. A solution is said to be a *Pareto optimal* (named after the Italian economist Vilfredo Pareto) if no player's payoff can be increased without decreasing the payoff to another player. Such solutions are also termed *socially efficient* or just *efficient.*

Definition 1.11. A strategy for player 1, σ_1, is *strictly dominated* by σ_1^* if

$$\pi_1(\sigma_1^*, \sigma_2, \ldots, \sigma_n) > \pi_1(\sigma_1, \sigma_2, \ldots, \sigma_n)\; \forall \sigma_i \in \boldsymbol{S_i},\; i = 2, 3, \ldots, n.$$

An equivalent definition can be given for any other player of the game.

A strategy for player 1, σ_1, is *weakly dominated* by σ_1^* if

$$\pi_1(\sigma_1^*, \sigma_2, \ldots, \sigma_n) \geq \pi_1(\sigma_1, \sigma_2, \ldots, \sigma_n)\; \forall \sigma_i \in \boldsymbol{S_i},\; i = 2, 3, \ldots, n,$$

An equivalent definition can be given for any other player of the game.

Observation 1.5. *From the definition we can extract the conclusion that an individual will never use a strategy that is strictly dominated by another one. We can't be so sure in the case of weakly dominated strategies, they will have to be studied in their particular game. Using this conclusion we can derive a process to solve a game using domination, the* "Iterated elimination of strictly dominated strategies" *(IESD). These process consists in checking the possible existing domination between strategies and removing the strategies that are dominated and, hence, will not be played in any case. This iterative process can end up giving us a solution for the game we are treating.*

Example 1.2. *Strategic Innovation(continued)*
We will solve our previous example with the IESD process. As the game is symmetric, if we find a strategy that is strictly dominated for any of the two players, the same will happen to the other player. Let's rewrite our initial matrix

	NI	LI	HI
NI	(500,500)	(400,600)	(250,750)
LI	(600,400)	(500,500)	(300,700)
HI	(750,250)	(700,300)	(500,500)

As we can see, the payoffs that player 1 obtains when using LI are always greater than the ones using NI, so we can remove this strategy for both players. Then the matrix looks like

	LI	HI
LI	(500,500)	(300,700)
HI	(700,300)	(500,500)

Now, we can observe that using HI gives a greater payoff in the two remaining cases so we will remove LI from the table and therefore we will obtain the result of the game

	HI
HI	(500,500)

Then, (HI,HI) is the solution of the game obtained by using the IESD process.

Observation 1.6. *Looking at the previous example there is something that the reader should clearly notice. While (500, 500) is the payoff every time both companies use the same strategy, only the case where both of them innovate is the solution of the game. Why does this happen? If the result obtained is the same in all three cases (NI,NI), (LI, LI) and (HI, HI) why only the last one remains as the solution of the game. We will explain this situation by defining the concepts of best response and Nash Equilibrium of a game.*

Definition 1.12. A strategy for player 1, σ_1^*, is a *best response* to some fixed strategy profile for the rest of the players, $(\sigma_2, \sigma_3, \ldots, \sigma_n)$, if

$$\pi_1(\sigma_1^*, \sigma_2, \sigma_3, \ldots, \sigma_n) \geq \pi_1(\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n)\ \forall \sigma_1 \in \boldsymbol{S_1}.$$

An equivalent definition can be given for any other player of the game.

Definition 1.13. A *Nash equilibrium* is a strategy profile $(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ such that

$$\pi_i(\sigma_1^*, \sigma_2^*, \ldots, \sigma_i^*, \ldots, \sigma_n^*) \geq \pi_i(\sigma_1^*, \sigma_2^*, \ldots, \sigma_i, \ldots, \sigma_n^*)\ \forall \sigma_i \in \boldsymbol{S_i}, i = 1, 2, \ldots, n.$$

That is, no player has incentive to unilaterally change his/her strategy because the payoff obtained would be lesser, i.e., they cannot do strictly better by adopting any other combination of strategies.

Using the previous definition of a best response, we can give an alternative definition for this concept. We say that $(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ is a *Nash equilibrium* of the game if each strategy σ_i^* is a best response to every possible strategy of the other players.

Definition 1.14. A *generic game* is one in which a small change to any of the payoffs does not introduce new Nash Equilibria or remove existing ones.

Observation 1.7. *From these definitions of a Nash equilibrium we can also derive a procedure to find the solution of a game. We have to find the best responses to every possible strategy of the other players. Finally, we have to look for the sets of strategies that are best responses to each other. There can be multiple or any Nash equilibrium in a game, it will depend on every particular game.*

Example 1.3. *Strategic innovation(continued)*
Going back to same example, this time we will solve it by highlighting the best responses for each player

	NI	LI	HI
NI	(500,500)	(400,600)	(250,**750**)
LI	(600,400)	(500,500)	(300,**700**)
HI	(**750**,250)	(**700**,300)	(**500**, **500**)

Again we end up having the same conclusion, (HI, HI) is the NE of the game because these are the only strategies that are a best response to each other.

Observation 1.8. *Now we can answer to the questions asked in the previous observation. Even if there are two more strategies that give exactly the same payoffs to both companies, they are not a solution of the game because they are not mutual best responses. For* *example, if company A decided to use the strategy NI, then company B would unilaterally move to LI or HI and obtain higher payoff. As the game is symmetric, exactly the same thing would happen if company B did not use the strategy HI, company A would deviate to improve its payoff.*

Theorem 1.2. Equality of Payoffs
Let $\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ *be a Nash Equilibrium, and let* $S_1^*(\sigma_1^*)$ *be the support of* σ_1^* *. Then*

$$\pi_1(s, \sigma_2^*, \ldots, \sigma_n^*) = \pi_1(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*) \ \forall s \in \S_1^*.$$

Proof. If the set S_1^* contains only one strategy, then the theorem is trivially true. Suppose now that the set S_1^* contains more than one strategy. We will suppose that the theorem is not true and we will end up with a contradiction. If the theorem is not true, then there is at least one strategy in the support that gives a higher payoff than $(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$; let s' be that strategy and let$(s', \sigma_2^*, \ldots, \sigma_n^*)$ be the combination of strategies that gives a higher payoff. Then

$$\begin{aligned}
\pi_1(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*) &= \sum_{s \in S^*} p^*(s)\pi_1(s, \sigma_2^*, \ldots, \sigma_n^*) \\
&= \sum_{s \neq s'} p^*(s)\pi_1(s, \sigma_2^*, \ldots, \sigma_n^*) + p^*(s')\pi_1(s', \sigma_2^*, \ldots, \sigma_n^*) \\
&< \sum_{s \neq s'} p^*(s)\pi_1(s', \sigma_2^*, \ldots, \sigma_n^*) + p^*(s')\pi_1(s', \sigma_2^*, \ldots, \sigma_n^*) \\
&= \pi(s', \sigma_2^*, \ldots, \sigma_n^*)
\end{aligned}$$

which contradicts the assumption that $(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ is a Nash Equilibrium. In the third line, to establish the inequality, we have used that $(s', \sigma_2^*, \ldots, \sigma_n^*)$ is the combination of strategies that gives strictly a higher payoff. □

Observation 1.9. *A question that we could ask after proving this theorem is, why does player 1 choose the randomising strategy* σ_1^* *if all pure strategies in the support end up giving him/her the same payoff? The reason is that if player 1 decides to deviate from this strategy, then the Nash Equilibrium disintegrates. As we have mentioned before, a NE is the combination of strategies where all the players obtain a result that cannot be improved without reducing the other players'. Therefore, if player 1 decided to change to any other possible strategy, the rest of the players would "counterattack" to improve their payoff and the equilibrium would be broken. That is the reason why randomising strategies will be very important from now on.*

Theorem 1.3. Nash′s Theorem
Every game that has a finite strategic form, i.e., with finite number of players and finite number of pure strategies for each player, has at least one Nash Equilibrium involving pure or mixed strategies.

This theorem was provided by John Forbes Nash Jr. as part of his PhD thesis in 1950 and it involves the use of a fixed point theorem(e.g., Brouwer's). The proof of the theorem is not an essential part for this thesis as we are going to be focusing on the topic of evolutionary game theory. However, it is important to mention it because the concept of Nash equilibrium is very important in game theory.

It is also important to mention that even if this theorem provides us with the existence of at least one Nash equilibrium for the type of games previously mentioned, it does not necessarily have to be unique.

Observation 1.10. *Then, one more important question that should arise from the concept of Nash Equilibrium is, what happens when we have multiple Nash Equilibrium in a game? Is there any way to determine which one of them is better? Unfortunately, there is no other concept or refinement of Nash Equilibrium that leads us to a unique equilibrium in a game. However, this does not mean that we cannot go a bit further to analyze the different equilibriums of a game and cast more light upon them. This is the point where evolutionary game theory comes into play.*

1.3 Qualitative Theory of Differential Equations

In this section we will introduce all the mathematical concepts, specially related to differential equations, that will be used later on in this work.

Definition 1.15. A *differential equation* is a functional relation that involves a function and its derivatives. An *ordinary differential equation* (ODE) is a differential equation in which the function contains only one independent variable. On the other hand, a *partial differential equation* (PDE) is an equation which imposes relations between the various partial derivatives of a multivariable function. A differential equation that does not explicitly depend on the independent variable is called an *autonomous differential equation.*

Example 1.4. A linear homogeneous 1-dimensional differential equation is for example

$$\dot{x}(t) = ax(t),\ x, a \in \mathbb{R}.$$

Which we can easily solve in the following way

$$\frac{dx}{x} = adt \iff \int \frac{dx}{x} = \int adt \iff \log(x(t)) = at + k \iff$$

$$\iff x(t) = \exp(at + k) \iff x(t) = k' \exp(at)$$

where the parameter $k' = exp(k) \in \mathbb{R}$ gives the "family of solutions" of the differential equation.

Definition 1.16. The normal formalization of the ODE can be written as $\dot{x} = f(t, x)$, where $t \in \mathbb{R}$, $x \in \mathbb{R}^n$ and $f : \mathbb{U} \in \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. Usually f is assumed to be continuous and Lipschitz with respect to the second variable. Then, we can define the *Cauchy problem* or *initial value problem* as the previous differential equation together with an initial condition $x(0) = x_0 \in \mathbb{R}^n$. That is,

$$\begin{cases} \dot{x} = f(t, x) \\ x(0) = x_0. \end{cases} \tag{1.1}$$

With this regularity, the Picard's theorem, which will be presented soon, guarantees the existence and uniqueness of a local solution of (1.1).

Example 1.5. Let's consider the linear differential equation $\dot{x} = A(t)x$, $t \in \mathbb{R}$, $x \in \mathbb{R}^n$. In this case, the *Cauchy problem* is,

$$\begin{cases} \dot{x} = A(t)x \\ x(0) = x_0. \end{cases}$$

Then, the solution of the initial value problem is written as a fundamental matrix $\phi(t)$ where each column of the matrix is a solution (linearly independent) of the equation. Therefore, solution of the previous initial value problem can be written as $\varphi(t) = \phi(t)\phi^{-1}(0)x_0$. If $A(t) \equiv A$, i.e., $A(t)$ is a constant matrix, the equation is called *homogeneous linear with constant coefficients.* In this case, the fundamental principal solution of the differential equation can be written as the exponential of the matrix, i.e., $\phi(t) = exp(tA)$ and the solution of the initial value problem is $\varphi(t) = \phi(t)x_0$.

Theorem 1.4. Picard′s Theorem
Let $f : \mathbb{U} \in \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz continuous function in x, and continuous in t in $\mathbb{U} = I_a \times B_b$, where $I_a = \{t; |t - t_0| \leq a\}$, $B_b\{x; |x - x_0| \leq b\}$, $a, b > 0$. Given M such that $f \leq M$ in $\mathbb{U}$, there is a unique solution of

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0. \end{cases}$$

in I_α, where $\alpha = min\{a, \frac{b}{M}\}$.

Definition 1.17. Let $\mathbb{U} \in \mathbb{R}^n$ be a on open set. A *vector field* is an application

$$\mathbb{X} : \mathbb{U} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

This vector field induces an autonomous differential equation

$$\dot{x} = \mathbb{X}(x).$$

The solution curves of $\dot{x} = \mathbb{X}(x)$ satisfy that at each point the tangent vector corresponds to the vector field defined by $\mathbb{X}$. We will assume that the vector field $\mathbb{X} \in \mathcal{C}^r(\mathbb{U})$, $r \geq 1$.

Let's now consider an autonomous 2-dimensional ODE

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y), \end{cases} \tag{1.2}$$

where $\mathbb{X} = (f, g) : \mathbb{U} \longrightarrow \mathbb{R}^2$ is the correspondent vector field, $\mathbb{X} \in \mathcal{C}^1(\mathbb{U})$. Let's present some theoretical concepts related to this vector field.

Definition 1.18. The vector field $\mathbb{X}$ is *integrable* if there is a function $H : \mathbb{U} \longrightarrow \mathbb{R}$, $H \in \mathcal{C}^1(\mathbb{U})$, named *first integral*, that is not constant in any open set but is constant over the solutions of (1.2). Then, for each solution $\varphi : \mathbb{I} \longrightarrow \mathbb{U}$ of (1.2), $H(\varphi(t)) = H(\varphi(0))$, $\forall t \in \mathbb{I}$.

Definition 1.19. The vector field $\mathbb{X}$ is *conservative* if $\text{div}\mathbb{X} = 0$, i.e., $\frac{\partial f}{\partial x}(x, y) + \frac{\partial g}{\partial y}(x, y) = 0$, $\forall (x, y) \in \mathbb{U}$.

Definition 1.20. An *integrant factor* of (1.2) is a non constant function $\mu : \mathbb{U} \longrightarrow \mathbb{R}$, $\mu \in \mathcal{C}^1(\mathbb{U})$, such that the vector field $\mu\mathbb{X}$ is conservative, i.e., $\forall (x,y) \in \mathbb{U}$,

$$f(x,y)\frac{\partial \mu}{\partial x}(x,y) + g(x,y)\frac{\partial \mu}{\partial y}(x,y) + \mathrm{div}\mathbb{X}(\mathrm{x},\mathrm{y})\mu(\mathrm{x},\mathrm{y}) = 0.$$

Definition 1.21. The vector field $\mathbb{X} = (f,g)$ is a *Hamiltonian* vector field if there exists a function $H(x,y)$ such that

$$\frac{\partial H}{\partial y}(x,y) = f(x,y), \frac{\partial H}{\partial x}(x,y) = -g(x,y).$$

We also say that vector field $\mathbb{X}$ is a *Hamiltonian* with *Hamiltonian* H.

We will follow up by writing two propositions that explain the relationships between conservative and integrable vector fields. We will not prove the second proposition since we think it is too technical and does not serve the purpose of this work.

Observation 1.11. *It is important to note that in both propositions we suppose that the open set U has particular characteristics. We suppose that U is a star convex set in respect to a certain point $x_0 \in U$, that is, for all $x \in U$ the segment between x_0 and x is contained in U.*

Proposition 1.1. *A $\mathcal{C}^1$ vector field $\mathbb{X} = (f,g)$ with a first integral H is integrable $\iff$ $DH(x,y)\mathbb{X}(x,y) = 0 \ \forall (x,y) \in \mathbb{U}$.*

Proof. $\Rightarrow$) Let's suppose that $\mathbb{X}$ is integrable. Then, we know that $\exists H$ first integral which is a function not constant on any open set but constant over the solutions of the system. That is, for any given solution $\varphi(t)$ of $\dot{x} = \mathbb{X}(x)$ we have $\frac{d}{dt}H(\varphi(t)) = 0$.

Now, we have that

$$\frac{d}{dt}H(\varphi_1(t),\varphi_2(t)) = 0 \iff H_x(\varphi_1(t),\varphi_2(t))\dot{\varphi}_1(t) + H_y(\varphi_1(t),\varphi_2(t))\dot{\varphi}_2(t) = 0 \iff$$

$$\iff < (H_x(\varphi_1(t),\varphi_2(t)), H_y(\varphi_1(t),\varphi_2(t))), (\dot{\varphi}_1(t),\dot{\varphi}_2(t)) >= 0 \iff$$

$$\iff DH(x,y)|_{(\varphi_1(t),\varphi_2(t))}\mathbb{X}(\varphi_1(t),\varphi_2(t)) = 0.$$

$\Leftarrow$) Reciprocally, let's suppose that $DH(x,y)\mathbb{X}(x,y) = 0$ for a given function $H \in \mathcal{C}^1(\mathbb{U})$ not locally constant. That is,

$$< (H_x(x,y), H_y(x,y)), (f(x,y), g(x,y)) >= 0 \iff H_x(x,y)f(x,y) + H_y(x,y)g(x,y) = 0.$$

We want to prove that H is a first integral of the field, i.e., that it is constant over the solutions $\varphi(t)$ of (1.2). Let's now calculate $\frac{d}{dt}H(\varphi(t))$,

$$\frac{d}{dt}H(\varphi(t)) = DH(x,y)|_{\varphi(t)}\frac{d}{dt}\varphi(t) = [*].$$

If we calculate both terms separately we have,

$$DH(x,y)|_{\varphi(t)} = H_x(x,y)|_{\varphi(t)} + H_y(x,y)|_{\varphi(t)}.$$

As $\varphi(t) = (\varphi_1(t), \varphi_2(t))$ is the solution of

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y), \end{cases}$$

we have that

$$\frac{d}{dt}\varphi_1(t) = f(\varphi_1(t), \varphi_2(t))$$

$$\frac{d}{dt}\varphi_2(t) = g(\varphi_1(t), \varphi_2(t)).$$

Consequently $\frac{d}{dt}\varphi(t) = \begin{pmatrix} \frac{d}{dt}\varphi_1(t) \\ \frac{d}{dt}\varphi_2(t) \end{pmatrix} = \begin{pmatrix} f(\varphi_1(t), \varphi_2(t)) \\ g(\varphi_1(t), \varphi_2(t)) \end{pmatrix}$.

Writing them together again, we have

$$[*] = H_x(\varphi_1(t), \varphi_2(t))f(\varphi_1(t), \varphi_2(t)) + H_y(\varphi_1(t), \varphi_2(t))g(\varphi_1(t), \varphi_2(t)) = 0.$$

□

Proposition 1.2. *A conservative field $\mathbb{X}$ is integrable, i.e., $\exists H$ function that is a first integral of the system.*

Now, after having defined the basic concepts of differential equations we can start to dig deeper into the qualitative theory.

Definition 1.22. Let $\mathbb{X} \in \mathcal{C}^1(U)$ be a vector field as previously described. Given $x \in \mathbb{U}$ we define the orbit of x as

$$\theta(x) = \{\varphi_t(x), t \in I(x)\}.$$

From the definition it is easy to see that $y \in \theta(x) \iff \theta(y) = \theta(x)$. The relationship $y \sim x \iff y \in \theta(x)$ is an equivalence relationship. Then, the set $\mathbb{U}$ and the partition in orbits that is induced by the relationship $\sim$ is called the *phase portrait*.

Definition 1.23. Let $\dot{x} = \mathbb{X}(x)$, and let $x^* \in U$. We say that x^* is a *fixed point* if $\mathbb{X}(x^*) = 0$. Otherwise, we say that x^* is a *regular point*. There are different types of fixed points:

1. x^* is a *hyperbolic* fixed point $\iff Re(\lambda) \neq 0, \forall \lambda \in \mathrm{Spec}(\mathrm{D}\mathbb{X}(\mathrm{x}^*))$.
2. x^* is an *elliptical* fixed point $\iff Re(\lambda) = 0, Im(\lambda) \neq 0, \forall \lambda \in \mathrm{Spec}(\mathrm{D}(\mathbb{X}(\mathrm{x}^*))$.
3. x^* is a *parabolic* fixed point $\iff 0 \in \mathrm{Spec}(\mathrm{D}(\mathbb{X}(\mathrm{x}^*))$.

Definition 1.24. Consider the differential equation $\dot{x} = Ax$, $A \in \mathcal{M}_{2x2}$, $x \in \mathbb{R}^2$. Let's suppose that $det(A) \neq 0$, so that the origin is the only fixed point. The phase portrait of the linear system is then determined by the eigenvalues λ_1, λ_2 of A. Concretely:

1. Case $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$. In a base of eigenvectors the system is reduced to the fundamental form
$$\dot{x} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} x.$$
The signs of λ_1, λ_2 determine the type of the point:

   (a) If $\lambda_1\lambda_2 > 0$ the origin is a *node*. We have a *stable node* if $\lambda_1,\ \lambda_2 < 0$ and an *unstable node* if $\lambda_1,\ \lambda_2 > 0$.
   (b) If $\lambda_1\lambda_2 < 0$ the origin is a *saddle point*.

2. Case $\lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1 = \lambda_2 = \lambda \neq 0$. We distinguish 2 cases:
   (a) If A diagonalizes, the system is reduced to the fundamental form (??). The origin is a *node*.
   (b) If A does not diagonalize, the system is reduced to the fundamental form
   $$\dot{x} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} x.$$
   The origin is a *node*.

3. Case $\lambda_1, \bar{\lambda}_2 = \alpha + i\beta,\ \beta \neq 0$ (conjugate complex eigenvalues). The system is reduced to the fundamental form
   $$\dot{x} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} x.$$
   We distinguish two cases:
   (a) If $\alpha = 0$ the origin is a *center*. It is a non-hyperbolic case.
   (b) If $\alpha \neq 0$ it is a *focus*. If $\alpha < 0$ the focus is *stable* and if $\alpha > 0$ the focus is *unstable*.

Example 1.6. Phase portraits of some of the cases that we have just defined, i.e., systems $\dot{x} = Ax$, where A is a 2×2 matrix with $det(A) \neq 0$.

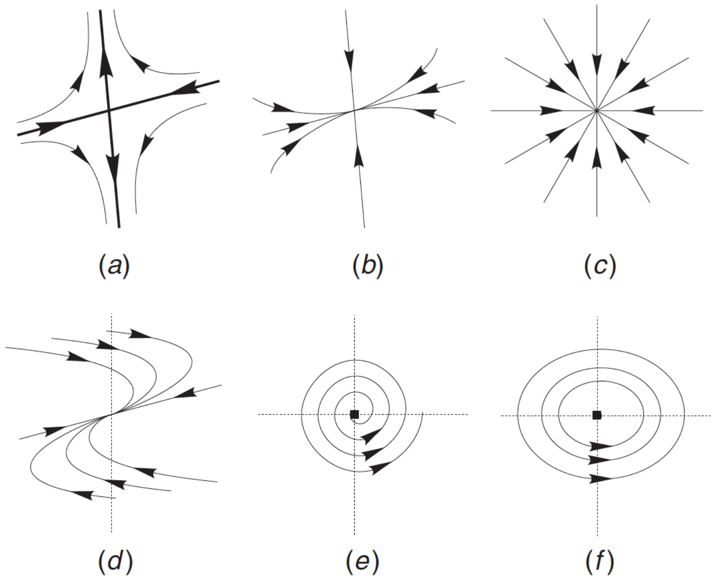

Figure 1.1: Phase portrait of systems $\dot{x} = Ax$. Source: 'Qualitative theory of planar differential systems', [2].

In case a) we have a saddle; case b) is a stable node; case c) is a node where $\lambda_1 = \lambda_2 = \lambda \neq 0$ and A diagonalizes; case d) is a node where $\lambda_1 = \lambda_2 = \lambda \neq 0$ and A does not diagonalize; case e) is an unstable focus; case f) is a center.

Definition 1.25. A *separatrix* is a trajectory separating two regions in which the behaviour of solutions as $t \to \infty$ or $t \to -\infty$ is different. The trajectories entering and leaving a saddle point are very often separatrices.

Theorem 1.5. Hartman – Grobman

Consider the vector field $\mathbb{X} : \mathbb{U} \longrightarrow \mathbb{R}^n$, $\mathbb{U} \subset \mathbb{R}^n$, $\mathbb{X} \in \mathcal{C}^1(\mathbb{U})$ *and let* x^* *be a fixed hyperbolic point. Let* $A = D\mathbb{X}(x^*)$ *and let* $\mathbb{Y} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be the linear field* $\mathbb{Y}(y) = Ay$. *Then, the vector fields* $\mathbb{X}$ *and* $\mathbb{Y}$ *are locally topologically conjugate. That is,* $\exists \mathbb{U}_*$ *neighbourhood of* x^* *and* $\exists \mathbb{V}_*$ *neighbourhood of* $y^* = 0$ *and there exists* $h : \mathbb{U}_* \longrightarrow \mathbb{V}_*$ *homeomorphism such that* $h(\varphi(t, x)) = \psi(t, h(x))$. *Where* φ *is the flow of the field* $\mathbb{X}$ *and* ψ *is the flow of the field* $\mathbb{Y}$.

Observation 1.12. *This theorem will be very important because using it we will be able to transfer the local study that we do about a linear field to the correspondent conjugate vector field. The map h has the property of sending orbits to orbits, i.e.,*

$$h(\theta_1(x)) = \bigcup_{t\in\mathbb{R}} h(\varphi_1(t, x)) = \bigcup_{t\in\mathbb{R}} \varphi_2(t, h(x)) = \theta_2(h(x)).$$

In particular, h sends fixed points to fixed points.

1.4 Poincaré Compactification

In order to study the behaviour of the trajectories of a planar differential system near infinity it is possible to use a compactification. A good approach to do so is using the so called Poincaré sphere. It has the advantage that the singular points along infinity are spread out along the equator of the sphere. In the Poincaré sphere it is possible to work only in one of the hemispheres that will be called the Poincaré disk. If the functions defining the vector field that we are studying are polynomials, we can apply the Poincaré compactification, which will tell us how to draw the phase portrait in a finite region. As we previously said, the construction of the Poincaré compactification is completely based on the fifth chapter of the book *"Qualitative Theory of Planar Differential Systems"*. To find more detailed information we refer the reader to [2].

Just as a reminder, we will define a differential geometry concept that will be used in the construction of the compactification.

Definition 1.26. Let M be an m-dimensional manifold. We say that (U, ϕ), where $U \subseteq M$ and $\phi : U \to \mathbb{R}^m$ is a map, is an *m-dimensional chart* on M if $\phi(U) \subseteq \mathbb{R}^m$ is open and ϕ is a bijection from U to $\phi(U)$.

Let's now consider the polynomial vector field $\mathbb{X} = (P(x_1, x_2), Q(x_1, x_2))$ where P and Q are polynomials of arbitrary degree in the variables x_1 and x_2. The degree of $\mathbb{X}$, d, will be the maximum of the degrees of P and Q.

The Poincaré compactification works as follows. First we consider $\mathbb{R}^2$ as the plane in $\mathbb{R}^3$ defined by $(y_1, y_2, y_3) = (x_1, x_2, 1)$. We consider the sphere $\mathbb{S}^2 = \{y \in \mathbb{R}^3 : y_1^2+y_2^2+y_3^2 = 1\}$ which is tangent to $\mathbb{R}^2$ at $(0, 0, 1)$ and we will call it *Poincaré sphere*. This sphere can be divided into the northern and southern hemisphere, which are respectively $H_+ = \{y \in \mathbb{S}^2 : y_3 > 0\}$ and $H_- = \{y \in \mathbb{S}^2 : y_3 < 0\}$. Then, the equator is $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$.

Now we consider the projection of the vector field $\mathbb{X}$ from $\mathbb{R}^2$ to $\mathbb{S}^2$ given by the central projection $f^+ : \mathbb{R}^2 \to \mathbb{S}^2$, which is the intersection of the straight line passing through the point $y = (x_1, x_2, 1)$ and the origin with the northern hemisphere of $\mathbb{S}^2$:

$$f^+(x) = \left(\frac{x_1}{\nabla(x)}, \frac{x_2}{\nabla(x)}, \frac{1}{\nabla(x)} \right),$$

where $\nabla(x) = \sqrt{x_1^2 + x_2^2 + 1}$. The southern projection, $f^-(x)$, can be similarly defined.

Observation 1.13. *As the projection $f^+(x)$ goes from the plane $\mathbb{R}^2$ to the sphere $\mathbb{S}^2$, then the differential of this projection sends the vector that we have in the point $x \in \mathbb{R}^2$ to a vector in the sphere.*

In this way we have induced two vector fields, one in each hemisphere, that are analytically conjugate to $\mathbb{X}$. The induced vector field on H_+ is $\bar{\mathbb{X}}(y) = Df^+(x)\mathbb{X}(x)$, where $y = f^+(x)$ and it is a vector field on $\mathbb{S}^2 \setminus \mathbb{S}^1$ that is everywhere tangent to $\mathbb{S}^2$. Notice that the points at infinity of $\mathbb{R}^2$ are in bijective correspondence with the points of the equator of $\mathbb{S}^2$.

Observation 1.14. *Clearly, the length of the vector field $\mathbb{X}$ will tend to infinity at the speed that is marked by the degree of the polynomials. What we will do in the construction is to stop the evolution to infinity by multiplying or dividing by a factor that "reduces the speed". We will be able to do this because, as we have a polynomial vector field, the speed with which $\mathbb{X}$ goes to infinity "uniformly controlled" by the degree of the field, which is the maximum degree of P and Q. The "uniformity of a polynomial field" lets us find a factor that slows down the speed of all its orbits.*

Now we will extend the induced vector field $\bar{\mathbb{X}}$ from $\mathbb{S}^2 \setminus \mathbb{S}^1$ to $\mathbb{S}^2$. To do so, we need to multiply the vector field by the factor $\rho(x) = x_3^{d-1}$ (which allows us to adequately do the extension) and the extended vector field on $\mathbb{S}^2$ is called the *Poincaré compactification* of the vector field $\mathbb{X}$ on $\mathbb{R}^2$, and it is denoted by $p(\mathbb{X})$.

Definition 1.27. The *finite singular points* of $\mathbb{X}$ or $p(\mathbb{X})$ are the singular points of $p(\mathbb{X})$ that lie in $\mathbb{S}^2 \setminus \mathbb{S}^1$. The *infinite singular points* of $\mathbb{X}$ or $p(\mathbb{X})$ are the singular points of $p(\mathbb{X})$ that lie in $\mathbb{S}^1$.

To make the calculations in a curved surface like this one, we will use the six local charts given by $U_k = \{y \in \mathbb{S}^2 : y_k > 0\}$, $V_k = \{y \in \mathbb{S}^2 : y_k < 0\}$ for $k = 1, 2, 3$. The corresponding local maps $\phi_k : U_k \to \mathbb{R}^2$ and $\psi_k : V_k \to \mathbb{R}^2$ are defined as $\phi_k(y) = -\psi_k(y) = (\frac{y_m}{y_k}, \frac{y_n}{y_k})$ for $m < n$ and $m, n \neq k$. We denote by $z = (u, v)$ the value of $\phi_k(y)$ or $\psi_k(y)$ for any k, such that (u, v) will play different roles depending on the local chart that we are considering. The points of $\mathbb{S}^1$ in any chart have $v = 0$.

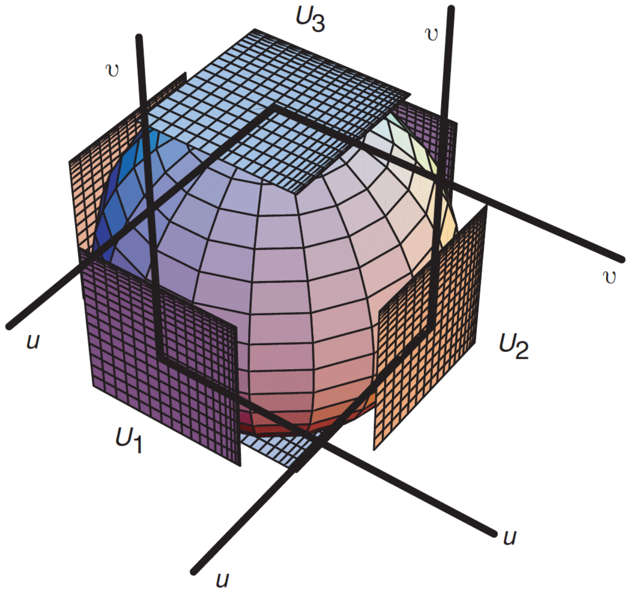

Figure 1.2: The local charts (U_k, ϕ_k) for $k = 1, 2, 3$ of the Poincaré sphere. Source: 'Qualitative theory of planar differential systems', [2].

We will not go into detail to work out the construction of the compactification, all the details and calculations can be found in [2]. However, we will present the main elements that we think the reader should pay more attention to.

We have

$$\bar{\mathbb{X}}|_{U_1} = v^2\left(-\frac{u}{v}P\left(\frac{1}{v},\frac{u}{v}\right)+\frac{1}{v}Q\left(\frac{1}{v},\frac{u}{v}\right), -P\left(\frac{1}{v},\frac{u}{v}\right)\right).$$

Now,

$$\rho(y) = y_3^{d-1} = \frac{1}{\nabla(x)^{d-1}} = \frac{v^{d-1}}{\nabla(z)^{d-1}} = v^{d-1}m(z),$$

where $m(z) = (1+u^2+v^2)^{\frac{1-d}{2}}$. It follows that

$$\rho(\bar{\mathbb{X}}|_{U_1})(z) = v^{d+1}m(z)\left(-\frac{u}{v}P\left(\frac{1}{v},\frac{u}{v}\right)+\frac{1}{v}Q\left(\frac{1}{v},\frac{u}{v}\right), -P\left(\frac{1}{v},\frac{u}{v}\right)\right).$$

Observation 1.15. *As we mentioned before, we have added a factor that makes the extension of $\rho\bar{\mathbb{X}}$ to $p(\mathbb{X})$ possible. Notice that while $\bar{\mathbb{X}}|_{U_1}$ is not well defined when $v =$ 0, $p(\mathbb{X})|_{U_1} = \rho\bar{\mathbb{X}}|_{U_1}$ is well defined along $v = 0$ since the multiplying factor v^{d+1} cancels any factor of v which could appear in the denominator. This is linked to the previous observation where we talked about slowing down the speed of the field as it got close to infinity.*

Now, in order to simplify the extended vector field we can make a change in the time variable and remove the factor $m(z)$. Finally, we can give the expressions of $p(\mathbb{X})$ in each local chart. In the local chart (U_1, ϕ_1) is given by

$$p(\mathbb{X})|_{U_1} = \begin{cases} \dot{u} = v^d\left[-uP\left(\frac{1}{v},\frac{u}{v}\right)+Q\left(\frac{1}{v},\frac{u}{v}\right)\right] \\ \dot{v} = v^{d+1}P\left(\frac{1}{v},\frac{u}{v}\right). \end{cases}$$

The expression for (U_2, ϕ_2) is

$$p(\mathbb{X})|_{U_2} = \begin{cases} \dot{u} = v^d \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right] \\ \dot{v} = v^{d+1} Q\left(\frac{u}{v}, \frac{1}{v}\right). \end{cases}$$

Finally, for (U_3, ϕ_3) we have

$$p(\mathbb{X})|_{U_3} = \begin{cases} \dot{u} = P(u, v) \\ \dot{v} = Q(u, v). \end{cases}$$

The expression for $p(\mathbb{X})$ in the charts (V_k, ψ_k) is the same as for (U_k, ϕ_k) multiplied by $(-1)^{d-1}$, for $k = 1, 2, 3$.

Observation 1.16. *As we previously said, it is not necessary to work with the complete sphere to study the initial vector field $\mathbb{X}$. It is sufficient to work in the Poincaré disk, i.e., $H_+ \cup \mathbb{S}^1$ and all the calculations can be done using the charts (U_k, ϕ_k), for $k = 1, 2, 3$.*

Observation 1.17. *The infinite singular points of the vector field will be spread along the equator of the sphere and the finite ones will be contained in the northern and southern hemispheres. It is important no note that if $y \in \mathbb{S}^1$ is an infinite singular point, then $-y$ is also a singular point. Since the local behaviour near $-y$ is the local behaviour near y multiplied by $(-1)^{d-1}$ it follows that the orientation of the orbits changes when the degree is even. For example, if d is odd and $y \in \mathbb{S}^1$ is a stable node of $p(\mathbb{X})$, then $-y$ is a stable node too. Then, it is just necessary to study half of the singular points and use the degree of the field to know the behaviour of the rest.*

Observation 1.18. *The tool used by the program P4&P5 for studying nonelementary singularities of a differential system in the plane consists on changes of variables called* blow-ups*. This technique is used to show that at isolated singularities, an analytic system has a finite sectorial decomposition. A consequence of a Theorem proved by Freddy Dumortier that we will not specify in this work, is that with a finite number of blow-ups we will eventually end up in a situation where all the singular points will be saddles, nodes or saddle-nodes. Then, as there is a finite number of blow-ups, when the desingularisation process is done to study the local behaviour around the point, that will consequently lead to a finite number of sectors. We will see several examples in the third chapter of this work that illustrate the sectorial decomposition of different systems.*

Chapter 2

Evolutionary Models and ESS

Evolution is a broad concept that can be viewed and approached from many different points. Our first interest will be to make a first contact through the biological approach, since we think it is the field where it has achieved a greatest development and it will therefore be a good starting point to establish our basis.

There are three main concepts that are involved in the process of evolution, these are selection, inheritance and mutation. Starting with the first one, *selection*, we can go back to the famous phrase referred to *Charles Darwin's* work, "the survival of the fittest". That is, selection is the force that determines the chances of survival and reproduction of a certain species with a certain behaviour versus the environment where it lives. Survival will clearly have a short-run effect in the evolution process and the repeated process through generations will end up having a long-run effect which will decide the species that populate the environment.

The concept that follows is *inheritance*. It is conceived as the force that establishes links between generations of individuals. That is, it is the way through which individuals behaviours' get transferred to their offspring. Linking this concept to selection, we could say that once we know which individuals will be "chosen" to continue living and reproducing, we would like to know to which extent does their behaviour get transferred to their offspring. This will also be a very important issue related to evolution since the behaviour of individuals is crucial to their survival and so it will be for the ones that come after them.

Finally, *mutation* plays also a very important role en evolutionary processes. Mutation acts as the creator and modifier of existing behaviours in the environment to enrich (or not) patterns of behaviour and make them "better against the environment". That is, mutation acts as a force that introduces new behaviours into the game and therefore it changes the adaption process of individuals against the environment they are facing or living in.

Hence, it will be our task to try to formalize these biological concepts and give them a proper and formal economic and mathematical form. We will have to establish a base where to start our assumptions and start building the theory of Evolutionary games.

The organization of this chapter has been inspired by the third part of the book *"Game Theory. Decisions, Interaction and Evolution"*, [17]. while also adding more detailed information provided by Vega-Redondo, [16], in his book *"Evolution, games, and economic behaviour"*. The first reference provides an easier and not so in depth approach to the topic and in the second one we can find more complex information to complement this work in some specific points.

2.1 Framework

In the first chapter we have presented all the basic concepts that are important to explain evolutionary game theory and we have done this for n-player games. However, from now on we will present the theory from the perspective of just a specific type of games, this will be 2 player symmetric games. Placing ourselves in this type of games allows us to give a different interpretation of Nash Equilibrium by seeing the game in a population context. If we encounter the situation of a population where all its members use the the Nash Equilibrium, σ^*, then the population will remain that way, it is in equilibrium. This concept was introduced by Nash in 1951, [12], as the "mass action interpretation". A question we must ask ourselves is whether the equilibrium is reached or not if the population is close to but not in equilibrium. Where does the evolutionary process lead? Does the "most popular" strategy remain or will it be "invaded" by mutations? Then, we will try to identify the ending points of evolutionary processes. This will be formalized later on with the concept of Evolutionary Stable Strategy (ESS).

Hence, we will consider a population of decision makers that keeps evolving. That is, the proportion of a population that has a certain behaviour or decides to carry out an action keeps changing. From a biological perspective we would consider a population of animals that present a certain behaviour that is transferred to their offspring. Fittest individuals will have more offspring and hence, their behaviour will have a better chance of survival over time. Economically speaking, the population does not change over generations but over repetition. That is, individuals keep evaluating the result of their actions and eventually may decide to change their behaviour in the seek of a better payoff.

2.2 Evolution and Stability

Definition 2.1. Consider an infinite population of individuals that can use some set of pure strategies, S. A *population profile* is a vector x that gives a probability $x(s)$ with which each strategy $s \in S$ is played in the population.

Before continuing we will go through an example to make a clear distinction between what a strategy and a population profile are, because they may lead to confusion at some point.

Example 2.1. Consider nowadays situation, where we are in the middle of a pandemic. Imagine that the global population can use two strategies, wearing a mask (M) and not wearing a mask (NM), i.e., $S = \{M, NM\}$. If every member of the population decides to randomise by wearing a mask 80% of the time and not wearing it 20% of the time, then the population profile is $x = (\frac{8}{10}, \frac{2}{10})$. In this case, the population profile is equivalent to the mixed strategy used by each individual because of the randomisation.

Let's now imagine that in this same population, 80% of the individuals wear a mask all the time and the rest never do. The population profile is exactly the same as before, $x = (\frac{8}{10}, \frac{2}{10})$. But in this case, the population profile is not the same as any strategy adopted by any member of the population. That is because every member plays one of the possible pure strategies without randomising.

Now, we will consider a specific individual using a strategy σ "against a population" that has a population profile x. We will specify later who the individual is playing against, but, for now, we just have to note that his/her payoff depends both on his/her strategy and the population profile.

Definition 2.2. Given x and σ the *payoff* to an agent will be denoted $\pi(\sigma, x)$ and calculated by

$$\pi(\sigma, x) = \sum_{s \in S} p(s)\pi(s, x)$$

where $S = (s_1, s_2, \ldots, s_n)$.

As we already mentioned before, the payoffs that each strategy gives to the individuals will affect their chances of survival, i.e., the payoffs determine the evolution of the population.

Before continuing, we have to make a distinction between the two types of games that will be present in the evolutionary context. The difference between them is who the opponent is. We will consider two situations, in the first one the player will be playing against the whole population or versus the environment, i.e., there is no specific player. Then, in the second case the player will be randomly matched with another player that belongs to a population with a given population profile. After making this distinction we will be able to give a more explanatory definition of how $\pi(s, x)$ behaves as a function of x.

Definition 2.3. A *Player versus Environment or PVE* game describes a situation in which a given individual plays against an nonspecific opponent. This type of games escape from classical Game Theory because, as we said, they consist in the interaction that occurs between a single player and the environment that surrounds him/her. In classical Game Theory all the situations consist of a certain number of players that choose a concrete strategy between all their options and the different combinations chosen produce all the possible outcomes for the players.

A *Player versus Player or PVP* game is the case where our individual plays against a random opponent extracted from a population with a population profile x. Therefore, the payoff obtained depends just on what both individuals do. This case will be translated as a 2-player game with its correspondent normal form and payoff matrix and it will be the type of game where we will mainly focus all our study.

Assuming that the population from where the individual has been selected has a population profile x and that the set of pure strategies available for players is S, we can write

$$\pi(\sigma, x) = \sum_{s \in S} \sum_{s' \in S} p(s)x(s')\pi(s, s').$$

Example 2.2. Turning our attention again to the current pandemic situation we can also give a very simple example of each type of game. Assuming that we are in the same situation as before, i.e., a population using a set of two pure strategies, $S = \{M, NM\}$ with an associated population profile $x = (\frac{8}{10}, \frac{2}{10})$.

A *PVE* game would consist in, for example, going to a concert in a disco full of people. In this case, the player will enter a situation where he/she is interacting against a great number of individuals that may or may not be wearing a mask with the described population profile.

Then, a *PVP* game would consist in, for example, a date over dinner between the player and a certain individual from the population with the previous population profile. Our player does not know if his/her partner will be wearing a mask but he/she knows what the population profile is, 80% of individuals wear a mask and the rest do not.

What we will be interested in, is studying if there is an end to the evolutionary process of the population. That is, we want to establish certain conditions under which the population does not evolve anymore, i.e., where it is stable. Clearly, there is a necessary condition for a strategy to be stable and this is that the strategy must give a better payoff than the rest of possible strategies available. In terms of the concepts that we have introduced in the first chapter of this work, we can say that a strategy needs to be a best response to the associated population profile. Looking back at the first chapter, we can also write an equivalent of Theorem 1.2.

Theorem 2.1. *Let σ^* be a strategy that generates a population profile x^*. Let S^* be the support of σ^*. If the population is stable, then*

$$\pi(s, x^*) = \pi(\sigma^*, x^*) \ \forall s \in S^*.$$

Proof. The proof is equivalent to the one used in Theorem 1.2. We would also start assuming that there is a strategy that gives a higher payoff and would end up contradicting the fact that the the strategy σ^* is a best response to the associated population profile. $\square$

Observation 2.1. *It is also clear that if there is a unique best response to the strategy profile, then the evolution of the population stops. Hence, the cases where we will have to drive our attention is those that have more than one best response and find a way to decide which one of them guides the population (if it does) to a stable situation.*

Definition 2.4. Let's assume that initially there is a population where all individuals adopt the same strategy σ^* and the associated population profile is x^*. Then suppose that a mutation occurs and a small portion of individuals, ϵ, starts using a different strategy, σ. The entrance of these individuals that adopt a different behaviour alters the composition of the population. We will refer to this new population as *invasive population* and the new population profile will be x_ϵ.

Definition 2.5. A strategy σ^* is an *ESS (Evolutionary Stable Strategy)* if there exists an ϵ' such that for every $0 < \epsilon < \epsilon'$ and every $\sigma \neq \sigma^*$

$$\pi(\sigma^*, x_\epsilon) > \pi(\sigma, x_\epsilon).$$

That is, a strategy is an ESS if the population that adopts any other different strategy ends up having a smaller payoff. In biological terms we would say that the mutant population gives fewer offspring by not using σ^* and therefore this new behaviour will not last.

2.3 ESS in PVE games

In this section we will go through an example to determine if a strategy is an ESS in the context of a PVE game. A classical example is the ratio of males and females in human and other animal populations. This ratio is 50:50 in this species because this situation is an ESS. More details and examples of this type of games can be found in [9], one of them is the classical Hawk-Dove game.

Example 2.3. *Males to females ratio*
Let's consider a game with the following initial conditions:

1. The proportion of males in the population is μ and the proportion of females is $1-\mu$.
2. Each female reproduces once and produces n offspring.
3. Each male reproduces $\frac{1-\mu}{\mu}$ times on average.
4. Only female genes affect the sex ratio, so females are the only ones that drive the evolution process.

Let's also suppose that there are two possible pure strategies available for each female, s_1= all male offspring and s_2= all female offspring. The population profile can be clearly expressed in terms of the proportion of males and females and therefore written as $x = (\mu, 1-\mu)$.

The number of offspring that each female gives birth to is always fixed at n, so we cannot use this number as a payoff to determine the evolution of the population. However, when we look at the "third generation", the number of grandchildren that each female from the first generation has is different. So, our payoff will be determined by this number. Now we can establish the payoffs given by every pure strategy with a population profile $x = (\mu, 1-\mu)$. We know that each female from the second generation gives birth to n grandchildren and each male from the second generation reproduces $\frac{1-\mu}{\mu}$ times:

$$\pi(s_1, x) = n * n\frac{1-\mu}{\mu} = n^2\frac{1-\mu}{\mu}$$

$$\pi(s_2, x) = n * n = n^2.$$

Observation 2.2. *What this payoffs try to explain is how many descendants will the second generation bring. If its all females, each of them will mate once and bring n children. If its all males, each of them will mate $\frac{1-\mu}{\mu}$ times and from each mating he will bring n children.*

Now we can also express the payoff of a general mixed strategy $\sigma = (p, 1-p)$ as:

$$\pi(\sigma, x) = pn^2\frac{1-\mu}{\mu} + (1-p)n^2 = n^2\left[(1-p) + p\frac{1-\mu}{\mu}\right].$$

As we are only interested in the sex ratio and not the total number of males and females, we can suppose that $n = 1$ because the ratio is independent of this number. Then the previous payoff can be rewritten as:

$$\pi(\sigma, x) = (1-p) + p\frac{1-\mu}{\mu}.$$

What we want to do now is prove that the only ESS is, as we have already mentioned, a 50% ratio, i.e., $\mu = \frac{1}{2}$. It is important to remember that several conditions have to hold for a strategy to be an ESS. Firstly, a necessary but not sufficient condition is that this strategy has to be a best response to the associated population profile. Then, following Theorem 2.1, if the population is stable then $\pi(s, x^*) = \pi(\sigma^*, x^*)\ \forall s \in S^*$. If this first condition is passed, then we have to check the definition of ESS, which is, that σ^* is an ESS if there exists an ϵ' such that for every $0 < \epsilon < \epsilon'$ and every $\sigma \neq \sigma^*$

$$\pi(\sigma^*, x_\epsilon) > \pi(\sigma, x_\epsilon).$$

The three cases that we have to consider are the following:

1. $$\mu < \frac{1}{2} \Rightarrow \frac{1-\mu}{\mu} < 1 \Rightarrow n^2 \frac{1-\mu}{\mu} > n^2.$$
   Clearly, females using s_1 have more grandchildren and μ would end up rising. Theorem 2.1 is violated and therefore we do not have an ESS in this case.

2. $$\mu > \frac{1}{2} \Rightarrow \frac{1-\mu}{\mu} > 1 \Rightarrow n^2 \frac{1-\mu}{\mu} < n^2.$$
   Clearly, females using s_2 have more grandchildren and μ would end up decreasing. Theorem 2.1 is violated and therefore we do not have an ESS in this case.

3. $$\mu = \frac{1}{2} \Rightarrow \frac{1-\mu}{\mu} = 1 \Rightarrow n^2 \frac{1-\mu}{\mu} = n^2.$$
   In this case, both s_1 and s_2 leave the same offspring, n^2 which is the same payoff that the mixed strategy $\sigma^* = (\frac{1}{2}, \frac{1}{2})$ gives. Theorem 2.1 is not violated this time, so we can proceed to and check if this strategy also meets the definition of ESS with a population profile $x = (\frac{1}{2}, \frac{1}{2})$.
   Let $\sigma = (p, 1-p)$ be another strategy and
   $$x_\epsilon = (1-\epsilon)\sigma^* + \epsilon\sigma$$
   $$\mu_\epsilon = \frac{1}{2}(1-\epsilon) + \epsilon p = \frac{1}{2} + \epsilon\left(p - \frac{1}{2}\right).$$
   We have to check that this mutation does not invade the population, i.e., we have to check if $\pi(\sigma^*, x_\epsilon) - \pi(\sigma, x_\epsilon) > 0$.
   $$\begin{aligned}\pi(\sigma^*, x_\epsilon) - \pi(\sigma, x_\epsilon) &= \frac{1}{2} + \frac{1}{2}\frac{1-\mu_\epsilon}{\mu_\epsilon} - (1-p) - p\frac{1-\mu_\epsilon}{\mu_\epsilon} \\ &= \left(\frac{1}{2} - p\right)\left(\frac{1-\mu_\epsilon}{\mu_\epsilon} - 1\right) \\ &= \left(\frac{1}{2} - p\right)\left(\frac{1-2\mu_\epsilon}{\mu_\epsilon}\right)\end{aligned}$$
   We just have to prove that this difference is positive $\forall \sigma = (p, 1-p) \neq \sigma^*$.
   (a) $p < \frac{1}{2} \Rightarrow \mu_\epsilon < \frac{1}{2} \Rightarrow \frac{1-2\mu_\epsilon}{\mu_\epsilon} > 0 \Rightarrow \left(\frac{1}{2} - p\right)\left(\frac{1-2\mu_\epsilon}{\mu_\epsilon}\right) > 0.$
   (b) $p > \frac{1}{2} \Rightarrow \mu_\epsilon > \frac{1}{2} \Rightarrow \frac{1-2\mu_\epsilon}{\mu_\epsilon} < 0 \Rightarrow \left(\frac{1}{2} - p\right)\left(\frac{1-2\mu_\epsilon}{\mu_\epsilon}\right) > 0.$

Then, as we said at the beginning of the example, the strategy $\sigma^* = (\frac{1}{2}, \frac{1}{2})$ is the only ESS and so it is the sex ratio 50:50.

2.4 ESS and Nash Equilibria

In this section we will try to link the concepts of Nash Equilibria and Evolutionary Stable Strategy in PVP games. As we said before, a necessary condition for a strategy to be an ESS is that it must be a best response to the population profile generated. Hence, there will be a connection between NE and ESS, as we will see in the following pages.

Observation 2.3. *From now on we will only be working with PVP games. The reason is that the concept of NE cannot be applied to PVE games as they do not involve an interaction between specific players(one of them is always "the nature"). In PVP games it is always possible to associate a two-player game to work with the concepts that we have defined so far.*

In a PVP game we have expressed the payoff like:

$$\pi(\sigma, x) = \sum_{s \in S} \sum_{s' \in S} p(s)x(s')\pi(s, s').$$

Theorem 2.2. *In a PVP game, σ^* is an ESS if and only if $\forall \sigma \neq \sigma^*$ either:*

1. $\pi(\sigma^*, \sigma^*) > \pi(\sigma, \sigma^*)$
2. $\pi(\sigma^*, \sigma^*) = \pi(\sigma, \sigma^*)$ *and* $\pi(\sigma^*, \sigma) > \pi(\sigma, \sigma)$.

Proof. The definition of ESS that we established in the last section was that σ^* is an ESS if, for ϵ sufficiently small,

$$\pi(\sigma^*, x_\epsilon) > \pi(\sigma, x_\epsilon)$$

where the population profile x_ϵ is $x_\epsilon = ((1-\epsilon), \epsilon)$. Now that we have already defined how the payoffs are expressed for PVP games, we can rewrite the previous condition for ESS as

$$(1-\epsilon)\pi(\sigma^*, \sigma^*) + \epsilon\pi(\sigma^*, \sigma) > (1-\epsilon)\pi(\sigma, \sigma^*) + \epsilon\pi(\sigma, \sigma). \tag{2.1}$$

$\Leftarrow$) If condition 1 holds, then equation (2.1) can be satisfied for ϵ sufficiently small. If condition 2 holds, then we can cancel the terms that are equal on both sides of (2.1) and then we can rewrite it like

$$\epsilon\pi(\sigma^*, \sigma) > \epsilon\pi(\sigma, \sigma)$$

which is clearly satisfied $\forall \epsilon \in (0, 1)$.

$\Rightarrow$) We will try to prove it using a counter-reciprocal argument. In this case, we assume that σ^* is an ESS and we want to prove that either condition (1) or (2) have to hold. If σ^* is an ESS, by definition, (2.1) holds. Let's assume that (1) and (2) do not hold, i.e., we assume that $\pi(\sigma^*, \sigma^*) < \pi(\sigma, \sigma^*)$. In this case, $\exists\epsilon$ sufficiently small that (2.1) is violated, i.e., σ^* is not an ESS. So we have

$$(2.1) \Rightarrow \pi(\sigma^*, \sigma^*) \geq \pi(\sigma, \sigma^*).$$

If $\pi(\sigma^*, \sigma^*) = \pi(\sigma, \sigma^*)$, then

$$(2.1) \Rightarrow \pi(\sigma^*, \sigma) > \pi(\sigma, \sigma).$$

□

Observation 2.4. *It is important to note that for a strategy σ^* to be an ESS, one of the previous conditions has to hold for any other possible strategy σ. Let's suppose that in a particular example we have a set of pure strategies $S = \{s_1, s_2, \ldots, s_n\}$. Then, any other mixed strategy σ is a linear combination of the pure strategies with the associated vector of probabilities as the parameters of the linear combination*

$$\sigma = \sum_{s \in S} p(s)s, \sum_{s \in S} p(s) = 1.$$

Then, in a practical example if we prove that either condition (1) or (2) of the theorem hold $\forall s \in S$, then it will also hold for any possible mixed strategy.

Observation 2.5. *It is also important to clarify that when we write the payoff function as $\pi(\sigma_1, \sigma_2)$, even if we do not use a sub-index, we are referring to the first player's payoff. That is, we are studying the performance of the established individual against the opponent selected from the population.*

As we can see, the definition of ESS is more restrictive than the definition of a NE, so the set of ESS will be a subset of the NE in the associated two-player game. There will be cases where we have a non-empty set of NE but none of them passes the test to be an ESS. Thanks to this theorem we can establish a procedure to find the set of ESS in a PVP game. Once we associate the correspondent two-player game, we just have to find the set of NE and test them with the conditions of the theorem. The ones that pass either condition 1 or 2 will be the ESS of the game.

As we have just mentioned, we will not always find strategies that end up being evolutionary stable as the set of ESS can be empty. But, is there any result that can ensure the existence of ESS? In fact, there is one for a very specefic type of game.

Theorem 2.3. *All generic, two-action, symmetric PVP games have an ESS.*

Proof. A symmetric two-player game has the following form

	A	B
A	(a,a)	(b,c)
B	(c,b)	(d,d)

By applying the previously defined affine transformations we can modify the matrix as follows. First we will transform payoffs of player 1,

	A	B
A	(a - c,a)	(0,c)
B	(0,b)	(d - b,d)

Then, similarilly, we will modify the payoffs of player 2,

	A	B
A	(a - c,a - c)	(0,0)
B	(0,0)	(d - b,d - b)

As it happens with Nash Equilibria, ESS conditions are not affected by applying affine transformations to the payoffs. The fact that we are considering generic games implies that $a \neq c$ and $d \neq b$. Now, we will consider the different possibilities that we have depending on the payoffs and we will see that in all the cases an ESS exists.

If either $a - c > 0$ or $d - b > 0$ then, $\pi(A, A) > \pi(B, A)$ or $\pi(B, B) > \pi(A, B)$ respectively. In both cases, condition (1) of Theorem 2.2 holds and then $\sigma^* = (A, A)$ or $\sigma^* = (B, B)$ are the respective ESS. If both $a - c > 0$ and $d - b > 0$ at the same time, then we have two ESS in the game.

So, there is one more case to cover, $a - c < 0$ and $d - b < 0$. In this case there are no symmetric NE in pure strategies, but, as we know, at least one NE exists. Therefore it will have to be a mixed strategy $\sigma^* = (p^*, 1 - p^*)$ that leads to a symmetric NE (σ^*, σ^*). Using the Equality of Payoffs theorem and the initial matrix without having applied affine transformations, we have that

$$\pi_1(A, \sigma^*) = \pi_1(B, \sigma^*) \iff ap^* + b(1-p^*) = cp^* + d(1-p^*) \iff p^* = \frac{b - d}{(c - a) + (b - d)}.$$

To be completely strict we should have done this calculation assuming that we had a NE (σ_1^*, σ_2^*) with $\sigma_1^* = (p^*, 1 - p^*)$ and $\sigma_2^* = (q^*, 1 - q^*)$ and we would have end up getting the same result and $p = q$, i.e., seeing that the NE is effectively symmetric. In this case we cannot prove that an ESS exists by using (1) of Theorem 2.2 because also using the Equality of Payoffs theorem we have that $\pi(\sigma^*, \sigma^*) = \pi(\sigma, \sigma^*)$. Then, we have to focus on the second option of the theorem. Now, for a mixed strategy $\sigma = (p, 1 - p)$

$$\pi(\sigma^*, \sigma) = p^* p(a - c) + (1 - p)(1 - p^*)(d - b)$$

and

$$\pi(\sigma, \sigma) = p^2(a - c) + (1 - p)^2(d - b).$$

So,

$$\begin{aligned}
\pi(\sigma^*, \sigma) - \pi(\sigma, \sigma) &= p(p^* - p)(a - c) + (1 - p)(p - p^*)(d - b) \\
&= (p^* - p)[p(a - c + d - b) - (d - b)] \\
&= -(a - c + d - b)(p^* - p)^2 \\
&> 0.
\end{aligned}$$

Then, σ^* is an ESS and we have proved that in all cases we can effectively find an ESS in a generic PVP 2-player, 2-action game. □

As we have seen in the last theorem, there is only a very special case where we will always find an ESS. In these cases where we cannot find it, we may have NE that do not end up being evolutionary stable, as we will see when we develop more examples in the next chapter. At this point we would not be able to say anything about the evolution of the population if there is no ESS. Our goal then will be to find new concepts that are weaker than ESS and can give us some information about the evolution of the population. To do that, we will shift our focus to the population instead of the strategies. Hopefully that will give us more information about evolution and stability than only the ESS concept.

Chapter 3

Replicator dynamics

In the previous chapter of this work we have studied the evolutionary games with the concept of ESS as the main argument. We have done it for both PVP and PVE games, although we have remarked that the first type of games will be the ones where we will mainly focus our attention, since these are the ones that are approached by classical game theory.

As we already established by the end of the previous chapter, the concept of ESS, while giving a clear concept of an evolutionary process, cannot be the only tool we have to measure or test evolution in a population. The first problem about the ESS is that we cannot always find a strategy that is so stable in the game. In this case, we have no information at all about the evolution of the system described by the game. That is the reason why we want to find weaker evolutionary concepts that can give us some information in case the ESS could not be reached. Second, the definition of ESS was presented in a very specific type of situation, we had a population where all individuals used the same strategy, i.e., we were dealing with a monomorphic population. Then, if we could find a strategy σ^* that was an ESS, all the other strategies in the support of σ^* would obtain the same payoff than the ESS.

At this moment the replicator dynamics come into play. We want to address the situations where we have a polymorphic population, i.e., populations where variations of the strategy exist and can be adopted by its members. What will happen then, can we find any stability of any kind? The replicator dynamics will approach the evolution and stability of the population from a different perspective that can cast more light upon the overall evolutionary process.

Again, we will follow [16] and [17] as a reference for the organization and development of this chapter. In the parts where we use more concepts related to differential equations we will again follow procedures and notation used in [14] and [2].

3.1 Framework

This time we will consider a population with the following characteristics. Firstly, there will be different type of individuals in the same population and every type of individuals will use a pre-programmed strategy for each game. Second, these pre-programmed strategy or behaviour is passed to descendants without modification.

All the individuals use strategies from a finite set $S = \{s_1, s_2, \dots, s_k\}$ and we will call n_i the number of individuals using each strategy s_i. Then, the total population size is

$$N = \sum_{i=1}^{k} n_i$$

and the proportion of individuals using each strategy s_i is

$$x_i = \frac{n_i}{N}$$

where clearly $\sum_{i=1}^{k} x_i = 1$. As we did in the previous chapter, we will define the *population profile* as the vector of probabilities with which each strategy is played as $x = (x_1, x_2, \dots, x_k)$.

We will consider a factor that represents the growth or decrease of the total number of individuals in the population that is independent of each game, ϵ. That is, ϵ represents the *rate of appearance and disappearance* of individuals in the population. However, the payoffs of each strategy will also affect the evolution of the population as those that give a better payoff have to be "rewarded" in some way. Having this in mind, we should conclude that the rate of change in the number of individuals using s_i is affected by a systematic component (independent of the game), ϵ, and an idiosyncratic component, $\pi(s_i, x)$. We can explicitly express the rate of change in the number of individuals using s_i as

$$\dot{n_i} = (\epsilon + \pi(s_i, x))n_i$$

and the rate of change in the total population as

$$\begin{aligned}
\dot{N} &= \sum_{i=i}^{k} \dot{n_i} \\
&= \sum_{i=i}^{k} (\epsilon + \pi(s_i, x))n_i \\
&= \epsilon \sum_{i=i}^{k} n_i + \sum_{i=i}^{k} \pi(s_i, x)n_i \\
&= \epsilon N + N \sum_{i=i}^{k} x_i \pi(s_i, x) \\
&= (\epsilon + \bar{\pi}(x))N
\end{aligned}$$

where

$$\bar{\pi}(x) = \sum_{i=i}^{k} \pi(s_i, x)x_i$$

is the *average payoff* of the population.

However, what are more interested in is the evolution of the number of individuals of each type that integrate the population, i.e., how the proportion of individuals using each strategy changes. Let's show how to find this variation.

$$n_i = Nx_i \Rightarrow \dot{n_i} = \dot{N}x_i + N\dot{x_i}.$$

Now, using the previous definitions,

$$N\dot{x_i} = \dot{n}_i - x_i\dot{N} = (\epsilon + \pi(s_i, x))n_i - (\epsilon + \bar{\pi}(x))x_iN = (\epsilon + \pi(s_i, x))x_iN - (\epsilon + \bar{\pi}(x))x_iN$$

and cancelling and dividing by N, we have

$$\begin{aligned}\dot{x_i} &= (\epsilon - \epsilon + \pi(s_i, x) - \bar{\pi}(x))x_i \\ &= (\pi(s_i, x) - \bar{\pi}(x))x_i.\end{aligned}$$

In conclusion, the increase and decrease in the proportion of individuals using strategy s_i depends on the payoff that this strategy gives compared to the average payoff. As we can see, the systematic component that we had previously defined, ϵ, has no effect in the rate of change of the proportion of individuals using s_i.

3.2 Stability in PVP games

In this section we will dig into PVP player games and offer new definitions related to the concept of stability that are weaker than the ESS concept. We will proceed as follows. First we will present all the concepts that will appear from now on. Then, we will develop some examples and theoretical results in two-strategy PVP games to give an initial view of how the evolutionary process is going to be approached. Finally, we will present all the results for n-strategy PVP games and give more examples for 3-strategy PVP games.

Definition 3.1. A *fixed point* of the replicator dynamics describes a population that is no longer evolving. That is, a population where the proportion of individuals using each strategy remains constant, i.e., $\dot{x_i} = 0\ \forall i$. A fixed point of the replicator dynamics is said to be *asymptotically stable* if any small deviations from that state are eliminated by the dynamics as $t \to \infty$.

Let's now define some sets that we will be frequently using in the section.

Definition 3.2. Assume that we are studying a PVP game and the correspondent dynamical system associated to this game (we will show in the next example how to connect both). Then,

1. $\boldsymbol{N}$ = set of symmetric Nash equilibria in the PVP game.
2. $\boldsymbol{E}$ = set of evolutionary stable strategies in the PVP game.
3. $\boldsymbol{F}$ = set of fixed points of the dynamical system.
4. $\boldsymbol{A}$ = set of asymptotically fixed points of the dynamical system.

As we saw in the previous section, $ESS \subseteq NE$. And, clearly, $A \subseteq F$. Our job from now on will be to link these concepts and establish relationships between them in PVP games and their respective associated dynamical systems.

Let's go through an example of a two-strategy PVP game first. In the case of a two-strategy PVP game we can rewrite the previous conditions as, $S = \{s_1, s_2\}$, $x = (x_1, x_2)$ where $x_2 = 1 - x_1$. Then,

$$\dot{x_1} = (\pi(s_1, x) - \bar{\pi}(x))x_1$$

$$\dot{x_2} = -\dot{x_1}$$

where,

$$\bar{\pi}(x) = x_1\pi(s_1, x) + (1 - x_1)\pi(s_2, x).$$

In conclusion,

$$\begin{aligned}\dot{x_1} &= (\pi(s_1, x) - x_1\pi(s_1, x) + (1 - x_1)\pi(s_2, x))x_1 \\ &= x_1(1 - x_1)(\pi(s_1, x) - \pi(s_2, x)).\end{aligned}$$

Observation 3.1. *All the calculations to find NE in both pure and mixed strategies, fixed points and ESS will be generally done with the help of a Python script that can be found in appendix A. In this first example we will do the calculations "by hand" to show how it should be done, but in future examples where the number of strategies increases we will apply the mentioned program.*

Example 3.1. Let's assume that we are in a two-strategy PVP game where both players can use strategies from the set $S = \{A, B\}$ with the following associated payoff matrix

Player 2		
	A	B
A	(3,3)	(0,0)
B	(0,0)	(1,1)

There are two Nash equilibria in pure strategies in this game, (A, A) and (B, B), where A and B can be represented as the mixed strategies $\sigma_A = (1, 0)$ and $\sigma_B = (0, 1)$ respectively. Then, to find out if there are more Nash equilibria in mixed strategies, let's suppose that player 1 is using a mixed strategy $\sigma_1 = (p, 1 - p)$ and player 2 is using a mixed strategy $\sigma_2 = (q, 1 - q)$. Then, the expected payoff for players is,

$$\pi_{\sigma_1} = p[3q + 0(1 - q)] + (1 - p)[0q + 1(1 - q)]$$

$$\pi_{\sigma_2} = q[3p + 0(1 - p)] + (1 - q)[0p + 1(1 - q)].$$

Player 1 is using a mixed strategy such that player 2 in indifferent between using the first or the second column(since the respective expected payoffs are the same). The same thing happens with the strategy of player 2 and the rows for player 1. Then, we have,

$$3q = 1 - q \Rightarrow q = \frac{1}{4}$$

$$3p = 1 - p \Rightarrow p = \frac{1}{4}.$$

Therefore, we have found one NE mixed strategy $\sigma = (\frac{1}{4}, \frac{3}{4})$ and we can write the set $N = \{(\sigma_1, \sigma_1), (\sigma_2, \sigma_2), (\sigma, \sigma)\}$. Now we can check if any of them is also an evolutionary stable strategy. We will do it by checking the definition of ESS for each one of them. Just as a reminder, σ^* is an ESS if and only if $\forall \sigma \neq \sigma^*$ either:

1. $\pi(\sigma^*, \sigma^*) > \pi(\sigma, \sigma^*)$

2. $\pi(\sigma^*, \sigma^*) = \pi(\sigma, \sigma^*)$ and $\pi(\sigma^*, \sigma) > \pi(\sigma, \sigma)$.

Let's check if any of the two conditions holds for each NE:

1. $\sigma_A = (1, 0)$. $\pi(\sigma_1, \sigma_1) = 3 > \pi(\sigma_2, \sigma_1) = 0$. Condition 1) holds, so σ_A is an ESS.
2. $\sigma_B = (0, 1)$. $\pi(\sigma_2, \sigma_2) = 1 > \pi(\sigma_1, \sigma_2) = 0$. Condition 1) holds, so σ_B is an ESS.
3. $\sigma = (\frac{1}{4}, \frac{3}{4})$.

$$\pi(\sigma, \sigma) = (\frac{1}{4})^2\pi(A, A) + 2\frac{1}{4}\frac{3}{4}\pi(A, B) + (\frac{3}{4})^2\pi(B, B) = \frac{7}{4} < \pi(B, B) = 3.$$

None of the conditions hold, so σ is not an ESS.

Then, $E = \{\sigma_A, \sigma_B\}$.

Let's now begin to work with the replicator dynamics. Let x and $1-x$ be the porportion of individuals using A and B respectively. Then,

$$\begin{aligned} \dot{x} &= x(1-x)(\pi(A, x) - \pi(B, x)) \\ &= x(1-x)(3x - (1-x)) \\ &= x(1-x)(4x - 1). \end{aligned}$$

We can clearly see that the set of fixed points is $F = \{0, \frac{1}{4}, 1\}$. Now we would like to check which of these points are asymptotically stable. Now,

$$\frac{d\dot{x}}{dx} = -12x^2 + 10x - 1$$

1. $x = 0$
$$\frac{d\dot{x}}{dx}(0) = -1.$$
So, $x = 0$ is a stable fixed point.
2. $x = \frac{1}{4}$
$$\frac{d\dot{x}}{dx}\left(\frac{1}{4}\right) = \frac{3}{4}.$$
So, $x = \frac{1}{4}$ is an unstable fixed point.
3. $x = 1$
$$\frac{d\dot{x}}{dx}(1) = -3.$$
So, $x = 1$ is a stable fixed point.

Then, $A = \{0, 1\}$.

Now that we have all the sets, we want to establish the links and relationships between them. It's easy to see that, $\dot{x} > 0$ if $x > \frac{1}{4}$ and $\dot{x} < 0$ if $x < \frac{1}{4}$. That is, if the population starts with a population profile where more than 25% of individuals use A, then all the population evolves until everyone uses A. In contrast, if the population starts with less than 25% of individuals using A, then all the population evolves until everyone uses B.

In conclusion, only the pure strategies are evolutionary end points and only the evolutionary end points (asymptotically stable) correspond to an ESS. So, falling into an abuse of notation that identifies a strategy with the population profile it generates, we can write all the relationships in this particular example as,

$$E = A \subseteq N = F.$$

Now that we have presented an example of how to work with two-strategy PVP games and their correspondent replicator dynamics, we will explain some results that highlight the relationships between the different sets in a general two-strategy PVP game.

Theorem 3.1. *Let $S = \{s_1, s_2\}$ and let $\sigma^* = (p^*, 1-p^*)$ be the strategy that uses s_1 with probability p^*. If (σ^*, σ^*) is a symmetric NE, then the population profile $x^* = (x^*, 1-x^*)$ with $x^* = p^*$ is a fixed point of the replicator dynamics*

$$\dot{x} = x(1-x)(\pi(s_1, x) - \pi(s_2, x)).$$

Proof. Suppose that the NE strategy σ^* is pure, i.e., every player in the population uses $s_j \in S$. Then, $x^* = 0$ or $x^* = 1$ and in both cases $\dot{x} = 0$.

Now suppose that the NE strategy σ^* is a mixed strategy, then the Equality of Payoffs theorem tells us that $\pi(s_1, \sigma^*) = \pi(s_2, \sigma^*)$. Now, in a PVP game,

$$\pi(s_i, \sigma^*) = p^*\pi(s_i, s_1) + (1-p^*)\pi(s_i, s_2) = x^*\pi(s_i, s_1) + (1-x^*)\pi(s_i, s_2) = \pi(s_i, x^*)$$

where we have used that $p^* = x^*$. Then, we also have that $\pi(s_1, x^*) = \pi(s_2, x^*)$ and consequently $\dot{x} = 0$. $\square$

Theorem 3.2. *For any two-strategy PVP game, a strategy is an ESS if and only if the corresponding fixed point in the replicator dynamics is asymptotically stable.*

Proof. Suppose that the set of pure strategies is $S = \{s_1, s_2\}$ and let x be the proportion of individuals using s_1. As we have already seen, the replicator dynamics system is

$$\dot{x} = x(1-x)(\pi(s_1, x) - \pi(s_2, x)).$$

Let's define $f(x) = \dot{x}$. If x^* is an equilibrium, we have that $f(x^*) = 0$ for $x \in [0, 1]$. We want to check the stability of x^* in the case of the pure strategies and the case of a mixed strategy that is different to pure ones. The point x^* is a solution of the Cauchy problem

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x^*. \end{cases}$$

Let's now compute the derivative of the function f

$$\begin{aligned}
\frac{d}{dx}f(x) &= \frac{d}{dx}\left[x(1-x)(\pi(s_1, x) - \pi(s_2, x))\right] = \\
&= \frac{d}{dx}\left[x(1-x)\right] * (\pi(s_1, x) - \pi(s_2, x)) + x(1-x) * \frac{d}{dx}\left[\pi(s_1, x) - \pi(s_2, x)\right] \\
&= (1-2x) * (\pi(s_1, x) - \pi(s_2, x)) \\
&+ x(1-x) * \frac{d}{dx}\left[x\pi(s_1, s_1) + (1-x)\pi(s_1, s_2) - x\pi(s_2, s_1) - (1-x)\pi(s_2, s_2)\right] \\
&= (1-2x) * (\pi(s_1, x) - \pi(s_2, x)) \\
&+ x(1-x) * \left[\pi(s_1, s_1) - \pi(s_1, s_2) - \pi(s_2, s_1) + \pi(s_2, s_2)\right].
\end{aligned}$$

Suppose that $\sigma^* = (1, 0)$, i.e., the pair (s_1, s_1) with the correspondant fixed point $x^* = 1$. Then, for any other $\sigma = (y, 1-y)$, σ^* is an ESS if and only if

$$\begin{aligned}
&\pi(s_1, x_\epsilon) - \pi(\sigma, x_\epsilon) > 0 \\
&\iff \pi(s_1, x_\epsilon) - y\pi(s_1, x_\epsilon) - (1-y)\pi(s_2, x_\epsilon) > 0 \\
&\iff (1-y)[\pi(s_1, x_\epsilon) - \pi(s_2, x_\epsilon)] > 0 \\
&\iff \pi(s_1, x_\epsilon) - \pi(s_2, x_\epsilon) > 0
\end{aligned}$$

because $y \in (0, 1)$. Let's consider $x = x^* - \epsilon$ and evaluate

$$\frac{d}{dx} f(1) = -(\pi(s_1, x_\epsilon) - \pi(s_2, x_\epsilon))) < 0,$$

because $(\pi(s_1, x_\epsilon) - \pi(s_2, x_\epsilon)) > 0$ when $\epsilon \to 0$. Then, the point $x^* = 1$ is an asymptotically stable fixed point. An analogous procedure is used to see that $\sigma^* = (0, 1)$, i.e., the pair (s_2, s_2), is an ESS if and only if the correspondent population $x^* = 0$ is asymptotically stable.

Let's now prove the same result for a mixed strategy $\sigma^* = (p^*, 1-p^*)$ that is different to the pure strategies, i.e., $p^* \in (0, 1)$. The Equality of Payoffs theorem tells us that $\pi(\sigma^*, \sigma^*) = \pi(\sigma, \sigma^*)$ for any other mixed strategy σ. Then, for σ^* to be an ESS we will have to check the second condition of Theorem 2.2, i.e., check if $\pi(\sigma^*, \sigma) > \pi(\sigma, \sigma)$ $\forall \sigma \neq \sigma^*$. As we have already mentioned earlier, it is enough to check that the condition holds for both pure strategies. Then, taking $\sigma = s_1$ and $\sigma = s_2$ the condition becomes the two conditions,

$$\pi(s_2, s_1) > \pi(s_1, s_1)\ ,\ \pi(s_1, s_2) > \pi(s_2, s_2).$$

Let's again evaluate

$$\frac{d}{dx} f(x^*) = x(1-x) * [\pi(s_1, s_1) - \pi(s_1, s_2) - \pi(s_2, s_1) + \pi(s_2, s_2)] < 0,$$

because the first term, $x(1-x)$, is positive as $x \in (0, 1)$ and the second one is negative because of the inequalities that we previously wrote. Then, the point x^* is an asymptotically stable fixed point.

□

Then, in the case of two-strategy PVP we have the following relationships(with the correspondent abuse of notation),

$$E = A \subseteq N \subseteq F.$$

Observation 3.2. *As we saw in the previous example, we had 2 pure strategies and in principle we had 2 equations to deal with, but using the constraint $x_1 + x_2 = 1$, we reduced the number of equations and left it in 1. Then, exactly the same thing would happen if we were dealing with a n-strategy PVP game. The initial number of equations would be n but we can reduce it using the constraint $\sum_{i=1}^n x_i = 1$ and then we would only have to deal with a $n-1$ equations dynamical system.*

Now we will present another example of a PVP game, but this time it will be a 3-strategy game. As we will see, this example is more complex and has more important points to study than the previous one. Also, even if the theory related to n-player games is similar to the one applied to 2-player games that we have already presented, there will be significant differences that will be proved after the example.

Example 3.2. Let's assume that we are in a 3-strategy PVP game where both players can use strategies from the set $S = \{A, B, C\}$ with the following associated payoff matrix.

	A	B	C
A	(0,0)	(3,3)	(1,1)
B	(3,3)	(0,0)	(1,1)
C	(1,1)	(1,1)	(1,1)

We can assume that player 1 is the row player and player 2 is the column player but it is not really important since the game is symmetric.

Firstly, we will present the NE of the game. As we said, the calculations will be done using the program developed for this purpose, so we will only see the results. There are 3 NE in pure strategies, $(A, B), (B, A)$ and(C, C), and there is one more NE in mixed strategies, (σ, σ), where $\sigma = (\frac{1}{2}, \frac{1}{2}, 0)$. Then, the set of symmetric NE is $N = \{(C, C), (\sigma, \sigma)\}$.

Now we will continue representing the dynamical system associated to the game. As a reminder, we had established earlier that the evolution in the proportion of individuals using each strategy s_i is $\dot{x}_i = (\pi(s_i, x) - \bar{\pi}(x))x_i$. Assuming that the population profile is the vector $x = (x_A, x_B, x_C)$ we have,

$$\begin{cases} \dot{x}_A = x_A(3x_B + x_C - \bar{\pi}(x)) \\ \dot{x}_B = x_B(3x_A + x_C - \bar{\pi}(x)) \\ \dot{x}_C = x_C(1 - \bar{\pi}(x)) \end{cases}$$

where $\bar{\pi}(x) = 6x_Ax_B + x_C(x_A + x_B) + x_C$.

Observation 3.3. *We will use x as the new variable and we had use it to define the population profile, but we think there will be no problem to distinguish which one we are using during the example.*

As we did in the 2-strategy case, writing $x_A = x, x_B = y$ and $x_C = 1 - x - y$, we have

$$\begin{cases} \dot{x} = x(1 - x + 2y - \bar{\pi}(x, y)) \\ \dot{y} = y(1 + 2x - y - \bar{\pi}(x, y)) \end{cases}$$

where $\bar{\pi}(x) = 1 + 4xy - x^2 - y^2$. Then, we can now represent our vector field

$$\mathbb{X} = \begin{cases} \dot{x} = x(x^2 + y^2 - 4xy - x + 2y) \\ \dot{y} = y(x^2 + y^2 - 4xy + 2x - y) \end{cases}$$

Now we can proceed calculating the fixed points of the system, the set of fixed points is $F = \{(0, 0), (1, 0), (0, 1), (\frac{1}{2}, \frac{1}{2})\}$. There are also 4 invariant lines in the system, these are the horizontal and vertical isoclines $y = 0$, $x = 0$ and $x - y = 0$ and $x + y = 1$.

Now we can proceed with the local study near the fixed points. Let's first write the matrix of partial derivatives corresponding to the vector field,

$$D\mathbb{X}(x,y)=\begin{pmatrix}3x^2+y^2-8xy+2y-2x & -4y^2+2xy+2y\\ -4x^2+2xy+2x & 3y^2+x^2-8xy+2y-2x\end{pmatrix}.$$

Let's now evaluate each point in the matrix and obtain the corresponding eigenvalues and eigenvectors to lineally classify the points.

$$D\mathbb{X}(0,0)=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

There is just one eigenvalue with multiplicity 2, $\lambda=0$. Then the origin is not a hyperbolic point and the study cannot go further since we cannot apply the Hartman-Grobman theorem.

$$D\mathbb{X}\left(\frac{1}{2},\frac{1}{2}\right)=\begin{pmatrix}-1 & \frac{1}{2}\\ \frac{1}{2} & -1\end{pmatrix}.$$

The eigenvalues are $\lambda_1=-\frac{1}{2},\lambda_2=-\frac{3}{2}$. Then this point is a stable node. The eigenvectors are $x=y$ and $x=-y$ respectively.

$$D\mathbb{X}(1,0)=\begin{pmatrix}1 & 0\\ -2 & 3\end{pmatrix}.$$

The eigenvalues are $\lambda_1=3,\lambda_2=1$. Then this point is a unstable node. The eigenvectors are $x=0$ and $x=y$ respectively.

$$D\mathbb{X}(0,1)=\begin{pmatrix}3 & -2\\ 0 & 1\end{pmatrix}.$$

The eigenvalues are $\lambda_1=3,\lambda_2=1$. Then this point is a unstable node. The eigenvectors are $y=0$ and $x=y$ respectively.

We have then that $A=\{(\frac{1}{2},\frac{1}{2})\}$. To get a clearer view of how the phase portrait will look like we can observe the behaviour of the system on the invariant lines.

1. Over $y=0$. We have that $\dot{x}=x(x^2-1)\Rightarrow\dot{x}<0$ if $x\in(0,1)$ and $\dot{x}>0$ if $x\in(-\infty,0)\cup(1,\infty)$.
2. Over $x=0$. We have that $\dot{y}=y(y^2-1)\Rightarrow\dot{y}<0$ if $y\in(0,1)$ and $\dot{y}>0$ if $y\in(-\infty,0)\cup(1,\infty)$.
3. Over $x-y=0$. We have that $\dot{x},\dot{y}>0$ if $x,y\in(0,\frac{1}{2})$.
4. Over $x+y=1$. We have that $\dot{x},\dot{y}<0$ if $x\in(0,\frac{1}{2})$ and $\dot{x},\dot{y}>0$ if $x\in(\frac{1}{2},1)$.

With all the information that we have gathered so far we can represent the phase portrait using the Poincaré compactification.

Observation 3.4. *The green points in the representation correspond to the saddle points, the red points correspond to unstable nodes and the blue points correspond to stable nodes. Also, following the same color code, we can see the stability of the invariant lines, blue corresponds to stable lines and red to unstable lines. With all this information its easier* *to see how each phase portrait will be divided into sectors. In each representation we will comment how many different sectors can be found and the type of each of them. The stability is over the invariant lines could be checked in every particular case as we have done in this example.*

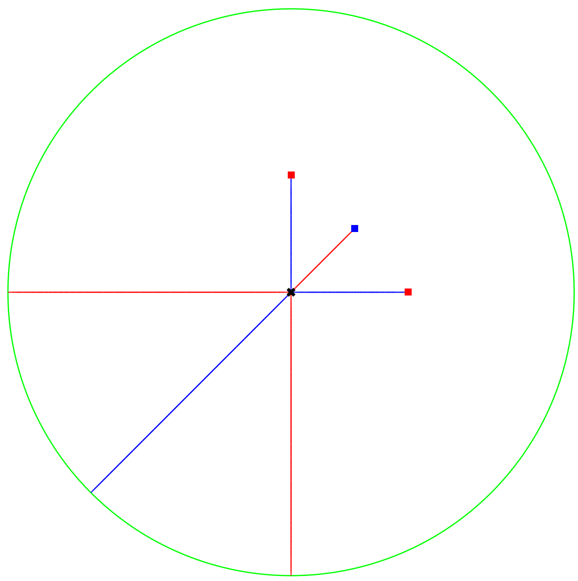

Figure 3.1: phase portrait of the vector field $\mathbb{X}$.

As we can see, infinity is a line of singularities in this case. In the finite representation, the invariant separatrices break the representation into 6 hyperbolic sectors. The sectors boundaries are the set of invariant separatrices, which is $S = \{x = 0; y = 0; x = y\}$.

Seeing the representation we can already guess which one of the two symmetric NE will be an ESS. It is clear that the orbits escape from the unstable fixes point $(0, 0)$ and end up going to the fixed point $(\frac{1}{2}, \frac{1}{2})$ so let's check if this last one effectively holds any of the conditions to be an ESS. Let's check if any of the two ESS conditions holds for each NE:

$$\sigma = \left(\frac{1}{2}, \frac{1}{2}, 0\right) = \pi(\sigma, \sigma) = \frac{3}{2} = \pi(A, \sigma) = \pi(B, \sigma).$$

We have to check if condition 2) holds.

$$\pi(A, \sigma) = \pi(B, \sigma) = \frac{3}{2} > \pi(A, A) = \pi(B, B) = 0.$$

It holds, so, as we already imagined, $\sigma = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ is the only ESS, i.e., $E = \{\sigma\}$.

Example 3.3. Let's modify the previous example by adding a parameter in the payoffs, leaving the matrix as follows

	A	B	C
A	(λ,λ)	(3,3)	(1,1)
B	(3,3)	(λ,λ)	(1,1)
C	(1,1)	(1,1)	(1,1)

We will just comment the parameter cases where there is a change in the dynamics. We will not go into every case in details but just present the results and the representation of the dynamical system.

$\lambda = 2$	Fixed Point Type
(0,0)	Not Hyperbolic
(1,0)	Saddle Point
(0,1)	Saddle Point
$(\frac{1}{2},\frac{1}{2})$	Stable Node

$$\mathbb{X}_2 = \begin{cases} \dot{x} = x(-x^2 - y^2 - 4xy + x + 2y) \\ \dot{y} = y(-x^2 - y^2 - 4xy + 2x + y) \end{cases}$$

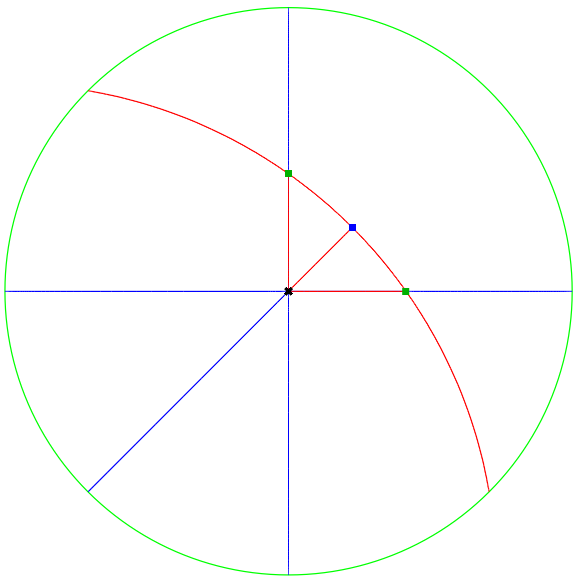

Figure 3.2: phase portrait of the vector field $\mathbb{X}_2$.

As we can see, infinity is a line of singularities in this case. In the finite representation, the invariant separatrices break the representation into 4 parabolic and 2 elliptic sectors. The sectors boundaries are the set of invariant separatrices, which is $S = \{x = 0; y = 0; x = y\}$. The first elliptic sector's boundaries are the separatrices $\{x = 0, y < 0\}$ and $\{y = 0, x > 0\}$ and for the second we have the separatrices $\{x = 0, y > 0\}$ and $\{y = 0, x < 0\}$.

In this case, the only ESS is still $\sigma = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$.

$\lambda = 4$	Fixed Point Type
(0,0)	Not Hyperbolic
(1,0)	Stable Node
(0,1)	Stable Node
$(\frac{1}{2},\frac{1}{2})$	Saddle Point

$$\mathbb{X}_4 = \begin{cases} \dot{x} = x(-3x^2 - 3y^2 - 4xy + 3x + 2y) \\ \dot{y} = y(-3x^2 - 3y^2 - 4xy + 2x + 3y) \end{cases}$$

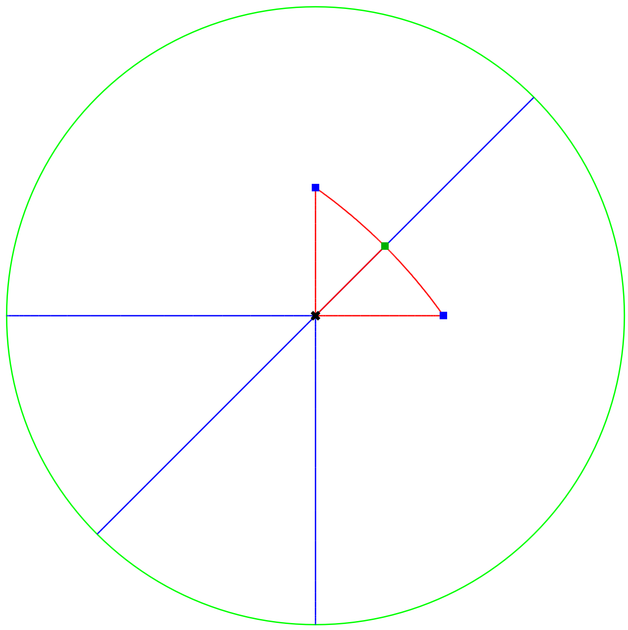

Figure 3.3: phase portrait of the vector field $\mathbb{X}_4$.

As we can see, infinity is a line of singularities in this case. In the finite representation, the invariant separatrices break the representation into 4 parabolic and 2 hyperbolic sectors. The sectors boundaries are the set of invariant separatrices, which is $S = \{x = 0; y = 0; x = y\}$. The first hyperbolic sector's boundaries are the separatrices $\{x = 0, y < 0\}$ and $\{y = 0, x > 0\}$ and for the second we have the separatrices $\{x = 0, y > 0\}$ and $\{y = 0, x < 0\}$.

In this case, we have two ESS, $\sigma_A = (1, 0, 0) = A$ and $\sigma_B = (0, 1, 0) = B$.

Observation 3.5. *Additional information about each vector field and its correspondent phase portrait representation can be found in appendix B of this work. This is the information and the calculus provided by the software P4&P5, [1], developed by "Grup de sistemes dinàmics de la UAB".*

As we did after the example of the 2-strategy PVP game, we will now prove the relationships between the four sets that we have previously mentioned. This time, we will end up having the following relationship (with the correspondent abuse of notation):

$$E \subseteq A \subseteq N \subseteq F.$$

Theorem 3.3. *If (σ^*, σ^*) is a symmetric Nash equilibrium, then the population state $x^* = \sigma^*$ is a fixed point of the replicator dynamics.*

Proof. First, let's suppose that the NE strategy σ^* is pure, i.e., every member of the population uses one pure strategy s_j from the set $S = \{s_1, s_2, \ldots s_n\}$. Then, clearly $x_i = 0$ for $i \neq j$ and $\pi(x^*) = \pi(s_j, x^*) \Rightarrow \dot{x}_i = (\pi(x_i, x) - \bar{\pi}(x))x_i = 0$. Then, x^* is a fixed point of the system.

Now, let's suppose that σ^* is a mixed strategy and let S^* be the support of σ^*. The Equality of Payoffs theorem tells us that $\pi(s, \sigma^*) = \pi(\sigma^*, \sigma^*)\ \forall s \in S^*$. Then, in the case of a polymorphic population with $x^* = \sigma^*$, i.e., where $x_j = p_j\ \forall s \in S^*$, we have that

$$\pi(s_i, x^*) = \sum_{j=1}^{k} \pi(s_i, s_j)x_j = \sum_{j=1}^{k} \pi(s_i, s_j)p_j = \pi(s_i, \sigma^*) = k,$$

where k is a constant.

Now,

$$\begin{aligned}
\dot{x}_j &= (\pi(s_j, x^*) - \bar{\pi}(x^*))x_j \\
&= \left(\pi(s_j, x^*) - \sum_{i=1}^{k} x_i \pi(s_j, x^*)\right) x_j \\
&= \left(k - k\sum_{i=1}^{k} x_i\right) x_j \\
&= (k-k)x_j = 0,
\end{aligned}$$

where we have used that $\sum_{i=1}^{k} x_i = 1$. For the strategies s_i that are not in the support of σ^* we have that $x_i = 0$ and hence, as before, $\dot{x}_i = 0$. Then, x^* is a fixed point of the system. □

Therefore, we have proved that $N \subseteq F$.

Observation 3.6. *What we can highlight after having proved the theorem is that, an evolutionary process can "produce Nash equilibriums". This is particularly important in this case because we are treating a population of non-rational individuals which are able to get to a rational behaviour, the Nash equilibrium. The result would not be so surprising if we had assumed that the population is more critical and rational. In that case individuals could only observe other's members behaviour and discard the options that would give them a worse result than the strategy they are actually following.*

Theorem 3.4. *If x^* is an asymptotically stable fixed point of the replicator dynamics, then the symmetric strategy pair (σ^*, σ^*) with $\sigma^* = x^*$ is a Nash equilibrium.*

Proof. First, if x^* is a fixed point with $x_i > 0\ \forall i$, then because of the Equality of Payoffs theorem, the payoff obtained by any pure strategy must be the same. And, with the correspondence of σ^* and x^* we can prove as we did in the previous theorem that

$\pi(s_i, x^*) = \pi(s_i, \sigma^*) = k$, $\forall s_i$ in the set of pure strategies, where k is a constant. Therefore, the pair (σ^*, σ^*) is a Nash Equilibrium, because if any individual deviated from this strategy, he/she would not get a better payoff.

Let's now suppose that S^* is the support of σ^*. Because x^* is a fixed point, we have that $\dot{x}_i = (\pi(s_i, x^*) - \bar{\pi}(x^*))x_i = 0 \ \forall i$. In the case of the strategies that have a non-null probability, we must have $\pi(s_i, x^*) - \bar{\pi}(x^*)$ because $x_i \neq 0$. Also, because of the Equality of Payoffs theorem, $\pi(s_i, x^*) = \pi(s_i, \sigma^*) = k$, $\forall s_i \in S^*$. Let's now suppose that the pair (σ^*, σ^*) is not a NE and we will get to a contradiction. Then, there must be some strategy $s' \notin S^*$ such that $\pi(s'_i, \sigma^*) > \pi(\sigma^*, \sigma^*)$ and consequently $\pi(s', x^*) > \bar{\pi}(x^*)$. Consider a population close x_ϵ to x^* but with a small proportion ϵ of players using s'. Then,

$$\begin{aligned}\dot{\epsilon} &= [\pi(s', x_\epsilon) - \bar{\pi}(x_\epsilon)]\epsilon \\ &= [\pi(s', x^*)(1-\epsilon) + \pi(s', s')\epsilon - \bar{\pi}(x^*)(1-\epsilon) - \bar{\pi}(\epsilon)\epsilon]\epsilon \\ &= [\pi(s', x^*) - \bar{\pi}(x^*)]\epsilon + O(\epsilon^2),\end{aligned}$$

where we have used that $\epsilon << 1$ and then we can ignore the terms with ϵ^n with $n > 1$.

So, the proportion of players using s' increases because $\pi(s', x^*) > \bar{\pi}(x^*)$ and that is a contradiction because we have assumed that x^* is an asymptotically stable fixed point. Then, we have proved that the pair (σ^*, σ^*) is a Nash Equilibrium. □

Observation 3.7. *What we have seen in this case is that if a point is stable locally, then the corresponding strategy leads to a Nash Equilibrium, i.e., a situation where all the players have no incentive to change their behaviour.*

Therefore, we have proved that $A \subseteq N$.

Before we prove the last inclusion, we need to give the definition of a specific type of function that we will use in the proof of the theorem.

Definition 3.3. Let $\dot{x} = f(x)$ be a dynamical system with a fixed point at x^*. Then, let $V(x)$ be a function, defined for all allowable states of the system close to x^*, such that

1. $V(x^*) = 0$
2. $V(x^*) > 0$ for $x \neq x^*$
3. $\frac{dV}{dt} < 0$ for $x \neq x^*$.

This is called a (strict) *Lyapunov function.* If such a function exists, then the fixed point x^* is asymptotically stable.

Theorem 3.5. *Every ESS corresponds to an asymptotically stable fixed point in the replicator dynamics. That is, if σ^* is an ESS, then the population with $x^* = \sigma^*$ is asymptotically stable.*

Proof. If σ^* is an ESS then, by definition, there exists an $\bar{\epsilon}$ such that for all $\epsilon < \bar{\epsilon}$, $\pi(\sigma^*, \sigma_\epsilon) > \pi(\sigma, \sigma_\epsilon)$, $\forall \sigma \neq \sigma^*$, where $\sigma_\epsilon = (1-\epsilon)\sigma^* + \epsilon\sigma'$, $\sigma' \neq \sigma^*$. In particular, this holds for $\sigma = \sigma_\epsilon$, so $\pi(\sigma^*, \sigma_\epsilon) > \pi(\sigma_\epsilon, \sigma_\epsilon)$. This implies that in the replicator dynamics we have, for $x^* = \sigma^*$, $x = (1-\epsilon)x^* + \epsilon x'$ and $\forall \epsilon < \bar{\epsilon}$, $\pi(\sigma^*, x) > \bar{\pi}(x)$.

Now, let's consider the function

$$V(x) = -\sum_{i=1}^{k} x_i^* ln\left(\frac{x_i}{x_i^*}\right).$$

Let's check the conditions described in the previous definition. Clearly, $V(x^*) = 0$. Now, using that the function $ln(x)$ is concave, and therefore $-ln(x)$ is convex, and using that $\sum_{i=1}^{k} x_i = 1$, we have

$$V(x) = -\sum_{i=1}^{k} x_i^* ln\left(\frac{x_i}{x_i^*}\right) \geq -ln\left(\sum_{i=1}^{k} x_i^* \frac{x_i}{x_i^*}\right) = -ln\left(\sum_{i=1}^{k} x_i\right) = -ln(1) = 0.$$

Where the equality is only reached in $x = x^*$. Finally, the time derivative of $V(x)$ along solution trajectories of the replicator dynamics is

$$\begin{aligned}
\frac{d}{dt}V(X) &= \frac{d}{dt}\left(-\sum_{i=1}^{k} x_i^* ln\left(\frac{x_i}{x_i^*}\right)\right) \\
&= -\sum_{i=1}^{k} x_i^* \dot{x}_i \frac{\partial}{\partial x_i} ln\left(\frac{x_i}{x_i^*}\right) \\
&= -\sum_{i=1}^{k} \frac{x_i^*}{x_i} \dot{x}_i \\
&= -\sum_{i=1}^{k} \frac{x_i^*}{x_i} (\pi(s_i, x) - \bar{\pi}(x)) x_i \\
&= -((\pi(\sigma^*, x) - \bar{\pi}(x)) \sum_{i=1}^{k} x_i^* \\
&= -((\pi(\sigma^*, x) - \bar{\pi}(x)) \\
&< 0.
\end{aligned}$$

Where we have used that $\dot{x}_i = (\pi(s_i, x) - \bar{\pi}(x))x_i$ and $\sum_{i=1}^{k} x_i^* = 1$. We have also used that $(\pi(\sigma^*, x) - \bar{\pi}(x)) > 0$, which why previously proved for σ^* ESS in a region near x^* for $x \neq x^*$.

Then, we have seen that $V(x)$ is a Lyapunov function for population profiles close to the fixed point. Consequently, the fixed point x^* is asymptotically stable. □

Observation 3.8. *From this theorem we can derive that there will be cases where we can find asymptotically stable fixed points in the replicator dynamics that do not correspond to an ESS. We will see a classical example of that later.*

So, we have proved that $E \subseteq A$.

In conclusion, we have proved the relationships that we previously established in n-strategy PVP games

$$E \subseteq A \subseteq N \subseteq F.$$

Example 3.4. *Rock-Paper-Scissors*

This game is represented by the following payoffs matrix. When both players use the same strategy they get a payoff of 0 and in the rest of the combinations the winner gets 1 and the loser gets -1. It is a good and simple example of a zero-sum game.

	R	P	S
R	(0,0)	(-1,1)	(1,-1)
P	(1,-1)	(0,0)	(-1,1)
S	(-1,1)	(1,-1)	(0,0)

As we can clearly see, there is no NE in pure strategies. The only NE of this game is the mixed strategy $\sigma = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Then, $NE = \{\sigma\}$.

Let's now write the replicator dynamics system,

$$\begin{cases} \dot{x}_1 = x_1(x_3 - x_2) \\ \dot{x}_2 = x_2(x_1 - x_3) \\ \dot{x}_3 = x_3(x_2 - x_1). \end{cases}$$

Then, as we did in the other example, we will write $x_1 = x$, $x_2 = y$ and $x_3 = 1 - x - y$. Then we can rewrite the previous equations,

$$\mathbb{X}_{RPS} = \begin{cases} \dot{x} = x(1 - x - 2y) \\ \dot{y} = y(-1 + 2x + y). \end{cases}$$

This system has 4 fixed points, $F = \{(0,0), (0,1), (1,0), (\frac{1}{3}, \frac{1}{3})\}$. Let's now write the matrix of partial derivatives,

$$D\mathbb{X}(x, y) = \begin{pmatrix} -2x - 2y + 1 & -2x \\ 2y & 2y + 2x - 1 \end{pmatrix}.$$

Let's evaluate the fixed points in the previous matrix to check its linear type.

$$D\mathbb{X}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$. Then, the origin is a saddle point.

$$D\mathbb{X}(1,0) = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$. Then, this point is a saddle point.

$$D\mathbb{X}(0,1) = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$. Then, this poing is a saddle point.

$$D\mathbb{X}\left(\frac{1}{3}, \frac{1}{3}\right) = \begin{pmatrix} \frac{-1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

There are two complex conjugate eigenvalues, $\lambda_1 = i\sqrt{\frac{1}{3}}$, $\lambda_2 = -i\sqrt{\frac{1}{3}}$. Then, this point in non-hyperbolic and we cannot apply the Hartman-Grobman theorem. To know if this point is either a focus or a center we can try to find a first integral of the system, let's proceed. Remember that our dynamical system is,

$$\begin{cases} \dot{x} = x(1 - x - 2y) \\ \dot{y} = y(-1 + 2x + y) \end{cases}.$$

Then, the orbital equation of the field is $\dot{y}dx + \dot{dy}$, which in our case can be written as

$$y(-1 + 2x + y)dx + x(1 - x - 2y)dy.$$

Now, we have to see if the field is conservative by checking if $\dot{y}dx + \dot{x}dy = 0$, i.e., if $\frac{d\dot{y}}{dy} = -\frac{d\dot{x}}{dx}$.

$$-\frac{d\dot{x}}{dx} = 2(x + y) - 1$$
$$\frac{d\dot{y}}{dy} = 2(x + y) - 1.$$

It is conservative, therefore we can try fo find a first integral of the system, i.e., a function $H(x, y)$ that satisfies the Pfaffian: $\frac{dH}{dx} = -\dot{y}$, $\frac{dH}{dy} = \dot{x}$. To do so, we will integrate the second equation and then derive it to find the integration constant,

$$H(x, y) = \int f(x, y)dy + C(x) = \int x(1 - x - 2y)dy + C(x) = xy - yx^2 - xy^2 + C(x)$$

$$\frac{dH}{dx} = y - 2yx - y^2 + C'(x) = y(1 - 2x - y) + C'(x)$$
$$\frac{dH}{dx} = -\dot{y} \iff C'(x) = 0 \iff C(x) = k$$

where k is a constant. Then, we have that $H(x, y) = xy(1 - x - y) + k$ is a first integral of the system.

Then, after having found a first integral of the system, we can assure that the point $(0, 0)$ is a center of the system

Observation 3.9. *We will just explain in broad terms why the point $(\frac{1}{3}, \frac{1}{3})$ cannot be a focus after having found the first integral. Let's suppose that $P = (\frac{1}{3}, \frac{1}{3})$ is a stable focus and that Q is a point very close to P. Then, the orbit of this point should be attracted to P after some time. At the same time, the orbit of Q lives on one of the contour lines of the first integral. Then, as the point Q is very close to P and the first integral is a continuous function, the images of these two points have to be also very close to each other, i.e., $H(Q) = H(P)$. As we have chosen any point Q that is sufficiently close to the origin, we can find a neighbourhood of the origin where the function H is constant. That is a contradiction, because by definition a first integral is a non-constant function. Then, the singular point $(\frac{1}{3}, \frac{1}{3})$ is a center.*

Having said all this, we can have a look at the phase portrait of the system

$$\mathbb{X}_{RPS} = \begin{cases} \dot{x} = x(1 - x - 2y) \\ \dot{y} = y(-1 + 2x + y) \end{cases}$$

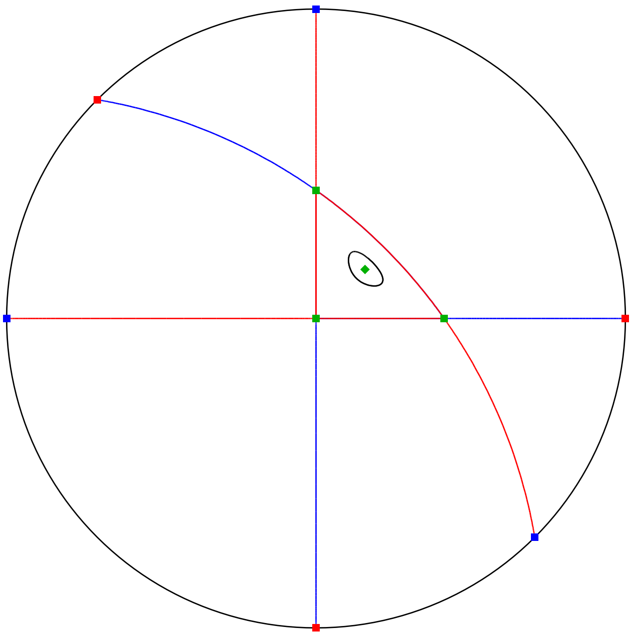

Figure 3.4: phase portrait of the vector field.

In this case we do have singular infinite points that lie in the equator, as we can clearly see in the representation. Remember that depending the on the degree of the vector field, opposite singularities had the same or opposite behaviour. In this case, as the degree of the vector field is $d = 2$ (the maximum degree of the polynomials) the behaviour of the opposite singularities changes. We have 3 stable nodes and their respective unstable nodes for the projections of this points.

Also, as we already proved when we found a first integral for the Hamiltonian system, the program tells us that $(\frac{1}{3}, \frac{1}{3})$ is a weak focus and therefore, we have a center. The other three singular finite points are saddle points.

Appendix A

Python code to solve games

This is the code that we used for the particular example that we developed in the third chapter of this book. To be able to use it in different games, it is just necessary to modify the dynamical system and the partial derivatives matrix. The rest of the code should work without making any more changes. As a reminder, the code is used to find Nash equilibria, fixed points and classify the linear type of the fixed points in PVP games. However, it could also be used in games that are not symmetric.

To create this code, the implementation of several Python libraries is required. These libraries and more complementary information can be found in the references as [4], [5], [6], [10] and [11].

```
#IMPORTING THE NECESSARY MODULES
import numpy as np
import pandas as pd
import sympy
import nashpy as nash
import matplotlib.pyplot as plt
from sympy import symbols, Eq, solve

#HERE WE DEFINE THE PAYOFF MATRICES OF BOTH PLAYERS. THE INPUT CAN BE ANY
N-DIMENSIONAL PAIR OF MATRICES
A = np.array([[0,3,1], [3,0,1], [1,1,1]])
B = np.array([[0,3,1], [3,0,1], [1,1,1]])

#HERE WE COMPUTE THE NE OF THE GAME
rps = nash.Game(A, B)
eqs = rps.support_enumeration()
print(list(eqs))
print()

#HERE WE DEFINE THE REPLICATOR DYNAMICS SYSTEM
x,y = symbols('x y')
eq1 = Eq(x*(x**2 + y**2 - 4*x*y + 2*y - x))
eq2 = Eq(y*(x**2 + y**2 - 4*x*y + 2*x - y))

#NOW WE FIND THE FIXED POINTS BY SOLVING THE SYSTEM
```

```
sol = solve((eq1, eq2), (x,y))
print(sol)

#HERE WE DEFINE THE MATRIX OF PARTIAL DERIVATIVES ASSOCIATED TO THE SYSTEM.
WE ALSO DEFINE ANOTHER MATRIX WHERE WE WILL HAVE THE RESULT OF EVALUATING THE
FIXED POINTS IN THE PARTIAL DERIVATIVES MATRIX
eq1_x = "3*x**2 + y**2 - 8*x*y + 2*y - 2*x"
eq2_x = "2*x*y - 4*y**2 + 2*y"
eq1_y = "2*x*y - 4*x**2 + 2*x"
eq2_y = "3*y**2 + x**2 - 8*x*y + 2*x - 2*y"

der_matrix = []
der_matrix_eval = [[0,0],[0,0]]
col_1 = [eq1_x, eq2_x]
col_2 = [eq1_y, eq2_y]
der_matrix.append(col_1)
der_matrix.append(col_2)

print(der_matrix)
print()

#EVALUATING EACH FIXED POINT IN THE PARTIAL DERIVATIVES MATRIX
for i in range(len(sol)):
    x = sol[i][0]
    y = sol[i][1]
    der_matrix_eval[0][0] = eval(der_matrix[0][0])
    der_matrix_eval[0][1] = eval(der_matrix[0][1])
    der_matrix_eval[1][0] = eval(der_matrix[1][0])
    der_matrix_eval[1][1] = eval(der_matrix[1][1])

    #TRANSFORMING THE MATRIX INTO FLOAT TO USE THE FUNCTION TO FIND EIGENVALUES
    AND EIGENVECTORS
    A = np.array(der_matrix_eval)
    A = A.astype('float64')
    w, v = np.linalg.eig(A) #w eigenvalues, v eigenvectors

    print("Fixed Point = ",sol[i])
    print(der_matrix_eval)

    print("Eigenvalues = ",w)
    if(w[1]*w[0] < 0):
        print("PUNT DE SELLA")
    if(w[1]*w[0] == 0):
        print("NOT HYPERBOLIC")
    if((w[1]*w[0] > 0) and (w[0] > 0)):
        print("UNSTABLE NODE")
    if((w[1]*w[0] > 0) and (w[0] < 0)):
        print("STABLE NODE")
    print("Eigenvectors. Column i corresponds to eigenvalue i:")
```

```
    print(v)
    print()
print()
```

Appendix B

P4&P5 phase portraits

In this appendix we will give more information about each vector field that we have studied and its correspondent phase portrait. The extra information if part of the output obtained by using the program P4&P5 that was created by "Grup de sistemes dinàmics de la UAB" and that is referenced in the bibliography, [1].

Regarding the phase portrait of figure 3.1:

```
LET US STUDY THE DIFFERENTIAL SYSTEM:
    x' = x*(x^2-4*x*y+y^2-x+2*y)
    y' = y*(x^2-4*x*y+y^2+2*x-y)
AT THE FINITE REGION
(0.,0.) is a non-elementary point.


The invariant separatrices are: (t>0)
[01]: [0, -1.*t] : unstable
hyperbolic sector
[02]: [t, 0] : stable
hyperbolic sector
[03]: [t, 1.*t] : unstable
hyperbolic sector
[04]: [0, t] : stable
hyperbolic sector
[05]: [-1.*t, 0] : unstable
hyperbolic sector
[06]: [-1.*t, -1.*t] : stable
hyperbolic sector
The singularity (0.,0.) has index -2
Sector representation (counterclockwise):
   |       |       |       |       |       |
   ^  HYP  v  HYP  ^  HYP  v  HYP  ^  HYP  v  HYP
   |       |       |       |       |       |
->-*-<- -<-*->- ->-*-<- -<-*->- ->-*-<- -<-*->-
  01      02      03      04      05      06

########################################################
```

```
(0.,1.) is an unstable node.
########################################################
(1.,0.) is an unstable node.
########################################################
(.500000000000000,.500000000000000) is a stable node.
########################################################
```

Regarding the phase portrait of figure 3.2:

```
LET US STUDY THE DIFFERENTIAL SYSTEM:
    x' = x*(-x^2-4*x*y-y^2+x+2*y)
    y' = y*(-x^2-4*x*y-y^2+2*x+y)
AT THE FINITE REGION
(0.,0.) is a non-elementary point.


The invariant separatrices are: (t>0)
[01]: [0, -1.*t] : stable
elliptic sector
[02]: [t, 0] : unstable
parabolic sector
[03]: [t, 1.*t] : unstable
parabolic sector
[04]: [0, t] : unstable
elliptic sector
[05]: [-1.*t, 0] : stable
parabolic sector
[06]: [-1.*t, -1.*t] : stable
parabolic sector
The singularity (0.,0.) has index 2
Sector representation (counterclockwise):
   |       |       |       |       |       |
   v  ELL  ^  PAR  ^  PAR  ^  ELL  v  PAR  v  PAR
   |       |       |       |       |       |
->-*-<- -<-*->- ->-*-<- -<-*->- ->-*-<- -<-*->-
  01      02      03      04      05      06

########################################################
(0.,1.) is a saddle point.
########################################################
(1.,0.) is a saddle point.
########################################################
(.500000000000000,.500000000000000) is a stable node.
########################################################
```

Regarding the phase portrait of figure 3.3:

```
LET US STUDY THE DIFFERENTIAL SYSTEM:
    x' = x*(-3*x^2-4*x*y-3*y^2+3*x+2*y)
    y' = y*(-3*x^2-4*x*y-3*y^2+2*x+3*y)
```

```
AT THE FINITE REGION
(0.,0.) is a non-elementary point.


The invariant separatrices are: (t>0)
[01]: [0, -1.*t] : stable
hyperbolic sector
[02]: [t, 0] : unstable
parabolic sector
[03]: [t, 1.*t] : unstable
parabolic sector
[04]: [0, t] : unstable
hyperbolic sector
[05]: [-1.*t, 0] : stable
parabolic sector
[06]: [-1.*t, -1.*t] : stable
parabolic sector
The singularity (0.,0.) has index 0
Sector representation (counterclockwise):
   |       |       |       |       |       |
   v  HYP  ^  PAR  ^  PAR  ^  HYP  v  PAR  v  PAR
   |       |       |       |       |       |
-<-*->- ->-*-<- -<-*->- ->-*-<- -<-*->- ->-*-<-
  01      02      03      04      05      06

########################################################
(0.,1.) is a stable node.
########################################################
(1.,0.) is a stable node.
########################################################
(.500000000000000,.500000000000000) is a saddle point.
########################################################
```

Regarding the phase portrait of figure 3.4:

```
LET US STUDY THE DIFFERENTIAL SYSTEM:
    x' = x*(1-x-2*y)
    y' = y*(-1+2*x+y)
AT THE FINITE REGION
(0.,0.) is a saddle point.
########################################################
(0.,1.) is a saddle point.
########################################################
(1.,0.) is a saddle point.
########################################################
(.333333333333333,.333333333333333) is a weak focus.
Since the system is hamiltonian, we have a center.
########################################################
```

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