Free inertial processes driven by Gaussian noise: Probability distributions, anomalous diffusion, and fractal behavior

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We study the motion of an unbound particle under the influence of a random force modeled as Gaussian colored noise with an arbitrary correlation function. We derive exact equations for the joint and marginal probability density functions and find the associated solutions. We analyze in detail anomalous diffusion behaviors along with the fractal structure of the trajectories of the particle and explore possible connections between dynamical exponents of the variance and the fractal dimension of the trajectories.

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I. INTRODUCTION

In the past decade there has been much interest in studying the effects of colored noise on a given dynamical system [1,2]. The kind of systems that have been treated are mainly first-order systems, that is, random processes $X(t)$ whose dynamical evolution is described in terms of first-order differential equations of the form

$$\dot{X} = f(X) + F(t), \tag{1.1}$$

where $F(t)$ is a random function with specified statistical characteristics. However, from the point of view of physics, the displacement $X(t)$ of a particle (which we assume to be of unit mass) under the influence of a deterministic force $f(X)$ and an external random force $F(t)$ is not properly given by the above equation but by Newton's law

$$\ddot{X} + \beta\dot{X} + f(X) = F(t). \tag{1.2}$$

Obviously Eq. (1.1) can be considered an approximation to Eq. (1.2) in the overdamped regime when inertial effects are negligible, i.e. $|\ddot{X}| \ll |\beta\dot{X}|$. Unfortunately, dynamical equations such as (1.2) are much more difficult to deal with even in the simplest cases. Nevertheless, in spite of the technical difficulties inherent in second-order dynamics, there has been recent work on inertial processes driven by colored noise. Thus we have found exact results for unbounded second-order processes [3–5], approximate results for free second-order processes driven by Ornstein-Uhlenbeck noise [6], and results on linear oscillators driven by colored noise, which can be internal [7,8], or external noise [9–11].

In this paper we will consider the simple (but nontrivial) case of an undamped free particle under the influence of a random acceleration

$$\ddot{X} = F(t), \tag{1.3}$$

where $F(t)$ is Gaussian colored noise, which we assume to be zero centered and with an arbitrary correlation function

$$k(t,t') = \langle F(t)F(t')\rangle.$$

We note that $F(t)$ will generally be a nonstationary and non-Markovian random process and, in many cases, $F(t)$ will posses a fractal structure [8,12]. Our objectives here are first, to obtain exact equations and expressions for the joint probability density function $p(x,v,t)$ of displacement x and velocity v of the particle and the marginal densities $p(v,t)$ and $p(x,t)$ of the velocity and the displacement, second, to find anomalous diffusive behaviors of process (1.3); and third, to study the fractal structure of the trajectories.

The paper is organized as follows. In Sec. II we obtain the joint and marginal probability density functions of process (1.3). Section III is devoted to stationary input noise with special emphasis on the only case of stationary colored noise that is both Gaussian and Markovian: Ornstein-Uhlenbeck noise. In Sec. IV we set a fairly general asymptotic analysis of the stationary case and study anomalous diffusion behaviors. In Sec. V we present a relevant case of nonstationary input noise and in Sec. VI we study the fractal behavior of the trajectories. We end with a brief summary and some conclusions in Sec. VII. Technical aspects are in the Appendixes.

II. PROBABILITY DENSITY FUNCTIONS

The joint probability density function $p(x,v,t)$ for a displacement x and a velocity v at time t of the process (1.3) is defined by

$$p(x,v,t) = \langle \delta(x - X(t))\delta(v - \dot{X}(t))\rangle, \tag{2.1}$$

where the symbol $\langle\ \rangle$ means the average over all possible realizations of the input noise $F(t)$ and

$$X(t) = x_0 + v_0 t + \int_0^t dt' \int_0^{t'} dt'' F(t''), \tag{2.2}$$

$$\dot{X}(t) = v_0 + \int_0^t dt' F(t'). \tag{2.3}$$

Now starting from Eq. (2.1) and following a standard procedure that involves functional derivatives and the use of Novikov's theorem, one can easily see that $p(x,v,t)$ obeys the equation (see Appendix A for details)

$$\frac{\partial p}{\partial t} = -v\frac{\partial p}{\partial x} + \left[\varphi(t)\frac{\partial^2 p}{\partial v^2} + \phi(t)\frac{\partial^2 p}{\partial x \partial v}\right], \qquad (2.4)$$

where

$$\varphi(t) = \int_0^t k(t,t')dt' \qquad (2.5)$$

and

$$\phi(t) = \int_0^t (t-t')k(t,t')dt'. \qquad (2.6)$$

The initial condition attached to Eq. (2.4) is

$$p(x,v,0) = \delta(x-x_0)\delta(v-v_0). \qquad (2.7)$$

The joint characteristic function of the process

$$\tilde{p}(\mu,\rho,t) = \int_{-\infty}^{\infty} dx e^{-i\mu x}\int_{-\infty}^{\infty} dv e^{-i\rho v} p(x,v,t) \qquad (2.8)$$

obeys the first-order partial differential equation

$$\left(\frac{\partial}{\partial t} - \mu\frac{\partial}{\partial \rho}\right)\tilde{p}(\mu,\rho,t) = -[\mu\rho\phi(t) + \rho^2\varphi(t)]\tilde{p}(\mu,\rho,t) \qquad (2.9)$$

with initial condition

$$\tilde{p}(\mu,\rho,0) = e^{-i(\mu x_0 + \rho v_0)}. \qquad (2.10)$$

In Appendix B we show that the solution to problem (2.9) and (2.10) is given by

$$\tilde{p}(\mu,\rho,t) = \exp\left\{ -i[\mu x_0 + (\rho+\mu t)v_0] - \frac{1}{2}\left[\rho^2\sigma_v^2(t) + 2\rho\mu\sigma_{xv}^2(t) + \mu^2\sigma_x^2(t)\right]\right\}, \qquad (2.11)$$

where

$$\sigma_v^2(t) = 2\int_0^t dt' \int_0^{t'} dt'' k(t',t''), \qquad (2.12)$$

$$\sigma_{xv}^2(t) = \int_0^t dt' \int_0^{t'} dt''(2t - t' - t'')k(t',t''), \qquad (2.13)$$

and

$$\sigma_x^2(t) = 2\int_0^t dt'(t-t') \int_0^{t'} dt''(t-t'')k(t',t''). \qquad (2.14)$$

The joint probability density function $p(x,v,t)$ can thus be written as

$$p(x,v,t) = \frac{1}{2\pi\sigma_v(t)S(t)}\exp\left\{-\frac{(v-v_0)^2}{2\sigma_v^2(t)} - \frac{\left[x - x_0 - v_0 t - (v-v_0)r^2(t)\right]^2}{2S^2(t)}\right\}, \qquad (2.15)$$

where

$$S(t) = \sqrt{\sigma_x^2(t) - \frac{\sigma_{xv}^4(t)}{\sigma_v^2(t)}} \qquad (2.16)$$

and

$$r(t) = \frac{\sigma_{xv}(t)}{\sigma_v(t)}. \qquad (2.17)$$

We note at this point that Eq. (2.15) can also be obtained following a different approach that takes into account the Gaussian character of $X(t)$, which, in turn, is a consequence of the linearity of Eq. (1.3). This approach consists of assuming a joint characteristic function of the form given by Eq. (2.11) and evaluating the variances $\sigma_x^2(t)$, $\sigma_{xv}^2(t)$, and $\sigma_v^2(t)$ from Eqs. (2.2) and (2.3). Nevertheless, this procedure works only for linear processes, while that of Appendixes A and B is more general and suitable for the approximate treatment of nonlinear processes.

The marginal characteristic function of the velocity, which is the Fourier transform of the marginal probability density of the velocity defined by

$$p(v,t) = \int_{-\infty}^{\infty} p(x,v,t)dx,$$

can be obtained from the joint characteristic function just by setting $\mu = 0$ in Eq. (2.9). We thus have

$$\tilde{p}(\rho,t) = \exp\left\{-i\rho v_0 - \frac{1}{2}\rho^2\sigma_v^2(t)\right\}. \qquad (2.18)$$

One easily sees from this equation that the marginal density of the velocity satisfies the diffusion equation

$$\frac{\partial}{\partial t}p(v,t) = \frac{1}{2}D_v(t)\frac{\partial^2}{\partial v^2}p(v,t), \qquad (2.19)$$

with a time varying diffusion coefficient given by

$$D_v(t) = \frac{d\sigma_v^2(t)}{dt} = 2\int_0^t k(t,t')dt'. \qquad (2.20)$$

If we now set $\rho = 0$ in Eq. (2.9) we obtain the marginal characteristic function of the displacement:

$$\tilde{p}(\mu,t) = \exp\left\{-i\mu(x_0 + v_0 t) - \frac{1}{2}\mu^2\sigma_x^2(t)\right\}. \qquad (2.21)$$

In this case the marginal density $p(x,t)$ obeys the somewhat more complicated diffusion equation

$$\frac{\partial}{\partial t}p(x+v_0 t,t) = \frac{1}{2}D_x(t)\frac{\partial^2}{\partial x^2}p(x+v_0 t,t), \qquad (2.22)$$

with a diffusion coefficient $D_x(t) = d\sigma_x^2(t)/dt$ given by

$$D_x(t) = 2\int_0^t dt' \int_0^{t'} dt''(2t - t' - t'')k(t', t''). \quad (2.23)$$

Equations (2.4) and (2.11)–(2.23) furnish a complete exact solution to the problem of free inertial processes driven by Gaussian colored noise. In the following sections we will study some relevant special cases and explore the consequences of having these exact expressions.

III. STATIONARY INPUT NOISE

In this section we assume that the input noise $F(t)$ is stationary, that is,

$$\langle F(t)F(t')\rangle = k(t - t'). \quad (3.1)$$

The equation satisfied by the joint density $p(x, v, t)$ is given by Eq. (2.4) and its solution by Eq. (2.15) with (see Appendix C)

$$\sigma_v^2(t) = 2\int_0^t dt' \int_0^{t'} dt'' k(t''), \quad (3.2)$$

$$\sigma_{xv}^2(t) = \frac{1}{2} t\sigma_v^2(t), \quad (3.3)$$

$$\sigma_x^2(t) = \int_0^t dt' t' \sigma_v^2(t'). \quad (3.4)$$

Analogously, the equations for the marginal density of velocity $p(v, t)$ and displacement $p(x, t)$ are given by Eqs. (2.19) and (2.22) with diffusion coefficients given by

$$D_v(t) = 2\int_0^t k(t')dt' \quad (3.5)$$

and

$$D_x(t) = t\int_0^t D_v(t')dt'. \quad (3.6)$$

Let us now assume that $F(t)$ is also a Markovian noise. In this case Doob's theorem [13] tell us that $F(t)$ is necessarily either white noise or exponentially correlated noise (also referred to as Ornstein-Uhlenbeck noise)

$$k(t - t') = \frac{D}{2\tau_c} e^{-|t-t'|/\tau_c}, \quad (3.7)$$

where D is the noise intensity and τ_c is the correlation time. The variance of the velocity now reads

$$\sigma_v^2(t) = D\tau_c \left(\frac{t}{\tau_c} - 1 + e^{-t/\tau_c}\right). \quad (3.8)$$

We note that when $t/\tau_c \gg 1$ the variance behaves like

$$\sigma_v^2(t) \sim Dt \qquad (t/\tau_c \gg 1), \quad (3.9)$$

that is, at sufficiently long times the velocity undergoes ordinary diffusion. On the other hand, when $t/\tau_c \ll 1$ we have

$$\sigma_v^2(t) \sim \frac{D}{2\tau_c} t^2 \qquad (t/\tau_c \ll 1), \quad (3.10)$$

which corresponds to ballistic motion. In this case the marginal density $p(v, t)$ obeys Eq. (2.19) with

$$D_v(t) = D\left(1 - e^{-t/\tau_c}\right). \quad (3.11)$$

The variance of the displacement is now given by

$$\sigma_x^2(t) = D\tau_c^3 \left[\frac{1}{3}\left(\frac{t}{\tau_c}\right)^3 - \frac{1}{2}\left(\frac{t}{\tau_c}\right)^2 + 1 - \left(1 + \frac{t}{\tau_c}\right) e^{-t/\tau_c}\right]. \quad (3.12)$$

When $t/\tau_c \gg 1$ we see that

$$\sigma_x^2(t) \sim \frac{D}{3} t^3 \qquad (t/\tau_c \gg 1), \quad (3.13)$$

which corresponds to superdiffusive behavior [3]. When $t/\tau_c \ll 1$ we obtain from Eq. (3.12)

$$\sigma_x^2(t) \sim \frac{D}{8\tau_c} t^4 \qquad (t/\tau_c \ll 1) \quad (3.14)$$

and this again corresponds to ballistic motion. We also note that the marginal density $p(x, t)$ obeys Eq. (2.22) with

$$D_x(t) = Dt\left[t - \tau_c\left(1 - e^{-t/\tau_c}\right)\right]. \quad (3.15)$$

We finally mention that the expansions of all of the above results (regarding Ornstein-Uhlenbeck noise) when $t/\tau_c \ll 1$ and $t/\tau_c \gg 1$ agree with the corresponding approximate expressions recently obtained by Heinrichs [6].

IV. ASYMPTOTIC BEHAVIOR AND ANOMALOUS DIFFUSION

We will now discuss with some generality the asymptotic behavior of the variances $\sigma_v^2(t)$ and $\sigma_x^2(t)$ in terms of the asymptotic behavior of the correlation function $k(t)$ of the stationary input noise. Let us start studying the behavior of the variances at very short times. When $t \to 0$ we can easily see from Eqs. (3.2) and (3.4) that

$$\sigma_v^2(t) \sim t^2, \quad \sigma_x^2(t) \sim t^4, \qquad (t \to 0), \quad (4.1)$$

which corresponds to ballistic motion. This is not a surprising result since at very short times colored noise keeps its initial value, i.e., $F(t) = F(0)$ (with probability 1), and the motion is ballistic.

We will now explore the more interesting case when t is large. Let us denote by $\hat{k}(s)$ the Laplace transform of the correlation function

$$\hat{k}(s) = \int_0^\infty e^{-st} k(t) dt$$

and assume that, for small values of s, $\hat{k}(s)$ has the behavior

$$\hat{k}(s) \sim s^{\alpha-1} \quad (s \to 0). \tag{4.2}$$

In such a case, Tauberian theorems tell us that as $t \to \infty$, the correlation function behaves as [14]

$$k(t) \sim t^{-\alpha}. \tag{4.3}$$

We note that the case where $\alpha > 0$ but not an integer corresponds to long-time tail correlation functions [8].

Let $\hat{\sigma}_v^2(s)$ be the Laplace transform of the variance $\sigma_v^2(t)$; then the Laplace transform of Eq. (3.2) and the use of Eq. (4.2) yield

$$\hat{\sigma}_v^2(s) = s^{\alpha-3} \quad (s \to 0),$$

whence

$$\sigma_v^2(t) \sim t^{2-\alpha} \quad (t \to \infty). \tag{4.4}$$

We observe that if $\alpha > 2$, then $\sigma_v^2(t) \to 0$ as $t \to \infty$. Therefore, at sufficiently long times, the velocity $\dot{X}(t)$ becomes "localized" at its mean value, that is, $\dot{X}(t) \sim \langle \dot{X}(t) \rangle$.

Starting from Eq. (3.4) and following an analogous reasoning, we also obtain the asymptotic estimate for the variance of the displacement

$$\sigma_x^2(t) \sim t^{4-\alpha} \quad (t \to \infty). \tag{4.5}$$

Note that if $\alpha > 4$, then $\sigma_x^2(t) \to 0$ as $t \to \infty$. Therefore, when $\alpha > 4$ both the velocity and the displacement become localized at their mean values. In what follows we will assume that $\alpha \le 2$.

Suppose that $k(t)$ is an integrable function for $t \in [0, \infty)$, then

$$\hat{k}(0) = \int_0^\infty k(t) dt < \infty$$

and from Eq. (4.2) we see that

$$\alpha \ge 1.$$

Since we also asume that $\alpha \le 2$, we conclude from Eqs. (4.4) and (4.5) that *if a long-tailed correlation function $k(t)$ is integrable, then, at sufficiently long times, the velocity $\dot{X}(t)$ is always subdiffusive while the displacement $X(t)$ is always superdiffusive.*

We now relax the assumption that $k(t)$ is an integrable function and only assume that the input noise $F(t)$ is ergodic. In this case its correlation function $k(t)$ satisfies the Slutski condition [15]

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t k(t') dt' = 0. \tag{4.6}$$

The Laplace transform of

$$\varphi(t) = \int_0^t k(t') dt'$$

is given by

$$\hat{\varphi}(s) = \frac{\hat{k}(s)}{s},$$

and by Eq. (4.2) we see that

$$\hat{\varphi}(s) \sim s^{\alpha-2} \quad (s \to 0).$$

In this case from the Tauberian theorem we obtain

$$\varphi(t) \sim t^{1-\alpha} \quad (t \to \infty).$$

Hence

$$\frac{1}{t} \int_0^t k(t') dt' \sim t^{-\alpha} \quad (t \to \infty)$$

and the Slutski condition Eq. (4.6) will be satisfied if and only if

$$\alpha > 0.$$

Since we have also assumed that $\alpha \le 2$, we conclude from Eqs. (4.4) and (4.5) that *if $F(t)$ is ergodic, the velocity presents a subdiffusive behavior when $2 > \alpha > 1$ and a superdiffusive behavior when $0 < \alpha < 1$, while the displacement is always superdiffusive* [16].

V. FRACTIONAL BROWNIAN MOTION

As a relevant example of nonstationary input noise we will study the case when $F(t) = B_\alpha(t)$ is the so-called fractional Brownian motion (FBM). The FBM process is a Gaussian fractal process and represents a generalization of ordinary Brownian motion in which the standard deviation of the increment $|B_\alpha(t+T) - B_\alpha(t)|$ goes as T^α with $0 < \alpha < 1$. When $\alpha = 1/2$ the FBM reduces to an ordinary Brownian motion. One important feature of these kinds of processes is that they show a strong interdependence between distant samples. This asymptotic dependence is the reason for their usefulness in modeling nonstationary time series [17].

A simple version of FBM is given by the following moving average [12]:

$$B_\alpha(t) = \frac{1}{\Gamma(\alpha + 1/2)} \int_0^t (t-\tau)^{\alpha-1/2} \xi(\tau) d\tau,$$

where $0 < \alpha < 1$ and $\xi(\tau)$ is Gaussian white noise. Assuming that $\xi(\tau)$ is zero centered we easily find that the correlation function $k_\alpha(t,t') = \langle B_\alpha(t) B_\alpha(t') \rangle$ of FBM reads

$$k_\alpha(t,t') = \frac{1}{\Gamma^2(\alpha+1/2)} \int_0^{\min(t,t')} [(t-\tau)(t'-\tau)]^{\alpha-1/2} d\tau. \tag{5.1}$$

We will first evaluate the variance of the velocity $\sigma_v^2(t)$. The substitution of Eq. (5.1) into Eq. (2.12) yields

$$\sigma_v^2(t) = \frac{2}{\Gamma^2(\alpha+1/2)} \times \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} d\tau [(t'-\tau)(t''-\tau)]^{\alpha-1/2}.$$

If we interchange the order of the integrals with respect to τ and t'' we finally get

$$\sigma_v^2(t) = \frac{1}{2(\alpha+1)\Gamma^2(\alpha+3/2)}\, t^{2(\alpha+1)}. \tag{5.2}$$

Since $0<\alpha<1$ we see that the behavior of the velocity is always superdiffusive.

Following an analogous reasoning we can also obtain, after some algebra, an expression for the variance of the displacement $\sigma_x^2(t)$. Nevertheless, it turns out to be easier to follow a more straightforward way. Indeed, from the point of view of generalized functions we can write $B_\alpha(t)$ to be the solution of the differential equation

$$\dot{B}_\alpha(t) = B_{\alpha-1}(t). \tag{5.3}$$

If we assume that $x_0 = v_0 = 0$, the velocity of the particle is then given by [cf. Eq. (1.3)]

$$\dot{X}(t) = B_{\alpha+1}(t) \tag{5.4}$$

and its displacement by

$$X(t) = B_{\alpha+2}(t). \tag{5.5}$$

Therefore the correlation function of the velocity is [cf. Eq. (5.1)]

$$\langle \dot{X}(t)\dot{X}(t')\rangle = k_{\alpha+1}(t,t') \tag{5.6}$$

and the correlation function of the displacement is

$$\langle X(t)X(t')\rangle = k_{\alpha+2}(t,t'). \tag{5.7}$$

The variance of the velocity $\sigma_v^2(t) = k_{\alpha+1}(t,t)$ is therefore given by Eq. (5.2) and the variance of the displacement $\sigma_x^2(t) = k_{\alpha+2}(t,t)$ reads [cf. Eq. (5.1)]

$$\sigma_x^2(t) = \frac{1}{2(\alpha+2)\Gamma^2(\alpha+5/2)}\, t^{2(\alpha+2)}. \tag{5.8}$$

We thus see that the displacement also exhibits a superdiffusive behavior.

As we have mentioned, when $\alpha = 1/2$ the FBM reduces to the Wiener process $W(t)$ for which $\langle W(t)W(t')\rangle = \min(t,t')$. In this case

$$\sigma_v^2(t) = \frac{1}{3}t^3$$

and

$$\sigma_x^2(t) = \frac{1}{5}t^5.$$

To our knowledge, a superdiffusive behavior with a dynamical exponent $\nu = 5$ has never been reported in the literature [18]. We finally observe that since Gaussian white noise $\xi(t)$ is the derivative (in the sense of generalized functions) of the Wiener process $W(t) = d\xi(t)/dt$, and if $V(t) = \dot{X}(t)$, then the variances above correspond to that of the random process $\ddot{V}(t) = \xi(t)$ [3], this immediately implies the superdiffusive behaviors t^3 and t^5.

VI. FRACTAL STRUCTURE OF THE TRAJECTORIES

As is well known, the trajectories of the Brownian motion (or Wiener) process (that is, the trajectories of unbounded first-order processes driven by Gaussian white noise) are fractal objects in the sense that they show self-similarity and have a Hausdorff (or fractal) dimension $d=2$ greater than the topological dimension [19,20]. We have shown elsewhere [12] that the trajectories of unbound first-order processes driven by arbitrary Gaussian noise may possess a fractal structure with Hausdorff dimensions greater than or equal to the topological dimension. In this section we will explore the possible fractal structure of the trajectories of the second-order process (1.3).

We first present a brief summary of the main results of our previous work [12]. Let $Z(t)$ be a random process whose dynamical evolution is governed by the equation

$$\dot{Z}(t) = F(t), \tag{6.1}$$

where $F(t)$ is Gaussian noise with zero mean and arbitrary correlation function

$$\langle F(t)F(t')\rangle = k(t,t'). \tag{6.2}$$

We will assume that $Z(0)=0$ and define

$$\Psi_z(t,t') \equiv \int_{t'}^t d\tau \int_{t'}^t d\tau' k(\tau,\tau'). \tag{6.3}$$

This function can also be written in the form

$$\Psi_z(t,t') = k_z(t,t) + k_z(t',t') - 2k_z(t,t'), \tag{6.4}$$

where $k_z(t,t)$ is the correlation function of the output $Z(t)$,

$$k_z(t,t') = \langle Z(t)Z(t')\rangle = \int_0^t d\tau \int_0^t d\tau' k(\tau,\tau'). \tag{6.5}$$

We now suppose an equal time behavior of the function $\Psi_z(t,t')$ in the form

$$\Psi_z(t,t+\epsilon) = A(t)\epsilon^\gamma + o(\epsilon^\gamma), \tag{6.6}$$

where $0 \le \gamma \le 2$ and $A(t)$ is an arbitrary function. Let us denote by d_z the fractal dimension of the curve $Z(t)$; then

$$d_z = \frac{2}{\gamma}, \tag{6.7}$$

that is, the fractal dimension of the trajectories of the process (6.1) is (with probability 1) proportional to the inverse of the exponent that governs the equal time behavior of the correlation function of the output process [12].

Before proceeding further let us note that if the input noise is stationary

$$\langle F(t)F(t')\rangle = k(t-t'),$$

then one can easily see that Eq. (6.3) can be written in the form

$$\Psi_z(t,t+\epsilon) = \int_0^\epsilon d\tau \int_0^\epsilon d\tau' k(\tau-\tau') \tag{6.8}$$

and the variance of $Z(t)$ can be written as

$$\sigma_z^2(t) = k_z(t,t) = \int_0^t d\tau \int_0^t d\tau' k(\tau-\tau').$$

Hence

$$\Psi_z(t,t+\epsilon) = \sigma_z^2(\epsilon). \tag{6.9}$$

Therefore, we see from Eqs. (6.6), (6.7), and (6.9) that *the fractal dimension of a first-order process driven by stationary Gaussian noise is governed by the short-time behavior of its variance.*

Let us now apply these results to our second-order process. We first observe that when the input noise is stationary the fractal dimension of the velocity trajectories will be governed by the short-time behavior of the variance

$$\Psi_v(t,t+\epsilon) = \sigma_v^2(\epsilon).$$

Thus, for Ornstein-Uhlenbeck driving noise we have [cfr. Eq. (3.8)]

$$\sigma_v^2(\epsilon) = \frac{D}{2\tau_c}\epsilon^2 + o(\epsilon^2).$$

In this case $d_v = 1$ and the fractal dimension of the velocity trajectories $\dot{X}(t)$ equals the topological dimension.

For stationary Gaussian $1/f$ noise with zero mean and power spectrum given by $1/f^a$, we have

$$k(t-t') = \frac{c}{|t-t'|^{1-a}}, \tag{6.10}$$

where c is a constant and $0 < a < 1$ [12] (this is a typical example of the kind of input noise with a long-time tail correlation that we discussed in Sec. IV). The variance of the velocity now reads

$$\sigma_v^2(t) = \frac{2c}{a(1+a)} t^{1+a},$$

whence

$$d_v = \frac{2}{1+a}.$$

Since $0 < a < 1$, we see that the fractal dimension of the trajectories of the velocity is any real number between 1 and 2 depending on the exponent of the power spectrum.

In order to evaluate the fractal dimension of the trajectories $X(t)$ of the free inertial process (1.3) we assume that $x_0 = v_0 = 0$ and write the dynamical equation (1.3) in the form

$$\dot{X}(t) = \int_0^t F(t')dt'. \tag{6.11}$$

Therefore, the correlation function of the displacement $k_x(t,t') = \langle X(t)X(t')\rangle$ is

$$k_x(t,t') = \int_0^t d\tau \int_0^{t'} d\tau' k_v(\tau,\tau'), \tag{6.12}$$

where $k_v(t,t')$ is the correlation function of the velocity

$$k_v(t,t') = \int_0^t d\tau \int_0^t d\tau' \langle F(\tau)F(\tau')\rangle \tag{6.13}$$

and the fractal dimension of $X(t)$ will now be given by the equal time behavior of the function

$$\Psi_x(t,t+\epsilon) = k_x(t,t) + k_x(t+\epsilon,t+\epsilon) - 2k_z(t,t+\epsilon). \tag{6.14}$$

For Gaussian $1/f$ noise the correlation function of the input noise $F(t)$ is given by Eq. (6.10) and we obtain, after some algebra, the following correlation function for $X(t)$:

$$k_x(t,t') = \frac{c}{a(1+a)(2+a)}\Bigg\{t^{2+a}t' + tt'^{2+a} - \frac{1}{3+a}\left[t^{3+a} + t'^{3+a} - |t-t'|^{3+a}\right]\Bigg\}. \tag{6.15}$$

If we substitute Eq. (6.15) into Eq. (6.14) we get

$$\Psi_x(t,t+\epsilon) = \frac{2c}{a(1+a)(2+a)} \times \left\{\epsilon\left[(t+\epsilon)^{2+a} - t^{2+a}\right] - \frac{1}{3+a}\epsilon^{3+a}\right\}. \tag{6.16}$$

The Taylor expansion of this equation around $\epsilon = 0$ yields

$$\Psi_x(t,t+\epsilon) = A(t)\epsilon^2 + O(\epsilon^3). \tag{6.17}$$

Therefore $d_x = 1$, i.e., the fractal dimension of the trajectory $X(t)$ equals the topological dimension and, contrary to the velocity, the displacement of the particle presents no fractal behavior.

For Ornstein-Uhlenbeck driving noise, with $D = \tau_c = 1$, we have

$$\langle F(t)F(t')\rangle = (1/2)\exp(-|t-t'|)$$

and the correlation function of the trajectory of the particle is [cf. Eqs. (6.12) and (6.13)]

$$k_x(t,t') = -\frac{1}{6}t^3 + \frac{1}{2}t^2t' - \frac{1}{2}tt' + \frac{1}{2}t' - \frac{1}{2}t - \frac{1}{2}(1+t')e^{-t'} - \frac{1}{2}(1+t)e^{-t} + \frac{1}{2}e^{-(t+t')} + \frac{1}{2}. \quad (6.18)$$

In this case, one can easily show that

$$\Psi_x(t,t+\epsilon) = A(t)\epsilon^2 + O\left(\epsilon^3\right), \quad (6.19)$$

whence $d_x = 1$ and, as in the case of the velocity, there is no fractal structure for the trajectories $X(t)$.

Let us finally assume, as an example of nonstationary input noise, that $F(t)$ is the fractional Brownian motion defined in Sec. V. In this case, the correlation function of the velocity is given by Eq. (5.6), where $k_\alpha(t,t')$ is defined in Eq. (5.1). Analogously, the correlation function of the displacement is given by Eq. (5.7). In order to obtain the fractal dimension of $\dot{X}(t)$ and $X(t)$ we must evaluate the equal time behavior of the function

$$\Psi_{\alpha+j}(t,t+\epsilon) = k_{\alpha+j}(t,t) + k_{\alpha+j}(t+\epsilon,t+\epsilon) - 2k_{\alpha+j}(t,t+\epsilon), \quad (6.20)$$

where $j = 1$ (2) for the velocity (displacement). The substitution of Eq. (5.1) into Eq. (6.20) yields

$$\Psi_{\alpha+j}(t,t+\epsilon) = \frac{1}{\Gamma^2(\alpha+j+\frac{1}{2})} \times \left\{ \frac{1}{2(\alpha+j)}\left[(t+\epsilon)^{2(\alpha+j)} + t^{2(\alpha+j)}\right] - 2\int_0^t \left[(t-\tau)(t+\epsilon-\tau)\right]^{\alpha+j-1/2} d\tau \right\}. \quad (6.21)$$

By expanding the right-hand side of this equation we can easily see that

$$\Psi_{\alpha+j}(t,t+\epsilon) = A(t)\epsilon^2 + O(\epsilon^3) \quad (6.22)$$

and by Eq. (6.7) we see that both the velocity and the displacement have the same fractal dimension

$$d_v = d_x = 1,$$

which is equal to the topological dimension.

We finally note that the results of this section seem to indicate that the fractal dimension of the trajectories for the displacement $X(t)$ of the inertial process (1.3) is always equal to the topological dimension regardless the form of the correlation function $k(t,t')$ of the driving noise. In fact, we can give a simple, although not rigorous, proof of this assertion. In effect, for very short time increments, the displacement of the particle is given by

$$X(t+\Delta t) = \dot{X}(t)\Delta t + O\left(\Delta t^2\right)$$

and $\langle X^2(t+\Delta t)\rangle \sim \Delta t^2$. Hence $d_x = 1$ and the fractal dimension equals the topological dimension. However, the fractal dimension of trajectories of the velocity $\dot{X}(t)$ is dependent of the correlation function of the driving noise.

VII. SUMMARY AND CONCLUSIONS

The problem of describing the motion of a free inertial processes in an unbounded space driven by arbitrary Gaussian colored noise has been analyzed in detail. We have derived exact equations for the joint and marginal densities and obtained the exact solutions for the initial-value problem.

Under a variety of circumstances the asymptotic behavior of a random process is that of ordinary diffusion, that is, the variance of the process grows linearly as time increases. Nevertheless, the consideration of inertial effects has been shown to change this situation and supperdiffusion appears [3]. On the other hand, a nonflat spectrum of the driving noise (i.e., the color) also changes that situation, for colored noise may produce not only superdiffusion but anomalous diffusion

$$\sigma^2(t) \sim t^\nu \qquad (\nu > 0),$$

where ν is the so-called dynamical exponent of the variance and $\nu < 1$ for subdiffusion and $\nu > 1$ for superdiffusion. Herein we have also analyzed the anomalous diffusion behavior of both the velocity and the displacement of the random particle with special emphasis on input noises that have long-time tail correlation functions. One interesting aspect of this behavior is the appearance of subdiffusion or superdiffusion depending on the power of the tail.

Fractal geometry seems to be a convenient tool for studying disorderd media since the fractal dimension is a quantitative measure of some degree of disorder [21]. On the other hand, it has been known for some time that some random processes possess a fractal structure with Hausdorff dimensions greater than the topological dimensions [19,12]. We have studied in detail the effects of a random acceleration on the fractal behavior of the trajectories of the particle. We have first shown that for first-order processes driven by stationary noise, there is a simple relation between the short-time behavior of the variance and the fractal dimension d of the process [cf. Eqs. (6.6), (6.7), and (6.9)]

$$\sigma^2(t) \sim t^{2/d} \quad (7.1)$$

($t \to 0$). This is an interesting expression since it constitutes an example of the interplay between statistical properties and the underlying geometry of the problem. In this sense, it resembles the classical Donsker-Varadhan approximation for the trapping problem where the average survival probability inside a volume of topological dimension d has the asymptotic property [22,23]

$$\ln\langle S(t)\rangle \sim t^{d/(d+2)}$$

($t \to \infty$). Unfortunately, there does not seem to be an equivalent relation to that of Eq. (7.1) either when the input noise is nonstationary or for the long-time behavior of the variance of the velocity and for the fractal dimension of the trajectories $X(t)$ of free inertial processes. Although, in the latter case, we have obtained a variety of fractal dimensions depending on the correlation function of the driving noise. Nevertheless, we have been unable to relate these fractal dimensions to any asymptotic property of the variance. This point is presently under investigation.

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APPENDIX A: DERIVATION OF EQ. (2.4)

We write $\delta(\mathbf{r}-\mathbf{R}(t)) \equiv \delta(x-X(t))\delta(v-\dot{X}(t))$. Then the derivative of Eq. (2.1) with respect to time and the use of Eq. (1.3) yield

$$\frac{\partial}{\partial t}p(x,v,t) = -v\frac{\partial}{\partial x}p(x,v,t) - \frac{\partial}{\partial v}\langle F(t)\delta(\mathbf{r}-\mathbf{R}(t))\rangle. \tag{A1}$$

We now use Novikov's theorem [24] to write

$$\langle F(t)\delta(\mathbf{r}-\mathbf{R}(t))\rangle = \int_0^t dt' k(t,t')\left\langle \frac{\delta}{\delta F(t')}\delta(\mathbf{r}-\mathbf{R}(t))\right\rangle,$$

but

$$\frac{\delta}{\delta F(t')}\delta(\mathbf{r}-\mathbf{R}(t)) = -\frac{\delta X(t)}{\delta F(t')}\frac{\partial}{\partial x}\delta(\mathbf{r}-\mathbf{R}(t)) - \frac{\delta \dot{X}(t)}{\delta F(t')}\frac{\partial}{\partial v}\delta(\mathbf{r}-\mathbf{R}(t)), \tag{A2}$$

where [cf. Eqs. (2.2) and (2.3)]

$$\frac{\delta X(t)}{\delta F(t')} = (t-t')\Theta(t-t'), \quad \frac{\delta \dot{X}(t)}{\delta F(t')} = \Theta(t-t'),$$

where $\Theta(x)$ is the Heaviside step function. Therefore,

$$\langle F(t)\delta(\mathbf{r}-\mathbf{R}(t))\rangle = -\left[\phi(t)\frac{\partial}{\partial x} + \varphi(t)\frac{\partial}{\partial v}\right]p(x,v,t), \tag{A3}$$

where $\varphi(t)$ and $\phi(t)$ are defined in Eqs. (2.5) and (2.6). The substitution of Eq. (A3) into Eq. (A1) immediately leads to Eq. (2.4).

APPENDIX B: DERIVATION OF EQ. (2.11)

The change of variables

$$\rho' = \rho, \quad \sigma = \rho + \mu t$$

turns Eq. (2.9) into the first-order ordinary differential equation

$$\frac{\partial}{\partial \rho}\tilde{p}(\mu,\rho,\sigma) = \frac{1}{\mu}\left[\mu\phi\left(\frac{\sigma-\rho}{\mu}\right) + \rho^2\varphi\left(\frac{\sigma-\rho}{\mu}\right)\right] \times \tilde{p}(\mu,\rho,\sigma) \tag{B1}$$

with initial condition

$$\tilde{p}(\mu,\rho=\sigma,\sigma) = e^{-i(\mu x_0+\sigma v_0)}. \tag{B2}$$

In the original variables (μ,ρ,t) the solution of problem (B1) and (B2) reads

$$\tilde{p}(\mu,\rho,t) = \exp\left\{ -i\left[\mu x_0 + (\rho+\mu t)v_0\right] - \int_\rho^{\rho+\mu t} y\phi\left(t+\frac{\rho-y}{\mu}\right)dy - \frac{1}{\mu}\int_\rho^{\rho+\mu t} y^2\varphi\left(t+\frac{\rho-y}{\mu}\right)dy\right\},$$

which after some manipulations results in Eq. (2.11).

APPENDIX C: DERIVATION OF EQS. (3.3) AND (3.4)

Taking into account that $k(t',t'') = k(t'-t'')$ we can write Eq. (2.12) in the form

$$\sigma_x^2(t) = 2\int_0^t dt'(t-t')\left[(t-t')\int_0^{t'} dt'' k(t'') + \int_0^{t'} dt'' t'' k(t'')\right].$$

Integrating by parts the last integral on the right-hand side of this equation we get

$$\sigma_x^2(t) = 2\int_0^t dt'(t-t')\left[t\varphi(t') - \int_0^{t'} dt''\varphi(t'')\right], \tag{C1}$$

where $\varphi(t)$ is defined by Eq. (2.5). Integrating by parts another time we have

$$\int_0^t dt' \int_0^{t'} dt'' \varphi(t'') = t \int_0^t dt' \varphi(t') - \int_0^t dt' t' \varphi(t').$$

The substitution of this equation into Eq. (C1) and some reorganization of terms yield

$$\sigma_x^2(t) = \int_0^t t' dt' \int_0^{t'} dt'' \varphi(t''),$$

that is [cf. Eq. (2.12)],

$$\sigma_x^2(t) = \int_0^t t' dt' \int_0^{t'} dt'' \int_0^{t''} dt''' k(t''')$$

and taking into account Eq. (3.2) we finally get Eq. (3.4). Following an analogous procedure one can also obtain Eq. (3.3).

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