On Convexity in Cooperative Games with Externalities

J.M. Alonso-Meijide[1], M. Álvarez-Mozos[2], M.G. Fiestras-Janeiro[3], and A. Jiménez-Losada[4]

[1] *Departamento de Estadística, Análisis Matemático y Optimización, Universidade de Santiago de Compostela, Spain.*

[2] *Departament de Matemàtica Econòmica, Financera i Actuarial and BEAT, Universitat de Barcelona, Spain.*

[3] *Departamento de Estatística e Investigación Operativa, Universidade de Vigo, Spain.*

[4] *Departamento de Matemática Aplicada II, Universidad de Sevilla, Sevilla, Spain.*

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Abstract

We introduce new notions of superadditivity and convexity for games with coalitional externalities. We show parallel results to the classic ones for transferable utility games without externalities. In superadditive games the grand coalition is the most efficient organization of agents. The convexity of a game is equivalent to having non decreasing contributions to larger embedded coalitions. We also see that convex games can only have negative externalities.

Keywords: Externalities; Superadditivity; Convexity; Contribution; Partition function; Lattice

1 Introduction

Cooperative game theory provides tools to study situations in which the coalitions are the main actors. In a cooperative game the details of the underlying interaction among players are omitted to build a robust model. The focus is on what coalitions will emerge and how to share the benefits of the cooperation. Even if these games are as old as game theory itself[1] their applications to economics have not been as successful as the ones of their non-cooperative counterpart (Maskin, 2016). The fact that, traditionally externalities have been overlooked in the literature may be a reason. Indeed, externalities are present in most economic examples where coalitions are the fundamental elements. For instance, when firms merge in a cartel or after a takeover bid, the expected profit will depend on the potential merging carried out by the rest of firms in the market. Jelnov and Tauman (2009) study a Cournot market where there is a patent holder using games with externalities. The cooperative game is defined using the equilibrium payoffs in a strategic game. The management of fishing resources is another problem where coalitions play a significant role. Liu et al. (2016) study the Norwegian spring-spawning herring fishing using a three players game, Norway, Russia and the union of Iceland, the Faroe Islands and the EU. In their model, what a player can get depends on weather the other two players reach an agreement or not. The focus is on the stability of the grand coalition. There is a large literature studying river sharing problems (Ambec and Sprumont, 2002) from a cooperative perspective. When agents utility functions have a satiation point, externalities across coalitions emerge naturally (Ambec and Ehlers, 2008). van den Brink et al. (2012) formalize a model where the utility that a coalition can get from water consumption depends on the whole coalition structure.

Thrall and Lucas (1963) introduced games in partition function form to describe situations in which coalitions generate externalities on one another. In this model, the main ingredient are not just coalitions but embedded coalitions, that consist of a coalition and a partition of the rest of agents. This enables a coalition to

[1]Their origin dates back to Von Neumann and Morgenstern (1944).

have different values depending on what partition it is embedded in. More recently, many important contributions have been published, most of them focusing on the problem of how to share the benefits of the cooperation. For instance, Macho-Stadler et al. (2007), de Clippel and Serrano (2008), McQuillin (2009), and Dutta et al. (2010) address the issue of how to extend the Shapley value and Kóczy (2007) and Bloch and van den Nouweland (2014) propose generalizations of the core to games with externalities. Fewer papers have explored the properties of the game itself. Hafalir (2007) points out that extending the classic properties of superadditivity and convexity is not a trivial task. He shows that superadditivity, as defined by Maskin (2003) is not a sufficient condition for the efficiency of the grand coalition in situations with negative externalities. Abe (2016) proposes alternative definitions of superadditivity that do the work when externalities are either positive or negative. Hafalir (2007) also introduced a notion of convexity that guarantees that the grand coalition is the most efficient configuration. With a different purpose Abe (2020) introduced another notion of convexity, logically independent to the previous one. A different branch of the literature follows a non-cooperative approach to study situations with coalitional externalities. For instance, Ray and Vohra (1999) use an extensive form bargaining game to find out the coalition structures that are likely to arise.

Here, we rely on a partial order among embedded coalitions implicitly defined by de Clippel and Serrano (2008). Alonso-Meijide et al. (2017) analyze the set of embedded coalitions endowed with this partial order and show that it has a lattice structure. Then, it is very natural to interpret the supremum and the infimum of two embedded coalitions as their union and intersection, respectively. The supremum is obtained taking the union of the coalitions and the intersection of the partitions, more precisely their infimum in the lattice of partitions of a finite set. That is, the two coalitions whose worth is being evaluated are merged while the rest of agents form the partition obtained by keeping the divisions of the two original partitions. The infimum works just the other way around, intersection of coalitions and union of partitions, which results in only keeping the divisions in which the two partitions agree. These operations allow us to generalize the classic definitions of superadditivity and convexity to games with externalities in a natural way.

To start with, we see that our properties imply the superadditivity proposed by Maskin (2003) and the convexity studied by Hafalir (2007). Our main result is the characterization of convexity through a condition that requires the contributions to embedded coalitions to be non decreasing with respect to their size. To define what is a contribution to an embedded coalition in a game with externalities we use the lattice structure again. Alonso-Meijide et al. (2019) employ these contributions to build a super family of Shapley values that contains the ones proposed in the previous references. Some intermediate results that we use are interesting on their own. For instance, we show that a convex game can only have negative externalities. Which means that coalitions' worth decrease when the partition of the complement becomes coarser. Finally, we also obtain some interesting implications of our property with respect to certain core notions.

The rest of the paper is organized as follows. Section 2 presents some discrete mathematical terms that we will employ. Then, the partial order among embedded coalitions in which we ground our results is introduced and some of the results of Alonso-Meijide et al. (2017) are adapted to our framework. In Section 3 we introduce our notions of superadditive and convex game with externalities and discuss their implications. Next, we present some interesting lemmata followed by our main result. Section 4 features some additional results on the cores of convex games with externalities. Finally, Section 5 concludes with a comparison of the different notions of convexity in the literature. The proofs and the examples are relegated to the Appendix.

2 A lattice of embedded coalitions

Let $(L, \leq)$ be a partially ordered set, with L being a finite set and $x, y \in L$.[2] The *supremum*, denoted by $x \vee y$, is the unique element of L such that $x, y \leq x \vee y$ and if $z \in L$ is such that $z \geq x, y$, then $z \geq x \vee y$. The *infimum*, denoted by $x \wedge y$, is the unique element of L such that $x \wedge y \leq x, y$ and if $z \in L$ is such that $z \leq x, y$, then $z \leq x \wedge y$.[3] A *finite lattice* is a finite partially ordered set in which every pair of elements have supremum and infimum. From now on, we assume that $(L, \leq)$ is a finite lattice.

A key notion for our paper is the covering relation. We say that x is *covered* by y or y *covers* x if $x < y$ and there is no $z \in L$ such that $x < z < y$. A *chain* $\mathcal{C}$ (between x_0 and x_k) is an ordered subset of L, $\mathcal{C} = \{x_0, x_1, \ldots, x_k\}$ such that x_{l+1} covers x_l, for every $l = 0, \ldots, k-1$. If $x \leq y$, we denote by $[x, y]_L$ the set of elements $z \in L$ such that $x \leq z \leq y$. If no confusion arises, we may just write $[x, y]$. Notice that $[x, y]$ is also a lattice (see for instance Topkis, 1998). A lattice is graded if all the chains between the infimum of L and a given element, $x \in L$, have the same cardinality, the *height of* x. We say that $(L, \leq)$ is *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, for every $x, y, z \in L$. $(L, \leq)$ is *lower semimodular* if whenever $x \vee y$ covers both x and y, then both x and y cover $x \wedge y$, for every $x, y \in L$. $(L, \leq)$ is *semimodular* or *upper semimodular* if whenever both x and y cover $x \wedge y$, then $x \vee y$ covers both x and y, for every $x, y \in L$.

The classic notion of convexity (Shapley, 1971) corresponds to the supermodularity of the characteristic function, which is a real function on the Boolean lattice of subsets. In general, a real function on $(L, \leq)$, f, is said to be supermodular (submodular) if for every $x, y \in L$, $f(x) + f(y) \leq (\geq) f(x \wedge y) + f(x \vee y)$.

Let N be a finite set, $n = |N|$, $S \subseteq N$, and $i \in N$. We denote $S \cup \{i\}$ by $S \cup i$ and $S \setminus \{i\}$ by $S \setminus i$. The family of partitions of N is denoted by $\Pi(N)$. Let, $P \in \Pi(N)$.

[2] We write $x = y$ if $x \leq y$ and $y \leq x$. Also, $x < y$ means that $x \leq y$ but $x \neq y$.

[3] The definition of supremum and infimum is extended to any finite subset of elements of L in the usual way.

We denote by $|P|$ the number of non-empty elements of P, that we also call blocks. The partition P_{-S} of $N \setminus S$ is given by $\{T \setminus S : T \in P\}$. The partition of singletons of S, $\{\{i\} : i \in S\}$, is denoted by $\lfloor S \rfloor$ and the partition of S in one block, $\{S\}$, is denoted by $\lceil S \rceil$. If $P \in \Pi(N \setminus i)$, we also denote $\{\{i\}\} \cup P$ by $\{i\} \cup P$. A well-known partial order on $\Pi(N)$ is the following:

$$P \preceq Q \text{ if and only if for every } S \in P \text{ there is some } T \in Q \text{ such that } S \subseteq T.$$

It is known that $(\Pi(N), \preceq)$ is a semimodular lattice. The height of an element, $P \in \Pi(N)$ is given by $r(P) = n - |P|$. If $P, Q \in \Pi(N)$, we denote by $P \bigwedge Q$ the infimum of P and Q; the supremum of P and Q is denoted by $P \bigvee Q$.

An *embedded coalition* of N is a pair $(S; P)$ with $S \subseteq N$ and $P \in \Pi(N \setminus S)$, i.e., $\{S\} \cup P \in \Pi(N)$. In particular, $(\emptyset; P)$ with $P \in \Pi(N)$ is also an (empty) embedded coalition. If all agents form the grand coalition we write $(N; \emptyset)$. That is, we consider that $\emptyset$ is the only partition in $\Pi(\emptyset)$. For simplicity we denote by $(S; N \setminus S)$ the embedded coalition $(S; \lceil N \setminus S \rceil)$, for every $S \subseteq N$. The family of all embedded coalitions of N is denoted by $\mathcal{EC}^N$.

Alonso-Meijide et al. (2017) studied the partial order outlined in de Clippel and Serrano (2008) over the set $(\mathcal{EC}^N \setminus \{(\emptyset; P) : P \in \Pi(N)\}) \cup \{\bot\}$, being $\bot$ a fictitious infimum. Here we consider this partial order over the whole set $\mathcal{EC}^N$. It is convenient to extend some of the results in Alonso-Meijide et al. (2017) to this framework. Next, we introduce the partial order formally.

Definition 2.1. *Let $(S; P), (T; Q) \in \mathcal{EC}^N$. We define the* inclusion *among embedded coalitions as follows:*[4]

$$(S; P) \sqsubseteq (T; Q) \text{ if and only if } S \subseteq T \text{ and } Q \preceq P_{-T}. \tag{1}$$

The inclusion relation describes two ways an embedded coalition can become larger. On the one hand, some agents could join the coalition. On the other hand,

[4] As usual, $(S; P) \sqsubset (T; Q)$ means that $(S; P) \sqsubseteq (T; Q)$ and $(S; P) \neq (T; Q)$.

the partition of the complement could become finer. Sometimes it may be convenient to use an equivalent formulation of the condition in Eq. (1):

$$(S;P) \sqsubseteq (T;Q) \text{ if and only if } S \subseteq T \text{ and } Q \cup \lfloor T \setminus S \rfloor \preceq P. \tag{2}$$

In the remainder of this section we describe some properties of the algebraic structure $\left(\mathcal{EC}^N, \sqsubseteq\right)$.

Proposition 2.1. *Let* $(S;P)$, $(T;Q) \in \mathcal{EC}^N$. *Then,*

1. $(S;P) \vee (T;Q) = (S \cup T; P_{-T} \bigwedge Q_{-S})$.
2. $(S;P) \wedge (T;Q) = (S \cap T; M)$, *with* $M = (P \cup \lfloor S \setminus T \rfloor) \bigvee (Q \cup \lfloor T \setminus S \rfloor)$.

From Proposition 2.1 we conclude that $\left(\mathcal{EC}^N, \sqsubseteq\right)$ is a lattice. The infimum of this structure is $(\emptyset; N)$ and the supremum is $(N; \emptyset)$. However, the lattice is not distributive as Example 5.1 shows.

Proposition 2.2. *The lattice* $\left(\mathcal{EC}^N, \sqsubseteq\right)$ *is graded, the height of any* $(S;P) \in \mathcal{EC}^N$ *is given by* $h(S;P) = |P| + 2|S| - 1$.

Notice that the height of every embedded coalition $(S;P) \in \mathcal{EC}^N$ can be described by means of the height of S in the Boolean lattice, $|S|$, and the height of $P \cup \lfloor S \rfloor$ in the partition lattice, $r(P \cup \lfloor S \rfloor)$ as follows:

$$h(S;P) = n - 1 - r(P \cup \lfloor S \rfloor) + |S|. \tag{3}$$

Since $(\Pi(N), \preceq)$ is a graded and semimodular lattice, the height is a submodular function. This fact and Equation (3) are used to prove the following result.

Proposition 2.3. *Let* $(S;P), (T;Q) \in \mathcal{EC}^N$. *Then,*

$$h((S;P) \vee (T;Q)) - h(T;Q) \geq h(S;P) - h((S;P) \wedge (T;Q)). \tag{4}$$

Given a pair of embedded coalitions, the length of the chains between one of them and their supremum is greater or equal the length of the chains between their infimum and the other one. Note that this implies that $\left(\mathcal{EC}^N, \sqsubseteq\right)$ is a lower semimodular lattice.

3 Superadditiviy and convexity

In this section, we extend some of the most important properties of a classic game to situations with coalitional externalities. Let N be a finite set. A *game (with externalities)* with player set N is defined by a partition function $v : \mathcal{EC}^N \longrightarrow \mathbb{R}$ such that $v(\emptyset; P) = 0$, for every $P \in \Pi(N)$. We denote by $\mathcal{G}^N$ the class of all games with player set N. Any partition function v satisfying $v(S; P) = v(S; Q)$, for every $S \subseteq N$ and $P, Q \in \Pi(N \setminus S)$ is called a *classic game*.[5] To begin with, we introduce the notion of superadditive game with externalities inspired by the inclusion relation studied in Section 2.

Definition 3.1. *Let $v \in \mathcal{G}^N$. We say that v is* superadditive *if and only if*

$$v((S; P) \vee (T; Q)) \geq v(S; P) + v(T; Q),$$

for every $(S; P), (T; Q) \in \mathcal{EC}^N$ such that $S \cap T = \emptyset$.

That is, for every pair of embedded coalitions whose intersection is an empty one, the worth of their supremum in $(\mathcal{EC}^N, \sqsubseteq)$ is greater or equal to the joint worths of the two embedded coalitions. Recall that $(S; P) \vee (T; Q) = (S \cup T; P_{-T} \bigwedge Q_{-S})$. In other words, if we evaluate the worths of two disjoint coalitions, each embedded in an arbitrary partition, this amount is weakly less than the worth of the union of the two coalitions embedded in the partition obtained by keeping all the divisions in the original partitions.

Definition 3.1 extends the classic notion of superadditivity of a game without externalities. Example 5.2 shows that this extension is not trivial as there are superadditive games which are not classic games.

[5] A game in characteristic function.

An important property of a game with externalities is the efficiency of the grand coalition. Let $v \in \mathcal{G}^N$. We say that v is *efficient* for the grand coalition if for every $P \in \Pi(N)$,

$$\sum_{S \in P} v(S; P_{-S}) \leq v(N; \emptyset).$$

It is easy to check that if a game is superadditive, then it is also efficient for the grand coalition. Hafalir (2007) points out that this fact does not happen with Maskin's definition of superaditivity (Maskin, 2003): $v \in \mathcal{G}^N$ is superadditive if for every $S, T \subseteq N$ with $S \cap T = \emptyset$ and $P \in \Pi(N \setminus (S \cup T))$,

$$v(S \cup T; P) \geq v(S; \lceil T \rceil \cup P) + v(T; \lceil S \rceil \cup P).$$

It is clear that any superadditive game in our sense is also a superadditive game in Maskin's sense, but the reverse does not hold (see Example 5.3).

We also compare our definition of superadditivity with *optimistic superadditivity* (Optimistic-SA) as defined by Abe (2016). A game $v \in \mathcal{G}^N$ is optimistic-SA if the associated optimistic game, v_{max}, is superadditive in the classic sense. The optimistic game is defined for every $S \subseteq N$ by $v_{max}(S) = \{v(S; P) : P \in \Pi(N \setminus S)\}$. It is easy to check that our notion of superadditivity implies optimistic-SA. Nevertheless, as Example 5.4 shows, the two notions are not equivalent.

Next, we formulate our notion of convexity for games with externalities as the supermodularity of a real function on the lattice $\left(\mathcal{EC}^N, \sqsubseteq\right)$.

Definition 3.2. *Let $v \in \mathcal{G}^N$. We say that v is* convex *if for all $(S; P), (T; Q) \in \mathcal{EC}^N$*

$$v((S; P) \vee (T; Q)) + v((S; P) \wedge (T; Q)) \geq v(S; P) + v(T; Q) \tag{5}$$

That is, for every pair of embedded coalitions, the sum of their worths is less than or equal to the sum of the worths of their supremum and infimum in $\left(\mathcal{EC}^N, \sqsubseteq\right)$. It is a very natural generalization of the classic definition (Shapley, 1971) if the supremum and infimum in $\left(\mathcal{EC}^N, \sqsubseteq\right)$ are understood as the union and intersection of embedded coalitions, respectively. As it happens when there are no externalities, any convex game is a superadditive game. In the literature there are several definitions of convexity for games with externalities. An important conceptual difference of our property with respect to others in the literature is the fact that it applies to coalitions which are embedded in potentially different partitions. In a sense, we evaluate worths of coalitions that can have different expectations on how the complementary coalition will be organized. Example 5.5 shows that this extension is not trivial.

Let us review the convexity notion of Hafalir (2007) and analyze its relationship with Definition 3.2. The game $v \in \mathcal{G}^N$ is *Hafalir convex* if and only if

$$v(S \cup T; P) + v(S \cap T; P \cup \lceil S \setminus T \rceil \cup \lceil T \setminus S \rceil) \geq v(S; P \cup \lceil T \setminus S \rceil) + v(T; P \cup \lceil S \setminus T \rceil)$$

for every $S, T \subseteq N$ and $P \in \Pi(N \setminus (S \cup T))$. Notice that for every $S, T \subseteq N$ and $P \in \Pi(N \setminus (S \cup T))$, we have $(S; P \cup \lceil T \setminus S \rceil) \vee (T; P \cup \lceil S \setminus T \rceil) = (S \cup T; P)$ and $(S; P \cup \lceil T \setminus S \rceil) \wedge (T; P \cup \lceil S \setminus T \rceil) = (S \cap T; P \cup \lceil S \setminus T \rceil \cup \lceil T \setminus S \rceil)$. This implies that our convexity implies Hafalir convexity. However, the reverse implication does not hold (see Example 5.6). In Example 5.7 we show that our notion of superadditivity does not imply Hafalir convexity.

In order to present our main result we first have to specify what a contribution[6] is in the presence of externalities. To that end, we use the lattice studied in Section 2. In classic games the contribution of an agent to a coalition corresponds to a link in the Boolean lattice of subsets $\left(2^N, \subseteq\right)$. Then, we consider that each link in the lattice $\left(\mathcal{EC}^N, \sqsubseteq\right)$ generates a contribution to the embedded coalition on its top. Note that this leads to two kinds of contributions. The first is the movement of a player who is isolated in the partition and joins the coalition. The second is the movement of a block in the partition that splits in two. Next, we present these contributions that were introduced in Alonso-Meijide et al. (2019) and explain what it means for a game to have non-decreasing contributions.

Let $v \in \mathcal{G}^N$ and $(S; P) \in \mathcal{EC}^N$ such that $\{i\} \in P$ for some $i \in N$. Then, we call

[6]What is many times called a marginal contribution.

agent i's contribution to the difference $v\left(S\cup i;P_{-i}\right)-v(S;P)$. Moreover, we say that agents' contributions are non-decreasing in v if

$$v(T\cup i;Q_{-i})-v(T;Q)\geq v\left(S\cup i;P_{-i}\right)-v(S;P), \tag{6}$$

for every $i\in N$ and $(S;P),(T;Q)\in\mathcal{EC}^N$ such that $(S;P)\sqsubseteq(T;Q)\neq(N;\emptyset)$ and $\{i\}\in P$.

Let $v\in\mathcal{G}^N$, $(S;P)\in\mathcal{EC}^N$, and $P'\in\Pi(N\setminus S)$ covering P in the partition lattice $(\Pi(N\setminus S),\preceq)$. Then, we call *external contribution* to the difference $v(S;P)-v(S;P')$.[7] Moreover, we say that external contributions are non-decreasing in v if

$$v(T;Q)-v(T;Q')\geq v(S;P)-v(S;P'), \tag{7}$$

for every $(S;P),(T;Q)\in\mathcal{EC}^N$ such that $(S;P)\sqsubseteq(T;Q)\neq(N;\emptyset)$, $P'\in\Pi(N\setminus S)$ covering P, $Q'\in\Pi(N\setminus T)$ covering Q, and $(S;P')\sqsubseteq(T;Q')$.

We state some auxiliary results that will be used to prove Theorem 3.1.

Lemma 3.1. *Let $v\in\mathcal{G}^N$ such that the external contributions are non-decreasing. Then, $v(S;P)\leq v(S;M)$, for every $(S;P),(S;M)\in\mathcal{EC}^N$ such that $(S;P)\sqsubseteq(S;M)$.*

That is, a game in which the external contributions are non-decreasing exhibits a monotonicity property in the sense that the worth of a coalition grows as the coalitions in the complement get more divided. In other words, it is a game with negative externalities (Hafalir, 2007).

Lemma 3.2. *Let $v\in\mathcal{G}^N$ such that the external contributions are non-decreasing. Then,*

$$v\left(T;P\bigwedge Q\right)+v\left(T;P\bigvee Q\right)\geq v(T;P)+v(T;Q), \tag{8}$$

for every $(T;P),(T;Q)\in\mathcal{EC}^N$.

[7]Notice that the external contribution is just the externality effect on the worth of coalition S when a coalition of $N\setminus S$ splits in two.

That is, for a fixed coalition T the partition function of a game with non-decreasing external contributions is a supermodular function on $\Pi(N \setminus T)$.

Lemma 3.3. *Let $v \in \mathcal{G}^N$ such that agents' contributions are non-decreasing. Then,*

$$v(T;Q) - v(S;Q \cup \lfloor T \setminus S \rfloor) \geq v(T;P) - v(S;P \cup \lfloor T \setminus S \rfloor), \tag{9}$$

for every $S \subseteq T$ and $P, Q \in \Pi(N \setminus T)$, with $Q \preceq P$.

The above result states that when agents' contributions are non-decreasing in a game, the incorporation of several agents that were singletons in the partition is more beneficial for larger embedded coalitions.

Lemma 3.4. *Let $v \in \mathcal{G}^N$ such that agents' contributions are non-decreasing. Then,*

$$v(S \cup T;P) + v(S \cap T;P \cup \lfloor S \setminus T \rfloor \cup \lfloor T \setminus S \rfloor) \geq v(S;P \cup \lfloor T \setminus S \rfloor) + v(T;P \cup \lfloor S \setminus T \rfloor), \tag{10}$$

for every $S, T \subseteq N$ and $P \in \Pi(N \setminus (S \cup T))$.

Notice that Equation (10) is very similar to Hafalir convexity. The only difference is the fact that here we consider that agents who only participate in one of the two coalitions are singletons.

We are now ready to present our main result, which is a characterization of convexity by non-decreasing contributions. That is, we generalize the characterization of classic convex games by Shapley (1971) to environments with externalities.

Theorem 3.1. *Let $v \in \mathcal{G}^N$. The following three items are equivalent.*

i) v is a convex game.

ii) Let $(S;P), (T;Q) \in \mathcal{EC}^N \setminus \{(N;\emptyset)\}$ such that $(T;Q)$ covers $(S;P)$. Then,

1. For every $i \in N$ with $\{i\} \in P$, we have

$$v(T \cup i;Q_{-i}) - v(T;Q) \geq v(S \cup i;P_{-i}) - v(S;P) \tag{11}$$

2. For every $P' \in \Pi(N \setminus S)$ covering P and $Q' \in \Pi(N \setminus T)$ covering Q such that $(T; Q')$ covers $(S; P')$, we have

$$v(T;Q) - v(T;Q') \geq v(S;P) - v(S;P') \tag{12}$$

iii) v has non-decreasing agents' and external contributions.

Observe that condition *ii)* is a weakening of *iii)* as it is only applied when the embedded coalition $(T; Q)$ covers $(S; P)$, in point *2.* it is also required that $(T; Q')$ covers $(S; P')$. This is parallel to the characterization of classic convex games where it is sufficient to check that the contributions are non-decreasing when one player is incorporated to the coalition. Hafalir (2007) also considered a similar weakening of his notion of convexity, which is obtained by requiring Inequality (10) only when $|T \setminus S| = |S \setminus T| \leq 1$. However, as he points out, this condition alone is not even sufficient for the efficiency of the grand coalition. Abe (2016) shows that for games with negative externalities, Hafalir's weak convexity implies the efficiency of the grand coalition. From Theorem 3.1 we can also conclude that it is enough to check that contributions are non-decreasing to coalitions that are just one link away from one another to guarantee that the grand coalition is efficient.

4 Convexity and the core

In this section we include some comments on the core of the optimistic and the pessimistic games[8] associated to a convex game. Both of them are classic games. First we recall the notion of the core of a classic game. Let $w \in \mathcal{G}^N$ be a classic game. The *core* of w is given by

$$Core(w) = \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = w(N), \sum_{i \in S} x_i \geq w(S), \text{ for every } S \subseteq N \right\}.$$

[8] Which are essentially the α-core and β-core (Hart and Kurz, 1983). More recently Dutta et al. (2010), Bloch and van den Nouweland (2014), and Abe (2016) also use these games.

In general, $Core(w)$ can be empty, but every convex classic game has a non-empty core. Besides, it is quite easy to determine its extreme points. Denote the set of permutations of N by $\Theta(N)$, i.e., $\sigma \in \Theta(N)$ if and only if σ is a bijective mapping $\sigma : N \longrightarrow \{1, \ldots, n\}$. Let $\sigma \in \Theta(N)$ and $i \in N$. The set of *predecessors* of i is $Pr(\sigma, i) = \{j \in N : \sigma(j) < \sigma(i)\}$. The *vector of marginal contributions* with respect to σ is given by $m^{\sigma}(w) \in \mathbb{R}^N$ such that $m_i^{\sigma}(w) = w(Pr(\sigma, i) \cup i) - w(Pr(\sigma, i))$, for every $i \in N$. It is well known that if w is a classic convex game, then the vectors of marginal contributions are the vertices of the core, i.e., $Core(w) = conv\,\{m^{\sigma}(w) : \sigma \in \Theta(N)\}$.

Let $v \in \mathcal{G}^N$. Recall that the optimistic game, denoted by v_{max}, is the classic game defined by $v_{max}(S) = \max\{v(S; P) : P \in \Pi(N \setminus S)\}$, for every $S \subseteq N$. The *pessimistic game*, denoted by v_{min}, is the classic game defined by $v_{min}(S) = \min\{v(S; P) : \ P \in \Pi(N \setminus S)\}$, for every $S \subseteq N$. Notice that $v_{max}(S) \geq v(S; P) \geq v_{min}(S)$, for every $(S; P) \in \mathcal{EC}^N$ and $v_{max}(N) = v_{min}(N) = v(N; \emptyset)$. Then, $Core(v_{max}) \subseteq Core(v_{min})$. And any plausible definition of the core of v should be in between the two. Abe (2016) proved that if v has negative externalities and satisfies the weak convexity condition, then $Core(v_{max})$ is non-empty as well as $Core(v_{min})$. Since a convex game according to Definition 3.2 satisfies the weak convexity condition, we already known that both $Core(v_{max})$ and $Core(v_{min})$ are non-empty sets when v is a convex game.

Definition 4.1. *Let $v \in \mathcal{G}^N$. For every $P \in \Pi(N)$, we define the classic game v^P by $v^P(S) = v(S; P_{-S})$, for every $S \subseteq N$.*

Notice that v^P is defined for every $S \subseteq N$ even if S is not a block in P. Besides, $v^P(S) = v^Q(S)$, for every $P, Q \in \Pi(N)$ and $S \subseteq N$ with $P_{-S} = Q_{-S}$. The optimistic game can then be defined by $v_{max}(S) = \max\left\{v^P(S) : \ P \in \Pi(N)\right\}$, analogously for the pessimistic game by $v_{min}(S) = \min\left\{v^P(S) : \ P \in \Pi(N)\right\}$, for every $S \subseteq N$.

Proposition 4.1. *Let $v \in \mathcal{G}^N$. Then,*

$$Core\left(v_{max}\right) = \bigcap_{P \in \Pi(N)} Core\left(v^P\right).$$

Next, we describe some properties of the classic games associated with a convex game and characterize the extreme points of the core of one of them, the pessimistic game.

Theorem 4.1. *Let $v \in \mathcal{G}^N$ be a convex game.*

1. *Let $P, Q \in \Pi(N)$ such that $Q \preceq P$. Then, $Core\left(v^Q\right) \subseteq Core\left(v^P\right)$.*
2. *For every $S \subseteq N$, $v_{max}(S) = v^{\lfloor N \rfloor}(S)$ and $v_{min}(S) = v^{\lceil N \rceil}(S)$.*
3. *The classic game v_{max} is convex and*

$$Core\left(v_{max}\right) = conv\left\{m^{\sigma}\left(v^{\lfloor N \rfloor}\right) : \sigma \in \Theta(N)\right\}.$$

As a consequence of Theorem 4.1, if v is convex the Externality-free value (de Clippel and Serrano, 2008) is the average of the extreme points of the core of v_{max} and it also belongs to the core of v_{min}. Finally, Example 5.8 shows that our definition of convexity is not enough to guarantee the convexity of the pessimistic game.

5 Conclusion

We have proposed new notions of superadditivity and convexity for games with externalities and compared them with the ones defined in the literature. Figure 1 summarizes the existing implications.

Our main result is a characterization of convexity by means of non-decreasing contributions to embedded coalitions of incresing size. We also offer another characterization that uses the standard convexity of some associated classic games. Our results may help identifying games with negative externalities that are convex. Recall that only games with negative externalities can be convex. They also offer alternative views on what a convex game with externalities is.

In the future, we would like to deepen the understanding on the different classes of games with externalities. For instance, we would like to explore the implications

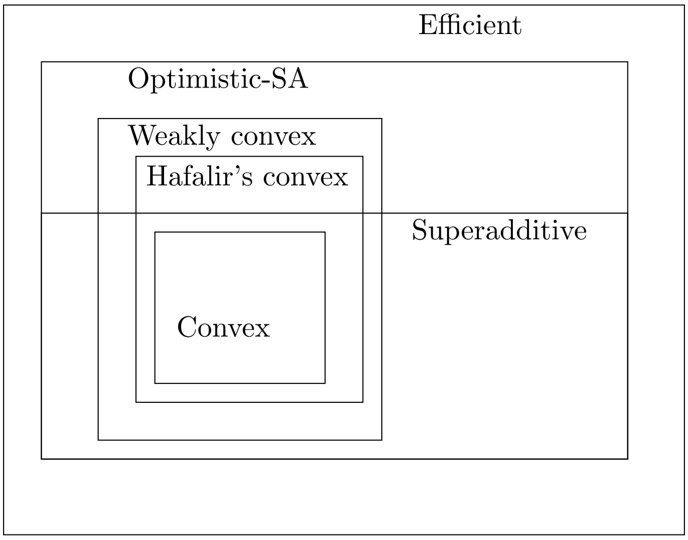

Figure 1: Relationship among several families of games with negative externalities.

of concave games with externalities, that can be defined by just reversing the inequality in the definition of convexity. Whether this implies that the game shows positive externalities is an open question. In this line, we would also like to analyze the structure of the core of the associated games and study if some of the existing generalizations of the Shapley value belong to it. Finally, we would also like to consider a definition of convexity based on blocking coalitions as Milgrom and Shannon (1996) do and see if their results carry on to situations with externalities.

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Appendix

Proof of Proposition 2.1.

1. The first item can be proven in the same way as Item 1 of Proposition 1 in Alonso-Meijide et al. (2017).

2. Take $(S \cap T; M)$ with

$$M = (P \cup \lfloor S \setminus T \rfloor) \bigvee (Q \cup \lfloor T \setminus S \rfloor).$$

Then, $(S \cap T; M) \sqsubseteq (S; P)$ and $(S \cap T; M) \sqsubseteq (T; Q)$. Let $(R; M') \in \mathcal{EC}^N$ such that $(R; M') \sqsubseteq (S; P)$ and $(R; M') \sqsubseteq (T; Q)$ then, it is easy to see that $(R; M') \sqsubseteq (S \cap T; M)$. □

Example 5.1. *The lattice* $\left(\mathcal{EC}^N, \sqsubseteq\right)$ *is not distributive.*

Let $N = \{1,2,3,4,5,6,7\}$, $(S; P) = (\{2,3\}; \{\{1,4\},\{5,6\},\{7\}\})$, $(T; Q) = (\{1,2\}; \{\{3,5\},\{4,6,7\}\})$, *and* $(L; M) = (\{1,3\}; \{\{2\},\{4,5\},\{6,7\}\})$. *Then*

$$\begin{aligned}
&(T; Q) \wedge (L; M) = (\{1\}; \{\{2\},\{3,4,5,6,7\}\}),\\
&(S; P) \vee ((T; Q) \wedge (L; M)) = (\{1,2,3\}; \{\{4\},\{5,6\},\{7\}\}),\\
&(S; P) \vee (T; Q) = (\{1,2,3\}; \lfloor N \setminus \{1,2,3\} \rfloor),\\
&(S; P) \vee (L; M) = (\{1,2,3\}; \lfloor N \setminus \{1,2,3\} \rfloor), \quad \textit{and}\\
&((S; P) \vee (T; Q)) \wedge ((S; P) \vee (L; M)) = (\{1,2,3\}; \lfloor N \setminus \{1,2,3\} \rfloor).
\end{aligned}$$

Then, $(S;P) \vee ((T;Q) \wedge (L;M)) \neq ((S;P) \vee (T;Q)) \wedge ((S;P) \vee (L;M))$.

Besides, $(S;P) \wedge ((T;Q) \vee (L;M)) \neq ((S;P) \wedge (T;Q)) \vee ((S;P) \wedge (L;M))$ *as we see next:*

$$
\begin{aligned}
&(T;Q) \vee (L;M) = (\{1,2,3\}; \lfloor 4,5 \rfloor \cup \{6,7\}\}),\\
&(S;P) \wedge ((T;Q) \vee (L;M)) = (\{2,3\}; \{\{1,4\},\{5,6,7\}\}),\\
&(S;P) \wedge (T;Q) = (\{2\}; \lceil N \setminus \{2\} \rceil),\\
&(S;P) \wedge (L;M) = (\{3\}; \lceil N \setminus \{2,3\} \rceil \cup \{2\}), \quad \textit{and}\\
&((S;P) \wedge (T;Q)) \vee ((S;P) \wedge (L;M)) = (\{2,3\}; \lceil N \setminus \{2,3\} \rceil).
\end{aligned}
$$

Proof of Proposition 2.2. The result follows immediately if $|N| \leq 2$. Let us assume that $|N| \geq 3$ and $(S;P) \in \mathcal{EC}^N$. We prove that all chains between the infimum of the lattice, $(\emptyset; N)$, and (S,P) have length $|P| + 2|S| - 1$. We proceed by induction on k, the length of such a chain.

If $k = 0$, then $(S;P) = (\emptyset; N)$ and $h(\emptyset; N) = 0 = |P|+2|S|-1$. Let us take $k = 1$. That is, we consider a chain of length $k = 1$ joining $(S;P)$ and $(\emptyset; N)$, this implies that $(S;P)$ covers $(\emptyset; N)$. Then, $(S;P) = (\emptyset; \{T, N \setminus T\})$ for some $T \notin \{\emptyset, N\}$, there is only one chain from $(\emptyset; N)$ to $(S;P)$, and $h(S;P) = h(\emptyset; \{T, N \setminus T\}) = 1 + h(\emptyset; N) = |P| + 2|S| - 1$.

Suppose that the result holds for every $(S;P)$ such that there is a chain of length $k > 0$ from $(\emptyset; N)$ to $(S;P)$. Let $(S;P) \in \mathcal{EC}^N$ such that there is a chain of length k joining $(S;P)$ and $(\emptyset; N)$. We distinguish two cases.

First, if $|P| \leq 1$, we have $|S| > 0$ because $k > 0$. Then, $(S;P)$ only covers embedded coalitions of type $(S \setminus i; P \cup \{i\})$, for every $i \in S$ and there is a chain of length $k-1$ from $(\emptyset; N)$ to $(S \setminus i; P \cup \{i\})$. By the induction hypothesis, all chains from $(\emptyset; N)$ to $(S \setminus i; P \cup \{i\})$ have length $k - 1 = h(S \setminus i; P \cup \{i\})$. Since $(S;P)$ covers $(S \setminus i; P \cup \{i\})$,

$$k = h(S;P) = 1 + h(S \setminus i; P \cup \{i\}) = 1 + 1 + |P| + 2|S \setminus i| - 1 = |P| + 2|S| - 1.$$

Second, let us assume that $|P| > 1$ and take $P = \{P_1, \ldots, P_m\}$, with $m \geq 2$.

Then, we can have $|S| = 0$ or $|S| > 0$. If $|S| = 0$, then $(S; P)$ only covers embedded coalitions of type $\left(\emptyset; P_{-P_j \cup P_l} \cup \lceil P_j \cup P_l \rceil\right)$ for every $j, l \in \{1, \ldots, m\}$ with $j \neq l$ and there is a chain of length $k - 1$ from $(\emptyset; N)$ to $(\emptyset; P_{-P_j \cup P_l} \cup \lceil P_j \cup P_l \rceil)$. By induction, all chains from $(\emptyset; N)$ to $(\emptyset; P_{-P_j \cup P_l} \cup \lceil P_j \cup P_l \rceil)$ have length $k - 1 = h(\emptyset; P_{-P_j \cup P_l} \cup \lceil P_j \cup P_l \rceil)$. Since $(S; P)$ covers $(\emptyset; P_{-P_j \cup P_l} \cup \lceil P_j \cup P_l \rceil)$,

$$k = h(S; P) = 1 + h(\emptyset; P_{-P_j \cup P_l} \cup \lceil P_j \cup P_l \rceil) = 1 + (|P| - 1) + 2|S| - 1 = |P| + 2|S| - 1.$$

Finally, if $|S| > 0$, $(S; P)$ covers embedded coalitions of two types, $(S \setminus i; P \cup \{i\})$, for every $i \in S$ and $(S; P_{-P_j \cup P_l} \cup \lceil P_j \cup P_l \rceil)$ for every $j, l \in \{1, \ldots, m\}$ with $j \neq l$. Using the induction hypothesis as before for each of the types of embedded coalitions we obtain that $k = h(S; P) = |P| + 2|S| - 1$. $\square$

Proof of Proposition 2.3. Let $(S; P), (T; Q) \in \mathcal{EC}^N$. First, recall that $(S; P) \vee (T; Q) = (S \cup T; P_{-T} \bigwedge Q_{-S})$ and $(S; P) \wedge (T; Q) = (S \cap T; (P \cup \lfloor S \setminus T \rfloor) \bigvee (Q \cup \lfloor T \setminus S \rfloor))$. Using Equation (3), Inequality (4) is equivalent to

$$\begin{aligned} & r\left(Q \cup \lfloor T \rfloor\right) - r\left(\left(P_{-T} \bigwedge Q_{-S}\right) \cup \lfloor S \cup T \rfloor\right) \\ & \geq r\left(\left((P \cup \lfloor S \setminus T \rfloor) \bigvee (Q \cup \lfloor T \setminus S \rfloor)\right) \cup \lfloor S \cap T \rfloor\right) - r\left(P \cup \lfloor S \rfloor\right). \end{aligned}$$

Taking $P \cup \lfloor S \rfloor$, $Q \cup \lfloor T \rfloor \in \Pi(N)$, it happens that

- $(P \cup \lfloor S \rfloor) \bigwedge (Q \cup \lfloor T \rfloor) = (P_{-T} \bigwedge Q_{-S}) \cup \lfloor S \cup T \rfloor$, and
- $(P \cup \lfloor S \rfloor) \bigvee (Q \cup \lfloor T \rfloor) = ((P \cup \lfloor S \setminus T \rfloor) \bigvee (Q \cup \lfloor T \setminus S \rfloor)) \cup \lfloor S \cap T \rfloor$.

Using the fact that the height of an element on the the partition lattice is a submodular function and taking $P \cup \lfloor S \rfloor$, $Q \cup \lfloor T \rfloor \in \Pi(N)$, we obtain

$$\begin{aligned} & r\left(P \cup \lfloor S \rfloor\right) + r\left(Q \cup \lfloor T \rfloor\right) \\ & \geq r\left((P_{-T} \bigwedge Q_{-S}) \cup \lfloor S \cup T \rfloor\right) + r\left(\left((P \cup \lfloor S \setminus T \rfloor) \bigvee (Q \cup \lfloor T \setminus S \rfloor)\right) \cup \lfloor S \cap T \rfloor\right) \end{aligned}$$

and the result follows. $\square$

Example 5.2. *There are superadditive games which are not classic games.*

Let $N = \{1, 2, 3\}$ *and consider the partition function* v *defined by*

$$v(N;\emptyset) = 8,\ v(\{1\};\lceil 2,3\rceil) = 3,\ v(\{1\};\lfloor 2,3\rfloor) = 0,$$
$$v(\{i\};\lfloor N\setminus i\rfloor) = v(\{i\};\lceil N\setminus i\rceil) = 2,\ \textit{for every } i \in N\setminus 1,\ \textit{and}$$
$$v(\{i,j\};N\setminus\{i,j\}) = 5,\ \textit{for every } i,j \in N, i \neq j.$$

Since the worth of coalition $\{1\}$ *depends on the coalition structure of the complement it is not a classic game. Moreover, it is easy to check that* v *is superadditive according to Definition 3.1.*

Example 5.3. *Maskin's definition of superadditivity is not equivalent to Definition 3.1*

Let $N = \{1, 2, 3\}$ *and* $v \in \mathcal{G}^N$ *such that*

$$v(N;\emptyset) = 7,\ v(\{1\};\lceil 2,3\rceil) = 3,\ v(\{1\};\lfloor 2,3\rfloor) = 0,$$
$$v(\{i\};\lfloor N\setminus i\rfloor) = v(\{i\};\lceil N\setminus i\rceil) = 2,\ \textit{for every } i \in N\setminus 1,\ \textit{and}$$
$$v(\{i,j\};N\setminus\{i,j\}) = 4,\ \textit{for every } i,j \in N, i \neq j.$$

It is easy to check that v *is superadditive in Maskin's sense. However, it is not superadditive according to Definition 3.1 as we can see taking* $(S;P) = (\{2\};\lceil 1,3\rceil)$ *and* $(T;Q) = (\{1\};\lceil 2,3\rceil)$*.*

Example 5.4. *There are Optimistic-SA games which are not superadditive according to Definition 3.1.*

Let $N = \{1, 2, 3, 4\}$ *and* $v \in \mathcal{G}^N$ *defined as follows:*

$$v(N;\emptyset) = 60,\ v(N\setminus i;\lfloor i\rfloor) = 45,\ \textit{for every } i \in N,$$
$$v(\{i,j\};\lceil h,k\rceil) = 29 \textit{ and } v(\{i,j\};\lfloor h,k\rfloor) = 30,\ \textit{for every } \{i,j,h,k\} = N,$$
$$v(\{i\};P) = 15,\ \textit{for every } (\{i\};P) \in \mathcal{EC}^N.$$

This game is an adaptation of Example 3.8 in Abe (2016). It is still superadditive in Maskin's sense, but it is not superadditive according to Definition 3.1 because, for instance,

$$v(\{1\};\{\{2,3\},\{4\}\}) + v(\{4\};\lceil N\setminus 4\rceil) = 15 + 15 = 30 > v(\{1,4\};\lceil 2,3\rceil) = 29.$$

The optimistic game associated to it, given by

$$\begin{aligned}
&v_{max}(N) = 60,\ \ v_{max}(N\setminus i) = 45,\ \ \textit{for every } i \in N,\\
&v_{max}(S) = 30,\ \ \textit{if } |S| = 2,\\
&v_{max}(S) = 15,\ \ \textit{if } |S| = 1,
\end{aligned}$$

is a classic superadditive game, which means that v is Optimistic-SA.

Example 5.5. *There are convex games which are not classic games.*

Let $N = \{1,2,3\}$ and $v \in \mathcal{G}^N$ defined as follows:

$$\begin{aligned}
&v(N;\emptyset) = 15, \quad v(N\setminus i;\{i\}) = 10,\ \textit{for every } i \in N,\\
&v(N\setminus\{i,j\};\lfloor i,j\rfloor) = 5, \textit{ for every } i,j \in N, i \neq j,\\
&v(N\setminus\{i,j\};\lceil i,j\rceil) = 4, \textit{ for every } i,j \in N, i \neq j,\\
&v(\emptyset;N) = 0.
\end{aligned}$$

Clearly, v is not a classic game. It is easy to check that it is convex according to Definition 3.2.

Example 5.6. *Hafalir's definition of convexity is not equivalent to our notion of convexity (see Definition 3.2).*

Let $N = \{1, 2, 3, 4\}$ *and* $v \in \mathcal{G}^N$ *be defined as follows:*

$v(N; \emptyset) = 12,\ v(\{1, 2, 3\}; \{4\}) = 7,\ v(\{1, 2, 4\}; \{3\}) = 6,\ v(\{1, 3, 4\}; \{2\}) = 3,$
$v(\{2, 3, 4\}; \{1\}) = 6,\ v(\{1, 2\}; \lfloor 3, 4 \rfloor) = 4,\ v(\{1, 4\}; P) = 1,$ *for every* $P \in \Pi(\{2, 3\})$,
$v(\{2, 3\}; \lfloor 1, 4 \rfloor) = 2,\ v(\{2, 3\}; \lceil 1, 4 \rceil) = 4,\ v(\{2, 4\}; P) = 2,$ *for every* $P \in \Pi(\{1, 3\})$,
$v(\{1, 3\}; \lfloor 2, 4 \rfloor) = 2,\ v(\{1\}; \lfloor 2, 3, 4 \rfloor) = 1,\ v(\{2\}; \lfloor 1, 3, 4 \rfloor) = 2,$
$v(S; P) = 0,$ *otherwise.*

This game is superadditive and Hafalir convex. But, it does not satisfy Inequality (5). For instance, if we take $(S; P) = (\{1, 2\}; \lfloor 3, 4 \rfloor)$, $(T; Q) = (\{2, 3\}; \lceil 1, 4 \rceil)$, *then* $(S; P) \vee (T; Q) = (\{1, 2, 3\}; \{4\})$, $(S; P) \wedge (T; Q) = (\{2\}; \{\{1, 4\}, \{3\}\})$, *and*

$$v(\{1, 2, 3\}; \{4\}) + v(\{2\}; \{\{1, 4\}, \{3\}\}) = 7 + 0 < 4 + 4 = v(\{1, 2\}; \lfloor 3, 4 \rfloor) + v(\{2, 3\}; \lceil 1, 4 \rceil).$$

Example 5.7. *Our notion of superadditivity (see Definition 3.1) does not imply Hafalir's definition of convexity.*

To see this we can modify Example 5.6 above as follows:

$$v(\{1, 2, 3\}; \{4\}) = 5,\ v(\{2, 3\}; \lfloor 1, 4 \rfloor) = 4.$$

This game is still superadditive according to Definition 3.1 but it is not Hafalir convex because

$$v(\{1, 2, 3\}; \{4\}) + v(\{2\}; \lfloor 1, 3, 4 \rfloor) = 5 + 2 < 4 + 4 = v(\{1, 2\}; \lfloor 3, 4 \rfloor) + v(\{2, 3\}; \lfloor 1, 4 \rfloor).$$

Proof of Lemma 3.1. Let $(S; P), (S; M) \in \mathcal{EC}^N$, such that $(S; P) \sqsubseteq (S; M)$ and $(S; P) \neq (S; M)$. If $S \in \{\emptyset, N\}$ or $P = M$, $v(S; P) = v(S; M)$ and the result follows immediately. Then, suppose that $S \notin \{\emptyset, N\}$ and $P \neq M$. Since $(S; P) \sqsubseteq (S; M) \neq (N; \emptyset)$ and $(S; P) \neq (S; M)$, $M \prec P$ holds. Take a chain $M = Q_0 \prec Q_1 \prec \cdots \prec Q_k = P$. Then, Q_r covers Q_{r-1}, for every $r = 1, \ldots, k$. Take the family of embedded coalitions $\{(\emptyset; \lfloor S \rfloor \cup Q_r) :\ r = 0, \ldots, k\}$. Let $r \in$

$\{0, \ldots, k-1\}$. Then, $\lfloor S \rfloor \cup Q_{r+1}$ covers $\lfloor S \rfloor \cup Q_r$ and $(\emptyset; \lfloor S \rfloor \cup Q_r) \sqsubseteq (S; Q_r) \neq (N; \emptyset)$. Applying Inequality (7) to $(\emptyset; \lfloor S \rfloor \cup Q_r) \sqsubseteq (S; Q_r)$, $\lfloor S \rfloor \cup Q_{r+1}$, and Q_{r+1}, we obtain $v(S; Q_r) - v(S; Q_{r+1}) \geq v(\emptyset; \lfloor S \rfloor \cup Q_r) - v(\emptyset; \lfloor S \rfloor \cup Q_{r+1})$. Since $v(\emptyset; \lfloor S \rfloor \cup Q_{r+1}) = v(\emptyset; \lfloor S \rfloor \cup Q_r) = 0$, we get $v(S; Q_r) \geq v(S; Q_{r+1})$. Thus,

$$v(S; M) = v(S; Q_0) \geq v(S; Q_1) \geq \cdots \geq v(S; Q_{k-1}) \geq v(S; Q_k) = v(S; P).$$

□

Proof of Lemma 3.2. Take $(T; P), (T; Q) \in \mathcal{EC}^N$. If $T \in \{N, \emptyset\} \cup \{N \setminus i : i \in N\}$ or $(T; P) \sqsubseteq (T; Q)$, Inequality (8) follows immediately. Let us assume that $(T; P)$ and $(T; Q)$ are not comparable, $0 < |T| < n - 1$, and w.l.o.g. we assume $h(T; Q) \geq h(T; P)$. Then, $|Q| \geq |P|$ and $P \bigvee Q \notin \{P, Q\}$. Let $P \bigwedge Q = P_0 \prec P_1 \prec \ldots \prec P_k \prec P_{k+1} = P$, with $k \geq 1$, be a chain that joins $P \bigwedge Q$ and P, and $P \bigwedge Q = Q_0 \prec Q_1 \prec \ldots \prec Q_r \prec Q_{r+1} = Q$, with $r \geq 1$, be a chain that joins $P \bigwedge Q$ and Q. Notice that P_j and Q_l are not comparable for every $j = 1, \ldots, k+1$, $l = 1, \ldots, r+1$. We distinguish four cases.

1. $h(T; P \bigwedge Q) - h(T; P) = 1$, $h(T; P \bigwedge Q) - h(T; Q) = 1$. That means both P and Q cover $P \bigwedge Q$. Figure 2 illustrates the situation.

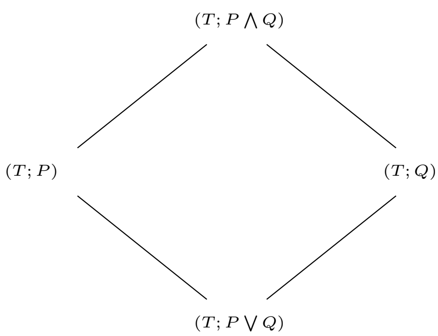

Figure 2: Case 1. solid line: one link.

Since $\Pi(N \setminus T)$ is semimodular, $P \bigvee Q$ covers both P and Q. Then, $(T; P) \vee (T; Q) = (T; P \bigwedge Q)$ covers both $(T; P)$ and $(T; Q)$. Since $(\mathcal{EC}^N, \sqsubseteq)$ is lower semimodular, then $(T; P)$ and $(T; Q)$ both cover $(T; P) \wedge (T; Q) = (T; P \bigvee Q)$. Applying

Inequality (7) to $(T;P)$, $(T;P\bigwedge Q)$, $(T;P\bigvee Q)$, and $(T;Q)$ we get Inequality (8).

2. $h(T;P\bigwedge Q)-h(T;Q)=1$, but $h(T;P\bigwedge Q)-h(T;P)>1$. Using Proposition 2.3 and the fact that $(T;P)$ and $(T;Q)$ are not comparable, $h(T;P)-h(T;P\bigvee Q)=1$. Then, $P\bigvee Q$ covers P, Q covers $P\bigwedge Q$, but P does not cover $P\bigwedge Q$ and $P\neq P\bigwedge Q$. Figure 3 illustrates the situation.

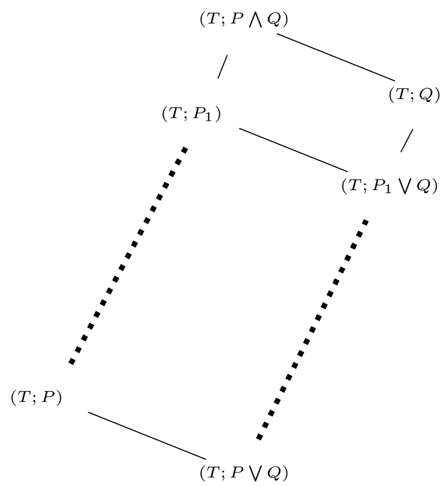

Figure 3: Case 2. solid line: one link; dashed line: more than one link.

Take a chain $P\bigwedge Q=P_0\prec P_1\prec\ldots\prec P_k\prec P_{k+1}=P$, with $k\geq 1$. Notice that $Q\prec Q\bigvee P_1\prec\ldots\prec Q\bigvee P_k\prec Q\bigvee P_{k+1}=Q\bigvee P$, with $k\geq 1$ is a chain from Q to $P\bigvee Q$. Let $j\in\{0,\ldots,k\}$. Then, $P_j\bigwedge Q=P\bigwedge Q$ and using Proposition 2.3 we have

$$1=h\left(T;P\bigwedge Q\right)-h(T;Q)=h\left(T;P_j\bigwedge Q\right)-h(T;Q)\geq h(T;P_j)-h\left(T;P_j\bigvee Q\right).$$

Since $P_j\neq P_j\bigvee Q$, $h(T;P_j)-h(T;P_j\bigvee Q)=1$. Besides, $(P_j\bigvee Q)\bigwedge P_{j+1}=P_j$. Using that $(\Pi(N),\preceq)$ is semimodular, $(P_j\bigvee Q)\bigvee P_{j+1}=P_{j+1}\bigvee Q$ covers both $P_j\bigvee Q$ and P_{j+1}. Then, using Item 1, we have

$$v(T;P_j)+v\left(T;P_{j+1}\bigvee Q\right)\geq v(T;P_{j+1})+v\left(T;P_j\bigvee Q\right).$$

Summing up the inequalities given above, we get

$$\sum_{j=0}^{k}\left[v(T;P_j)+v\left(T;P_{j+1}\bigvee Q\right)\right]\geq\sum_{j=0}^{k}\left[v(T;P_{j+1})+v\left(T;P_j\bigvee Q\right)\right].$$

Rearranging this inequality, we obtain Inequality (8).

3. $h(T;P)-h(T;P\bigvee Q)=1$, but $h(T;P\bigwedge Q)-h(T;Q)>1$. This means $P\bigvee Q$ covers P, but Q does not cover $P\bigwedge Q$. Figure 4 illustrates the situation.

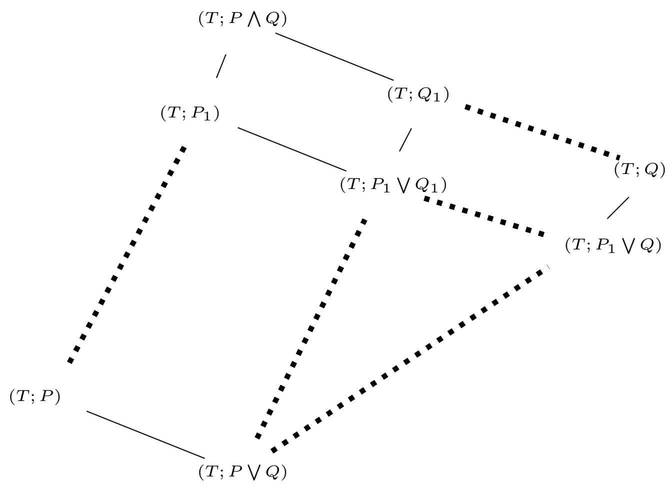

Figure 4: Case 3. solid line: one link; dashed line: one or more links.

Since $h(T;Q)\geq h(T;P)$ and $h(T;P\bigwedge Q)-h(T;Q)>1$, we have $h(T;P\bigwedge Q)-h(T;P)>1$. Take $P\bigwedge Q=P_0\prec Q_1\prec\ldots\prec Q_r\prec Q_{r+1}=Q$, with $r\geq 1$, a chain that joins $P\bigwedge Q$ and Q. By the choice of Q_1, we have

- $(T;Q_1)\sqsubseteq(T;P\bigwedge Q)$, $(T;P)\sqsubseteq(T;P\bigwedge Q)$, $h(T;P\bigwedge Q)-h(T;Q_1)=1$, and the fact that $(T;P)$ and $(T;Q_1)$ are not comparable imply that $(T;P\bigwedge Q)=(T;P\bigwedge Q_1)$.
- $(T;P\bigvee Q)\sqsubseteq(T;P\bigvee Q_1)\sqsubseteq(T;P)$ and $h(T;P)-h(T;P\bigvee Q)=1$. Then, $P=P\bigvee Q_1$ or $P\bigvee Q_1=P\bigvee Q$. If $P=P\bigvee Q_1$ we have $Q_1\preceq P$ and $(T;P)\sqsubseteq(T;Q_1)$, but this fact contradicts that $(T;P)$ and $(T;Q_1)$ are not comparable. Then, $(T;P\bigvee Q)=(T;P\bigvee Q_1)$.

As a consequence of all this, applying Item 2 to $(T;P)$ and $(T;Q_1)$, we obtain

$$v\left(T;P\bigwedge Q\right)+v\left(T;P\bigvee Q\right)\geq v(T;P)+v(T;Q_1) \qquad (13)$$

Since v satisfies Inequality (7) and $Q_1 \prec Q$, using Lemma 3.1, we get $v(T;Q_1) \geq v(T;Q)$ and

$$v\left(T;P\bigwedge Q\right)+v\left(T;P\bigvee Q\right)\geq v(T;P)+v(T;Q),$$

concluding the proof of this item.

4. $h(T;P)-h(T;P\bigvee Q)>1$ and $h(T;P\bigwedge Q)-h(T;Q)>1$. Then, $h(T;P\bigwedge Q)-h(T;P)\geq h(T;P\bigwedge Q)-h(T;Q)>1$. That means $P\bigvee Q$ does not cover P nor does Q cover $P\bigwedge Q$. Figure 5 illustrates the situation.

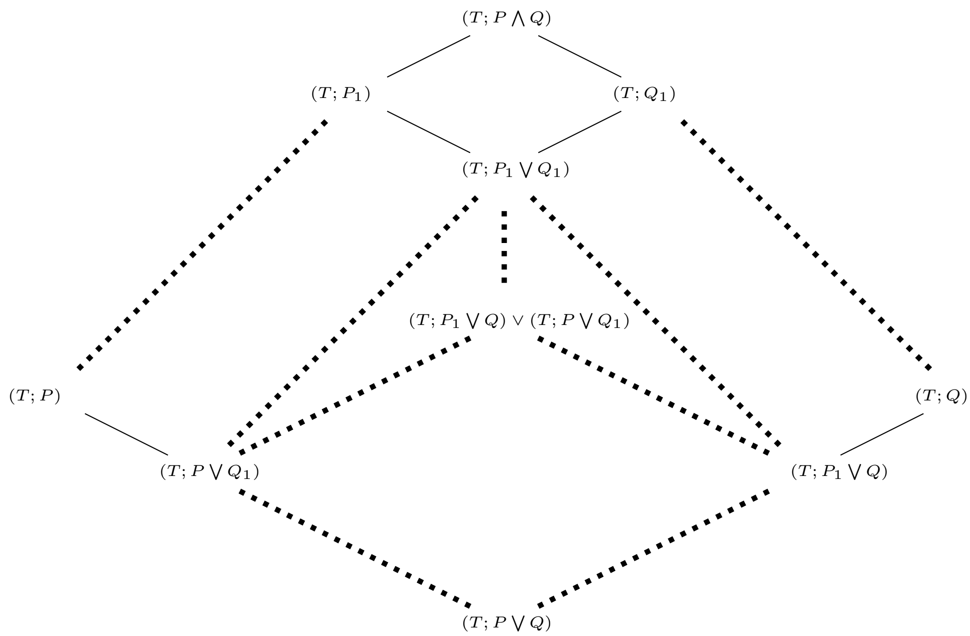

Figure 5: Case 4. solid line: one link; dashed line: more than one link.

We proceed by induction on $h(T;P)-h(T;P\bigvee Q)$. The case $h(T;P)-h(T;P\bigvee Q)=1$ corresponds to Item 3. Let us assume that the result holds if $1\leq h(T;P)-h(T;P\bigvee Q)<l$. Take $(T;Q)$ and $(T;P)$ with $h(T;P)-h(T;P\bigvee Q)=l$. Take

$P \bigwedge Q = P_0 \prec P_1 \prec \ldots \prec P_k \prec P_{k+1} = P$, with $k \geq 1$, a chain that joins $P \bigwedge Q$ and P and $P \bigwedge Q = Q_0 \prec Q_1 \prec \ldots \prec Q_r \prec Q_{r+1} = Q$, with $r \geq 1$, a chain that joins $P \bigwedge Q$ and Q. Applying Item 1 to $(T; P_1))$ and $(T; Q_1)$ because $(T; P_1) \vee (T; Q_1) = (T; P \bigwedge Q)$, $h(T; P \bigwedge Q) - h(T; P_1) = 1 = h(T; P \bigwedge Q) - h(T; Q_1)$, we get

$$v\left(T; P \bigwedge Q\right) + v\left(T; P_1 \bigvee Q_1\right) \geq v(T; P_1) + v(T; Q_1) \tag{14}$$

Due to the choice of P_1 and Q_1, we have $(T; Q) \vee (T; P_1 \bigvee Q_1) = (T; Q_1)$, $h(T; Q_1) - h(T; P_1 \bigvee Q_1) = 1$, and $h(T; Q_1) - h(T; Q) \geq 1$. Then, applying Item 1 if $h(T; Q_1) - h(T; Q) = 1$ and applying Item 2 if $h(T; Q_1) - h(T; Q) > 1$ we get

$$v(T; Q_1) + v\left(T; P_1 \bigvee Q\right) \geq v(T; Q) + v\left(T; P_1 \bigvee Q_1\right). \tag{15}$$

In a similar way if we take $(T; P)$ and $(T; P_1 \bigvee Q_1)$, we get

$$v(T; P_1) + v\left(T; P \bigvee Q_1\right) \geq v(T; P) + v\left(T; P_1 \bigvee Q_1\right). \tag{16}$$

Finally, we take $(T; P_1 \bigvee Q)$ and $(T; P \bigvee Q_1)$. Then, $(T; P_1 \bigvee Q) \wedge (T; P \bigvee Q_1) = (T; P \bigvee Q)$ and $(T; P \bigvee Q_1) \vee (T; P_1 \bigvee Q) \sqsubseteq (T; P_1 \bigvee Q_1)$. Besides, $h(T; P \bigvee Q_1) - h(T; P \bigvee Q) = l - 1 < l$. We apply the induction hypothesis and obtain

$$v\left(\left(T; P \bigvee Q_1\right) \vee \left(T; P_1 \bigvee Q\right)\right) + v\left(T; P \bigvee Q\right) \geq v\left(T; P \bigvee Q_1\right) + v\left(T; P_1 \bigvee Q\right). \tag{17}$$

Adding up Inequalities (14), (15), (16), and (17), and using Lemma 3.1 applied to $(T; P \bigvee Q_1) \vee (T; P_1 \bigvee Q) \sqsubseteq (T; P_1 \bigvee Q_1)$, we obtain

$$v\left(T; P \bigwedge Q\right) + v\left(T; P \bigvee Q\right) \geq v(T; P) + v(T; Q),$$

concluding the proof. □

Proof of Lemma 3.3. Take $S \subseteq T$, $P, Q \in \Pi(N \setminus T)$ with $Q \preceq P$. We proceed by induction on $|T \setminus S|$. If $|T \setminus S| = 0$, Inequality (9) follows immediately. Let us assume that $|T \setminus S| = 1$, i.e., $T \setminus S = \{i\}$ for some $i \in N$. Then, $(S; \{i\} \cup P) \sqsubseteq (S; \{i\} \cup Q)$. Applying Inequality (6) to i, $(S; \{i\} \cup P)$, and $(S; \{i\} \cup Q)$ we get

$$v(S \cup i; Q) - v(S; \{i\} \cup Q) \geq v(S \cup i; P) - v(S; \{i\} \cup P).$$

Now, let us assume that the result holds for every $S \subseteq T$, $P, Q \in \Pi(N \setminus T)$ with $Q \preceq P$ and $|T \setminus S| < k$. Take $S \subseteq T$, $P, Q \in \Pi(N \setminus T)$ with $Q \preceq P$ and $|T \setminus S| = k$. Take $i \in T \setminus S$, $(T \setminus i; \{i\} \cup P)$, and $(T \setminus i; \{i\} \cup Q)$. It is clear that $T \setminus \{i\} \subseteq T$, $\{i\} \cup Q \preceq \{i\} \cup P$ and $|T \setminus (T \setminus \{i\})| = 1$. As we have just seen

$$v(T; Q) - v(T \setminus i; \{i\} \cup Q) \geq v(T; P) - v(T \setminus i; \{i\} \cup P). \tag{18}$$

Notice that $S \subseteq T \setminus i$. Take $P' = \{i\} \cup P$, and $Q' = \{i\} \cup Q$. Since $|T \setminus (S \cup i)| = k - 1 < k$, $\{i\} \in P'$, and $Q' \preceq P'$, applying the induction hypothesis we get

$$v(T \setminus i; Q') - v(S; \lfloor T \setminus S \rfloor \cup Q) \geq v(T \setminus i; P') - v(S; \lfloor T \setminus S \rfloor \cup P). \tag{19}$$

Adding up Inequalities (18) and (19) we get the result. □

Proof of Lemma 3.4. Let $S, T \subseteq N$, $P \in \Pi(N \setminus (S \cup T))$. If $S \in \{\emptyset, N\}$ or $T = \emptyset$, Inequality (10) follows immediately. Let us assume that both S and T are proper non-empty subsets of N. If $S \subseteq T$, Inequality (10) follows immediately. Then, let us assume that S and T are not comparable and $S \setminus T = \{i_1, \ldots, i_r\}$. Let $A_0 = S \cap T$ and $B_0 = T$. For each $j = 1, \ldots, r$, take

- $(A_j; P'_j) \in \mathcal{EC}^N$ given by $A_j = A_{j-1} \cup \{i_j\}$, $P'_j = P \cup \lfloor T \setminus S \rfloor \cup \lfloor S \setminus A_j \rfloor$, and
- $(B_j; Q'_j) \in \mathcal{EC}^N$ given by $B_j = B_{j-1} \cup \{i_j\}$, $Q'_j = P \cup \lfloor S \setminus B_j \rfloor$.

For every $j = 0, \ldots, r$, we have $(A_j; P'_j) \sqsubseteq (B_j; Q'_j)$. Thus, for every $j = 0, \ldots, r-1$, applying Inequality (6) to i_{j+1}, $(A_j; P'_j)$ and $(B_j; Q'_j)$, we obtain $v(B_j \cup \{i_{j+1}\}; Q'_{j+1}) -$

$v(B_j;Q'_j) \geq v(A_j \cup \{i_{j+1}\};P'_{j+1}) - v(A_j;P'_j)$. Adding up these r inequalities, we get

$$\sum_{j=0}^{r-1}[v(B_j \cup \{i_{j+1}\};Q'_{j+1}) - v(B_j;Q'_j)] \geq \sum_{j=0}^{r-1}[v(A_j \cup \{i_{j+1}\};P'_{j+1}) - v(A_j;P'_j)].$$

Hence,

$$v(S \cup T;P) - v(T;\lfloor S \setminus T \rfloor \cup P) \geq v(S;\lfloor T \setminus S \rfloor \cup P) - v(S \cap T;\lfloor T \setminus S \rfloor \cup \lfloor S \setminus T \rfloor \cup P),$$

concluding the proof. □

Proof of Theorem 3.1. First, we proof that *i)* implies *ii)*. Let $v \in \mathcal{G}^N$. Let us assume that v is a convex game. Take $(S;P),(T;Q) \in \mathcal{EC}^N$ such that $(S;P) \sqsubseteq (T;Q) \neq (N;\emptyset)$ and $(T;Q)$ covers $(S;P)$. If there is $\{i\} \in P$, then $\{i\} \in Q$ since $(S;P) \sqsubseteq (T;Q)$. Notice that $(T;Q) \bigvee (S \cup i;P_{-i}) = (T \cup i;Q_{-i})$ and $(T;Q) \bigwedge (S \cup i;P_{-i}) = (S;P)$. Applying Inequality (5) to $(T;Q)$ and $(S \cup i;P_{-i})$ and rearranging terms, we obtain

$$v\left(T \cup i;Q_{-i}\right) - v(T;Q) \geq v\left(S \cup i;P_{-i}\right) - v(S;P).$$

Let us take $P' \in \Pi(N \setminus S)$, $Q' \in \Pi(N \setminus T)$ such that P' covers P, Q' covers Q, and $(T;Q')$ covers $(S;P')$. Then, $(S;P)$ covers $(S;P')$ and $(T;Q)$ covers $(T;Q')$. Besides, $(S;P) \bigvee (T;Q') = (T;Q)$ and $(S;P) \bigwedge (T;Q') = (S;P')$. Then, applying Inequality (5) to $(S;P)$ and $(T;Q')$ and rearranging terms, we obtain

$$v(T;Q) - v\left(T;Q'\right) \geq v(S;P) - v\left(S;P'\right).$$

Second, we prove that *ii)* implies *iii)*. Let $(S;P),(T;Q) \in \mathcal{EC}^N$ with $(S;P) \sqsubseteq (T;Q) \neq (N;\emptyset)$. If $h(T;Q) - h(S;P) = 0$, then Inequalities (6) and (7) hold immediately because $(S;P) = (T;Q)$. If $h(T;Q) - h(S;P) = 1$, Inequalities (6) and (7) hold because v satisfies Inequalities (11) and (12). In the following, we assume that $h(T;Q) - h(S;P) > 1$. We divide the proof in two parts, the first to check Inequality (6) and the second to check Inequality (7). Figure 6 illustrates the scheme of the proof of the first part.

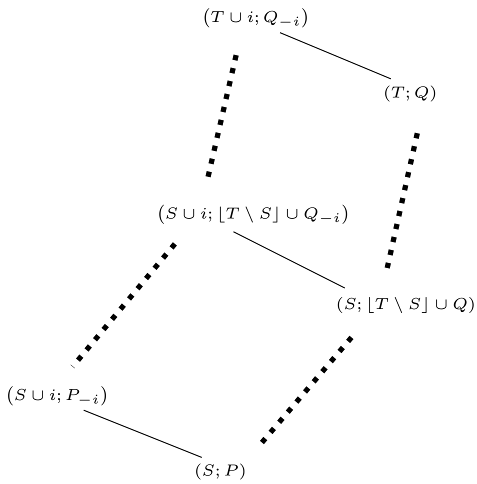

Figure 6: Inequality (6). Solid line: one link; dashed line: one or more links.

Let us assume that $h(T;Q) - h(S;P) = k > 1$. If there is some $\{i\} \in P$, then $\{i\} \in Q$. Take a chain $\lfloor T \setminus S \rfloor \cup Q = Q_0 \prec Q_1 \prec \ldots \prec Q_m = P$ with $m > 1$ in the lattice of partitions $(\Pi(N), \preceq)$. Notice that $\{i\} \in Q_j$, for every $j = 0, \ldots, m$. Note also that for every $j = 0, \ldots, m-1$, $(S; Q_j)$ covers $(S; Q_{j+1})$. Then, we can apply Inequality (11) to $(S; Q_{j+1})$ and $(S; Q_j)$ to get $v\left(S \cup i; (Q_j)_{-i}\right) - v(S; Q_j) \geq v\left(S \cup i; (Q_{j+1})_{-i}\right) - v(S; Q_{j+1})$, for every $j = 0, \ldots, m-1$. Thus,

$$\sum_{j=0}^{m-1} \left[v\left(S \cup i; (Q_j)_{-i}\right) - v(S; Q_j)\right] \geq \sum_{j=0}^{m-1} \left[v\left(S \cup i; (Q_{j+1})_{-i}\right) - v(S; Q_{j+1})\right],$$

which yields

$$v\left(S \cup i; \lfloor T \setminus S \rfloor \cup Q_{-i}\right) - v(S; \lfloor T \setminus S \rfloor \cup Q) \geq v\left(S \cup i; P_{-i}\right) - v(S; P). \tag{20}$$

If $T \setminus S = \emptyset$, Inequality (20) is Inequality (6) and the proof is finished. If $T \setminus S \neq \emptyset$, let us assume that $T \setminus S = \{i_1, \ldots, i_r\}$ and take $R_j = \{i_1, \ldots, i_j\}$, for every $j = 1, \ldots, r$ and $R_0 = \emptyset$. Now, for every $j \in \{0, \ldots, r-1\}$, $(S \cup R_j \cup i; \lfloor T \setminus (S \cup R_j) \rfloor \cup Q_{-i})$ covers $(S \cup R_j; \lfloor T \setminus (S \cup R_j) \rfloor \cup Q)$. We apply Inequality (11) to $(S \cup R_j; \lfloor T \setminus (S \cup R_j) \rfloor \cup Q) \sqsubseteq (S \cup R_j \cup i; \lfloor T \setminus (S \cup R_j) \rfloor \cup Q_{-i})$ and $i_{j+1} \in T \setminus S$, obtaining

$$\begin{aligned} & v(S \cup R_{j+1} \cup i; \lfloor T \setminus (S \cup R_{j+1}) \rfloor \cup Q_{-i}) - v(S \cup R_j \cup i; \lfloor T \setminus (S \cup R_j) \rfloor \cup Q_{-i}) \\ & \geq v(S \cup R_{j+1}; \lfloor T \setminus (S \cup R_{j+1}) \rfloor \cup Q) - v(S \cup R_j; \lfloor T \setminus (S \cup R_j) \rfloor \cup Q). \end{aligned}$$

Adding up these r inequalities, we get

$$v(T \cup i; Q_{-i}) - v(S \cup i; \lfloor T \setminus S \rfloor \cup Q_{-i}) \geq v(T; Q) - v(S; \lfloor T \setminus S \rfloor \cup Q). \tag{21}$$

Adding up Inequalities (20) and (21), and rearranging terms, we obtain

$$v(T \cup i; Q_{-i}) - v(T; Q) \geq v(S \cup i; P_{-i}) - v(S; P).$$

Then, Inequality (6) holds.

We check that Inequality (7) also holds. Figure 7 illustrates the scheme of the proof.

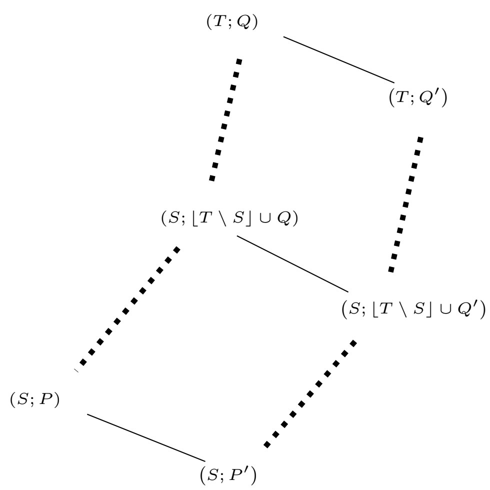

Figure 7: Inequality (7). Solid line: one link; dashed line: one or more links.

Let P' be a partition that covers P in $(\Pi(N \setminus S), \preceq)$, Q' be a partition that covers Q in $(\Pi(N \setminus T), \preceq)$, such that $(S; P') \sqsubseteq (T; Q')$. Take a pair of chains $Q_0 =$

$\lfloor T\setminus S\rfloor\cup Q\prec Q_1\prec\ldots\prec Q_m=P$ and $Q'_0=\lfloor T\setminus S\rfloor\cup Q'\prec Q'_1\prec\ldots\prec Q'_m=P'$ in the lattice of partitions $(\Pi(N),\preceq)$, such that Q'_j covers Q_j, for every $j=0,\ldots,m$ with $m>1$. Notice that both chains have the same length because P' covers P, Q' covers Q, $Q\prec P_{-T}$, and $Q'\prec P'_{-T}$. For every $j=0,\ldots,m$, $(S;Q_j)$ covers $(S;Q'_j)$. Then we apply Inequality (12) to $(S;Q_{j+1})\sqsubseteq(S;Q_j)$, Q'_{j+1}, and Q'_j, obtaining $v(S;Q_j)-v(S;Q'_j)\geq v(S;Q_{j+1})-v(S;Q'_{j+1})$, for every $j=0,\ldots,m-1$. Adding up these m inequalities, we get

$$v(S;\lfloor T\setminus S\rfloor\cup Q)-v(S;\lfloor T\setminus S\rfloor\cup Q')\geq v(S;P)-v(S;P'). \tag{22}$$

If $T\setminus S=\emptyset$, we finish the proof. If $T\setminus S\neq\emptyset$, we proceed as we did above in order to obtain Inequality (21) with $(S;\lfloor T\setminus S\rfloor\cup Q')\sqsubseteq(S;\lfloor T\setminus S\rfloor\cup Q)$ until we get $(T;Q')$ and $(T;Q)$. Hence,

$$v(T;Q)-v(S;\lfloor T\setminus S\rfloor\cup Q)\geq v(T;Q')-v(S;\lfloor T\setminus S\rfloor\cup Q') \tag{23}$$

Adding up Inequalities (22) and (23), we get $v(T;Q)-v(T;Q')\geq v(S;P)-v(S;P')$, concluding this part of the proof.

Finally, we check that *iii)* implies *i)* using Lemma 3.3, Lemma 3.4, and Lemma 3.2.

Let $(S;P),(T;Q)\in\mathcal{EC}^N$. If $(S;P)\sqsubseteq(T;Q)$ it is trivial to check Inequality (5). Let us assume $(S;P)$ and $(T;Q)$ are not comparable. We prove Inequality (5) using the disaggregation of the Hasse diagram among $(S;P)$, $(T;Q)$, $(S;P)\wedge(T;Q)$, and $(S;P)\vee(T;Q)$ depicted in Figure 8. Label **I** corresponds to a situation analized in Lemma 3.3, label **II** corresponds to a situation analized in Lemma 3.2, and label **III** corresponds to a situation analized in Lemma 3.4.

I.1 Apply Lemma 3.3 to $S\cap T\subseteq S$, $\lfloor T\setminus S\rfloor\cup(P_{-T}\bigvee Q_{-S})$, and $P\bigvee(\lfloor T\setminus S\rfloor\cup Q_{-S})$ because $\lfloor T\setminus S\rfloor\cup(P_{-T}\bigvee Q_{-S})\preceq P\bigvee(\lfloor T\setminus S\rfloor\cup Q_{-S})$. Then,

$$\begin{aligned}&v(S;\lfloor T\setminus S\rfloor\cup(P_{-T}\textstyle\bigvee Q_{-S}))+v(S\cap T;\lfloor S\setminus T\rfloor\cup(P\bigvee(Q_{-S}\cup\lfloor T\setminus S\rfloor))\geq\\&v(S;P\textstyle\bigvee(\lfloor T\setminus S\rfloor\cup Q_{-S}))+v(S\cap T;\lfloor S\setminus T\rfloor\cup\lfloor T\setminus S\rfloor\cup(P_{-T}\bigvee Q_{-S})).\end{aligned} \tag{24}$$

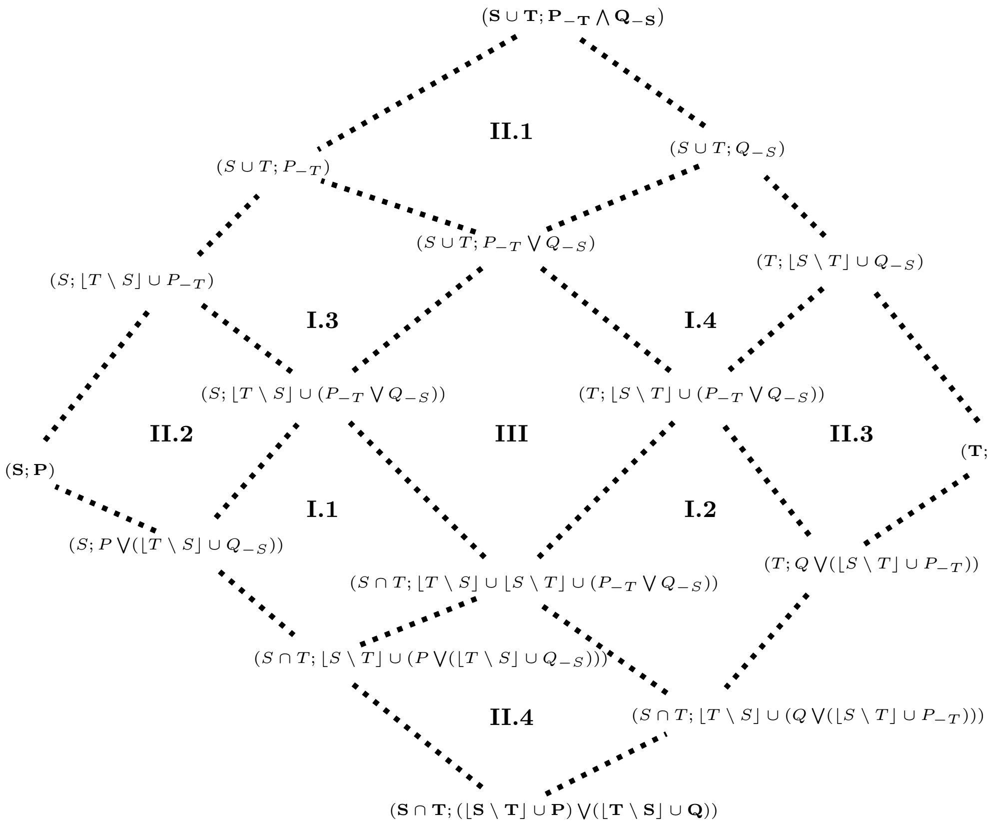

Figure 8: The structure of the proof.

I.2 Apply Lemma 3.3 to $S \cap T \subseteq T$, $\lfloor S \setminus T \rfloor \cup (P_{-T} \bigvee Q_{-S})$, and $Q \bigvee (\lfloor S \setminus T \rfloor \cup P_{-T})$ because $\lfloor S \setminus T \rfloor \cup (P_{-T} \bigvee Q_{-S}) \preceq Q \bigvee (\lfloor S \setminus T \rfloor \cup P_{-T})$. Then,

$$v(T; \lfloor S \setminus T \rfloor \cup (P_{-T} \bigvee Q_{-S})) + v(S \cap T; \lfloor T \setminus S \rfloor \cup (Q \bigvee (P_{-T} \cup \lfloor S \setminus T \rfloor)) \geq v(T; Q \bigvee (\lfloor S \setminus T \rfloor \cup P_{-T})) + v(S \cap T; \lfloor S \setminus T \rfloor \cup \lfloor T \setminus S \rfloor \cup (P_{-T} \bigvee Q_{-S})). \tag{25}$$

I.3 Apply Lemma 3.3 to $S \subseteq S \cup T$, P_{-T}, and $P_{-T} \bigvee Q_{-S}$ because $P_{-T} \preceq P_{-T} \bigvee Q_{-S}$. Then,

$$v(S \cup T; P_{-T}) + v(S; \lfloor T \setminus S \rfloor \cup (P_{-T} \bigvee Q_{-S})) \geq v(S \cup T; P_{-T} \bigvee Q_{-S}) + v(S; \lfloor T \setminus S \rfloor \cup P_{-T}). \tag{26}$$

I.4 Apply Lemma 3.3 to $T \subseteq S \cup T$, Q_{-S}, and $P_{-T} \bigvee Q_{-S}$ because $Q_{-S} \preceq$

$P_{-T} \bigvee Q_{-S}$. Then,

$$v(S \cup T; Q_{-S}) + v(T; \lfloor S \setminus T \rfloor \cup (P_{-T} \bigvee Q_{-S})) \geq v(S \cup T; P_{-T} \bigvee Q_{-S}) + v(T; \lfloor S \setminus T \rfloor \cup Q_{-S}). \tag{27}$$

II.1 Apply Lemma 3.2 to $(S \cup T; P_{-T})$ and $(S \cup T; Q_{-S})$. Then,

$$v(S \cup T; P_{-T} \bigwedge Q_{-S}) + v(S \cup T; P_{-T} \bigvee Q_{-S}) \geq v(S \cup T; P_{-T}) + v(S \cup T; Q_{-S}). \tag{28}$$

II.2 Apply Lemma 3.2 to $(S; P)$ and $(S; \lfloor T \setminus S \rfloor \cup (P_{-T} \bigvee Q_{-S}))$. Then,

$$v(S; \lfloor T \setminus S \rfloor \cup P_{-T}) + v(S; P \bigvee (\lfloor T \setminus S \rfloor \cup Q_{-S})) \geq v(S; P) + v(S; \lfloor T \setminus S \rfloor \cup (P_{-T} \bigvee Q_{-S})). \tag{29}$$

II.3 Apply Lemma 3.2 to $(T; Q)$ and $(T; \lfloor S \setminus T \rfloor \cup (P_{-T} \bigvee Q_{-S}))$. Then,

$$v(T; \lfloor S \setminus T \rfloor \cup Q_{-S}) + v(T; Q \bigvee (\lfloor S \setminus T \rfloor \cup P_{-T})) \geq v(T; Q) + v(T; \lfloor S \setminus T \rfloor \cup (P_{-T} \bigvee Q_{-S})). \tag{30}$$

II.4 Apply Lemma 3.2 to $(S \cap T; \lfloor T \setminus S \rfloor \cup (Q \bigvee (P_{-T} \cup \lfloor S \setminus T \rfloor))$ and $(S \cap T; \lfloor S \setminus T \rfloor \cup (P \bigvee (Q_{-S} \cup \lfloor T \setminus S \rfloor))$. Then,

$$\begin{gathered} v(S \cap T; \lfloor S \setminus T \rfloor \cup \lfloor T \setminus S \rfloor \cup (P_{-T} \bigvee Q_{-S})) + v(S \cap T; (\lfloor S \setminus T \rfloor \cup P) \bigvee (\lfloor T \setminus S \rfloor \cup Q)) \geq \\ v(S \cap T; \lfloor T \setminus S \rfloor \cup (Q \bigvee (P_{-T} \cup \lfloor S \setminus T \rfloor)) + v(S \cap T; \lfloor S \setminus T \rfloor \cup (P \bigvee (Q_{-S} \cup \lfloor T \setminus S \rfloor)). \end{gathered} \tag{31}$$

III Apply Lemma 3.4 to S, T, and $P_{-T} \bigvee Q_{-S}$. Then,

$$\begin{gathered} v(S \cup T; P_{-T} \bigvee Q_{-S}) + v(S \cap T; (\lfloor S \setminus T \rfloor \cup \lfloor T \setminus S \rfloor) \cup (P_{-T} \bigvee Q_{-S})) \geq \\ v(S; \lfloor T \setminus S \rfloor \cup (P_{-T} \bigvee Q_{-S})) + v\left(T; \lfloor S \setminus T \rfloor \cup (P_{-T} \bigvee Q_{-S})\right). \end{gathered} \tag{32}$$

Adding up Inequalities (26), (28), and (29), we obtain

$$v(S \cup T; P_{-T} \bigwedge Q_{-S}) + v(S; P \bigvee (\lfloor T \setminus S \rfloor \cup Q_{-S})) \geq v(S \cup T; Q_{-S}) + v(S; P). \tag{33}$$

Adding up Inequalities (24), (27), and (32), we obtain

$$v(S \cup T; Q_{-S}) + v(S \cap T; \lfloor S \setminus T \rfloor \cup (P \bigvee (Q_{-S} \cup \lfloor T \setminus S \rfloor)) \geq \\ v(T; \lfloor S \setminus T \rfloor \cup Q_{-S}) + v(S; P \bigvee (\lfloor T \setminus S \rfloor \cup Q_{-S})). \tag{34}$$

Adding up Inequalities (25), (30), and (31), we obtain

$$v(T; \lfloor S \setminus T \rfloor \cup Q_{-S}) + v(S \cap T; (\lfloor S \setminus T \rfloor \cup P) \bigvee (\lfloor T \setminus S \rfloor \cup Q)) \\ \geq v(T; Q) + v(S \cap T; \lfloor S \setminus T \rfloor \cup (P \bigvee (Q_{-S} \cup \lfloor T \setminus S \rfloor)). \tag{35}$$

Finally, adding up Inequalities (33), (34), and (35), we obtain

$$v\left(S \cup T; P_{-T} \bigwedge Q_{-S}\right) + v\left(S \cap T; (\lfloor S \setminus T \rfloor \cup P) \bigvee (\lfloor T \setminus S \rfloor \cup Q)\right) \geq v(S; P) + v(T; Q).$$

Summarizing all the previous results, we have the characterization of convexity for games with externalities given in Theorem 3.1. □

Proof of Proposition 4.1. Let $x \in Core(v_{max})$ and $P \in \Pi(N)$. Then,

$$\sum_{i \in N} x_i = v(N; \emptyset) = v^P(N).$$

For every $S \subseteq N$,

$$\sum_{i \in S} x_i \geq v_{max}(S) \geq v^P(S).$$

Then, $x \in Core\left(v^P\right)$. Let $x \in \bigcap_{P \in \Pi(N)} Core\left(v^P\right)$. Let $(S, Q) \in \mathcal{EC}^N$ be such that $v_{max}(S) = v(S; Q)$. Take, for instance, $P = Q \cup \lceil S \rceil$. It is clear that $P_{-S} = Q$ and $v^P(S) = v(S; Q) = v_{max}(S)$. Since $x \in Core\left(v^P\right)$, we have

$$\sum_{i \in S} x_i \geq v^P(S) = v(S; Q) = v_{max}(S).$$

Thus, $x \in Core(v_{max})$. □

Proof of Theorem 4.1. Let $v \in \mathcal{G}^N$ be a convex game.

1. Let $P, Q \in \Pi(N)$ such that $Q \preceq P$. Recall that, $v^P(N) = v^Q(N)$. Let $x \in Core\left(v^Q\right)$, then
$$x_S = \sum_{i \in S} x_i \geq v(S; Q_{-S}) \geq v(S; P_{-S}),$$
where the first inequality follows because $x \in Core(v^Q)$ and the second inequality because $(S; P_{-S}) \sqsubseteq (S; Q_{-S})$ and Lemma 3.1 holds. Then, $x \in Core(v^P)$.

2. Let $S \subseteq N$. Since v is convex and according to Lemma 3.1, we have $v(S;Q) \geq v(S;P)$, for every $(S;P), (S;Q) \in \mathcal{EC}^N$ with $(S;P) \sqsubseteq (S;Q)$. Notice that $(S; \lceil N \setminus S \rceil) \sqsubseteq (S;Q) \sqsubseteq (S; \lfloor N \setminus S \rfloor)$, for every $(S;Q) \in \mathcal{EC}^N$ and there is no $(S;M) \in \mathcal{EC}^N$ such that $(S;M) \sqsubset (S; \lceil N \setminus S \rceil) \sqsubseteq (S;Q)$ nor $(S;M') \in \mathcal{EC}^N$ with $(S;Q) \sqsubseteq (S; \lfloor N \setminus S \rfloor) \sqsubset (S;M')$. As a consequence of all this, $v_{max}(S) = \max\left\{v^Q(S) : Q \in \Pi(N)\right\} = v^{\lfloor N \rfloor}(S)$ and $v_{min}(S) = \min\left\{v^Q(S) : Q \in \Pi(N)\right\} = v^{\lceil N \rceil}(S)$.

3. First, we see that $v_{max} = v^{\lfloor N \rfloor}$ is a convex game. Let $i \in N$, $S \subseteq T \subseteq N \setminus i$. We prove that
$$v^{\lfloor N \rfloor}(T \cup i) - v^{\lfloor N \rfloor}(T) \geq v^{\lfloor N \rfloor}(S \cup i) - v^{\lfloor N \rfloor}(S). \tag{36}$$
Notice that $\{i\} \in \lfloor N \rfloor_{-S}$ and $(S; \lfloor N \setminus S \rfloor) \sqsubseteq (T; \lfloor N \setminus T \rfloor) \neq (N; \emptyset)$. According to Item *iii.1)* in Theorem 3.1, we have $v\left(T \cup i; \lfloor N \setminus (T \cup i) \rfloor\right) - v(T; \lfloor N \setminus T \rfloor) \geq v(S \cup i; \lfloor N \setminus (S \cup i) \rfloor) - v(S; \lfloor N \setminus S \rfloor)$, or equivalently, $v^{\lfloor N \rfloor}(T \cup i) - v^{\lfloor N \rfloor}(T) \geq v^{\lfloor N \rfloor}(S \cup i) - v^{\lfloor N \rfloor}(S)$. Thus, Inequality (36) holds and $v^{\lfloor N \rfloor}$ is a convex game.

□

Example 5.8. *There are convex games whose associated pessimistic game is not a classic convex game.*

We revisit the game used in the Proof of Proposition 5.5. In this case, we have

$$v_{min}(N) = 15,\ v_{min}(S) = 10, \textit{ for every } S \subset N \textit{ with } |S| = 2, \textit{ and}$$
$$v_{min}(S) = 4, \textit{ for every } S \subset N \textit{ with } |S| = 1.$$

This classic game is not convex. For instance, if we take $S = \{1\} \subseteq T = \{1, 2\}$ *and* $i = 3$*, we have*

$$v_{min}(N) - v_{min}(T) = 15 - 10 = 5 < 6 = 10 - 4 = v_{min}(S \cup i) - v_{min}(S).$$