Properties of resonant activation phenomena

Marián Boguñá, Josep M. Porrà, and Jaume Masoliver
Departament de Física Fonamental, Universitat de Barcelona, Diagonal 647, 08028 Barcelona, Spain

Katja Lindenberg
Department of Chemistry and Biochemistry 0340 and Institute for Nonlinear Science, University of California–San Diego, La Jolla, California 92093-0340

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The phenomenon of resonant activation of a Brownian particle over a fluctuating barrier is revisited. We discuss the important distinctions between barriers that can fluctuate among ''up'' and ''down'' configurations, and barriers that are always ''up'' but that can fluctuate among different heights. A resonance as a function of the barrier fluctuation rate is found in both cases, but the nature and physical description of these resonances is quite distinct. The nature of the resonances, the physical basis for the resonant behavior, and the importance of boundary conditions are discussed in some detail. We obtain analytic expressions for the escape time over the barrier that explicitly capture the minima as a function of the barrier fluctuation rate, and show that our analytic results are in excellent agreement with numerical results. [S1063-651X(98)11304-1]

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I. INTRODUCTION

Noise-induced nonequilibrium phenomena in nonlinear systems have recently attracted a great deal of attention in a variety of contexts [1]. In general, these phenomena involve a response of the system that is not only produced or enhanced by the presence of the noise, but that is *optimized* for certain values of the parameters of the noise. One example is the phenomenon of stochastic resonance [2], wherein the response of a nonlinear system to a signal is enhanced by the presence of noise and maximized for certain values of the noise parameters. Another involves nonequilibrium ratchets, wherein intrinsically unbiased Brownian motion in stochastic asymmetric potentials leads to a systematic drift motion whose magnitude and even direction can be tuned by the parameters of the noise [3,4]. A third is the recent discovery of a re-entrant noise-induced phase transition in a nonlinear coupled array, that is, a transition that is only observed for certain finite ranges of noise parameters [5]. A fourth such phenomenon, the one of interest to us in this paper, has been called ''resonant activation,'' and was first identified by Doering and Gadoua [6] and further studied by a number of other authors. Here the mean escape time of a particle driven by (usually white) noise over a barrier of randomly fluctuating height exhibits a minimum as a function of the parameters of the barrier fluctuations.

Our interest in this problem first arose because it seemed to us that for sufficiently simple potentials it should be possible to find *analytic* dependences of the escape rate on the system parameters (or at least good approximations to them) and, more specifically, that it should be possible to find analytic expressions for the parameter combinations that lead to the minimum in the escape rate. Some analytic results are available [6–11], including those in the original work of Doering and Gadoua that apply to a very specific circumstance discussed in more detail below. In general, however, most available results are numerical [11]. Analytic results are scarce, and usually apply only to one parameter regime *or* another and are thus unable to explicitly capture the occurrence of a minimum in the escape rate. A minimum in the escape rate usually arises from these approximations only by inference, and the approximations provide no way to locate the minimum specifically, except as an intersection point of two unrelated approximations. They also do not provide a way to determine the dependence of the minimum on the system parameters.

In this paper we accomplish our goal, that is, we obtain a number of analytic results for moments of the first passage time over a fluctuating barrier for the particular model system (a triangular potential barrier subject to dichotomous fluctuations) used in a number of studies of resonant activation. In particular, we obtain analytic approximations that explicitly capture not only the minimum in the escape rate but that allow us to study the variability of the escape rate in parameter space, that is, the depth and width of this minimum.

In the process of obtaining these results, we have also accomplished a number of important clarifications on the nature of models that have been presented under the common ''resonant activation'' rubric, and on the nature of resonant activation itself. Some of these models in fact differ from one another in essential respects. We discuss these clarifications and differences in some detail, and thus shed some light on the role played by the interplay of the white noise and the barrier fluctuations on the escape process. We anticipate some of our findings.

(i) A distinction must be made between situations in which the fluctuating barrier can be ''up'' or ''down'' (i.e., can go from being a barrier to being flat or even a well), and situations in which there is always a barrier. Although resonant behavior can be observed in all cases, the physical picture underlying this behavior is different in different cases.

(ii) Boundary conditions play an extremely important role in the problem.

(iii) The qualitative physical description of the resonance in the fluctuating barrier problem is as follows. When the barrier fluctuates extremely slowly within a range $0<v_{\min}\leqslant v\leqslant v_{\max}$, the mean first passage time to the top of the barrier is extremely long because it is dominated by those realizations for which the barrier starts in the high position. The mean first passage time is then proportional to $e^{v_{\max}/D}$, where D is the intensity of the white noise. Indeed, if the barrier fluctuation rate is smaller than the inverse of the mean first passage time to the highest barrier, the barrier is essentially quasistatic throughout the process. At the other extreme, if the barrier fluctuates very rapidly, the mean first passage time is determined by the average barrier v_0, i.e., it is proportional to $e^{v_0/D}$. Between these extremes, and over a broad range of barrier fluctuation rates, passage over the barrier occurs primarily when the barrier is low, and the mean first passage time is then proportional to $e^{v_{\min}/D}$. This dependence is quite robust, and the prefactor determines the actual minimum within this broad range.

(iv) This behavior does not require that the barrier fluctuate; an oscillatory variation of the barrier height yields essentially the same results.

In Sec. II we provide a detailed statement of the resonance activation problem. Section III discusses the analytic solution of the ''up-down'' case; we show that the resonance flipping rate and the resonance activation in this case are independent of the white noise intensity. In Sec. IV the significance of the white noise and of the boundary conditions in this ''up-down'' problem are discussed in detail. Section V deals with the case of barrier fluctuations when the barriers are always high. We obtain a single analytic expression for the mean first passage time that exhibits a minimum as a function of the barrier fluctuation rate and that in fact quantitatively captures the correct behavior over most of parameter space, as determined by comparison with numerical results. With this result we are able to determine the resonance frequency analytically, and also the range of barrier fluctuation rates over which the mean first passage time is essentially flat. In Sec. VI we discuss the case of a barrier that oscillates (rather than fluctuates). This case also exhibits resonant activation, although some of the quantitative details of the problem are slightly modified. Finally, we conclude with a summary and some final points in Sec. VII.

II. STATEMENT OF THE PROBLEM

Consider a process that evolves in a bistable potential and is driven by weak Gaussian white noise, so that the process is occasionally able to cross from one minimum of the bistable potential to the other. If the parameters of the system are fixed in time, the rate at which the process crosses from one well to the other under a variety of conditions is well known (e.g., the Kramers rate). Suppose now that the height of the barrier separating the two minima of the bistable potential fluctuates in time. We wish to explore the effect of the barrier fluctuations on the rate of passage of the process from one well to the other. More specifically, it is known that there is an optimal barrier fluctuation rate that minimizes the passage time from one well to the other for given parameter values [6–11]. This minimum identifies the phenomenon of *resonant activation*. We are interested in the analytic properties of the resonant activation phenomenon. Note that the barrier fluctuations here are such that the energy difference

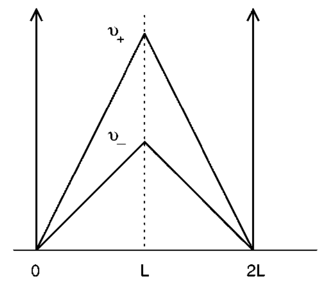

FIG. 1. Schematic of the fluctuating potential barrier problem.

between the potential minima remains constant — only the barrier *height* fluctuates. This is to be contrasted with the phenomenon of stochastic resonance, where the energy difference is modulated by a small periodic signal.

We adhere to the overdamped regime, and hence the process $y(\tau)$ evolves according to the Langevin equation

$$\dot{y}(\tau)=-V'(y)-g'(y)\eta(\tau)+\xi(\tau). \tag{1}$$

Here $\xi(\tau)$ is zero-centered Gaussian white noise with correlation function

$$\langle\xi(\tau)\xi(\tau')\rangle=2D\delta(\tau-\tau'). \tag{2}$$

One can think of the white noise as arising from a heat bath, in which case the diffusion coefficient D is proportional to the bath temperature T. Time is measured in units of the friction coefficient, which has been set to unity in Eq. (1).

The potential $V(y)$ is a bistable potential, typically with isoenergetic minima. Doering and Gadoua [6] introduced the triangular potential shown in Fig. 1. The potential barrier is defined by

$$V(y)=\begin{cases} v_0 y/L, & 0\leqslant y<L\\ -v_0 y/L+2v_0, & L\leqslant y\leqslant 2L,\end{cases} \tag{3}$$

and the potential rises to infinity at $y=0$ and at $y=2L$. In the absence of the contribution $g'(y)\eta(\tau)$ in Eq. (1), this represents a standard problem where the rate κ at which the process crosses the barrier at $x=L$ is related to the mean first passage time $\widetilde{T}_1$ from the bottom of one of the wells, say the one at $y=0$, to the top of the barrier: $\kappa=1/2\widetilde{T}_1$. To calculate the mean first passage time one assumes a reflecting boundary condition at $y=0$ and an absorbing boundary condition at $y=L$.

In the resonant activation problem we have, in addition, the contribution $g'(y)\eta(\tau)$. Here $\eta(\tau)$ is a nonequilibrium noise that, coupled to $g'(y)$, causes the potential barrier to fluctuate. It is a nonequilibrium noise because there is no dissipative contribution in the equation of motion associated with this fluctuating term, and hence the system is open. The noise $\eta(\tau)$ is usually taken to be exponentially correlated, the most ubiquitous choices being Ornstein-Uhlenbeck noise [9–11] and Markovian dichotomous noise [8,11]. Here we deal only with the latter: $\eta(\tau)$ takes on the values ±1, and the change from one to the other is distributed in time according to the exponential density function

$$\phi(\tau)=\gamma e^{-\gamma\tau}, \tag{4}$$

so that the flipping rate of the dichotomous noise is γ. The fluctuating barrier is accomplished by picking for $g(y)$ the function

$$g(y)=\begin{cases}\alpha y/L, & 0\leq y<L\\ -\alpha y/L+2\alpha, & L\leq y\leq 2L,\end{cases} \tag{5}$$

and zero otherwise. The addition of the random potential term $g(y)\eta(\tau)$ causes the potential barrier to switch between the two values $v_+=v_0+\alpha$ and $v_-=v_0-\alpha$.

We wish to calculate the rate at which the process crosses the point $y=L$, which in turn is related, as before, to the mean first passage time $\widetilde{T}_1$ from $y=0$ to $y=L$ when a reflecting boundary is located at $y=0$ and an absorbing boundary at $y=L$. (The distortions in the potential profile that may be caused by multiplicative noise, and the implications on the appropriate definition of an escape time, are well known and have been widely discussed in the literature; see, e.g., Ref. [11]. The potentials used here do not exhibit such distortions.) In particular, we wish to establish analytically the dependence of $\widetilde{T}_1$ on the flipping rate γ, and to identify the flipping rate for which $\widetilde{T}_1$ is a minimum.

Doering and Gadoua [6] calculated the mean first passage time for this model in the absence of the potential $V(y)$, that is, when the ''barrier'' flips between being a true barrier (of height $+\alpha$) to being a well $(-\alpha)$, and they obtained a resonance phenomenon, that is, the mean first passage time from 0 to L exhibits a minimum at a particular value of the flipping rate γ. Doering and Gadoua also presented simulation results for the case $v_0=\alpha$, that is, when the ''barrier'' flips between being a true barrier (height 2α) and there being no barrier. Bier and Astumian [7] considered the true barrier case, that is, the case where there is always a barrier (in fact, they took $v_0\gg\alpha$), and obtained numerical results that show a resonance. Their analytic barrier crossing rate results are obtained separately for low flipping rates (small γ) and for high flipping rates (large γ). Neither result in itself exhibits a minimum, although one can infer the existence of a minimum (but not its dependence on the system parameters; see also Ref. [11]) from their combination.

With this general statement of the problem we can be more precise about the results that we present in this paper. First, we consider the $\pm\alpha$ barrier-well case of Doering and Gadoua, reproduce their analytic results for the mean first passage time, and also obtain analytic results for the resonant mean first passage time, the resonant flipping rate, and the second moment of the first passage time distribution. We argue that the $\pm\alpha$ barrier case represents a situation that is completely different from the ''true barrier'' case considered by Bier and Astumian. Both exhibit resonance behavior, but via different mechanisms. We explore these differences and interpret the Doering-Gadoua case on the basis of an even simpler model. Furthermore, we obtain analytic results for the high barrier case considered by Bier and Astumian that yield an *explicit* minimum in the mean first passage time as a function of the flipping rate. We present analytic results for the mean first passage time at resonance, for the resonant flipping rate, and we analyze the behavior of the system away from this point to assess how sharp this resonance might be.

We shall present our analysis and results in terms of the dimensionless variables

$$t\equiv\tau D/L^2,\quad x\equiv y/L,\quad T_1\equiv\widetilde{T}_1 D/L^2, \tag{6}$$

and the dimensionless parameters

$$a\equiv\alpha/D,\quad V_0\equiv v_0/D,\quad \lambda\equiv\gamma L^2/D. \tag{7}$$

The differential equation for the mean first passage time in all cases considered in this paper is given by

$$\frac{d^4T_1}{dx^4}-2V_0\frac{d^3T_1}{dx^3}+(V_0^2-a^2-2\lambda)\frac{d^2T_1}{dx^2}+2\lambda V_0\frac{dT_1}{dx}=2\lambda, \tag{8}$$

with the boundary conditions

$$\left.\frac{dT_1}{dx}\right|_{x=0}=0, \tag{9}$$

$$\left.\frac{d^2T_1}{dx^2}\right|_{x=0}=-1 \tag{10}$$

at the reflecting boundary, and

$$T_1(x=1)=0, \tag{11}$$

$$\left[\frac{d^3T_1}{dx^3}-2V_0\frac{d^2T_1}{dx^2}+(V_0^2-a^2)\frac{dT_1}{dx}\right]_{x=1}=V_0 \tag{12}$$

at the absorbing boundary. A brief description of how this equation and boundary conditions arise is given in the Appendix.

III. ANALYTICAL SOLUTION OF THE DOERING-GADOUA MODEL

Consider the mean first passage time to $x=1$ $(y=L)$ when the mean barrier height is $V_0=0$. The solution as a function of the initial position x can in this case be given analytically:

$$\begin{aligned}T_1(x)=&(x-1)\left[\frac{2\lambda a^2}{\mu^3}\frac{\mu-\sinh(\mu)}{a^2+2\lambda\cosh(\mu)}-\frac{\lambda}{\mu^2}(x+1)\right]\\&-\frac{2a^2}{\mu^4}\frac{\sinh[\mu(x-1)/2]}{a^2+2\lambda\cosh(\mu)}\{a^2\sinh[\mu(x+1)/2]\\&+2\lambda\sinh[\mu(x-1)/2]+2\mu\lambda\cosh[\mu(x+1)/2]\},\end{aligned} \tag{13}$$

where we have introduced the symbol

$$\mu=\sqrt{a^2+2\lambda}. \tag{14}$$

This result has been previously reported for the particular initial value $x=0$ [6]. We know from Ref. [6] that $T_1(x=0)\equiv T_1$ exhibits a resonance with respect to λ; we wish to establish the resonance flipping rate λ_{res}, and the behavior of the mean first passage time at this resonance point.

The expression for the mean first passage time simplifies considerably when the dimensionless quantity μ is large ($\mu \gg 1$), which is the physically interesting weak-white-noise regime. Indeed, the only way that μ could be small is if α and λ are small, that is, if (in dimensioned units) the white noise intensity D is greater than the barrier height α and greater than $2\gamma L^2$. The first condition renders the problem uninteresting — if barriers are on average lower than the noise then one has an essentially free diffusion problem. The second condition requires a small system with a low flipping rate, again a very specific situation that is not particularly interesting in this context. The customarily interesting physical situation occurs when the white noise is weak compared to the barrier height, that is, when $a \gg 1$ and this in turn leads to $\mu \gg 1$. We use these two statements of the ''interesting regime'' interchangeably.

When μ is sufficiently large, the following approximations are valid:

$$\cosh \mu - 1 \sim \cosh \mu \sim \sinh \mu - \mu \sim \sinh \mu \sim \frac{1}{2} e^{\mu}. \tag{15}$$

If, in addition,

$$\lambda \gg a^2 e^{-a}, \tag{16}$$

then the result simplifies even further, and one finally obtains the following much simpler approximate expression:

$$T_1(x) \sim \frac{a^2(a^2 - 2\lambda e^{-\mu x})}{2\lambda(a^2+2\lambda)^2} + \frac{a^2(2-x)}{(a^2+2\lambda)^{3/2}} + \frac{\lambda(1-x^2)}{a^2+2\lambda} + O(a^2 e^{-\mu}). \tag{17}$$

It can be shown that this expression as a function of λ has a minimum at a finite value λ_{res} that to leading order is of the form $\lambda_{\mathrm{res}} \sim a$. This minimum is identified among the roots of $dT_1(x)/dx = 0$ as the one that coincides with the minimum of the complete expression of $T_1(x)$ [Eq. (13)] as $a \to \infty$. Explicitly, following Ref. [6], we set $x=0$ to simplify the analysis further:

$$T_1 \sim \frac{(a^2-2\lambda)a^2}{2\lambda(a^2+2\lambda)^2} + \frac{2a^2}{(a^2+2\lambda)^{3/2}} + \frac{\lambda}{a^2+2\lambda} + O(a^2 e^{-\mu}). \tag{18}$$

The extrema of T_1 as a function of λ obey the equation

$$\frac{\lambda^2}{a^2}(a^2+2\lambda) - \frac{2\lambda}{a^2}(a^2-2\lambda) - \frac{1}{2}(a^2+2\lambda) - \frac{6\lambda^2}{a^2}(a^2+2\lambda)^{1/2} = 0. \tag{19}$$

This equation can be solved perturbatively by taking $\lambda \sim a(\lambda_0 + \lambda_1 a^{-1} + \lambda_2 a^{-2})$, an expansion consistent with the fact that $a \gg 1$. Once this expansion is substituted into Eq. (19), the following result is obtained for the resonant flipping rate as a function of a:

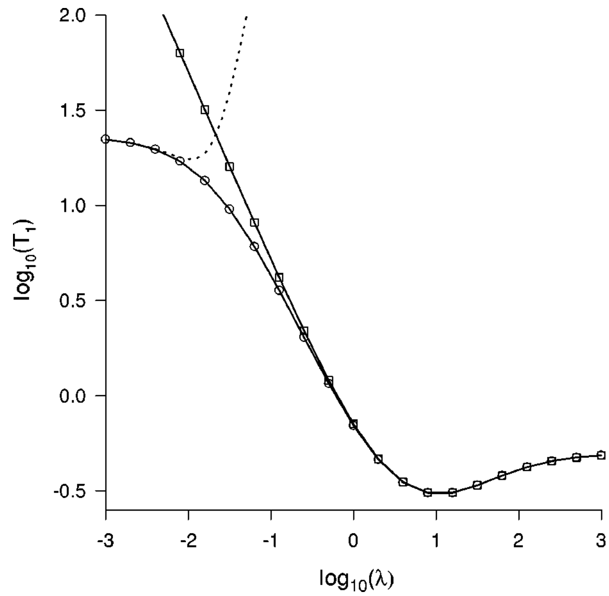

FIG. 2. Mean first passage time as a function of barrier fluctuation rate for the Doering-Gadoua model. The barrier fluctuates between the up and down positions with slopes a and $-a$, respectively. Solid curve with circles: exact mean first passage time obtained numerically. Solid curve with squares: our analytical result (18). Dotted curve: the low-frequency approximation (22). The parameter $a=8$.

$$\lambda_{\mathrm{res}} \sim \frac{1}{\sqrt{2}} a + \left(1 + \frac{3}{\sqrt{2}}\right) + \frac{3}{2}\left(3 + \frac{7}{\sqrt{2}}\right)\frac{1}{a}, \quad a \gg 1. \tag{20}$$

Note that the condition $\lambda \gg a^2 e^{-a}$ is thus satisfied. The value of T_1 at the resonant flipping rate is

$$T_1(\lambda = \lambda_{\mathrm{res}}) \equiv T_{\mathrm{res}} \sim (2+\sqrt{2})\frac{1}{a} - (4+3\sqrt{2})\frac{1}{a^2} + \left(\frac{3}{2} + \frac{1}{\sqrt{2}}\right)\frac{1}{a^3}, \tag{21}$$

provided a is large. For small values of λ ($\lambda \ll a^2 e^{-a}$) one readily finds from Eq. (13) with $x=0$ that

$$T_1 \sim \frac{1}{2a^2} e^{a} - \frac{\lambda}{2a^4} e^{2a} + \frac{\lambda^2}{2a^6} e^{3a} + O\left(\frac{\lambda^3}{a^6} e^{4a}\right), \tag{22}$$

when $a \gg 1$.

In Fig. 2, the exact expression of T_1 is compared with approximation (22) when $\lambda \to 0$, and with approximation (18) when $\lambda \to \infty$.

The second-order moment of the first passage time distribution, T_2, also exhibits a resonance. However, the resonant frequency of T_2 does not coincide with that of T_1. This means that there does not exist a unique resonant frequency or universal scaling associated with the first passage time distribution. T_2 can be calculated in a similar way (albeit even more expansively) as T_1. However, the full expression for $T_2(x)$ is too long to be included here. Instead, we only reproduce the expression for T_2 at $x=0$ when $a \gg 1$,

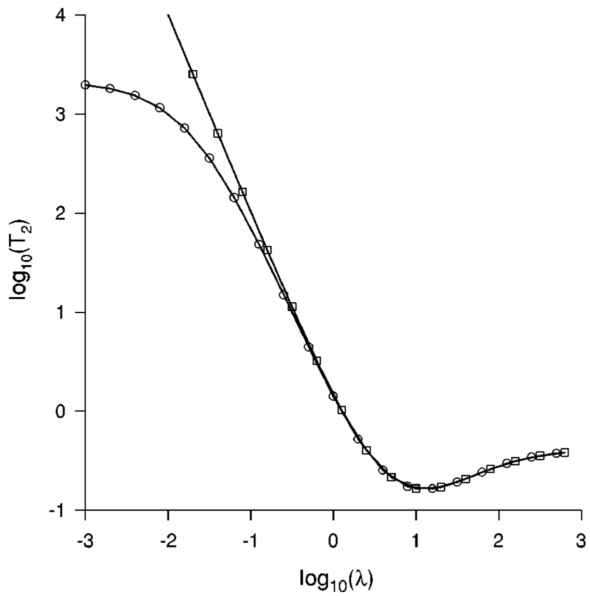

FIG. 3. Second moment of the first passage time distribution for the Doering-Gadoua model as a function of barrier fluctuation rate, with $a=8$. Solid curve with circles: exact second moment obtained numerically. Solid curve with squares: our analytical result (23).

$$T_2 \sim \frac{a^4(a^4+a^2\lambda-4\lambda^2)}{\lambda^2\mu^8}+\frac{4a^2(a^2-\lambda^2)}{\lambda\mu^7}+\frac{2a^2(4a^2-\lambda)}{\mu^6}+\frac{20\lambda a^2}{3\mu^5}+\frac{5\lambda^2}{3\mu^4}+O(a^4e^{-\mu}). \quad (23)$$

In Fig. 3, this approximate result is compared with the exact expression for T_2 when $a=8$. The behavior of the resonant frequency $\lambda_{\mathrm{res},2}$ of T_2 can be obtained from the expression for T_2 when $a\to\infty$. The calculation yields

$$\lambda_{\mathrm{res},2} \sim b_0 a + b_1 + O(a^{-1}), \quad (24)$$

where b_0 is the positive real solution of the equation $5b_0^4 + 10b_0^3 - 6b_0 - 3 = 0$. Numerical solution of this equation leads to the value $b_0 = 0.825\,724\ldots$. With this value, the coefficient b_1 can also be evaluated numerically, and one obtains $b_1 = 3.560\,57$. Note that $\lambda_{\mathrm{res},2} > \lambda_{\mathrm{res}}$. Finally, T_2 at the resonant frequency behaves as $1/a^2$ when $a\to\infty$:

$$T_2(\lambda_{\mathrm{res},2}) \sim \frac{c_0}{a^2}+\frac{c_1}{a^3}+O(a^{-4}), \quad (25)$$

with $c_0 = 20.9521\ldots$ and $c_1 = -104.244\ldots$.

Equations (18), (20), (21), and (22) are the main results of this section — they give the mean first passage time as a function of the barrier height (or well depth) a, provided $a \ll e^a$, for all values of λ. The resonant behavior of the mean first passage time as a function of the flipping rate of the barrier is clear and dramatic. It is useful to exhibit explicitly the limiting results in the original units so that the dependence on system parameters is clear. When the flipping rate is large ($\lambda \gg a^2$ or $\gamma \gg \alpha^2/DL^2$) the mean first passage time from $y=0$ to $y=L$ grows as

$$\widetilde{T}_1 \to \widetilde{T}_{\mathrm{white}} = \frac{L^2}{2D_{\mathrm{eff}}}, \quad (26)$$

as seen from Eq. (18). Here

$$D_{\mathrm{eff}} \equiv D + \frac{\alpha^2}{2\gamma L^2}. \quad (27)$$

This is the well-known result for the mean first passage time from 0 to L for a freely diffusing particle with diffusion constant D_{eff}. In this limit the flipping barrier behaves simply as an additional source of white noise of intensity $\alpha^2/2\gamma L^2$. At the other extreme, when the flipping rate is very low, $\lambda\to 0$, the barrier never flips as the process moves from 0 to L. If the barrier is initially ''down'' [$\eta(0)=-1$], then it remains down and the process is simply diffusively driven toward the absorbing barrier by a constant force; as $\alpha/D \to\infty$, the motion of the system becomes increasingly ballistic. If the barrier is initially ''up'' [$\eta(0)=1$], on the other hand, it remains up and the process moves between the reflective barrier at $y=0$ and the absorbing barrier at $y=L$ *against* a constant opposing force. The mean first passage time for such a process grows exponentially with the barrier height as $D^2e^{\alpha/D}/\alpha^2$ [12]. In our calculations either initial configuration is equally likely. The average of these two possibilities is the leading term in Eq. (22) (the ballistic contribution is negligible):

$$\widetilde{T}_1 \to \widetilde{T}_{\mathrm{static}} = \frac{L^2 D}{2\alpha^2} e^{\alpha/D}. \quad (28)$$

Between these two limits lies a regime in which the mean first passage time is a minimum. The minimum value occurs at the resonant flipping rate whose leading term for large values of α/D is

$$\gamma_{\mathrm{res}} \sim \frac{\alpha}{\sqrt{2}L^2}. \quad (29)$$

The leading contribution to the mean first passage time at this flipping rate is

$$\widetilde{T}_{\mathrm{res}} \sim (2+\sqrt{2})\frac{L^2}{\alpha}, \quad (30)$$

and thus decreases with increasing α. This result has the L dependence of a diffusive process, but the effective diffusion coefficient here is α and not D. Note that the product $\gamma_{\mathrm{res}}\widetilde{T}_{\mathrm{res}} = O(1)$.

IV. A SIMPLER MODEL SHOWING RESONANT ACTIVATION

A surprising observation about the results of the Doering-Gadoua model is that the resonant frequency γ_{res} and the mean first passage time at this resonant frequency, $\widetilde{T}_{\mathrm{res}}$, do not depend on the white noise intensity D to leading order in α. This means that as $\alpha/D\to\infty$, the resonant properties become increasingly independent of the white noise intensity. Indeed, the resonance therefore appears unaffected by and unrelated to the white noise; in particular, the minimum in the mean first passage time in the Doering-Gadoua model appears not to arise from the coupling usually invoked between the white noise and the random dynamics of the potential.

We have used somewhat equivocal language in this description because the situation is in fact somewhat subtle. The resonance in the Doering-Gadoua model arises from two features: (1) the random dynamics of the potential (that is, the random switching between barrier up and barrier down) and, in particular, the initial average over these dynamics, and (2) the nature of the reflecting boundary at $x=0$. It is this latter feature, subtly influenced by the white noise, that is especially noteworthy: the resonance characteristics of the Doering–Gadoua result when $D\to 0$ are *not* reproduced by simply setting $D=0$ to begin with in the model equations.

To pursue this issue in more detail, let us consider the same model equations as did Doering and Gadoua but now in the absence of white noise from the outset. In place of Eq. (1), the system thus evolves according to the simpler Langevin equation

$$\dot{y}(\tau)=-g'(y)\eta(\tau). \tag{31}$$

The solution of this mean first passage time problem is most easily found by splitting $\tilde{T}(y)$ into two components [13]: $\tilde{T}^{+}(y)$, the mean first passage time to $y=L$ when $\eta(0)=+1$, and $\tilde{T}^{-}(y)$, the mean first passage time to $y=L$ when $\eta(0)=-1$. The convenience of this representation lies in the ease of expression of the boundary conditions in terms of $\tilde{T}^{\pm}$.

The boundary condition (9) is completely equivalent to the so called ''immediate reinjection'' condition [14,15]

$$\tilde{T}^{+}(y=0)=\tilde{T}^{-}(y=0). \tag{32}$$

In this case, whenever the system reaches the boundary at $y=0$, the velocity immediately changes its sign, that is, the driving noise $\eta(\tau)$ changes its value from -1 to $+1$. Note that here the boundary condition directly affects the dynamics of the dichotomous barrier fluctuations since arrival at the boundary causes the noise to change its value. It is easy to ascertain that in terms of $\tilde{T}(y)=(\tilde{T}^{+}+\tilde{T}^{-})/2$, the ''immediate reinjection'' reflecting boundary condition indeed translates to the Doering-Gadoua condition (9), i.e.,

$$\left.\frac{d\tilde{T}(y)}{dy}\right|_{y=0}=0. \tag{33}$$

With this boundary condition (together with the absorbing condition at $y=L$), the mean first passage time from $y=0$ to $y=L$ if the initial values $\eta(0)=\pm 1$ are equally probable is [14]

$$\tilde{T}_{1,ir}=\frac{\gamma L^4}{\alpha^2}+\frac{L^2}{\alpha}. \tag{34}$$

Note that $\tilde{T}_{1,ir}$ is a monotonically increasing function of γ and thus exhibits no resonance. Clearly, this solution is *not* the one approached by the Doering-Gadoua model when D is set to zero in the solution of the latter.

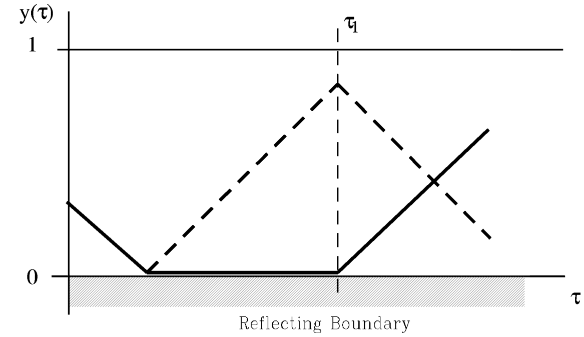

FIG. 4. Typical trajectories near the reflecting boundary for the two types of boundaries discussed in the text. Dashed lines: ''immediate rejection'' boundary. Solid lines: ''natural'' boundary.

There is another way to think of a reflecting boundary, namely, to assume that the boundary only limits the region of movement of the system without interfering with the dynamics of the dichotomous barrier fluctuations. The dichotomous noise evolves according to its own dynamics, and changes its value at random times that are independent of where the process $y(\tau)$ happens to be. Thus, if the system reaches the boundary $y=0$ when the noise happens to be $\eta=-1$, the noise may retain this value according to its own statistical properties. The process simply waits at the boundary, until the noise switches to $\eta=1$ in the natural course of events. We call this condition a ''natural'' reflecting boundary condition. This behavior is implemented via the following boundary condition for the mean first passage time components:

$$\tilde{T}^{-}(y=0)=\tilde{T}^{+}(y=0)+\frac{1}{\gamma}, \tag{35}$$

or, in terms of $\tilde{T}(y)$,

$$\left.\frac{d\tilde{T}(y)}{dy}\right|_{y=0}=-\frac{L}{\alpha}. \tag{36}$$

The solution for the mean first passage time is now

$$\tilde{T}_{1,n}=\frac{\gamma L^4}{\alpha^2}+\frac{2L^2}{\alpha}+\frac{1}{2\gamma}. \tag{37}$$

It is easily seen that $\tilde{T}_{1,n}$ has a minimum at $\gamma_{\rm res}=\alpha/\sqrt{2}L^2$ [see Eq. (29)]. In Fig. 4, we plot a realization of the process $y(\tau)$ for the two reflecting boundary conditions, the ''immediate reinjection'' and ''natural.'' From this figure, it is clear that the two boundary conditions lead to different results for the mean first passage time.

The interesting point to note is that the $D\to 0$ limit of the mean first passage time in the Doering-Gadoua model is $\tilde{T}_{1,n}$, that of the ''natural boundary,'' and *not* $\tilde{T}_{1,ir}$, although the reflecting boundary condition used for the solution of the Doering-Gadoua model is Eq. (9). In the Doering-Gadoua model, no matter how weak the white noise, its effects become dominant near the reflecting boundary. The white noise allows reversal of the trajectory even infinitesimally close to

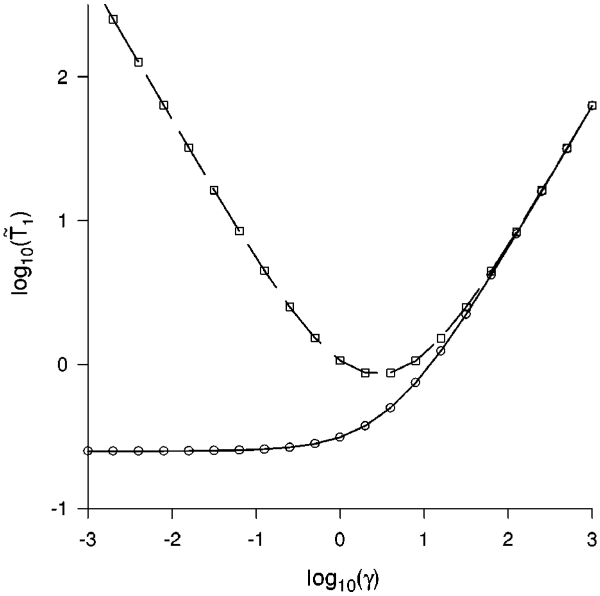

FIG. 5. Mean first passage time to $L=1$ as a function of barrier fluctuation rate for the two types of reflecting boundaries discussed in the text. Dashed curve with squares: ''immediate rejection'' boundary. Solid curve with circles: ''natural'' boundary.

the reflecting boundary without triggering a flip of the barrier, thus removing the dynamical interference between the boundary and the dichotomous flipping process. There is therefore a profound difference between the situation in which white noise is present, albeit as weak as one wishes, and the situation in which there is no white noise in the first place — the limit $D\to 0$ is discontinuous. In the Doering-Gadoua model there is a resonance in the mean first passage time to absorption because, through the action of the white noise (no matter how weak) the process *can* delay (increasingly as the white noise intensity decreases) arrival at the absorbing state if the barrier is up and does not flip of its own dynamics. However, the characteristics of the resonant flipping rate and the mean first passage time at resonance do not explicitly depend on the intensity of the white noise. In Fig. 5, we exhibit the two mean first passage times, $\tilde{T}_{1,ir}$ and $\tilde{T}_{1,n}$.

V. BIER-ASTUMIAN MODEL

The main conclusion that follows from the discussion of the preceding sections is that the resonant effect in the ''toy'' model of Doering and Gadoua is not of the same nature as the resonant activation in systems where the activation process is exclusively due to the presence of white noise (i.e., nonzero temperature). In order to study the resonant process in this latter situation, we return to the full model introduced by Doering and Gadoua but now with $V_0>a$, so that there is always a barrier. This problem was first analytically studied by Bier and Astumian [7]. The approximation developed by these authors coincides with the so called kinetic approximation introduced in Ref. [16]. The main limitation of this method for the present purposes is that it leads to a mean first passage time that does not exhibit a minimum.

We have developed an approximation for the mean first passage time to the absorbing boundary for high average barriers ($V_0\gg 1$) that does lead to a minimum, and hence can be used to describe the resonance phenomenon analytically. We return to the Langevin equation (1) with Eqs. (2)–(5). Now, however, we take $v_0\pm\alpha\gg D$ or, in dimensionless quantities, $V_0\pm a\gg 1$. The barrier thus flips between two large values.

The general solution to Eq. (8) is

$$T(x)=\frac{1}{V_0}(x-1)+A_1(e^{q_1x}-e^{q_1})+A_2(e^{q_2x}-e^{q_2})+A_3(e^{q_3x}-e^{q_3}),\tag{38}$$

where the coefficients q_i are the three roots of the polynomial equation

$$q^3-2V_0q^2+(V_0^2-a^2-2\lambda)q+2\lambda V_0=0,\tag{39}$$

and the constants A_i have to be found from the boundary conditions (9)–(12). It can be demonstrated that for $V_0>a$ the roots of Eq. (39) are all real, two of them positive and the other one negative. The full expressions for the constants A_i are complicated and too long to be included here. However, it is possible to derive shorter useful expressions for them as a series in λ. In this case, the roots q_i can be written as

$$q_1=-\frac{2V_0}{V_+V_-}\lambda+\frac{4V_0(a^2+V_0^2)}{(V_+V_-)^3}\lambda^2+O(\lambda^3),$$

$$q_2=V_-+\frac{\lambda}{V_-}-\frac{V_+}{2aV_-^3}\lambda^2+O(\lambda^3),\tag{40}$$

$$q_3=V_++\frac{\lambda}{V_+}+\frac{V_-}{2aV_+^3}\lambda^2+O(\lambda^3),$$

where

$$V_\pm\equiv V_0\pm a.\tag{41}$$

When these expressions are introduced into Eq. (38), the following result is obtained for the mean first passage time to order λ^2:

$$T(x=0)=\frac{N_1e^{V_-}+N_2e^{V_+}+N_3e^{2V_0}}{D_1+D_2e^{V_-}+D_3e^{V_+}},\tag{42}$$

where the coefficients in the numerator are

$$N_1=V_+^2-\lambda\left(\frac{V_-V_+}{V_0}-\frac{V_0^2}{aV_-}-\frac{3aV_+}{V_-}+\frac{a^2V_+}{V_0V_-}-\frac{a(V_0-5a)}{V_-^2}\right),$$

$$N_2=V_-^2-\lambda\left(\frac{V_-V_+}{V_0}+\frac{V_0^2}{aV_+}+\frac{3aV_-}{V_+}+\frac{a^2V_-}{V_0V_+}+\frac{a(V_0+5a)}{V_+^2}\right),\tag{43}$$

$$N_3=4\lambda\left\{1+\lambda\left[\frac{(2V_0-1)}{V_+V_-}-\frac{4a^2}{(V_+V_-)^2}\right]\right\},$$

and those of the denominator are

$$D_1=2(V_+V_-)^2+2\lambda(a^2+3V_0-2V_0V_+V_-),$$

$$D_2=2\lambda\left[V_+^2+\frac{\lambda}{aV_-^2}(aV_+^2V_--V_+V_-^2+2a^3-6a^2V_0)\right], \tag{44}$$

$$D_3=2\lambda\left[V_-^2+\frac{\lambda}{aV_+^2}(aV_-^2V_++V_-V_+^2+2a^3+6a^2V_0)\right].$$

This approximation is one order higher in λ than the one derived in Ref. [7], which is equivalent to the so-called kinetic approximation [16]. The advantage of our approximation is that it shows a minimum as function of the frequency λ. When $V_0 \gg 1$ the resonant frequency can be calculated explicitly:

$$\lambda_{\text{res}}\sim\left[\frac{a(e^{2a}-1)^2V_0^3}{4e^{2a}(1+a-e^{2a}+ae^{2a})}\right]^{1/2}\exp\left(-\frac{V_-}{2}\right), \tag{45}$$

and the associated minimal mean first passage time reads

$$T_{\text{res}}\sim\frac{2}{V_-^2+e^{-2a}V_+^2}e^{V_-}. \tag{46}$$

The analytic expressions (42)–(46) are the principal results of this paper.

As we did in the Doering-Gadoua case, it is useful to exhibit explicitly various limiting results in the original units so that the dependence on system parameters is clarified.

When the flipping rate is large ($\lambda \gg a^2$ or $\gamma \gg \alpha^2/DL^2$), result (42) reduces to

$$T_1 \to T_{\text{white}}=\frac{2+a^2/\lambda}{2V_0^2}\ \exp\left(\frac{2V_0}{2+\dfrac{a^2}{\lambda}}\right), \tag{47}$$

or, in the original units,

$$\tilde{T}_1 \to \tilde{T}_{\text{white}}=\frac{L^2D_{\text{eff}}}{v_0^2}e^{v_0/D_{\text{eff}}}, \tag{48}$$

where D_{eff} is the effective diffusion coefficient defined in Eq. (27). This is the appropriate and familiar result for activation over a barrier of height v_0 with diffusion coefficient D_{eff}. At the other extreme, as λ becomes small, the kinetic approximation [7,16] is valid and the mean first passage time (42) reduces to

$$T_1 \to T_{\text{kin}}=\frac{2\lambda+(k_++k_-)/2}{k_+k_-+\lambda(k_++k_-)}, \tag{49}$$

where

$$k_\pm=V_\pm^2e^{-V_\pm}. \tag{50}$$

If λ becomes so small that the time scale of barrier fluctuations is much slower than the escape time, then this further simplifies to

$$T_1 \to T_{\text{kin}}\sim\frac{1}{2}\left(\frac{1}{k_-}+\frac{1}{k_+}\right), \tag{51}$$

which is just the arithmetic mean associated with the two possible initial barrier configurations [see the discussion surrounding Eq. (28)]. In the original units,

$$\tilde{T}_1 \to \tilde{T}_{\text{kin}}=\frac{L^2D}{2v_+^2}e^{v_+/D}+\frac{L^2D}{2v_-^2}e^{v_-/D}. \tag{52}$$

Between these two limits lies the resonance regime where the mean first passage time is shorter than either the ''white noise'' or ''static noise'' results. In the original units the mean first passage time at resonance [Eq. (46)] reads

$$\tilde{T}_{\text{res}}\sim\frac{2L^2D}{(v_-^2+e^{-2a}v_+^2)}e^{v_-/D}\sim\frac{2L^2D}{v_-^2}e^{v_-/D}, \tag{53}$$

where the second expression, valid if $a \gg 1$, serves to stress the point that the resonant mean first passage time is essentially the usual passage time over the lower of the two barriers. It is not particularly instructive to exhibit the full expression (45) for the resonance frequency in the original units, but, if $a \gg 1$ we can display the shorter expression

$$\gamma_{\text{res}}\sim\frac{v_0^{3/2}}{2L^2D^{1/2}}e^{-v_-/2D}. \tag{54}$$

It should be noted that both the resonant mean first passage and the resonant frequency depend on the intensity of the white noise, as does their product. This dependence appears in the exponents as well as prefactors.

A general feature of our solution and, more generally, of the resonant activation phenomenon is that with increasing barrier height the resonance phenomenon becomes less and less sharp: a long flat region develops around the resonant frequency, a fact that has been explicitly noted in earlier work [17]. Analysis of Eq. (42) makes it possible to estimate analytic bounds of this flat region, which spans the range

$$\frac{V_+^2e^{-2a}+V_-^2}{4}e^{-V_-} \ll \lambda \ll \frac{V_+V_-}{2V_0}. \tag{55}$$

Thus, rather than stressing the resonance aspect of the problem, it might be more accurate to describe the time scale of the activation process as relatively insensitive to the parameters of the system except in the limits of very low and very high barrier fluctuation rates. As noted above, if the barrier fluctuations are sufficiently slow, then an initially high barrier remains that way essentially forever, and the system on average crosses it before the barrier flips. Passage over the higher barrier then dominates the mean first passage time. At the other extreme, when the barrier fluctuations are very rapid, crossing occurs essentially over the average barrier. However, over most parameter ranges the mean first passage time is essentially determined by passage over the lower barrier — the system can avoid passage over the higher barrier

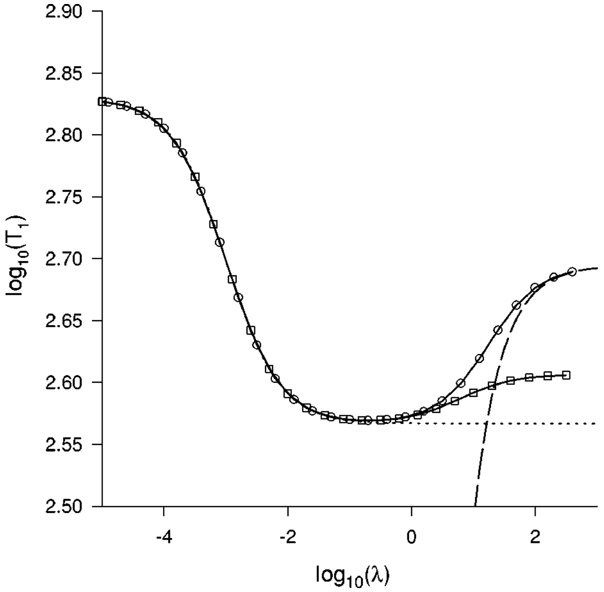

FIG. 6. Mean first passage time as a function of barrier fluctuation rate for the Bier-Astumian model. The barrier fluctuates between the high value V_0+a and the lower value V_0-a, with $V_0=11$ and $a=1$. Solid curve with circles: exact mean first passage time obtained numerically. Dotted curve: kinetic approximation (49). Dashed curve: white noise approximation (47). Solid curve with squares: our result (42).

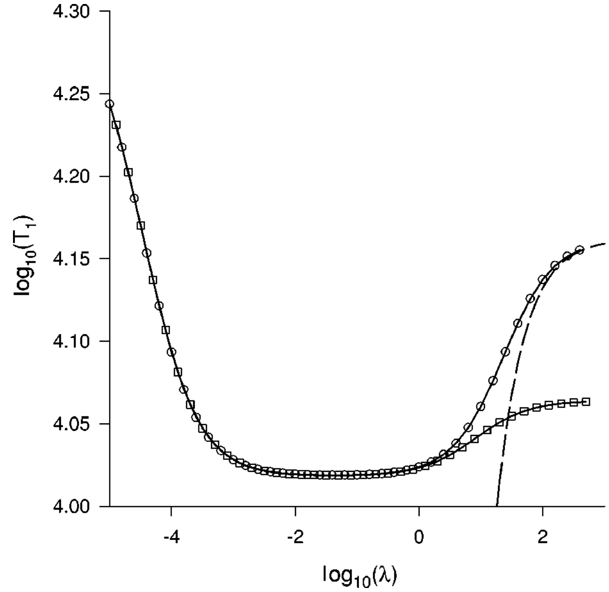

FIG. 7. Mean first passage time as a function of barrier fluctuation rate for the Bier-Astumian model with $V_0=15$ and $a=1$. Solid curve with circles: exact mean first passage time obtained numerically. Dashed curve: white noise approximation (47). Solid curve with squares: our result (42).

by ''waiting'' for it to flip. Provided the waiting time is shorter than the time it would take the system to cross the high barrier, flipping will occur first and the system will cross when the barrier is lower (unless flipping is too rapid). This process is most efficient (but not dramatically more efficient — hence the flat behavior) at the resonance frequency.

In Fig. 6, the mean first passage time and the different approximations explained above have been plotted for $V_0=11$ and $a=1$. Our approximation clearly captures the resonance behavior extremely accurately and for that matter the behavior of the mean first passage time over a broad range of barrier fluctuation rates. Figure 7 again shows the mean first passage time, but now with $V_0=15$ and $a=1$. This figure serves to reconfirm the agreement of our results with the numerical ones, and also illustrates the flattening of the region around the resonance as the height of the barrier increases. Figure 8 shows the resonance frequency as a function of the barrier height. Again, our approximation clearly captures the exact results extremely well for sufficiently high barriers.

VI. ACTIVATION DRIVEN BY A SQUARE WAVE FUNCTION

It is interesting to explore whether the resonant activation phenomenon requires that the barrier fluctuate stochastically, or whether it also occurs when a noisy process crosses a barrier that changes periodically. Indeed, stochastic fluctuation of the barrier is not a requirement.

To investigate the activation process when the barrier oscillates periodically between higher and lower values, we replace the dichotomous noise $\eta(\tau)$ in Eq. (1) with a square wave $w(\tau)$, a periodic function that alternately takes on the values $+1$ and -1. The changes from one to the other occur at a constant frequency γ. The period of the square wave function is thus $2/\gamma$.

The Fokker-Planck equation describing the evolution of the probability for the system now includes a time-periodic potential. The problem can be tackled analytically using Floquet theory. We simply state qualitatively the results that one obtains with this exact approach, but then follow a simpler approach to arrive at some quantitative conclusions.

We continue our discussion in terms of dimensionless variables and parameters. Exact solution of the problem does, as noted above, also lead to resonant activation when the barrier changes from higher to lower periodically, with a resonance behavior very similar to that of the stochastic case. In other words, the mean first passage time is large when the period of oscillation is very slow and also when it is very fast. As before, and for the same physical reasons, in the former case the mean first passage time is dominated by the high barrier V_+, and in the latter case it is determined by the average barrier V_0. Again as before, between these two limits there is a flat region (i.e., rather insensitive to the parameter values) where the mean first passage time is determined primarily by the lower barrier V_-. The only difference between this problem and the stochastic one lies in the detailed way in which the mean first passage time changes from one behavior to the other.

To find the mean first passage time at the slow-barrier-modulation end of the problem (where the difference between stochastic and periodic modulation is most pronounced), we recall that for a fixed barrier of height V the probability that the process has *not* yet crossed the barrier at time t (i.e., the survival probability at time t), is exponential [18], e^{-kt}, where the crossing rate $k=V^2e^{-V}$ [cf., Eq. (50)]. If the barrier is not fixed, but instead changes slowly from

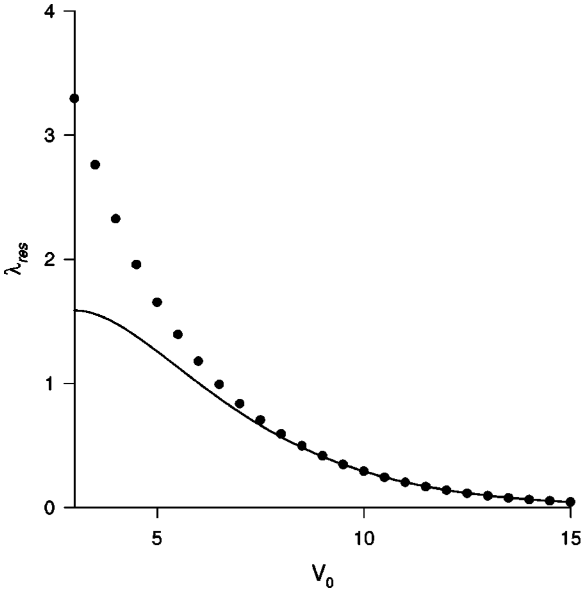

FIG. 8. Resonance frequency as a function of average barrier height for the Bier-Astumian model with $a=1$. Circles: exact numerical results. Solid curve: our result (45).

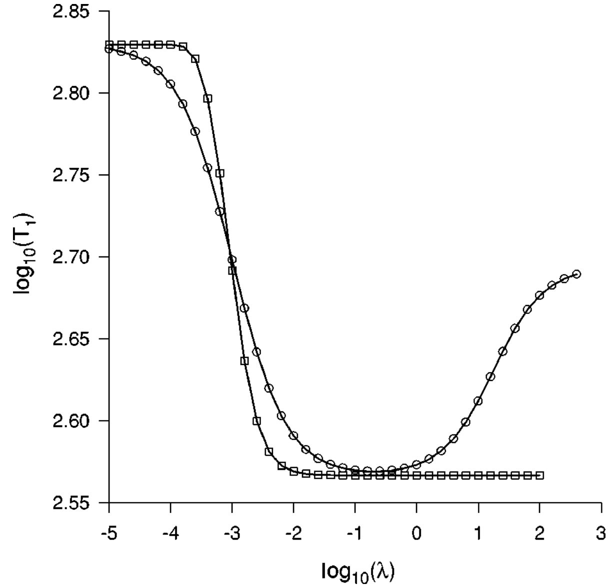

FIG. 9. Mean first passage time as a function of barrier oscillation-fluctuation rate. Square symbols: kinetic approximation for square wave barrier oscillations. Circles: fluctuating barrier. Barrier parameter values: $V_0=11$ and $a=1$.

one value to the other at predetermined periodic time intervals $\Delta\equiv\lambda^{-1}$ with $\lambda\lesssim 1$, we can track the trajectory of the system explicitly and write down an expression for the survival probability $S(t)$ that the process has not yet crossed the barrier at time t. If initially the barrier is V_+ and the time $t=0$ corresponds to the beginning of a period, then

$$\begin{aligned} S^+ &= \mathrm{Prob}\{T^+>t\}=q_+^n q_-^n e^{-k_+(t-2n\Delta)}, \\ & 2n\Delta<t\leqslant(2n+1)\Delta, \\ S^+ &= \mathrm{Prob}\{T^+>t\}=q_+^{n+1} q_-^n e^{-k_-[t-(2n+1)\Delta]}, \\ & (2n+1)\Delta<t\leqslant(2n+2)\Delta, \end{aligned} \tag{56}$$

where $n=0,1,2,\ \dots$, and q_+ and q_- are the probabilities that a crossing event does not happen when the barriers are, respectively, V_+ and V_-. The assumption about the statistics of the crossing events yields

$$q^{\pm}=e^{-k_{\pm}/\lambda}. \tag{57}$$

The mean first passage time T^+ can then be calculated directly as a moment of this probability. The survival probability S^- and associated mean first passage time T^- when the barrier is initially V_- is similarly obtained. To compare most directly with the stochastic results, we assume that initially the barrier is equally likely to be V_+ or V_-. A short calculation then leads to the following result for the mean first passage time when $\lambda\lesssim 1$:

$$T=\frac{T^+ + T^-}{2}=\frac{1}{2}\left(\frac{1}{k_+}+\frac{1}{k_-}\right)+\frac{1}{2}\left(\frac{1}{k_+}-\frac{1}{k_-}\right)\frac{q_+-q_-}{1-q_+q_-}. \tag{58}$$

This result corresponds to the same level of approximation as the kinetic result (49). At very low frequencies, $\lambda\to 0$, the mean escape time is correctly given by an average of the escape time $1/k_+$ when the barrier is V_+ and $1/k_-$ when the barrier is V_-. This kinetic approximation also does not exhibit a minimum because it does not behave correctly when $\lambda\to\infty$; instead, it converges to the same value as the kinetic approximation (49), that is, to $(k_+ + k_-)/2$.

In Fig. 9, approximation (58) to the mean escape time for the activation process driven by a periodic signal is compared with the escape time for the same system driven by dichotomous noise. The difference between the two is noticeable in the decrease of the mean first passage time with increasing flipping rate — the dependence on the flipping rate is considerably sharper in the periodic case than in the random case. A similar effect was observed recently in systems that exhibit coherent stochastic resonance [19]. The minimum first passage time and resonance flipping rate are essentially identical in the two cases.

VII. CONCLUSION

We have revisited the problem of resonant activation, that is, of the mean escape time of a particle driven by white noise of intensity D over a barrier of randomly fluctuating height. The initial position of the particle is $y=0$, and the barrier is at $y=L$. A substantial recent literature [6–9,11,16] deals with this problem, but the results to be found in the literature are almost exclusively numerical. The distribution of barrier fluctuations is typically taken to be either dichotomous (i.e., the barrier fluctuates between two values) or Gaussian. The correlation function of the barrier fluctuations is usually assumed to be exponential and thus characterized by a rate parameter γ. The quantity of interest is the mean escape time $\widetilde{T}_1$ of the particle over the barrier as a function of γ. It is observed that $\widetilde{T}_1$ vs γ exhibits a minimum, i.e., there is an optimal barrier fluctuation rate that minimizes the escape time of the particle. This minimum defines the resonant activation phenomenon.

In this paper we have concentrated on dichotomous fluctuations and on triangular potential barriers, so our quantitative results are restricted to these cases. However, we believe that our results provide insights beyond these specific conditions. In particular, they provide insights for barriers whose fluctuations are bounded between an upper value v_+ and a lower value v_-. The applicability of our conclusions to Gaussian fluctuations is therefore less certain, but below we will present some conjectures for this case as well.

A variety of approaches to the problem of the escape over a fluctuating barrier of bounded variation provide excellent and consistent analytic approximations to the escape rate in the limiting cases of very slow barrier fluctuations and of very fast barrier fluctuations. In the slow fluctuation case, the so-called ''kinetic approximation'' [16] captures the behavior of the system very well. In the limit of very slow fluctuations ($\gamma\to 0$) the barrier retains its initial height throughout the process. The mean first passage time for the ensemble is then just the mean first passage time averaged over the initial distribution of barrier heights. For example, in the dichotomous case if the height of the high barrier is v_+ then the mean escape time as $\gamma\to 0$ is determined by the mean escape time over this high barrier (the mean escape time over the lower barrier being negligible in comparison):

$$\widetilde{T}_{\text{static}} \sim \frac{L^2 D}{2v_+^2} e^{v_+/D}. \tag{59}$$

This is the result captured, for instance, in Eqs. (28) and (52). Clearly, this result is determined in part by the assumption (generally made in the literature) that an initial average over an ensemble of barrier heights is appropriate. The entire discussion that follows, including the occurrence of a resonance, is dependent on such an initial average or at least on the assumption that a finite fraction of realizations begin with a barrier configuration that is higher than the lowest barrier.

At the opposite extreme, when the barrier fluctuations are very rapid ($\gamma\to\infty$), the main effect of the flipping barrier is to increase the effective intensity of the white noise. The escape then occurs over the average barrier, with a diffusion coefficient D_{eff} which exceeds D by an amount determined by the detailed distribution of barrier fluctuations. If the average barrier height is $v_0>0$, then the mean escape time in this limit is

$$\widetilde{T}_{\text{white}} \sim \frac{L^2 D_{\text{eff}}}{v_0^2} e^{v_0/D_{\text{eff}}}. \tag{60}$$

If the average barrier height is zero, then

$$\widetilde{T}_{\text{white}} = \frac{L^2}{2D_{\text{eff}}}. \tag{61}$$

These are the results captured in Eqs. (26) and (48). In any case, the escape time is clearly smaller in the fast barrier fluctuation limit than in the slow barrier fluctuation limit.

In the literature, each of the above approximations had been carried sufficiently far to deduce the behavior of the escape time as one moves away from the strict limits. Thus, within the kinetic approximation, it can be shown that the escape time decreases with increasing γ. At the opposite limit, it can be shown that the escape time decreases with decreasing γ. These two results clearly point to a minimum for some finite value of γ, but neither approximation is sufficient to actually capture the minimum. Our goal here has been to develop a single approximation to capture this minimum, and in this we succeeded.

However, we found in the process that a distinction needs to be made between two cases that lead to a different physical origin and parameter dependences for the resonant flipping rate and the associated escape time. In the literature, these two cases have been treated more or less as one because both involve dichotomous fluctuations, but they are in fact very different. One of these is the case in which the ''barrier'' fluctuates between an ''up'' or positive (barrier) configuration of height $v_+=\alpha$ and a ''down'' or negative (valley) configuration of height $v_-=-\alpha$. We have called this the Doering-Gadoua model [6]. In the other case, the barrier fluctuates between a high value $v_+=v_0+\alpha$ and a lower (but still positive) value $v_-=v_0-\alpha$. We have called this the Bier-Astumian model [7]. For each model we found a single expression for the mean first passage time that has a minimum, and we compared our results with exact ones obtained numerically. The agreement in both cases is excellent for almost the entire range of flipping rates, and in particular over a broad range surrounding the resonance.

The distinctive aspect of the Doering-Gadoua model is the fact that part of the time the ''barrier'' is really a valley, so that the particle can essentially roll rather than climb toward L during these times. We found an explicit expression for the resonant flipping rate and the resonant mean first passage time in this case:

$$\gamma_{\text{res}} \sim \frac{\alpha}{\sqrt{2}L}, \tag{62}$$

$$\widetilde{T}_{\text{res}} \sim (2+\sqrt{2})\frac{L^2}{\alpha}. \tag{63}$$

The noteworthy fact about these results is that neither the resonant flipping rate nor the resonant escape time depend explicitly on the intensity D of the white noise. This fact seems not to have been noted before. We then went on to explore whether in fact this resonance is observed in a process defined by the Doering-Gadoua model with no white noise from the outset, and found that there is no resonance for such a model. We explained this apparent contradiction by noting a discontinuity in the $D\to 0$ limit of the problem and by presenting a modified set of boundary conditions that does lead to a resonance (precisely the Doering-Gadoua resonance) in the absence of white noise.

In order to obtain a result for the escape time in the Bier-Astumian model that captures the resonant behavior, we found that we had to retain terms in our solutions to one power higher in γ than had been done previously (the lower orders yielded only the kinetic approximation) [7]. With this, we identified the resonant frequency and escape times as

$$\gamma_{\text{res}} \sim \frac{v_0^{3/2}}{2L^2 D^{1/2}} e^{-v_-/2D} \tag{64}$$

and (more complete results are found in Sec. V)

$$\tilde{T}_{\rm res} \sim \frac{2L^2 D}{v_-^2} e^{v_-/D}. \tag{65}$$

We noted that in this case both the resonance frequency and the mean first passage time at the resonance frequency depend on the intensity of the white noise, as does their product. At resonance the escape over the barrier occurs primarily when the barrier is at its lowest. We also noted that the dependence on the flipping rate, especially for high barriers, is very flat: there is a broad range of flipping rates where passage over the barrier occurs primarily when the barrier is low. In this broad range of flipping rates the escape time over the high barrier is so long that the barrier is likely to flip to its lower height before the escape is completed.

We also discussed the fact that the resonant activation phenomenon does not require a fluctuating barrier — it also occurs if the barrier oscillates periodically between the high and low values. The behavior of the escape time at low and high oscillation periods is the same as in the dichotomous fluctuation case, and at intermediate oscillation periods a resonance effect is also observed.

Finally, we note that our analysis does not address the case of Gaussian barrier fluctuations, that is, of Ornstein-Uhlenbeck barrier fluctuations [11]. The results for such barrier fluctuations with fixed variance [10] should be similar to our results for dichotomous noise. In particular, the escape time for the model analogous to that of Doering and Gadoua ($V_0=0$) will show a minimum even in the absence of white noise.

ACKNOWLEDGMENTS

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APPENDIX: SURVIVAL PROBABILITY AND FIRST PASSAGE TIME MOMENTS

The moments of the first passage time from $x=0$ to $x=1$ can be obtained from the survival probability $S(x,t)$ that the system evolving according to Eq. (1) (appropriately scaled to dimensionless variables) with a reflecting boundary at $x=0$ and an absorbing boundary at $x=1$ has not left the interval (0,1) at time t. This survival probability obeys the following partial differential equation (a detailed derivation and original references for the survival probability in an interval terminated by two absorbing boundaries are presented in Ref. [20]):

$$\mathcal{L}^2 S + 2\lambda \mathcal{L} S = a^2 \frac{\partial^2 S}{\partial x^2}, \tag{A1}$$

where $\mathcal{L}$ is the differential operator

$$\mathcal{L} \equiv \frac{\partial}{\partial t} + V_0 \frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2}. \tag{A2}$$

Note that Eq. (A1) is a second-order partial differential equation in the time variable and a fourth-order partial differential equation in the state variable. Therefore, two initial conditions and four boundary conditions are needed to solve it. The initial conditions are

$$S(x,0) = 1, \tag{A3}$$

$$\left.\frac{\partial S}{\partial t}\right|_{t=0} = 0. \tag{A4}$$

For the absorbing trap at $x=1$, the boundary conditions read

$$S(1,t) = 0, \tag{A5}$$

$$\left(\mathcal{L} + V_0 \frac{\partial}{\partial x} - V_0^2 + a^2\right) \left.\frac{\partial S}{\partial x}\right|_{x=1} = -V_0 \delta(t), \tag{A6}$$

and for the reflecting boundary at $x=0$ they are

$$\left.\frac{\partial S}{\partial x}\right|_{x=0} = 0, \tag{A7}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) S|_{x=0} = 0. \tag{A8}$$

The first passage time moments T_n are related to the survival probability according to

$$T_n(x) = n \int_0^\infty t^{n-1} S(x,t)\, dt. \tag{A9}$$

Clearly $T_0(x)=1$ by normalization. $T_1(x)$ is the mean first passage time to 1 for a process that starts at $X(0)=x$; $T_2(x)$ is the second moment of the distribution, so that the variance of the distribution of mean first passage times is $\sigma^2 \equiv T_2 - T_1^2$.

Equations for the first passage time moments can be obtained by multiplying Eq. (A1) by t^{n-1} and integrating over time by parts. The following recursive-differential equation is easily found:

$$\mathcal{L}_D^2 T_n - 2\lambda \mathcal{L}_D T_n - a^2 \frac{d^2 T_n}{dx^2} = g_n, \tag{A10}$$

where

$$g_n \equiv n(2\lambda - 2\mathcal{L}_D) T_{n-1} - n(n-1) T_{n-2}, \tag{A11}$$

and $\mathcal{L}_D$ is the Fokker-Planck operator,

$$\mathcal{L}_D \equiv -V_0 \frac{d}{dx} + \frac{d^2}{dx^2}, \tag{A12}$$

with $T_{-1} \equiv 0$. The boundary conditions can be obtained directly from those of the survival probability:

$$(1) \quad T_n(1) = 0, \tag{A13}$$

$$(2) \quad \left(\mathcal{L}_D - V_0 \frac{d}{dx} + V_0^2 - a^2 \right) \frac{dT_n}{dx} \bigg|_{x=1} = -n \frac{dT_{n-1}}{dx} \bigg|_{x=1} + V_0 \delta_{n,1}, \tag{A14}$$

$$(3) \quad \frac{dT_n}{dx} \bigg|_{x=0} = 0, \tag{A15}$$

$$(4) \quad \frac{d^2 T_n}{dx^2} \bigg|_{x=0} = -n T_{n-1}(0). \tag{A16}$$

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