Magnetic moments of leptons beyond the Standard Model

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Abstract: The anomalous magnetic dipole moment of a lepton (a_ℓ) can be measured experimentally and also computed theoretically within and beyond the Standard Model. Currently, there are discrepancies between measurements and Standard Model predictions for both the electron and the muon magnetic moments. We perform an analysis of a_ℓ beyond the Standard Model in terms of a low-energy effective field theory, and include the effects of effective operators of dimension five and six. A final expression for a_ℓ is given that allows to interpret the discrepancy in terms of non-standard contributions to the coefficients of these effective operators.

I. INTRODUCTION

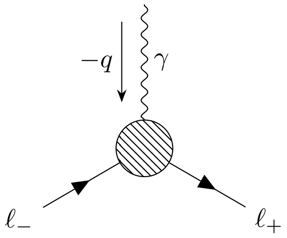
The anomalous magnetic dipole moment of a fermion is a contribution of effects of quantum mechanics to the magnetic moment of that particle. In quantum electrodynamics, the Dirac magnetic moment, that corresponds to tree-level Feynman diagrams, is usually expressed in terms of the g-factor, which from the Dirac equation its value equals $g = 2$.

This result differs from the observed value for particles such as the electron or muon by a small fraction due to quantum corrections. This difference from the g-factor is the so-called anomalous magnetic dipole moment, defined as:

$$a_\ell = \frac{g-2}{2} \ . \tag{1}$$

The electron anomalous magnetic moment is also deeply related to the fine-structure constant α, which during this work we will try to determine using the experimentally observed values, and the computation of a_ℓ using an effective field theory (EFT). An effective field theory is a tool conceived to analyse phenomena at a certain low energy scale parameterizing the dynamics of physics at high energies in full generality by a few effective coefficients.

Currently, the measurements for the anomalous magnetic moments of electrons and muons disagree to some level with the Standard Model (SM) predictions. In this project we investigate the possibility that this difference is due to beyond Standard Model (BSM) physics. The anomalous magnetic and electric dipole moments of a lepton are defined through the vertex function:



$$= -ie\overline{u}_\ell(p')\Gamma(p,p')u_\ell(p)$$

The vertex function can be decomposed in terms of Lorentz-invariant form factors [1, 2] as:

$$\begin{aligned}\Gamma(p,p') \;&=\; \gamma^\mu F_E(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m_\ell}F_M(q^2) \\ &+\; \frac{\sigma^{\mu\nu}q_\nu}{2m_\ell}\gamma_5 F_D(q^2) + \frac{q^2\gamma^\mu - q^\mu \not{q}}{m_\ell^2}\gamma_5 F_A(q^2) \ .\end{aligned} \tag{2}$$

The anomalous magnetic moment of a lepton is then defined as the magnetic form factor at $q^2 = 0$ [1]:

$$a_\ell = \frac{(g_\ell - 2)}{2} = F_M(q^2 = 0) \ . \tag{3}$$

Thus we need to calculate the vertex function $\Gamma(p,p')$ and isolate the magnetic form factor F_M at zero momentum transfer.

II. LOW ENERGY EFFECTIVE FIELD THEORY

A. Effective Lagrangian

We consider dimesion-five and -six terms in the LEFT Lagrangian, that include both SM and BSM contributions [3]:

$$\mathcal{L} = \mathcal{L}_I + \mathcal{L}_{II} + \mathcal{L}_{III} \tag{4}$$

where

$$\begin{aligned}\mathcal{L}_I \;&=\; L_{e\gamma}(\overline{e}_L\sigma^{\mu\nu}e_R)F_{\mu\nu} + \text{h.c.} \ , \\ \mathcal{L}_{II} \;&=\; \sum_\ell L^{S,RR}_{e\ell\ell e}(\overline{e}_L\ell_R)(\overline{\ell}_L e_R) + \text{h.c.} \ , \\ \mathcal{L}_{III} \;&=\; \sum_q \Big[L^{S,RL}_{eq}(\overline{e}_L e_R)(\overline{q}_R q_L) \\ &\qquad + L^{S,RR}_{eq}(\overline{e}_L e_R)(\overline{q}_L q_R)\Big] + \text{h.c.} \ ,\end{aligned} \tag{5}$$

where $\ell = \{e,\mu,\tau\}$ and $q = \{u,d,s,c,b\}$, and the subindexes R,L are such that $\psi_{R,L} \equiv P_{R,L}\psi$, where

$P_{R/L} = (1 \pm \gamma^5)/2$ are the right- and left-handed chirality projectors.

B. Contribution from effective operators

We begin considering the dimension-five dipole contributions in $\mathcal{L}_I$, and expand the field-strength tensor in terms of photon fields in momentum space:

$$\begin{aligned}\mathcal{L}_I &= L_{e\gamma}(\overline{e}_L\sigma^{\mu\nu}e_R)F_{\mu\nu} + \text{h.c.}\\ &= -iL_{e\gamma}\overline{e}\sigma^{\mu\nu}P_R q_\mu A_\nu e + iL_{e\gamma}\overline{e}\sigma^{\mu\nu}P_R q_\nu A_\mu e + \text{h.c.}\\ &= 2iL_{e\gamma}\overline{e}\sigma^{\mu\nu}P_R q_\nu A_\mu e + \text{h.c.} \qquad (6)\end{aligned}$$

The hermitian conjugation leads only to conjugation of the Wilson coefficient $L_{e\gamma}$, leading to the dependence

$$L_{e\gamma} + (L_{e\gamma})^* = 2\text{Re}\,(L_{e\gamma})\ . \qquad (7)$$

Then the contribution to the vertex function from $\mathcal{L}_I$ is

$$\begin{aligned}&-ie\overline{u}_e(p')\Gamma_I(p,p')u_e(p)\\ &\quad= -i\Big[4i\text{Re}\,(L_{e\gamma})\,\overline{u}_e\sigma^{\mu\nu}P_R q_\nu u_e\Big]\\ &\quad= -ie\overline{u}_e\Big[\frac{4i}{e}\text{Re}\,(L_{e\gamma})\,\sigma^{\mu\nu}P_R q_\nu\Big]u_e \qquad (8)\\ &\quad= -ie\overline{u}_e\Big[\frac{2i}{e}\text{Re}\,(L_{e\gamma})\,\sigma^{\mu\nu}q_\nu + \frac{2i}{e}\text{Re}\,(L_{e\gamma})\,\sigma^{\mu\nu}\gamma_5 q_\nu\Big]u_e\ .\end{aligned}$$

Identifying the magnetic form factor in our expression and assuming the only impact of the new physics comes from this dimension-five operator $L_{e\gamma}$, we have:

$$a_e = \frac{\alpha}{2\pi} + \frac{4m_e\text{Re}\,(L_{e\gamma})}{e} + \mathcal{O}(\alpha^2)\ . \qquad (9)$$

The first term corresponds to the magnetic form factor in the Standard Model to order α, first time computed by Schwinger in 1948 [4].

Assuming that the standard model is valid the most precise determination of α is [5, 6]:

$$\alpha^{-1} = 137.035\ 999\ 1570(334)\ . \qquad (10)$$

This value is computed from

$$a_e(\text{theory}) = a_e(\text{experiment})\ , \qquad (11)$$

where the theoretical a_e(theory) goes up to order α^5 and the experimental value was a measurement obtained by the Harvard group using a one-electron quantum cyclotron [6].

Using our Eq. 9 we can plot the "correct" value of α depending on the value of $L_{e\gamma}$, as shown in Figure 1. The best α available at present is the one from the experiment that measures the recoil velocity of a rubidium atom that absorbs a photon by a matter-wave interferometry [7],

$$\alpha^{-1}(\text{Rb}) = 137.035\ 999\ 206(11)\ . \qquad (12)$$

On the other hand, the latest measurement of the anomalous electron magnetic moment is [6]

$$a_e^{exp} = 0.001\ 159\ 652\ 180\ 73(28)\ . \qquad (13)$$

From these values, using the relation of Eq.(9) with the QED correction up to α^5[5], we can compute a valid value of the effective coefficient $L_{e\gamma}$:

$$L_{e\gamma} = 0.34(26)\times 10^{-11}\ \text{GeV}^{-1}\ . \qquad (14)$$

For the second term in the Lagrangian we have a four-lepton operator with a coefficient of dimension-six:

$$\mathcal{L}_{II} = \sum_\ell L^{S,RR}_{e\ell\ell e}(\overline{e}_L\ell_R)(\overline{\ell}_L e_R) + \text{h.c.} \qquad (15)$$

The contribution to the vertex function from this dimension-six operator (see Fig. 2) is given by the integral over the loop momentum k of

$$\begin{aligned}&L^{S,RR}_{e\ell\ell e}\left(\overline{u}_e P_R\frac{\not{k}+m_\ell}{k^2-m_\ell^2}\gamma^\mu\frac{\not{k}+\not{q}+m_\ell}{(k+q)^2+m_\ell^2}P_R u_e\right) \qquad (16)\\ &\quad= \frac{m_\ell}{[k^2-m_\ell^2][(k+q)^2+m_\ell^2]}\overline{u}_e\left(\Gamma_1+\Gamma_2\right)P_R u_e\ ,\end{aligned}$$

where

$$\begin{aligned}\Gamma_1+\Gamma_2 &= \gamma^\mu\not{k} + \gamma^\mu\not{q} + \not{k}\gamma^\mu \qquad (17)\\ &= 2k^\mu - \not{k}\gamma^\mu + \gamma^\mu\not{q} + \not{k}\gamma^\mu = 2k^\mu + \gamma^\mu\not{q}\ .\end{aligned}$$

So two integrals will have to be computed:

$$S_1 = 2\int\frac{d^4k}{(2\pi)^4}\frac{k^\mu}{[k^2-m_\ell^2][(k+q)^2+m_\ell^2]}\ , \qquad (18)$$

$$S_2 = \int\frac{d^4k}{(2\pi)^4}\frac{1}{[k^2-m_\ell^2][(k+q)^2+m_\ell^2]}\ . \qquad (19)$$

Computing these integrals, we have

$$S_1 = 2\int\frac{d^4\ell}{(2\pi)^4}\int_0^1 dx\frac{\ell^\mu - xq^\mu}{[\ell^2+x(1-x)q^2-m_\ell^2]^2} = 2\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\tilde{\mu}^{2\epsilon}\frac{\ell^\mu - xq^\mu}{[\ell^2-\Delta]^2} \qquad (20)$$

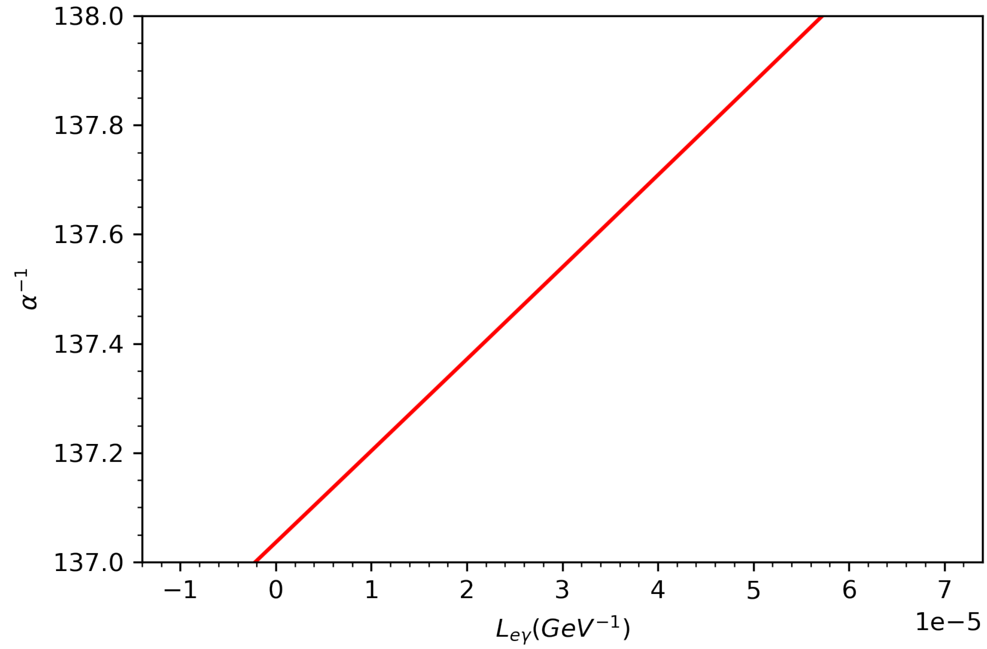


Figure 1: Determination of the fine structure constant α as a function of the effective coefficient $L_{e\gamma}$.

$$= -2\int_0^1 dx \frac{ixq^\mu \tilde{\mu}^{2\epsilon}}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} = -2\int_0^1 dx \frac{ixq^\mu}{(4\pi)^2}\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{\Delta}\right) \overset{q^2\to 0}{=} \frac{-iq^\mu}{(4\pi)^2}\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{m_\ell^2}\right) ,$$

$$S_2 = \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2-\Delta]^2} = \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d}\tilde{\mu}^{2\epsilon}\frac{1}{[\ell^2-\Delta]^2} = \int_0^1 dx \frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{\Delta}\right)$$
$$\overset{q^2\to 0}{=} \frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{m_\ell^2}\right) , \tag{21}$$

where $\ell = k + xq$ and $\Delta = m_\ell^2 - x(1-x)q^2$. Also, we have used dimensional regularization in $d = 4 - 2\epsilon$ dimensions in order to regularize ultraviolate divergences. The renormalization scale μ is defined in the $\overline{\text{MS}}$-scheme (modified minimal substraction) in order to absorb the divergent part plus some constants that always arise alongside the divergence in Feynman diagram calculations. This is implemented by rescaling the renomalization scale: $\tilde{\mu}^2 = \frac{\mu^2 e^\gamma}{4\pi}$ [8].

Putting everything together, the contribution to the vertex function from $\mathcal{L}_{II}$ is:

$$-ie\overline{u}_e(p')\Gamma_{II}(p,p')u_e(p) = \frac{i\,m_\ell}{(4\pi)^2}L_{e\ell\ell e}^{S,RR}\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{m_\ell^2}\right)\overline{u}_e(\gamma^\mu \not{q} - q^\mu)P_R u_e \ . \tag{22}$$

Using that

$$-q^\mu + \gamma^\mu \not{q} = -q^\mu + \frac{1}{2}(\gamma^\mu\not{q} + 2q^\mu - \not{q}\gamma^\mu) \tag{23}$$
$$= \frac{1}{2}[\gamma^\mu\not{q} - \not{q}\gamma^\mu] = -i\frac{i}{2}[\gamma^\mu,\gamma^\nu]q_\nu = -i\sigma^{\mu\nu}q_\nu ,$$

we can identify the contribution from the lepton l to the magnetic form factor at zero momentum transfer,

$$F_M(q^2=0) = \frac{2m_e m_\ell}{(4\pi)^2}\log\frac{\mu^2}{m_\ell^2}L_{e\ell\ell e}^{S,RR} . \tag{24}$$

For the total contribution to the anomalous magnetic moment of the four lepton operator, it has to be taken in account the hermitian conjugate and summed up over all the leptons:

$$a_e^{4l} = m_e \sum_{\ell=e,\mu,\tau}\frac{m_\ell}{4\pi^2}\log\frac{\mu^2}{m_\ell^2}\text{Re}(L_{e\ell\ell e}^{S,RR}) . \tag{25}$$

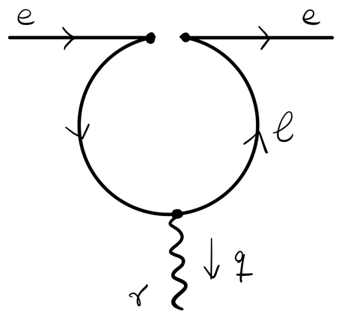


Figure 2: One loop contribution to a_e from the four-lepton operator with coefficient $L^{S,RR}_{e\ell\ell e}$.

Regarding the one loop contributions from the third term in the EFT Lagrangian, the two coefficient terms and their hermitian conjugates give the same contribution to the vertex function, so we will only show the computation for the first one.

$$\mathcal{L}_{III} = \sum_q L^{S,RL}_{eq} (\bar{e}_L e_R)(\bar{q}_R q_L) + \text{h.c.} \tag{26}$$

The contribution to the vertex function of this third term is also given by the following integral over the loop momentum k:

$$\begin{aligned}
&-ie\bar{u}_e(p')\Gamma_{III}(p,p')u_e(p) = \\
&\quad -ieL^{S,RL}_{eq}(\bar{u}_e P_R u_e)\int \frac{\tilde{\mu}^{2\epsilon} d^d k}{(2\pi)^d}(-1)\times \\
&\quad \times \text{Tr}\left[\frac{i[(\not{q}+\not{k})+m_q]}{(q+k)^2-m_q^2} P_L \frac{i(\not{k}+m_q)}{k^2-m_q^2}\gamma^\mu\right] \\
&\quad = 2ieL^{S,RL}(\bar{u}_e P_R u_e)\int \frac{d^d k}{(2\pi)^d}\frac{\tilde{\mu}^{2\epsilon} m_q (q^\mu + 2k^\mu)}{[(q+k)^2-m_q^2][k^2-m_q^2]} \\
&\quad = \int_0^1 dx \frac{i m_q (1-2x) q^\mu}{(4\pi)^2}\left(\frac{1}{\epsilon} - \log\frac{\mu}{\Delta}\right) = 0 \ .
\end{aligned} \tag{27}$$

The difference with respect the four-lepton operators is that this is a *closed* fermion loop (as opposed to Fig.2, which is an *open* loop). Thus there are no one-loop contributions proportional to $L^{S,XY}_{eq}$.

III. TOTAL CONTRIBUTION TO THE ANOMALOUS MAGNETIC MOMENT

After all the computation for each term of the LEFT Lagrangian and identifying the magnetic form factor, a_e can be expressed up to one-loop order as:

$$\begin{aligned}
a_e \;=\; &\frac{\alpha}{2\pi} + \frac{4m_e Re\,(L_{e\gamma})}{e} \\
&+ m_e \sum_{\ell=e,\mu,\tau} \frac{m_\ell}{4\pi^2}\text{Re}(L^{S,RR}_{e\ell\ell e})\log\frac{\mu^2}{m_\ell^2} + \mathcal{O}(\alpha^2) \ .
\end{aligned} \tag{28}$$

The first term comes from the one-loop QED correction, the second term is the contribution from the photonic dipole operator (see Eq. 9) and the third term is the contribution from the four-lepton operator (see Eq. 25).

IV. CONCLUSIONS

Using the Low-Energy Effective Field Theory Lagrangian, we have calculated the one loop contribution to the anomalous magnetic moment of the electron. Apart from the pure QED correction (we have only considered the one-loop contribution of order α, but it is known up to α^5 [5]) we have provided the contribution from effective coefficients of higher dimension that include the possible effects of physics beyond the Standard Model.

A future work could include the study of the values of these coefficients in order to explain the anomalies between the theory and experimental value of a_e and a_μ. For example, assuming that just one of the coefficients has a non-zero value, or allowing combinations of them. In this work we have considered the case of $L_{e\gamma}$ as the only non-zero coefficient, obtaining a value of

$$L_{e\gamma} = 0.34(26) \times 10^{-11}\,\text{GeV}^{-1} \ , \tag{29}$$

using the currently most precise measurements of α and a_e.

As a conclusion, it has been observed that the discrepancy Δa_e can be explained within the LEFT with suitable BSM contributions to the coefficients of dipole and four-lepton operators. It remains to see if these values are allowed by the measurements of other leptonic observables.

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