Numerical simulations of hyperbolic encounters of black holes

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Abstract: The objective of this project is to study simulations of hyperbolic encounters of black holes (BHs) using a 1.5 Post-Newtonian approximation and assess its validity by comparing the data calculated on the movement and on gravitational wave (GW) radiation with the results provided by numerical relativity simulations. The PN approximations are known to not always converge, but they are suitable in some regions of parameters space. For the region of parameters explored, we find that the best results are obtained if the expansion of the GW radiation is truncated at the leading order.

I. INTRODUCTION

Since the LIGO-Virgo collaboration detected the GW signal of a collision of two BHs for the first time ever in 2015 [1], the interest of the scientific community in this research field has boomed. All the binary BHs observations done by LIGO-Virgo were analysed with wave-forms generated or informed by numerical relativity simulations [2]. These simulations are based on numerical calculation methods to solve the dynamical field equations of General Relativity, when the initial parameters are given. They are very useful to create *template banks* of waveforms to have a better understanding of the observations. Numerical relativity simulations of BH binaries became a reality in 2005 with the *moving puncture approach* formalism proposed by Baumgarte-Shapiro-Shibata-Nakamura [3], and along the following years the software named `Einstein Toolkit` was developed [4].

These simulations provide very accurate results of the relativistic dynamic equations and the physics of GWs. But they have a drawback, which is that they take up to several days to run in a supercomputer. The latter fact motivates the community to find approximate calculations to reduce the duration of the simulation to a few seconds. A common method to do so is to perform Post-Newtonian approximations of the General Relativity equations. In recent years, the community has devoted significant effort to model GW beyond the quasi-circular mergers that lead to the first detections. Recently, Juan García-Bellido *et al.* have published a paper with a 1.5 Post-Newtonian approximation for the particular problem of a hyperbolic encounter (i.e. interaction without merger) of two spinning BHs [5]. The purpose of my project was to check the calculations performed in this paper and to develop an independent code in order to reproduce the results of [5] and compare them with numerical relativity calculations performed with the software `Einstein Toolkit`. We had access to the code used in [5], so we could directly cross-check with it too, finding perfect agreement.

As it is well-known, the Post-Newtonian expansion is asymptotic, e.g. depending on the parameters, subleading terms can dominate over the leading ones [6]. We have observed this behavior for the parameters analyzed, finding that the leading order was the one that captured best the results of numerical relativity.

II. SIMULATION DETAILS

A. 1.5 PN Approximation

With this method, one is able to study the general case in which the BHs can spin, but it has been chosen to describe the non-spinning case for brevity. Taking into consideration a binary system of total mass m with two non-spining BHs of masses m_1 and m_2 and reduced mass $\mu = \frac{m_1 m_2}{m}$, the reduced Hamiltonian $H = \mathcal{H}/\mu$ in the centre of masses reference frame that has to be solved is: [5]

$$H = H_N(\vec{r}, \vec{p}) + H_{1\mathrm{PN}}(\vec{r}, \vec{p}) \tag{1}$$

Where $\vec{r} = \vec{\mathcal{R}}/Gm$ is the relative distance vector of the bodies, $\vec{p} = \vec{\mathcal{P}}/G\mu$ is the conjugate momentum and:

$$H_N(\vec{r}, \vec{p}) = \frac{p^2}{2} - \frac{1}{r} \tag{2}$$

$$H_{1\mathrm{PN}}(\vec{r}, \vec{p}) = \frac{1}{c^2}\left(\frac{1}{8}(3\eta - 1)(p^2)^2 - \frac{1}{2}[(3+\eta)p^2 + \eta(\hat{n}\cdot\vec{p})^2]\frac{1}{r} + \frac{1}{2r^2}\right) \tag{3}$$

Where it has been defined $\eta = \mu/m$ and $\hat{n} = \vec{r}/r$. Since the BHs do not spin, their trajectories are going to be confined in a plane, which simplifies the problem, allowing one to reduce the dynamics to three coupled ODEs recorded in the Appendix.

Solving the radial equation $\dot{r} = \{\vec{r}, H\} \cdot \hat{n}$ neglecting terms $O(1/c^4)$ we get:

$$\dot{\bar{r}}^2 = A + \frac{2B}{\bar{r}} + \frac{C}{\bar{r}^2} \tag{4}$$

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With $\overline{r} = r - \frac{D}{2L^2}$ and where A, B, C, D are constants in terms of the constants of movement:

$$L^2 = |\vec{r} \times \vec{p}|^2 \tag{5}$$

$$E = \frac{p^2}{2} - \frac{1}{r} + O(1/c^2) \tag{6}$$

Equation (4) has a parametric solution for hyperbolic orbits with an auxiliary variable v that has to be evaluated at each time step of the simulation:

$$\begin{cases} \overline{n}(t - t_0) = e_t \sinh v - v \\ r = \overline{a}_r(e_r \cosh v - 1) \end{cases} \tag{7}$$

Where:

$$e_r = e_t \left(1 + (3\eta - 8)\frac{E}{c^2}\right) \tag{8}$$

$$\overline{a}_r = \frac{1}{c^2 \overline{\xi}^{2/3}} \left(1 - \overline{\xi}^{2/3} \frac{(\eta - 9)}{3}\right) \tag{9}$$

The dynamical variables can be chosen to be $\overline{n} = \overline{\xi} c^3$ and e_t, related to radial motion, in addition to an angular variable Φ. They satisfy a system of coupled ODEs (see Appendix). The solution for the trajectory can then be used to evaluate the GWs by using the expressions for the strain given in equations (71a) and (71b) in [5]. The simulations using this algorithm have been programmed with `Wolfram Mathematica`.

B. Numerical Relativity

Numerical relativity simulations have implemented the hydrodynamic and the field equations of General Relativity in its algorithms that can be solved when a set of initial conditions is given: masses of the BHs, positions, momentum, ADM energy and ADM angular momentum. The evolution is computed using the 3+1 decomposition of the space-time [7], based on the projection of vectors and tensors along the normal direction (time vector) or onto the spatial slices to obtain the evolution of the system. To extract the physics of the GW, it is useful to employ the Newman-Penrose formalism, that uses the 3+1 split of space-time to project the components of the Riemann tensor (Gauss-Codazzi-Ricci equations) to a later contraction with a certain complex null tetrad, obtaining the Weyl scalar ψ_4 [8]. This scalar is interpreted as the outgoing gravitational radiation, and it is related to the amplitudes of polarization of the GWs (observable strain) with the following expression:

$$\psi_4 = \ddot{h}_+ - i\ddot{h}_\times \tag{10}$$

The ψ_4 admits an expansion in spherical harmonics:

$$\psi_4(t, r, \theta, \phi) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \psi_4^{\ell m}(t, r)_{-2}Y_{\ell m}(\theta, \phi) \tag{11}$$

Where the coefficients $\psi_4^{\ell m} = \ddot{Q}^+_{\ell m} - i\ddot{Q}^\times_{\ell m}$. In practice, the ψ_4 is calculated by just considering the dominant mode ($\ell = 2, m = 2$):

$$\psi_4(t, r, \theta, \phi) = \psi_4^{(2,2)}(t, r)_{-2}Y_{22}(\theta, \phi) \tag{12}$$

The `Einstein Toolkit` [4] is a package developed with `Python` that has implemented the formalism and the numerical methods described. The ψ_4 data (11) is obtained by the modules `WeylScal4` and `Multipole`.

III. RESULTS AND ANALYSIS

Four different simulations have been performed considering $G = c = 1$, with the same resolution and BHs of equal masses. All the magnitudes are expressed in geometric units.

A. Initial data

To perform a proper comparison of the numerical relativity and the 1.5PN approximation, the initial conditions to be input in the `Einstein Toolkit` have to be converted into the variables used in the PN simulation. The initial values of the dynamical variables defined in equations (7) are:

$$\overline{\xi}_0 = \frac{\overline{n}_0}{c^3} = (2E)^{3/2} \left(1 - \frac{1}{4}(\eta - 15)E\right) \tag{13}$$

$$e_{t_0} = [1 + 2EL^2 - \left(4(\eta - 1) + (7\eta - 17)EL^2\right) E]^{1/2} \tag{14}$$

Rewriting the constants of movement E and L^2 in terms of the initial conditions that are set in the numerical relativity simulation input, the initial data for $\overline{\xi}$ and e_t can be obtained.

$$E = \frac{E_{\text{ADM}} - m}{\mu}, \qquad L^2 = \left(\frac{J_{\text{ADM}}}{m\mu}\right)^2 \tag{15}$$

Where $m = m_1 + m_2$ with $m_1 = m_2 = 0.5$ and J_{ADM} the ADM angular momentum, which is chosen to be in the $\hat{z}$ direction. Then, as mentioned before, the movement is going to be confined in a plane (in this case the (x, y) plane), with a constant direction of the angular momentum. The initial value for the dynamical angular variable is set to $\Phi_0 = 0$. The initial separation between the BHs has been set to $d = 35$ for the four simulations, with initial position vectors $\vec{x}_1 = (17.5, 0)$ and $\vec{x}_2 = (-17.5, 0)$.

SIMULATION	$\vec{p}_1$	$\vec{p}_2$	E_{ADM}	$J^{\hat{z}}_{\mathrm{ADM}}$	$\bar{\xi}_0$	e_{t_0}
BBH_hyp_1	$(-0.0939, 0.0343)$	$(0.0939, -0.0343)$	1.010956	1.2	0.03014	1.956
BBH_hyp_2	$(-0.0928, 0.0371)$	$(0.0928, -0.0371)$	1.010959	1.3	0.03016	2.072
BBH_hyp_3	$(-0.0917, 0.0400)$	$(0.0917, -0.0400)$	1.010962	1.4	0.03017	2.191
BBH_hyp_4	$(-0.0904, 0.0429)$	$(0.0904, -0.0429)$	1.010965	1.5	0.03019	2.312

TABLE I: Initial conditions for the four simulations. The momenta are estimated in order to produce a hyperbolic orbit without a merger.

B. Trajectories

The evolution of the position of the BHs is determined by using the numerical relativity simulation with the `TwoPunctures` and the `McLachlan` modules [9] from the `Einstein Toolkit`.

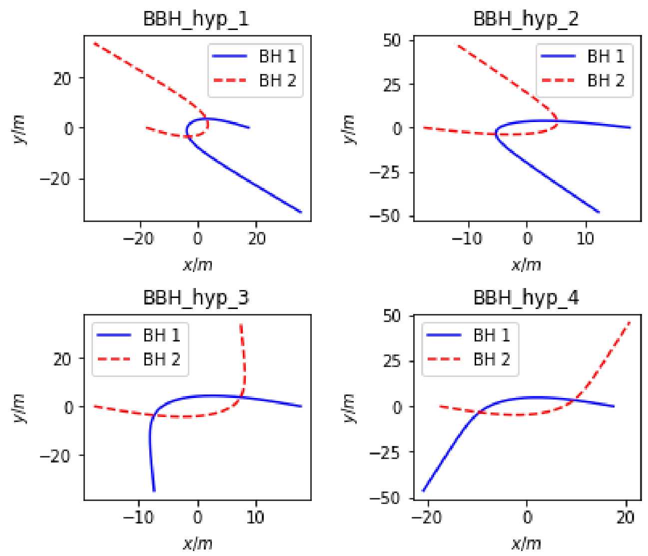

FIG. 1: Computed trajectories in the (x, y) plane for the BHs in each simulation.

It is shown that the orbits produced by the two BHs are indeed hyperbolic and without a merger for all simulations, which means that the momenta has been correctly estimated and the rest of the initial conditions are suitable for the situation proposed. Therefore, it allows a description employing the 1.5PN approximation algorithm with the correspondent initial parameters. It can be computed the time evolution of the separation distance between the BHs by substituting equations (8) and (9) into r in (7) neglecting terms $O(1/c^4)$:

$$r = \frac{1}{\bar{\xi}^{2/3}}\left(e_t \cosh v - 1 + \bar{\xi}^{2/3}\frac{(7\eta - 6)e_t \cosh v + 2(\eta - 9)}{6}\right) \qquad (16)$$

Where it has been imposed $c = 1$, and v has to be evaluated at each time step. With the data of the positions obtained with the `Einstein Toolkit`, the separation distance can be calculated trivially, and performing the proper time shift correction between simulations booth data can be compared:

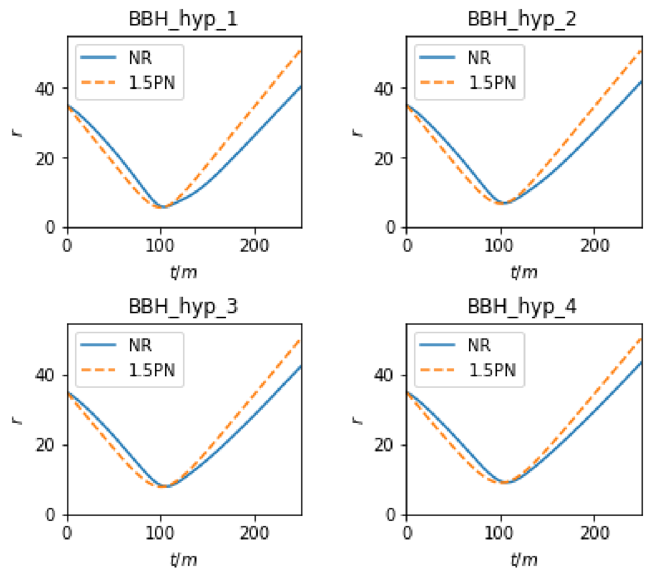

FIG. 2: Time evolution of the separation distance between BHs for the four simulations, comparing the results obtained with numerical relativity and the 1.5PN algorithm.

The last plot shows that the 1.5PN approximation doesn't match exactly with numerical relativity, though the tendency is similar. In the PN approximation the BHs apparently "move faster" and there is symmetry with respect to the point of maximum approach. That is because of the conservation of energy in this approximation.

C. Gravitational waves

The most efficient way to compare the physical information of the GWs emitted from both methods is studying the time evolution of the Weyl scalar ψ_4. The output of the 1.5PN algorithm provides the amplitudes of polarization (strain) of the GWs, and recalling equation (10), one only has to perform two consecutive time derivatives to h_+ and $h_\times$ to get the ψ_4, since $h = h_+ - ih_\times$ and then compare them with numerical relativity results. It could be done the opposite way, by integrating twice the ψ_4 obtained with the `Einstein Toolkit`, and compare the results with the strain calculated with the PN method. But usually, trouble appears when performing this time integrals due to the arbitrary constants of integration, producing a drift as a consequence of numerical noise (junk radiation) [10].

As a matter of convenience, the ψ_4 is extracted centering the point of maximum approach at $t = 0$:

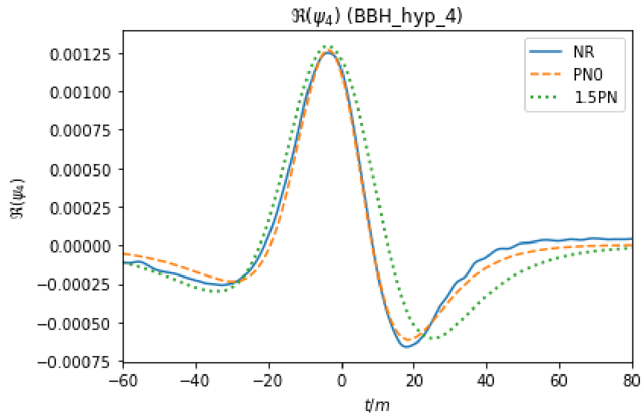

FIG. 3: Time evolution of the real part of ψ_4 for the BBH_hyp_4 simulation, comparing the numerical relativity results (solid blue line) with the complete 1.5PN expansion (green dotted line) and with its leading term (orange dashed line).

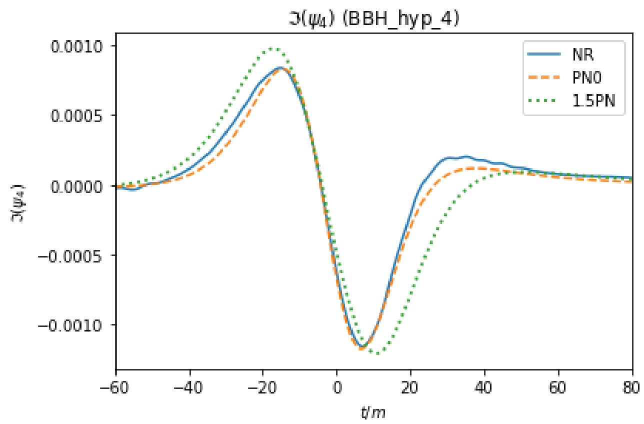

FIG. 4: Time evolution of the imaginary part of ψ_4 for the BBH_hyp_4 simulation, comparing the numerical relativity results (solid blue line) with the complete 1.5PN expansion (green dotted line) and with its leading term (orange dashed line).

The results show that the 1.5PN complete expansion does not match as good as the leading term with the numerical relativity data. The expansion consists of three sub-leading terms (equations (71a) and (71b) in [5]): the first and the third terms are null because $m_1 = m_2$ and the BHs are not spinning. Hence, the second sub-leading term contribution worsens the results given by the leading term, which means that the expansion does not converge at that order with those parameters. In fact, for the other simulations the results are not better, meaning that this expansion is not suitable in general. As mentioned before, the Post-Newtonian expansions are not always convergent [6]. This suggests that more terms of the expansion should be considered.

In addition, it turned out that for some simulations the second sub-leading term even changes the curvature of the waveform, giving trouble when performing the derivatives. Nevertheless, the results show how the leading term is a fine approximation to numerical relativity in those situations:

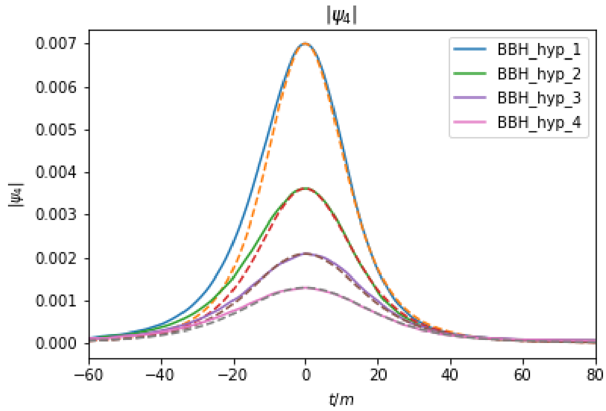

FIG. 5: Time evolution of the modulus of ψ_4 for the four simulations. The solid lines are the results obtained using numerical relativity, and the dashed lines are the corresponding PN approximation considering just the leading term.

The leading term of the 1.5PN expansion matches remarkably well with the numerical relativity results for all the simulations performed. It has the following expressions considering the amplitudes of polarization in the TT (transverse and traceless) gauge:

$$\mathrm{Re}(\psi_4^{\mathrm{PN0}}) \approx \frac{d^2}{dt^2}\left[\left((\hat{q}\cdot\hat{n})^2 - (\hat{p}\cdot\hat{n})^2\right)\frac{1}{r} + (\hat{p}\cdot\dot{\vec{r}})^2 - (\hat{q}\cdot\dot{\vec{r}})^2\right] \quad (17)$$

$$\mathrm{Im}(\psi_4^{\mathrm{PN0}}) \approx \frac{d^2}{dt^2}\left(-(\hat{p}\cdot\hat{n})(\hat{q}\cdot\hat{n})\frac{1}{r} + (\hat{p}\cdot\dot{\vec{r}})(\hat{q}\cdot\dot{\vec{r}})\right) \quad (18)$$

Where r is given by equation (16), $\hat{n} = \vec{r}/r$, $\dot{\vec{r}} = \partial_t\vec{r}$, $\hat{p} = -\hat{e}_y$ and $\hat{q} = \cos\Theta\hat{e}_x - \sin\Theta\hat{e}_z$, where Θ is the angle of inclination of the orbit with respect to the observer. In this case $\Theta = 0$ since with the `Einstein Toolkit` the orbit is described in a plane perpendicular to the direction of observation.

IV. CONCLUSIONS

- The results obtained in the range of parameters studied in this project corroborate that algorithms that use simplifications of the Einstein's Theory of Relativity such as the Post-Newtonian approximation are valid to describe processes like encounters of black holes. These algorithms are a complementary approach to numerical relativity due to their simplicity, the accuracy of the results and the short time of calculation required. Therefore, these algorithms provide an efficient way to build template banks of waveforms to help the astrophysics community to identify gravitational waves in their observations and the post-processing of the data.

- More specifically, in the case studied in this project, the results show that the 1.5 Post-Newtonian approximation is able to describe the dynamical evolution of a system of non-spinning black holes with a hyperbolic orbit. The obtained results compared qualitatively to numerical relativity are promising. The description with the complete expansion of the physics of the outgoing gravitational radiation is not accurate enough to be considered as an acceptable result. But, if the expansion is truncated at the leading term (for the case of equal mass black holes), then the obtained results reasonably match with the numerical relativity calculations in the parameter range considered. A more careful study is required in order to identify a range of initial parameters more suitable to apply the 1.5PN approximation, to master more generic cases with spinning BHs and also to provide a quantitative comparison with numerical relativity.

- This project has allowed me to learn more about General Relativity and to understand how it can be implemented using algorithms to solve situations in the context of real astrophysical phenomena. I am also glad for having the opportunity to take place in a research project starting from scratch and to learn how to use programming tools like `Wolfram Mathematica` and `Python` to solve mathematical problems and to analyse the outputs. Finally, these topics are in the cutting edge of research, and other different methods are being developed to get better results for more generic scenarios. Therefore, this memory is just the beginning of a major project, which is going to be developed in my *Master*.

V. APPENDIX

Evolution equations from the 1.5PN algorithm:

$$\frac{d\bar{\xi}}{dt} = \frac{c^3\bar{\xi}^{11/3}8\eta}{5\beta^7}(-49\beta^2 - 32\beta^3 + 35(e_t^2-1)\beta - 6\beta^4 + 9e_t^2\beta^2) \tag{19}$$

$$\frac{de_t}{dt} = \frac{c^3\bar{\xi}^{8/3}8\eta(e_t^2-1)}{15\beta^7 e_t}(-49\beta^2 - 17\beta^3 + 35(e_t^2-1)\beta - 3\beta^4 + 9e_t^2\beta^2) \tag{20}$$

$$\frac{d\Phi}{dt} = \frac{c^3\bar{\xi}\sqrt{e_t^2-1}}{\beta^2}\left[1 - \bar{\xi}^{2/3}\left(\frac{\eta-4}{\beta} - \frac{\eta-1}{e_t^2-1}\right)\right] \tag{21}$$

Where $\beta = e_t \cosh v - 1$.

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