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BRIEF REPORTS

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Reentrant transition induced by multiplicative noise in the time-dependent Ginzburg-Landau model

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An effect of multiplicative noise in the time-dependent Ginzburg-Landau model is reported, namely, that noise at a relatively low intensity induces a phase transition towards an ordered state, whereas strong noise plays a destructive role, driving the system back to its disordered state through a reentrant phase transition. The phase diagram is calculated analytically using a mean-field theory and a more sophisticated approach and is compared with the results from extensive numerical simulations. [S1063-651X(96)06412-4]

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In recent years, it has been demonstrated that noise can play a constructive, ordering role in nonequilibrium situations through its interaction with the nonlinearities of a system [1,2]. A first example is the case of stochastic resonance [2,3], in which the signal-to-noise ratio of the system reaches a maximum for a given value of the noise intensity. Another example is the pattern-forming transition (from a homogeneous phase to roll structures) controlled by multiplicative noise in the Swift-Hohenberg model [4,5]. A disorder-order transition induced by multiplicative noise has also been found in the Ginzburg-Landau model [5–7].

Recently, a spatially extended model has been reported that exhibits a genuine noise-induced nonequilibrium phase transition [8]. In this case another interesting phenomenon has been observed: once the noise-induced transition has brought the system to an ordered state, a reentrant second-order phase transition takes place at a higher value of the multiplicative noise intensity, disordering the system again. In other words, the order parameter is exactly zero for small and large noise intensities, but nonzero and going through a maximum in a window of intermediate intensities. This phenomenon resembles stochastic resonance in that the order-producing effect of noise is optimal for a specific intermediate value of the noise intensity. For larger values of the intensity, the noise resumes its more familiar order-destroying role. In this Brief Report we present another example of such a phenomenon, namely, the existence of a reentrant transition under the influence of multiplicative noise in the time-dependent Ginzburg-Landau model [9], which is a generic model describing phase transitions and critical phenomena in both equilibrium and nonequilibrium situations.

The time-dependent Ginzburg-Landau model is described by the field equation

$$\dot{\psi}(\mathbf{r},t)=-\alpha\psi(\mathbf{r},t)-\psi(\mathbf{r},t)^3+D\nabla\psi(\mathbf{r},t)+\xi_a(\mathbf{r},t), \quad (1)$$

where $\psi(\mathbf{r},t)$ is a scalar field and $\xi_a(\mathbf{r},t)$ is a spatially uncorrelated Gaussian white noise that accounts for the thermal fluctuations in the system. For simplicity, we consider a discretized version of the Ginzburg-Landau model [6,7,10] on a square lattice. We assume that the control parameter, i.e., the coefficient α of the linear term, is subject to fluctuations that are also white in space and time. The system is then described by the scalar field variable $\psi(\mathbf{r},t)$, with $\mathbf{r}$ defined on a square lattice, obeying the coupled set of stochastic differential equations

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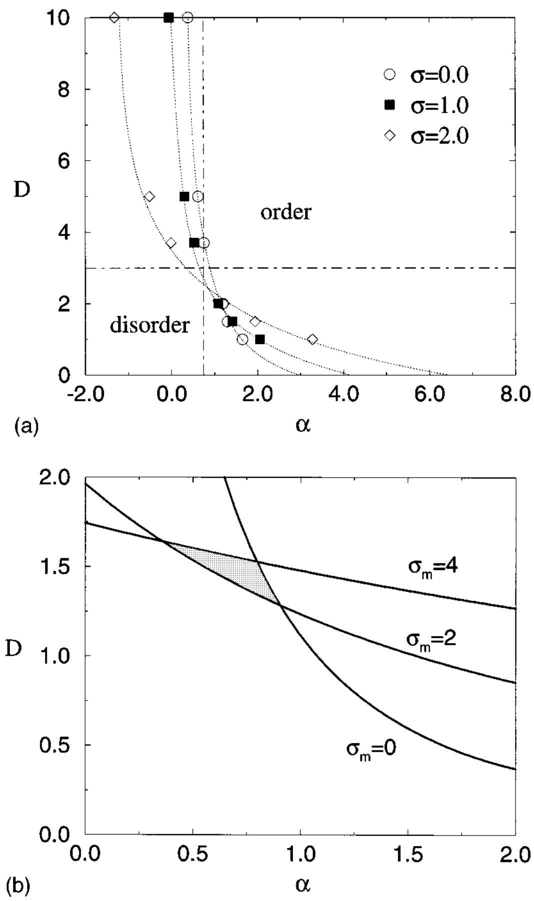

FIG. 1. Phase diagram in the D vs α plane. (a) Numerical results and (b) mean-field theory. The dark region in (b) illustrates the occurrence of a reentrant phase transition.

$$\dot{\psi}(\mathbf{r},t) = -[\alpha + \xi_m(\mathbf{r},t)]\psi(\mathbf{r},t) - \psi(\mathbf{r},t)^3 + \frac{D}{2d}\sum_{n(\mathbf{r})} [\psi(n(\mathbf{r}),t) - \psi(\mathbf{r},t)] + \xi_a(\mathbf{r},t). \quad (2)$$

The sum over $n(\mathbf{r})$ runs over the nearest neighbors of $\mathbf{r}$ and the corresponding term stands for the discretized form of the diffusion operator. $\xi_a(\mathbf{r},t)$ and $\xi_m(\mathbf{r},t)$ represent independent Gaussian white noises with zero mean value and correlation

$$\langle \xi_{a,m}(\mathbf{r},t)\xi_{a,m}(\mathbf{r}',t')\rangle = \sigma_{a,m}^2 \delta_{\mathbf{r},\mathbf{r}'}\delta(t-t'). \quad (3)$$

The additive noise term $\xi_a(\mathbf{r},t)$ models the presence of thermal fluctuations, while the multiplicative noise $\xi_m(\mathbf{r},t)$ represents the effect of a parametric noise on the control variable α. They will be interpreted according to the Stratonovich calculus [1]. We have chosen the coefficient of the third-power term and the intensity of the additive noise σ_a to be equal to 1 in Eq. (2). This can always be achieved by an appropriate choice of the units for ψ and t (see Ref. [10] for a review of other parametrizations).

Even in the absence of the multiplicative noise $\sigma_m^2 = 0$, the exact phase diagram for the Ginzburg-Landau model is not known analytically, but the location of the phase boundary between the disordered phase $\langle \psi(\mathbf{r},t)\rangle = 0$ and the ordered phase $\langle \psi(\mathbf{r},t)\rangle \neq 0$ has been evaluated through extensive numerical simulations [10]. We have simulated the model defined by Eqs. (2) and (3) on a cubic lattice in spatial dimension $d=2$, using Heun's method, for system sizes up to 50×50, and have obtained the phase boundaries when the multiplicative noise is present. Our results are plotted in Fig. 1(a). The qualitative form of this boundary can be obtained theoretically on the basis of a Weiss mean-field theory, which replaces in (2) the value of the field at $n(\mathbf{r})$ by its mean density [6,11], and is reproduced in Fig. 1(b). The mean-field theory predicts that the location of the critical-α value, at which order sets in, is shifted by the multiplicative noise in a nontrivial way: for a large value of the spatial coupling D, the transition is advanced, while it is delayed for small values of D. A closer inspection of the figure suggests that the transitions can be reentrant as a function of the intensity of the multiplicative noise. For example, those points within the dark region in Fig. 1(b) belong, according to the mean-field theory, to the disordered phase for $\sigma_m = 0$ and $\sigma_m = 4$, but to the ordered region for $\sigma_m = 2$. The main purpose of this paper is to confirm this somewhat surprising existence of reentrant transitions by extensive simulations.

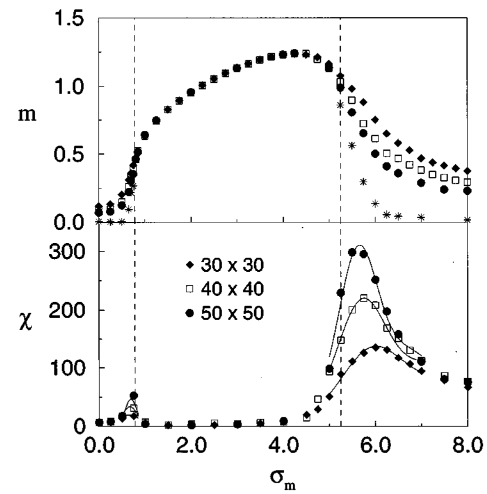

FIG. 2. Order parameter m and relative fluctuations χ as a function of the intensity of the external noise σ_m for $\sigma_a = 1$, $\alpha = 0.75$, $D=3$, and different lattice sizes. The stars in the upper figure are an extrapolation of the finite-size results and the vertical lines are the estimates of the critical boundaries.

The main results are collected in Figs. 1(a) and 2. Fig. 1(a) represents the phase boundary for three different values of the intensity of the multiplicative noise. The ordered and disordered regions are separated by a line of critical points. The figure confirms the general tendencies predicted by the mean-field theory, cf. Fig. 1(b). It also serves as a guideline for identifying the parameter values at which a reentrant transition occurs, while at the same time taking into account accuracy and computation limits of the simulations. The best result was obtained for $D=3, \alpha=0.75$, corresponding to the point where the two straight lines in Fig. 1(a) intersect. This point belongs to the disordered region for $\sigma_m = 0$ and lies in the ordered region for the values $\sigma_m = 1$ and $\sigma_m = 2$. How-

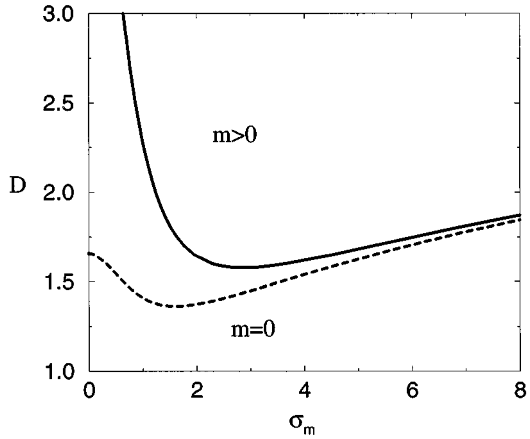

FIG. 3. Phase diagram in the (D,σ_m) plane for $\sigma_a=1$ and $\alpha=0.75$: mean-field theory (dashed line) and CFA (solid line). In the CFA the line corresponds to those points where the disorder solution $m=0$ becomes unstable.

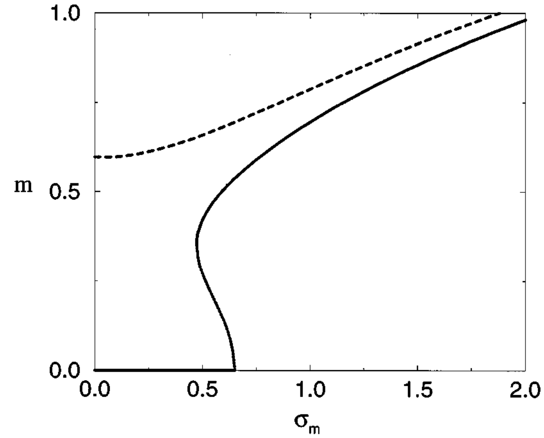

FIG. 4. Order parameter m as a function of the intensity of the external noise σ_m, for $\sigma_a=1$, $\alpha=0.75$, and $D=3$: mean-field theory (dashed line) and CFA (solid line for the stable branches and dotted line for the unstable branch). Note that the CFA predicts a first-order phase transition.

ever, in view of the shape of the critical lines for increasing σ_m, one can expect that the point will again belong to a disordered region for slightly higher values of σ_m. To verify this expectation, we have measured the order parameter m as a function of the noise intensity. m is defined as usual [10] in terms of the field variable $\psi(\mathbf{r},t)$,

$$m=\frac{\langle\Psi\rangle}{L^2}, \tag{4}$$

where $\Psi=|\Sigma_{\mathbf{r}}\psi(\mathbf{r},t)|$, with the sum running over the entire lattice, and L^2 is the number of sites. The relative fluctuations of the order parameter have also been evaluated:

$$\chi=\frac{\langle\Psi^2\rangle-\langle\Psi\rangle^2}{L^2}. \tag{5}$$

In a disorder-order transition we expect that m will be very small in the disordered region, and going to zero as the system size increases, while it converges to a finite nonzero value, almost independent of the system size, in the ordered phase. In analogy to equilibrium critical phenomena, the transition can also be characterized by a singularity of the relative fluctuations χ. As is clear from Fig. 2, these critical properties are observed twice, once at an entrant transition, estimated numerically by a finite-size analysis to be located at $\sigma_m\simeq0.78$, and again at the reentrant transition, taking place at $\sigma_m\simeq5.25$.

Figure 3 presents analytical results for the phase diagram in the (D,σ_m) plane for $\alpha=0.75$ obtained by using the mean-field theory and the correlation function approach (CFA) introduced in [8], whereas Fig. 4 shows the order parameter m as a function of the intensity of the noise σ_m for $D=3$ and $\alpha=0.75$. In Fig. 3, the plotted transition line for the CFA corresponds to those points where the disorder solution $m=0$ becomes unstable. Notice that in the absence of external noise ($\sigma_m=0$) and in the limit $D\to\infty$, the mean-field approximation, which is exact in this limit, predicts the system to be in the ordered phase for any positive α [6]. Consequently, the transition line in the (D,σ_m) plane (see Fig. 3) starts at a finite value of D, $D_c(\sigma_m=0)$. For $\alpha=0.75$, the mean-field theory predicts $D_c(\sigma_m=0)\simeq1.66$ below $D=3$ and therefore it does not account for the *entrant* transition, as is also clear from Fig. 4. On the other hand, in the CFA $D_c(\sigma_m=0)$ is above $D=3$ and this theory gives a critical value $\sigma_m\simeq0.645$ at the entrant transition, which is not too far from the observed value of 0.70. Note, however, that the theory predicts a first order transition (jump to a finite value of m) with hysteresis effects (see Fig. 4), which are not observed in the simulations. We expect that this is an artifact of the approximations involved in the theoretical approach, but further numerical and theoretical work is needed to settle this issue. For strong noise, both mean-field theory and the CFA present large deviations from the numerical results (cf. Figs. 2 and 4).

In conclusion, we have presented clear evidence of a reentrant phase transition in the Ginzburg-Landau model induced by parametric noise on its control parameter, thereby demonstrating once more the dual role of the noise intensity as an order-producing and order-destroying control parameter. These noise-induced reentrant transitions are not particular to this model and can be expected to occur also in pattern-forming transitions [4,12] (in this case from roll structures to an homogeneous phase).

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[1] W. Horsthemke and R. Lefever, *Noise-Induced Transitions* (Springer-Verlag, Berlin, 1984).
[2] *Noise in Nonlinear Dynamical Systems*, edited by F. Moss and P.V.E. McClintock (Cambridge University Press, Cambridge, 1989).
[3] J.F. Lindner, B.K. Meadows, W.L. Ditto, M.E. Inchiosa, and A.R. Bulsara, Phys. Rev. Lett. **75**, 3 (1995).
[4] J. García-Ojalvo, A. Hernández-Machado, and J.M. Sancho, Phys. Rev. Lett. **71**, 1542 (1993).
[5] A. Becker and L. Kramer, Phys. Rev. Lett. **73**, 955 (1994).
[6] C. Van den Broeck, J.M.R. Parrondo, J. Armero, and A. Hernández-Machado, Phys. Rev. E **49**, 2639 (1994).
[7] P. Luque, J. García-Ojalvo, and J.M. Sancho, in *Fluctuation Phenomena: Disorder and Nonlinearity*, edited by A.R. Bishop, S. Jiménez, and L. Vázquez, (World Scientific, Singapore, 1995), p. 75.
[8] C. Van den Broeck, J.M.R. Parrondo, and R. Toral, Phys. Rev. Lett. **73**, 3395 (1994).
[9] S.K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, Reading, MA, 1976).
[10] R. Toral and A. Chakrabarti, Phys Rev. B **42**, 2445 (1990).
[11] A.D. Bruce, Adv. Phys. **29**, 111 (1980).
[12] J. M. R. Parrondo, C. Van den Broeck, J. Buceta, and J. de la Rubia, Physica A **224**, 153 (1996).