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The Dirichlet problem and Kakutani's theorem

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Abstract

In this memoir we prove a weak version in $\mathbb{R}^2$ of Kakutani's theorem which gives a solution to the Dirichlet problem.

The Dirichlet problem is a classical problem in partial differential equations with many applications in various fields. Given a bounded domain $D \subset \mathbb{R}^d$ and a function f continuous at ∂D, the Dirichlet problem consists in finding an harmonic function u on D, which matches the values of f on the boundary.

It is known that for very general domains the solution exists and is unique.

The solution given by Kakutani in 1944 is based in the use of probabilistic methods, specifically in the properties of Brownian motion, which will play an important role throughout this memoir.

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Contents

Chapter 1

Introduction

The Dirichlet problem is a classical problem in partial differential equations that arises naturally in the studies of the flow of heat, electricity, fluid dynamics and many other areas.

The Dirichlet problem. *Let $D \subset \mathbb{R}^d$ be a regular bounded domain and let $f \in \mathcal{C}(\partial D)$ be a continuous function on the boundary of D. Does there exist a function $u \in \mathcal{C}^2(\overline{D})$ such that*

$$\begin{cases} \Delta u = 0 \text{ in } D \\ u|_{\partial D} = f \ ? \end{cases}$$

Here $\Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}$ is the Laplacian in $\mathbb{R}^d$, and the functions $u \in \mathcal{C}^2(D)$ with $\Delta u = 0$ are called harmonic in D.

For the moment we may think that a regular domain is one with nice boundary; the precise technical conditions will be given later in Chapter 3.

The problem is named for the 19th century mathematicians Peter Gustav Lejeune Dirichlet, who suggested the first general method for solving this class of problems. Many prominent mathematicians, such as K.F. Gauss, Lord Kelvin, B. Riemann and D. Hilbert, worked in this problem.

Nowadays the Dirichlet problem can be solved in a variety of ways. The classical solution is given in terms of the so-called Poisson Kernel of the domain (the normal derivative of the Green function), with the help of Green's formula.

In this memoir we present, in a modern language, the proof given by Kakutani in 1944, which is based on probabilistic methods, specifically in the use of Brownian motion.

The Brownian motion in $\mathbb{R}$ (or Wiener process) may be thought as a random continuous trajectory $\{\mathbb{B}_t(\omega)|t \geq 0\}$ starting at 0 and such that:

1) The increments $\mathbb{B}_{t+h} - \mathbb{B}_t$ $(t \geq 0,\ h > 0)$ follow a Gaussian distribution of mean 0 and variance h, denoted $\mathbb{B}_{t+h} - \mathbb{B}_t \sim N(0,h)$. In particular $\mathbb{B}_t \sim N(0,t)$.

2) For $0 = t_0 \leq t_1 \leq \cdots \leq t_n$ the increments $\mathbb{B}_{t_n} - \mathbb{B}_{t_{n-1}}, \ldots, \mathbb{B}_{t_2} - \mathbb{B}_{t_1}$ are independent.

Brownian motion in $\mathbb{R}^d$ is defined just as $\mathbb{B}_t = (\mathbb{B}_t^1, \ldots, \mathbb{B}_t^d)$, where $\mathbb{B}_t^j$ are independent one-dimensional random motions. Therefore $\mathbb{B}_t$ in $\mathbb{R}^d$ satisfies properties 1) and 2) above as well, if $N(0,h)$ is interpreted as the multivariate Gaussian distribution of mean $0 \in \mathbb{R}^d$ and covariance matrix $h\mathrm{Id}$.

Brownian motion in $\mathbb{R}^d$ can also be viewed as a limit of a random walk on a lattice $\epsilon_n \mathbb{Z}^d$ as $\epsilon_n \to 0$ (see e.g Chapter 5 in [Y-P]).

A key property of $\mathbb{B}_t$ is its lack of memory (Markov's property, Theorem 3.3): the behaviour of $\mathbb{B}_t$ after a fixed time s is the same as a new Brownian motion starting at the point $\mathbb{B}_s$, regardless of the path taken to reach $\mathbb{B}_s$.

Another important property of $\mathbb{B}_t$ in $\mathbb{R}^d$, given by the independence of the different $\mathbb{B}_t^i$ such that $\mathbb{B}_t = (\mathbb{B}_t^1, \ldots, \mathbb{B}_t^d)$, is the isotropy: the density function of $\mathbb{B}_t$ depends only on the distance to 0, $||x||$, but not on the direction, so that the distribution of $\mathbb{B}_t$ is isotropic, it is invariant by rotations around 0.

Kakutani's theorem is stated in terms of Brownian motion starting at a given point $x \in \mathbb{R}^d$. This is just defined as $\mathbb{B}_t^x = x + \mathbb{B}_t$, where $\mathbb{B}_t$ is the standard Brownian motion (starting at 0).

Given a bounded domain D and $z \in D$ let τ_D denote the exiting time of $\mathbb{B}_t^z$, that is

$$\tau_D = \inf\{t > 0 | \ \mathbb{B}_t^z \notin D\}.$$

By the properties of the Brownian motion $\tau_D < \infty$ almost surely, so that $\mathbb{B}_{\tau_D}^z$ is a point of the boundary ∂D with probability one.

Kakutani's theorem. *Let $D \subset \mathbb{R}^d$ be a bounded regular domain and let $f \in \mathcal{C}(\partial D)$. Then*

$$u(z) = \mathbb{E}[f(\mathbb{B}_{\tau_D}^z)]$$

is the unique solution to the Dirichlet problem, that is, u is harmonic in D and $u(\zeta) = f(\zeta)$ for all $\zeta \in \partial D$.

In this memoir we prove Kakutani's theorem only for $d = 2$. We do so to avoid hiding the main ideas in technicalities. As it will be clear in the proofs the arguments work as well for any dimension d.

The proof has naturally three parts: showing that u is harmonic in D, proving that the boundary values of u are f and proving the uniqueness of the solution.

In order to show that u is harmonic it is enough to prove that it is continuous and satisfies the mean value property. That u is continuous follows easily from the definition of u and the properties of the Brownian motion.

To prove the mean value property, let $x \in D$ and let $U = \mathbb{D}(x, r)$ be an open disk with $r > 0$, such that $\overline{U} \subset D$. Let

$$\tau_U := \inf\{t > 0 | \mathbb{B}_t^x \notin U\}$$

be the first time that $\mathbb{B}_t^x$ reaches the boundary ∂U. Then, by the continuity of Brownian motion clearly $\tau_U \leq \tau_D$. In addition, by the isotropy property mentioned previously, the probability of leaving U through a point $\zeta \in \partial U$, i.e $\mathbb{B}_{\tau_U}^x = \zeta$, is uniformly distributed over ∂U. In other words, for any measurable subset $A \subset \partial U$ we have

$$\mathbb{P}(\mathbb{B}_{\tau_U}^x \in A) = \frac{|A|}{|\partial U|},$$

where $|\cdot|$ denotes the standard Lebesgue measure on ∂U.

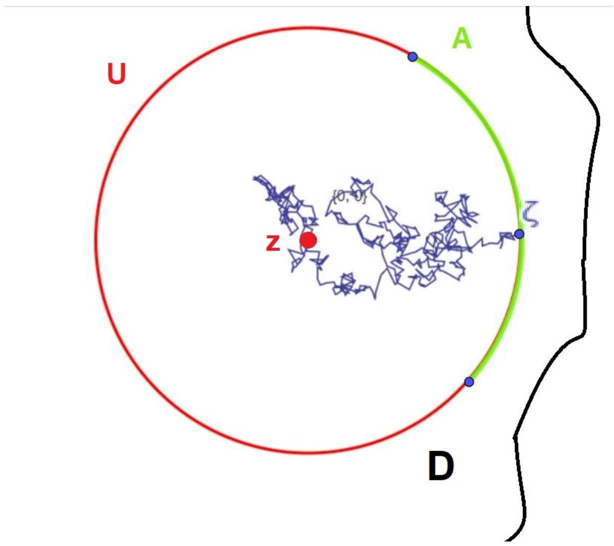


Figure 1.1: Brownian motion started at $z \in D$, the center of the disk U, with $\mathbb{B}^z_{\tau_U} = \zeta \in A \subset \partial U$.

Since $\mathbb{B}^x_t$ has to hit ∂U before hitting ∂D this shows that

$$u(x) = \mathbb{E}[f(\mathbb{B}^x_{\tau_D})] = \frac{1}{|\partial U|}\int_{\partial U} \mathbb{E}[f(\mathbb{B}^x_{\tau_D})|\mathbb{B}^x_{\tau_U} = y]\, \mathrm{d}y.$$

By the lack of memory of $\mathbb{B}^x_t$ (Markov's property) the conditional property in the integral does not depend on the paths taken to reach y, that is

$$\mathbb{E}[f(\mathbb{B}^x_{\tau_D})|\mathbb{B}^x_{\tau_U} = y] = \mathbb{E}[f(\mathbb{B}^y_{\tau_D})].$$

This finishes this part of the proof, since then

$$u(x) = \frac{1}{|\partial U|}\int_{\partial U} \mathbb{E}_y[f(\mathbb{B}^y_{\tau_D})]\mathrm{d}y = \frac{1}{|\partial U|}\int_{\partial U} u(y)\mathrm{d}y.$$

The second part of the proof consists in showing that

$$\lim_{\substack{z\to\zeta\\ z\in D}} u(z) = f(\zeta) \quad \forall \zeta \in \partial D.$$

By the hypothesis on D, and since $f(\zeta)$ is constant, this is equivalent to

$$\lim_{\substack{z\to\zeta\\ z\in D}} \mathbb{E}_z[|f(\mathbb{B}^z_{\tau_D}) - f(\zeta)|] = 0. \tag{1.1}$$

Fix $r > 0$ and take the disk $\mathbb{D}(\zeta, r)$, let τ_r be the stopping time of $\mathbb{B}_t^z$ of the disk, that is,

$$\tau_r = \inf\left\{t \geq 0 | \mathbb{B}_t^z \notin \mathbb{D}(\zeta, r)\right\}.$$

Separating the estimate in cases, depending on whether $\tau_r < \tau_D$ or $\tau_r \geq \tau_D$, we see that

$$\begin{aligned}\mathbb{E}_z[|f(\mathbb{B}^z_{\tau_D}) - f(\zeta)|] &= \mathbb{E}_z[|f(\mathbb{B}^z_{\tau_D}) - f(\zeta)|, \tau_r < \tau_D] \\ &\quad + \mathbb{E}_z[|f(\mathbb{B}^z_{\tau_D}) - f(\zeta)|, \tau_r \geq \tau_D] \\ &\leq 2\dot{|}|f||_\infty \mathbb{P}_z(\tau_r < \tau_D) + \sup_{\substack{|\zeta-\eta|\leq r \\ \eta\in\partial D}} |f(\eta) - f(\zeta)|.\end{aligned}$$

The estimate of the first summand is direct and the estimate of the second one follows because when $\tau_r \geq \tau_D$ one has $\mathbb{B}^z_{\tau_D} \in \mathbb{D}(\zeta, r)$.

The probability $\mathbb{P}(\tau_r < \tau_D)$ that $\mathbb{B}_t^z$ exists $\mathbb{D}(z, r)$ before exiting D tends to 0 as z approaches ζ. Also, by the continuity of f the supreme in the previous estimate tends to 0 or r tends to 0. This proves (1.1), and u has boundary values f.

The memoir is essentially devoted to provide the rigorous definitions and proofs of the sketch given in this introduction.

The first chapter studies harmonic functions. Its main result is that for a continuous function being harmonic is equivalent to satisfying the mean value property. Chapter 2 is devoted to introduce Brownian motion and the properties that are necessary in the proofs. Here we assure many properties, without proofs, since a serious study of Brownian motion is well beyond the scope of this work. The main goal of the chapter is to state and proof the strong Markov property (Theorem 3.3). In the final chapter we discuss the regularity conditions on D and give the detailed proof of Kakutani's theorem.

Chapter 2

Harmonic functions

In this chapter we define and study some properties of harmonic functions in $\mathbb{R}^2$. We use its close relationship with holomorphic functions in $\mathbb{C} \simeq \mathbb{R}^2$.

The main goal is to show that for a continuous function the harmonicity is equivalent to satisfying the mean value property and to prove the Maximum Principle. These are the properties that we will need in the proof of Kakutani's theorem.

Definition 1. *Given a domain $D \subset \mathbb{R}^d$, a function $f : D \longrightarrow \mathbb{R}^d$ is called harmonic on D if $f \in \mathcal{C}^2(D)$ and verifies Laplace's equation $\Delta f = 0$, i.e,*

$$\Delta f = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2} = 0.$$

Since we work in $\mathbb{R}^2$, we have

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

The next result details the relationship between harmonic and holomorphic functions.

Theorem 2. *Let $D \subset \mathbb{C}$ be a domain, i.e an open, connected set. Then:*

1. *If F is holomorphic on D the function $f = \operatorname{Re} F$ is harmonic on D.*

2. *If f is harmonic on D and if D is simply connected, then $f = \operatorname{Re} F$ for some function F holomorphic on D. Moreover F is unique up to adding a constant.*

Proof. 1) Let F be a holomorphic function on D and let it $f = \operatorname{Re} F$. Then we can write F as $F = f + ik$, and since F is holomorphic it must verify the Cauchy-Riemann equations

$$\begin{cases} f_x = k_y \\ f_y = -k_x. \end{cases}$$

Therefore,

$$\Delta f = f_{xx} + f_{yy} = k_{yx} - k_{xy} = 0,$$

that is, f is harmonic on D.

2) In order to prove the second statement we must see that if f =$\operatorname{Re} F$ for a holomorphic function F on D, then there exists another function k, called harmonic conjugate of f, such that $F = f + ik$ is holomorphic in D. Such F must therefore satisfy the Cauchy-Riemann equations. In particular

$$\begin{aligned} F_x &= f_x + ik_x = f_x - if_y \\ F_y &= f_y + ik_y = f_y + if_x. \end{aligned}$$

Thus F' is completely determined by h, and since D is simple connected, it is uniquely determined, up to addition of constants.

Let us see this in more detail. Define $g : \mathbb{D} \longrightarrow \mathbb{R}^2$ by

$$g = f_x - if_y.$$

Then $g \in \mathcal{C}^2(D)$ and it is holomorphic, since it satisfies the Cauchy-Riemman equations

$$\begin{cases} f_{xx} = -f_{yy} \\ f_{xy} = f_{yx}. \end{cases}$$

Fix $z_0 \in D$ and define $F : D \longrightarrow \mathbb{R}^2$ by

$$F(z) = f(z_0) + \int_{z_0}^{z} g(w)\mathrm{d}w,$$

where the integral is taken over any path in D from z_0 to z; the integral is independent of the particular path by Cauchy theorem, since D is simply connected. Then F is holomorphic on D and $F' = g$.

Finally let's see the uniqueness of this function. Given $t = \mathrm{Re}\, F$ we have

$$t_x - it_y = F' = f_x - if_y,$$

and it follows that $(t-f)_x = 0$ and $(t-f)_y = 0$. So $t - f$ is constant on D and evaluating at $z = z_0$ we have that this constant must be 0. Therefore $t = f$, as desired. □

We state below some theorems that we will need to prove the uniqueness in Kakutani's theorem.

Identity Principle. *Let f and g be harmonic functions on a domain $D \subset \mathbb{C}$. If $f = g$ on a non-empty subset $U \subset D$, then $f = g$ throughout D.*

Proof. We can suppose without loss of generality that $g = 0$, which is equivalent to consider the difference between the functions $f - g$ and the constant 0 function.

Set a function $F = f_x - if_y$. Then, as in the proof of Theorem 1, F is holomorphic on D and also $F = 0$ on U since $f = 0$ on U.

Therefore, applying the identity principle for holomorphic functions, it follows that $F = 0$ on all D. Then $f_x = f_y = 0$, which means that f is constant and, since $f = 0$ on U, we have that $f = 0$ in all D, as we wanted to prove. □

Maximum Principle. *Let f be a harmonic function on a domain $D \subseteq \mathbb{C}$.*

1. *If f has a local maximum in D, then f is constant.*
2. *If f extends continuously to $\overline{D}$ and $f \leq 0$ on ∂D, then $f \leq 0$ on D.*

Proof. 1) Suppose that f attains a local maximum at $z \in D$. Then for some $r > 0$ we have that $f(w) \leq f(z)$ for all w in the disk $\mathbb{D}(z, r)$.

By Theorem 1 there exists a function F holomorfic on $\mathbb{D}(z,r)$ such that f =Re F on the disk. So the function $|e^F| = e^f$ attains a local maximum at z and, by the maximum principle for holomorphic functions, it follows that e^F must be constant. Then, f is constant on $\mathbb{D}(z,r)$, and hence on the whole D, by Theorem 2.

2) As $\overline{D}$ is compact, there a point $z \in \overline{D}$ such that

$$f(z) = \max_{\overline{D}} f.$$

If $z \in \partial D$, then $f(z) \leq 0$ by assumption and we see that $f \leq 0$ on D.

If $z \in D$, then by 1) we have that f is constant on D, hence on $\overline{D}$, and therefore $f \leq 0$ on D. □

2.1 The mean value property

In this section we will prove that for continuous functions the harmonicity is characterized by the mean value property.

Theorem 3. *Let f be a harmonic function on a open neighbourhood of the closed disk $\mathbb{D}(z,\rho)$, $\rho > 0$. Then f verifies the mean-value property in $\partial\mathbb{D}(z,\rho)$, i.e*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+\rho e^{i\theta})\mathrm{d}\theta.$$

Proof. Let us choose $r > \rho$ so that f is harmonic on the open disk $\mathbb{D}(z,r)$. Then, applying Theorem 1, there exists a holomorphic function F such that f =Re F on the disk. By the Cauchy's integral formula it follows that

$$F(z) = \frac{1}{2\pi i}\int_{|\zeta - z| = \rho} \frac{F(\zeta)}{\zeta - z}\mathrm{d}\zeta = \frac{1}{2\pi}\int_0^{2\pi} F(z+\rho e^{i\theta})\mathrm{d}\theta.$$

Taking the real part on both sides of the equation we finally see that

$$f(z) = \mathrm{Re}\ F(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\ F(z+\rho e^{i\theta})\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z+\rho e^{i\theta})\mathrm{d}\theta.$$

□

Finally, let us prove the equivalence between harmonicity and the mean-value property.

Theorem 4. *Let f be a harmonic function in a domain $D \subseteq \mathbb{C}$. Then, the following statements are equivalent:*

1. *f is continuous on D and it satisfies the mean value property.*
2. *$f \in \mathcal{C}^\infty(D)$ and $\Delta f = 0$, i.e, f is harmonic.*

Proof. Notice that the implication 2) $\implies$ 1) is direct by Theorem 3.

Thus, assume that 1) holds, i.e, that for $\overline{\mathbb{D}(z,\rho)} \subseteq D$

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + \rho e^{i\theta}) \mathrm{d}\theta.$$

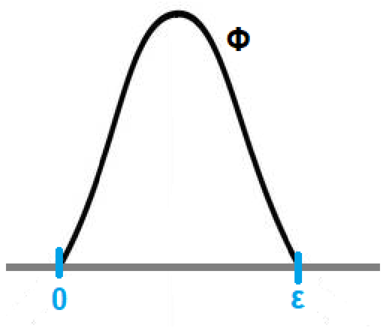
The proof of 2) will consist in two parts: showing that $f \in \mathcal{C}^\infty(D)$ and proving that $\Delta f = 0$.

Let us start proving that $f \in \mathcal{C}^\infty$. Choose a radial function $\phi \in \mathcal{C}^\infty(\mathbb{R})$ such that:

(a) supp $\phi \subseteq [0, \epsilon]$,

(b) $\phi > 0$ in $t \in (0, \epsilon)$,

(c) $\int_0^\epsilon \phi = 1$.



Then we fix $z \in D$ and we consider the following integral, which as we shall see approximates $f(z)$:

$$\begin{aligned}\int_{\mathbb{R}^2} \phi(|z-w|^2) f(w)\mathrm{d}w &= \int_{\mathbb{R}^2} \phi(|u|^2) f(u+z)\mathrm{d}u \\ &= \int_0^\infty \int_0^{2\pi} \phi(r^2) f(z+re^{i\theta}) r\mathrm{d}\theta\mathrm{d}r \qquad (2.1) \\ &= \int_0^\infty \phi(r^2) r \int_0^{2\pi} f(z+re^{i\theta})\mathrm{d}\theta\mathrm{d}r\end{aligned}$$

where we made the change to polar coordinates around z. Since, by assumption, f satisfies the mean value property we have that

$$\int_0^{2\pi} f(z+re^{i\theta})\mathrm{d}\theta = 2\pi f(z),$$

so we can rewrite the first integral (2.1) as

$$\begin{aligned}\int_{\mathbb{R}^2} \phi(|z-w|^2) f(w)\mathrm{d}w = 2\pi f(z) \int_0^\infty \phi(r^2) r\mathrm{d}r &= \pi f(z) \int_0^\infty \phi(t)\mathrm{d}t \\ = \pi f(z) \int_0^\epsilon \phi(t)\mathrm{d}t &= \pi f(z).\end{aligned}$$

Then

$$f(z) = \frac{1}{\pi} \int_{\mathbb{R}^2} \phi(|z-w|^2) f(w)\mathrm{d}w.$$

Observe that $\phi(|z-w|^2) \in C^\infty$ in the variable $z \in D$, so that $f(z)$ is infinitely differentiable and

$$\frac{\mathrm{d}^k f(z)}{\mathrm{d}z^k} = \frac{\mathrm{d}^k}{\mathrm{d}z^k} \frac{1}{\pi} \int_{\mathbb{R}^2} \phi(|z-w|^2) f(w)\mathrm{d}w = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\mathrm{d}^k \phi(|z-w|^2)}{\mathrm{d}z^k} f(w)\mathrm{d}w,$$

for any $k \geq 1$.

This is because we are in the conditions of the differentiation theorem under the integral sign. Let $g(z,w) = \frac{1}{\pi}\phi(|z-w|^2) f(w)$, so that

$$f(z) = \int_{\mathbb{R}^2} g(z,w)\mathrm{d}w.$$

Denoting $M = \max \phi$, and since the support of $\phi(|z-w|^2)$ is contained in $\mathbb{D}(z, \sqrt{\epsilon})$, we have

$$|g(z,w)| \leq M \mathcal{X}_{\mathbb{D}(z,\sqrt{\epsilon})}(w) |f(w)| \leq M \mathcal{X}_{\mathbb{D}(z,\sqrt{\epsilon})}(w) \max_{|w-z|<\sqrt{\epsilon}} |f(w)|$$

and

$$\begin{aligned}|\frac{\mathrm{d}g(z,w)}{\mathrm{d}z}| &= (\max|\phi'(w)|)|\overline{z-w}||f(w)|\mathcal{X}_{\mathbb{D}(z,\sqrt{\epsilon})}(w) \\ &\leq (\max|\phi'(w)|)\sqrt{\epsilon}(\max_{|w-z|<\sqrt{\epsilon}}|f(w)|)\mathcal{X}_{\mathbb{D}(z,\sqrt{\epsilon})}(w).\end{aligned}$$

Observe that the right hand side of these two estimates gives a function which is trivially integrable in a neighbourhood of any fixed $z_0 \in \mathbb{C}$. Therefore, we can apply the differentiation theorem.

It remains to show that $\Delta f = 0$ when $f \in \mathcal{C}^\infty(D)$ and it verifies the mean value property. We do so by applying Green's theorem.

Green's theorem. *Let $U \subseteq \mathbb{R}^2$ be a simply connected region with a positively oriented curve boundary ∂U. If $F = (P, Q) : U \longrightarrow \mathbb{R}^2$ is a vector field with continuous partial derivatives in an open region containing U, then*

$$\oint_{\partial U} P\mathrm{d}x + Q\mathrm{d}y = \iint_U (\frac{\mathrm{d}Q}{\mathrm{d}x} - \frac{\mathrm{d}P}{\mathrm{d}y})\mathrm{d}m$$

where m denotes the Lebesgue measure in $\mathbb{R}^2$.

Take $U = \mathbb{D}(z, r)$, the disk of center $z \in D$ and radius $r > 0$ and define $F = (P, Q)$ as follows:

$$\begin{cases} P = -\frac{\mathrm{d}f}{\mathrm{d}y} \\ Q = \frac{\mathrm{d}f}{\mathrm{d}x}. \end{cases}$$

Then

$$\Delta f = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 f}{\mathrm{d}y^2} = Q_x - P_y.$$

Writing x and y in polar coordinates around $z = (x_0, y_0)$ we have

$$\begin{cases} x - x_0 = r\cos\theta \\ y - y_0 = r\sin\theta \end{cases} \qquad \begin{cases} \mathrm{d}x = -r\sin\theta\mathrm{d}\theta \\ \mathrm{d}y = r\cos\theta\mathrm{d}\theta. \end{cases} \tag{2.2}$$

It follows that

$$\begin{cases} r = \sqrt{(x-x_0)^2 + (y-y_0)^2} \\ \theta = \arctan\frac{y-y_0}{x-x_0} \end{cases}$$

and therefore

$$\begin{cases} \frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{-(y-y_0)}{(x-x_0)^2+(y-y_0)^2} = -\frac{\sin(\theta)}{r} \\ \frac{\mathrm{d}\theta}{\mathrm{d}y} = \frac{x-x_0}{(x-x_0)^2+(y-y_0)^2} = \frac{\cos(\theta)}{r}. \end{cases}$$

Since on $\partial\mathbb{D}(z,r)$ the radius is constant we deduce that $\frac{\mathrm{d}r}{\mathrm{d}x} = \frac{\mathrm{d}r}{\mathrm{d}y} = 0$. Then, by chain rule, we can write $\frac{\mathrm{d}f}{\mathrm{d}x}$ and $\frac{\mathrm{d}f}{\mathrm{d}y}$ as follows:

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}x} &= \frac{\mathrm{d}f}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x} = -\frac{\mathrm{d}f}{\mathrm{d}\theta}\frac{\sin(\theta)}{r} \\ \frac{\mathrm{d}f}{\mathrm{d}y} &= \frac{\mathrm{d}f}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}y} = \frac{\mathrm{d}f}{\mathrm{d}\theta}\frac{\cos(\theta)}{r}. \end{aligned} \tag{2.3}$$

Seen this, we proceed to compute the integral of Δf on $\mathbb{D}(z,r)$ by applying Green's theorem. By (2.2) and (2.3).

$$\begin{aligned} \iint_{\mathbb{D}(z,r)} \Delta f \mathrm{d}x\mathrm{d}y &= \int_{\partial\mathbb{D}(z,r)} -\frac{\mathrm{d}f}{\mathrm{d}y}\mathrm{d}x + \frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}y = \int_{\partial\mathbb{D}(z,r)} (-\frac{\mathrm{d}f}{\mathrm{d}y}, \frac{\mathrm{d}f}{\mathrm{d}x})(\mathrm{d}x, \mathrm{d}y) \\ &= \int_0^{2\pi} (-\frac{\mathrm{d}f}{\mathrm{d}\theta}\frac{\cos(\theta)}{r}, -\frac{\mathrm{d}f}{\mathrm{d}\theta}\frac{\sin(\theta)}{r})(-r\sin\theta\mathrm{d}\theta, r\cos\theta\mathrm{d}\theta) \\ &= \int_0^{2\pi} (\cos\theta\sin\theta - \sin\theta\cos\theta)\mathrm{d}\theta = 0. \end{aligned} \tag{2.4}$$

It remains to see that if $\iint_{\mathbb{D}(z,r)} \Delta f \mathrm{d}x\mathrm{d}y = 0$ for all $z \in \mathbb{R}^2$ and $r > 0$ then necessarily $\Delta f \equiv 0$.

Since $f \in \mathcal{C}^\infty(D)$, we have in particular that Δf is continuous on D. If $\Delta f \neq 0$ somewhere on D, then there exists a point $x_0 \in D$ and $r_0 > 0$ with $\mathbb{D}(x_0, r_0) \subset D$ where $\Delta f > 0$. However, that means that

$$\iint_{\mathbb{D}(z_0,r_0)} \Delta f \mathrm{d}x\mathrm{d}y > 0,$$

which contradicts (2.4).

□

Chapter 3

Brownian motion

In this chapter we define Brownian motion and gather some of its properties, emphasizing the ones we will use in the proof of Kakutani's theorem. We also prepare concepts and results related to the strong Markov property, whose consequence is the lack of memory of a Brownian motion.

The main goal of the chapter is to show that Brownian motion, with a stopping time for an open or bounded set, satisfies the Strong Markov property Theorem 3.3, which will be crucial in order to prove Kakutani's theorem.

3.1 Brownian motion

There are many ways to define Brownian motion. Here we choose the so-called canonical model: In order to define the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider first

$$\Omega = \mathcal{C}_0[0,\infty) := \{\omega : [0,\infty) \to \mathbb{R} |\ \omega \text{ is continuous and } \omega(0) = 0\}.$$

For each $t \in [0,\infty)$ fixed consider also the random variables

$$\mathbb{B}_t : \Omega \longrightarrow \mathbb{R} \quad \text{defined by} \quad \mathbb{B}_t(\omega) := \omega(t).$$

Observe that $\mathbb{B}_t$ is continuous in t and that $\mathbb{B}_0(\omega) = 0$ for all $\omega \in \Omega$.

Let $\mathcal{B}(\mathbb{R})$ denote the Borel σ-algebra in $\mathbb{R}$, that is, the σ-algebra generated by the open subsets of $\mathbb{R}$. Then we define $\mathcal{F}$ as the σ-algebra of subsets of $\mathbb{R}$ generated by $\mathbb{B}_t^{-1}(A)$, with $A \in \mathcal{B}(\mathbb{R})$ and $t \in [0, \infty)$.

Finally, let $\mathbb{P}$ be the unique probability measure on $(\Omega, \mathcal{F})$ such that:

1. $\mathbb{P}(\mathbb{B}_0(\omega) = 0) = \mathbb{P}(\omega(0) = 0) = 1$.

2. For any $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$ the increments
$$\mathbb{B}_{t_n} - \mathbb{B}_{t_{n-1}}, \mathbb{B}_{t_{n-1}} - \mathbb{B}_{t_{n-2}}, \ldots, \mathbb{B}_{t_2} - \mathbb{B}_{t_1}, \mathbb{B}_{t_1}$$
are independent random variables.

3. For all $t \geq 0$ and $h > 0$ we have
$$\mathbb{B}_{t+h} - \mathbb{B}_t \sim N(0, h),$$
where $N(0, h)$ denotes the standard normal distribution with mean 0 and variance h. In particular $\mathbb{B}_t \sim N(0, h)$.

We recall here that a continuous random variable X is called normal of mean μ and variance σ^2, denoted $X \sim N(\mu, \sigma^2)$, if its density function is
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}, \;\; x \in \mathbb{R}.$$

Remark 3.1. The construction of this probability $\mathbb{P}$ is far from trivial, but it is well-known and can be found in many references (see e.g [Krot] or [Y-P]). In any case, it is by no means in the scope of this memoir.

Definition 5. *The one-dimensional standard Brownian motion is the 4-tuple*
$$(\Omega, \mathcal{F}, \mathbb{P}, \{\mathbb{B}_t | t \in [0, \infty)\}).$$

Remark 3.2. In almost every case dealing with Brownian motion we have to consider a general starting point $x \in \mathbb{R}$, which can be different from 0. For that reason, we define the Brownian motion starting at a point x as
$$\mathbb{B}_t^x := \mathbb{B}_t + x.$$
This satisfies conditions 2) and 3) above, and has $\mathbb{B}_0^x = x$ almost surely. Moreover, we define the corresponding probabilities $\mathbb{P}_x$ also by translating $\mathbb{P}$:
$$\mathbb{P}_x(A) := \mathbb{P}((t \longmapsto \mathbb{B}_t + x) \in A) \quad A \in \mathcal{F}.$$

One may think of Brownian motion as a random path starting at 0 and governed by the probability distribution $\mathbb{P}$, which is what really determines its behaviour.

In order to study the properties of stopping times and prove the strong Markov property we need the notion of filtration adapted to the Brownian motion.

Definition 6. *A filtration $(\mathcal{F}_t)_{t\geq 0}$ is an increasing family of sub-σ-algebras $\mathcal{F}_t$ of the σ-algebra $\mathcal{F}$.*

In our case it would be natural to choose, for each $t \geq 0$, the smallest σ-algebra for which all the variables $\{\mathbb{B}_s | s \leq t\}$ are measurable, denoted by $\mathcal{F}_t^0$. Intuitively, we can interpret $\mathcal{F}_t^0$ as the information that we have of $\{\mathbb{B}_t | t \geq 0\}$ up to time t. However, the filtration $(\mathcal{F}_t^0)_{t\geq 0}$ is not right-continuous and we will need continuity in some proofs.

Instead define

$$\mathcal{F}_t := \bigcap_{r>t} \mathcal{F}_r^0 = \bigcap_{s>t} (\bigcap_{r>s} \mathcal{F}_r^0) = \bigcap_{s>t} \mathcal{F}_s.$$

3.1.1 Brownian motion basic properties

In this section we recall several properties of Brownian motion. Some of them are introduced just to have a better intuition of how $\mathbb{B}_t$ behaves, and are not proved. We emphasize (and prove) the ones we use in the proof of Kakutani's theorem.

1. Almost surely, for all $0 < a < b < \infty$, Brownian motion is **not monotone** on the interval $[a, b]$. Roughly speaking, $\mathbb{B}_t$ goes back and forth all the time.

2. Almost surely Brownian motion is **not differentiable** at any $t \geq 0$.

3. **Law of larges numbers**: almost surely,

$$\lim_{t\to\infty} \frac{\mathbb{B}_t}{t} = 0.$$

   This property, shows that t goes to infinity before Brownian motion.

4. **Scaling invariance**: suppose $\{\mathbb{B}_t^x | t \geq 0\}$ a Brownian motion starting

at $x \in \mathbb{R}$ and let $a > 0$. Then $\{X_t | t \geq 0\}$ defined by $X_t = \frac{1}{a}\mathbb{B}_{a^2 t}$ is also a Brownian motion starting at x.

5. **Levy's modulus of continuity**: almost surely,

$$\lim_{h \to 0} \sup_{t \in [0,1-h]} \frac{|\mathbb{B}_{t+h} - \mathbb{B}_t|}{\sqrt{2h \log(\frac{1}{h})}} = 1.$$

6. It $\alpha \in (0, \frac{1}{2})$, then, almost surely, for any $t \geq 0$ exist $\epsilon > 0$ and $c > 0$ such that

$$|\mathbb{B}_t - \mathbb{B}_s| \leq c|t-s|^{\alpha}$$

for any $s \geq 0$ with $|t-s| < \epsilon$

In order to work in $\mathbb{R}^2$ we define d-dimensional Brownian motion.

Definition 7. *The Brownian motion in $\mathbb{R}^d$, still denoted $\{\mathbb{B}_t | \ t \geq 0\}$, is defined as*

$$\mathbb{B}_t = (\mathbb{B}_t^1, \mathbb{B}_t^2, \dots, \mathbb{B}_t^d)$$

where $\mathbb{B}_t^i$, $i \in \{1, \dots, d\}$, are independent one-dimensional Brownian motions. Similarly, the Brownian motion in $\mathbb{R}^d$ starting at $x = (x_1, \dots, x_d)$ is $\mathbb{B}_t^x = (\mathbb{B}_t^1, \mathbb{B}_t^2, \dots, \mathbb{B}_t^d)$, where $\mathbb{B}_t^{x_i}$ are one-dimensional independent Brownian motions starting at $x_i \in \mathbb{R}$.

Remark 3.3. As we define Brownian motion in $\mathbb{R}^2$, we have $\mathbb{B}_t = (\mathbb{B}_t^1, \mathbb{B}_t^2)$ where $\mathbb{B}_t^1, \mathbb{B}_t^2$ are independent one-dimensional Brownian motions. Observe that then $\mathbb{B}_t \sim N(0,t)$, where now $N(0,t)$ is the 2-dimensional Gaussian of mean 0 and covariance matrix tId, that is, the Gaussian distribution with density: for $w = (x,y) \in \mathbb{R}^2$

$$F(w) = F(x,y) = \frac{1}{\sqrt{2\pi(t)}} e^{-\frac{x^2}{2(t)}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} = \frac{1}{2\pi t}\, e^{-\frac{1}{2}\frac{x^2+y^2}{t}}.$$

Similarly for $w \in \mathbb{R}^2$ and $\mathbb{B}_t^z \sim N(z,t)$, the density function is

$$F_z(w) = \frac{1}{2\pi t}\, e^{-\frac{1}{2}\frac{|w-z|^2}{t}}. \tag{3.1}$$

From now on $\mathbb{B}_t^z$ will always denote a two-dimensional Brownian motion starting at $z = (x, y) \in \mathbb{R}^2$.

Observe that $F(x, y)$ above depends only on the distance from (x, y) to 0. This leads to an important property, which will be used in the proof of Kakutani's theorem.

Lemma 3.4 (Isotropy of Brownian motion). *Let $U = \mathbb{D}(z, r)$, with $r > 0$, be a disk centered at $z \in \mathbb{R}^2$ and let $\mathbb{B}_t^z$ be a Brownian motion started at the center of the disk. Then, the probability that $\mathbb{B}_t^z$ leaves U is uniformly distributed over ∂U, i.e, if $\zeta \in \partial U$ denotes the first point of ∂U hit by $\mathbb{B}_t^z$ and $A \subset \partial U$ is measurable, then*

$$\mathbb{P}_z(\zeta \in A) = \frac{|A|}{|\partial U|},$$

where $|\cdot|$ denotes the Lebesgue measure on ∂U.

Proof. By Remark (3.1) this probability is invariant by rotations, so it must be the normalized measure on ∂U.

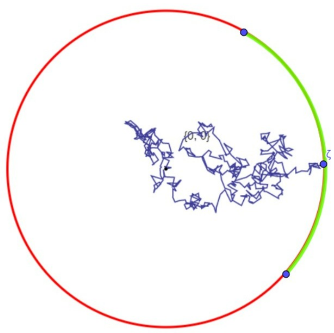


Figure 3.1: Brownian motion started at the center of the disk U, with $\mathbb{B}_{\tau_U}^z = \zeta \in A \subset \partial U$. The set A has green color.

□

3.1.2 Preliminaries for the strong Markov's property

We begin here the preparations to state and prove the strong Markov property. We also state some other results related to the fact that Brownian motion has no memory.

Markov's property of memory loss intuitively can be understood as the fact that the future behaviour of a Brownian motion $\{\mathbb{B}_t^x | t \geq 0\}$, with $x \in \mathbb{R}$, that at time $s > 0$ is at point y ($\mathbb{B}_s^x = y$) does not depend on the path $\mathbb{B}_t^x$, $t < s$. Moreover, for time $t > s$ its behavior is equivalent to a new Brownian $\mathbb{B}_t^y$ starting from y. In other words, it means that the path before we get to a point is irrelevant in what follows.

Theorem 8 (Weak Markov's property)**.** *Let $\{\mathbb{B}_t^x | t \geq 0\}$ be a Brownian motion starting at $x \in \mathbb{R}^2$. Given $s > 0$, the process $\left\{\mathbb{B}_{t+s}^x - \mathbb{B}_s^x | t \geq 0\right\}$ is a Brownian motion starting at the origin, which is independent of the process $\{\mathbb{B}_t^x | 0 \leq t \leq s\}$.*

Proof. First let us see that $X_t = \mathbb{B}_{t+s}^x - \mathbb{B}_s^x$ verifies the conditions of Brownian motion (see Section 3.1):

- X_t is continuous in t because it is the subtraction of continuous functions.
- For $t = 0$ we have $X_0 = \mathbb{B}_s^x - \mathbb{B}_s^x = 0$.
- Given $t, s, h \geq 0$

$$\begin{aligned} X_{t+h} - X_t &= \mathbb{B}_{t+s+h}^x - \mathbb{B}_s^x - (\mathbb{B}_{t+s}^x - \mathbb{B}_s^x) \\ &= \mathbb{B}_{t+s+h}^x - \mathbb{B}_{t+s}^x \sim N(0, t+s+h-(t+s)) = N(0,h). \end{aligned}$$

- For $0 = t_0 \leq t_1 \leq \cdots \leq t_n$

$$\begin{aligned} X_{t_i} - X_{t_{i-1}} &= \mathbb{B}_{t_i+s}^x - \mathbb{B}_s^x - (\mathbb{B}_{t_{i-1}+s}^x - \mathbb{B}_s^x) \\ &= \mathbb{B}_{t_i+s}^x - \mathbb{B}_{t_{i-1}+s}^x, \end{aligned}$$

  so the increments $X_{t_n} - X_{t_{n-1}}, \ldots, X_{t_1} - X_{t_0}$ are independent.

□

Remark 3.5. Notice that the fact that the increments $\left\{\mathbb{B}_{t+s}^x - \mathbb{B}_s^x | t \geq 0\right\}$ are also a new Brownian motion means that the information of $\mathbb{B}_t^x$ at time

$t < s$ is irrelevant to the future, since its behavior corresponds to a new Brownian. In other words, as a consequence of Theorem 8 we have that $\mathbb{B}^x_{t+s} - \mathbb{B}^x_s$ is independent of the filtration $\mathcal{F}^0_s$.

The next result tells us a little bit more than Theorem 8: it proves that $\mathbb{B}^x_{t+s} - \mathbb{B}^x_s$ is actually independent of $\mathcal{F}_s$ as well.

Theorem 9. *For all $s \geq 0$ the process $\left\{\mathbb{B}^x_{t+s} - \mathbb{B}^x_s \middle|\ t \geq 0\right\}$ is independent of $\mathcal{F}_s = \bigcap_{r>s} \mathcal{F}^0_r$.*

Proof. Take $\{s_n\}_n$ decreasing to s. By continuity

$$\mathbb{B}^x_{t+s} - \mathbb{B}^x_s = \lim_{n\to\infty} (\mathbb{B}^x_{t+s_n} - \mathbb{B}^x_{s_n}).$$

Since $\{s_n\}_n$ decreases to s and $\mathcal{F}_s = \bigcap_{r>s} \mathcal{F}^0_r \subseteq \mathcal{F}^0_{s_n}$, we have that $\mathbb{B}^x_{t+s_n} - \mathbb{B}^x_{s_n}$ is independent of $\mathcal{F}^0_{s_n}$ (Remark 3.5), which means that it is independent of $\mathcal{F}_s$ as well.

Also, by Remark 3.5 for all $t_1, \ldots, t_m \geq 0$ the vector

$$(\mathbb{B}^x_{t_1+s} - \mathbb{B}^x_s, \ldots, \mathbb{B}^x_{t_m+s} - \mathbb{B}^x_s) = \lim_{j\to\infty} (\mathbb{B}^x_{t_1+s_j} - \mathbb{B}^x_{s_j}, \ldots, \mathbb{B}^x_{t_m+s_j} - \mathbb{B}^x_{s_j})$$

is independent of $\mathcal{F}_s$. Then by continuity, the process $\left\{\mathbb{B}^x_{t+s} - \mathbb{B}^x_s \middle|\ t \geq 0\right\}$ is independent of $\mathcal{F}_s$ too. □

A consequence of the previous result is the following:

Blumenthal's 0–1 law. *Let $x \in \mathbb{R}^2$ and $A \in \mathcal{F}_0$. Then $\mathbb{P}_x(A)$ is either 0 or 1.*

Proof. Applying the previous theorem with $s = 0$ we have that $\mathbb{B}^x_t - \mathbb{B}^x_0$ is independent of $\mathcal{F}_0 = \bigcap_{r>0} \mathcal{F}^0_r$, hence $A \in \mathcal{F}_0$ is independent of itself.

Therefore

$$\mathbb{P}(A) = \mathbb{P}(A \cap A) = \mathbb{P}(A)\mathbb{P}(A).$$

This is only possible if $\mathbb{P}(A)$ is either 0 or 1. □

3.2 Stopping times

In this section we deal with stopping times and some of their properties, in particular those which are used in the proofs of the strong Markov property and of Kakutani's theorem.

Definition 10. *A random variable $\tau : \Omega \longrightarrow [0, \infty)$ is called a stopping time with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.*

Notice that one example are the constant variables $\tau(\omega) = s$, which determine a stopping time because $\{s \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

There is another example of stopping time that we will work with called hitting time. The hitting time for a set $D \subseteq \mathbb{R}^2$ is the first time that a Brownian motion hits D and we denote it as τ^D.

Lemma 11. *Let $D \subset \mathbb{R}^2$ be an open set and define $\tau^D = \inf\{t > 0 | \mathbb{B}_t \in D\}$. Then τ^D is a stopping time.*

Proof. We need to see that $\left\{\tau^D \leq t\right\} \in \mathcal{F}_t$ for all $t \geq 0$, and for that, it is sufficient to show that

$$\left\{\tau^D < t\right\} \in \mathcal{F}_t \text{ for all } t \geq 0.$$

Take the countable set $\mathbb{Q} \cap (0, t)$; then by continuity of Brownian motion we have

$$\left\{\tau^D < t\right\} = \bigcup_{s \in \mathbb{Q} \cap (0,t)} \{\mathbb{B}_s \in D\} \in \mathcal{F}_t.$$

□

Stopping times are stable by increasing limits.

Proposition 3.6. *Let $(\tau_n)_n$ be an increasing sequence of stopping times converging to τ. Then τ is a stopping time.*

Proof. Let us fix t and let us prove that $\{\tau \leq t\} \in \mathcal{F}_t$. That the sequence increases implies that $\tau_n \leq \tau$ and, since $\lim_{n \to \infty} \tau_n = \tau$,

$$\{\tau \leq t\} = \bigcap_{n=1}^{\infty} \{\tau_n \leq t\} \in \mathcal{F}_t.$$

□

Next we prove the analogue of Lemma 11 for the hitting time of a closed set.

Proposition 3.7. *Let $H \subset \mathbb{R}^2$ be a closed set and define $\tau^H = \inf\{t \geq 0 | \mathbb{B}_t \in H\}$. Then τ^H is a stopping time.*

Proof. Define the open set

$$G(n) = \left\{ x \in \mathbb{R}^2 | \ \exists y \in H \text{ with } |x-y| < \frac{1}{n} \right\} = \left\{ x \in \mathbb{R}^2 | d(x,H) < \frac{1}{n} \right\}.$$

Since H is closed we have that

$$H = \bigcap_{n=1}^{\infty} G(n).$$

Consider now the hitting times of the open set $G(n)$: $\tau_n = \tau^{G(n)}$. By Lemma 11 these τ_n are stopping times. Then, by proposition Proposition 3.6 the limit is also a stopping time, since clearly τ_n increases to τ^H. Since

$$\tau^H = \inf \left\{ t \geq 0 | \mathbb{B}_t \in \bigcap_{n=1}^{\infty} G(n) \right\} = \inf\{t \geq 0 | \mathbb{B}_t \in H\},$$

the proof is finished. □

In the proof of Kakutani's theorem we use exiting times, instead of hitting times. The exiting time of $D \subseteq \mathbb{R}^2$ is just the hitting time of the complementary D^c, therefore it is well defined for both open and closed sets. We denote

$$\tau_D = \tau^{D^c} = \{t \leq 0 | \mathbb{B}_t \notin D\}.$$

Remark 3.8. Given two sets, open or closed, such that $A \subset B$ in $\mathbb{R}^2$, their respective exiting times τ_A and τ_B satisfy $\tau_A \leq \tau_B$.

3.2.1 Regularity of some stopping times

The next proposition bounds the exiting times of sets of finite measure and will be useful in Chapter 3.

Proposition 3.9. *For an open or closed subset $U \subset \mathbb{R}^2$ with Lebesgue measure $m(U)$, we have*

$$\mathbb{P}_z(\tau_U > t) \le \frac{m(U)}{2\pi t}, \ \ t \ge 0.$$

In particular, if $m(U) < \infty$, then

$$\mathbb{P}_z(\tau_U = \infty) = \lim_{t\to\infty} \mathbb{P}_z(\tau_U > t) = 0.$$

Proof. If $\tau_U > t$, then $\mathbb{B}_t^z \in U$ and therefore $\{\tau_U > t\} \subset \{\mathbb{B}_t^z \in U\}$. Hence, since $\mathbb{B}_t^z \sim N(z,t)$ has density given by (3.1):

$$\mathbb{P}_z(\tau_U > t) \le \mathbb{P}_z(\mathbb{B}_t^z \in U) = \int_U \frac{e^{\frac{-||z-y||^2}{2t}}}{2\pi t} \mathrm{d}y \le \int_U \frac{1}{2\pi t} \mathrm{d}y = \frac{m(U)}{2\pi t},$$

as desired. □

Remark 3.10. The closure of a bounded open set D is always in a disk. That implies that it has finite measure, and therefore a Brownian motion starting at $z \in D$ leaves D in finite time with probability 1. (This also proofs that $\mathbb{B}_t^z$ goes to infinity with probability 1, since it leaves any disk $\mathbb{D}(0,n), n \ge 1$).

A technical property of stopping times, required in the proof of Kakutani's theorem, is that of upper semicontinuity. One might think that the function $h(t) = \mathbb{P}(\tau_D > t)$, for a given domain D, is always continuous. It turns out that there are domains with irregular points where the function does strange things. Fortunately, we can ensure its semicontinuity.

Definition 12. *A real-valued function $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is upper semi continuous at a point $z \in \mathbb{R}^2$ if*

$$\limsup_{x\to z} h(x) \le h(z).$$

Analogously, it is lower semi continuous if

$$\liminf_{x\to z} h(x) \ge h(z).$$

The specific property that we will need is the following.

Lemma 13. *Let $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence of continuous functions in $\mathbb{R}^2$ such that*

$$\lim_{n\to\infty} f_n(z) = f(z), \ \ z \in \mathbb{R}^2.$$

Then, f is lower semi continuous for all $z \in \mathbb{R}^2$, meaning that

$$\liminf_{x \to z} f(x) \geq f(z), \;\; z \in \mathbb{R}^2.$$

Proof. Since $(f_n)_{n\in\mathbb{N}}$ is increasing, it holds that $f(z) \geq f_m(z)$ for all $m \in \mathbb{N}$ and $z \in \mathbb{R}^2$. Given a fixed $z \in \mathbb{R}^2$ and $m \in \mathbb{N}$, it follows that

$$\liminf_{x \to z} f(x) \geq \liminf_{x \to z} f_m(x) = f_m(z).$$

as desired. □

Lemma 14. *Let $D \subset \mathbb{R}^2$ be a domain. For any fixed $t > 0$ the function $f(x) = \mathbb{P}_x(\tau_D \leq t)$ is lower semi continuous on $\mathbb{R}^2$.*

Proof. We want to prove that for all $z \in \mathbb{R}^2$

$$\liminf_{x \to z} \mathbb{P}_x(\tau_D \leq t) \geq \mathbb{P}_z(\tau_D \leq t).$$

We will construct an increasing sequence of continuous functions that converges pointwise to $\mathbb{P}_x(\tau_D \leq t)$ and then apply the previous lemma (Lemma 13). Fix $0 < s < t$. By the Markov property (Theorem 8), we can write

$$\begin{aligned}
\mathbb{P}_x(\exists u \in (s,t] : \mathbb{B}_u^x \in D^c) &= \mathbb{E}_x[\mathbb{P}_x(\exists u \in (0, t-s] : \mathbb{B}_{u+s}^x \in \mathbb{D}^c | \mathcal{F}_s)] \\
&= \mathbb{E}_x[\mathbb{P}_{\mathbb{B}_s^x}(\exists u \in (0, t-s] : \mathbb{B}_u^{\mathbb{B}_s^x} \in \mathbb{D}^c] \\
&= \mathbb{E}_x[\mathbb{P}_{\mathbb{B}_s^x}(\tau_D \leq t-s)],
\end{aligned}$$

where $\mathbb{E}_x$ denotes the expectation of a Brownian motion started at x measured by $\mathbb{P}_x$.

This expectation can be expressed as the integral of the conditional expectations given that $\mathbb{B}_s^x = y$, for $y \in \mathbb{R}^2$. Since $\mathbb{B}_s^x \sim N(x,s)$ has density function given by (3.1), we get

$$\begin{aligned}
\mathbb{E}_x[\mathbb{P}_{\mathbb{B}_s^x}(\tau_D \leq t - s)] &= \int_\Omega \mathbb{P}_{\mathbb{B}_s^x}(\tau_D \leq t - s)\mathrm{d}\mathbb{P} \\
&= \int_\Omega \int_{\mathbb{R}^2} \mathbb{P}_{\mathbb{B}_s^x}(\tau_D \leq t - s|\mathbb{B}_s^x = y)F_x(y)\mathrm{d}y\mathrm{d}\mathbb{P} \\
&= \int_\Omega \int_{\mathbb{R}^2} \mathbb{P}_y(\tau_D \leq t - s)F_x(y)\mathrm{d}y\mathrm{d}\mathbb{P} \\
&= \int_{\mathbb{R}^2} \mathbb{P}_y(\tau_D \leq t - s)F_x(y)\mathrm{d}y \\
&= \int_{\mathbb{R}^2} \frac{1}{2\pi s} e^{-\frac{|x-y|^2}{2s}}\ \mathbb{P}_y(\tau_D \leq t - s)\mathrm{d}y.
\end{aligned}$$

Since the right hand side of the equation is continuous in x, then the left side is continuous as well.

It remains to show that $\mathbb{P}_x(\tau_D \leq t)$ is the increasing limit

$$\lim_{s\to 0} \mathbb{P}_x(\exists u \in (s, t] : \mathbb{B}_u \in D^c).$$

Take a sequence $(s_n)_{n\in\mathbb{N}}$ such that $s_n \searrow 0$ as $n \to \infty$, and note that the sequence of sets $(A_n)_{n\in\mathbb{N}}$ defined by $A_n := \{\exists u \in (s_n, t]|\mathbb{B}_u \in D^c\}$, is increasing to τ_D and

$$\{\tau_D \leq t\} = \{\exists u \in (0, t]|\mathbb{B}_u \in D^c\} = \bigcap_{n\in\mathbb{N}} A_n.$$

By the continuity of the measure $\mathbb{P}_x$ it follows that

$$\lim_{n\to\infty} \mathbb{P}_x(A_n) = \mathbb{P}_x(\tau_D \leq t),$$

and by Lemma 13 the function $f(x) = \mathbb{P}_x(\tau_D \leq t)$ is lower semi continuous. $\square$

Remark 3.11. A reformulation of the previous Lemma 14 is that

$$\mathbb{P}_z(\tau_D > t) = 1 - \mathbb{P}_z(\tau_D \leq t)$$

is upper semicontinuous, that is,

$$\limsup_{x\to z} \mathbb{P}_x(\tau_D > t) \leq \mathbb{P}_z(\tau_D > t).$$

3.3 Strong Markov's property

As we have seen, Brownian motion has no memory (Theorem 8 and Theorem 9), meaning that for a Brownian motion $\mathbb{B}_t^z$ that passes across a point y at a given time s, the following path of this motion will have the same distribution as $\mathbb{B}_{t-s}^y$. Here we prove the same result as in Theorem 8, but when the time s is randomized.

Let us observe that a stopping time τ can be discretized in the following way: let

$$\tau_n = \frac{m+1}{2^n} \quad \text{if} \quad \tau \in [\frac{m}{2^n}, \frac{m+1}{2^n}) \;\; m \in \mathbb{Z}. \tag{3.2}$$

Then, by construction $0 \leq \tau_n - \tau \leq \frac{1}{2^n}$.

Strong Markov's property. *For any τ finite stopping time, the process*

$$\{\mathbb{B}_{\tau+t} - \mathbb{B}_\tau | t \geq 0\}$$

is a Brownian motion independent of $\mathcal{F}_\tau$.

Proof. We will first see the independence for the Brownian motion defined with the discretization $(\tau_n)_n$ of τ, so that passing to the limit we will have the independence for the τ case. Then, we will finish the prove by showing that the that Brownian motion passing to the limit is also Brownian motion.

We begin by using the discretization τ_n defined in (3.2) to define the following processes. Let $\mathbb{B}^k = \{\mathbb{B}_t^k | t \geq 0\}$ where

$$\mathbb{B}_t^k = \mathbb{B}_{t+\frac{k}{2^n}} - \mathbb{B}_{\frac{k}{2^n}},$$

and let $\mathbb{B}^* = \{\mathbb{B}_t^* | t \geq 0\}$ be defined by

$$\mathbb{B}_t^* = \mathbb{B}_{t+\tau_n} - \mathbb{B}_{\tau_n}.$$

It is immediate to see that $\mathbb{B}_t^k$ and $\mathbb{B}_t^*$ verify the Brownian motion properties detailed in Section 3.1.

We want to see here that $\mathbb{B}_t^*$ is independent of $\mathcal{F}_{\tau_n}$. It will be enough to see that for all $t \geq 0$, $A \in \mathcal{F}$ and $E \in \mathcal{F}_{\tau_n}$

$$\mathbb{P}(\{\mathbb{B}^* \in A\} \cap E) = \mathbb{P}(\{\mathbb{B}^* \in A\})\mathbb{P}(E).$$

Since $\left\{\mathbb{B}^k \in A\right\}$ is independent of $E \cap \left\{\tau_n = \frac{k}{2^n}\right\} \in \mathcal{F}_{\frac{k}{2^n}}$, by Theorem 9 with time $s = \frac{k}{2^n}$, we have that

$$\begin{aligned}\mathbb{P}(\{\mathbb{B}^* \in A\} \cap E) &= \sum_{k=0}^{\infty} \mathbb{P}(\left\{\mathbb{B}^k \in A\right\} \cap E \cap \left\{\tau_n = \frac{k}{2^n}\right\}) \\ &= \sum_{k=0}^{\infty} \mathbb{P}(\left\{\mathbb{B}^k \in A\right\})\mathbb{P}(E \cap \left\{\tau_n = \frac{k}{2^n}\right\}).\end{aligned}$$

Moreover, by the Weak Markov property (Theorem 8) we have that independently of k, $\mathbb{P}(\left\{\mathbb{B}^k \in A\right\}) = \mathbb{P}(\{\mathbb{B}_t \in A\}) = \mathbb{P}(\{\mathbb{B}^* \in A\})$, so

$$\begin{aligned}\sum_{k=0}^{\infty} \mathbb{P}(\left\{\mathbb{B}^k \in A\right\})\mathbb{P}(E \cap \left\{T_n = \frac{k}{2^n}\right\}) &= \mathbb{P}(\{\mathbb{B}_t \in A\}) \sum_{k=0}^{\infty} \mathbb{P}(E \cap \left\{\tau_n = \frac{k}{2^n}\right\}) \\ &= \mathbb{P}(\{\mathbb{B}^* \in A\}) \sum_{k=0}^{\infty} \mathbb{P}(E \cap \left\{\tau_n = \frac{k}{2^n}\right\}).\end{aligned}$$

We can rewrite the discrete summation as

$$\sum_{k=0}^{\infty} \mathbb{P}(E \cap \left\{\tau_n = \frac{k}{2^n}\right\}) = \mathbb{P}(E),$$

so it follows that

$$\mathbb{P}(\{\mathbb{B}^* \in A\} \cap E) = \mathbb{P}(\{\mathbb{B}^* \in A\})\mathbb{P}(E).$$

We have seen that $\mathbb{B}^*$ is a Brownian motion independent of the filtration $\mathcal{F}_{\tau_n}$ and now we will see this for time τ.

Since $\tau_n \geq \tau$ and $(\tau_n)_n$ converges to τ as $n \to \infty$, we have $\mathcal{F}_{\tau_n} \supset \mathcal{F}_\tau$ and

$$\mathbb{B}_{s+t+\tau} - \mathbb{B}_{t+\tau} = \lim_{n\to\infty} \mathbb{B}_{s+t+\tau_n} - \mathbb{B}_{t+\tau_n}. \tag{3.3}$$

Notice that we define $\mathbb{B}^*$ as $\mathbb{B}_{s+t+\tau} - \mathbb{B}_{t+\tau}$, which is a Brownian motion independent of $\mathcal{F}_{\tau_n}$ and $\mathcal{F}_{\tau_n} \supset \mathcal{F}_\tau$, that is, $\{\mathbb{B}_{t+\tau} - \mathbb{B}_\tau | t \geq 0\}$ is independent of $\mathcal{F}_\tau$.

To finish the proof, it only remains to show that $\{\mathbb{B}_{t+\tau} - \mathbb{B}_\tau | t \geq 0\}$ is a Brownian motion, but this is routinary. Let be $X_t = \mathbb{B}_{t+\tau} - \mathbb{B}_\tau$. We have to see that X_t verifies the conditions of Brownian motion (see Section 3.1):

1. For $t = 0$ we have $X_0 = \mathbb{B}_\tau - \mathbb{B}_\tau = 0$.

2. Given $t, s, h \geq 0$

$$\begin{aligned} X_{t+s} - X_t &= \mathbb{B}_{\tau+t+s} - \mathbb{B}_\tau - (\mathbb{B}_{\tau+t} - \mathbb{B}_\tau) \\ &= \mathbb{B}_{\tau+t+s} - \mathbb{B}_{\tau+t} \end{aligned}$$

   which follows a normal distribution $N(0, s)$.

3. For $0 = t_0 \leq t_1 \leq \cdots \leq t_n$

$$\begin{aligned} X_{t_i} - X_{t_{i-1}} &= \mathbb{B}_{t_i+s} - \mathbb{B}_s - (\mathbb{B}_{t_{i-1}+s} - \mathbb{B}_s) \\ &= \mathbb{B}_{t_i+s} - \mathbb{B}_{t_{i-1}+s}, \end{aligned}$$

   so we get the increments $X_{t_n} - X_{t_{n-1}}, \ldots, X_{t_1} - X_{t_0}$ are independent.

$\square$

Chapter 4

Kakutani's theorem

In this final chapter we give the detailed proof of Kakutani's theorem. We begin with a discussion about the domains where the Dirichlet problem can be solved.

4.1 Regular domains

The Dirichlet problem has solution for most bounded domains $D \subset \mathbb{R}^2$ but not all. If the boundary ∂D isn't good enough and has complicated parts is not possible to solve it.

Definition 15. *Let $D \subset \mathbb{R}^2$ be an open set and let $\zeta \in \partial D$. The point ζ is called regular if $\mathbb{P}_\zeta(\tau_D = 0) = 1$. The domain D is regular if all points $\zeta \in \partial D$ are regular.*

Remark 4.1. By Blumenthal's 0-1 law (Section 3.1.2), the probability of the event $\{\tau_D = 0\} \in \mathcal{F}_0$ must be 0 or 1. Therefore $\mathbb{P}_\zeta(\tau_D = 0) > 0$ is equivalent to $\mathbb{P}_\zeta(\tau_D = 0) = 1$.

Unfortunately, there is no clear geometrical description of regular points or domains. We state a rather general geometrical condition that implies regularity.

Definition 16. *A point $\zeta \in \partial D$ satisfies the truncated cone condition if there exists a truncated cone V with vertex at ζ such that $V \subset D^c$.*

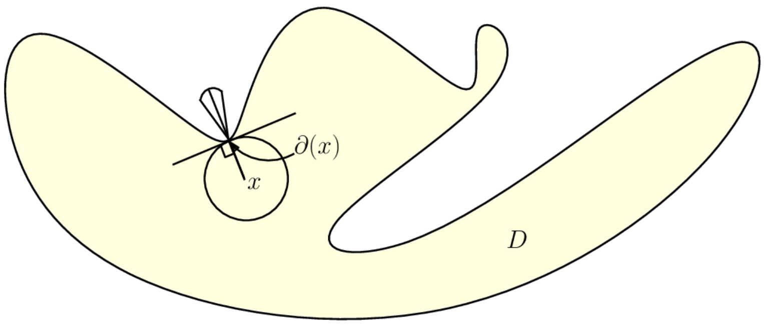


Figure 4.1: Domain verifying the cone condition at the point $\partial(x)$.

Proposition 4.2. *If D satisfies the cone condition at $\zeta \in \partial D$, then ζ is a regular point of ∂D.*

Proof. As pointed out in Remark 4.1, it is enough to show that $\mathbb{P}_\zeta(\tau_D = 0) > 0$. Let V be a cone contained in D^c with vertex ζ. Since $t \leq \tau_D$ implies $\mathbb{B}_t^\zeta \in D$ notice that

$$\mathbb{P}_\zeta(t > \tau_D) \geq \mathbb{P}_\zeta(\mathbb{B}_t^\zeta \notin D) \geq \mathbb{P}_\zeta(\mathbb{B}_t^\zeta \in V \cap \mathbb{D}(\zeta, r))$$

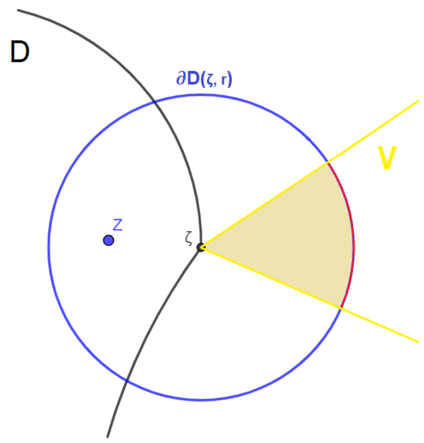


Figure 4.2: The orange colour is $V \cap \mathbb{D}(\zeta, r)$ and the red $V \cap \partial\mathbb{D}(\zeta, r)$.

where $r > 0$. Then, due to the Brownian motion isotropy (Lemma 3.4), we can rewrite the last term in the following way. Let m denote the Lebesgue measure in $\mathbb{R}^2$; then

$$\mathbb{P}_\zeta(\mathbb{B}_t^\zeta \in V \cap \mathbb{D}(\zeta, r)) = \frac{m(V \cap \partial\mathbb{D}(\zeta, r))}{m(\partial\mathbb{D}(\zeta, r))}\mathbb{P}_\zeta(\mathbb{B}_t^\zeta \in \mathbb{D}(\zeta, r)),$$

where $C(V) := \frac{m(V \cup \partial\mathbb{D}(\zeta, r))}{m(\partial\mathbb{D}(\zeta, r))}$ is a positive constant. Then

$$\mathbb{P}_\zeta(t > \tau_D) \geq C(V)\mathbb{P}_\zeta(\mathbb{B}_t^\zeta \in \mathbb{D}(\zeta, r)).$$

By the previous remark, it remains to see that $\mathbb{P}_\zeta(\mathbb{B}_t^\zeta \in \mathbb{D}(\zeta, r))$ is bounded below by a positive constant. To prove it, we will use that $\mathbb{B}_t^\zeta \sim N(\zeta, t)$ and write the density function $F_\zeta(w)$ of (3.1) in polar coordinates:

$$\begin{aligned}\mathbb{P}_\zeta(\mathbb{B}_t^\zeta \in \mathbb{D}(\zeta, r)) &= \int_0^r \int_0^{2\pi} \frac{1}{2\pi t} e^{-\frac{\rho^2}{2t}} \rho \mathrm{d}\theta \mathrm{d}\rho = \int_0^r \frac{1}{t} e^{-\frac{\rho^2}{2t}} \rho \mathrm{d}\rho \\ &= \int_0^{\frac{r}{\sqrt{t}}} e^{-\frac{u^2}{2}} u \mathrm{d}u = 1 - e^{-\frac{r^2}{2t}}.\end{aligned}$$

This tends to 1 as $t \to 0$, as desired. Thus, passing to the limit as t tends to zero, it follows that

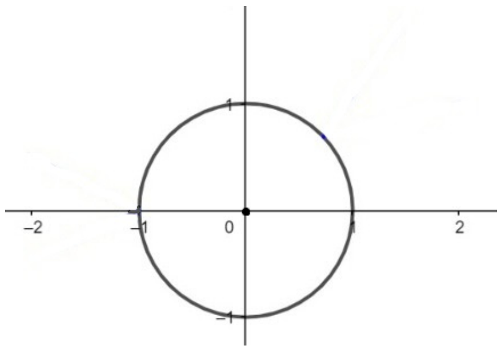
$$\begin{aligned}\mathbb{P}_\zeta(\tau_D = 0) = \mathbb{P}_\zeta(\bigcap_{n=1}^{\infty} \left\{\tau_D < \frac{1}{n}\right\}) &= \lim_{n\to\infty} \mathbb{P}_\zeta(\tau_D < \frac{1}{n}) \\ &\geq \lim_{n\to\infty} C(V)\mathbb{P}_\zeta(\mathbb{B}_{\frac{1}{n}}^\zeta \in \mathbb{D}(\zeta, r)) = C(V) > 0.\end{aligned}$$

□

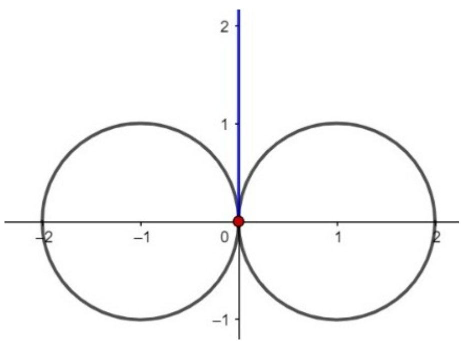
Let us see some examples of domains satisfying (or not) the cone condition:

1. Convex domains are regular. For every $\zeta \in \partial D$, for D convex, there exists a line through ζ (tangent to D) so that D is on one side of the line. Any cone V on the other side proves that ζ is regular.

2. The punctured disk $\Omega = \mathbb{D} \setminus \{0\}$ is clearly not regular at $z = 0$:



3. The open set $D = \mathbb{D}(1,1) \cup \mathbb{D}(-1,1)$ does not verify the cone condition at $z = 0$. Notice that there is only one tangent line to D through 0 and therefore is not possible to let it be the vertex of a truncated cone in D^c:



4.2 Proof of Kakutani's theorem

Let us recall the statement.

Kakutani's theorem. *Let $D \subset \mathbb{R}^2$ be a bounded regular domain and let $f \in \mathcal{C}(\partial D)$. Then*

$$u(z) = \mathbb{E}_z[f(\mathbb{B}^z_{\tau_D})]$$

is the unique solution to the Dirichlet problem with data function f, that is

$$\begin{cases} \Delta u = 0 \quad \text{in } D, \\ u|_{\partial D} = f. \end{cases}$$

Proof. First we shall to prove that $u(z)$ is harmonic and that its boundary values, $u(\zeta)$, $\zeta \in \partial D$, coincide with $f(\zeta)$. We will finish by proving the uniqueness.

1. u is harmonic:
First, notice that $f(\mathbb{B}^z_{\tau_D})$ is continuous at any $z \in \overline{D}$, since it is composed by $\mathbb{B}^z_{\tau_D} \in \partial D$, which is continuous by definition, and f continuous at the boundary points of D. So, as the expectation of a continuous function is continuous, $u(z) = \mathbb{E}_z[f(\mathbb{B}^z_{\tau_D})]$ is continuous.

Since $u(z)$ is continuous its enough to check the mean value property (see Theorem 4).

Thus, given $z \in D$ and $U = \mathbb{D}(z, r)$ a disk centered in z and such that $\overline{U} \subseteq D$, we want to prove that:

$$u(z) = \mathbb{E}[f(\mathbb{B}^z_{\tau_D})] = \frac{1}{|\partial U|} \int_{\partial U} \mathbb{E}[f(\mathbb{B}^w_{\tau_D})] |\mathrm{d}w| = \frac{1}{|\partial U|} \int_{\partial U} u(w) |\mathrm{d}w|,$$

where $|\mathrm{d}w|$ indicates the arc length of ∂U.

Consider the exiting time of U,

$$\tau_U = \inf\{t > 0 | \mathbb{B}^z_t \notin U\}$$

(see Lemma 11 and 3.2). Since $U \subset D$ we have $\tau_U < \tau_D < \infty$ almost surely (see Remark 3.8 and Proposition 3.9).

By the isotropy (Lemma 3.4) at the Brownian motion, the probability that $\mathbb{B}^z_{\tau_D}$ leaves U through $A \subset \partial U$ is uniformly distributed in ∂U, that is:

$$\mathbb{P}_z \left(\mathbb{B}^z_{\tau_U} \in A \right) = \frac{|A|}{|\partial U|}.$$

Then, since $\mathbb{B}^z_t$ has to leave U before leaving D, we can write:

$$u(z) = \mathbb{E}[f(\mathbb{B}^z_{\tau_D})] = \int_{\partial U} \mathbb{E}[f(\mathbb{B}^z_{\tau_D}) | \mathbb{B}^z_{\tau_U} = w] \frac{|\mathrm{d}w|}{|\partial U|}$$

By the strong Markov property (Theorem 3.3) the distribution of $\mathbb{B}^z_t$ after hitting ∂U at a point $w \in \partial U$ is the same as $\mathbb{B}^w_t$, and therefore:

$$\mathbb{E}[f(\mathbb{B}^z_{\tau_D}) | \mathbb{B}^z_{\tau_U} = w] = \mathbb{E}[f(\mathbb{B}^w_{\tau_D})].$$

Therefore

$$u(z) = \mathbb{E}[f(\mathbb{B}^z_{\tau_D})] = \int_{\partial U} \mathbb{E}[f(\mathbb{B}^w_{\tau_D})] \frac{|\mathrm{d}w|}{|\partial U|} = \frac{1}{|\partial U|} \int_{\partial U} u(w)|\mathrm{d}w|,$$

as desired.

2. u has boundary values f:
Let $\zeta \in \partial U$ fixed. We want to prove that

$$\lim_{\substack{z \to \zeta \\ z \in D}} u(z) = f(\zeta).$$

Since ζ is a regular boundary point of D we have $\mathbb{P}_\zeta(\tau_D = 0) = 1$, so:

$$u(\zeta) = \mathbb{E}_\zeta[f(\mathbb{B}^\zeta_{\tau_D})] = \mathbb{E}_\zeta[f(\mathbb{B}^\zeta_0)] = \mathbb{E}_\zeta[f(\zeta)] = f(\zeta).$$

Moreover, since $f(\zeta)$ is a constant it follows that

$$\mathbb{E}_z[f(\mathbb{B}^z_{\tau_D})] - f(\zeta) = \mathbb{E}_z[f(\mathbb{B}^z_{\tau_D}) - f(\zeta)].$$

Then,

$$|u(z) - f(\zeta)| = |\mathbb{E}_z[f(\mathbb{B}^z_{\tau_D}) - f(\zeta)]| \leq \mathbb{E}_z[|f(\mathbb{B}^z_{\tau_D}) - f(\zeta)|],$$

so its suffices to show that

$$\lim_{\substack{z \to \zeta \\ z \in D}} \mathbb{E}_z[|f(\mathbb{B}^z_{\tau_D}) - f(\zeta)|] = 0. \tag{4.1}$$

Let $\epsilon > 0$. We want to find $r > 0$ so that if $|z - \zeta| < r$ and $z \in D$ then $\mathbb{E}_z[|f(\mathbb{B}^z_{\tau_D}) - f(\zeta)|] < \epsilon$.

Fix $r > 0$ and take the disk $\mathbb{D}(\zeta, r)$. Let τ_r be the stopping time $\tau_{\mathbb{D}(\zeta,r)}$.

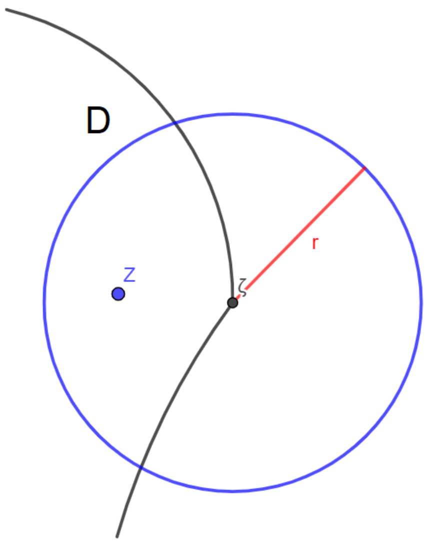


Figure 4.3: Disk centered at $\zeta \in \partial D$ with radius r such that $z \in \mathbb{D}(\zeta, r)$. The black line indicates ∂D.

In order to prove (4.1) we separate two cases, depending on whether $\mathbb{B}_t^z$ leaves first $\mathbb{D}(\zeta, r)$ or the domain D:

$$\begin{aligned}\mathbb{E}_z[|f(\mathbb{B}^z_{\tau_D}) - f(\zeta)|] = \mathbb{E}_z[&|f(\mathbb{B}^z_{\tau_D}) - f(\zeta)|, \tau_r \leq \tau_D] \\ &+ \mathbb{E}_z[|f(\mathbb{B}^z_{\tau_D}) - f(\zeta)|, \tau_D \leq \tau_r].\end{aligned}$$

For the first term, we estimate brutally

$$\begin{aligned}\mathbb{E}_z[|f(\mathbb{B}^z_{\tau_D}) - f(\zeta)|, \tau_r \leq \tau_D] &= \int_{\{\tau_r \leq \tau_D\}} |f(\mathbb{B}^z_{\tau_D}) - f(\zeta)| \mathrm{d}\mathbb{P}(\omega) \\ &\leq 2 \cdot ||f||_\infty \cdot \mathbb{P}_z(\tau_r \leq \tau_D).\end{aligned}$$

For the second term, since $\tau_D < \tau_r$ and therefore $\mathbb{B}^z_{\tau_D} \in \mathbb{D}(\zeta, r)$, we can estimate as follows:

$$\begin{aligned}\mathbb{E}_z[|f(\mathbb{B}^z_{\tau_D}) - f(\zeta)|, \tau_D < \tau_r] &\leq \sup_{\substack{|\zeta - \eta| \leq r \\ \eta \in \partial D}} |f(\zeta) - f(\eta)| \cdot \mathbb{P}_z(\tau_D < \tau_r) \\ &\leq \sup_{\substack{|\zeta - \eta| \leq r \\ \eta \in \partial D}} |f(\zeta) - f(\eta)|.\end{aligned} \tag{4.2}$$

It remains to show that these two terms tend to 0 as z tends to ζ. The first one will go to zero because $\mathbb{P}_z(\tau_r \leq \tau_D)$ tends to 0 as z tends to ζ, whereas the second one will vanish because of the continuity of f.

Let us prove first that the second term (4.2) tends to zero.

By the continuity of f, given $\epsilon > 0$ there exists $r > 0$ such that for all $\eta \in \mathbb{D}(\zeta, r) \cap \partial D$ we have $|f(\zeta) - f(\eta)| < \epsilon/2$. Since $\tau_D < \tau_r$, we have $\mathbb{B}^z_{\tau_D} \in \mathbb{D}(\zeta, r)$ and therefore

$$\sup_{\substack{|\zeta-\eta|\leq r \\ \eta\in\partial D}} |f(\zeta) - f(\eta)| \leq \frac{\epsilon}{2},$$

as desired.

Let us see now that the first term can also be made smaller than $\epsilon/2$. In order to show $\mathbb{P}_z(\tau_r \leq \tau_D)$ tends to zero, assume that $z \in D$ is close enough to ζ so that $|z - \zeta| < \frac{r}{2}$.

So, if we consider the disk centered at z with radius $\frac{r}{2}$ then $\mathbb{D}(z, \frac{r}{2}) \subset \mathbb{D}(\zeta, r)$ and so $\tau_{\mathbb{D}(z,\frac{r}{2})} \leq \tau_r$. Therefore,

$$\mathbb{P}_z(\tau_r \leq \tau_D) \leq \mathbb{P}_z(\tau_{\mathbb{D}(z,\frac{r}{2})} \leq \tau_D),$$

and it suffices to estimate $\mathbb{P}(\tau_{\mathbb{D}(z,\frac{r}{2})} \leq \tau_D)$ as z tends to ζ (i.e, as $r \to 0$).

For any small $t > 0$ the event $E = \{\tau_{\mathbb{D}(z,\frac{r}{2})} \leq \tau_D\}$ is the union of $E_1 = E \cap \{\tau_D \leq t\}$ and $E_2 = E \cap \{\tau_D > t\}$. Since clearly $E_1 \subseteq \{\tau_{\mathbb{D}(z,\frac{r}{2})} \leq t\}$ and $E_2 \subseteq \{\tau_D > t\}$ we deduce that

$$E = \{\tau_{\mathbb{D}(z,\frac{r}{2})} \leq \tau_D\} \subseteq \{\tau_{\mathbb{D}(z,\frac{r}{2})} \leq t\} \cup \{\tau_D > t\}.$$

It follows that

$$\mathbb{P}_z(\tau_{\mathbb{D}(z,\frac{r}{2})} \leq \tau_D) \leq \mathbb{P}_z(\tau_{\mathbb{D}(z,\frac{r}{2})} \leq t) + \mathbb{P}_z(\tau_D > t).$$

Therefore, it is enough to see that both probabilities tend to zero as $t \to 0$. For that, we bound the first term using distribution of $\mathbb{B}^z_t$ as t tends to 0 and, for the second term, we take the limit with z approaching ζ.

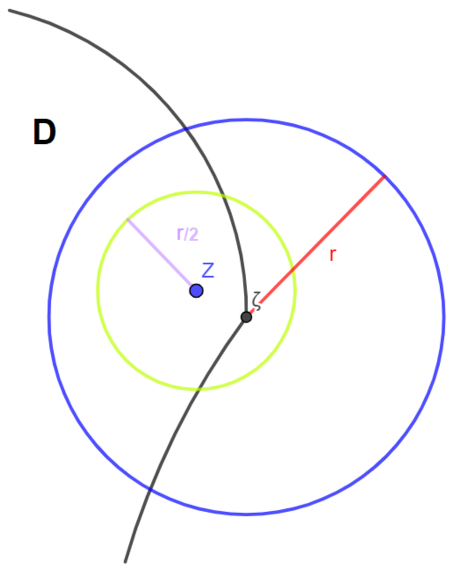


Figure 4.4: Disks $\mathbb{D}(z,\frac{r}{2}) \subset \mathbb{D}(\zeta, r)$ such that $\zeta \in \mathbb{D}(z,\frac{r}{2})$.

Let $\mathbb{P}_z(\tau_{\mathbb{D}(z,\frac{r}{2})} \leq t)$ and notice that its value clearly increases as t increases, so

$$\begin{aligned}\mathbb{P}_z(\tau_{\mathbb{D}(z,\frac{r}{2})} \leq t) = \mathbb{P}_z(\exists s < t | \mathbb{B}^z_s \notin \mathbb{D}(z, \frac{r}{2})) &\leq \max_{s \leq t} \mathbb{P}_z(\mathbb{B}^z_s \notin \mathbb{D}(z, \frac{r}{2})) \\ &= \mathbb{P}_z(\mathbb{B}^z_t \notin \mathbb{D}(z, \frac{r}{2})).\end{aligned}$$

Moreover, since $\mathbb{B}^z_t \sim N(z,t)$, using the density function $F_z(w)$ of (3.1) in polar coordinates we have that

$$\begin{aligned}\mathbb{P}_z(\tau_{\mathbb{D}(z,\frac{r}{2})} \leq t) \leq \mathbb{P}_z(\mathbb{B}^z_t \notin \mathbb{D}(z, \frac{r}{2})) &= \int_{\frac{r}{2}}^{\infty} \int_0^{2\pi} \frac{1}{2\pi t} e^{-\frac{\rho^2}{2t}} \rho \mathrm{d}\theta \mathrm{d}\rho \\ &= \int_{\frac{r}{2}}^{\infty} \frac{1}{t} e^{-\frac{\rho^2}{2t}} \rho \mathrm{d}\rho = e^{-\frac{r^2}{8t}}\end{aligned}$$

Therefore

$$\lim_{t \to 0} \mathbb{P}_z(\tau_{\mathbb{D}(z,\frac{r}{2})} \leq t) \leq \lim_{t \to 0} e^{-\frac{r^2}{8t}} = 0.$$

Thus for t small enough we have

$$\mathbb{P}_z(\tau_{\mathbb{D}(z,\frac{r}{2})} \leq \tau_D) < \frac{\epsilon}{4} + \mathbb{P}_z(\tau_D > t).$$

So it remains to prove that for any fixed t, $\lim_{z\to\zeta} \mathbb{P}_z(\tau_D > t) = 0$. Applying Remark 3.11 we see that

$$\limsup_{z\to\zeta} \mathbb{P}_z(\tau_D > t) \leq \mathbb{P}_\zeta(\tau_D > t) = 1 - \mathbb{P}_\zeta(\tau_D \leq t),$$

where $\mathbb{P}_\zeta(\tau_D \leq t)$ increases as t does. So

$$\mathbb{P}_\zeta(\tau_D = 0) \leq \mathbb{P}_\zeta(\tau_D \leq t) \leq 1,$$

since $\mathbb{P}_\zeta(\tau_D = 0) = 1$ by the assumption of regularity on ζ. Then, taking z close enough to ζ we have that $\mathbb{P}_z(\tau_D > t) < \epsilon/4$.

Finally, it follows that

$$\mathbb{E}_z[|f(\mathbb{B}^z_{\tau_D}) - f(\zeta)|, \tau_D < \tau_r] \leq \frac{\epsilon}{2} + (\frac{\epsilon}{4} + \frac{\epsilon}{4}) \leq \epsilon,$$

and therefore (4.1) holds.

It only remains to show that the solution is unique. To prove the uniqueness we will not require probabilistic methods, since this is given by the fact that the solution u is harmonic.

Suppose that $h \in \mathcal{C}^2(\overline{D})$ is another solution to the Dirichlet problem. Then, u and h harmonic on $\overline{D}$ and we have that the function $(u-h)$ is also harmonic since

$$\Delta(u-h) = \Delta u - \Delta h = 0.$$

Moreover, $(u-h) = 0$ on ∂D because $f(\zeta) = u(\zeta) = h(\zeta)$ for all $\zeta \in \partial D$.

Applying the maximum principle, Theorem 2, of harmonic functions to $u-h$, we have that

$$\max_{z\in\overline{D}}(u-h)(z) = \max_{\zeta\in\partial D}(u-h)(\zeta) = 0.$$

Therefore, $h = u$ on $\overline{D}$ as we wanted to prove. $\square$

Conclusions

We conclude by emphasizing that solving the Dirichlet problem via Brownian motion is an example of how powerful are the probabilistic methods in the study of many problems in different areas of Mathematics.

In this memoir we illustrate how the properties of Brownian motion lead to an explicit solution to a classical partial differential equation with boundary values.

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