Universitat de Barcelona
Facultat de Matemàtiques i Informàtica

GRAU DE MATEMÀTIQUES

Treball final de grau

Isoperimetric inequalities in the plane

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Realitzat a: Departament de Matemàtiques i Informàtica

Barcelona, 13 de juny de 2022

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Abstract

The main goal of this work is to study different geometric inequalities in the plane. In particular, we will work on the isoperimetric, the Saint-Venant and the Faber-Krahn inequalities for simple connected domains. We will use two different approaches: first a classic one by complex analysis, and then a more recent one by operator theory, bounding the commutator of Toeplitz operators in the Hardy-Smirnov space E_2 and the Bergman space A^2. We will also study these spaces and how they relate with geometric quantities. Finally, we will talk about functions of bounded variation in order to extend the classical isoperimetric inequality for any domain in the plane.

Resum

Estudiarem diverses desigualtats geomètriques en el pla per dominis simplement connexos, en concret, la desigualtat isoperimètrica clàssica, la de Saint-Venant i la de Faber-Krahn. Ho farem de dues maneres, primer una més clàssica utilitzant anàlisi complexa i després una més recent utilitzant teoria d'operadors, utilitzant els commutadors dels operadors de Toeplitz en l'espai de Hardy-Smirnov E_2 i en l'espai de Bergman A^2. També estudiarem aquests espais i la seva relació amb diverses quantitats geomètriques. Finalment, parlarem sobre les funcions de variació acotada per tal d'estendre la desigualtat isoperimètrica per a tot domini del pla.

2020 Mathematics Subject Classification. 26D10, 30A10, 30H10, 30H20, 47B35

Agraïments

Primer de tot, m'agradaria agrair al meu tutor, Dr. Joaquim Ortega Cerdà, per la seva incansable ajuda i pels seus consells en les quedades setmanals que hem fet durant aquest semestre. Moltes gràcies per explicar-me una vegada i una altra els conceptes que em fallaven més i haver-me ajudat a entendre els més difícils. Moltes gràcies per la teva paciència, les teves correccions, la teva resposta immediata quan tenia qualsevol dubte, per la teva disponibilitat i flexibilitat a l'hora de quedar per molts problemes que hi hagués i, en general, per tot el teu temps invertit en aquest treball. He après molt i ha estat una experiència increïble.

També vull agrair als meus pares que m'han ajudat tots aquests anys des de la distància i a totes les persones que han estat al meu costat a Barcelona, tant les noves com les de sempre, especialment als meus companys de pis. Per acabar, no podria haver arribat fins aquí sense el suport del Joan, el Ton i l'Edu que m'han acompanyat aquests quatre anys tant a dins com a fora de la universitat.

Chapter 1

Introduction

At last they landed, where from far your eyes
May view the turrets of new Carthage rise;
There bought a space of ground, which Byrsa call'd,
From the bull's hide, they first inclos'd, and wall'd.

Virgil, *The Aeneid*, Book I

Imagine we have a simple non-elastic rope that we close by the two ends. Now we have a simple and closed curve and we can play with it doing all kind of diferent shapes (crossings are not allowed), noticing the perimeter always stays the same while the area is changing. One natural question is to ask which shape maximizes the area: this is the classical isoperimetric problem [23]. The solution is the circle and was already known by the greeks. This leads to an inequality: given any shape Ω, if we know the perimeter, then its area must be less than a circle D with the same perimeter, which is the classical isoperimetric inequality:

$$\text{Area}(\Omega) \leq \text{Area}(D) = \frac{\text{Per}(\Omega)}{4\pi}$$

This is probably one of the oldest problems in the Calculus of Variations, with its first appearance in Book V of the *Collection* of Pappus of Alexandria in the IV century AD [8]. Although there is plenty of proofs of it, the first proof was not given until 1841 by Jakob Steiner [23]. After that, with the use of symmetrization, Schwartz extended the inequality for a ball in 3 dimensions in 1884 and at the begining of the 20th century Hurwitz used trigonometric series and the method was also extended to upper dimensions [2]. Our purpose is to study different proofs that use complex analysis and operator theory techniques, and therefore we will focus on dimension 2. This allows to work with complex numbers and use tools such as the Riemann mapping theorem or the Hilbert spaces $H^2(\mathbb{D})$ and $A^2(\mathbb{D})$. We will show how with few properties of these tools we can prove three different inequalities and how they are closely related.

Apart from the area or the perimeter, there are many other geometrical quantities. They are geometrical in the sense that they only depend on the shape and size of Ω, and they have a wide application in physics. In fact, we will even reduce the dependence on the size by considering only shapes with the same area, so that we get properties inherent to the shape. The first quantity we will study is the torsional rigidity, which is a constant from mechanics that quantifies the resistance to be twisted of a cylindrical object with cross-section equal to Ω (although it has also several hydrodynamical and electrodynamical applications [21, p. 3]). The other quantity we will study is the principal frequency, which is the frequency of the gravest proper tone of a drum, that is, a uniform elastic membrane. This is also an interesting mathematical quantity, corresponding to the lowest eigenvalue of the Laplacian and again, it only depends on Ω. For these quantities there are only a few cases of Ω in which an explicit formula has been found, so we can only get an approximation and try to estimate the error, and here is when the inequalities gain importance. There is a lot of inequalities relating these quantities (see [21, pp. 17–19]), but here we will work on the most important ones: the Saint-Venant and the Faber-Krahn inequalities, which do not depend on the size and bound the torsional rigidity and the principal frequency by the area, respectively.

In order to prove this inequalities, we will first present some classical proofs working with the coefficients of the Riemann mapping (following the method of Polya and Szego in [21]) and with Fourier series. Then we will focus on operator theory, using a particular operator called Toeplitz operator. As we will observe, Hardy and Bergman spaces will come up even in the most classical proofs, with a natural association with the perimeter and the area. This relation has developed to the point that they are important spaces to study on its own, and nowadays we know a lot about its properties and they have been generalized in several ways. This classical inequalities motivate to keep going in the right direction and they help for discovering new inequalities involving more complicated mathematical objects with more generality. For example, our strategy using operator theory will be finding upper and lower bounds for the commutator of Toeplitz operators, which would have been hard to obtain if Bell, Ferguson and Lundberg would not have tried to prove the Saint-Venant inequality.

In the last chapter, in contrast with the others, we will consider the isoperimetrical inequality for general domains (we will not assume they have smooth boundary). To do this, we will have to be more rigorous and talk about the exact definition of perimeter and what implies to have a boundary with finite length. After seeing some consequences, we will observe that it is easy to obtain the isoperimetrical inequality for any connected domain in the plane.

The structure of the work is as follows. In Chapter 2 we introduce the Hardy and Bergman spaces in the disk and we show two classical proofs of the isoperimetrical inequality. Chapter 3 is dedicated to the torsional rigidity and we proof the Saint-Venant inequality in two ways, following the ideas of the proofs in Chapter 2. In Chapter 4 we extend the Hardy and Bergman spaces in order to prove the isoperimetric inequality and the Saint-Venant inequality. We also introduce the principal frequency and show a close approximation to the Faber-Krahn inequality with Toeplitz operators.

Chapter 2

The classical isoperimetric inequality

We begin by stating our first version of the isoperimetric inequality, where we consider regions in the plane with smooth borders:

Theorem 2.1 (Isoperimetric Inequality I)**.** *Let $\Omega \subseteq \mathbb{R}^2$ such that $\partial\Omega$ is a smooth curve, then*

$$A(\Omega) \leq \frac{L(\partial\Omega)^2}{4\pi} \tag{2.1}$$

and we have the equality if and only if Ω is a disk. [22]

In this section we will prove it in two ways, but first of all we need to introduce the Hardy and Bergman spaces in the disk and some of its properties since we will use them extensively not only in our first proof, but also later when proving other isoperimetrical inequalities.

2.1 Hardy and Bergman spaces in the disk

Definition 2.2. *Let $f : \mathbb{D} \to \Omega$ be an holomorphic function, if $0 \leq r < 1$ we can write the functions in $\mathbb{S}^1$ as $f_r(e^{i\theta}) = f(re^{i\theta})$, which are the restriccions to the circumference of radius r and center 0. Let μ be the normalized Lebesgue measure in $\mathbb{S}^1$, we can define for $0 < p < \infty$:*

$$\|f\|_{H^p} := \sup_{0\leq r<1} \|f_r\|_{L^p(\mu)} = \sup_{0\leq r<1} \left(\int_{\mathbb{S}^1} |f_r|^p d\mu \right)^{1/p} = \sup_{0\leq r<1} \int_0^{2\pi} \left(|f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}$$

$$\|f\|_{H^\infty} := \sup_{0\leq r<1} \|f_r\|_{L^\infty(\mu)} = \sup_{0\leq r<1} \left(\sup_{\theta} |f(re^{i\theta})| \right)$$

and we define the Hardy space H^p for $0 < p \leq \infty$ as:

$$H^p := \{f \in H(\mathbb{D}) : \|f\|_{H^p} < \infty\}$$

Since $\|f_r\|_{H^p}$ is a non-decreasing function of r for all $0 < p \leq \infty$ [19, p. 338], exists the limit as r approaches 1 from the right so we can write the norm of H^p as:

$$\|f\|_{H^p} = \lim_{r\to 1^-} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}$$

Since we will need to take powers of f (with real exponent), we have to study the zeros of H^p functions. In order to do this, we will factorize f in the Blaschke product (which have all the zeros of f) and a function without zeros where we will be able to take powers without problem. We will see that the zeros of any H^p function have a well-defined Blaschke product (and viceversa, if we have the zeros of a Blaschke product, since the Blashcke product itself is bounded, we have a function in $H^\infty \subseteq H^p$ with these zeros [19, pp. 310–311]) and we will state and prove some usefull properties of it. First of all, let's see that indeed the Blaschke product is well-defined:

Definition 2.3. *Let $\{z_n\} \subseteq \mathbb{D}$ such that $z_n \neq 0$ and $\sum_{n=1}^{\infty}(1-|z_n|) < \infty$, let k be a non negative integer, the Blaschke product of $\{z_n\}$ is*

$$B(z) = z^k \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n} z}, \quad z \in \mathbb{D} \tag{2.2}$$

Proposition 2.4. *The Blaschke product B is well-defined and B has zeros only in the z_n (and in the origin if $k > 0$). Moreover, $|B(z)| < 1\,\forall z \in \mathbb{D}$ (and since it is bounded, belongs to H^∞).*

Proof. We will use that if $f_n \in H(\Omega), f_n \not\equiv 0\ \forall n \geq 1$ and $\sum_{n=1}^{\infty} |1 - f_n(z)|$ converges uniformly in compact sets of Ω then $f(z) = \prod_{n=1}^{\infty} f_n$ converges uniformly in compact sets of Ω [19, Teorema 15.6]. Therefore, we only have to see that

$$\sum_{n=1}^{\infty} \left| 1 - \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n} z} \right|$$

is uniformly convergent in compact sets of $\mathbb{D}$. To do this, we will use the Weierstrass M-test [1, p. 37]. Let be $|z| \leq r < 1$,

$$\begin{aligned} \left| 1 - \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n} z} \right| &= \left| \frac{z_n - |z_n|^2 z - |z_n| z_n + |z_n| z}{(1 - \overline{z_n} z) z_n} \right| = \\ &= \left| \frac{z_n(1 - |z_n|) + z|z_n|(1 - |z_n|)}{(1 - \overline{z_n} z) z_n} \right| = \frac{|z_n + z|z_n||}{|(1 - \overline{z_n} z) z_n|} (1 - |z_n|) \\ &\leq \frac{(1 + r)|z_n|}{|1 - \overline{z_n} z||z_n|} (1 - |z_n|) \leq \frac{1 + r}{1 - r} (1 - |z_n|) = M_n \end{aligned}$$

and $\sum_{n=1}^{\infty} M_n < \infty \Leftrightarrow \frac{1+r}{1-r}\sum_{n=1}^{\infty}(1-|z_n|) < \infty$ which is our hypothesis. Furthermore, it will only be zero if $z_n - z = 0 \Rightarrow z = z_n$ (since the denominator never vanishes because $|z| \leq 1$ and $|z_n| < 1$, which implies $|\overline{z_n}z| < 1 \Rightarrow \overline{z_n}z \neq 1$).

Finally, we will see that if $|z| < 1$ then $|B(z)| \leq 1$. Since

$$|B(z)| = |z|^k \prod_{n=1}^{\infty} \left| \frac{z_n - z}{1 - \overline{z_n}z} \right|$$

it is enough that each factor is less (or equal) than 1 $\forall z \in \mathbb{D}$. Clearly $|z|^k < 1$, so we just have to see that $\left|\frac{z_n-z}{1-\overline{z_n}z}\right|^2 = \left|\left(\frac{z_n-z}{1-\overline{z_n}z}\right)^2\right| \leq 1$ (which implies $\left|\frac{z_n-z}{1-\overline{z_n}z}\right| \leq 1$). Evaluating at the boundary,

$$\left|\frac{z_n - z}{1 - \overline{z_n}z}\right|^2 = \frac{|z|^2 - z_n\overline{z} - z\overline{z_n} + |z_n|^2}{1 - \overline{z_n}z - \overline{z}z_n + |z_n|^2|z|^2} = 1, \quad |z| = 1$$

and as we have an analytic function in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$, by the maximum modulus principle [1, p. 134], we have the inequality for $|z| < 1$. □

Now we have to prove the reciprocal, that is, given a function in H^p we can construct its Blaschke product. For this, we will study the Nevanlinna class, which is a bigger function space than H^p where we can always find a Blaschke product of the zeros of its functions.

Definition 2.5. *Let $t \in \mathbb{R}^+$, we denote $\log^+(t) = \log(t)$ if $t \geq 1$ and $\log^+(t) = 0$ if $0 < t < 1$. The Nevanlinna class is*

$$N = \left\{ f \in H(\mathbb{D}) : \sup_{0<r<1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} < \infty \right\}$$

This class includes all the Hardy spaces:

$$H^\infty \subseteq H^p \subseteq H^q \subseteq N, \quad 0 < q < p < \infty$$

and the following theorem tells us that if $f \in N$ then the zeros of f have the condition that allows us to construct the Blaschke product (and therefore we will be able to construct it for all $f \in H^p$, with $p > 0$).

Theorem 2.6. *Let be $f \in N$ such that $f \not\equiv 0$ in $\mathbb{D}$, and $\{z_n\}$ are the zeros of f, repeated taking into account its multiplicity, then*

$$\sum_{n=1}^{\infty}(1 - |z_n|) < \infty$$

To prove it we first need the Jensen formula:

Proposition 2.7. *Let $f \in H(D(0;R)), f(0) \neq 0, 0 < r < R$, let $z_1, ..., z_N$ be the zeros of f in $\overline{D(0;r)}$, then*

$$|f(0)| \prod_{n=1}^{N} \frac{r}{|z_n|} = \exp\left\{ \int_0^{2\pi} \log|f(re^{i\theta})| \frac{d\theta}{2\pi} \right\} \tag{2.3}$$

Proof. We arrange the zeros (taking into account the multiplicities) such that $|z_j| < r$ if $j = 1, ..., m$ and $|z_{m+1}| = ... = |z_N| = r$. Let $\epsilon > 0$ such that $f \in H(D(0;r+\epsilon))$ has the same zeros in $D(0;r+\epsilon)$, we construct the following holomorphic function and without zeros in $D(0;r+\epsilon)$:

$$g(z) = f(z) \prod_{n=1}^{m} \frac{r^2 - \overline{z_n} z}{r(z_n - z)} \prod_{n=m+1}^{N} \frac{z_n}{z_n - z} \tag{2.4}$$

that indeed is not zero because $z_n \neq 0 \ \forall n$ and if $|z| \leq r, |z_n| < r$ then $r^2 > |\overline{z_n} z|$. Hence, it exists a branch of the logarithm of g in $D(0;r+\epsilon)$ [19, Teorema 13.11] and we have $\mathrm{Re}(\log(g(z))) = \log(|g(z)|)$. With the Integral Cauchy Formula [1, p. 119] in the disk $D(0;r)$ we get

$$\log(g(0)) = \int_0^{2\pi} \log(g(re^{i\theta})) \frac{d\theta}{2\pi}$$

so

$$\log(|g(0)|) = \mathrm{Re}(\log(g(z))) = \int_0^{2\pi} \mathrm{Re}(\log(g(re^{i\theta}))) \frac{d\theta}{2\pi} = \int_0^{2\pi} \log(|g(re^{i\theta})|) \frac{d\theta}{2\pi} \tag{2.5}$$

On one hand,

$$g(0) = f(0) \prod_{n=1}^{m} \frac{r}{z_n} \Rightarrow |g(0)| = |f(0)| \prod_{n=1}^{m} \frac{r}{|z_n|}$$

On the other hand, if $|z| = r$, for $n = 1, ..., m$,

$$\left| \frac{r^2 - \overline{z_n} z}{r(z_n - z)} \right|^2 = \left(\frac{r^2 - \overline{z_n} z}{r(z_n - z)} \right) \left(\overline{\frac{r^2 - \overline{z_n} z}{r(z_n - z)}} \right) = \frac{r^4 - r^2 z_n \overline{z} - r^2 \overline{z_n} z + |z_n|^2 |z|^2}{r^2 |z|^2 - r^2 z_n \overline{z} - r^2 \overline{z_n} z + r^2 |z_n|^2} = 1$$

so we have $\left| \frac{r^2 - \overline{z_n} z}{r(z_n - z)} \right| = 1$ and the integrant becomes, writing the zeros $z_n = re^{i\theta_n}$:

$$\begin{aligned} \log(|g(re^{i\theta})|) &= \log\left(|f(re^{i\theta})| \prod_{n=m+1}^{N} \left| \frac{re^{i\theta_n}}{re^{i\theta_n} - re^{i\theta}} \right| \right) = \\ &= \log\left(|f(re^{i\theta})| \prod_{n=m+1}^{N} \frac{1}{|1 - re^{i(\theta - \theta_n)}|} \right) = \\ &= \log(|f(re^{i\theta})|) - \sum_{n=m+1}^{N} \log(|1 - re^{i(\theta - \theta_n)}|) \end{aligned}$$

Finally, we just need to prove that the integral of the summation is 0, so that the integral in (2.4) does not change if we substitute g for f and we would have:

$$\int_0^{2\pi} \log|1-e^{i\theta}|\frac{d\theta}{2\pi} = 0 \Rightarrow \int_0^{2\pi} \log(|g(re^{i\theta})|)\frac{d\theta}{2\pi} = \int_0^{2\pi} \log(|f(re^{i\theta})|)\frac{d\theta}{2\pi} \tag{2.6}$$

and doing the change of variable $\theta = \theta' - \theta_n$ we would get the integral we were looking for.

Let $\Omega = \{z \in \mathbb{C} : \mathrm{Re}(z) < 1\}$, since it is simply connected and $1 - z \neq 0$ in Ω, exists $h \in H(\Omega)$ such that $e^{h(z)} = 1 - z$ and it is unique if $h(0) = 0$ [19, Teorema 13.11]. If $w \in \Omega$, the image of $1 - w$ is the half-plane $\{\mathrm{Im}(z) > 0\}$ so that $|\mathrm{Im}(h(w))| = |\arg(1-w)| < \frac{\pi}{2}$. Therefore,

$$\mathrm{Re}(h(z)) = \log|1-z|, \quad |\mathrm{Im}(h(z))| < \frac{\pi}{2}, \quad \forall z \in \Omega \tag{2.7}$$

Now let $\delta > 0$ be small enough, let $\Gamma(\theta) = e^{i\theta}$ with $\theta \in [\delta, 2\pi - \delta]$ and let γ be the path that goes from $e^{i\theta}$ to $e^{-i\theta}$, following the circumferencia of center 1 and radius $\sqrt{2(1-\cos\delta)}$ through the inside of $\mathbb{D}$. Utilitzant (2.7),

$$\int_\delta^{2\pi-\delta} \log|1-e^{i\theta}|\frac{d\theta}{2\pi} = \mathrm{Re}\left(\int_\delta^{2\pi-\delta} h(e^{i\theta})\frac{d\theta}{2\pi}\right) = \mathrm{Re}\left(\int_\Gamma \frac{h(z)}{z}\frac{dz}{2\pi i}\right)$$

and due to the Cauchy's Theorem [19, Teorema 10.35] in Ω we have:

$$\int_\Gamma \frac{h(z)}{z}\frac{dz}{2\pi i} = \int_\gamma \frac{h(z)}{z}\frac{dz}{2\pi i}$$

and since we can bound the length of γ by $\delta\pi$, $|\gamma| > 1/2$ for a small enough δ and $|1-\gamma| = \sqrt{2(1-\cos\delta)}$, we get

$$\begin{aligned}\left|\int_\gamma \frac{h(z)}{z}\frac{dz}{2\pi i}\right| &\le L(\gamma)\sup_{z\in\gamma}\left|\frac{h(z)}{2\pi i z}\right| \le \frac{\delta}{2}\sup_{z\in\gamma}\left|\frac{h(z)}{z}\right| \le \\ &\le \frac{\delta}{2}\frac{\sup_{z\in\gamma}|\mathrm{Re}(h(z))|}{1/2} \le \delta|\log(\sqrt{2(1-\cos\delta)})| = \\ &= \frac{\delta}{2}\log\left(\frac{1}{2(1-\cos\delta)}\right) \xrightarrow{\delta\to 0} 0\end{aligned}$$

Therefore,

$$\int_0^{2\pi} \log|1-e^{i\theta}|\frac{d\theta}{2\pi} = \lim_{\delta\to 0}\int_\delta^{2\pi-\delta} \log|1-e^{i\theta}|\frac{d\theta}{2\pi} = 0$$

Considering (2.4), (2.5) and (2.6), we get

$$\begin{aligned}\log\left(|f(0)|\prod_{n=1}^{N}\frac{r}{|z_n|}\right) &= \int_0^{2\pi}\log|f(re^{i\theta})|\frac{d\theta}{2\pi} \Rightarrow \\ &\Rightarrow |f(0)|\prod_{n=1}^{N}\frac{r}{|z_n|} = \exp\left\{\int_0^{2\pi}\log|f(re^{i\theta})|\frac{d\theta}{2\pi}\right\}\end{aligned}$$

□

Now we can prove the main theorem:

Proof. Let z_n be the zeros of $f \in N$, we can assume they are ordered such that $|z_n| \leq |z_{n+1}|$ and that f has infinit zeros (if f has a finite number of zeros, then it is clera that the sum is finite). Moreover, we can assume that $f(0) \neq 0$ because if f has a root of order m in the origin, then $g = z^{-m}f(z) \in N$ and has the same zeros of f, so if g fulfills the theorem, f also does it.

We put $n(r)$ the number of roots of f in $\overline{D(0;r)}$, and given a $k \in \mathbb{N}$, we take $r < 1$ such that $n(r) > k$. Since $1 \leq \frac{r}{z_n}$ and $\log(t) \leq \log^+(t)$, by the Jensen formula (2.3),

$$|f(0)| \prod_{n=1}^{k} \frac{r}{|z_n|} \leq |f(0)| \prod_{n=1}^{n(r)} \frac{r}{|z_n|} = \exp\left(\int_0^{2\pi} \log|f(re^{i\theta})| \frac{d\theta}{2\pi}\right) \leq \exp\left(\int_0^{2\pi} \log^+|f(re^{i\theta})| \frac{d\theta}{2\pi}\right)$$

Since $f \in N$, we know that $\exists\, C > 0$ such that

$$\exp\left(\int_0^{2\pi} \log^+|f(re^{i\theta})| \frac{d\theta}{2\pi}\right) \leq C$$

so

$$|f(0)| \prod_{n=1}^{k} \frac{r}{|z_n|} = |f(0)| \frac{r^k}{\prod_{n=1}^{k} |z_n|} \leq C \Rightarrow \frac{|f(0)| r^k}{C} \leq \prod_{n=1}^{k} |z_n| \quad \forall k \in \mathbb{N}$$

Therefore, taking $r \to 1$ and then $k \to \infty$,

$$\prod_{n=1}^{\infty} |z_n| \geq \frac{|f(0)|}{C} > 0$$

(we know the limit exists when $k \to \infty$ because $|z_n| < 1$ and therefore we have a non-decreasing succession in k).

Now we assume that $\sum_{n=1}^{\infty}(1 - |z_n|) = \infty$, using $h(x) = e^{x-1} - x \geq 0\ \forall x \in \mathbb{R}$ (h has an absolute minimum in $x = 1$ and $h(1) = 0$),

$$\prod_{n=1}^{\infty} |z_n| \leq \prod_{n=1}^{k} |z_n| \leq \prod_{n=1}^{k} e^{|z_n|-1} = \exp\left\{-\sum_{n=1}^{k} 1 - |z_n|\right\} \xrightarrow{k\to\infty} 0$$

and so $\prod_{n=1}^{\infty} |z_n| = 0$, leading to contradiction. □

The property of the Blaschke product we will use is the following:

Proposition 2.8. *Let $p > 0$, let $f \in H^p$ with B the Blaschke product of the roots of f. If $g = f/B$ then $g \in H^p$ with $\|g\|_{H^p} = \|f\|_{H^p}$ and $g(z) \neq 0\ \forall z \in \mathbb{D}$.*

Proof. First of all, since f and B have the same roots in $\mathbb{D}$ (taking into account multiplicities), g is holomorphic in $\mathbb{D}$ and has no roots in it. We have already seen in Proposition 2.4 that $|B(z)| < 1$ for $z \in \mathbb{D}$ and therefore

$$|g| \geq |f| \Rightarrow \|g\|_{H^p} \geq \|f\|_{H^p} \tag{2.8}$$

Now let B_n the finite Blaschke product of the first n roots of f, let be $g_n = f/B_n$, if we take $r \to 1$ for all n, we have that $|B_n(re^{i\theta})|$ converges uniformly to 1 so that $\|g_n\|_{H^p} = \|f\|_{H^p}$. Since $|g_n|$ tends to $|g|$ and is non-decreasing, we can apply the Monotone Convergence Theorem [19, Teorema 1.26]:

$$\lim_{n\to\infty} \int_0^{2\pi} |g_{r,n}(re^{i\theta})|^p \frac{d\theta}{2\pi} = \int_0^{2\pi} \lim_{n\to\infty} |g_{r,n}(re^{i\theta})|^p \frac{d\theta}{2\pi} = \int_0^{2\pi} |g(re^{i\theta})|^p \frac{d\theta}{2\pi}$$

so that

$$\|g_r\|_{H^p} = \lim_{n\to\infty} \|g_{n,r}\|_{H^p}$$

and it could be at most $\|f\|_{H^p}$ so taking $r \to 1$:

$$\|g\|_{H^p} \leq \|f\|_{H^p}$$

Together with (2.8) we have the equality. □

Definition 2.9. *Let be $f : \mathbb{D} \to \Omega$ an holomorphic function, let α be the area Lebesgue mesure normalized in $\mathbb{D}$, we define the Bergman norm for $0 < p \leq \infty$ as*

$$\|f\|_{A^p} := \|f\|_{L^p(\alpha)} = \left(\int_{\mathbb{D}} |f(z)|^p \frac{dA(z)}{\pi}\right)^{1/p}$$

and the Bergman space A^p is

$$A^p = \{f \in H(\mathbb{D}) : \|f\|_{A^p} < \infty\}$$

There is a very special case, and that is when $p = 2$. Since $f = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$ and z^n are orthogonal in H^2 and A^2, we can characterize the Hardy and Bergman norms in terms of the coefficients a_n:

Proposition 2.10. *Let $f \in H(\mathbb{D})$ such that $f = \sum_{n=0}^{\infty} a_n z^n$, then*

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2, \quad \|f\|_{A^2}^2 = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}$$

Proof. Let $0 \leq r < 1$, we have uniform convergence in $|z| = r$ de $f(re^{i\theta})$ and $\overline{f(re^{i\theta})}$, so that

$$|f(re^{i\theta})|^2 = f(re^{i\theta})\overline{f(re^{i\theta})} = \sum_{n,m=0}^{\infty} a_n \overline{a_m} r^{n+m} e^{i(n-m)\theta}$$

also converges uniformly in $|z| = r$ and therefore we can exchange the sum and the integral. Since we have $\int_0^{2\pi} e^{i(n-m)\theta} d\theta = 0$ if $n \neq m$, then

$$\|f\|^2_{H^2(\mu)} = \lim_{r\to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} = \lim_{r\to 1^-} \int_0^{2\pi} \sum_{n,m=0}^{\infty} a_n \overline{a_m} r^{n+m} e^{i(n-m)\theta} \frac{d\theta}{2\pi}$$
$$= \lim_{r\to 1^-} \sum_{n,m=0}^{\infty} a_n \overline{a_m} r^{n+m} \int_0^{2\pi} e^{i(n-m)\theta} \frac{d\theta}{2\pi} = \lim_{r\to 1^-} \sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \sum_{n=0}^{\infty} |a_n|^2$$

$$\|f\|^2_{A^2(\alpha)} = \int_{\mathbb{D}} |f(z)|^2 \frac{dA(z)}{\pi} = \int_0^1 \int_0^{2\pi} \sum_{n,m=0}^{\infty} a_n \overline{a_m} r^{n+m} e^{i(n-m)\theta} r \frac{d\theta}{\pi} dr$$
$$= \int_0^1 \sum_{n,m=0}^{\infty} a_n \overline{a_m} r^{n+m+1} \left(\int_0^{\pi} e^{i(n-m)\theta} \frac{d\theta}{2\pi} \right) dr =$$
$$= \sum_{n=0}^{\infty} |a_n|^2 \int_0^1 r^{2n+1} dr = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}$$

□

We will also use the following version of the Cauchy-Schwarz inequality:

Theorem 2.11 (Cauchy-Schwarz). *Let be $u, v \in \mathbb{C}^{n+1}$, then*

$$|\langle u, v\rangle|^2 = \left| \sum_{k=0}^{n} u_k \overline{v_k} \right|^2 \leq \langle u, u\rangle \langle v, v\rangle = \sum_{k=0}^{n} |u_k|^2 \sum_{k=0}^{n} |v_k|^2$$

and we have the equality if and only if $\exists C \in \mathbb{C}$ such that $u_k = Cv_k$ $\forall k = 0, ..., n$.

2.2 A proof of the isoperimetrical inequality using power series

Let $\Omega \subseteq \mathbb{R}^2$ be a domain such that $\partial\Omega$ is a rectificable Jordan curve (that is, a simply connected curve) and smooth (we will only need $\mathcal{C}^1$), then from the Riemann Mapping Theorem [19] we have a conformal, analytical and bijective $F : \mathbb{D} \to \Omega$ and we can relate the Hardy and Bergman norms with the perimeter and area of Ω, respectively: let be $\gamma_r(\theta) = F(re^{i\theta})$,

$$L(\partial\Omega) = \lim_{r\to 1^-} L(F(\{|z| = r\})) = \lim_{r\to 1^-} \int_0^{2\pi} |\gamma_r'(\theta)| d\theta =$$
$$= \lim_{r\to 1^-} \int_0^{2\pi} |rie^{i\theta} F'(re^{i\theta})| d\theta = \lim_{r\to 1^-} r \int_0^{2\pi} |F'(re^{i\theta})| d\theta = 2\pi \|F'\|_{H^1}$$

$$A(\Omega) = \int_\Omega 1 \cdot dA(w) = \int_\Omega 1 \cdot \frac{dw \wedge d\bar{w}}{-2i} = \int_{\mathbb{D}} \frac{F'(z)dz \wedge \overline{F'(z)}d\bar{z}}{-2i} =$$
$$= \int_{\mathbb{D}} |F'(z)|^2 \frac{dz \wedge d\bar{z}}{-2i} = \int_{\mathbb{D}} |F'(z)|^2 dA(z) = \pi \|F'\|^2_{A^2}, \tag{2.9}$$

where we have made the change of variable $w = F(z)$ and we have used that $dz \wedge d\bar{z} = (dx + idy) \wedge (dx - idy) = -i(dx \wedge dy) + i(dy \wedge dx) = -2idx \wedge dy = -2i\, dA(z)$.

The Hardy i Littlewood theorem (1932) that we will use is the following [5]:

Theorem 2.12. *Let $p \in (0, \infty)$, if $f \in H^p$ then $\|f\|_{A^{2p}} \le \|f\|_{H^p}$ and we have an equality if and only if*

$$f(z) = C \cdot \left(\frac{1}{1 - \lambda z}\right)^{2/p}, \quad |\lambda| < 1, \quad C \in \mathbb{C}$$

Proof. We will first take $p = 2$. We have

$$\|f\|^4_{A^4} = \int_{\mathbb{D}} |f(z)|^4 \frac{dA(z)}{\pi} = \int_{\mathbb{D}} |f(z)^2|^2 \frac{dA(z)}{\pi} = \|f^2\|^2_{A^2}$$

Now, using twice that

$$\left(\sum_{k=0}^\infty a_k z^k\right)\left(\sum_{m=0}^\infty a_m z^m\right) = \sum_{k=0}^\infty \sum_{m=0}^\infty a_k a_m z^{k+m} =$$
$$= \sum_{k=0}^\infty \sum_{n=k}^\infty a_k a_{n-k} z^n = \sum_{n=0}^\infty \sum_{k=0}^n a_k a_{n-k} z^n \tag{2.10}$$

and the Cauchy-Schwarz inequality for the vectors $v = (a_0 a_n, a_1 a_{n-1}, ..., a_n a_0)$ and $u = (\frac{1}{\sqrt{n+1}}, ..., \frac{1}{\sqrt{n+1}})$,

$$\|f^2\|^2_{A^2} = \left\| \left(\sum_{k=0}^\infty a_k z^k\right)\left(\sum_{m=0}^\infty a_m z^m\right) \right\|^2_{A^2} =$$
$$= \left\| \sum_{n=0}^\infty \left(\sum_{k=0}^n a_k a_{n-k}\right) z^n \right\|^2_{A^2} =$$
$$= \sum_{n=0}^\infty \frac{|\sum_{k=0}^n a_k a_{n-k}|^2}{n+1} \le \sum_{n=0}^\infty \sum_{k=0}^n |a_k|^2 |a_{n-k}|^2 =$$
$$= \left(\sum_{m=0}^\infty |a_m|^2\right)\left(\sum_{n=0}^\infty |a_n|^2\right) = \|f\|^4_{H^2},$$

Therefore, $\|f\|_{A^4} \leq \|f\|_{H^2}$. We have an equality if and only if the Cauchy-Schwarz inequality is an equality for every $n \geq 2$, that is, if $\forall n \geq 2$,

$$a_k a_{n-k} = \frac{C_n}{\sqrt{n+1}} = D_n, \qquad C_n, D_n \in \mathbb{C}, \quad \forall k = 0, ..., n \tag{2.11}$$

Now, if $a_0 = 0$ we have $f \equiv 0$, since $a_n^2 = a_0 \cdot a_{2n} = 0 \; \forall n \in \mathbb{N}$. Therefore, we can assume $a_0 \neq 0$, and since we have from (2.11) that

$$a_0 a_n = a_1 a_{n-1} \Rightarrow a_n = \frac{a_1}{a_0} a_{n-1} \quad \forall n = 1, 2, 3, ...$$

then $a_n = \left(\frac{a_1}{a_0}\right)^n a_0$ and as we wanted to see,

$$f(z) = \sum_{n=0}^{\infty} a_0 \left(\frac{a_1}{a_0}\right)^n z^n = \frac{a_0}{1 - \lambda z}.$$

Now we take $p \in (0, \infty)$, we assume $f(z) \neq 0 \; \forall z \in \mathbb{D}$, since $\mathbb{D}$ is simply connected we can choose a branch of $\log(f)$ [19, Teorema 13.11] and therefore of $f^{p/2} = \exp(\frac{p}{2} \log(f))$. Then,

$$\|f\|_{A^{2p}}^{p/2} = \left(\int_{\mathbb{D}} |f(z)|^{2p} \frac{dA(z)}{\pi}\right)^{1/4} = \|f^{p/2}\|_{A^4} \leq$$

$$\leq \|f^{p/2}\|_{H^2} = \lim_{r \to 1^-} \left(\int_0^{2\pi} |f(re^{i\theta})^{\frac{p}{2}}|^2 \frac{d\omega}{2\pi}\right)^{1/2} = \|f\|_{H^p}^{p/2}$$

and we have the equality if and only if $f^{p/2} = \left(\frac{a_0}{1-\lambda z}\right) \Leftrightarrow f = \left(\frac{a_0}{1-\lambda z}\right)^{2/p}$.

Finally, if f has roots (z_n) in $\mathbb{D}$, by the Proposition 2.8 we can factorize $f = Bg$ with B the Blaschke product and g is a function without roots such that $\|f\|_{H^p} = \|f/B\|_{H^p} = \|g\|_{H^p}$. Since $|B(z)| < 1$ if $|z| < 1$, we get

$$\frac{\|f\|_{A^{2p}}}{\|f\|_{H^p}} = \frac{\|Bg\|_{A^2p}}{\|g\|_{H^p}} < \frac{\|g\|_{A^2p}}{\|g\|_{H^p}} \leq 1$$

□

Isoperimetric Inequality I. In order to obtain the isoperimetric inequality we only have to use the theorem with $p = 1$ and $f = F'$:

$$A(\Omega) = \pi \|F'\|_{A^2}^2 \leq \pi \|F'\|_{H^1}^2 = \frac{L(\partial\Omega)^2}{4\pi}$$

and we have an equality if and only if

$$F'(z) = \frac{C}{(1 - \lambda z)^2} \Rightarrow F(z) = \frac{D}{1 - \lambda z} + A, \quad A, C, D, \lambda \in \mathbb{C}$$

which is a Möbius trasformation and a translation and therefore F sends disks to disks or half-planes, so $\Omega = F(\mathbb{D})$ is a disk. □

2.3 Another proof of the isoperimetrical inequality using Fourier series

Another proof of the isoperimetrical inequality using Fourier series [6]. In particular we will prove:

Theorem 2.13 (Desigualtat isoperimètrica II). *Let be $\Omega \subseteq \mathbb{R}^2$ such that $\partial\Omega$ is a smooth Jordan curve (here we need $\mathcal{C}^3$), then*

$$A(\Omega) \leq \frac{L(\partial\Omega)^2}{4\pi}$$

and we have a equality if and only if Ω is a disk.

Proof. Let $\gamma([0,2\pi]) = \partial\Omega$ be parameterized by arc length and we assume $L(\partial\Omega) = 2\pi$ (we are able to do it because we can scale by any λ and the inequality holds, $A(\bar{\Omega}) = \lambda^2 A(\Omega) \leq \frac{(\lambda L(\partial\Omega))^2}{4\pi} = \frac{L(\partial\bar{\Omega})^2}{4\pi}$). Therefore, since $\gamma \in L^2([0,2\pi])$,

$$\partial\Omega = \{\gamma(s) = \sum_{-\infty}^{\infty} c_n e^{ins}\}, \quad |\gamma'(s)| = 1, \quad c_n \in \mathbb{C}$$

with $c_n = \int_0^{2\pi} \gamma(s) e^{-ins} \frac{ds}{2\pi}$.

The coefficients $|c_n|$ are bounded by $C/|n|^3$: if we integrate by parts (we know $\gamma(s)$ has continuous derivatives) and using that $\gamma(0) = \gamma(2\pi)$),

$$\int_0^{2\pi} \gamma(s) e^{-ins} ds = \gamma(s) \frac{e^{-ins}}{-in}\Big|_0^{2\pi} - \int_0^{2\pi} \gamma'(s) \frac{e^{-ins}}{-in} ds =$$
$$= \frac{\gamma(2\pi) - \gamma(0)}{-in} + \int_0^{2\pi} \gamma'(s) \frac{e^{-ins}}{in} ds = \int_0^{2\pi} \gamma'(s) \frac{e^{-ins}}{in} ds$$

so

$$\int_0^{2\pi} \gamma(s) e^{-ins} ds = \int_0^{2\pi} \gamma'(s) \frac{e^{-ins}}{in} ds = \int_0^{2\pi} \gamma''(s) \frac{e^{-ins}}{-n^2} ds = \int_0^{2\pi} \gamma'''(s) \frac{e^{-ins}}{-in^3} ds$$

Therefore,

$$|c_n| \leq \int_0^{2\pi} \left| \gamma'''(s) \frac{e^{-ins}}{-in^3} \right| ds = \frac{1}{|n|^3} \int_0^{2\pi} \left| \gamma'''(s) \right| ds \leq \frac{2\pi}{|n|^3} \sup_{s \in [0,2\pi]} |\gamma'''(s)| < \frac{C}{|n|^3}$$

with $C \in \mathbb{R}$. This allows us to prove that $\gamma'(s) = \sum_{n=1}^{\infty} c_n i n e^{ins} - \sum_{n=1}^{\infty} c_n i n e^{-ins}$ is uniformly convergent by the Weierstrass M-test [1, p. 37], since

$$|c_n i n e^{ins}| = |c_n n| \leq \frac{1}{n^2}, \quad |c_n i n e^{-ins}| \leq \frac{1}{n^2} := M_n$$

and $\sum_{n=1}^{\infty} M_n < \infty$. For the same reason, $\overline{\gamma'(s)}$ and $\overline{\gamma(s)} = \sum_{-\infty}^{\infty} \overline{c_n} e^{-ins}$ also converge uniformly.

Now we can swap sums and integrals. On one hand,

$$1 = \int_0^{2\pi} |\gamma'(s)|^2 \frac{ds}{2\pi} = \int_0^{2\pi} \gamma'(s)\overline{\gamma'(s)} \frac{ds}{2\pi} = \int_0^{2\pi} \left(\sum_{-\infty}^{\infty} c_n ine^{ins} \right) \overline{\left(\sum_{-\infty}^{\infty} c_m ime^{ims} \right)} \frac{ds}{2\pi}$$

and using the uniform convergence of $\gamma'(s)$ and $\overline{\gamma'(s)}$ in $\mathbb{D}$,

$$\begin{aligned} 1 &= \int_0^{2\pi} \left(\sum_{-\infty}^{\infty} c_n ine^{ins} \right) \overline{\left(\sum_{-\infty}^{\infty} c_n ine^{ins} \right)} \frac{ds}{2\pi} = \\ &= \int_0^{2\pi} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n \overline{c_m} nme^{i(n-m)s} \frac{ds}{2\pi} = \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n \overline{c_m} nm \int_0^{2\pi} e^{i(n-m)s} \frac{ds}{2\pi} = \sum_{-\infty}^{\infty} |c_n|^2 n^2 \qquad (2.12) \end{aligned}$$

On the other hand, by Green's Theorem to find the area,

$$\begin{aligned} A(\Omega) &= \frac{1}{2} \int_{\partial\Omega} xdy - ydx = \frac{1}{2} \operatorname{Im} \left(\int_{\partial\Omega} (x - iy)(x'(s) + iy'(s))ds \right) = \\ &= \frac{1}{2} \operatorname{Im} \left(\int_{\partial\Omega} \bar{z}(s) z'(s) ds \right) = \qquad (2.13) \\ &= \frac{1}{2} \operatorname{Im} \left(\int_0^{2\pi} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (\overline{c_n} e^{-ins})(c_m ime^{ims}) ds \right) = \\ &= \frac{1}{2} \operatorname{Im} \left(\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \overline{c_n} c_m im \int_0^{2\pi} e^{i(m-n)s} ds \right) = \\ &= \frac{1}{2} \sum_{n=-\infty}^{\infty} n|c_n|^2 2\pi = \pi \sum_{n=-\infty}^{\infty} n|c_n|^2 \qquad (2.14) \end{aligned}$$

and we have used the orthogonality of the base $\{e^{ins}\}_{n\in\mathbb{Z}}$.

Since $n \le n^2$, if we compare both (2.12) and (2.13) we see that

$$A(\Omega) \le \pi = \frac{(2\pi)^2}{4\pi} = \frac{L(\partial\Omega)^2}{4\pi}$$

and we have an equality if and only if $c_n = 0\ \forall n \ne 0, 1$, that is, $\partial\Omega = \{c_0 + c_1 e^{ins}\}$ which is a circle of center c_0 and radius $|c_1|$. □

Chapter 3

The Saint-Venant inequality

In this chapter we introduce the Saint-Venant inequality, a geometrical inequality that relates the area with the torsional rigidity, another geometrical quantity that depends only on the shape of a simply connected domain. We start by defining the torsional rigidity in several ways and showing they are indeed equivalent. Then we will prove the Saint-Venant inequality in two ways that follow the same idea as the proofs in Chapter 2, one using the coefficients of the Riemann map and a more heuristic one using a specific parameterization and Green's theorem.

3.1 The Torsional Rigidity

We consider Ω a plane simply connected domain. We have to conceive it as the cross-section of a uniform and isotropic cylinder twisted around an axis perpendicular to Ω. The resistance to the twist offered by the shape of the cross-section is what is called torsional rigidity and it is a purely geometric quantity [21, p. 2], independent of the units of mass and time. This means that given any shape we can associate this constant with it. There are several ways to define this quantity mathematically, the most classical being a variational definition among all the smooth functions that cancel at the boundary $\partial\Omega$:

Definition 3.1. *Let $f : \Omega \to \mathbb{R}$ smooth enough, the torsional rigidity of Ω is*

$$\rho_\Omega = \sup_{f(\partial\Omega)=0} \frac{4\left(\int_\Omega f dxdy\right)^2}{\int_\Omega f_x^2 + f_y^2 dxdy} \tag{3.1}$$

Another definition (and the one that we will use the most) is a definition in terms of a partial differential equation:

Definition 3.2. *Let Ω be a simply connected domain, let v be the solution of*

$$\begin{cases} \Delta v = v_{xx} + v_{yy} = -2 \\ v|_{\partial\Omega} = 0 \end{cases} \tag{3.2}$$

then the torsional rigidity of Ω *is*

$$\rho'_\Omega = 2\int_\Omega v dxdy$$

In fact, this function is the one that maximizes (3.1) in domains with a smooth boundary and the supremum turns into a maximum. Let's see that indeed both definitions are equivalent [18]:

Proposition 3.3. *Let* ρ_Ω *be the torsional rigidity of a simply connected domain* Ω *with smooth boundary as in (3.1), then if* v *is the solution of (3.2),*

$$\rho_\Omega = 2\int_\Omega v dxdy$$

Proof. Let $f : \Omega \to \mathbb{R}$ be a smooth function such that $f(\partial\Omega) = 0$. First of all, we observe that integrating by parts and by the Cauchy-Schwarz inequality we have

$$\begin{aligned}\int_\Omega 2f\ dxdy = \int_\Omega (-\Delta v) f\ dxdy =\\ = \int_\Omega \nabla v \nabla f\ dxdy \leq \left(\int_\Omega |\nabla v|^2 dxdy \int_\Omega |\nabla f|^2 dxdy\right)^{1/2}\end{aligned}$$

and therefore

$$\frac{4\left(\int_\Omega f\ dxdy\right)^2}{\int_\Omega |\nabla f|^2 dxdy} \leq \int_\Omega |\nabla v|^2 dxdy$$

Since $\Delta v = -2$ and integrating by parts we have the identity:

$$2\int_\Omega v\ dxdy = -\int_\Omega v\Delta v\ dxdy = \int_\Omega |\nabla v|^2\ dxdy$$

Then we have seen that $\rho_\Omega \leq 2\int_\Omega v\ dxdy$. On the other hand, if we choose $f = v$ we obtain:

$$\rho_\Omega \geq \frac{4\left(\int_\Omega v\ dxdy\right)^2}{\int_\Omega |\nabla v|^2\ dxdy} = \frac{4\left(\int_\Omega v\ dxdy\right)^2}{2\int_\Omega v\ dxdy} = 2\int_\Omega v\ dxdy$$

□

We can write v as $v = \phi - \frac{1}{2}(x^2 + y^2)$ with ϕ such that

$$\begin{cases} \Delta\phi = \phi_{xx} + \phi_{yy} = 0 & \text{en } \Omega, \\ \phi|_{\partial\Omega} = \frac{x^2+y^2}{2} \end{cases} \tag{3.3}$$

then ϕ is unique (because if there was a ϕ_2 that also works, then $g = \phi - \phi_2$ is 0 at the boundary and $\Delta g = \Delta\phi - \Delta\phi_2 = 0$ so by the maximum modulus principle for harmonic functions g has to be identically 0, and therefore $\phi = \phi_2$) and we can check that in effect (if it exists):

$$\begin{cases} \Delta(\phi - \frac{1}{2}(x^2+y^2)) = \phi_{xx} - 1 + \phi_{yy} - 1 = -2 \\ v_{|\partial\Omega} = \phi_{|\partial\Omega} - \frac{1}{2}(x^2+y^2) = 0 \end{cases}$$

Then, putting the torsional rigidity in terms of the ϕ we obtain:

$$\rho_\Omega = \int_\Omega 2\phi dxdy - \int_\Omega (x^2+y^2)dxdy \tag{3.4}$$

The strategy of the first proof will be writing ρ_Ω and the area with the coefficients of the function given by the Riemann mapping theorem, and then comparing directly the coefficients. This is why we are interested in writing another definition of the torsional rigidity with this coefficients. Given a simply connected domain, we will always refer to the analytical, bijective and conformal function given by the Riemann mapping theorem as $F : \mathbb{D} \to \Omega$ and its coefficients will be $F(z) = \sum_{n=0}^{\infty} a_n z^n$. Moreover, we assume it is uniformly convergent for $|z| \leq 1$ (see the discussion in Chapter 5, where we see that given F we can find an extension $\overline{F}$ in $\overline{\mathbb{D}}$ that is uniformly continuous and such that the image of $\{|z| = 1\}$ is $\partial\Omega$). First we will put ϕ in terms of the coefficients (a_n):

Lemma 3.4. *Let ϕ be the function that satisfies (3.3), let $F(z) = a_n z^n$ be the Riemann map just described, let be $z = re^{i\theta}$, then*

$$2\phi(z) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_k \overline{a_l} r^{|k-l|} e^{i(k-l)\theta} \tag{3.5}$$

Proof. We can find ψ such that

$$\begin{cases} \psi_x = \phi_y \\ \psi_y = -\phi_x \end{cases}$$

which are the Cauchy-Riemann equations and therefore ϕ is the real part of an analytic function G totally characterized by ϕ, except for an imaginary additive constant. We put it as $G = \phi + i\psi = \sum_{n=0}^{\infty} u_n z^n$. We can take the constant such that $u_0 \in \mathbb{R}$ making that $\psi(0) = 0$. Since $2\phi(z) = |z|^2$ in $\partial\Omega$ because of the definition of ϕ in (3.3), we can link it with $F(z)$. Let $|z| = 1$,

$$|F(e^{i\theta})|^2 = |z|^2 = 2\phi(z) = 2 \cdot \frac{G(z) + \overline{G(z)}}{2} = 2u_0 + \sum_{n=1}^{\infty} u_n z^n + \sum_{n=1}^{\infty} \overline{u_n}\, \overline{z}^n$$

and taking $a_m = 0$ if $m < 0$,

$$|F(e^{i\theta})|^2 = F(e^{i\theta})\overline{F(e^{i\theta})} = \sum_{k=0}^{\infty}\sum_{r=0}^{\infty} a_k \overline{a_r} e^{i(k-r)\theta} = \sum_{n=-\infty}^{\infty} \left(\sum_{k-r=n} a_k \overline{a_r} \right) e^{in\theta}$$

By the unicity of the Fourier series, the coefficients of the two series must be equal and we obtein the coefficients u_n:

$$2u_0 = \sum_{k-r=0} a_k\overline{a_r} = \sum_{k=0}^{\infty} |a_k|^2, \qquad u_n = \sum_{k-r=n} a_k\overline{a_r} \quad \forall n \in \mathbb{N}$$

which in effect satisfy $\overline{u_n} = \sum_{k-r=n} \overline{a_k}a_r = \sum_{r-k=-n} a_r\overline{a_k}$, that are the coefficients with negative n. Therefore, if $|z| \leq 1$,

$$\begin{aligned}
2\phi(re^{i\theta}) &= 2u_0 + \sum_{n=1}^{\infty} u_n z^n + \overline{u_n z^n} = \\
&= \sum_{k=0} |a_k|^2 + \sum_{\substack{k-r=n\\ n\in\mathbb{N}}} a_k\overline{a_r}r^n e^{in\theta} + \sum_{\substack{k-r=-n\\ n\in\mathbb{N}}} a_k\overline{a_r}r^n e^{-in\theta} = \\
&= \sum_{\substack{k-r=n\\ n\in\mathbb{Z}}} a_k\overline{a_r}r^{|n|}e^{in\theta} = \sum_{k=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} a_k\overline{a_r}r^{|k-r|}e^{i(k-r)\theta} = \\
&= \sum_{k=0}^{\infty}\sum_{r=0}^{\infty} a_k\overline{a_r}r^{|k-r|}e^{i(k-r)\theta}
\end{aligned}$$

since $a_m = 0$ if $m < 0$. □

Now we can define the torsional rigidity in terms of the corfficients of F:

Proposition 3.5. *Let Ω be a simply connected domain, we take the analytic function $F(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{D}$ given by the Riemann mapping theorem, then*

$$\rho_\Omega = \sum_{\substack{\alpha,\beta,\gamma,\delta>0,\\ \alpha+\beta=\gamma+\delta}} \min(\alpha,\beta,\gamma,\delta)a_\alpha a_\beta\overline{a_\gamma a_\delta} \tag{3.6}$$

Proof. We will start from the definition (3.4),

$$\rho_\Omega = \int_\Omega 2\phi dxdy - \int_\Omega x^2 + y^2 dxdy = \int_\Omega 2\phi dxdy - I_0$$

where I_0 is a quantity called polar moment of inertia of Ω about the origin. We will calculate the two terms of the sum independently. Developing the first integral and making the change of variable $w = F(z)$,

$$\int_\Omega 2\phi(w)dw = \int_{\mathbb{D}} 2\phi(F(z))|F'(z)|^2 dz = \int_0^1 \int_{-\pi}^{\pi} 2\phi(re^{i\theta})|F'(re^{i\theta})|^2 r dr d\theta =$$

$$= \int_0^1 \int_{-\pi}^{\pi} \left(\sum_{\alpha=0}^{\infty} \sum_{\gamma=0}^{\infty} a_\alpha \overline{a_\gamma} r^{|\alpha-\gamma|} e^{i(\alpha-\gamma)\theta} \right) F'(re^{i\theta})\overline{F'(re^{i\theta})} r dr d\theta =$$

$$= \int_0^1 \int_{-\pi}^{\pi} \left(\sum_{\alpha=0}^{\infty} \sum_{\gamma=0}^{\infty} a_\alpha \overline{a_\gamma} r^{|\alpha-\gamma|} e^{i(\alpha-\gamma)\theta} \right) \left(\sum_{\beta=0}^{\infty} \beta a_\beta r^{\beta-1} e^{i(\beta-1)\theta} \right) \left(\sum_{\delta=0}^{\infty} \delta \overline{a_\delta} r^{\delta-1} e^{-i(\delta-1)\theta} \right) =$$

$$= \int_0^1 \int_{-\pi}^{\pi} \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \sum_{\gamma=0}^{\infty} \sum_{\delta=0}^{\infty} \beta\delta a_\alpha a_\beta \overline{a_\gamma a_\delta} r^{|\alpha-\gamma|+\beta+\delta-1} e^{i(\alpha+\beta-\gamma-\delta)\theta} d\theta dr =$$

$$= 2\pi \sum_{\substack{\alpha,\beta,\gamma,\delta\geq 0 \\ \alpha+\beta=\gamma+\delta}} a_\alpha a_\beta \overline{a_\gamma a_\delta} \frac{\beta\delta}{|\alpha-\delta|+\beta+\delta} \tag{3.7}$$

Now if $\alpha \geq \gamma$, using that $\alpha + \beta = \gamma + \delta$ we get

$$\frac{2\beta\delta}{|\alpha-\gamma|+\beta+\delta} = \frac{2\beta\delta}{\alpha-\gamma+\beta+\delta} = \frac{2\beta\delta}{2\delta} = \beta$$

If $\alpha < \gamma$,

$$\frac{2\beta\delta}{|\alpha-\gamma|+\beta+\delta} = \frac{2\beta\delta}{\gamma-\alpha+\beta+\delta} = \frac{2\beta\delta}{2\beta} = \delta$$

Therefore, from both cases we arrive to:

$$\frac{2\beta\delta}{|\alpha-\delta|+\beta\delta} = \min(\beta,\delta) \Rightarrow P + I_0 = \pi \sum_{\substack{\alpha,\beta,\gamma,\delta\geq 0 \\ \alpha+\beta=\gamma+\delta}} a_\alpha a_\beta \overline{a_\gamma a_\delta} \min(\beta,\delta)$$

Now, making combinations we get 4 different expressions of (3.5) substituting β for α and δ for γ:

$$\rho_\Omega + I_0 = \pi \sum_{\substack{\alpha,\beta,\gamma,\delta\geq 0 \\ \alpha+\beta=\gamma+\delta}} a_\alpha a_\beta \overline{a_\gamma a_\delta} \min(\beta,\delta) = \pi \sum_{\substack{\alpha,\beta,\gamma,\delta\geq 0 \\ \alpha+\beta=\gamma+\delta}} a_\alpha a_\beta \overline{a_\gamma a_\delta} \min(\beta,\gamma) =$$

$$= \pi \sum_{\substack{\alpha,\beta,\gamma,\delta\geq 0 \\ \alpha+\beta=\gamma+\delta}} a_\alpha a_\beta \overline{a_\gamma a_\delta} \min(\alpha,\gamma) = \pi \sum_{\substack{\alpha,\beta,\gamma,\delta\geq 0 \\ \alpha+\beta=\gamma+\delta}} a_\alpha a_\beta \overline{a_\gamma a_\delta} \min(\alpha,\delta)$$

and we obtain the following expression:

$$4(\rho_\Omega + I_0) = \pi \sum_{\substack{\alpha,\beta,\gamma,\delta\geq 0 \\ \alpha+\beta=\gamma+\delta}} a_\alpha a_\beta \overline{a_\gamma a_\delta} (\min(\beta,\delta) + \min(\beta,\gamma) + \min(\alpha,\gamma) + \min(\alpha,\delta)) \tag{3.8}$$

For the second integral, that we called polar moment of inertia, changing to polar coordinates we obtain

$$I_0 = \int_0^1 \int_{-\pi}^{\pi} |F(z)F'(z)|^2 r d\theta dr \tag{3.9}$$

From (2.10) we can write $F(z)^2$ as:

$$F(z)^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) z^n$$

Then we can find the derivative respect z in both sides of the equality so that

$$2F(z)F'(z) = \sum_{n=0}^{\infty} n \sum_{k=0}^{n} a_k a_{n-k} z^{n-1}$$

and substituting in (3.7)

$$\begin{aligned} I_0 &= \int_0^1 \int_{-\pi}^{\pi} \frac{1}{4} \left(\sum_{n=0}^{\infty} n \sum_{k=0}^{n} a_k a_{n-k} z^{n-1} \right) \overline{\left(\sum_{m=0}^{\infty} m \sum_{k'=0}^{m} a_{k'} a_{m-k'} z^{m-1} \right)} r d\theta dr = \\ &= \int_0^1 \int_{-\pi}^{\pi} \frac{1}{4} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} nm \left(\sum_{k=0}^{n} a_k a_{n-k} \right) \left(\sum_{k'=0}^{n} \overline{a'_k a_{n-k'}} \right) r^{n+m-2} e^{i(n-m)\theta} d\theta r dr = \\ &= \int_0^1 \frac{\pi}{2} \sum_{n=0} n^2 \left(\sum_{k=0}^{n} a_k a_{n-k} \right) \overline{\left(\sum_{k'=0}^{n} a_{k'} a_{n-k'} \right)} r^{2n-1} dr = \\ &= \frac{\pi}{4} \sum_{n=0}^{\infty} n \left| \sum_{k=0}^{n} a_k a_{n-k} \right|^2 \end{aligned}$$

We can write this last expression as a summation like (3.6):

$$I_0 = \frac{\pi}{4} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{k'=0}^{n} n a_k a_{n-k} \overline{a_{k'} a_{n-k'}} = \frac{\pi}{4} \sum_{\substack{\alpha,\beta,\gamma,\delta \geq 0 \\ \alpha+\beta=\gamma+\delta}} (\alpha+\beta) a_\alpha a_\beta \overline{a_\gamma a_\delta} \tag{3.10}$$

Finally, from (3.8) and (3.10), we obtain the expression for the torsional rigidity:

$$\rho_\Omega = \frac{\pi}{4} \sum_{\substack{\alpha,\beta,\gamma,\delta \geq 0 \\ \alpha+\beta=\gamma+\delta}} a_\alpha a_\beta \overline{a_\gamma a_\delta} (\min(\alpha,\gamma) + \min(\alpha,\delta) + \min(\beta,\gamma) + \min(\beta,\delta) - \alpha - \beta)$$

Now by taking cases and using that $\alpha + \beta = \gamma + \delta$ we have:

$$\min(\alpha,\gamma) + \min(\alpha,\delta) + \min(\beta,\gamma) + \min(\beta,\delta) - \alpha - \beta = 2\min(\alpha,\beta,\gamma,\delta)$$

And therefore we get the expression we were searching:

$$\rho_\Omega = \frac{\pi}{2} \sum_{\substack{\alpha,\beta,\gamma,\delta \geq 0 \\ \alpha+\beta=\gamma+\delta}} \min(\alpha,\beta,\gamma,\delta) a_\alpha a_\beta \overline{a_\gamma a_\delta}$$

□

3.2 The Saint-Venant inequality

Our first proof of the Saint-Venant inequality will follow the same basic idea of the first proof of the isoperimetric inequality, using the coefficients of the Rieman mapping. As in the first chapter, we let Ω be a simply connected domain such that $\partial\Omega$ is a smooth Jordan curve. Then we can consider the Riemann mapping $F : \mathbb{D} \to \Omega$ that is conformal, analytic and bijective, and we write it as $F(z) = \sum_{n=0}^{\infty} a_n z^n$. We have already seen in (2.9) that we can put the area of Ω in terms of the coefficients:

$$A(\Omega) = \pi \|F'\|_{A^2}^2 = \pi \sum_{n=1}^{\infty} n|a_n|^2$$

and we will use (3.3) and (3.4) to bound the torsional rigidity with the area.

Theorem 3.6. *Let Ω be a domain like the one just described, let ρ_Ω and $A(\Omega)$ be the torsional rigidity and the area of Ω respectively, then*

$$2\pi\rho_\Omega \leq A^2$$

and we have the equality if and only if Ω is a disk.

Proof. If we choose $u_k = \min(\alpha,\beta,\gamma,\delta)^{1/2} a_\alpha a_\beta$ and $v_k = \min(\alpha,\beta,\gamma,\delta)^{1/2} a_\gamma a_\delta$ for all the $\alpha,\beta,\gamma,\delta \in \mathbb{N}$ satisfying $\alpha+\beta = \gamma+\delta$ then by Cauchy-Schwarz we have

$$\left| \sum_{\substack{\alpha,\beta,\gamma,\delta>0,\\ \alpha+\beta=\gamma+\delta}} \min(\alpha,\beta,\gamma,\delta) a_\alpha a_\beta \overline{a_\gamma a_\delta} \right|^2 \leq$$

$$\leq \left(\sum_{\substack{\alpha,\beta,\gamma,\delta>0,\\ \alpha+\beta=\gamma+\delta}} \min(\alpha,\beta,\gamma,\delta)|a_\alpha a_\beta|^2 \right) \left(\sum_{\substack{\alpha,\beta,\gamma,\delta>0,\\ \alpha+\beta=\gamma+\delta}} \min(\alpha,\beta,\gamma,\delta)|a_\gamma a_\delta|^2 \right) =$$

$$= \left(\sum_{\substack{\alpha,\beta,\gamma,\delta>0,\\ \alpha+\beta=\gamma+\delta}} \min(\alpha,\beta,\gamma,\delta)|a_\alpha a_\beta|^2 \right)^2$$

Developing the sum inside the square,

$$\sum_{\substack{\alpha,\beta,\gamma,\delta>0,\\ \alpha+\beta=\gamma+\delta}} \min(\alpha,\beta,\gamma,\delta)|a_\alpha a_\beta|^2 = \sum_{\alpha=1}^{\infty}\sum_{\beta=1}^{\infty} |a_\alpha a_\beta|^2 \left(\sum_{\gamma=1}^{\alpha+\beta} \min(\alpha,\beta,\gamma,\alpha+\beta-\gamma) \right) \tag{3.11}$$

Now we can remove the dependance on γ by developing the summation: we can assume without loss of generality that $\alpha \leq \beta$, then

$$\begin{aligned}\sum_{\gamma=1}^{\alpha+\beta} \min(\alpha,\beta,\gamma,\alpha+\beta-\gamma) &= \sum_{\gamma=1}^{\alpha-1}\gamma + \sum_{\gamma=\alpha}^{\beta}\alpha + \sum_{\gamma=\beta+1}^{\alpha+\beta}(\alpha+\beta-\gamma) = \\ &= \frac{(\alpha-1)\alpha}{2} + \alpha(\beta-\alpha+1) + \sum_{\gamma=1}^{\alpha-1}\gamma = \\ &= \frac{(\alpha-1)\alpha}{2} + \alpha\beta - \alpha(\alpha-1) + \frac{(\alpha-1)\alpha}{2} = \alpha\beta\end{aligned}$$

Therefore, (3.11) is left as:

$$\sum_{\alpha=1}^{\infty}\sum_{\beta=1}^{\infty}|a_\alpha a_\beta|^2\alpha\beta = \left(\sum_{\alpha=1}^{\infty}\alpha|a_\alpha|^2\right)\left(\sum_{\beta=1}^{\infty}\beta|a_\beta|^2\right) = \left(\sum_{n=1}^{\infty}n|a_n|^2\right)^2 = A(\Omega)^2$$

Finally, since we know ρ_Ω is a real positive number, we get the inequality we wanted:

$$\frac{2\rho_\Omega}{\pi} = \left|\sum_{\substack{\alpha,\beta,\gamma,\delta>0,\\ \alpha+\beta=\gamma+\delta}} \min(\alpha,\beta,\gamma,\delta)a_\alpha a_\beta \overline{a_\gamma a_\delta}\right| \leq A(\Omega)^2$$

For the equality, we do the same as in the first proof of the isoperimetric inequality: since we only used the Cauchy-Schwarz inequality, we know we have an equality if and only if the coefficients are proportional, that is, if

$$u_k = Cv_k \Rightarrow a_\alpha a_\beta = Ca_\gamma a_\delta, \qquad \text{for } \alpha+\beta=\gamma+\delta; \quad \alpha,\beta,\gamma,\delta = 1,2,3,...$$

In particular, for all n we have $a_n a_1 = Ca_{n-1}a_2$. On one hand, if $a_1 = 0$, since $a_1 a_{2n-1} = a_n^2$ we have $a_n = 0\ \forall n$. Then we can assume $a_1 \neq 0$ and we can find the a_n:

$$a_n = \left(\frac{a_2}{a_1}\right)a_{n-1} = \left(\frac{a_2}{a_1}\right)^2 a_{n-2} = ... = \left(\frac{a_2}{a_1}\right)^{n-1} a_1, \qquad n = 2,3,...$$

obtaining the function

$$F(z) = a_0 + a_1\sum_{n=0}^{\infty}\left(\frac{a_2}{a_1}\right)^n z^n = a_0 + \frac{a_1 z}{1-(a_2/a_1)z}$$

which a function that sends disks to disks or half-planes, and we know the image can not be a half-plane so the image must be a disk (and therefore, we have the equality if and only if Ω is a circle. □

3.3 Another proof of the Saint-Venant inequality

We will give another proof, without using the Riemann mapping, for simply connected domains with smooth boundary, based in a proof of Makai (see [11]). In this case, we will use the variational definition of the torsional rigidity (3.1) and we will develop the integrals. We recall that if $\partial\Omega$ is smooth (it is enough to be $\mathcal{C}^1$) then exists $v(x,y)$ such that we reach the supremum and therefore

$$\rho_\Omega = \frac{4\left(\int_\Omega v dxdy\right)^2}{\int_\Omega v_x^2 + v_y^2 dxdy} \tag{3.12}$$

and since this function must satisfy (3.3), it can not have local minimums and has to be positive inside Ω.

We can parameterize the level curves of $v(x,y)$ in the following way: given the level curve $v(x,y)=t$, we consider the domain $D=\{(x,y)\in\mathbb{R}^2 | f(x,y)>t\}$ and for each $\tau\in[0,A(\Omega)]$ we consider D_τ such that its area is τ. We will denote by C_τ the level curve of the boundary of D_τ and we define $\chi(\tau)=v(x,y)$ for $\tau\in[0,A]$, which is an increasing function and $\chi(A(\Omega))=0$. On the other hand, let s be the arc-length parameter of C_τ (which can be not connected but the different components must be connected because it can not have local minima) going from 0 to $L(C_\tau)$, the length of C_τ. We will assume that all these parameters are well-defined, but we should see that they are indeed differentiable except on a set of measure 0 and that it is not a problem for the proof. In particular, the biggest problem is in defining the parameter s, because we can have different components and when the parameter cross from one connected component to another it is not differentiable. Because of this, the proof is not completely rigorous. That said, we introduce these new coordinates τ and s instead of x and y, and we put the Jacobian as

$$\varDelta = \begin{vmatrix} \frac{\partial x}{\partial \tau} & \frac{\partial y}{\partial \tau} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{vmatrix} = \begin{vmatrix} x_\tau & y_\tau \\ x_s & y_s \end{vmatrix}$$

Notice that from these coordinates we have the following:

$$\begin{aligned} \frac{\partial v}{\partial \tau} &= \frac{\partial \chi}{\partial \tau} = \chi'(\tau) = v_x x_\tau + v_y y_\tau \\ \frac{\partial v}{\partial s} &= \frac{\partial \chi}{\partial s} = 0 = v_x x_s + v_y y_s \end{aligned} \tag{3.13}$$

and since it is the arc-length parameter, $x_s^2+y_s^2=1$. From this three expressions, developing (3.13) we have

$$\begin{aligned} 0 &= (v_x x_s + v_y y_s)^2 = v_x^2 x_s^2 + v_y^2 y_s^2 + 2 v_x v_y x_s y_s = \\ &= (x_s^2+y_s^2)(v_x^2+v_y^2) - v_y^2 x_s^2 - v_x^2 y_s^2 + 2 v_x v_y x_s y_s = \\ &= (x_s^2+y_s^2)(v_x^2+v_y^2) - (v_y x_s - v_x y_s)^2 \end{aligned}$$

and since $x_s^2 + y_s^2 = 1$, we get:

$$v_x^2 + v_y^2 = (v_y x_s - v_x y_s)^2$$

By expanding and using (3.13) and $x_s^2 + y_s^2 = 1$ we get

$$(v_y x_s - v_x y_s)^2 \Delta^2 = (v_y x_s - v_x y_s)^2 (x_\tau y_s - x_s y_\tau)^2 = ... = (v_x y_s + v_y y_\tau)^2 = \chi'(\tau)^2$$

so

$$v_x^2 + v_y^2 = (v_y x_s - v_x y_s)^2 = \frac{\chi'(\tau)^2}{\Delta^2}$$

We can finally make the change of variables and we get the integrals:

$$\int_\Omega v(x,y)dxdy = \int_0^A \int_0^{L(C_\tau)} \chi(\tau)|\Delta|dsd\tau = \int_0^A \chi(\tau) \int_0^{L(C_\tau)} |\Delta|dsd\tau \qquad (3.14)$$

$$\int_\Omega v_x^2 + v_y^2 dxdy = \int_0^A \int_0^{L(C_\tau)} \frac{\chi'(\tau)^2}{\Delta^2}|\Delta|dsd\tau = \int_0^A \int_0^{L(C_\tau)} \frac{\chi'(\tau)^2}{|\Delta|}dsd\tau \qquad (3.15)$$

which can be improper integrals because Δ can vanish at the boundary of Ω but in any case it can only vanish in a countably set of points.

Now we will find two inequalities and one equality that involve integrals and functions of the parameterization in τ and s (see [10]). The first one, using the Cauchy-Schwarz inequality for integrals we get

$$L(C_\tau)^2 = \left(\int_0^{L(C_\tau)} ds\right)^2 \leq \left(\int_0^{L(C_\tau)} |\Delta|ds\right)\left(\int_0^{L(C_\tau)} \frac{1}{|\Delta|}ds\right) \qquad (3.16)$$

On the other hand, we take τ_0 and we consider the curves $L(C_{\tau_0})$ and $L(C_{\tau_0+\epsilon})$. We know the area of the ring (or rings) $R_{\tau_0,\epsilon}$ between both curves are exactly $\tau_0 + \epsilon - \tau_0 = \epsilon$, so that

$$\epsilon = \int_{R_{\tau_0,\epsilon}} dxdy = \int_{\tau_0}^{\tau_0+\epsilon} \int_0^{L(C_\tau)} |\Delta|dsd\tau$$

By the mean value theorem for integrals, exists $c \in (\tau_0, \tau_0 + \epsilon)$ such that

$$\int_{\tau_0}^{\tau_0+\epsilon} \left(\int_0^{L(C_{\tau_0})} |\Delta|ds\right) d\tau = \left(\int_0^{L(C_c)} |\Delta|ds\right)(\tau_0 + \epsilon - \tau_0) = \epsilon \int_0^{L(C_c)} |\Delta|ds$$

Making ϵ go to 0 we get the equality we wanted:

$$\epsilon = \epsilon \int_0^{L(C_c)} |\Delta|ds \Rightarrow 1 = \int_0^{L(C_c)} |\Delta|ds \quad \xrightarrow{\epsilon\to 0} \quad 1 = \int_0^{L(C_{\tau_0})} |\Delta|ds \qquad (3.17)$$

Finally, the last inequality we will use involves the radius of the biggest circle inscribed in Ω that we will call $\sigma = \sigma(A(\Omega))$ (in the same way, we will call $\sigma(\tau)$ to the radius of the biggest cercle inscribed in D_τ) and S will be the area of this circle (that is, $S = \pi\sigma(\tau)^2$). On one hand, by the isoperimetric inequality we have

$$L(C_\tau)^2 \geq 4\pi\tau \quad \Rightarrow \quad L(C_\tau) \geq \sqrt{4\pi\tau}, \qquad 0 \leq \tau \leq A_\Omega \tag{3.18}$$

On the other hand, we have the geometrical inequality $L(C_\tau) \geq \frac{\tau}{\sigma(\tau)} + \pi\sigma(\tau)$ (see [2, p. 3]) and because of how we defined τ, if $\tau' \leq \tau''$ then $D_{\tau'} \subseteq D_{\tau''}$. Therefore, $\sigma(\tau') \leq \sigma(\tau'')$. We put $g(x) = x + \frac{1}{x}$ and we notice that it is non-decreasing for $x > 1$. If $\tau \geq S = \pi\sigma^2$, then

$$\frac{\tau}{\pi\sigma(\tau)^2} \geq \frac{\tau}{\pi\sigma^2} \geq 1 \quad \Rightarrow \quad \frac{\sqrt{\tau}}{\sqrt{\pi}\sigma(\tau)} \geq \frac{\sqrt{\tau}}{\sqrt{\pi}\sigma} \geq 1$$

so

$$\begin{aligned} L(C_\tau) &\geq \frac{\tau}{\sigma(\tau)} + \pi\sigma(\tau) = \\ &= \sqrt{\pi\tau}\, g\left(\frac{\sqrt{\tau}}{\sqrt{\pi}\sigma(\tau)}\right) \geq \sqrt{\pi\tau}\, g\left(\frac{\sqrt{\tau}}{\sqrt{\pi}\sigma}\right) = \frac{\tau}{\sigma} + \pi\sigma, \qquad \tau \geq S \end{aligned} \tag{3.19}$$

Taking into account (3.18) and (3.19) we get the following inequality:

$$L(C_\tau) \geq M(\tau) = \begin{cases} \sqrt{4\pi\tau} & \text{si } 0 \leq \tau \leq S \\ \frac{\tau}{\sigma} + \pi\sigma & \text{si } S \leq \tau \leq A \end{cases} \tag{3.20}$$

Now, we can develop (3.14) and (3.15) by using (3.16), (3.17) and (3.20) (and integrating by parts in (3.14)):

$$\int_\Omega v(x,y)dxdy = \int_0^A \chi(\tau)d\tau = A \cdot \chi(A) - 0 \cdot \chi(0) - \int_0^A \tau\chi'(\tau)d\tau = -\int_0^A \chi'(\tau)d\tau$$

$$\begin{aligned} \int_\Omega v_x^2 + v_y^2 dxdy &= \int_0^A \chi'(\tau)^2 \int_0^{L(C_\tau)} \frac{1}{|\Delta|} dsd\tau = \\ &= \int_0^A \chi'(\tau)^2 \int_0^{L(C_\tau)} \frac{1}{|\Delta|} \left(\int_0^{L(C_\tau)} |\Delta| ds\right) dsd\tau \geq \\ &\geq \int_0^A \chi'(\tau)^2 L(C_\tau)^2 d\tau \geq \int_0^A \chi'(\tau)^2 M(\tau)^2 d\tau \end{aligned}$$

Directly from the definition (3.12) and using in the last step the Cauchy-Schwarz inequality we get:

$$\begin{aligned} \rho_\Omega &= \frac{4\left(\int_\Omega v dxdy\right)^2}{\int_\Omega v_x^2 + v_y^2 dxdy} \leq \frac{4\left(\int_0^A \tau\chi'(\tau)d\tau\right)^2}{\int_0^A \chi'(\tau)^2 M(\tau)^2 d\tau} = \\ &= \frac{4\left(\int_0^A \frac{\tau}{M(\tau)}\chi'(\tau)M(\tau)d\tau\right)^2}{\int_0^A \chi'(\tau)^2 M(\tau)^2 d\tau} \leq 4\int_0^A \left(\frac{\tau}{M(\tau)}\right)^2 d\tau \end{aligned}$$

and from the definition of $M(\tau)$ we can calculate the integral so that (we recall $S = \pi\sigma^2$):

$$\int_0^A \left(\frac{\tau}{M(\tau)}\right)^2 d\tau = \int_0^S \frac{\tau}{4\pi} d\tau + \int_S^A \left(\frac{\tau}{\frac{\tau}{\sigma} + \pi\sigma}\right)^2 d\tau$$
$$= \frac{1}{8\pi} S^2 + \frac{1}{(\pi\sigma)^2} \int_S^A \frac{\tau^2}{(\frac{\tau}{S}+1)^2} d\tau = \frac{S^2}{8\pi} + \frac{S^2}{\pi} \int_1^{\frac{A}{S}} \frac{\tau^2}{(\tau+1)^2} d\tau$$

A primitive of the last integral is

$$g(x) = x - \frac{1}{x+1} - 2\log(|x+1|)$$

so that, because $g(1) = 1 - \frac{1}{2} - 2\log(2) = \frac{1}{2} - 2\log(2)$,

$$\int_0^A \left(\frac{\tau}{M(\tau)}\right)^2 = \frac{S^2}{\pi} \left(\frac{1}{8} + g\left(\frac{A}{S}\right) + 2\log(2) - \frac{1}{2}\right) =$$
$$= \frac{S^2}{\pi} \left(-\frac{3}{8} + \frac{A}{S} - \frac{1}{\frac{A}{S}+1} - 2\log(\frac{\frac{A}{S}+1}{2})\right)$$

Therefore, writing the terms of the Saint-Venant inequality we get:

$$A^2 - 2\pi\rho_\Omega \geq A^2 - 8S^2 \left(-\frac{3}{8} + \frac{A}{S} - \frac{1}{\frac{A}{S}+1} - 2\log(\frac{\frac{A}{S}+1}{2})\right) =$$
$$= S^2 \left(\frac{A^2}{S^2} + 3 - 8\frac{A}{S} + \frac{8}{\frac{A}{S}+1} + 16\log(\frac{\frac{A}{S}+1}{2})\right) = S^2 h\left(\frac{A}{S}\right) \quad (3.21)$$

with $h(x) = x^2 - 8x + 3 + \frac{8}{x+1} + 16\log(\frac{x+1}{2})$. Since $h(1) = 0$ and

$$h'(x) = 2x - 8 - \frac{8}{(x+1)^2} + \frac{16}{x+1} = \frac{2(2x-8)(x+1)^2 - 8 + 16(x+1)}{(x+1)^2} = \frac{2x(x-1)^2}{(x+1)^2}$$

we can write (3.21) as an integral and we get

$$A^2 - 2\pi\rho_\Omega \geq S^2(h\left(\frac{A}{S}\right) - h(1)) =$$
$$= S^2 \int_1^{\frac{A}{S}} h'(x)dx = S^2 \int_1^{\frac{A}{S}} \frac{2x(x-1)^2}{(x+1)^2} dx \geq 0$$

which is what we wanted to see. Moreover, we see that it is an equality only if $A/S = 1$, that is, if $A = S$ and since S is the area of the biggest inscribed circle, if they are the same must be because Ω is a circle.

Chapter 4

An operator theory approach

We have already proved the isoperimetric inequality and the Saint-Venant inequality in two ways, using complex analysis techniques. Our goal will be using operator theory to prove both inequalities and we will see how they are very related and that with the same tools we can easily prove another inequality involving the principal frequency, that we will explain further. In particular, we will use the commutator of the operator $T_z(f) := zf$ in different spaces (we will extend the Hardy and Bergman spaces to any Ω) to obtain different bounds and get the inequalities we want. In this chapter we will strongly use the fact that H^2 and A^2 (and the extensions we will use) are Hilbert spaces, which is what allow us to work easily with them.

4.1 Preliminaries

First of all, lets introduce the notation we will use and some basic definitions. Let $T : H \to H$ be a lineal bounded operator wehre H is a Hilbert space with the inner product $\langle , \rangle : H \times H \to \mathbb{C}$ and we denote $\|\cdot\|$ the usual norm induced in H by the inner product. Let $I : H \to H$ be the identity operator. We call the spectrum of T to $sp(T) = \{z \in \mathbb{C} : T - zI \text{ no és invertible}\}$. We denote by T^* the adjoint operator of T, which is the one that satisfies $\langle Tx, y\rangle = \langle x, T^*y\rangle \ \forall x, y \in H$, and then we can define the commutator of T as the operator $[T^*, T] := T^*T - TT^*$. An operator is normal if $T^*T = TT^*$ (that is, if $\|[T^*, T]\| = 0$) so we can see $\|[T^*, T]\|$ as a measure of the abnormality of T. We will say that T is positive if $\langle Tx, x\rangle \geq 0$ for all $x \in H$ and we will write it as $T \geq 0$. For normal operators we have the following property:

Lemma 4.1. *Let $T : H \to H$ be a normal operator, then*

$$\left\|T^2\right\| = \|T\|^2$$

Proof. Let be $y \in H$, we observe that $\|Ty\|^2 = \langle Ty, Ty\rangle = \langle y, T^*Ty\rangle = \langle y, T^*Ty\rangle = \langle T^*y, T^*y\rangle = \|T^*y\|^2$, so $\|T\|^2 = \|T^*\|^2$ and therefore $\|T\| = \|T^*\|$. If we apply this to $y = Tx$ we get $\left\|T^2x\right\| = \|T^*Tx\|$ so $\left\|T^2\right\| = \|T^*T\|$. Now on one hand, $\left\|T^2\right\| \leq \|T\|^2$. On the other hand, $\forall x \in H$,

$$\|Tx\|^2 = \langle x, T^*Tx\rangle \leq \|x\| \|T^*Tx\| \leq \|T^*T\| \|x\|^2 \Rightarrow \|T\|^2 \leq \|T^*T\| = \left\|T^2\right\|$$

□

We observe that if T is normal, then T^2 is also normal: since $\langle T^2x, y\rangle = \langle x, T^{*2}y\rangle$, then $(T^2)^* = T^{*2}$ and therefore

$$T^2(T^2)^* = T^2T^{*2} = TTT^*T^* = (TT^*)(TT^*) = T^*(TT^*)T = T^{*2}T^2 = (T^2)^*T^2$$

By reiterating the last equality, we get $\left\|T^{2^n}\right\| = \|T\|^{2^n}$ $\forall n \in \mathbb{N}$.

The following lemma is another property of normal operators that we will use later:

Lemma 4.2. *Let $T : H \to H$ be a lineal normal operator with H a Hilbert space, then*

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$$

Proof. We set $M := \sup_{\|x\|=1} |\langle Tx, x\rangle|$. On one hand, by the Cauchy-Schwarz inequality, let $x \in H$ such that $\|x\| = 1$,

$$|\langle Tx, x\rangle|^2 \leq |\langle Tx, Tx\rangle\langle x, x\rangle| = |\langle T\frac{x}{\|x\|}, T\frac{x}{\|x\|}\rangle| \|x\|^4 \leq \|T\|^2 \|x\|^4 = \|T\|^2$$

so $\|T\| \geq \sup_{\|x\|=1} |\langle Tx, x\rangle|$.

On the other hand, we notice that for all $y \in H$ we have $|\langle Ty, y\rangle| \leq \sup_{\|x\|=1} |\langle Tx, x\rangle| \|y\|^2 = M \|y\|^2$. Therefore, if $x, y \in H$ with $\|x\| = \|y\| = 1$, by the parallelogram law we obtain the inequality

$$\begin{aligned}|\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle| &\leq |M \|x+y\|^2| + |M \|x-y\|^2| = \\ &= M(\|x+y\|^2 + \|x-y\|^2) = \\ &= 2M(\|x\|^2 + \|y\|^2) = 4M\end{aligned}$$

In particular, taking $y = \frac{Tx}{\|Tx\|}$ in the inequality we have just seen, and developing we get:

$$\begin{aligned}4M &\geq |\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle| = \\ &= |2\langle Tx, y\rangle + 2\langle Ty, x\rangle| = 2\left|\langle Tx, \frac{Tx}{\|Tx\|}\rangle + \langle \frac{T^2x}{\|Tx\|}, x\rangle\right| = \\ &= 2\left|\|Tx\| + \frac{1}{\|Tx\|}\langle T^2x, x\rangle\right|\end{aligned}$$

Let $\theta \in [0, 2\pi)$ be such that $\langle T^2 x, x\rangle = |\langle T^2 x, x\rangle| e^{i\theta}$, since T is normal then $e^{-i\theta/2}T$ is also normal and since $|\langle e^{-i\theta/2}Tx, x\rangle| = |\langle Tx, x\rangle|$ we can use it:

$$4M \geq 2\left| |e^{-i\theta/2}|\, \|Tx\| + \frac{1}{|e^{-i\theta/2}|\, \|Tx\|} |\langle T^2 x, x\rangle| \right| \Rightarrow$$
$$\Rightarrow 2M \geq \|Tx\| + \frac{1}{\|Tx\|} |\langle T^2 x, x\rangle| \geq \|Tx\|$$

Taking the supremum on $\|x\| = 1$ we get

$$\|T\| \leq 2M = 2 \sup_{\|x\|=1} |\langle Tx, x\rangle| \tag{4.1}$$

However, if we keep manipulating the inequality we get

$$0 \leq 2M\,\|Tx\| - \|Tx\|^2 - |\langle T^2 x, x\rangle| =$$
$$= -(M - \|Tx\|)^2 + M^2 - |\langle T^2 x, x\rangle| \leq M^2 - |\langle T^2 x, x\rangle|$$

and taking the supremum on $\|x\| = 1$ we obtain

$$\sup_{\|x\|\leq 1} |\langle T^2 x, x\rangle| \leq M^2 = (\sup_{\|x\|\leq 1} |\langle Tx, x\rangle|)^2 \tag{4.2}$$

Therefore, combining (4.1) and (4.2), taking into account what we have seen in Lemma 4.1 that for normal operators $\left\|T^{2^n}\right\| = \|T\|^{2^n}$ $\forall n \in \mathbb{N}$ (in fact, it is true that $\|T^n\| = \|T\|^n$ for all $n \in \mathbb{N}$),

$$\|T\|^{2^n} = \left\|T^{2^n}\right\| \leq 2 \sup_{\|x\|=1} |\langle T^{2^n} x, x\rangle| \leq 2 \sup_{\|x\|=1} |\langle Tx, x\rangle|^{2^n} = 2M^{2^n}$$

and finally

$$\|T\| \leq 2^{-2^n} M \quad \xrightarrow{n\to\infty} \quad \|T\| \leq M$$

□

We will also need to generalize the Hardy and Bergman spaces for more general domains. To do this, we need to talk about rectifiable curves, which are curves that have finite length. We will talk more rigorously about it in Chapter 5, but we notice that in previous chapters we were assuming this by considering domains with smooth boundary. The domains we will consider are simply connected domains $\Omega \subseteq \mathbb{C}$ such that $\partial\Omega$ is a rectifiable simple closed curve. In particular, we will generalize the Hilbert spaces $H^2(\mathbb{D})$ and $A^2(\mathbb{D})$ to $E_2(\Omega)$ and $A^2(\Omega)$.

At the beggining of the 30s, the theory of Hardy spaces had been well developed in the unit disk and it was natural to look for generalizations. It was Smirnov [20] who found a suitable extension to domains like the Ω we just described, also called Jordan domains. This extension is called the Smirnov class:

Definition 4.3. *Let Ω be a simply connected domain such that $\partial\Omega$ is a Jordan curve (that is, a simple closed curve), let $p > 0$, let $f : \Omega \to f(\Omega)$ be an analytic function, f belongs to the Smirnov class $E_p(\Omega)$ if it exists a sequence $\{C_n\}$ of rectifiable Jordan curves that approches the boundary of Ω such that, if Ω_n is the domain with boundary C_n, then $\Omega_n \subset \Omega_{n+1}$ and also*

$$\sup_n \int_{C_n} |f(z)|^p |dz| < \infty$$

In fact, this is equivalent to define $E_p(\Omega)$ in terms of the level curves of an arbitrary conformal mapping of $\mathbb{D}$ to Ω [3] (this is what we did working with the Riemann mapping!). We also notice that $E_2(\Omega)$ is a closed subspace of $L^2(ds)$, with ds the Lebesgue measure in $\partial\Omega$ and therefore, the perimeter of Ω is $P(\Omega) = \int ds$.

We also extend the Bergman spaces to domains like Ω with the following definition:

Definition 4.4. *Let Ω be a simply connected domain such that $\partial\Omega$ is a rectifiable closed curve and we consider the area measure $dA = dxdy/\pi$. We call Bergman space in Ω to the (closed) subspace of holomorphic functions in $L^2(\Omega, dA)$, that is,*

$$A^2(\Omega) = H(\Omega) \cap L^2(\Omega, dA)$$

Just for the interested reader, these two spaces are in fact not so different and if we extend the Bergman spaces with a weight, in the limit, we can reach to the Smirnov class $E_2(\Omega)$ (see [18]).

4.2 The isoperimetric inequality via Toeplitz operators

In this section we present the Toeplitz operators in the Smirnov class and we will use it to proof the isoperimetrical inequality (in fact, we will use the shift operator T_z that is a particular case of Toeplitz operator). With the same operators, changing only the space where we consider them, we will get the Saint-Venant inequality and almost the Faber-Krahn inequality. We start by defining them:

Definition 4.5. *Let Ω be a simply connected domain such that $\partial\Omega$ is a rectifiable simply closed curve, let ψ be analytic in Ω, the Toeplitz operator in $E_2(\Omega)$ with analytic symbol ψ is $T_\psi : E_2(\Omega) \to E_2(\Omega)$ such that $T_\psi(f) = \psi \cdot f$.*

In order to prove the isoperimetrical inequality we follow a proof of Khavinson [7], which is based in two inequalities of the commutator of Toeplitz operators. The first one is the Putnam inequality, an upper bound for positive commutators (which is a more general set of operators). The second inequality is a lower bound of the commutator specific for Toeplitz operators. We will only prove the second one, since in the next sections we will prove an upper bound specific for Toeplitz operators that it is better than Putnam's inequality.

As we said, the first inequality we will use is Putnam's inequality [16], valid for any lineal bounded operator with positive commutator:

Theorem 4.6 (Putnam's inequality)**.** *Let $T : H \to H$ be a bounded lineal operator such that $[T^*, T] \geq 0$, then*

$$\|[T^*, T]\| \leq \frac{Area(sp(T))}{\pi} \tag{4.3}$$

The other inequality is a lower bound of $\left\|[T_\psi^*, T_\psi]\right\|$ for Toeplitz operators in $E_2(\Omega)$:

Theorem 4.7. *Let ψ be an analytic and bijective function in a neighbourhood of $\overline{\Omega}$, let be $T_\psi : E_2(\Omega) \to E_2(\Omega)$ the Toeplitz operator with analytic symbol ψ, then*

$$\left\|[T_\psi^*, T_\psi\right\| \geq \frac{4Area(sp(T_\psi))^2}{\|\psi'\|^2_{E_2(\Omega)} P(\Omega)} \tag{4.4}$$

Proof. Let $\Pi : L^2(\partial\Omega, ds) \to E_2(\Omega)$ be the orthogonal projection (we can consider it because $E_2(\Omega)$ is a closed subspace), we have for all $g, h \in E_2(\Omega)$:

$$\langle T_\psi^*(g), h\rangle = \langle g, T_\psi(h)\rangle = \langle g, \psi h\rangle = \langle \overline{\psi} g, h\rangle = \langle \Pi(\overline{\psi} g), h\rangle \tag{4.5}$$

and therefore, $T_\psi^*(g) = \Pi(\overline{\psi} g)$. Now since $[T_\psi^*, T_\psi]$ is a normal operator in $E_2(\Omega)$, using Lemma 4.2 we have

$$\left\|[T_\psi^*, T_\psi]\right\| = \sup_{\substack{g\in E_2(\Omega)\\ \|g\|=1}} \langle [T_\psi^*, T_\psi]g, g\rangle$$

Let $g \in E_2(\Omega)$ with $\|g\| = 1$, then by the orthogonal projection theorem:

$$\begin{aligned}\langle (T_\psi^* T_\psi - T_\psi T_\psi^*)g, g\rangle &= \langle T_\psi(g), T_\psi(g)\rangle - \langle T_\psi^*(g), T_\psi^*(g)\rangle = \\ &= \|\psi g\|^2 - \left\|\Pi(\overline{\psi} g)\right\|^2 = \left\|\overline{\psi} g\right\|^2 - \left\|\Pi(\overline{\psi} g)\right\|^2 = \\ &= \text{dist}_{L^2}(\overline{\psi} g, E_2(\Omega))^2\end{aligned}$$

So by the definition of distance,

$$\left\|[T_\psi^*, T_\psi]\right\| = \sup_{\substack{g\in E_2(\Omega)\\ \|g\|=1}} \text{dist}_{L^2}(\overline{\psi} g, E_2(\Omega))^2 = \sup_{\substack{g\in E_2(\Omega)\\ \|g\|=1}} \left(\inf_{f\in E_2(\Omega)} \left\|\overline{\psi} g - f\right\|\right)^2$$

Now taking $g = \frac{1}{\sqrt{P(\Omega)}}$, since in effect $\|g\|^2 = \frac{1}{P(\Omega)}\|1\|_{L^2(ds)} = \frac{1}{P(\Omega)}\int ds = 1$, we find the lower bound

$$\left\|[T_\psi^*, T_\psi]\right\| \geq \inf_{f\in E_2(\Omega)} \left\| \frac{\overline{\psi}}{\sqrt{P(\Omega)}} - f \right\|^2 =$$
$$= \frac{1}{P(\Omega)}\left(\inf_{f\in E_2(\Omega)} \left\| \overline{\psi} - \frac{f}{\sqrt{P(\Omega)}} \right\|\right)^2 = \frac{1}{P(\Omega)}\left(\inf_{f\in E_2(\Omega)} \left\|\overline{\psi} - f\right\|\right)^2 \quad (4.6)$$

Let be $f \in E_2(\Omega)$, by the Riesz theorem we have

$$\left\|\overline{\psi} - f\right\|_{L^2} = \sup_{\substack{h\in L^2 \\ \|h\|=1}} \left|\langle \overline{\psi} - f, h\rangle\right| = \sup_{\substack{h\in L^2 \\ \|h\|=1}} \left|\int_C (\overline{\psi} - f)\overline{h} ds\right| \quad (4.7)$$

Since we want to have ψ' in the integral and integrate respect dz, a good option for h would be $h(z) = \frac{\overline{\psi'(z)}}{\|\psi'\|}\frac{d\overline{z}}{ds}$. Let's see that indeed $\|h\|_{E^2(\Omega)} = 1$:

$$\|h\|^2_{E^2(\Omega)} = \frac{1}{\|\psi'\|^2}\int_{\partial\Omega} |\psi'(z)|^2 \left|\frac{d\overline{z}}{ds}\right|^2 ds$$

and therefore it is enough that $|\frac{d\overline{z}}{ds}| = 1$. Let be $\phi : [0, L(\partial\Omega)] \to \mathbb{C}$ with $\phi(s) = (x(s), y(s))$ the arc-length parameterization of $\partial\Omega$ (we recall that $x'(s)^2 + y'(s)^2 = 1$). Since $d\overline{z} = dx - idy$, at the boundary $\partial\Omega$ we have $d\overline{z} = (x'(s) - iy'(s))ds$, so we get

$$|d\overline{z}| = |x'(s) - iy'(s)||ds| = \sqrt{x'(s)^2 + y'(s)^2}|ds| \Rightarrow |d\overline{z}| = |ds|$$

Now if we plug in h back in (4.7),

$$\left\|\overline{\psi} - f\right\| \geq \frac{1}{\|\psi'\|}\left|\int_{\partial\Omega} (\overline{\psi} - f)\psi' dz\right| = \frac{1}{\|\psi'\|}\left|\int_{\partial\Omega} \overline{\psi}\psi' dz - \int_{\partial\Omega} \psi' f dz\right|$$

We observe that f and ψ' belong to $E_2(\Omega)$, so $f\psi' \in E_2(\Omega)$ so by Cauchy's theoremi the integral over $\partial\Omega$ is 0. Moreover, using Stokes' theorem and making the change of variable $w = \psi(x + iy)$ we get (using that ψ is bijective):

$$\begin{aligned}
\left\|\overline{\psi} - f\right\| &\geq \frac{1}{\|\psi'\|}\left|\int_{\partial\Omega} \overline{\psi}\psi' dz\right| = \\
&= \frac{1}{\|\psi'\|}\left|\int_{\Omega} \frac{\partial(\overline{\psi}\psi')}{\partial\overline{z}} d\overline{z}\wedge dz\right| = \\
&= \frac{1}{\|\psi'\|}\left|\int_{\Omega} \overline{\psi'}\psi' d\overline{z}\wedge dz\right| = \\
&= \frac{1}{\|\psi'\|}\left|2i\int_{\Omega} |\psi|^2 dx\wedge dy\right| = \left|2i\int_{\psi(\Omega)} dxdy\right| = 2\grave{\text{A}}\text{rea}(\psi(\Omega))
\end{aligned}$$

Therefore, taking into account the inequality (4.6) we obtain

$$\left\|[T_\psi^*, T_\psi]\right\| \geq \frac{4\grave{\text{A}}\text{rea}(\psi(\Omega))^2}{\|\psi'\|^2_{E_2(\Omega)}\, P(\Omega)}$$

This inequality already works if we want to proof the isoperimetric inequality, but in order to find the inequality announced in the theorem we still have to see that $sp(T_\psi) = \overline{\psi(\Omega)}$. Directly from the definition, let be $f \in E^2(\Omega)$, if $G_\psi(f)(z) = (\psi(z) - \lambda)f(z)$ then $f(z) = G^{-1}(G(f))(z) = \frac{G(f)(z)}{\psi(z)-\lambda}$. Now if $\lambda \notin \overline{\psi(\Omega)}$ then G^{-1} is well-defined and is invertible so $\lambda \notin sp(T_\psi)$. For the reciprocal, if $\lambda \notin sp(T_\psi)$ then G^{-1} is invertible, and therefore $\forall z \in \psi(\overline{\Omega})$, $\psi(z) - \lambda \neq 0$, that is, $\lambda \notin \psi(\overline{\Omega}) = \overline{\psi(\Omega)}$ □

To prove the isoperimetric inequality we just have to take $\psi(z) = z$ and apply both inequalities (4.3) and (4.4). We get $sp(T_z) = \overline{\Omega}$ and $P(\Omega) = \|1\|^2 = \|\psi'(z)\|^2$ so that:

$$\frac{\text{Area}(\Omega)}{\pi} \geq \|[T_z^*, T_z]\| \geq \frac{4\text{Area}(\Omega)^2}{P(\Omega)^2} \quad \Rightarrow \quad \frac{P(\Omega)^2}{4\pi} \geq \text{Area}(\Omega)$$

4.3 Saint-Venant via Hankel operators: a lower bound

Our purpose is proving the Saint-Venant inequality using operator theory. In fact, by following the same strategy as in the isoperimetrical inequality we can prove it except for a factor of $\frac{1}{2}$. Notice that we proved the isoperimetrical inequality by finding upper and lower bounds to the commutator of Toeplitz operators in the Smirnov space. It turns out that using the same operators but in the Bergman space $A^2(\Omega)$ we get the inequality relating to the torsional rigidity we want. The first we will do is finding the lower bound in a very similar way to Theorem 4.7 and like Bell, Ferguson and Lundberg (see [9]) with Putnam's inequality as an upper bound we almost get the isoperimetric inequality. The definition for Toeplitz operators in the Bergman space is needless to say the analog of those defined in the Smirnov class:

Definition 4.8. *Let Ω be a simply connected domain such that $\partial\Omega$ is a rectifiable simply closed curve, if ψ is analytic in Ω, the Toeplitz operator in $A^2(\Omega)$ with analytic symbol ψ is $T_\psi : A^2(\Omega) \to A^2(\Omega)$ such that $T_\psi(f) = \psi \cdot f$.*

Moreover, in an analogous way to (4.5) we know that $T_\psi^* f = \Pi_{A^2(\Omega)}(\overline{\psi} f)$. We start with a lemma that formulates the norm of the commutator of the Toeplitz operators as a supremum:

Lemma 4.9. *Let T_ψ be the Toeplitz operator with analytic symbol ψ, let be $h \in A^2(\Omega)$, then*

$$\left\|[T_\psi^*, T_\psi]\right\| = \sup_{h \in A^2(\Omega)} \left(\frac{1}{\|h\|^2} \sup_{g \in A^2(\Omega)^\perp} \frac{|\langle \overline{\psi} h, g\rangle|^2}{\|g\|^2} \right)$$

Proof. By Lemma 4.2, we just have to prove that

$$|\langle [T_\psi^*, T_\psi]h, h\rangle| = \sup_{g\in A^2(\Omega)^\perp} \frac{|\langle \overline{\psi}h, g\rangle|^2}{\|g\|^2}$$

On one hand, expanding the inner product we obtain:

$$\begin{aligned}\langle [T_\psi^*, T_\psi]h, h\rangle &= \langle T_\psi h, T_\psi h\rangle - \langle T_\psi^* h, T_\psi^* h\rangle = \\ &= \|T_\psi h\|^2 - \left\|T_\psi^* h\right\|^2 = \|\psi h\|^2 - \left\|\Pi_{A^2(\Omega)}(\overline{\psi}h)\right\|^2 = \\ &= \left\|\overline{\psi}h\right\|^2 - \left\|\Pi_{A^2(\Omega)}(\overline{\psi}h)\right\|^2 = d(\overline{\psi}h, A^2(\Omega)) = \\ &= \inf_{f\in A^2(\Omega)} \left\|\overline{\psi}h - f\right\|^2 = \inf_{f\in A^2(\Omega)} \sup_{g\in L^2} \frac{|\langle \overline{\psi}h - f, g\rangle|^2}{\|g\|^2}\end{aligned}$$

We can take $g \in A^2(\Omega)^\perp$ so that we obtain the inequality

$$\langle [T_\psi^*, T_\psi]h, h\rangle \geq \inf_{f\in A^2(\Omega)} \sup_{g\in A^2(\Omega)^\perp} \frac{|\langle \overline{\psi}h, g\rangle - \langle f, g\rangle|^2}{\|g\|^2} = \sup_{g\in A^2(\Omega)^\perp} \frac{|\langle \overline{\psi}h, g\rangle|^2}{\|g\|^2}$$

But the extremum is attained by $g = \overline{\psi}h - \Pi_{A^2(\Omega)}(\overline{\psi}h)$ (which belongs to $A^2(\Omega)^\perp$ by the orthogonal projection theorem), and hence we get the identity we wanted: since $\Pi_{A^2(\Omega)}(\overline{\psi}h) \in A^2(\Omega)$ then $\langle \Pi_{A^2(\Omega)}(\overline{\psi}h), g\rangle = 0$ and therefore:

$$\begin{aligned}\langle [T_\psi^*, T_\psi]h, h\rangle &\geq \frac{|\langle \overline{\psi}h, g\rangle|^2}{\|g\|^2} = \frac{|\langle \overline{\psi}h, g\rangle - \langle \Pi_{A^2(\Omega)}(\overline{\psi}h), g\rangle|^2}{\|g\|^2} = \\ &= \frac{|\langle g, g\rangle|^2}{\|g\|^2} = \left\|\overline{\psi}h - \Pi_{A^2(\Omega)}(\overline{\psi}h)\right\|^2 = \\ &= \left\|\overline{\psi}h\right\|^2 - \left\|\Pi_{A^2(\Omega)}(\overline{\psi}h)\right\|^2 = \langle [T_\psi^*, T_\psi]h, h\rangle\end{aligned}$$

□

With this lemma we can get the lower bound for the commutator we were searching and by taking T_z we get an exact inequality that will lead to the Saint-Venant inequality:

Theorem 4.10. *Let Ω be a simply connected domain such that $\partial\Omega$ is a rectifiable Jordan curve, let $T_z : A^2(\Omega) \to A^2(\Omega)$ be the Toeplitz operator with symbol z, let ρ_Ω be the torsional rigidity, then*

$$\|[T_z^*, T_z]\| \geq \frac{\rho_\Omega}{Area(\Omega)}$$

Proof. The strategy of the proof is consider the closure of $B := \{\frac{\partial\phi(z,\bar{z})}{\partial z} : \phi \in \mathcal{C}^\infty(\overline{\Omega}), \phi_{|\partial\Omega} = 0\}$ instead of the space $A^2(\Omega)^\perp$ in order to obtain ρ_Ω. We take $\frac{\partial\psi}{\partial z} \in \overline{B}$ and $f \in A^2(\Omega)$, since $\overline{\psi}_{\partial\Omega} = 0$ and using Stokes' theorem,

$$0 = \int_{\partial\Omega} f\overline{\psi}dz = \int_\Omega \frac{\partial(f\overline{\psi})}{\partial\overline{z}}d\overline{z}\wedge dz = \int_\Omega (\frac{\partial f}{\partial\overline{z}}\overline{\psi} + f\frac{\partial\overline{\psi}}{\partial\overline{z}})d\overline{z}\wedge dz =$$
$$= \int_\Omega f\frac{\partial\overline{\psi}}{\partial\overline{z}}d\overline{z}\wedge dz = 2i\int_\Omega f\overline{\frac{\partial\psi}{\partial z}}dx\wedge dy = 2i\langle f, \frac{\partial\psi}{\partial z}\rangle$$

Therefore, since $f \perp \frac{\partial\psi}{\partial z}$ $\forall f \in A^2(\Omega)$ we have that $\overline{B} \subseteq A^2(\Omega)^\perp$ (in fact, both sets are equal but we only need this inclusion). Using Lemma 4.9 and taking $h \equiv 1$,

$$\|[T_z^*, T_z]\| \geq \sup_{g\in A^2(\Omega)^\perp} \frac{|\langle\overline{z}, g\rangle|^2}{\|1\|^2\|g\|^2} \geq \sup_{\substack{\psi\in\mathcal{C}^\infty(\overline{\Omega})\\ \psi_{|\partial\Omega}=0}} \frac{|\langle\overline{z}, \frac{\partial\psi}{\partial z}\rangle|^2}{\|1\|^2\left\|\frac{\partial\psi}{\partial z}\right\|^2} \geq \sup_{\substack{\psi\in\mathcal{C}^\infty(\overline{\Omega})\\ \psi(\overline{\Omega})\subseteq\mathbb{R}\\ \psi_{|\partial\Omega}=0}} \frac{|\langle\overline{z}, \frac{\partial\psi}{\partial z}\rangle|^2}{\|1\|^2\left\|\frac{\partial\psi}{\partial z}\right\|^2}$$

Now using that ψ vanish at the boundary and Stokes' theorem we can rewrite:

$$|\langle\overline{z}, \frac{\partial\psi}{\partial z}\rangle| = \left|\int_\Omega \overline{z}\overline{\frac{\partial\psi}{\partial z}}dx\wedge dy\right| = \left|\int_\Omega \left(\frac{\partial(\overline{z\psi})}{\partial\overline{z}} - \overline{\psi}\right)dx\wedge dy\right| =$$
$$= \left|\frac{1}{2i}\int_\Omega \frac{\partial(\overline{z\psi})}{\partial\overline{z}}d\overline{z}\wedge dz - \int_\Omega \overline{\psi}dx\wedge dy\right| =$$
$$= \left|\frac{1}{2i}\int_{\partial\Omega} \overline{z\psi}dz - \int_\Omega \overline{\psi}dx\wedge dy\right| = \left|\int_\Omega \overline{\psi}dx\wedge dy\right|$$

And using that $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$,

$$\left|\frac{\partial\psi}{\partial z}\right|^2 = \frac{\partial\psi}{\partial z}\overline{\frac{\partial\psi}{\partial z}} = \frac{1}{4}\left(\frac{\partial\psi}{\partial x} - i\frac{\partial\psi}{\partial y}\right)\left(\frac{\partial\psi}{\partial x} + i\frac{\partial\psi}{\partial y}\right) = \frac{1}{4}\left(\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right)$$

Finally, we can obtain ρ_Ω finishing the proof:

$$\|[T_z^*, T_z]\| \geq \sup_{\substack{\psi\in\mathcal{C}^\infty(\overline{\Omega})\\ \psi(\overline{\Omega})\subseteq\mathbb{R}\\ \psi_{|\partial\Omega}=0}} \frac{|\langle\overline{z}, \frac{\partial\psi}{\partial z}\rangle|}{\|1\|^2\left\|\frac{\partial\psi}{\partial z}\right\|^2} =$$
$$= \frac{1}{\|1\|^2}\sup_{\substack{\psi\in\mathcal{C}^\infty(\overline{\Omega})\\ \psi(\overline{\Omega})\subseteq\mathbb{R}\\ \psi_{|\partial\Omega}=0}} \frac{4\left|\int_\Omega \overline{\psi}dx\wedge dy\right|^2}{\int_\Omega(\partial_x\psi^2 + \partial_y\psi^2)dx\wedge dy} = \frac{\rho_\Omega}{\grave{\mathrm{A}}\mathrm{rea}(\Omega)} \tag{4.8}$$

□

We observe that with this bound, together with Putnam's inequality (4.3) we get:

$$\frac{\text{Area}(sp(T))}{\pi} \geq \|[T^*, T]\| \geq \frac{\rho_\Omega}{\text{Area}(\Omega)} \Rightarrow \rho_\Omega \leq \frac{\text{Area}(\Omega)^2}{\pi}$$

which is the Saint-Venant inequality except for a factor of $\frac{1}{2}$. We will solve this in the next section.

4.4 Saint-Venant via Hankel operators: an upper bound

Bell, Ferguson and Lundberg conjectured that it was possible to improve by a factor of 2 the Putnam's inequality for Toeplitz operators, in order to arrive to the Saint-Venant inequality. Olsen and Reguera answered in the positive ([18]) proving a better upper bound for the Toeplitz operators in the Bergman spaces. To do this, we have to introduce the Hankel operators and we will immediately see its direct relation with Toeplitz operators and its commutator.

Definition 4.11. *Let be ψ holomorphic, the Hankel operator with symbol ψ is*

$$\begin{aligned} H_\psi : A^2(\Omega) &\longrightarrow A^2(\Omega)^\perp \\ f &\longmapsto (Id - \Pi_{A^2(\Omega)})(\psi f) = \psi f - \Pi_{A^2(\Omega)}(\psi f) \end{aligned}$$

The Hankel operator sends f to the orthogonal of $T_\psi(f)$, and in fact $H_\psi + T_\psi = \psi f$ (which is the operator in $L^2(\Omega, dA)$ that multiplies by ψ to given a $f \in A^2(\Omega)$).

Studying the norm of the Hankel operators i the same as studying the commutator of Toeplitz operators, as we see in the following lemma:

Lemma 4.12. *Let be ψ analytic in Ω, let $T_\psi : A^2(\Omega) \to A^2(\Omega)$ and $H_\psi : A^2(\Omega) \to A^2(\Omega)^\perp$ be the Toeplitz and the Hankel operator respectively, then*

$$\left\|[T_\psi^*, T_\psi]\right\| = \left\|H_{\overline{\psi}}\right\|^2$$

Proof. We check it directly, using Lemma 4.2 and the orthogonal projection theorem (in Lemma 4.9 we did a very similar reasoning):

$$\begin{aligned}\left\|[T_\psi^*, T_\psi]\right\| &= \sup_{\substack{h\in A^2(\Omega)\\ \|h\|=1}} \langle (T_\psi^* T_\psi - T_\psi T_\psi^*)h, h\rangle = \sup_{\substack{h\in A^2(\Omega)\\ \|h\|=1}} \big(\langle T_\psi h, T_\psi h\rangle - \langle T_\psi^* h, T_\psi^* h\rangle\big) = \\ &= \sup_{\substack{h\in A^2(\Omega)\\ \|h\|=1}} \left(\|T_\psi h\|^2 - \left\|T_\psi^* h\right\|^2\right) = \sup_{\substack{h\in A^2(\Omega)\\ \|h\|=1}} \left(\|\psi h\|^2 - \left\|\Pi_{A^2(\Omega)}(\overline{\psi} h)\right\|^2\right) = \\ &= \sup_{\substack{h\in A^2(\Omega)\\ \|h\|=1}} \left(\left\|\overline{\psi} h\right\|^2 - \left\|\Pi_{A^2(\Omega)}(\overline{\psi} h)\right\|^2\right) = \\ &= \sup_{\substack{h\in A^2(\Omega)\\ \|h\|=1}} \left\|(Id - \Pi_{A^2(\Omega)})(\overline{\psi} h)\right\|^2 = \left\|H_{\overline{\psi}}\right\|^2\end{aligned}$$

□

From now on, we will talk indifferently of the norms of $H_{\overline{\psi}}$ and the commutator of T_ψ. We will also need the integral expression of the orthogonal projection $\Pi_{A^2(\Omega)}$. When $\Omega = \mathbb{D}$, we can give directly: if $f \in L^2(\mathbb{D})$,

$$\Pi_{A^2(\mathbb{D})}(f(z)) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^2} dA(w)$$

However, for a general domain Ω we need the Riemann mapping. As before, let $F : \mathbb{D} \to \Omega$ be the analytic and bijective Riemann mapping, then

$$\Pi_{A^2(\Omega)}(f(z)) = \int_{\Omega} f(w) \frac{(F^{-1})'(z)\overline{(F^{-1})'(w)}}{(1 - F^{-1}(z)\overline{F^{-1}(w)})^2} dA(w) \tag{4.9}$$

We are finally able to prove the upper inequality for the commutator of Toeplitz operators that improves Putnam inequality. First we will consider only the case $\Omega = \mathbb{D}$ (later we will be able to generalize it for all Ω).

Theorem 4.13. *Let be ψ analytic in $\mathbb{D}$ such that $\psi' \in A^2(\mathbb{D})$. Then,*

$$\left\|H_{\overline{\psi}}\right\|^2_{A^2(\mathbb{D})\to A^2(\mathbb{D})^\perp} \leq \frac{\|\psi'\|^2_{A^2(\mathbb{D})}}{2}$$

Proof. We consider $\psi(z) = \sum_{n\geq 1} c_n z^n$ (we can assume $c_0 = 0$ without loss of generality). Let be $f = \sum_{n\geq 0} a_n z^n \in A^2(\mathbb{D})$, the strategy of the proof is work with the coefficients and in fact we will only use the ineuality $0 \leq a^2 + b^2 - 2ab$.

Therefore, the first part of the proof consists in writing $\left\|H_{\overline{\psi}}(f)\right\| = \left\|\overline{\psi} f - \Pi_{A^2(\Omega)}(\overline{\psi} f)\right\|$ in terms of the coefficients. Since we are in $\Omega = \mathbb{D}$ we have an explicit expression of the projeccion in $A^2(\mathbb{D})$ so we can compute $\Pi_{A^2(\mathbb{D})}(\overline{\psi} f)$: using $\sum_{l\geq 0}(n+1)x^n = \frac{1}{(1-x)^2}$ and the orthogonality of the base,

$$\begin{aligned}
\Pi_{A^2(\mathbb{D})}(\overline{\psi} z^n) &= \sum_{k\geq 1} \overline{c}^k \int_{\mathbb{D}} \frac{\overline{w}^k w^n}{(1-z\overline{w})^2} dA(w) = \\
&= \sum_{k\geq 1} \overline{c}^k \left(\sum_{l\geq 0} (l+1) z^l \int_{\mathbb{D}} \overline{w}^l \overline{w}^k w^n \frac{dx\wedge dy}{\pi} \right) = \\
&= \sum_{k\geq 1} \overline{c}^k \left(\sum_{l\geq 0} (l+1) z^l \int_0^1 \int_0^{2\pi} r^{l+k+n} e^{i(n-(l+k))\theta} \frac{r d\theta dr}{\pi} \right) = \\
&= \sum_{k=1}^{n} \overline{c}^k (n-k+1) z^{n-k} 2 \int_0^1 r^{2n+1} dr = \\
&= \sum_{k=1}^{n} \overline{c}_k \frac{n-k+1}{n+1} z^{n-k} = \sum_{k=0}^{n-1} \frac{k+1}{n+1} \overline{c}_{n-k} z^k
\end{aligned}$$

With this, we get $H_{\overline{\psi}}(f)$ as we want:

$$H_{\overline{\psi}}(f)(z) = \overline{\psi}(z) f(z) - \Pi_{A^2(\mathbb{D})}(\overline{\psi} f)(z) = \underbrace{\sum_{l\geq 1}\sum_{n\geq 0} \overline{c}_l a_n \overline{z}^l z^n}_{(*)} - \sum_{n\geq 1}\sum_{k=0}^{n-1} \frac{k+1}{n+1} a_n \overline{c}_{n-k} z^k \tag{4.10}$$

Now to obtain the norm we just have to take the modulus to the squared and integrate. To do this, we will rearrange several times the sums. First, to integrate by $\int_0^{2\pi} d\theta/\pi$ we will separate it in two sums of powers of z and $\overline{z}$:

$$\begin{aligned}
(*) &= \sum_{l\geq 1} \overline{c}_l a_0 \overline{z}^l + \sum_{n\geq 1} a_n \left(\sum_{1\leq l\leq n} \overline{c}_l |z|^{2l} z^{n-l} + \sum_{l>n} \overline{c}_l |z|^{2n} \overline{z}^{l-n} \right) = \\
&= \sum_{l\geq 1} \overline{c}_l a_0 \overline{z}^l + \sum_{n\geq 1} a_n \left(\sum_{k=0}^{n-1} \overline{c}_{n-k} |z|^{2(n-k)} z^k + \sum_{k\geq 1} \overline{c}_{n+k} |z|^{2n} \overline{z}^k \right)
\end{aligned}$$

Plugging it in equation (4.10) we get

$$\begin{aligned}
&\sum_{k\geq 1} \left(a_0 \overline{c}_k + \sum_{n\geq 1} a_n \overline{c}_{n+k} |z|^{2n} \right) \overline{z}^k + \sum_{n\geq 1}\sum_{k=0}^{n-1} a_n \overline{c}_{n-k} \left(|z|^{2(n-k)} - \frac{k+1}{n+1} \right) z^k = \\
&\qquad = \sum_{k\geq 0} \overline{z}^k \sum_{n\geq 0} a_n \overline{c}_{n+k} |z|^{2n} + \sum_{k\geq 0} z^k \sum_{n\geq k+1} a_n \overline{c}_{n-k} \left(|z|^{2(n-k)} - \frac{k+1}{n+1} \right)
\end{aligned}$$

Substituting $z = re^{i\theta}$ and taking the modulus squared, since $\int_0^{2\pi} e^{in\theta}, e^{im\theta} d\theta = 0$ if $n \neq m$, then integrating by $\int_0^{2\pi} d\theta/\pi$ we obtain:

$$\sum_{k\geq 1}\left|\sum_{n\geq 0} a_n\bar{c}_{n+k}r^{2n}\right|^2 \int_0^{2\pi} r^{2k}\frac{d\theta}{\pi} + \sum_{k\geq 0}\sum_{n\geq k+1} a_n\bar{c}_{n-k}\left|r^{2(n-k)} - \frac{k+1}{n+1}\right|^2 \int_0^{2\pi} r^{2k}\frac{d\theta}{\pi} =$$

$$= \underbrace{2\sum_{k\geq 1} r^{2k}\left|\sum_{n\geq 0} a_n\bar{c}_{n+k}r^{2n}\right|^2}_{(I)} + \underbrace{2\sum_{k\geq 0} r^{2k}\sum_{n\geq k+1} a_n\bar{c}_{n-k}\left|r^{2(n-k)} - \frac{k+1}{n+1}\right|^2}_{(II)}$$

Finally, we just have to integrate over $\int_0^1 rdr$. For simplicity, we will work separately with (I) and (II) but the steps are analogous. We start expanding the square and integrating after:

$$(I) = 2\sum_{k\geq 1}\sum_{n,m\geq 0} a_n\bar{a}_m c_{m+k}\overline{c_{n+k}}r^{2n+2m+2k} \Rightarrow$$

$$\Rightarrow (I') = 2\int_0^1 \sum_{k\geq 1}\sum_{n,m\geq 0} a_n\bar{a}_m c_{m+k}\overline{c_{n+k}}r^{2n+2m+2k+1}dr = \sum_{k\geq 1}\sum_{n,m\geq 0}\frac{a_n\bar{a}_m c_{m+k}\overline{c_{n+k}}}{n+m+k+1}$$

Now we make a change of coefficients by setting $a_n = b_{n+1}(n+1)$ and we adjust the indices of summation slightly:

$$\begin{aligned}(I') &= \sum_{k\geq 1}\sum_{n,m\geq 0} b_{n+1}\bar{b}_{m+1}c_{m+k}\bar{c}_{n+k}\frac{(n+1)(m+1)}{n+m+k+1} = \\ &= \sum_{k\geq 0}\sum_{n,m\geq 1} b_n\bar{b}_m c_{k+m}\bar{c}_{k+n}\frac{nm}{n+m+k}\end{aligned}$$

We can easily check that $\forall z, w \in \mathbb{C},\ 2\,\mathrm{Re}\,(zw) = z\overline{w} + \overline{z}w \leq |z|^2 + |w|^2$ expanding the inequality $|z-w|^2 \geq 0$ (this is the only inequality we use in the proof) and therefore,

$$\begin{aligned}(I') &= \sum_{k\geq 0}\sum_{n,m\geq 1} b_n c_{k+m}\overline{b_m c_{k+n}}\frac{nm}{n+m+k} = \\ &= \sum_{k\geq 0}\frac{1}{2}\left(\sum_{n,m\geq 1} 2\,\mathrm{Re}(b_n c_{k+m}\overline{b_m c_{k+n}})\frac{nm}{n+m+k}\right) \leq \\ &\leq \sum_{k\geq 0}\frac{1}{2}\left(\sum_{n,m\geq 1}(|b_n c_{k+m}|^2 + |b_m c_{k+n}|^2)\frac{nm}{n+m+k}\right) = \\ &= \sum_{k\geq 0}\sum_{n,m\geq 1}|b_n c_{k+m}|^2\frac{nm}{n+m+k} := (I'_*)\end{aligned}$$

where in the last equality we used that n and m are symmetric.

For (II) we repeat exactly the same procedure:

$$(II) = 2\sum_{k\geq 0}\sum_{n,m\geq k+1} a_n\overline{a}_m c_{m-k}\overline{c}_{n-k}\left(r^{2n+2m-2k} - r^{2n}\frac{k+1}{m+1}\right.$$
$$\left. -r^{2m}\frac{k+1}{n+1} + r^{2k}\frac{(k+1)^2}{(n+1)(m+1)}\right)$$

so

$$\begin{aligned}(II') &= \sum_{k\geq 0}\sum_{n,m\geq k+1} a_n\overline{a}_m c_{m-k}\overline{c}_{n-k}\left(\frac{(n+1)(m+1)-(k+1)(m+n-k+1)}{(n+1)(m+1)(n+m-k+1)}\right) = \\ &= \sum_{k\geq 1}\sum_{n,m\geq 1} b_{n+k}\overline{b}_{m+k}c_m\overline{c}_n\frac{mn}{n+m+k} \leq \\ &\leq \sum_{k\geq 1}\frac{1}{2}\sum_{n,m\geq 1}\left(|b_{n+k}c_m|^2 + |b_{m+k}c_n|^2\right)\frac{mn}{n+m+k} = \\ &= \sum_{k\geq 1}\sum_{n,m\geq 1}|b_{n+k}c_m|^2\frac{nm}{n+m+k} := (II'_*)\end{aligned}$$

Rearranging the summations and using that for an arithmetic sum we have $\sum_{n=1}^{k} na_0 = \frac{a_0 k(1+k)}{2}$,

$$\begin{aligned}(I'_*) &= \sum_{n\geq 1}\sum_{m\geq 1}\sum_{k\geq m}|b_n c_k|^2\frac{nm}{n+k} = \sum_{n\geq 1}\sum_{k\geq 1}\sum_{m=1}^{k}|b_n c_k|^2\frac{nm}{n+k} = \\ &= \sum_{n\geq 1}\sum_{k\geq 1}|b_n c_k|^2\sum_{m=1}^{k}\frac{nm}{n+k} = \sum_{n\geq 1}\sum_{k\geq 1}|b_n c_k|^2\frac{nk(1+k)}{2(n+k)}\end{aligned}$$

$$\begin{aligned}(II'_*) &= \sum_{m\geq 1}\sum_{n\geq 1}\sum_{k\geq n+1}|b_k c_m|^2\frac{nm}{m+k} = \sum_{m\geq 1}\sum_{k\geq 2}|b_k c_m|^2\sum_{n=1}^{k-1}\frac{nm}{m+k} = \\ &= \sum_{m\geq 1}\sum_{k\geq 2}|b_k c_m|^2\frac{m(k-1)k}{2(m+k)} = \sum_{m\geq 1}\sum_{k\geq 1}|b_k c_m|^2\frac{m(k-1)k}{2(m+k)}\end{aligned}$$

Finally, we obtain the upper bound for $\left\|H_{\overline{\psi}}(f)\right\|$ in terms of the coefficients a_n and c_n by substituting $a_n = b_{n+1}(n+1)$, getting the norms of f and ψ:

$$\begin{aligned}(I'_*) + (I''_*) &= \sum_{n\geq 1}\sum_{k\geq 1}|b_n c_k|^2\left(\frac{nk(1+k)}{2(n+k)} + \frac{nk(n-1)}{2(n+k)}\right) = \sum_{n\geq 1}\sum_{k\geq 1}|b_n c_k|^2\frac{nk}{2} = \\ &= \sum_{n\geq 1}\sum_{k\geq 1}|a_{n-1}c_k|^2\frac{k}{2n} = \frac{1}{2}\sum_{n\geq 0}\sum_{k\geq 1}|a_n c_k|^2\frac{k}{n+1} = \\ &= \frac{1}{2}\left(\sum_{n\geq 0}\frac{|a_n|^2}{n+1}\right)\left(\sum_{k\geq 1}|c_k|^2 k\right) = \frac{1}{2}\|f\|^2_{A^2(\mathbb{D})}\left\|\psi'\right\|^2_{A^2(\mathbb{D})}\end{aligned}$$

□

We can easily extend this theorem to domains like Ω so that we obtain the following corollary:

Corollary 4.14. *Let Ω be a simply connected domain such that $\partial\Omega$ is a rectifiable simply closed curved, let ψ be analytic, then*

$$\left\| H_{\overline{\psi}} \right\|^2_{A^2(\Omega)\to A^2(\Omega)^\perp} \leq \frac{\|\psi'\|^2_{A^2(\Omega)}}{2}$$

Proof. To prove it, we have to go from a Ω to $\mathbb{D}$ and then use Theorem 4.13. To do this, we will need the projeccion of $A^2(\Omega)$ that we have seen in (4.9). Then, let be $F : \mathbb{D} \to \Omega$ the Riemann mapping, if $f \in A^2(\Omega)$ then we have $\Pi_{A^2(\Omega)}(f) = f$ (because it is a projeccion). Therefore, we can write explicitly the Hankel operator:

$$\begin{aligned} H_{\overline{\psi}}(f)(z) &= \overline{\psi(z)}\, \Pi_{A^2(\Omega)} f(z) - \Pi_{A^2(\Omega)}(\overline{\psi} f)(z) = \\ &= \int_\Omega f(w) \frac{(F^{-1})'(z)\overline{(F^{-1})'(w)}}{(1 - F^{-1}(z)\overline{F^{-1}(w)})^2} (\overline{\psi(z)} - \overline{\psi(w)}) \frac{dw \wedge d\overline{w}}{-2i\pi} \end{aligned}$$

If we do the changes of variables $z = F(\xi)$ and $w = F(\tau)$, we obtain

$$\begin{aligned} H_{\overline{\psi}}(f \circ F)(\xi) = \int_{\mathbb{D}} f \circ F(\tau) &\frac{(F^{-1})'(F(\xi))\overline{(F^{-1})'(F(\tau))}}{(1 - \xi\overline{\tau})^2} \\ &(\overline{\psi \circ F(\xi)} - \overline{\psi \circ F(\tau)}) \frac{F'(\tau) d\tau \wedge \overline{F'(\tau) d\tau}}{-2i\pi} = \\ = \int_{\mathbb{D}} f \circ F(\tau) &\frac{(\overline{\psi \circ F(\xi)} - \overline{\psi \circ F(\tau)}) F'(\tau)\overline{F'(\tau)}}{F'(F^{-1} \circ F(\xi))\overline{F'(F^{-1} \circ F(\tau))}(1 - \xi\overline{\tau})^2} dA(\tau) = \\ = \int_{\mathbb{D}} f \circ F(\tau) &\frac{F'(\tau)}{F'(\xi)} \frac{1}{(1 - \xi\overline{\tau})^2} (\overline{\psi \circ F(\xi)} - \overline{\psi \circ F(\tau)}) dA(\tau) \end{aligned}$$

Therefore,

$$F'(\xi) H_{\overline{\psi}}(f \circ F)(\xi) = H_{\psi \circ F}(F(\tau)(f \circ F)(\tau))$$

so by taking norms we have the identity

$$\left\| H_{\overline{\psi}} \right\|_{A^2(\Omega)\to A^2(\Omega)} = \left\| H_{\overline{\psi \circ F}} \right\|_{A^2(\mathbb{D})\to A^2(\mathbb{D})}$$

Finally, we just have to apply this equality in order to go from Ω to $\mathbb{D}$ and then use Theorem 4.13:

$$\left\| H_{\overline{\psi \circ F}} \right\|_{A^2(\mathbb{D})\to A^2(\mathbb{D})} \leq \frac{\|(\psi \circ F)'\|^2_{A^2(\mathbb{D})}}{2} = \frac{\|\psi'\|^2_{A^2(\mathbb{D})}}{2}$$

□

With this corollary and with Theorem 4.10, by taking $\psi = z$ we directly obtain exactly the Saint-Venant inequality:

$$\frac{\grave{\text{A}}\text{rea}(\Omega)}{2\pi} = \frac{\|1\|^2_{A^2(\Omega)}}{2} \geq \|H_{\overline{z}}\|^2 = \|[T_z^*, T_z]\| \geq \frac{\rho_\Omega}{\grave{\text{A}}\text{rea}(\Omega)} \quad \Rightarrow \quad \frac{\grave{\text{A}}\text{rea}(\Omega)^2}{2\pi} \geq \rho_\Omega$$

4.5 An approximation to the Faber-Krahn inequality

With the same strategy, using Toeplitz operators and all the tools we developed, we can still get very close to another important geometric inequality: the Faber-Krahn inequality, which bounds the principal frequency of a domain (which we will immediately explain) in terms of its area. The principal frequency of a domain Ω is the lowest frequency that a drum of shape Ω and uniform density and tension would sound. Mathematically, given a domain $\Omega \subseteq \mathbb{R}^2$, the oscilation frequencies of the drum are given by the eigenvalues of the Laplacian $-\Delta_\Omega$ with the Dirichlet boundary conditions, that is, with eigenfunctions that vanish at the boundary. Therefore, we have the following definition:

Definition 4.15. *Let Ω be a simply connected domain such that $\partial\Omega$ is a simply closed smooth curve, the principal frequency of Ω is the smallest eigenvalue λ_Ω such that for some $u : L^2(\Omega, dA) \to L^2(\Omega, dA)$ satisfies*

$$\begin{cases} \Delta u + \lambda_\Omega u = 0 \\ u|_{\partial\Omega} = 0 \end{cases}$$

We can also give a variational definition like we did with the torsional rigidity:

$$\lambda'_\Omega := \inf_{u \in \mathcal{C}_0^\infty(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} \tag{4.11}$$

For the sake of completeness we will prove the equivalence of the two definitions:

Proposition 4.16. *Let Ω be a simply connected domain such that $\partial\Omega$ is a simply closed smooth curve, let λ_Ω be the principal frequency of Ω, if the infimum is attained then it is attained by the eigenfunction of λ_Ω and $\lambda_\Omega = \lambda'_\Omega$.*

Proof. On one hand, let ψ_Ω be the eigenfunction of λ_Ω, since it is smooth enough (see [12]), by Green's first identity and using that ψ_Ω vanishes at the boundary,

$$\Delta\psi_\Omega + \lambda_\Omega\psi_\Omega = 0 \Rightarrow \int_\Omega (\Delta\psi_\Omega + \lambda_\Omega\psi_\Omega)\psi_\Omega dA = 0 \Rightarrow$$
$$\Rightarrow \lambda_\Omega \int_\Omega \psi_\Omega^2 dA = -\int_\Omega \psi_\Omega \Delta\psi_\Omega dA = \int_\Omega |\nabla\psi_\Omega|^2 dA \Rightarrow \lambda_\Omega = \frac{\int_\Omega |\nabla\psi_\Omega|^2 dA}{\int_\Omega \psi_\Omega^2 dA}$$

On the other hand, if there is a function ψ that attaines the infimum, we know that for any function $w \in \mathcal{C}_0^\infty(\Omega)$ and for any $\epsilon > 0$ we must have

$$\frac{\int_\Omega |\nabla\psi|^2}{\int_\Omega \psi^2} = \mathcal{J}(\psi) \leq \mathcal{J}(\psi + \epsilon w) = \frac{\int_\Omega |\nabla(\psi + \epsilon w)|^2}{\int_\Omega (\psi + \epsilon w)^2}$$

where $\mathcal{J}(f) := \frac{\int_\Omega |\nabla f|^2}{\int_\Omega f^2}$. In particular, if we see $\mathcal{J}(\psi + \epsilon w)$ as a function of ϵ it has a minimum at $\epsilon = 0$. Let us now compute this idea more explicitly:

$$\mathcal{J}(\psi + \epsilon w) = \frac{\int_\Omega |\nabla(\psi + \epsilon w)|^2}{\int_\Omega (\psi + \epsilon w)^2} = \frac{\int_\Omega |\nabla\psi|^2 + 2\epsilon \int_\Omega \nabla\psi \cdot \nabla w + \epsilon^2 \int_\Omega |\nabla w|^2}{\int_\Omega \psi^2 + 2\epsilon \int_\Omega \psi w + \epsilon^2 \int_\Omega w^2}$$

so that

$$\begin{aligned} 0 &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathcal{J}(\psi + \epsilon w) = \\ &= \frac{(2\int_\Omega \nabla\psi \cdot \nabla w)(\int_\Omega \psi^2) - (\int_\Omega |\nabla\psi|^2)(2\int_\Omega \psi w)}{(\int_\Omega \psi^2)^2} = 2\frac{\int_\Omega \nabla\psi \cdot \nabla w - \mathcal{J}(\psi)\int_\Omega \psi w}{\int_\Omega \psi^2} \end{aligned}$$

Therefore, since $\lambda'_\Omega = \mathcal{J}(\psi)$,

$$0 = \int_\Omega \nabla\psi \cdot \nabla w - \lambda'_\Omega \int_\Omega \psi w, \qquad \forall w \in \mathcal{C}_0^\infty(\Omega)$$

Integrating by parts we have

$$0 = -\int_\Omega \Delta\psi \cdot w - \lambda'_\Omega \int_\Omega \psi w = -\int_\Omega (\Delta\psi + \lambda'_\Omega \psi) w, \quad \forall w \in \mathcal{C}_0^\infty(\Omega)$$

And since we have it for any function $w \in \mathcal{C}_0^\infty(\Omega)$, it implies that

$$\Delta\psi + \lambda'_\Omega \psi = 0$$

Therefore, we have $\lambda_\Omega = \lambda'_\Omega$ because if $\lambda'_\Omega < \lambda_\Omega$ then λ_Ω is not the smallest eigenvalue, and viceversa, if $\lambda'_\Omega > \lambda_\Omega$ then λ'_Ω it is not the minimum, arriving to contradiction. □

In the same way as the area with the isoperimetric inequality and the torsional rigidity with the Saint-Venant inequality, we can bound the principal frequency λ_Ω of a general Ω in terms of the principal frequency of the unit disk, $\lambda_\mathbb{D}$. We have then the exact inequality:

$$\lambda_\Omega \geq \lambda_\mathbb{D} = \frac{j_0^2 \pi}{\grave{\mathrm{A}}\mathrm{rea}(\Omega)}$$

with $j_0 \simeq 2.405$ the first positive root of the Bessel function $J_0(x)$. This inequality was conjectured by Lord Rayleight in his work on the theory of sound [17] at the end of the 19th century and was proved independently by G.Faber and E.Krahn (see [8]). With the operator theory tools we developed we will get very close to this Faber-Krahn inequality, but it is still an open problem if we can prove it in this way. We will still use the Toeplitz operator in the Bergman space with symbol z and the improved Putnam's inequality by Olsen and Reguera, and we only have to find a new lower bound for the commutator of T_z involving λ_Ω. The closest one has been proved by Bell, Ferguson and Lundberg [9]:

Theorem 4.17. *Let Ω be a simply connected domain such that $\partial\Omega$ is a rectifiable simple closed curve, let $T_z : A^2(\Omega) \to A^2(\Omega)$ be the Toeplitz operator in $A^2(\Omega)$ with symbol z, let λ_Ω be the principal frequency of Ω, then*

$$\|[T_z^*, T_z]\| \geq \frac{4^2\pi}{\lambda_\Omega^2 \grave{A}rea(\Omega)}$$

Proof. If we start from the proof of theorem 4.10, we just have to choose another ψ for the supremum. Therefore, by (4.8) we have:

$$\|[T_z^*, T_z]\| \geq \sup_{\substack{\psi\in\mathcal{C}_0^\infty(\overline{\Omega}) \\ \psi(\overline{\Omega})\subseteq\mathbb{R}}} \frac{4|\int_\Omega \overline{\psi} dA|^2}{\|\nabla\psi\|^2\,\|1\|^2} = \sup_{\substack{\psi\in\mathcal{C}_0^\infty(\overline{\Omega}) \\ \psi(\overline{\Omega})\subseteq\mathbb{R}}} \frac{4|\int_\Omega \psi dA|^2}{\|\nabla\psi\|^2\,\|1\|^2} \tag{4.12}$$

We choose $\psi = \psi_\Omega$ to be the eigenfunction of λ_Ω for the Laplacian $-\nabla_\Omega$ with Dirichlet conditions as in proposition 4.16. We have seen that minimizes the variational definition (4.11) of the principal frequency so it satisfies:

$$\lambda_\Omega = \frac{\int_\Omega |\nabla\psi_\Omega|^2}{\int_\Omega \psi_\Omega^2}$$

We will still need another property of ψ_Ω: we can bound the integral $\int_\Omega \psi_\Omega dA$ in terms of the norm $\|\psi_\Omega\|_{A^2(\Omega)}$. We have the following inequality (see [14]):

$$\left(\int_\Omega \psi_\Omega dA\right)^2 \geq \frac{4\pi}{\lambda_\Omega} \|\psi_\Omega\|^2_{A^2(\Omega)}$$

Substituting in 4.12 we get the exact inequality we wanted (it is attained if Ω is a disk):

$$\|[T_z^*, T_z]\| \geq \frac{4|\int_\Omega \psi_\Omega dA|^2}{\|\nabla\psi_\Omega\|^2\,\|1\|^2} \geq \frac{4\pi}{\lambda_\Omega\ \grave{\text{A}}\text{rea}(\Omega)} \frac{\|\psi_\Omega\|^2}{\|\nabla\psi_\Omega\|^2} = \frac{4^2\pi}{\lambda_\Omega^2 \grave{\text{A}}\text{rea}(\Omega)}$$

□

Now together with corollary 4.14, that we recall gives us an upper bound of T_z, we have left a expression like the Faber-Krahn inequality:

$$\frac{\grave{\text{A}}\text{rea}(\Omega)}{2\pi} \geq \|[T_z^*, T_z]\| \geq \frac{4^2\pi}{\lambda_\Omega^2 \grave{\text{A}}\text{rea}(\Omega)} \Rightarrow \lambda_\Omega \geq \frac{4\sqrt{2}\pi}{\grave{\text{A}}\text{rea}(\Omega)} \simeq 0.978 \frac{j_0^2\pi}{\grave{\text{A}}\text{rea}(\Omega)}$$

which only differs in a small constant, very close to the original inequality.

Chapter 5

Extension to rectifiable curves

We have already proved the isoperimetric inequality for domains with smooth boundary (for example, we have proved it for $\mathcal{C}^1$ in section 2.2 and for $\mathcal{C}^3$ in section 2.3). In this section we will always consider Jordan curves, which we recall they are simple and closed curves in the plane. We would like to extend it to any domain with a rectifiable boundary, and from there it is easy to extend it to any domain in $\mathbb{R}^2$ (see Corollari 5.7). We recall that a curve is rectifiable if it has finite lenght, but since we don't have a smooth curve, we need to define properly what we understand by the *length* of a given contiuous curve. A good way to define it is, given a set of points, construct a polygon "following" the curve and consider its length, then the length of the curve will be the supremum of the lengths taken over all possible polygons. Specifically, from [4, p. 44],

Definition 5.1. *Let C be a Jordan curve, let $w : [0, 2\pi] \to \mathbb{R}^2$ be a continuous parameterization of C and we consider $T = \{t_0 < ... < t_n\}$ as all the finite partitions of $[0, 2\pi]$ with a given n. We define the length of C as the supremum of the length of the inscribed polygonals, that is,*

$$L = \sup_{\substack{\{t_0,...,t_n\} \\ n\in\mathbb{N}}} \sum_{k=1}^{n} ||w(t_k) - w(t_{k-1})||$$

Definition 5.2. *We say C is rectifiable if $L < \infty$.*

If we consider this property in functions on a closed interval instead of curves, we say they are of *bounded variation.* In fact, both definitions are analogous:

Definition 5.3. *Sigui $f : [a, b] \to \mathbb{C}$ una funció contínua, direm que és de variació acotada o BV (bounded variation) si considerant totes les particions finites $\{x_0, ..., x_n\}$ de $[a, b]$ amb n arbitrari tenim*

$$\sup_{\substack{\{x_0,...,x_n\} \\ n\in\mathbb{N}}} \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| < \infty$$

We see then that if we have a parameterization $w : [0, 2\pi] \to \mathbb{C}$ of C, asking for C rectifiable is the same as asking w to be of bounded variation. The funtions in this class are a.e. differentiable but not absolutely continuous, and in fact the absolutely continuous functions are of BV.

One important property of this class is that being of BV for a function $f : \mathbb{S}^1 \to \mathbb{C}$ is exactly the necessary condition to characterize an harmonic function in the disk (in fact we can construct the whole function in $\mathbb{D}$) and viceversa, given an harmonic function in the disk, its restriction to the boundary must be of BV.

We will need an important characterization of the H^p spaces for $p \in [1, \infty]$: they are subspaces of L^p, in particular, the subspace of functions in L^p without negative Fourier coefficients, that is,

$$c_{-n} = \int_0^{2\pi} e^{in\theta} F(e^{i\theta}) d\theta = 0 \tag{5.1}$$

We can identify it like this because given an analytic function in the disk, the Fourier coefficients of the restriction to $\mathbb{S}^1$ are the same as the Taylor coefficients in the disk [4, p. 38], and the way to go from one to the other is with the Poisson-Stieltjes integral. For our purpose we only need to define it for analytic functions (in fact we only need a the first derivative to be continuous) but we can extend it to functions of BV with the Riemann-Stieltjes integral.

Definition 5.4. *Let $f(t) : [0, 2\pi] \to \mathbb{C}$ be an analytic function, we define the Poisson kernel for $0 < r < 1$ as*

$$P(re^{i\theta}) = P(r, \theta) = \sum_{-\infty}^{\infty} r^{|n|} e^{in\theta} = \frac{1 - r^2}{1 - 2r cos\theta + r^2}$$

and the Poisson-Stieltjes integral is

$$u(re^{i\theta}) = \int_0^{2\pi} P(r, \theta - t) f'(t) \frac{dt}{2\pi}$$

and we have $u(e^{i\theta}) = f(\theta)$ (see [4, p. 2]).

Now we take as always $\Omega \in \mathbb{C}$ such that $\partial\Omega$ is a Jordan curve and let $F : \mathbb{D} \to \Omega$ be the Riemann conformal mapping, if $\partial\Omega$ is rectifiable, we can guarantee a certain regularity of the derivatives of F in $\mathbb{D}$:

Theorem 5.5. *Let be $F(z) : \mathbb{D} \to \Omega$ conformal such that $\partial\Omega$ is a Jordan curve. If $\partial\Omega$ is rectifiable then $F' \in H^1$.*

Proof. Given $F(z)$ conformal in the disk $|z| < 1$ in Ω by a Carathéodory's theorem [15] we can always find a continuous and bijective extension in $|z| \leq 1$. Let's see how, because of being a function of BV at the boundary (we have that $\partial\Omega$ is rectifiable), this extension is absolutely continuous at the boundary. Let $F(e^{i\theta}) = \mu(\theta)$ be the continuous extension at the boundary with $\mu(0) = \mu(2\pi)$ and let $F(z) = \sum_{n=0}^{\infty} a_n z^n$, which belongs to H^1 for being analytic in $\mathbb{D}$. We have already said in (5.1) that the negative Fourier coefficients of $F(e^{i\theta})$ are 0. Let's prove it: on one hand,

$$a_{-n} := \frac{1}{2\pi}\int_0^{2\pi} F(re^{it})r^n e^{int}dt = \frac{1}{2\pi}\sum_0^{\infty} a_n r^{2n}\int_0^{2\pi} e^{2int}dt = 0 \qquad n = 1, 2, 3, ...$$

and the negative Fourier coefficients are

$$c_{-n} = \frac{1}{2\pi}\int_0^{2\pi} e^{int}F(e^{it})dt$$

Therefore,

$$|r^{-n}a_{-n} - c_{-n}| \leq \frac{1}{2\pi}\left|\int_0^{2\pi} e^{int}(F(re^{it}) - F(e^{it}))dt\right| \leq \frac{1}{2\pi}\int_0^{2\pi} |F(re^{it}) - F(e^{it})|dt \xrightarrow{r\to 1} 0$$

so indeed $c_{-n} = 0$, that is,

$$c_{-n} = \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}F(e^{i\theta})d\theta = 0 \quad \forall n = 1, 2, 3, ...$$

Now, integrating by parts, for $n = 1, 2, 3, ...$ we have

$$0 = \int_0^{2\pi} e^{in\theta}F(e^{i\theta}) = \frac{e^{in\theta}}{in}F(e^{i\theta})\Big|_0^{2\pi} - \int_0^{2\pi}\frac{e^{in\theta}}{in}dF(e^{i\theta}) = \frac{1}{in}\int_0^{2\pi} e^{in\theta}dF(e^{i\theta})$$

which implies

$$\int_0^{2\pi} e^{in\theta}dF(e^{i\theta}) = 0, \qquad \forall n = 1, 2, 3, ...$$

Since $F(e^{i\theta})$ is of BV, then by the F. and M. Riesz theorem we have that $F(e^{i\theta})$ is absolutely continuous [4, p. 41].

We have $F(z)$ analytic in $|z| < 1$ and continuous in $|z| \leq 1$, then the Poisson-Stieltjes integral of its boundary function is

$$F(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P(r, \theta - t)F(e^{it})dt$$

Differentiating with respect to θ,

$$ire^{i\theta}F'(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial P(r, \theta - t)}{\partial\theta}F(e^{it})dt = \frac{1}{2\pi}\int_0^{2\pi} -\frac{\partial P(r, \theta - t)}{\partial t}F(e^{it})dt$$

and integrating by parts, using that $F(e^{it})$ is absolutely continuous and taking $z = re^{i\theta}$,

$$izF'(z) = \int_0^{2\pi} P(r, \theta - t)ie^{it}F'(e^{it})dt$$

So we have $izF'(z) \in H^1$ which implies $F'(z) \in H^1$. □

Since for each $r \in (0,1)$ we have

$$L_r = r\int_0^{2\pi} |F'(re^{i\theta})|d\theta < \infty$$

and these integrals are non-decreasing (because $|F'(re^{i\theta})| \leq |F'(r'e^{i\theta})|$ if $r < r'$, for being F' analytic in $|z| \leq 1$), then exists the limit when $r \to 1$. Finally, we just have to see that indeed

$$L = \lim_{r\to 1} L_r$$

but we have already seen in the proof of theorem 5.5 that $F(e^i\theta)$ is in fact absolutely continuous, so the length of $\partial\Omega$ is actually given by (see [13, p. 231]):

$$L = \int_0^{2\pi} |F'(e^{i\theta})|d\theta$$

so finally,

$$A_r(\Omega) \leq \frac{L_r^2}{4\pi} \xrightarrow{r\to 1} A(\Omega) \leq \frac{L^2}{4\pi}$$

We have proved that

Theorem 5.6 (Isoperimetric Inequality II)**.** *Let be $\Omega \subseteq \mathbb{R}^2$ such that $\partial\Omega$ is a rectifiable Jordan curve, then*

$$A(\Omega) \leq \frac{L(\partial\Omega)^2}{4\pi} \tag{5.2}$$

and we have an equality if and only if Ω is a disk. [22]

Moreover, is easy to extend it for any domain of the plane:

Corollary 5.7. *Let $\Omega \subseteq \mathbb{R}^2$ be any bounded domain, rectifiable or not, and with possible holes, then we still have the isoperimetrical inequality (5.2).*

Proof. If it is not rectifiable, since the perimeter is infinite, we have the inequality trivially. On the other hand, if it is a rectifiable domain and we make any hole, the perimeter increases while the area decreases, and hence the inequality still holds. □

Conclusion

We have seen with the isoperimetric inequality how a simple question about shapes can be approached in so many ways, and although it has been around since the Greeks it is still of interest proving it with new mathematical tools. This classic isoperimetric inequality leads us to consider inequalities of other quantities with the same basic property: the circle is the domain that reaches the equality. We have also seen how the Hardy and Bergman spaces arise naturally when working on the plane and how they are the natural spaces to work when considering the perimeter or the area, and in consequence other geometrical quantities that depend on the shape of the domain. In particular, we considered the torsional rigidity and the principal frequency, which are substantially more complex and in general we can only find approximations of them.

After proving some inequalities in the classical way, our goal was to prove them again with several recent results that involve Toeplitz operators. In consequence, we have seen how with really smooth changes we can prove the first two inequalities and almost the third one. To do this, we had to work again with the Hardy and Bergman spaces but using the structure of Hilbert space of the case $p = 2$. One kind of operators that we have been found to be important is precisely the Toeplitz operators, and in this work we have seen its close relation with geometric quantities. These operators and spaces are still a wide area of interest and we have only presented a tiny portion of its applications.

Although the principal goals of the work have been achieved, one possible extension would be proving the Faber-Krahn inequality with operator theory (this is still an open problem) in the same way we proved the other inequalities, which would be given by a lower bound of the commutator of Toeplitz operators. Another natural extension is dealing with higher dimensions, and we notice how in this case we could not work like the first proof due to its dependence of the dimension 2. However, it is easy to consider Bergman spaces of a domain $\Omega \subseteq \mathbb{C}^n$ and take similar arguments [9].

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