UNIVERSITAT DE BARCELONA

Three Essays on Dynamic Games with Asymmetric Agents

Joakim Alderborn

PhD in Economics

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PhD candidate:

Joakim Alderborn

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Jesús Marín Solano

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Chapter 1

Introduction

1.1 Some Notes on Dynamic Games

The content of this thesis falls within the field of dynamic games. In this introductory chapter, we will briefly describe some basic game theoretical concepts which will be appear frequently throughout the thesis, and which may be unfamiliar to the reader. We will also provide a summary of the content of the thesis. For a general introduction to dynamic games with applications in economics, see Dockner *et al* (2000).

The key aspect of a dynamic game is the presence of time and repeated action. As a rule, each agent (or player) has preferences over his own individual actions, both present and future, and over the "state of the game", which evolves over time as a result of the actions of all agents. In a noncooperative solution to the game (also known as a competitive solution), each agent chooses the decision rule (also known as a control rule or a strategy) that he considers to be optimal, given the decision rules of the other agents. In a cooperative solution, the agents act jointly, by forming a coalition such that the decision rule of each member is decided upon by the coalition as a whole. It is said that the coalition is stable if each agent is at least as well off (as measured by his own subjective preferences) in the cooperative solution than in the noncooperative solution. An important part of research in dynamic games is finding the conditions under which, for particular games, coalitional stability is obtained.[1]

In economics, there's a long-standing tradition of analyzing dynamic games by using the so-called discounted utility model. This framework goes back at least to Samuelson (1937). It assumes that each agent, in each time period, obtains some amount of utility (or happiness) from his actions and from the state of the game. Utility obtained at some future point in time is discounted relative to present utility at some constant rate, which reflects the impatience inherent in human decision making: other things being equal, it is preferable to receive a given

[1] Some papers analyze coalitions that contain some but not all of the agents in the game. This is sometimes referred to as partial cooperation (see e.g. Breton and Keoula (2014)). The coalition that contains all agents is then referred to as the global coalition. We shall not deal with this type of solution in this thesis.

amount of utility sooner than later. The sum of discounted utilities that an agent expects to receive is referred to as his intertemporal utility.

Mathematically, we can describe this as follows. Let time be indexed by $t \in 0, 1, ..., T$. The (discrete time) intertemporal utility function of agent i at time t can be represented as

$$J_{i,t}(x_t, c_s) = E_t \left[\sum_{s=t}^{T} \theta_{i,s-t} U_i(x_s, c_s) \mid x_t \right] .$$

All the intertemporal utility functions that we use in this thesis can be written in this general form.[2] The notation is as follows.

- x_t is a vector that captures the state at time t of the system (of the game) within which the agent acts. It contains all the state variables.
- c_t contains the control variables at time t, which the agents have direct control over, and which they use to steer the system. We will write $c_{i,t}$ to refer to the particular variables that are chosen by agent i at time t.
- $U_i(x_s, c_s)$ is the agent's utility function, which determines the amount of utility received at time s, as a function of the state and control variables.
- $\theta_{i,t} \in (0, 1]$ is the agent's discount function, which determines the discounting of utility obtained t periods of time into the future.
- T is the end of the planning horizon, the time at which the game 'ends'. If $T \to \infty$, we say that the planning horizon is infinite.

In many applications, x_t is not a vector but a single variable, which captures the state of the system in a single number. In the applications of this thesis, we will interpret state variables as the gross domestic product and rate of price inflation of a country, as the wealth owned by a household and as the biomass of fish in a territory of water. The economic interpretation generally imposes constraints on which values the variables are allowed to take. For example, a variable that represents a quantity must necessarily be nonnegative.

The law which determines how the state develops across time can be written as

$$X_{t+1} = f(x_t, c_t, Z_{t+1}) ,$$

where Z_{t+1} are random events that affect the development of the state variables. Random, in this context, means that the outcome of the event is unknown to the agents up until the point in time when the event takes place. This set of difference equations can be referred to as the dynamics of the game.[3] Depending on the application, the utility functions and the dynamics

[2] If time is continuous, we would write an integral instead of a summation sign.

[3] If time is continuous, we obtain the corresponding set of stochastic differential equations.

might have other names. In the business cycle model of Chapter 2 of this thesis, we will refer to the utility functions as loss functions. In the fish war model of Chapter 4 we will refer to the dynamics as a growth function.

1.2 Solution Concepts

In each chapter of this thesis, we will look for cooperative and noncooperative solutions to particular dynamic games. Hence, we need to specify precisely what we mean by these terms.

The decision rules that solve the game determine a particular control variable $c^*_{i,t}$ for each agent and time period. In this thesis, we will invariably look for so-called Markov decision rules, which means that the decision rule is a function of the current state and time. More precisely, the information which the agents use to make decisions is completely captured by the current state and time. Hence, we will write $c^*_{i,t} = \phi_{i,t}(x_t)$. If the decision rules are stationary, i.e. independent of time, we write $c^*_i = \phi_i(x_t)$. A Markov decision rule has convenient properties that simplify our analysis. They imply that past events effect current behavior only through the current state, which is an intuitive property of the solution. For example, for a person who is drawing up a plan for future business expenditures, what's relevant is his current budget, not how that budget developed in the past.

In a noncooperative solution, we will impose that the decision rules $\phi_{i,t}(x_t)$ have the property of being a subgame perfect Nash equilibrium to the game. This implies, first, that the solution must be an *intertemporal equilibrium*, meaning that at no potential state and time (that is, no (x_t, t)) does agent i have the incentive to substitute some other decision rule for $\phi_{i,t}(x_t)$. It also implies that the solution must be an *intratemporal equilibrium*, meaning that within each time period, the action of each agent is the optimal response to the actions of the other agents. It follows that $\phi_{i,t}(x_t)$ must satisfy

$$\phi_{i,t}(x_t) = \arg\sup_{c_{i,t}} J_{i,t}(x_t, c_{i,t}, \phi_{-i,t}(x_t), \phi_s(x_s))$$

for all (i, x_t, t), where $s = t+1, ..., T$. That is, for each agent and in each time period, it is optimal to act in accordance with the decision rule $\phi_{i,t}(x_t)$, conditional on that every other agent does so in the present, and that all agents do so in the future.

In a cooperative solution, the criteria for a solution are different. This is because when the agents form a coalition and act jointly, the "entity that acts" is in effect the coalition rather than the individual agents. That is, there is in effect only one agent. When there is only one agent in the game, the concept of an intratemporal equilibrium breaks down. Hence, the solution only needs to satisfy the criterium for an intertemporal equilibrium. The intertemporal utility function of the coalition can be written as

$$J_t(x_t; c_s) = \sum_{i=1}^{N} \lambda_i J_{i,t}(x_t, c_s) \ , \tag{1.1}$$

which is a weighted sum of the intertemporal utility functions of the N number of agents. The relative sizes of the weights λ_i determine the relative influence of the agents over the behavior of the coalition. The vector $\phi_t(x_t)$, which contains the equilibrium decision rules for each member of the coalition must then satisfy

$$\phi_t(x_t) = \arg\sup_{c_t} J_t(x_t, c_t, \phi_s(x_s)) \;, \tag{1.2}$$

where $s = t+1, ..., T$. That is, in each time period, it is optimal for the coalition to act in accordance with the decision rule $\phi_t(x_t)$, conditional on that the coalition does so in every future period.

In most of the games we solve in this thesis, we will restrict ourselves to finding decision rules in which the control variables are linear in the state variable(s) and independent of time (other than indirectly through the state). These are known are *linear decision rules.* Within this group of decision rules, we will find in our models that there are unique decision rules that satisfies the properties of cooperative and noncooperative solutions.

The intuition of an equilibrium solution for noncooperation and cooperation, as laid out above, is straightforward in discrete time models. When time is continuous, it is less obvious how an equilibrium solution shall be defined. The reason is that when a time period collapses into a single point, any action taken during that period has no impact on the course of events (i.e. on the development of the state variables x_t), nor on the intertemporal utility functions. An elegant way around this problem was formulated in Ekeland and Pirvu (2008) and Ekeland and Lazrak (2010). They allowed the agent at time t to control action over the interval $[t, t+\epsilon]$, thus allowing him to gain some influence over the development of the state of the system for an extended period. Then, by taking the limit $\epsilon \to 0$, the period vanishes, but the limit is constructed in such a way that the essence of the behavior is preserved. The mathematical details of this solution concept are quite involved, and we shall not bother with them at this stage of our discussion.[4]

An alternative definition of an equilibrium solution was suggested by Karp (2007). His approach is to first discretize the entire time interval $[0, T]$ into periods of length ϵ. Then, the problem can be solved as a discrete time problem. The variable ϵ will appear in the solution, and by taking the formal limit $\epsilon \to 0$ one will (hopefully) obtain valid expressions for a solution in continuous time. In general, it seems that decision rules based on one of these solution concepts coincides with that of the other.[5]

In Chapter 2 and Chapter 3 of this thesis, we solve dynamic games in continuous time. In both cases we will rely on the definition of an equilibrium solution due to Ekeland and Pirvu (2008) and Ekeland and Lazrak (2010). In Chapter 4, we solve models in discrete time.

[4] We will look at this in more detail in Chapter 3.

[5] See also the discussion on this in Marín-Solano and Navas (2010).

1.3 The Concept of Inconsistent Time Preferences

We've seen that the concept of intertemporal equilibrium is at the core of how we will characterize the solutions to dynamic games in this thesis. An alternative type of solution is to choose actions in each time period in such a way as to maximize the agent's sum of utilities as discounted back to the initial time period. That is, the intertemporal utility function of agent i is

$$J_{i,t}(x_t, c_s) = E_t\left[\sum_{s=t}^{T} \theta_{i,s} U_i(x_s, c_s) \mid x_t\right] .$$

This is referred to as a commitment (or precommitment) solution, the intuition being that the agent at time $t = 0$ commits to a set of actions that, from the perspective of that time, appears optimal, and follows them until the end of the planning horizon has been reached and the problem ends.

In general, a commitment solution is not an intertemporal equilibrium, because the actions that appear to be optimal at time $t = 0$ might not be optimal at time $t > 0$. That is, even though the agent's discount function remains the same, it might be the case the agent's current position in time affects what decision rule is prefered. The decision rule that at $t = 0$ appears optimal for $t = 1$ may no longer be optimal when $t = 1$ is the present. When this is the case, we say that the agent has inconsistent time preferences. He is inconsistent because he prefers to change his decision rule rather than to stick to the one that he decided upon earlier. If the agent's time preferences are such that the same decision rule is always considered to be optimal, regardless of the current time period, we say that the agent's time preferences are consistent.

The first paper to systematically treat the issue of inconsistent time preferences (within the context of a discounted utility model) seems to have been Strotz (1956). He showed that the time preferences of agent i are consistent if and only if his discount function is given by the exponential function

$$\theta_{i,t} = \beta_i^t ,$$

where $\beta \in (0, 1]$.[6] This implies that the function changes in time at the constant rate $\beta - 1$. Hence, consistent time preferences are characterized by the fact that the discount rate is constant. A coalition with several members has consistent time preferences if and only if all the members have the same constant discount rate. If this is the case, commitment and equilibrium decision rules coincide.

Experimental research from behavioral economics has consistently shown that people typically do not have consistent time preferences. See Frederick et al (2002) for a survey on the literature on this subject up to the year 2002. Moreover, it suggests that people's discount rates tend to decrease in magnitude over time. Hence, a more realistic model would have $\theta_{i,t}$ decrease at a rate which decreases in magnitude. Despite this, much of the literature on dynamic games

[6] His model was in continuous time, but the result extends to discrete time.

has assumed that all agents have constant and identical discount rates. Most likely, this is for reasons of tractability, although economic justifications have been provided as well.[7] As a rule, a commitment solution can be found by using standard techniques in optimal control, such as Hamilton-Jacobi-Bellman equations and Pontrayagin's Maximum Principle. An intertemporal equilibrium with general discount functions, on the other hand, requires more elaborate dynamic programming techniques. Moreover, in some common economic models, analytic solutions can be obtained only in the case of commitment. For example, Marín-Solano and Navas (2010) solve the classic Merton model (see Merton (1971)) for an agent with a general discount function, and show that in the case of a power utility function, an explicit solution can be obtained in the commitment solution but not in the intertemporal equilibrium solution. However, if discount functions are exponential, it is enough to solve for the commitment solution, because it coincides with the intertemporal equilibrium solution.

1.4 Summary of The Thesis

The thesis is divided into three chapters (excluding the concluding chapter and this introduction). Each of them falls within a separate field in Economics. The first is on business cycle theory, a field within Macroeconomics. The second is on life insurance models, which is generally regarded as a field within the actuarial sciences. The third is on the well-known fish war model, a topic within Environmental and Resource Economics.

In Chapter 2, we construct a Macroeconomic differential game where the agents are the fiscal and monetary authorities of an economy, i.e. the government and the central bank. The authorities use the government deficit and the nominal interest rate as policy instruments to steer the economy, the state of which is captured by aggregate output (or real gross domestic product) and price inflation. The objective of the agents is to maintain those variables at some given desired levels, which we think of as the economy's general equilibrium. As is standard in the literature, we assume that the use of policy instruments brings a cost, which must be balanced against the cost of remaining in disequilibrium. We investigate the development of the economy under the assumption that the government and the central bank discount future utilities at constant but different rates, which implies that a coalition formed by the two has inconsistent time preferences. We compute both noncooperative and cooperative solutions.

The main finding in Chapter 2 is that in the noncooperative solution, the authorities will enact policies that are mutually contradictory. By this we mean that the government and the central bank try to steer aggregate output in opposite directions, and therefore "cancel out" the actions of one another. This implies that there is room for Pareto improvement. Importantly, we show that this situation is a result of the asymmetry in the discount rates, and that in the cooperative solution the mutually contradictory policies are removed. Based on this, we make

[7]For an economic argument in favor of using constant discount rates, see Rubinstein (2003).

the claim that policy coordination between the government and the central bank is especially useful when they operate under different discount rates. Since, in general, there's no reason to assume that governments and central banks operate under the same discount rates, our findings have practical implications for the conduct of fiscal and monetary policy. For example, we suggest that in countries with a democratic political system, the government may have a higher discount rate because it is controlled by elected politicians, while the central bank is controlled by unelected technocrats.

In Chapter 3, we construct a continuous time life insurance model in the tradition of Richard (1975) and Pliska and Ye (2007). We model two agents that share a common household budget, but have separate expenditures on consumption and life insurance. At each point in time, each agent withdraws money from the budget at some rate to spend on his consumption expenditures and insurance-premium. At some randomly determined point in time one agent dies, and the survivor receives the life insurance payment and continues to consume until he too dies. The unusual feature of our model is, as in Chapter 2, that the agents have different discount rates of future events, meaning that the household as a whole has inconsistent time preferences. Thus, we combine two different features which up to now have been developed separately within the literature on life insurance models: on the one hand, households with multiple agents, and one the other hand, households with inconsistent time preferences. As is standard in life insurance models, we only compute the cooperative solution. That is, it is assumed that no agent will leave the household under any conditions.

The model of Chapter 3 contains quite a large number of parameters for describing the properties of the agents and the environment in which they act. Hence, there is plenty of opportunities for making comparative dynamics. The most interesting result, perhaps, is that if one agent becomes less patient (i.e. his discount rate increases), the household reacts by shifting spending to his insurance premium from that of the other insurance premiums. We provide an economic interpretation of this and several other results.

In Chapter 4, we investigate the fish war model due to Levhari and Mirman (1980), under the assumption that the agents have asymmetric quasi-hyperbolic discount functions (also known as $(\beta - \delta)$ preferences), which implies that the agents have inconsistent time preferences. Hence, this chapter differs from Chapter 3 and Chapter 4 in that the individual agents have nonconstant discount rates. We look at three types of fish war models. In the first, the agents have logarithmic utility functions and the growth function is nonlinear, which is the standard case. What is nonstandard are the discount functions. The behaviors under noncooperation and cooperation are compared. We find that the stability of the coalition in the cooperative solution is not sensitive to the inconsistency in time preferences introduced by quasi-hyperbolic discounting. In the second model, we build a nonstandard fish war game with power utility functions, quasi-hyperbolic discount functions and a linear stochastic autocorrelated growth function. In contrast to the previous model, we find that stability breaks down quickly when the inconsistency in time preference increases. Then, as our third model, we extend the second model by allowing asymmetric utility functions and a nonlinear growth function. Here, we make some observations on how cooperative solution is affected by the presence of risk and by asymmetry in risk aversions. Chapter 4 may be regarded as the "experimental" chapter of the thesis: we develop a new set of dynamic programming algorithms and apply them to solve specific problems.

1.5 The Mathematical Methodology

We've seen that the thesis covers diverse topics. What is common to all three of the main chapters is the underlying mathematical theory: each chapter features a dynamic game with two agents that may differ in terms of their discount functions and/or utility functions.

In Chapter 2, the model we build takes the form of an infinite horizon linear-quadratic game[8] in continuous time with two agents that have asymmetric utility and discount functions. In order to solve the model, we make use of a dynamic programming algorithm developed in de Paz et al (2013), which is based on the solution concept due to Ekeland and Lazrak (2010). Chapter 3 follows the same basic framework, except that we extend the dynamic programming algorithm to the case of stochastic dynamics. Specifically, we use a Brownian motion to model the random development of the dynamics, and exponential distributions for the event of the death of an agent. The utility functions are modeled as power functions. In Chapter 4, the agents are not only asymmetric but also have inconsistent time preferences. To handle this, we develop a set of dynamic programming algorithms for handling the case of stochastic discrete time games with general asymmetric utility and discount functions. The algorithms also allow for autocorrelation in the stochastic process that drives the dynamics, and we make use of this property in one of the fish war models.

An important aspect of the thesis is the repeated use of numeric methods, especially in Chapter 4. As a rule, we attempt to solve the models using analytic methods as far as this can be done. When the analytic methods can take us no further, we assign parameter values and solve the model numerically. In the models of Chapter 2 and Chapter 3, this implies that we use dynamic programming to derive a system of nonlinear equations, which is then solved numerically. However, in some cases in Chapter 4 we need to employ numeric methods at a much earlier stage of the process. Our reliance on such methods tends to increase in the complexity of the model.

It can certainly be argued that numerical methods are undesirable because they don't provide "clean expressions" for the solutions. It is true that analytic solutions, if they are sufficiently simple, provide direct information about the causal relationships between parameter,

[8]This means that the utility functions are second degree polynomials and the dynamics are first degree polynomials.

state variables and control variables, in a way that numeric solutions do not. But it's also true that if the analytic expressions are long and complicated, as they tend to be in complex models, then numeric methods may provide more information about said causal relationships than the expressions do. In any case, if analytic expressions are unattainable, the numeric methods are the only way of understanding the solution, other than mathematical intuition.

Chapter 2

The Impact of Coordination on Fiscal and Monetary Policy under Asymmetric Time Preferences

2.1 Introduction

Since the 1980's, there has been a tendency for countries to move towards increased central bank independence. This "divorce" of monetary and debt management, as it's referred to in Laurens and Piedra (1998), was motivated primarily by the need for price stability. It was believed that by removing monetary policy decisions from the political process, inflation could be stabilized, because policy would no longer be subject to the whims of voters and electoral cycles. Therefore, to the extent that inflation is damaging the performance of the real economy, central bank independence would increase overall welfare. Nowadays there exists a large body of evidence, both theoretical and empirical, in support of the claim that central bank independence is an effective method for controlling inflation.[1]

Notwithstanding the positive consequences of the reforms, the separation of monetary and fiscal policy decisions also created reasons for concern. When the major instruments for stabilizing the economy - the fiscal deficit and the steering nominal interest rate - are controlled by two separate institutions that are (more or less) independent from each other, there arises a need for policy coordination. In deciding the proper fiscal policy, the government will need to take into consideration how the central bank will respond, and vice versa. Consequently, both the monetary and fiscal authorities might find it useful to consult each other before they pursue their individual objectives, that is, to coordinate their decisions. As noted in Lambertini and Rovelli (2004), the need for coordination arises primarily because the policy instruments of the government and the central bank have similar effects on aggregate demand, and hence

[1] An Overview of the research on this topic since the 1980's and onwards can be found in Parkin (2012).

also on aggregate output and inflation. Fiscal and monetary expansions alike will, other things equal, increase output and inflation to levels above where they would have been otherwise. Contractions will have the opposite effect.

In general, the monetary and fiscal authorities will work under a set of objectives or guidelines, such as the commonly used inflation targeting. One may then measure the benefits of policy coordination by the extent to which it helps the authorities to reach those objectives. Moreover, the benefits of coordination will likely depend on the specific set of objectives that are being used (such as, for example, inflation targeting), how easy they are to reach without coordination, and what type of disequilibrium the policy makers are responding to[2]. In the absence of coordination, the authorities may pursue different policies because they prioritize different aggregates or because they have different time preferences. For example, when reacting to a shock in aggregate demand, one authority may prefer a fast return to equilibrium while the other prefers a slow and gradual return. Thus, Hanif and Arby (2003) make the point that the objectives of government and central bank policy may be contradictory, in the sense that the authorities, when they pursue policy independently without coordination, undermine the actions of each other. This concern is also mentioned in Abdel-Haleim (2016). In such situations, coordination may have substantial benefits for both authorities.

In this chapter, we analyze the argument that the need for policy coordination may crucially depend on the time preferences of the government and the central bank. By policy coordination we will mean a situation where the government and the central bank are actively consulting and communicating with each other, with the purpose of achieving a Pareto optimal outcome. We do not mean a situation where they act individually and obtain that same outcome because their individual objectives happen to produce it.[3]

The following example illustrates why policy coordination may be important when the government and the central bank have different time preferences. Assume that the disequilibrium faced by the authorities is such that inflation and output are both above their equilibrium levels. Then, they will want to enact proper contractionary policies to produce a smooth and stable return of aggregate output and inflation to their equilibrium levels. Moreover, they will want to do so without making aggressive use of their policy instruments, such as the government deficit and the steering nominal interest rate. However, while output can be affected by policy quite directly, for instance by changing public demand, the control of inflation is more indirect. Fiscal policy and monetary policy both affect inflation mainly *through* output. As discussed in Blanchard *et al* (2010), decreasing inflation might require driving output temporarily down below its equilibrium level. This is, for instance, what motivated the contractionary policies in

[2]Equilibrium, in this context, means that all the variables in the economy are at what is considered to be their business cycle neutral levels. That is, the economy is neither in a recession or in a state of overheating.

[3]A different definition of policy coordination among the fiscal and monetary authorities is offered in Abdel-Haleim (2016): "Coordination is defined as the necessary arrangements that assure that the decisions taken by both authorities are not contradictory".

the United Kingdom in the 1980's. Hence, it might not be sufficient for the authorities to steer output back to its equilibrium level. They will have to go further and decrease output below equilibrium if that is the only way to reduce inflation. Here, there's a trade-off to be made between losses due to high inflation, and losses due to the use of policy instruments. Moreover, there's an intertemporal aspect to the trade-off. By driving down output far below its equilibrium level, the authorities suffer large losses in the short term due to the aggressive use of their policy instruments. But as a result of bringing down inflation quickly, they avoid losses in the long term that would occur if inflation stayed high. How they choose to manage this trade-off will depend on their time preferences. However, if they have different time preferences, they will have different preferences regarding how to manage the return to equilibrium, and there will be a conflict of interest, possibly leading to contradictory policies. For example, if the government believes that the central bank is being too aggressive in pursuing contractionary policies, it can counter those policies by pursuing its own expansionary policies. The conflict can potentially be resolved by coordination of monetary and fiscal policy.

To analyze the issue described above, we will in this chapter solve a simple dynamic game. We will model the government and the central bank as agents in an infinite horizon policy differential game and solve for the noncooperative and cooperative solutions. Throughout the chapter, we will think of the noncooperative solution of the game as the situation where the fiscal and monetary authorities act independently from one another and don't use policy coordination. Likewise, we will think of the cooperative solution as reflecting policy coordination. The numeric solutions to the game use standard parameter values, and the gains from coordination are measured with loss functions of the sort that are commonly used in the literature on monetary and fiscal policy. As is typically the case, we will assume the authorities have asymmetric loss functions, which reflects the fact that they may target different variables in their policies.

As is standard in the literature, we will assume that the agents discount future losses at some constant rate. In other words, the discount functions are of the standard exponential form. However, we deviate from the standard case by allowing asymmetric discount rates. This implies that the government and central bank may have different preferences in terms of how to balance present against future losses. More precisely, we will assume in the numeric part of the chapter that the government has a higher discount rate than the central bank. This may be motivated by the fact that in countries where the political system is democratic, the government is forced to focus on short term effects in order to win elections, while the central bank is freer to focus on the long run. In countries that are not democratic, the government might still be more shorted sighted than the central bank, since it is eager to avoid public discontent that can lead to social disorder and revolutions. Such differences in time perspective, as we shall see, create reasons for coordination.

As was mentioned in the introductory chapter, asymmetric discount rates imply that the coalition in the cooperative solution has inconsistent time preferences. Consequently, we need to look for intertemporal equilibrium solutions, since they differ from commitment solutions. The theory that we employ for this purpose has been developed quite recently (see de Paz *et al* 2013). To the best of our knowledge, the present study is the first to apply it to a fiscal and monetary policy game.

The rest of this chapter is organized as follows. Section 2 contains a literature review and Section 3 presents the model that we will use. In Section 4 and Section 5 we solve the noncooperative and cooperative solutions respectively. (Parts of the solutions are left for the appendix). Section 6 employs numeric methods to show the effects of asymmetric discounting on the benefits of policy coordination. The results are illustrated with graphs and tables. Section 7 is a brief extension that illustrates how our model can be used to investigate other asymmetries, such as the consequences of when the central bank operates under inflation targeting and the government does not. Section 8 concludes the chapter.

2.2 Literature Review

One early contribution to the literature on differential policy games is Turnovsky *et al* (1988). They build a two-country model to analyze the effects of monetary policy on welfare. The agents of the game are the monetary authorities of the two countries, and both are charged with steering their respective economies toward equilibrium. The initial disequilibrium is created by a shock to the real exchange rate, which creates an imbalance in trade flows and hence affects output. Since monetary policy affects the real exchange rate, the central banks can reduce the loss in welfare caused by the shock. The authors conclude that policy coordination reduces welfare losses by between 6 and 10 percent.

An example of a game with interaction between both a monetary and a fiscal authority, acting in the same country, can be found in Dixit and Lambertini (2003). They build a model with monopolistic competition, which creates inefficiently low output, and gives policy makers an incentive to intervene in the economy even in the absence of any external shocks. Moreover, both the government and the central bank can operate under either discretion or commitment, and this has implications for the strategic interaction of the authorities. The game is solved for the noncooperative (Nash equilibrium) and leadership (Stackelberg) solutions. One conclusion of the paper is that, from a welfare point of view, fiscal leadership is generally better than monetary leadership, although both may be worse than the noncooperative solution.

Another paper with interaction between fiscal and monetary authorities is Di Bartolomeo and Di Gioacchino (2008). They too include a leadership version of the game, and solve for the cases of monetary leadership and fiscal leadership alongside the noncooperative solution. The game is solved in two stages. The first one establishes whether the differential game is sequential and which of the two agents that will have leadership. The paper also includes a discussion on what type of leadership scenario may be the most realistic.

A somewhat different – and creative – approach comes from Cellini and Lambertini (2006), where a game with two agents is set up between the central bank and the representative household of the private sector. The paper is concerned with the discretion vs commitment debate in monetary policy, and attempts to analyze under what conditions the central bank can exploit a first-mover advantage over the private sector to gain a costless reduction in inflation. For this purpose, a leadership solution is found, with the central bank as the leader.

Leeper (1991) and Bonan and Lukkezen (2019) investigate the need for policy coordination when public debt is taken into concern. In particular, the latter of these papers is concerned with the fact that active use of fiscal policy can lead to large public deficit and hence a risk of a default on the debt. Their paper was motivated by the fact that during the Great Recession, the inefficiency of monetary policy prompted some countries to run large deficits, which resulted in a breakdown of the market for government bonds. They find that when government bonds are considered a risky investment, active fiscal policy increases interest rates and leads to large crowding out effects of private investment, and that these undesirable effects can be mitigated by active monetary policy.

In the late 90's, the creation of the Euro raised interest in models for monetary unions. Van Aarle *et al* (2002) models two countries that have the same currency and hence the same monetary policy. A game with three agents involving the joint monetary authority and the fiscal authority for each country, is solved for a sophisticated macroeconomic model which includes dynamics for output, inflation, wages and unemployment. Moreover, the authors differentiate between a Neoclassical regime, in which excess unemployment is the result of high wages, and a Keynesian regime, in which excess unemployment is due to low aggregate demand. The game is solved for both the noncooperative and the cooperative cases, and the welfare effects of policy coordination are analyzed. The paper has been an important source of ideas for the present study.

A recent contribution to the literature on dynamic games with monetary and fiscal authorities is Anevlalis *et al* (2019). In their model the government and the central bank aim to keep government debt at a given level, by using the fiscal deficit and the rate of money growth as instruments. The paper is similar to the present study in that the model is an example of a linear-quadratic game, and in that both noncooperative and cooperative solutions are derived.

In the papers mentioned above, there are certain "standard questions" faced by the authors. One such question is which specific objectives one assumes that the fiscal and monetary authorities pursue. (That is, what do the intertemporal utility functions look like.) Another is which variables one assumes to be under the direct control of policy makers, as opposed to being steered only indirectly. (That is, which variables are state variables and which are control variables.) For the government, an obvious and common choice for a control variable is the fiscal deficit. For the central bank, a common but in no way dominating approach is to let the central bank control the nominal interest rate. This approach is used in Saulo *et al* (2012), Galí

(2008) and Woodford (2003).[4] On the other hand, Di Bartolomeo and Di Gioacchino (2008) and Van Aarle *et al* (2002) build models where the central bank controls the monetary base. Finally, in Cellini and Lambertini (2006), the central bank is given direct control over inflation itself.

As for the intertemporal utility functions, i.e. the aims of policy, it is generally assumed that utility is measured by loss function, which depend on the deviation of a set of macroeconomic aggregates from their equilibrium levels. It is also standard to let the loss function be quadratic in all its arguments. This is convenient because, if in addition the dynamics are linear, there are well known methods for solving the model. There are also intuitive arguments for the quadratic form: the marginal cost of deviation increases in the deviation and, importantly, small deviations in several variables from their optimal levels are preferable to a large deviation in a single variable. Typical arguments are output, inflation, the government deficit and the nominal interest rate. Hence, in a seminal paper by Kydland and Prescott (1977), the central bank's loss function depends on the deviation of inflation and output from their equilibrium levels. The same setup is adopted in Dixit and Lambertini (2003) and Cellini and Lambertini (2004). As pointed out in Woodford (2003), this simple form is widely used in the literature on monetary policy.

Other setups are also common. In Lambertini and Rovelli (2004), the central bank's loss function includes the real interest rate rather than output, while the government's loss function includes both of those variables along with the fiscal deficit. More extensive loss functions can be found in van Aarle *et al* (2002) where both the central bank and the government attempts to control deviations in output, inflation and unemployment. Another extensive setup is offered in Engwerda (2006), where the monetary base and the government debt enter the loss functions.

In general, the choice of loss functions will depend on how much one assumes that the fiscal and monetary authorities "care" about each variable. For instance, since most central banks have the main objective of ensuring price stability, inflation is an obvious choice for the monetary authority's loss function. The government, on the other hand, might be more concerned with unemployment and output.

As the preceding paragraphs make clear, it is common to include in the loss functions not just major aggregates like output and inflation, but also the policy instruments. What is the economic justification for this? In general, governments are advised not to deviate too much from running a balanced budget. Low deficits may require cuts in welfare services, which may be unpopular among the population of the country, while high deficits tend to crowd out private investment. Moreover, as noted in van Aarle *et al* (2002), for members of the European Union large deficits may result in sanctions on the country, as specified in the union's Stability and Growth Pact. It is less clear why monetary policy activism is costly, but this assumption is

[4]In Galí (2008), there's also a short section discussing the case in which the central bank controls the monetary base.

nevertheless common in the literature. One can, perhaps, imagine that large volatility in the nominal interest rate makes financial returns unpredictable and discourages investment, which hurts overall economic development. Van Aarle *et al* (2002), discussing the European Central Bank, note that "other things equal policy makers prefer a constant level of their instrument rather than to undertake changes all the time".

Finally, an issue which is often ignored in the literature is what exactly the loss functions are a measure of. In models on monetary policy, the loss function of the central bank is sometimes derived from underlying optimization problems describing the behavior of households and firms. This is the case in Galí (2008), Woodford (2003) and more generally in the New Keynesian Model. In those cases, the loss function can be interpreted as reflecting welfare or social utility. However, problems arise when the model contains both a government and a central bank with separate loss functions, as is the case in policy games, since there is no obvious way to decide which of the two loss functions that reflect welfare. In Dixit and Lambertini (2003) it is assumed that the government's loss function is the welfare loss, while the central bank has a loss function that differs from welfare. Van Aarle *et al* (2002) seem to assume that both loss functions capture welfare losses, although they do not explain whether this means that there are two separate measures of welfare. Another approach, however, is to assume that both loss functions reflect just the objectives that the authorities follow when deciding on policy, with no reference to welfare. The magnitude of the loss is then simply the extent to which an authority fails to reach its objectives. That is the interpretation we will use here.

2.3 The Model

We will now define the loss functions and the dynamics that make up our model. We will assume that the government controls the fiscal deficit and that the central bank controls the nominal interest rate. The two other variables in our model, which are output and inflation, are state variables that the authorities can impact indirectly.

The loss functions for the fiscal authority and for the monetary authority are given by

$$L_{F,t}(x_t, \pi_t, g_t) = \lambda_{Fx} x_t^2 + \lambda_{F\pi} \pi_t^2 + \frac{\lambda_{Fg}}{2} g_t^2$$

and

$$L_{M,t}(x_t, \pi_t, i_t) = \lambda_{Mx} x_t^2 + \lambda_{M\pi} \pi_t^2 + \frac{\lambda_{Mi}}{2} i_t^2 \ .$$

Here, x_t is output at time $t \in [0, \infty)$, π_t is inflation, g_t is the fiscal deficit and i_t is the nominal interest rate. The λ-parameters are all positive. Hence, our setup is the most standard one with output and inflation as arguments, except for the addition of the policy instruments. Future losses are discounted at the rates ρ_F for the government and ρ_M for the central bank. Hence, the intertemporal loss functions are

$$J_{F,t}(x_t, \pi_t, g_s) = \int_t^{\infty} e^{-\rho_F(s-t)} \left(\lambda_{Fx} x_s^2 + \lambda_{F\pi} \pi_s^2 + \frac{\lambda_{Fg}}{2} g_s^2 \right) ds$$

and

$$J_{M,t}(x_t, \pi_t, i_s) = \int_t^{\infty} e^{-\rho_M(s-t)} \left(\lambda_{Mx} x_s^2 + \lambda_{M\pi} \pi_s^2 + \frac{\lambda_{Mi}}{2} i_s^2 \right) ds \ .$$

Note that each authority obtains losses only from its own instrument, since it is not responsible for the actions of the other authority. As is standard, output and the fiscal deficit are measured in terms of log deviations from their desired levels. This means that all four variables have (approximately) a percentage interpretation and not a quantity interpretation. Henceforth when we write output or government deficit, we are referring to the log deviations of those variables from their equilibrium levels.

We will assume that the inflation target, to which inflation will be steered after a shock, is equal to zero. The assumption of a zero inflation in equilibrium is in line with Galí (2008), where the inflation target is assumed to be "close to zero". In practice, central banks tend to prefer inflation levels slightly above zero because deflation is viewed as more harmful than inflation. In our model, there is no difference between inflation and deflation in terms of impact on the loss function, and hence a target of zero is natural.[5]

The λ-coefficients determine how much loss is incurred from one variable relative to the others. We can therefore think of them as determining whether an authority operates under output targeting or inflation targeting. For instance, if $\lambda_{M\pi} > \lambda_{Mx}$ the central bank will prioritize inflation stability over output stability, because deviation in inflation from its equilibrium level is more costly. Thus, the parameters of the loss function determine which policy regime is followed.

Next, we will define the dynamics that govern how output and inflation evolve as functions of the policy decisions. For output, we will assume the relationship

$$\frac{dx_t}{dt} = \alpha g_t - \beta(i_t - \pi_t) \ , \tag{2.1}$$

where α and β are positive parameters. This is essentially a continuous time Investment-Saving equation of the kind that is common in models for monetary policy evaluation.[6] The two terms on the right hand side relate output to the fiscal deficit and the real interest rate (which is approximately equal to the nominal interest rate minus inflation): an increase in the fiscal deficit is expansionary while an increase in the real interest rate is contractionary.

[5] It is certainly possible to introduce a non-zero inflation target in our framework. The loss from inflation would then be calculated from the deviation of inflation from the target. This would make no qualitative difference to our model, but it would introduce an additional parameter.

[6] Most equations of this type are in discrete time and have no fiscal deficit, since the purpose is usually to analyze monetary policy only. See Clarida *et al* (1999) and Saulo *et al* (2012) for examples of this. For a version in continuous time, see Cochrane (2017).

The parameters α and β can be viewed as measures of the fiscal and monetary multipliers. For instance, an increase in the government deficit with one percent will, other things equal, increase output by approximately α percent.

For inflation, we assume the relationship

$$\frac{d\pi_t}{dt} = \kappa x_t \ , \tag{2.2}$$

where κ is a positive parameter. This tells us that inflation increases whenever output is above its equilibrium level, and decreases whenever output is below its equilibrium level. It's based on the discrete time Philips Curve as derived in Blanchard *et al* (2010), which has the form $\pi_t = \mu\pi_{t-1} - \kappa(u_t - u_n)$, where u_t is the unemployment rate and u_n its equilibrium level. By assuming a simple linear relationship between unemployment deviation and output deviation, the continuous time version of this Philips curve can be transformed into (2.2).[7]

Notice that the fiscal deficit and the nominal interest rate appear in equation (2.1), but not in equation (2.2). This implies that the policy makers can control output through the fiscal deficit and the nominal interest, while their control over inflation is only *through* output. That is, their control over output is more direct than is their control over inflation. In particular, inflation will decrease if, and only if, output is below zero, regardless of policy. Hence, in the case of excess inflation, the benefits of bringing inflation down must be balanced against the cost of low output. This trade-off will be important when we interpret the numeric results of our model. We stress that this key assumption is not unique for our model. Similar assumptions can be found in Galí (2008), Woodford (2003) and Blanchard *et al* (2010). The trade-off is often captured mathematically by a "sacrifice ratio", which denotes how much excess unemployment (or low output) is needed to bring inflation down by one percent. See Blanchard *et al* (2010) for a discussion on this.

Given the decision rules on the government deficit and inflation, the system given by the differential equations (2.1) and (2.2) has a unique solution only if a fixed point on output and inflation is provided. To this end, we will assume some initial condition on (x_0, π_0).

Remark 3.1: We will now describe a potential problem with the dynamics that we have assumed. It may be reasonable to assume that, other things equal, deviations in the level of

[7]Assuming $\mu = 1$, which would mean rational expectations, we get the difference equation $\pi_t - \pi_{t-1} = -\kappa(u_t - u_n)$. To get from unemployment to output, we apply the following argument, which is also from Blanchard *et al* (2010). If output deviation is currently positive, unemployment deviation must be negative, since additional labor is needed to produce the extra goods. The opposite is true if output deviation is negative. This relation is generally known as Okun's Law. Now, a decrease in unemployment shifts the balance of power in the labor market in the favor of the suppliers of labor, which puts upwards pressure on wages. Firms must react to the higher wages by either lowering profits or increasing the growth rate of the prices on their goods. In competitive markets where profit margins are low, the latter option must dominate, and so the growth in prices accelerates. Hence, we may write the Philips relation as $\pi_t - \pi_{t-1} = -\kappa(-x_t)$, which in continuous time becomes (2.2).

output from its equilibrium level (not the percentage deviation) is linear in deviations in the level of the government's budget deficit. We can write this as

$$X_t - \bar{X} = \gamma(G_t - \bar{G})$$

where X_t is the level of output, G_t is the level of government spending and $(\bar{X}, \bar{G})$ are the equilibrium levels. Every euro of government deficit results in γ euros of extra output. Rewrite this as

$$\frac{X_t - \bar{X}}{\bar{X}} = \gamma \frac{\bar{G}}{\bar{X}} \frac{G_t - \bar{G}}{\bar{G}}$$

to obtain an expression relating the percentage deviation of output to the percentage deviation of the government deficit. Using the log deviation notation from before, this can be written approximately as

$$x_t = \gamma \frac{\bar{G}}{\bar{X}} g_t$$

which in differential form is

$$dx_t = \gamma \frac{\bar{G}}{\bar{X}} dg_t \ .$$

This is probably the most intuitive relationship between output and the government deficit. But in (2.1) we have assumed that the differential of output depends on the *level* of government deficit, not the differential. Our assumption is convenient because it allows for easily expressing the policy game as a linear quadratic model. It is a questionable assumption made for reasons of tractability. □

2.4 The Noncooperative Solution

In this section we derive the noncooperative solution to the model. This is the case that we interpret as no policy coordination. We will find that a fully analytic solution cannot be obtained, which motivates the numeric solutions of section 5. The solution that we obtain is stationary (i.e. the decision rules and intertemporal utility functions are independent of time), and for notational simplicity we can therefore omit the time index in some of the following equations. We will let (g^*, i^*) refer to the decision rules that solve the problem.

Each agent solves a separate maximization problem. For each $t \in [0, \infty)$, the government's dynamic programming equation is

$$0 =$$

$$\max_g \left\{ -\lambda_{Fx} x^2 - \lambda_{F\pi} \pi^2 - \frac{\lambda_{Fg}}{2} g^2 - \rho_F V_F(x, \pi) + \frac{\partial V_F}{\partial x}(x, \pi)(\alpha g - \beta(i - \pi)) + \frac{\partial V_F}{\partial \pi}(x, \pi) \kappa x \right\}$$

and the central bank's dynamic programming equation is

$$0 =$$

$$\max_i \left\{ -\lambda_{Mx}x^2 - \lambda_{M\pi}\pi^2 - \frac{\lambda_{Mi}}{2}i^2 - \rho_M V_M(x,\pi) + \frac{\partial V_M}{\partial x}(x,\pi)(\alpha g - \beta(i-\pi)) + \frac{\partial V_M}{\partial \pi}(x,\pi)\kappa x \right\}$$

where $V_F(x,\pi)$ and $V_M(x,\pi)$ are the value functions (i.e. the intertemporal utility functions when we apply the decision rules (g^*, i^*)). The maximization problems yield the decision rules

$$g^* = \frac{\alpha}{\lambda_{Fg}}\frac{\partial V_F}{\partial x} \tag{2.3}$$

and

$$i^* = \frac{-\beta}{\lambda_{Mi}}\frac{\partial V_M}{\partial x} \ . \tag{2.4}$$

We now need to find the value functions. To this end, we substitute the decision rules into the partial differential equations of the value functions to obtain

$$0 = -\lambda_{Fx}x^2 - \lambda_{F\pi}\pi^2 - \rho_F V_F(x,\pi) + \frac{\partial V_F}{\partial x}(x,\pi)\beta\pi + \frac{\partial V_F}{\partial \pi}(x,\pi)\kappa x$$
$$\frac{\partial V_F}{\partial x}(x,\pi)\left(\frac{\partial V_F}{\partial x}(x,\pi)\frac{\alpha^2}{2\lambda_{Fg}} + \frac{\partial V_M}{\partial x}(x,\pi)\frac{\beta^2}{\lambda_{Mi}}\right)$$

and

$$0 = -\lambda_{Mx}x^2 - \lambda_{M\pi}\pi^2 - \rho_M V_M(x,\pi) + \frac{\partial V_M}{\partial x}(x,\pi)\beta\pi + \frac{\partial V_M}{\partial \pi}(x,\pi)\kappa x$$
$$\frac{\partial V_M}{\partial x}(x,\pi)\left(\frac{\partial V_M}{\partial x}(x,\pi)\frac{\beta^2}{2\lambda_{Mi}} + \frac{\partial V_F}{\partial x}(x,\pi)\frac{\alpha^2}{\lambda_{Fg}}\right) \ .$$

To proceed, we need to make a conjecture on the value functions. Since our model is linear-quadratic, we will look for a solution among all decision rules such that the value functions are quadratic in (x,π), that is

$$V_k(x,\pi) = A_k x^2 + B_k \pi^2 + C_k x\pi + D_k x + E_k \pi + F_k \ ,$$

where $k \in \{F, M\}$. Moreover, a close investigation of our model shows that the value functions have no constant nor linear terms, and that the quadratic terms must be nonpositive. Hence, it suffices to consider value functions of the form

$$V_k(x,\pi) = \frac{A_k}{2}x^2 + \frac{B_k}{2}\pi^2 + C_k x\pi$$

where A_k and B_k are negative for both k. To see that $F_k = 0$ must hold, it is enough to recognize that if the current state of the system is $(x,\pi) = (0,0)$, then the value function must be zero because no losses will occur. To see that $D_k = E_k = 0$, it is enough to recognize that any deviation of the state of the system from $(x,\pi) = (0,0)$, no matter how small or in what direction, must make the value function negative because losses occur.

Given the above conjectures and the system of equations in the value functions, we can derive a system of nonlinear equations in the parameters of the value functions. The system can only be solved numerically. The derivation of the system is left for the appendix.

Even though we can't find the value functions analytically, we can derive conditions for global asymptotic stability of the system in (x, π) in the noncooperative solution.[8] If we substitute (2.3) and (2.4) into the dynamics for output and inflation, we obtain a system of differential equations given by

$$\frac{dx_t}{dt} = \frac{\alpha^2}{\lambda_{Fg}}(A_F x_t + C_F \pi_t) - \frac{\beta^2}{\lambda_{Mi}}(A_M x_t + C_M \pi_t) + \beta\pi_t$$

and

$$\frac{d\pi_t}{dt} = \kappa\pi_t \ .$$

The system is globally asymptotically stable if, and only if,

$$\frac{\alpha^2}{\lambda_{Fg}}A_F + \frac{\beta^2}{\lambda_{Mi}}A_M < 0$$

and

$$-\frac{\alpha^2}{\lambda_{Fg}}C_F - \frac{\beta^2}{\lambda_{Mi}}C_M - \beta > 0 \ .$$

(i.e. the trace is negative and the determinant is positive). When we solve the system in the variables A_F, A_M, B_F, B_M, C_F and C_M numerically, we restrict ourselves to solutions such that the resulting system of differential equations in x_t and π_t is globally asymptotically stable. This makes sense because otherwise one or several variables may explode towards infinity. One consequence of this restriction is that although there can in principle be many solutions to the model, some may be ruled out.

2.5 The Cooperative Solution

In this section we derive the cooperative solution of the model. This is the case that we interpret as policy coordination. The solution is found by employing the dynamic programming algorithm of de Paz *et al* (2013). As in the noncooperative solution, we are unable to find a fully analytic solution.

The coalition formed by the government and the central bank has the intertemporal loss function

$$J_t(x_t, \pi_t) = \bar{\lambda} \int_t^\infty e^{-\rho_F(s-t)} L_{F,s}(x_s, \pi_s, g_s) ds + (1-\bar{\lambda}) \int_t^\infty e^{-\rho_M(s-t)} L_{M,s}(x_s, \pi_s, i_s) ds \ .$$

and is subject to the dynamics given by (2.1) and (2.2) and initial constraints on x_0 and π_0. Here, $\bar{\lambda}$ is a parameter in the open unit interval. Its value determines which agent has more influence over the decision making of the coalition.

[8] For conditions on global asymptotic stability in systems of differential equations, see Sydsæter *et al* (2008).

The dynamic programming equation of the coalition is

$$\max_{g,i}\left\{\bar{\lambda}\left(-\lambda_{Fx}x^2-\lambda_{F\pi}\pi^2-\frac{\lambda_{Fg}}{2}g^2+\frac{\partial V_F}{\partial x}(x,\pi)(\alpha g-\beta(i-\pi))+\frac{\partial V_F}{\partial \pi}(x,\pi)\kappa x\right)\right. \tag{2.5}$$

$$\left.+(1-\bar{\lambda})\left(-\lambda_{Mx}x^2-\lambda_{M\pi}\pi^2-\frac{\lambda_{Mi}}{2}i^2+\frac{\partial V_M}{\partial x}(x,\pi)(\alpha g-\beta(i-\pi))+\frac{\partial V_M}{\partial \pi}(x,\pi)\kappa x\right)\right\}$$

$$=(1-\bar{\lambda})\rho_M V_M(x,\pi)+\bar{\lambda}\rho_F V_F(x,\pi)+(1-\bar{\lambda})\frac{dV_M(x,\pi)}{dt}+\bar{\lambda}\frac{dV_F(x,\pi)}{dt}\,,$$

where $V_F(x,\pi)$ and $V_M(x,\pi)$ are again the value functions. The decision rules that solve the maximization problem in the dynamic programming equation is an equilibrium solution in the sense defined by Ekeland and Lazrak (2010).[9]

The function to be maximized in (2.5) is strictly concave in (g,i) and it is straightforward to derive

$$g^*=\frac{\alpha}{\bar{\lambda}\lambda_{Fg}}\left(\bar{\lambda}\frac{\partial V_F}{\partial x}(x,\pi)+(1-\bar{\lambda})\frac{\partial V_M}{\partial x}(x,\pi)\right) \tag{2.6}$$

and

$$i^*=\frac{-\beta}{(1-\bar{\lambda})\lambda_{Mi}}\left(\bar{\lambda}\frac{\partial V_F}{\partial x}(x,\pi)+(1-\bar{\lambda})\frac{\partial V_M}{\partial x}(x,\pi)\right)\,. \tag{2.7}$$

Note that the government deficit and the nominal interest rate always have different signs. This implies that if the government employs an expansionary policy, the central bank does the same. The same applies for a contractionary policy. Put differently, the authorities are never pushing the economy in different directions at the same time. This result was not obtained in the noncooperative solution.

If we substitute the controls into the partial differential equations of the value functions, we obtain

$$0=-\lambda_{Fx}x^2-\lambda_{F\pi}\pi^2-\rho_F V_F(x,\pi)+\frac{\partial V_F}{\partial x}(x,\pi)\beta\pi+\frac{\partial V_F}{\partial \pi}(x,\pi)\kappa x$$
$$+\left(\theta\frac{\partial V_F}{\partial x}(x,\pi)+(1-\theta)\frac{\partial V_M(x,\pi)}{\partial x}(x,\pi)\right)$$
$$\left(\left(\frac{\alpha^2}{2\bar{\lambda}\lambda_{Fg}}+\frac{\beta^2}{(1-\bar{\lambda})\lambda_{Mi}}\right)\frac{\partial V_F}{\partial x}(x,\pi)-\frac{(1-\bar{\lambda})\alpha^2}{2\theta^2\lambda_{Fg}}\frac{\partial V_M}{\partial x}(x,\pi)\right)$$

and

$$0=-\lambda_{Mx}x^2-\lambda_{M\pi}\pi^2-\rho_M V_M(x,\pi)+\frac{\partial V_M}{\partial x}(x,\pi)\beta\pi+\frac{\partial V_M}{\partial \pi}(x,\pi)\kappa x$$
$$+\left(\bar{\lambda}\frac{\partial V_F}{\partial x}(x,\pi)+(1-\bar{\lambda})\frac{\partial V_M(x,\pi)}{\partial x}(x,\pi)\right)$$
$$\left(\left(\frac{\beta^2}{2(1-\bar{\lambda})\lambda_{Mi}}+\frac{\alpha^2}{\bar{\lambda}\lambda_{Fg}}\right)\frac{\partial V_M}{\partial x}(x,\pi)-\frac{\bar{\lambda}\beta^2}{2(1-\bar{\lambda})^2\lambda_{Mi}}\frac{\partial V_F}{\partial x}(x,\pi)\right)\,.$$

[9] See de Paz *et al* (2013) for mathematical details and a proof. For a proof of the corresponding stochastic case, see the appendix of Chapter 3 to this thesis.

We now need to make a conjecture on the value function. As in the noncooperative solution, we will assume a quadratic form. We can then derive a system of equations in the parameters of the value functions. This is left for the appendix.

We next want to check for global asymptotic stability in the system in (x_t, π_t) in the cooperative solution. To this end, we substitute the decision rules (2.6) and (2.7) into the dynamics for output and inflation to obtain a system of differential equations given by

$$\frac{dx_t}{dt} = \left(\frac{\alpha^2}{\bar{\lambda}\lambda_{Fg}} + \frac{\beta^2}{(1-\bar{\lambda})\lambda_{Mi}}\right)((A_F x_t + C_F \pi_t) + (A_M x_t + C_M \pi_t)) + \beta \pi_t$$

and

$$\frac{d\pi_t}{dt} = \kappa x_t \ .$$

This system is globally asymptotically stable if, and only if,

$$\left(\frac{\alpha^2}{\bar{\lambda}\lambda_{Fg}} + \frac{\beta^2}{(1-\bar{\lambda})\lambda_{Mi}}\right)(A_F + A_M) < 0$$

and

$$-\left(\frac{\alpha^2}{\bar{\lambda}\lambda_{Fg}} + \frac{\beta^2}{(1-\bar{\lambda})\lambda_{Mi}}\right)(C_F + C_M) - \beta > 0 \ .$$

As in the noncooperative solution, we look for numeric solutions such that the system of partial differential equations in x_t and π_t satisfies asymptotic global stability.

2.6 The Effects of Asymmetric Time Preferences

In this section we solve the noncooperative and cooperative solutions numerically and investigate the benefits of coordination when the government has a higher discount rate than the central bank. We assign parameter values that are standard in the literature and investigate the response of the economy to disequilibrium. We restrict ourselves to solutions such that the resulting system of differential equations in output and inflation is globally asymptotically stable. Even then, there can in principle be more than one solution that satisfies the restrictions. In that case, some other argument must be employed to select one particular solution and discard the others. For all the parameterizations that we use here, there is only solution that satisfies the restriction of global asymptotic stability. Hence, no additional argument is necessary.

In all our numeric solutions, the initial conditions on output and inflation are set to $x_0 = 0.02$ and $\pi_0 = 0.05$. That is, output is initially two percent above its equilibrium level and inflation is initially five percent above its equilibrium level. The fact that the initial values have the same sign is significant. Recall that inflation decreases if, and only if, output is negative. Since output is initially positive, the government and central bank can only bring inflation down by first driving output down into the negative, i.e. by causing a recession. This creates a trade-off between the need to bring down inflation and the need to stabilize output.

We will first look at a base parameterization. In this scenario, the government and the central bank have the same discount rates. This parameterization will be our benchmark. We will then investigate a deviation from the benchmark, in which the discount rates differ. In each scenario, we will compare the noncooperative and cooperative solutions.

2.6.1 The Base Parameterization

The base parameterization uses the following values. For the coefficient on the fiscal deficit, we set $\alpha = 0.5$. Thus, a one percent increase in the fiscal deficit increases output by half a percent. For the coefficient on the nominal interest rate, we set $\beta = 0.5$. In the New Keynesian model in Galí (2008), this variable is the intertemporal elasticity of substitution in the optimization problem of the household. Hence, we should set β close to empirical estimates of that preference parameter. In a meta-analysis by Havanek *et al* (2013) it is concluded that a typical value of the intertemporal elasticity of substitution is around one half, which motivates our decision.

We set $\kappa = 0.25$ and $\rho_F = \rho_M = 0.1$. These values are from van Arlee *et al* (2001) and van Arlee *et al* (2002). We also set $\bar{\lambda} = 0.5$, so that the government and the central bank have equal influence over the cooperative solution. Finally, we set $\lambda_{Fx} = \lambda_{F\pi} = \lambda_{Fg} = \lambda_{Mx} = \lambda_{M\pi} = \lambda_{Mi} = 1$. This implies that output stability and inflation stability are equally prioritized by both agents.

Figure 2.1 shows the response functions under this parameterization. The corresponding Table 2.1 shows the solutions of the value functions and their parameters. In both solutions, output initially decreases sharply and becomes negative. This causes inflation to gradually decrease, and both state variables slowly converge to the equilibrium levels. The initial decrease in output is caused by a negative government deficit and a positive nominal interest rate. In the cooperative solution, the government and central bank are somewhat less aggressive in using their policy instruments, and hence the drop in output is slower.

In this simple parameterization, there are certain symmetries in the model ($\alpha = \beta$, $\rho_F = \rho_M$, etc). Because of this, policy coordination does not have a major effect on the behaviour of the economy. Moreover, there are no significant gains from policy coordination; the value functions do not change (after rounding) when coordination is introduced.

Also, notice that among the parameters in the value functions, given in Table 2.1, the magnitude is far greater for B_F and B_M than for the others. This is because of the trade-off discussed above. Deviation in inflation is much more costly than deviation in output, because inflation can only be stabilized by suffering costs in terms of output deviation.

Figure 2.1: The Base Parameterization

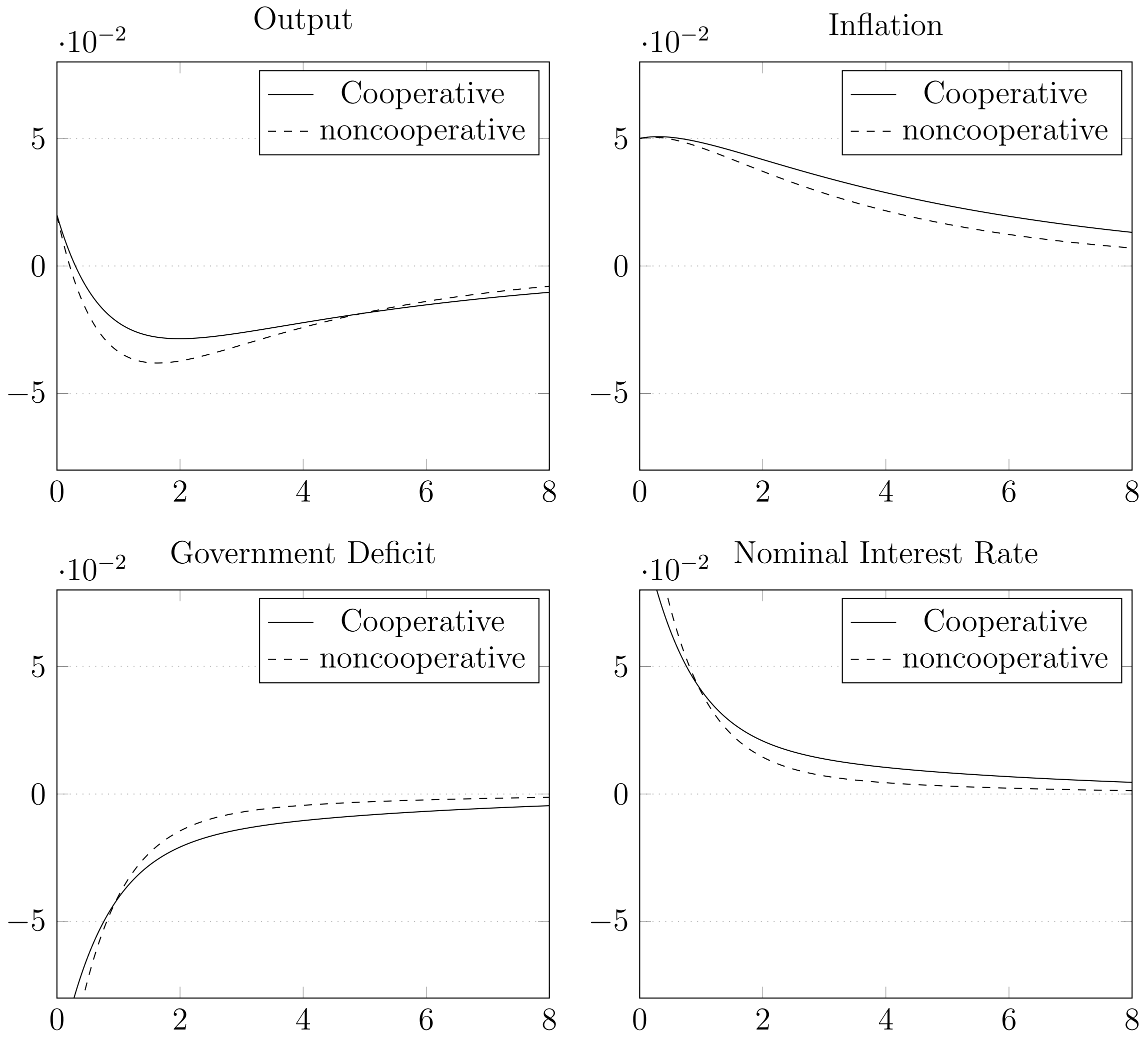

Table 2.1: The Base Parameterization

	Cooperative	noncooperative
V_F	-0.013	-0.013
V_M	-0.013	-0.013
A_F	-1.50	-1.69
A_M	-1.50	-1.69
B_F	-8.47	-8.14
B_M	-8.47	-8.14
C_F	-1.55	-2.10
C_M	-1.55	-2.10

2.6.2 Asymmetric Discount Rates

We now depart from the base parameterization by setting $\rho_F = 0.4$. This implies that the government has a higher discount rate than the central bank and is therefore more focused on losses in the near future than losses in the far future. The response functions are given in Figure 2.2.

We now see a striking result in the noncooperative solution. The central bank is much more aggressive in using its policy instrument than is the government. This implies that the central bank is suffering most of the burden in bringing output down into the negative. The reason is that the central bank is willing to suffer a high initial cost in terms of a high nominal interest rate in order to bring down future costs in terms of high inflation. The government, on the other hand, cares less about future cost and more about the present, and is therefore not prepared to make that sacrifice to the same extent. As a result, the central bank has a much greater loss in its value function.

Moreover, the authorities enact contradictory policies in their attempt to stabilize the economy. Specifically, they disagree on how short run losses in terms of output should be balanced against long run losses in terms of inflation. The central bank is eager to avoid losses in the long term from excess inflation and is willing to, in the short term, tolerate a large sacrifice in terms of low output as a method of bringing inflation down. Hence, it attempts to push the economy into a short but severe recession by enacting contractionary monetary policy. The government, on the other hand, being more preoccupied with short run losses, is not prepared to allow a severe recession in the short run. Consequently, it embarks on an expansionary fiscal policy to counter the central bank. The authorities are, in effect, "cancelling out" each other's actions. Since any use of policy instruments is costly, there is clearly room for Pareto improvements in the noncooperative solution.

In the cooperative solution, the situation is quite different. Here the government and the central bank share the burden of stabilizing the economy much more equally. As a result, the government's loss increases and the central bank's loss decreases. The sum of losses decreases, so policy coordination is beneficial.

In the introduction to this chapter, we referred to the literature for discussions on the fact that without policy coordination, the government and the central bank may pursue contradictory policies when their time preferences differ. We now see that our model can capture this scenario. In the noncooperative solution, contradictory policies were used because the government and central bank have different discount rates, and in the cooperative solution this issue was resolved.

Figure 2.2: Asymmetric Discount Rates

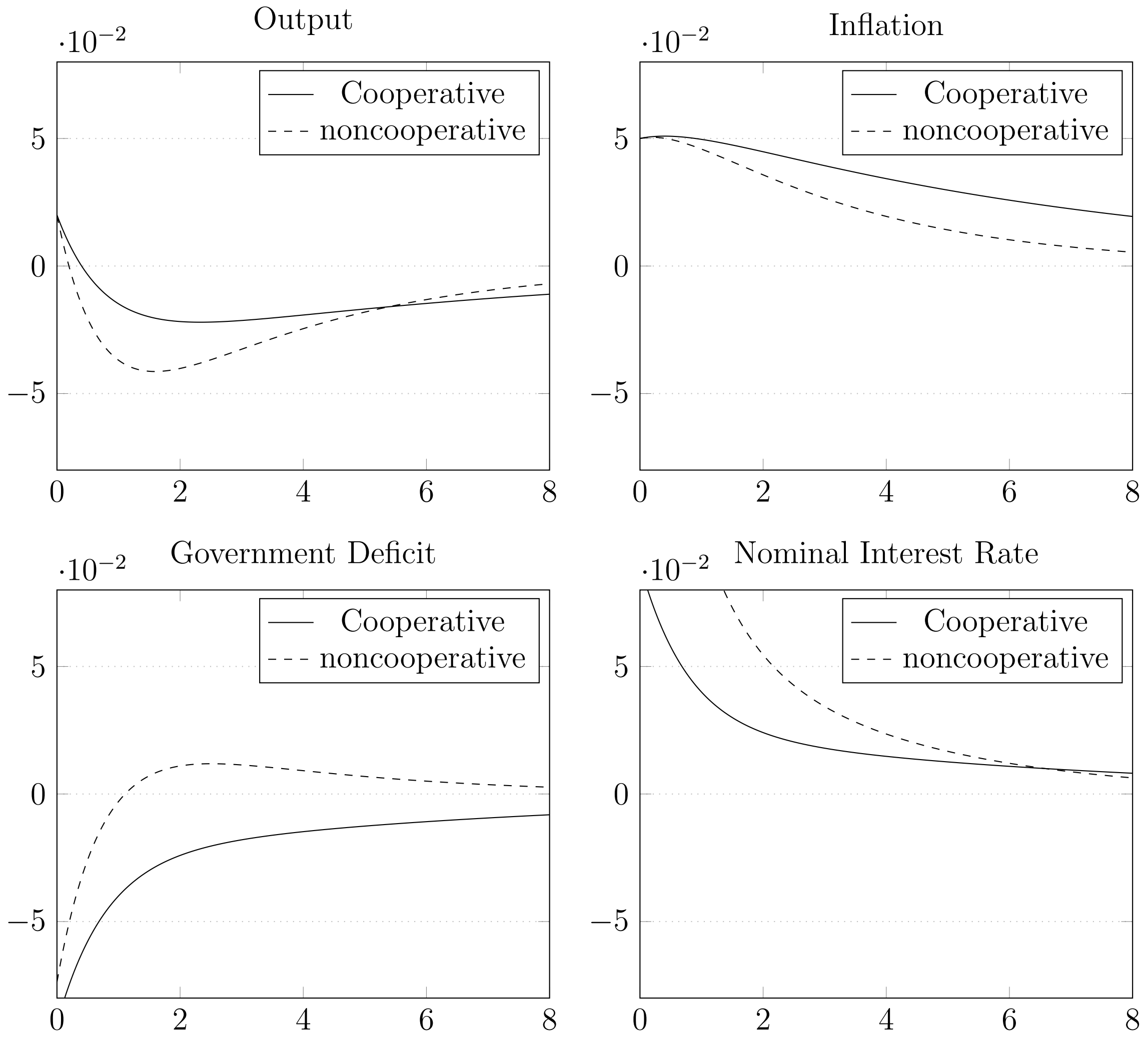

Table 2.2: Asymmetric Discount Rates

	Cooperative	noncooperative
V_F	-0.007	-0.006
V_M	-0.014	-0.019
A_F	-1.26	-1.05
A_M	-1.59	-2.29
B_F	-5.16	-3.97
B_M	-9.33	-11.60
C_F	-0.74	-0.36
C_M	-1.72	-4.10

2.7 Further Numeric Results

In this section, we provide some additional numeric results, which are not related to time preferences. We look at the effect of coordination under three other types of asymmetries. Unless mentioned otherwise, we always make use of the base parameterization values from the previous section.

2.7.1 Asymmetric Variable Targets

In this scenario, we depart from the base parameterization by setting $\lambda_{F\pi} = 0.2$ and $\lambda_{Mx} = 0.2$. This implies that the government now prioritizes output stability over inflation stability, and vice versa for the central bank. The response functions are given in Figure 2.3.

In the noncooperative solution, the government deficit is initially negative, but becomes positive after some time has passed. Meanwhile, the nominal interest rate remains positive throughout. This implies that for some time, the government and the central bank are steering output in different directions. The reason is that once output is sufficiently negative, the government will want to increase it because it prioritizes output stability. The central bank, on the other hand, will want to keep output low in order to decrease inflation, because it prioritizes inflation stability. The result is that they are canceling out each other's actions, as in the case of asymmetric discount rates. In the cooperative solution, the government deficit is negative the whole time. Thus, no such inefficiencies occur. The result is that the sum of the value functions improves, from -0.025 in the noncooperative solution to -0.019 in the cooperative solution.

2.7.2 Asymmetric Influence

We now look at the case where the weight $\bar{\lambda}$ is increased to 0.8, so that the government has a greater influence over the cooperative solution than does the central bank. The response functions are given in Figure 2.4.

The noncooperative solution is now identical to what it is in the base parameterization, as $\bar{\lambda}$ does not matter in that type of solution. In the cooperative solution, the central bank suffers most of the loss from stabilization. Hence, its loss increases relative to the noncooperative solution whereas the government's loss decreases. This advantage arises from the fact that in the cooperative solution, most of the trajectory of the government deficit and the nominal interest rate is determined by the government. Since the government does not suffer losses from the nominal interest rate, it prefers to use that instrument rather than its own.

2.7.3 Asymmetric Instrument Efficiencies

Finally, we look at the case when the government is more efficient in stabilizing the economy than is the central bank. We achieve this by setting $\alpha = 1.0$. The government deficit now has a greater effect on output than has the nominal interest rate. The response functions are given in Figure 2.5. We see that in both the noncooperative and cooperative solutions, the government deficit is employed to a greater extent than the nominal interest rate. This is because the payoff in terms of output movement is greater when the government deficit is used than when the nominal interest rate is used.

Figure 2.3: Asymmetric Variable Targets

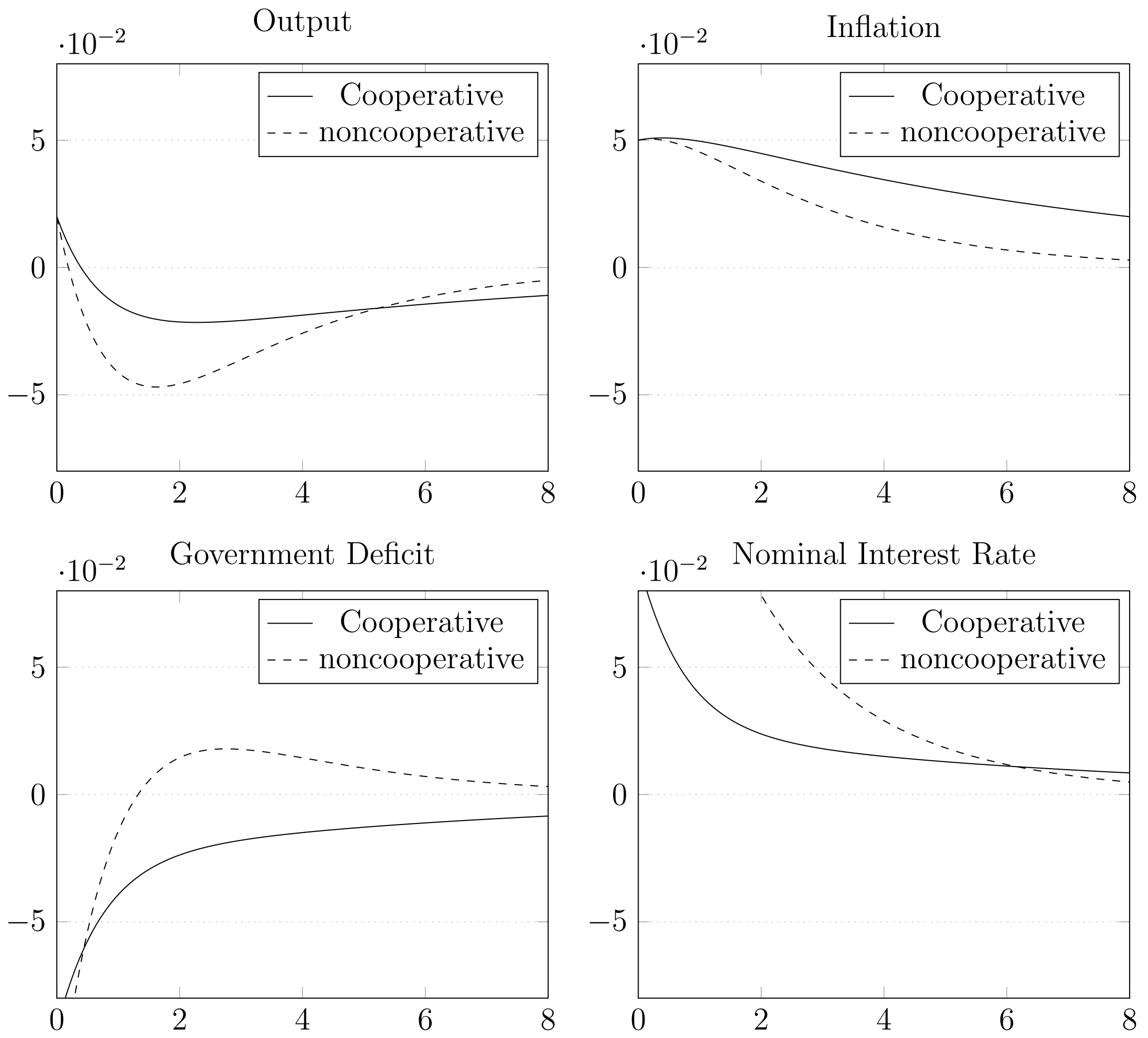

Table 2.3: Asymmetric Variable Targets

	Cooperative	noncooperative
V_F	-0.006	-0.005
V_M	-0.013	-0.020
A_F	-1.39	-1.27
A_M	-1.58	-1.70
B_F	-4.74	-3.48
B_M	-8.81	-12.18
C_F	-0.73	-0.04
C_M	-1.75	-4.60

Figure 2.4: Asymmetric Influence

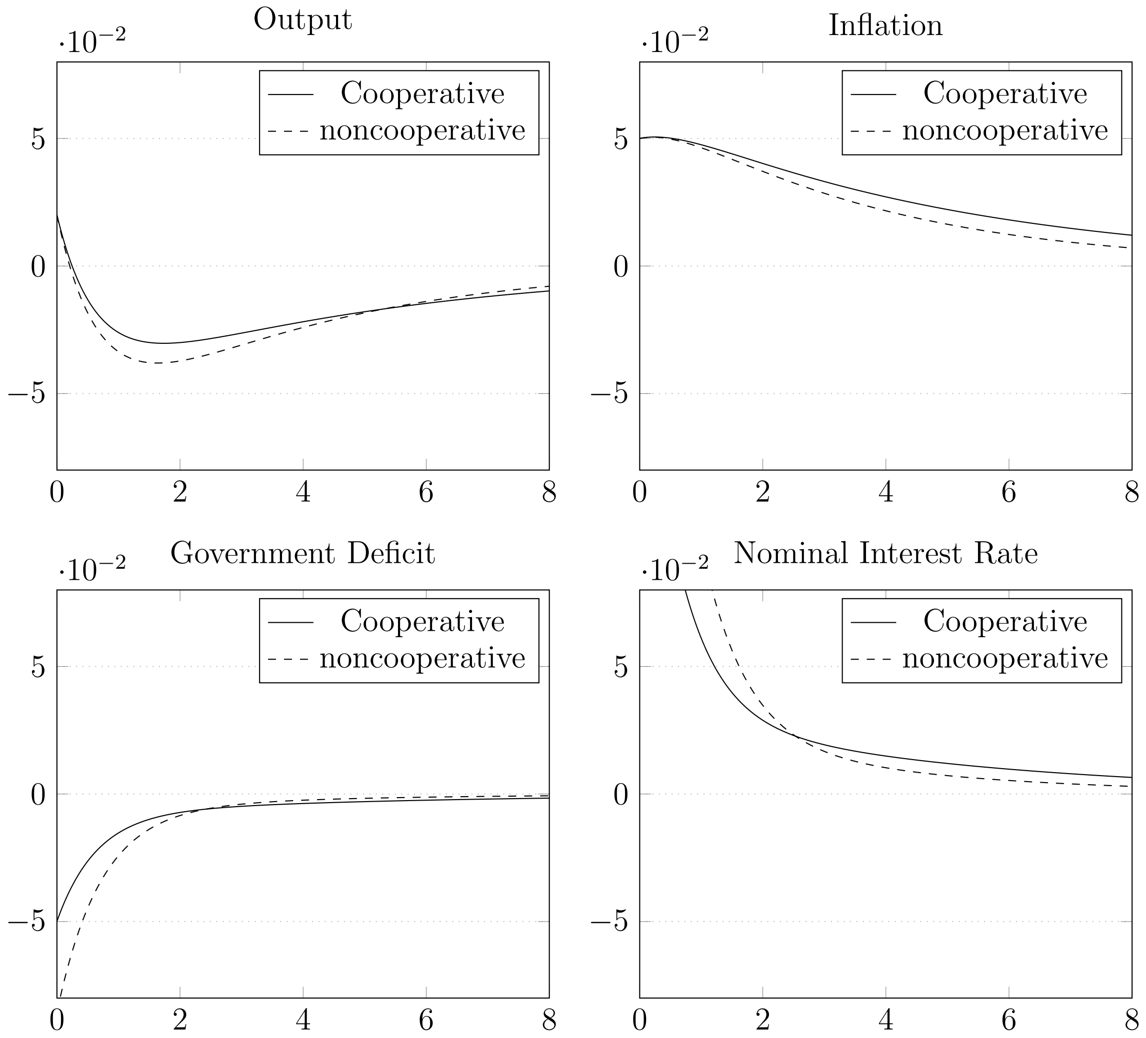

Table 2.4: Asymmetric Influence

	Cooperative	noncooperative
V_F	-0.010	-0.013
V_M	-0.016	-0.013
A_F	-0.76	-1.70
A_M	-2.67	-1.69
B_F	-7.34	-8.13
B_M	-10.07	-8.15
C_F	-0.68	-2.11
C_M	-2.98	-2.07

Figure 2.5: Asymmetric Instrument Efficiencies

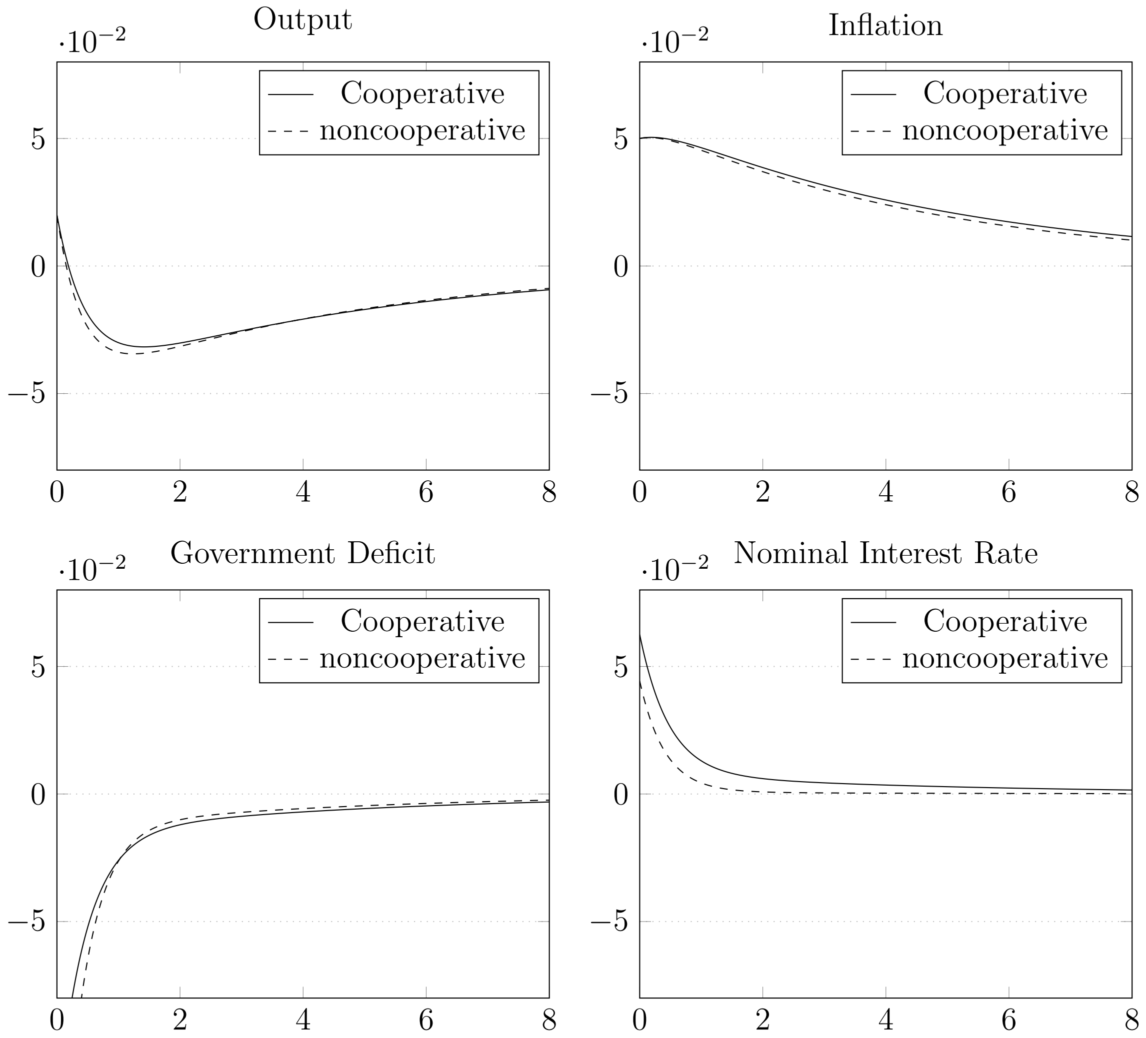

Table 2.5: Asymmetric Instrument Efficiencies

	Cooperative	noncooperative
V_F	-0.016	-0.012
V_M	-0.010	-0.010
A_F	-1.23	-1.45
A_M	-0.62	-0.69
B_F	-8.09	-8.15
B_M	-7.52	-6.99
C_F	-1.22	-1.49
C_M	-0.54	-0.61

2.8 Conclusion

In this chapter, we constructed a differential game where the government and the central bank act to steer the economy to equilibrium. Our objective was to compute the noncooperative and cooperative solutions when the government and the central bank have different time preferences. Our numeric results show that when time preferences differ, the noncooperative solution has room for Pareto improvement, and that such improvements are obtained by cooperation. We also showed that the contradictory policies enacted in the noncooperative solution are removed in the cooperative solution. Our results underline the importance of policy coordination, previously discussed in Hanif and Arby (2003) and Abdel-Haleim (2016). Our study seems to be the first to model in a differential game setting the issue of policy coordination when the government and the central bank have different time preferences.

There are some drawbacks to our model. One is the ambiguity of what the loss functions are a measure of. If the loss functions are a measure of welfare, we face the problem that there are two separate measures of welfare, one used by government and one used by the central bank. This issue was discussed in the introduction. Another problem, discussed in Remark 3.1 is an unintuitive aspect of the dynamics that we assume.

An extension to our model could follow van Aarle *et al* (2002) in having more state variables. For example, the model could include unemployment and real wages.

2.9 Appendix

2.9.1 The Noncooperative Solution

Under our assumptions, the partial differential equations in the value functions become

$$0 = -\lambda_{Fx}x^2 - \lambda_{F\pi}\pi^2 - \rho_F\left(\frac{A_F}{2}x^2 + \frac{B_F}{2}\pi^2 + C_F x\pi\right) + \frac{\alpha^2}{2\lambda_{Fg}}(A_F x + C_F\pi)^2$$

$$+\frac{\beta^2}{\lambda_{Mi}}(A_M x + C_M\pi)(A_F x + C_F\pi) + (B_F\pi + C_F x)\kappa x + (A_F x + C_F\pi)\beta\pi$$

and

$$0 = -\lambda_{Mx}x^2 - \lambda_{M\pi}\pi^2 - \rho_M\left(\frac{A_M}{2}x^2 + \frac{B_M}{2}\pi^2 + C_M x\pi\right) + \frac{\beta^2}{2\lambda_{Mi}}(A_M x + C_M\pi)^2$$

$$+\frac{\alpha^2}{\lambda_{Fg}}(A_M x + C_M\pi)(A_F x + C_F\pi) + (B_M\pi + C_M x)\kappa x + (A_M x + C_M\pi)\beta\pi\ .$$

All terms are multiplied by x^2, π^2 or $x\pi$, so we can derive a system in six equations in the variables A_F, A_M, B_F, B_M, C_F and C_M. Due to nonlinearities in the system, no analytic solution can be obtained, so we need to employ numeric solutions.

The system in six equations is the following. The first three equations are terms multiplied by x^2, π^2 and $x\pi$ respectively in the partial differential equation of $V_F(x,\pi)$. The last three equations are terms multiplied by x^2, π^2 and $x\pi$ respectively in the partial differential equation of $V_M(x,\pi)$. We obtain

$$0 = -\lambda_{Fx} - \frac{\rho_F}{2}A_F + \kappa C_F + \frac{\alpha^2}{2\lambda_{Fg}}A_F^2 + \frac{\beta^2}{\lambda_{Mi}}A_F A_M \ ,$$

$$0 = -\lambda_{F\pi} - \frac{\rho_F}{2}B_F + \beta C_F + \frac{\alpha^2}{2\lambda_{Fg}}C_F^2 + \frac{\beta^2}{\lambda_{Mi}}C_M C_F \ ,$$

$$0 = -\rho_F C_F + \kappa B_F + \beta A_F + \frac{\alpha^2}{\lambda_{Fg}}A_F C_F + \frac{\beta^2}{\lambda_{Mi}}(A_M C_F + A_F C_M) \ ,$$

$$0 = -\lambda_{Mx} - \frac{\rho_M}{2}A_M + \kappa C_M + \frac{\beta^2}{2\lambda_{Mi}}A_M^2 + \frac{\alpha^2}{\lambda_{Fg}}A_M A_F \ ,$$

$$0 = -\lambda_{M\pi} - \frac{\rho_M}{2}B_M + \beta C_M + \frac{\beta^2}{2\lambda_{Mi}}C_M^2 + \frac{\alpha^2}{\lambda_{Fg}}C_M C_F$$

and

$$0 = -\rho_M C_M + \kappa B_M + \beta A_M + \frac{\beta^2}{\lambda_{Mi}}A_M C_M + \frac{\alpha^2}{\lambda_{Fg}}(A_M C_F + A_F C_M) \ .$$

We solve it numerically, given the restriction of global asymptotic stability, as discussed in Section 4.

2.9.2 The Cooperative Solution

Under our assumptions, the partial differential equations in the value functions become

$$0 = -\lambda_{Fx}x^2 - \lambda_{F\pi}\pi^2 - \rho_F\left(\frac{A_F}{2}x^2 + \frac{B_F}{2}\pi^2 + C_F x\pi\right) + (B_F\pi + C_F x)\kappa x + (A_F x + C_F\pi)\beta\pi$$
$$+ \left(\bar{\lambda}(A_F x + C_F\pi) + (1-\bar{\lambda})(A_M x + C_M\pi)\right)$$
$$\left(\left(\frac{\alpha^2}{2\bar{\lambda}\lambda_{Fg}} + \frac{\beta^2}{(1-\bar{\lambda})\lambda_{Mi}}\right)(A_F x + C_F\pi) - \frac{(1-\bar{\lambda})\alpha^2}{2\theta^2\lambda_{Fg}}(A_M x + C_M\pi)\right)$$

and

$$0 = -\lambda_{Mx}x^2 - \lambda_{M\pi}\pi^2 - \rho_M\left(\frac{A_M}{2}x^2 + \frac{B_M}{2}\pi^2 + C_M x\pi\right) + (B_M\pi + C_M x)\kappa x + (A_M x + C_M\pi)\beta\pi$$
$$+ \left(\bar{\lambda}(A_F x + C_F\pi) + (1-\bar{\lambda})(A_M x + C_M\pi)\right)$$
$$\left(\left(\frac{\beta^2}{2(1-\bar{\lambda})\lambda_{Mi}} + \frac{\alpha^2}{\bar{\lambda}\lambda_{Fg}}\right)(A_M x + C_M\pi) - \frac{\bar{\lambda}\beta^2}{2(1-\bar{\lambda})^2\lambda_{Mi}}(A_F x + C_F\pi)\right) \ .$$

In each equation, all terms are multiplied by x^2, π^2 or $x\pi$. Hence, these two equations contain a system in six equations in the variables A_F, A_M, B_F, B_M, C_F and C_M. Due to nonlinearities in the system, no analytic solution can be obtained, so we need to employ numeric solutions.

The system in six equations is the following. The first three equations are terms multiplied by x^2, π^2 and $x\pi$ respectively in the partial differential equation of $V_F(x,\pi)$. The last three equations are terms multiplied by x^2, π^2 and $x\pi$ respectively in the partial differential equation of $V_M(x,\pi)$. We obtain

$$0 = -\lambda_{Fx} - \frac{\rho_F}{2}A_F + \kappa C_F + \left(\frac{\alpha^2}{2\bar{\lambda}\lambda_{Fg}} + \frac{\beta^2}{(1-\bar{\lambda})\lambda_{Mi}}\right)A_F^2 - \frac{(1-\bar{\lambda})\alpha^2}{2\bar{\lambda}^2\lambda_{Fg}}A_M^2$$
$$+\left(\frac{(2\bar{\lambda}-1)\alpha^2}{2\bar{\lambda}^2\lambda_{Fg}} + \frac{\beta^2}{(1-\bar{\lambda})\lambda_{Mi}}\right)A_F A_M \ ,$$
$$0 = -\lambda_{F\pi} - \frac{\rho_F}{2}B_F + \beta C_F + \left(\frac{\alpha^2}{2\bar{\lambda}\lambda_{Fg}} + \frac{\beta^2}{(1-\bar{\lambda})\lambda_{Mi}}\right)C_F^2 - \frac{(1-\bar{\lambda})\alpha^2}{2\bar{\lambda}^2\lambda_{Fg}}C_M^2$$
$$+\left(\frac{(2\bar{\lambda}-1)\alpha^2}{2\bar{\lambda}^2\lambda_{Fg}} + \frac{\beta^2}{(1-\bar{\lambda})\lambda_{Mi}}\right)C_F C_M \ ,$$
$$0 = \kappa B_F + \beta A_F - \rho_F C_F + 2\left(\frac{\alpha^2}{2\bar{\lambda}\lambda_{Fg}} + \frac{\beta^2}{(1-\bar{\lambda})\lambda_{Mi}}\right)A_F C_F - \frac{(1-\bar{\lambda})\alpha^2}{\bar{\lambda}^2\lambda_{Fg}}A_M C_M$$
$$+\left(\frac{(2\bar{\lambda}-1)\alpha^2}{2\bar{\lambda}^2\lambda_{Fg}} + \frac{\beta^2}{(1-\bar{\lambda})\lambda_{Mi}}\right)(A_M C_F + A_F C_M) \ ,$$
$$0 = -\lambda_{Mx} - \frac{\rho_M}{2}A_M + \kappa C_M + \left(\left(\frac{\beta^2}{2(1-\bar{\lambda})\lambda_{Mi}} + \frac{\alpha^2}{\bar{\lambda}\lambda_{Fg}}\right)A_M^2 - \frac{\bar{\lambda}\beta^2}{2(1-\theta)^2\lambda_{Mi}}A_F^2\right.$$
$$\left.+\left(\frac{(1-2\bar{\lambda})\beta^2}{2(1-\bar{\lambda})^2\lambda_{Mi}} + \frac{\alpha^2}{\bar{\lambda}\lambda_{Fg}}\right)A_M A_F \ ,\right.$$
$$0 = -\lambda_{M\pi} + \beta C_M - \frac{\rho_M}{2}B_M + \left(\left(\frac{\beta^2}{2(1-\bar{\lambda})\lambda_{Mi}} + \frac{\alpha^2}{\bar{\lambda}\lambda_{Fg}}\right)C_M^2 - \frac{\bar{\lambda}\beta^2}{2(1-\bar{\lambda})^2\lambda_{Mi}}C_F^2\right.$$
$$\left.+\left(\frac{(1-2\bar{\lambda})\beta^2}{2(1-\bar{\lambda})^2\lambda_{Mi}} + \frac{\alpha^2}{\bar{\lambda}\lambda_{Fg}}\right)C_M C_F\right.$$

and

$$0 = \kappa B_M + \beta A_M - \rho_M C_M + 2\left(\left(\frac{\beta^2}{2(1-\bar{\lambda})\lambda_{Mi}} + \frac{\alpha^2}{\bar{\lambda}\lambda_{Fg}}\right)A_M C_M - \frac{\bar{\lambda}\beta^2}{(1-\bar{\lambda})^2\lambda_{Mi}}A_F C_F\right.$$
$$\left.+\left(\frac{(1-2\bar{\lambda})\beta^2}{2(1-\bar{\lambda})^2\lambda_{Mi}} + \frac{\alpha^2}{\bar{\lambda}\lambda_{Fg}}\right)(A_M C_F + A_F C_M) \ .\right.$$

We solve it numerically, given the restriction of global asymptotic stability, as discussed in Section 5.

Chapter 3

A Life Insurance Model with Multiple Agents and Asymmetric Time Preferences

3.1 Introduction

Over the last 15 years, many papers have been published in continuous time life insurance models. Much of the work has been based on the seminal paper by Pliska and Ye (2007), although the basic framework goes back to Yaari (1965) and Richard (1975). The models in this literature typically set up three separate decisions or trade-offs for an agent to make. First, there's the trade-off between consumption in the present and consumption in the future (i.e. savings in the present). Secondly, there's the investment of savings into either an asset with a fixed return or an asset with a random return. Thirdly, there's the expenditure on an insurance premium which, at the time of the agent's death, results in an insurance being paid out to his descendants. Hence, the continuous time life insurance model can appropriately be referred to as a consumption-investment-premium model. We will refer to it simply as a life insurance model.

The papers that are part of the literature on the life insurance model, can generally be classified into two categories. On the one hand, there are those papers that expand on the basic models by Richard (1975) and Pliska and Ye (2007), by allowing the agent to have inconsistent time preferences. On the other hand, there are those papers that expand on the basis model by having a household that consists of several agents, each of which may purchase his own life insurance.

The issue of inconsistent time preferences, and its implications, has already been discussed in the introduction to this thesis. In the case of a household with multiple agents, it is natural to define the household's intertemporal utility function as a weighted sum of the intertemporal utility functions of the individual agents. However, this task is complicated by the fact that in life insurance models, agents are mortal. There is the issue of determining how the household's intertemporal utility function changes when one agent dies so that the set of household members changes. There is also the issue of determining how the knowledge that the set of members will change at some point in the future affects the behavior in the present.

In this chapter, we construct a life insurance model which combines the two features of, on the one hand, households with inconsistent time preferences and, on the other hand, households with multiple agents. Specifically, we assume that the household consists of two agents, and that each agent has his own unique discount rate. This implies that although each agent has time preferences that are consistent, the household taken as a whole does not. Thus, we will derive a cooperative solution where the decision rules determine expenditures on consumption, investment and premium for a household that has inconsistent time preferences because the members of the household discount future utilities at different rates. This type of model seems to not have been investigated previously in the literature on life insurance models.

We will assume that the agents have constant but different mortality rates. The assumption of constant mortality rates is made for reasons of tractability. It ensures that the solutions that we derive will be independent of time, as was the case in the Macroeconomic model of the previous chapter. This means that the probability of dying at a particular distance of time into the future is the same regardless of what is the current time. Hence, if the solution satisfies a given property at a given point in time, that property will hold for every other point in time as well. This simplifies the task of deriving properties of the solution, because the decision rules and value functions are independent of time.

Constant mortality rates are nonstandard in the literature. The most common approach is that no specific mortality rate is assumed in the analytic section of the paper, and that the mortality rate given by Gompertz law is assumed in the numeric section. According to Gompertz law, the mortality rate at time t is $e^{\frac{t-m}{n}}/n$, where n and m are constants. This implies that the solution will be a function of time, meaning that the behavior of an agent is affected by his age. One exception to this is found in Bayraktar and Young (2013), where the assumption is that the members of the household have the same constant discount and mortality rates. The present study extends their setup by allowing asymmetric discount and mortality rates.

The rest of this chapter is organized as follows. In Section 2 we provide a literature review on life insurance models. In Section 3 we formally describe the model and in Section 4 we try to solve the model analytically. We shall find that a fully analytic solution can be obtained only in special cases, which motivates the numeric solutions of Section 5. In Section 6, we provide some concluding remarks.

3.2 Literature Review

3.2.1 The Standard Models

We've already mentioned the early contribution of Richard (1975). He builds upon the model of Merton (1971), in which an agent withdraws money for consumption expenditures from his budget over a fixed time interval $[0, T]$, while simultaneously distributing his savings amongst an asset with a fixed return and a number of assets with random return. Richard (1975) added to this model that the agent dies at a random time $\tau \in [0, T]$, turning the problem into one of random terminal time. He also added a running wage income and gave the agent the option of purchasing life insurance by continuously paying a premium that is withdrawn from the budget, hence turning the model into a life insurance model. At the time of death, a payment is made to the descendants of the agent as a function of the rate at which the premium is being paid at that time. The agent is made interested in purchasing life insurance by the introduction of a bequest function which is activated at the time of death. The bequest function takes as an argument the current wealth plus the insurance payment, i.e. the total amount of money that the deceased agent leaves behind to his descendants.

The wealth dynamics of the model of Richard (1975) can be written as[1]

$$dx_t = (r + \epsilon_t(\mu - r))x_t dt + (I_t - c_t - q_t)dt + \epsilon_t \sigma x_t dz_t$$

and the agent's intertemporal utility function can be written as

$$J_t(x_t, c_s, q_s, \epsilon_s) = E_t \left[\int_t^\tau U(s, c_s) ds + B(x_\tau + q_\tau/\eta_\tau, \tau) \mid x_t \right]$$

where, for simplicity, we have assumed that the number of risky assets is one.[2] Here, x_t is the wealth, c_t is the rate at which money is spent on consumption, q_t is the rate at which the insurance premium is paid, and ϵ_t is the fraction of wealth that is invested in the risky asset (as opposed to in the safe asset). Out of those, wealth is the state variable and the remaining three are control variables. Utility is obtained from consumption through the function $U(s, c_s)$ and from the money left for the descendants (the bequest) through the function $B(x_\tau + q_\tau/\eta_\tau, \tau)$. As for the parameters, r is the rate of return of the safe asset, μ is the expected rate of return of the risky asset and σ is the volatility parameter of the risky asset. The functions I_t and η_t are exogenously determined nonrandom functions for the rate of wage income and the premium-insurance ratio. The latter is defined as the rate at which money is being spent on the insurance premium divided by the amount of money that is paid out in case of death.

[1] The notation given here is that which has become standard in the literature on life insurance models. It differs somewhat from that in Richard's paper.

[2] One may equally well assume an arbitrary number of risky assets, as did Merton (1971), although the calculations become more complicated. In general, the presence of multiple risky assets does not contribute anything meaningful to a model on life insurance.

The wealth dynamics in the model of Richard (1975) have been the basis for much of the literature on life insurance models, with only slight variations added by later authors. As for the intertemporal utility function, there's been more variation. An important extension comes from Pliska and Ye (2007). They removed the assumption of τ being bounded to the interval $[0, T]$, thus allowing the agent to live beyond time T. This solved a problem inherent in the model of Richard (1975), first pointed out by Leung (1994), namely that when τ is bounded to be smaller than or equal to T, the model does not have an interior solution that lasts until T. Instead, Pliska and Ye (2007) interpret T as a time of retirement. The agent may live beyond this time, but he no longer receives a wage income and makes no more decisions with respect to consumption, investment and life insurance. In fact, their intertemporal utility function is constructed in such a way that, if the agent is still alive at time T, a terminal function is activated and the problem ends. This terminal function takes wealth as an argument, but not an insurance payment. That is, it is assumed that the agent only purchases life insurance up to the point of his retirement. If he dies after retirement, no insurance is paid out because no premium is being paid anymore.

The intertemporal utility function in the model of Pliska and Ye (2007) can be written as

$$J_t(x_t, c_s, q_s, \epsilon_s) = E_t \left[\int_t^{\min(\tau,T)} U(s, c_s) ds + B(x_\tau + q_\tau/\eta_\tau, \tau) 1_{\tau<T} + L(x_T) 1_{\tau>T} \mid x_t \right]$$

where $L(x_T)$ is the terminal function and 1_A is an indicator function for event A. Thus, if the agent dies before time T, an insurance payment is received (provided that a premium is being paid).

One downside of the model of Pliska and Ye (2007) is that the terminal function is assumed to be independent of the probability distribution of τ. This is problematic because since the model effectively ends at T, but the agent may still be alive, it is reasonable to assume that $L(x_T)$ represents the intertemporal utility function at time T, the sum of utilities obtained over the interval $[T, \tau]$, discounted back to T. But if this is so, then the expected length of that interval should affect the value of that discounted sum of utilities. That is, the function $L(x_T)$ should depend on the probability distribution of τ. It should also depend on the other parameters of the model. This is typically not the case in papers that have used this type of intertemporal utility function. Instead, $L(x_T)$ is assumed to depend on a separate parameter, and it is not imposed that this parameter has any particular relationship with the other parameters.

One way around this issue is to assume that there is no terminal time T at which the problem ends (T is infinite), and that the agent continues to consume, invest and purchase life insurance until his death, whenever it may come. That is, there is no retirement. Thus, some authors have tackled the life insurance problem by assuming an intertemporal utility function with an infinite planning horizon. See for example Bruhn and Steffensen (2011), Bayraktar and Young (2013) and Koo and Lim (2021) for models of this kind. Of course, assuming that the agent may potentially live forever is unrealistic,[3] and one can still specify a maximum age, beyond which the agent cannot survive, by using a truncated probability distribution for τ. However, problems arise here too. When time approaches the maximum time of death, the agent knows for sure that he will die soon, and the incentive to buy ever more life insurance explodes.[4] Moreover, using probability distributions over the whole interval $[0, \infty)$ can be useful for reasons of tractability, as we shall see. For these reasons, the present study uses an infinite planning horizon and assumes that the time of death is distributed over the whole interval $[0, \infty)$.

3.2.2 Models with Multiple Agents

In the literature on life insurance models with more than one agent, an important contribution is Bruhn and Steffensen (2011). They developed a procedure for finding the cooperative solution for a household with an arbitrary number of agents. Each agent has his own mortality rate and wage income, and each agent purchases his own life insurance. When one agent dies, the life insurance that he has purchased is added to the household's budget, to be used by the remaining agents until, eventually, all agents have died. They also assume that, unlike in the framework of Richard (1975) and Pliska and Ye (2007), there is no utility from a bequest. Instead, the incentive to purchase life insurance arises from the fact that it is in the interest of each agent that every other agent is insured. Given that the household acts as a whole (i.e. that the solution is cooperative), and that each agent has at least some influence over the decision rules decided upon, it follows that the agents can prompt each other to purchase life insurance. Of course, when all agents except one has died, the survivor has no incentive to purchase any life insurance.

Bayraktar and Young (2013) solve a model with a household consisting of two agents, assuming that there is just a single premium and a single quantity of consumption that both agents benefit from. They also solve the model for when life insurance is paid for with a single lump sum payment, rather than the continuously paid premium that is standard. On the other hand, Kwak *et al* (2011) assume that the household consists of one parent and one child, and that only the parent receives a wage income and may purchase life insurance. An interesting aspect of this model is that the parent and the child have power utility functions with different coefficients.

A recent contribution is Wei *et al* (2020). They consider a household with two agents

[3] If we interpret the "agent" of the model not as one individual but as a series of individuals, each one of a new generation, than this problem goes away. But another problem arises, namely that we may have infinitely many generations, which is also unrealistic.

[4] From the point of view of a purchaser of life insurance, this may not be an unreasonable assumption. After all, a person who suspects he will die soon does indeed have incentive to purchase a very large life insurance. But of course, from the perspective of the insurance company, the incentive runs in the opposite direction, which puts a limit on the ability of the agent to purchase ever more insurance when his time of death approaches.

whose respective times of death are positively correlated. This is a realistic assumption for a household that consists of two people of similar age. Interestingly, they find that the stronger the correlation, the less life insurance is purchased by each agent. This is likely because after the first agent has died, the probability that the second agent will die soon increases (due to the correlation), which gives the second agent less time to transform the increase in wealth brought forth by the insurance payment into utility. Hence, the value of the life insurance payment in terms of the expected intertemporal utility that it yields is lower when the times of death are correlated.

3.2.3 Models with Inconsistent Time Preferences

All the multiple agent models described above assume that the household has consistent time preferences. That is, all members of the household are assumed to have exponential discount functions with the same coefficient. Hence, the issue of inconsistent time preferences is not dealt with.

Regarding life insurance models with inconsistent time preferences, Marín-Solano *et al* (2013) found the intertemporal equilibrium for a single agent with an arbitrary discount function. That paper can be regarded as an extension of the consumption-investment model in Marín-Solano and Navas (2010) into life insurance. One of their findings is that the amount of life insurance that is purchased increases in the agent's risk aversion. Similar models were solved by de Paz *et al* (2014) and Chen and Li (2020). All of those papers use the intertemporal utility function of Pliska and Ye (2007). As for the intertemporal utility function of Richard (1975), intertemporal equilibrium under inconsistent time preferences were derived in Purcal *et al* (2020) and Lim and Koo (2021). The latter of these also investigated the effects of taxation on the purchase of life insurance. Several papers find that inconsistent time preferences reduce the purchase of life insurance.

3.2.4 Literature Summary

Table (3.1) summarizes essential aspects of some of the papers mentioned above. The columns, starting from the left, gives the paper, the utility function, the time preferences (consistent or inconsistent), the mortality rate and the intertemporal utility function.

3.3 The Model

Our model contains a household with two agents. Each agent $i \in \{1, 2\}$ has a time of death $\tau_i \in [0, \infty]$ that follows an exponential distribution with mortality rate (or intensity) λ_i. That is,

$$P(\tau_i < t) = 1 - e^{-\lambda_i t} \ . \tag{3.1}$$

Table 3.1: Literature Summary

Paper	Utility F.	Time Pref.	Mortality R.	Int. Utility F.
Richard (1975)	Power	Consistent	Not Specified	Richard
Pliska and Ye (2007)	Power	Consistent	Linear	Pliska and Ye
Huang and Milevsky (2008)	Power	Consistent	Gompertz Law	Other
Kwak *et al* (2011)	Power	Consistent	Linear	Multiple Agents
Bruhn and Steffensen (2011)	Power	Consistent	Gompertz Law	Multiple Agents
Pirvu and Zhang (2012)	Power	Consistent	Gompertz Law	Pliska and Ye
Bayraktar and Young (2013)	Exponential	Consistent	Constant	Multiple Agents
Marín-Solano *et al* (2014)	Power / Exp.	Inconsistent	Gompertz Law	Pliska and Ye
de Paz *et al* (2014)	Power / Exp.	Inconsistent	Gompertz Law	Pliska and Ye
Purcal *et al* (2018)	Power	Inconsistent	Other	Richard
Chen and Li (2020)	Logarithm	Inconsistent	Gompertz Law	Pliska and Ye
Wei *et al* (2020)	Power	Consistent	Gompertz Law	Multiple Agents
Purcal *et al* (2020)	Power	Inconsistent	Other	Richard
Koo and Lim (2021)	Power	Inconsistent	Constant	Inf. Horizon
This Study	Power	Inconsistent	Constant	Multiple Agents

Hence, to place our model in its proper place in the literature, we have an infinite horizon model with no retirement and no terminal function. This is different from the commonly used intertemporal utility functions in Pliska and Ye (2007) and in Richard (1975), but similar to, for example, Bruhn and Steffensen (2011).

We assume that the agents share control over the household budget, and that they both add to the budget through their individual wage incomes. At each point in time, each agent withdraws money from the budget to use for his own individual consumption. Moreover, the two members of the household jointly decide how much of their wealth to allocate to investment in the risky asset, as opposed to in the safe asset. There are no constraints on the investments: the household may freely borrow in one asset in order to invest more than their whole net worth in the other asset.

Each agent purchases his own life insurance by continuously paying a premium. Thus, money is withdrawn from the budget to pay for two separate premiums, one for each agent. At time $\bar{\tau} = \min(\tau_1, \tau_2)$, one of the agents die and the other will be the sole member of the household until he too dies. If the agent who dies first is paying a premium, the insurance payment will be added to the household's wealth. Hence, the wealth process jumps at $\bar{\tau}$, with the jump being equal to the insurance payment. Specifically, if agent i dies first, so that $\bar{\tau} = \tau_i$, we have

$$x(\bar{\tau}) = x(\bar{\tau}^-) + q_i(\bar{\tau}^-)/\eta_i \ .$$

Also, notice that since this is a continuous time model, the event that the agents die at the same time, i.e. $\tau_1 = \tau_2$, has zero probability and can therefore be ignored.[5]

It is useful to think of the "state of the household" at any time t to be one of four potential states. In the initial state, which lasts for $t \in [0, \bar{\tau}]$, both agents are alive. We will refer to this as state A. Then there are two states in which one agent is alive and the other is dead. Which one of these states that is realized depends on which agent dies first. We will refer to these as states B. Finally, there's the state in which both agents are dead. See Table 3.2 for a summary of the potential states.

Table 3.2: Potential Household States

	$t < \tau_1$	$t > \tau_1$
$t < \tau_2$	Both agents are alive	Only agent 2 is alive
$t > \tau_2$	Only agent 1 is alive	Both agents are dead

Let's first look at state A. The intertemporal function of the household is given by

$$J_t^A(x_t, c_{1,s}, c_{2,s}, q_{1,s}, q_{2,s}, \epsilon_s) \tag{3.2}$$

$$= E_t \Bigg[\int_t^{\bar{\tau}} e^{-\rho_1(s-t)} \frac{c_{1,s}^{1-\gamma}}{1-\gamma} ds + e^{-\rho_1(\bar{\tau}-t)} V_1^B(x_{\bar{\tau}} + q_{2,\bar{\tau}}/\eta_2) 1_{\tau_2 = \bar{\tau}}$$

$$+ \int_t^{\bar{\tau}} e^{-\rho_2(s-t)} \frac{c_{1,s}^{1-\gamma}}{1-\gamma} ds + e^{\rho_2(\bar{\tau}-t)} V_2^B(x_{\bar{\tau}} + q_{1,\bar{\tau}}/\eta_1) 1_{\tau_1 = \bar{\tau}} \mid \bar{\tau} > t, x_t \Bigg] ,$$

which is the sum of the intertemporal utility functions of the two agents. The wealth dynamics are given by

$$dx_t = (r + \epsilon_t(\mu - r))x_t dt - (c_{1,t} + c_{2,t} + q_{1,t} + q_{2,t})dt + (I_1 + I_2)dt + \epsilon_t \sigma x_t dz_t , \tag{3.3}$$

where $t \in [0, \bar{\tau}]$. The notation is as follows.

- x_t is the wealth, i.e. the net worth of household. It is the only state variable in the model.
- $c_{i,t}$ is the rate at which money is withdrawn for consumption by agent i.
- $q_{i,t}$ is the rate at which the premium is being paid by agent i. The amount that is insured is then given by $q_{i,t}/\eta_i$, where η_i is the premium-insurance ratio.
- I_i is the fixed rate of wage income of agent i.

[5]The case of simultaneous deaths is analyzed in Wei *et al* (2020).

- ϵ_t is the fraction of wealth that is invested in the risky asset.[6]
- ρ_i is the discount rate of agent i. If $\rho_1 \neq \rho_2$, then the household as a whole has inconsistent time preferences.
- z_t is a standard Brownian motion.
- r, μ and σ determine the return of the safe asset, the expected return of the risky asset and the volatility of the risky asset.[7]
- 1_A is an indicator variable for the event A.
- $V_i^B(\cdot)$ is the intertemporal utility that agent i expects to receive when and if he is the sole survivor (after time $\bar{\tau}$) because the other agent dies first, given that the equilibrium decision rule is followed after time $\bar{\tau}$. It is the value function of the model in the state B.

Hence, our model has three different sources of randomness: the Brownian motion z_t and the exponentially distributed random variables τ_1 and τ_2. Note that given how we have defined these variables, they are mutually independent. For example, observing z_t provides no information that can be used to forecast τ_1 and τ_2. Moreover, the property of memorylessness of the exponential probability distribution implies that the additional amount of time that agent i expects to live is the same for every $t < \tau_i$.

Let's now look at state B, in which only one agent is alive. If only agent i is alive, he has the intertemporal utility function

$$J_t^B(x_t, c_i, \epsilon) = E_t \left[\int_t^{\tau_i} e^{-\rho_i(s-t)} \frac{c_{i,s}^{1-\gamma}}{1-\gamma} ds \mid \tau_i > t \, , \, x_t \right] \, , \tag{3.4}$$

and the dynamics are

$$dx_t = (r + \epsilon_t(\mu - r))x_t dt - c_{i,t}dt + I_1 dt + \epsilon_t \sigma x_t dz_t \, , \tag{3.5}$$

where $t \in [\bar{\tau}, \tau_i]$. In this case there are only two decisions to be made: the rate at which to spend money on consumption and how to allocate investments between the risky and the safe asset. No life insurance is purchased because there is no descendant who can benefit from it. Hence, there is no variable $q_{i,t}$. Also, in contrast to the state where both agents are alive, there is now only one wage income because the deceased agent receives no wage income.

Finally, in the fourth state, no agent is alive so no decisions are made, and there are no decision rules to be found.

[6] In Wei *et al* (2020) it is assumed that each of the two agents makes a separate investment decision. That is, there is one control variable $\epsilon_{1,t}$ and one $\epsilon_{2,t}$. They show that each of these variables can only be solved for in terms of the other, so that what matters is the total investment $\epsilon_{1,t} + \epsilon_{2,t}$. Hence, one may equally well make the substitution $\epsilon_t = \epsilon_{1,t} + \epsilon_2$, which is the approach we have taken here.

[7] We are assuming that the price of a share in the risky asset follows a geometric Brownian motion where μ and σ are the drift and volatility coefficients. See Chang (2004) or Merton (1971) for how this assumption impacts the dynamics of the wealth process.

3.4 Solving The Model

We will now find the cooperative solution to the model described above. To solve the model, we apply the method that was used by Bruhn and Steffensen (2011) and Wei *et al* (2020). First, we solve the problem for state B, where only one agent is alive. Then, having obtained the value functions for those two problems, we insert them into the intertemporal utility function of state A, where both agents are alive, equation (3.2), and solve the problem for that state. Hence, the model is solved using a kind of two-stage backward induction.

3.4.1 The Household With One Agent

We want to solve the problem with the intertemporal utility function (3.4) and the wealth dynamics (3.5). To proceed, we use the trick of transforming the intertemporal utility function from one with random terminal time into one with no terminal time (i.e. infinite horizon). Various versions of this trick is used in much of the literature on life insurance models, and goes back at least to Yaari (1965). The purpose is to remove one source of randomness, the time of death, thus turning the problem into one of a more familiar kind, in which the planning horizon is either fixed or, as in our case, infinite. The set of steps necessary to transform the intertemporal utility function into the desired form differs somewhat between the various papers in literature depending on the particular assumptions of the models. For the present model, the transformed intertemporal utility function of the states when one agent is alive is given below in Proposition 1, which is proved in the appendix.

Proposition 1 *Suppose that the intertemporal utility function is given by* (3.4) *and that the time of death* τ_i *follows the exponential distribution* (3.1). *Then the intertemporal utility function can also be written as*

$$J_t^B(x_t, c_{i,s}, \epsilon_s) = E_t \left[\int_t^\infty e^{-(\rho_i+\lambda_i)(s-t)} \frac{c_{i,s}^{1-\gamma}}{1-\gamma} ds \mid x_t \right] . \tag{3.6}$$

This intertemporal utility function can be interpreted as an infinite sum of utilities discounted at the rate of $\rho_i + \lambda_i$. We will say that $\rho_i + \lambda_i$ is the *effective discount rate* of the problem with one agent. The intertemporal utility function (3.6) has an effective discount rate that is constant. This implies that a commitment solution is an intertemporal equilibrium. Hence, this is a trivial problem to solve. The equilibrium decision rules are

$$c_{i,t} = \left(\frac{\lambda_i + \rho_i - (1-\gamma)r}{\gamma} - \frac{1-\gamma}{2} \frac{(\mu - r)^2}{(\sigma\gamma)^2} \right) (x_t + I_i/r)$$

and

$$\epsilon_t = \frac{\mu - r}{\gamma\sigma^2} \frac{x_t + I_i/r}{x_t} \ ,$$

which implies that the value function for this problem is

$$V_i^B(x_t) = \left(\frac{\lambda_i + \rho_i - (1-\gamma)r}{\gamma} - \frac{1-\gamma}{2} \frac{(\mu - r)^2}{(\sigma\gamma)^2} \right)^{-\gamma} \frac{(x_t + I_i/r)^{1-\gamma}}{1-\gamma} . \tag{3.7}$$

We also note that the stochastic process for wealth becomes

$$dx_t = \left(\frac{1 - \rho_i - \lambda_i}{\gamma} + \frac{(\mu - r)^2}{\gamma\sigma^2} \frac{1+\gamma}{2\gamma} \right) x_t dt + \tag{3.8}$$

$$\left(\frac{1 - \rho_i - \lambda_i}{\gamma r} + \frac{(\mu - r)^2}{\gamma\sigma^2} \frac{1-\gamma}{2\gamma r} \right) I_i dt + \frac{\mu - r}{\gamma\sigma} x_t dz_t .$$

Due to the presence of the second term on the right hand side of (3.8), there's no guarantee that wealth will stay positive. On the other hand, if we set $I_i = 0$ (so that there is no wage income), the term disappears and we obtain a geometric Brownian motion, which implies that wealth stays positive almost surely.

We note that the rate at which money is withdrawn for consumption, and the amount of money invested in the risky asset, are both linear in $x_t + I_i/r$. Moreover, since

$$\frac{1}{r} = \int_t^\infty e^{-r(s-t)} ds$$

for any t, we see that I_i/r is the present value of the future sum of wage incomes, as discounted by the rate of return of the safe asset. Richard (1975) refers to this term as the agent's "human capital". We can think of $x_t + I_i/r$ as the present value of wealth that is not contingent upon any investment decision.

3.4.2 The Household With Two Agents

We now proceed to the model for state A, where both agents are alive. That is, when the intertemporal utility function is (3.2) and the wealth dynamics are (3.3). As before, we transform the intertemporal utility function from one with random terminal time into one with infinite horizon. The result is given in Proposition 2, which is proved in the appendix.

Proposition 2 *Suppose that the intertemporal utility function is given by* (3.3) *and that, for each i, the time of death τ_i follows the exponential distribution* (3.1). *Then the objective function can also be written as*

$$J_t^A(x_t, c_{1,s}, c_{2,s}, q_{1,s}, q_{2,s}, \epsilon_s) \tag{3.9}$$

$$= E_t \left[\int_t^\infty e^{-(\rho_1+\lambda_1+\lambda_2)(s-t)} \frac{c_{1,s}^{1-\gamma} + \lambda_2 C_1^{-\gamma}(x_s + I_1/r + q_{2,s}/\eta_2)^{1-\gamma}}{1-\gamma} ds \right.$$

$$\left. + \int_t^\infty e^{-(\rho_2+\lambda_1+\lambda_2)(s-t)} \frac{c_{2,s}^{1-\gamma} + \lambda_1 C_2^{-\gamma}(x_s + I_2/r + q_{1,s}/\eta_1)^{1-\gamma}}{1-\gamma} ds \mid x_t \right] ,$$

where

$$C_i = \frac{\lambda_i + \rho_i - (1-\gamma)r}{\gamma} - \frac{1-\gamma}{2} \frac{(\mu - r)^2}{(\sigma\gamma)^2} .$$

We see that the intertemporal utility function can be interpreted as the sum of the two sums of discounted utility, each with a rather complicated utility function which depends on both consumptions and wealth, and effective discount rates of $\rho_i + \lambda_1 + \lambda_2$. Provided that $\rho_1 \neq \rho_2$, the two sums have different discount rates, which implies that a commitment solution to the problem will not be an intertemporal equilibrium.

We want to find an intertemporal equilibrium solution for the problem with the intertemporal utility function $J_t^A(x_t, c_{1,s}, c_{2,s}, q_{1,s}, q_{2,s}, \epsilon_s)$ and wealth dynamics given by (3.3). In the following, we will let $\{c^*_{1,t}, c^*_{2,t}, q^*_{1,t}, q^*_{2,t}, \epsilon^*_t\}$ denote the decision rules that solve this problem. Moreover, we will let

$$V_i^A(x_t) = E_t\left[\int_t^\infty e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)} \frac{c^*_{i,s}1-\gamma + \lambda_{-i}C_i^{-\gamma}(x^*_s + I_i/r + q^*_{-i,s}/\eta_{-i})^{1-\gamma}}{1-\gamma} ds \mid x_t\right] \quad (3.10)$$

denote the value function of agent i, i.e. the expected sum of discounted utility of agent i when the decision rules $\{c^*_{1,t}, c^*_{2,t}, q^*_{1,t}, q^*_{2,t}, \epsilon^*_t\}$ are applied. We will now present a proposition that gives the dynamic programming equation for this problem. Since the intertemporal utility function given by equation (3.9) is the sum of two sums of discounted utility, each with its own discount rate, the dynamic programming equation is non-standard. The proof is in the appendix.

Proposition 3 *Let the intertemporal utility function of the household be given by* (3.9)*, the wealth dynamics by* (3.3) *and the value functions by* (3.10)*. Then the solution to the dynamic programming equation*

$$\max_{c_1,c_2,q_1,q_2,\epsilon}\left\{\frac{c_1^{1-\gamma} + \lambda_2 C_1^{-\gamma}(x + I_1/r + q_2/\eta_2)^{1-\gamma}}{1-\gamma} + \frac{c_2^{1-\gamma} + \lambda_1 C_2^{-\gamma}(x + I_2/r + q_1/\eta_1)^{1-\gamma}}{1-\gamma}\right.$$
$$+\left(\frac{dV_1^{BA}}{dx}(x) + \frac{dV_2^{BA}}{dx}(x)\right)(rx + \epsilon(\mu - r)x - c_1 - c_2 - q_1 - q_2 + I_1 + I_2)$$
$$\left.+\left(\frac{d^2V_1^A}{dx^2}(x) + \frac{d^2V_2^A}{dx^2}(x)\right)\frac{(\epsilon\sigma x)^2}{2}\right\}$$
$$= E_t\left[\frac{dV_1^A}{dt}(x_t)\,|\,x_t\right] + (\rho_1 + \lambda_1 + \lambda_2)V_1^A(x_t) + E_t\left[\frac{dV_2^A}{dt}(x_t)\,|\,x_t\right] + (\rho_2 + \lambda_1 + \lambda_2)V_2^A(x_t)$$

is an equilibrium decision rule.

The decision rule implicit in Proposition 3 is an equilibrium decision rule in the sense that it satisfies the definition due to Ekeland and Pirvu (2008) and Ekeland and Lazrak (2010). In the introductory chapter to this thesis, we mentioned the intuition behind the solution concept that they suggested. The mathematical details of the definition are discussed in the appendix in Section 7.

To proceed with the solution, we have to make a conjecture on the value functions. Given the structure of the dynamics and the utility functions, it is natural to make the conjecture[8]

$$V_i^A(x) = \frac{A_i^{-\gamma}(x+B_i)^{1-\gamma}}{1-\gamma}. \tag{3.11}$$

This implies that the value functions for the model when both agents are alive have the same functional form as the value functions for the problems when one agent is alive. Indeed, the conjecture we make here is standard in the literature when the utility function for each agent is given by a power function. (See, for example, Brunh and Steffensen (2011) and Wei *et al* (2020)). Of course, in general, we should expect to obtain $A_1 \neq A_2$ and $B_1 \neq B_2$.

With the conjecture (3.11), the left hand side of the dynamic programming equation in Proposition 3 is

$$\max_{c_1,c_2,q_1,q_2,\epsilon}\left\{\frac{c_1^{1-\gamma}+\lambda_2 C_1^{-\gamma}(x+I_1/r+q_2/\eta_2)^{1-\gamma}}{1-\gamma}+\frac{c_2^{1-\gamma}+\lambda_1 C_2^{-\gamma}(x+I_2/r+q_1/\eta_1)^{1-\gamma}}{1-\gamma}\right.$$

$$+((A_1(x+B_1))^{-\gamma}+(A_2(x+B_2))^{-\gamma})(rx+\epsilon(\mu-r)x-c_1-c_2-q_1-q_2+I_1+I_2)$$

$$\left.-\gamma((A_1^{-\gamma}(x+B_1))^{1-\gamma}+(A_2^{-\gamma}(x+B_2))^{1-\gamma})\frac{(\epsilon\sigma x)^2}{2}\right\}.$$

The maximization problem implicit in this equation yields the decision rules

$$c_i^* = ((A_1(x+B_1))^{-\gamma}+(A_2(x+B_2))^{-\gamma})^{-1/\gamma},$$

$$q_i^* = \eta_i\left(\frac{\lambda_i}{\eta_i}\right)^{-1/\gamma}\left(\left(\frac{C_{-i}}{A_1(x+B_1)}\right)^{\gamma}+\left(\frac{C_{-i}}{A_2(x+B_2)}\right)^{\gamma}\right)^{-1/\gamma}-\eta_i(x+i_{-i}/r)$$

and

$$\epsilon^* = \frac{\mu-r}{\gamma\sigma^2 x}\frac{(A_1(x+B_1))^{-\gamma}+(A_2(x+B_2))^{-\gamma}}{(A_1(x+B_1))^{-\gamma-1}+(A_2(x+B_2))^{-\gamma-1}}.$$

Thus, we have obtained expressions for $\{c_1^*, c_2^*, q_1^*, q_2^*, \epsilon^*\}$ in terms of the constants $\{A_1, A_2, B_1, B_2\}$, which are functions of the parameters of the model.

In order to find the equilibrium decision rules, what remains for us is to find the set of constants $\{A_1, A_2, B_1, B_2\}$. Specifically, we would like to derive a system of equations that can be solved analytically or (if necessary) numerically. This is the method we employed in Chapter 2. However, this is not straightforward, and we have not been able to solve this problem for the general case. Instead, in the remainder of this chapter, we will focus on the particular case of when there is no wage income, so that $I_1 = I_2 = 0$. For the interested reader, the appendix includes a discussion on the difficulties of finding the set of constants for the general case.

[8] Chang (2004) has an extensive discussion on how to infer the general form of the value functions from the dynamics and the utility functions.

3.4.3 The Case of No Wage Income

When there is no wage income, it can be inferred that the correct conjecture on the value functions is such that $B_1 = B_2 = 0$.[9] In other words, we obtain

$$V_i^A(x) = \frac{A_i^{-\gamma} x^{1-\gamma}}{1-\gamma} \ .$$

As a consequence, the decision rules $\{c_1^*, c_2^*, q_1^*, q_2^*, \epsilon^*\}$ become

$$c_i^* = (A_1^{-\gamma} + A_2^{-\gamma})^{-1/\gamma} x \ ,$$

$$q_i^* = \eta_i C_{-i}^{-1} (\lambda_i/\eta_i)^{1/\gamma} (A_1^{-\gamma} + A_2^{-\gamma}) x - \eta_i x$$

and

$$\epsilon^* = \frac{\mu - r}{\gamma \sigma^2} \ .$$

We now only have to find two constants, namely A_1 and A_2. For this, we need a system with two equations. To proceed, we take the time derivatives of the value functions, which gives us the two equations that we need.[10] The equations that we obtain are

$$0 = \frac{(c_{i,t}^*)^{1-\gamma} + \lambda_{-i,t} C_i^{-\gamma} (x_t + I_1/r + q_{-i,t}^*/\eta_{-i,t})^{1-\gamma}}{1-\gamma} - (\rho_i + \lambda_1 + \lambda_2) V_{i,t}^A(x_t) + E\left[\frac{dV_{i,t}^A(x)}{dt} \mid x_t\right]$$

for $i \in \{1, 2\}$. If we insert our conjectures on the value functions and the optimal strategies (as functions of A_1 and A_2), and divide by $x_t^{1-\gamma}/(1-\gamma)$, we obtain the system

$$0 = \left(1 + \lambda_{-i} C_i^{-1} (\lambda_{-i}/\eta_{-i})^{\frac{1-\gamma}{\gamma}}\right) A_i^{-\gamma} (A_1^{-\gamma} + A_2^{-\gamma})^{\frac{1-\gamma}{\gamma}} \tag{3.12}$$

$$-(1-\gamma)\left(2 + \eta_1 C_2^{-1} (\lambda_1/\eta_1)^{1/\gamma} + \eta_2 C_1^{-1} (\lambda_2/\eta_2)^{1/\gamma}\right) (A_1^{-\gamma} + A_2^{-\gamma})^{-1/\gamma}$$

$$+ \left((1-\gamma)(r + \eta_1 + \eta_2) - (\rho_i + \lambda_1 + \lambda_2) + (1-\gamma)\frac{(\mu - r)^2}{2\gamma\sigma^2}\right) \ .$$

In this system, the variable x has been removed, leaving us with only the parameters of the model, and with A_1 and A_2 (which are functions of the parameters). Hence, although the system yields no analytic solution, we can use it to solve the model numerically.

Since we cannot find an explicit solution for A_1 and A_2, not much can be said analytically about the impact of the parameters of the model on the decision rules $\{c_1^*, c_2^*, q_1^*, q_2^*, \epsilon^*\}$. We need to defer such investigations to the numeric analysis in the next section. However, we can make the following observations (for the case of no wage income).

[9] Again, see Chang (2004) for how to make inferences on the form of the value functions.

[10] If the agents have the same discount rate ($\rho_1 = \rho_2$), one can solve the problem with two agents by working with a single value function for the whole household. This is the approach taken in, for example, Wei *et al* (2020).

- The two agents have the exact same consumption, even though their discount rates differ. Hence, differences in discount rates do not result in differences in consumption. The same result is found in a similar but deterministic model in de Paz *et al* (2013).

- Consumption and premium are both linear in the wealth, and the fraction of wealth invested in the risky asset is constant. These are standard results in the literature.

- The premiums are potentially negative. A negative premium implies that the household receives a flow of income in exchange for making a lump sum payment to the insurance company at the time of death of the first agent. This is an undesirable result. However, in the numeric solutions in Section 5, the parameter values are such that the premiums are always positive.

3.4.4 The Case of No Wage Income and Logarithmic Utility Functions

In closing this section, we will make some observations on the special case $\gamma = 1$, which corresponds to logarithmic utility functions. In this case, the system (3.12) simplifies to

$$0 = (1 + \lambda_{-i} C_i^{-1}) A_i^{-1} - (\rho_i + \lambda_1 + \lambda_2) \ ,$$

and the constant $C_i = \lambda_i + \rho_i$. This yields the analytic solution

$$A_i = \rho_i + \lambda_i \ .$$

Hence, the equilibrium decision rules become

$$c_i^* = ((\rho_1 + \lambda_1)^{-1} + (\rho_2 + \lambda_2)^{-1})^{-1} x \ ,$$

$$q_i^* = \frac{\lambda_i}{\rho_{-i} + \lambda_{-i}} ((\rho_1 + \lambda_1)^{-1} + (\rho_2 + \lambda_2)^{-1})^{-1} x - \eta_i x$$

and

$$\epsilon^* = \frac{\mu - r}{\sigma^2}.$$

We see that in this case, consumption depends only on the discount rates and mortality rates, and is independent of all other parameters, for example the premium-insurance ratio. The premium q_i decreases in η_i, as should be expected, but is independent of η_{-i}.

3.5 Numeric Solutions

In this section, we use numeric solutions to investigate the behavior of the household when both agents are alive (state A) and there is no wage income. That is, we infer properties of the decision rules $\{c_1^*, c_2^*, q_1^*, q_2^*, \epsilon^*\}$ when $I_1 = I_2 = 0$. The reason for imposing that there is no wage income is, as was mentioned in the previous section, that only under this condition are we able to derive a system of equations that can be used to find numeric expressions for the control variables.

In looking for a numeric solution, we can impose the conditions $A_1 > 0$ and $A_2 > 0$, which must hold because the value functions must have the appropriate sign. This is so because if $0 < \gamma < 1$, the instantaneous utility function is always positive, which means that the value functions must be positive, while if $\gamma > 1$ the instantaneous utility function is always negative, so the value functions must be negative. Hence, we can limit our search to solutions such that A_1 and A_2 are positive.

For our base specification, we have taken parameter values primarily from Koo and Lim (2021), which is a recent paper that also utilizes constant mortality rates and an infinite planning horizon. The parameter values are $\gamma = 1.2$, $\rho_1 = \rho_2 = 0.7$, $\lambda_1 = \lambda_2 = 0.1$, $\eta_1 = \eta_2 = 0.04$, $r = 0.3$, $\mu = 0.06$ and $\sigma = 0.02$. Also, we set $x = 1$ throughout the analysis.[11] Hence, in our base specification, the two agents are identical in that they have the same discount rate, mortality rate and premium-insurance ratio. In the following numeric solutions, we will create asymmetries by varying one or several parameters at a time, keeping all others fixed to their benchmark values, which allows us to investigate the effects of specific asymmetries on the behavior of the household.

3.5.1 Asymmetric Discount Rates

In order to investigate the effects of asymmetric discount rates on the behavior of the household, we vary ρ_1, the discount rate of agent 1. The results are given in Table 3.3. The column, starting from the left, gives the discount rate of agent 1, followed by the consumptions (recall that $c_1 = c_2$), the premiums, the investment in the risky asset and the expected rate of change in wealth (i.e. the drift of the wealth process). We also provide the value functions of state A for each agent. First, we note that consumption increases in ρ_1, which is to be expected. The

Table 3.3: Asymmetric Discount Rates.

ρ_1	c_1, c_2	q_1	q_2	ϵ	$E(dx)$	$V_1^A(x)$	$V_2^A(x)$
0.4	0.370678	1.14154	1.83948	0.625	-3.67363	-9.81568	-6.63464
0.5	0.419134	1.296	1.73556	0.625	-3.82108	-7.94254	-6.25283
0.6	0.464632	1.44103	1.65025	0.625	-3.97179	-6.63464	-5.90945
0.7	0.51072	1.58793	1.58793	0.625	-4.14856	-5.59913	-5.59913

[11] Notice that since the decision rules are linear, the wealth dynamics will have the form of the a geometric Brownian Motion, meaning that, in general, there will be no steady state value of wealth.

more impatient is the household, the more it will consume. As a result, wealth will decrease faster, as can be seen in the column for the expected rate of change in wealth.

Secondly, we note that when ρ_1 increases, the premium of agent 1 increases. The agent spends more on life insurance when he is less patient. For agent 2, on the other hand, the premium decreases. We see that when one agent becomes less patient, the household shifts spending to the premium of that agent from the premium of the other agent. The two premiums converge when $\rho_1 = 0.7$, because in this case the agents have the same discount rate. We also note that the total amount of spending on premiums, $q_1 + q_2$, increases in ρ_1, although the effect is not that large. Hence, we may say that when the household as a whole becomes less patient, it spends more money on both consumption and premiums, which leads to a faster depletion of its wealth.

Thirdly, we note that the change in the discount rate of agent 1 has no impact on the fraction of wealth that is invested in the risky asset. This we already knew from the discussion in Section 4.

Remark 5.1: The fact that the household purchases less life insurance when it becomes more patient may seem counter-intuitive. Afterall, shouldn't more patience imply more willingness to invest into a future life insurance payment? The answer to this seeming paradox is that given how the wealth dynamics of our model are set up, the life insurance payment depends only on the premium at the time of death, and is independent of the premium at any previous time. Hence, what the agent is purchasing with the premium is the life insurance what would be paid out should he die *right now* (as opposed to at anytime in the future). This is reflected by the fact that, in the transformed objective, the life insurance payment appears in what is effectively the utility function. The premium is therefore, strictly speaking, no different from consumption: the agent withdraws money from the budget to receive utility in the present. This interesting aspect of the model is present already in Richard (1975) and in Pliska and Ye (2007), as well as in most papers based on those two.[12] However, it has rarely been discussed in the literature. □

3.5.2 Asymmetric Mortality Rates

In order to investigate the effects of asymmetric mortality rates on the behavior of the household, we vary λ_1, the mortality rate of agent 1. The results are given in Table 3.4. First, we see that consumption increases in the mortality rate. The sooner an agent expects to die, the more he consumes. This is an intuitive result. What is, perhaps, less intuitive is that, if one agent's mortality rate increases, the other agent also consumes more. (Again, recall that the agents always have the same consumption).

[12] As was mentioned earlier, a different approach can be found in Bayraktar and Young (2013), where the premium is a lump-sum payment.

Table 3.4: Asymmetric Mortality Rates.

λ_1	c_1, c_2	q_1	q_2	ϵ	$E(dx)$	$V_1^{BA}(x)$	$V_2^{BA}(x)$
0.05	0.487987	0.832979	1.61809	0.625	-3.3783	-6.07669	-5.75045
0.1	0.51072	1.58793	1.58793	0.625	-4.14856	-5.59913	-5.59913
0.15	0.53291	2.3415	1.55966	0.625	-4.91822	-5.18614	-5.45495

Secondly, the premium of agent 1 increases quickly in λ_1, while the premium of agent 2 decreases, and much more slowly. To understand why this happens, we need to grasp that an increase in the mortality rate of agent 1 affects the premiums through three different channels.

- The probability that agent 1 will die before agent 2 increases. This provides incentive to increase q_1 so that agent 2 will receive a higher life insurance payment.
- There's a decrease in $V_1^A(x)$, the intertemporal utility that agent 1 expects to receive when and if he becomes the sole survivor. This provides incentive to decrease q_2, since the life insurance payment received by agent 1 is less valuable in terms of utility. That is, when agent 1 expects to live a shorter amount of time, then he has less time to transform the insurance payment that he receives when agent 2 dies into utility. Mathematically, this effect is due to the presence of the constant C_1 in the objective function (3.9).
- The effective discount rates increase, which provides incentive to increase both premiums.

The change in the premiums that we observe in Table 3.4 is the net effect of those three separate effects. The first contributes to the increase in q_1, and the second effect is the cause of the decrease in q_2. The third effect contributes to the fact that the total spending on premiums, $q_1 + q_2$, increases.

3.5.3 Asymmetric Premium-Insurance Ratios

In order to investigate the effects of asymmetric premium-insurance ratios on the behavior of the household, we vary η_1, the premium-insurance ratio of agent 1. The results are given in Table 3.5. Again, we see that consumption increases, although only slightly. This is because when life insurance for agent 1 becomes more expensive, the household shifts spending from premiums to consumption. Note that this is not an intertemporal shift, i.e. the household is not trading future wealth for present consumption. As we discussed above, the premium is effectively the purchase of present utility, as is consumption. Hence, when the household reduces the premium of agent 1 in order to increase consumption, it is shifting spending from one source of present utility to another. Moreover, the household is not just shifting money to consumption, but also to life insurance for agent 2, although the effect is small.

Table 3.5: Asymmetric Insurace-Premium Ratios.

η_1	c_1, c_2	q_1	q_2	ϵ	$E(dx)$	$V_1^{BA}(x)$	$V_2^{BA}(x)$
0.04	0.51072	1.58793	1.58793	0.625	-4.14856	-5.59913	-5.59913
0.08	0.51351	0.838637	1.59683	0.625	-3.41373	-5.52617	-5.59913
0.12	0.516265	0.538758	1.60561	0.625	-3.12815	-5.45495	-5.59913

The main effect of the increase in η_1 is, however, a decrease in overall spending on present utility, i.e. a decrease in $c_1 + c_2 + q_1 + q_2$, and consequently an increase in saving. Hence, the drift term of the wealth process becomes smaller (in absolute terms). The household is consuming its wealth at a slower rate.

3.5.4 Isolating the Effect of Which Agent Dies First

Suppose that we increase the parameter λ_1, keeping all other parameters constant. As we discussed above, this change affects the household through three different channels. However, if we increase λ_1 and simultaneously decrease ρ_1 by the same amount, two of these channels are mostly neutralized. This is because the effective discount rate of agent 1 is unchanged (including the one which in (3.9) appears inside the constant C_1, which is the effective discount of the state when only agent 1 is alive). For agent 2, the effective discount rate which appears inside the constant C_2 is also unchanged, while the one which appears explicitly in (3.9), namely $\rho_2+\lambda_1+\lambda_2$ is not. However, the channel that is not affected at all is the probability that agent 1 will die before agent 2. This implies that a change in the behavior of the household will be primarily the result of a change in that probability. Thus, we will have (almost) isolated the effect of one of the three channels.

In Table 3.6 we vary λ_1 and ρ_1 in such a way that the sum $\lambda_1 + \rho_1$ is constant. Hence, the effective discount rate of agent 1 is not changing. When the discount rate increases, agent 1 has incentive to increase consumption. But at the same time his life expectancy is increasing, which provides incentive to increase saving. The net effect is a very modest change in consumption, which is due to the fact that one of the effective discount rates of agent 2 has decreased slightly (since it contains λ_1). In contrast, we see a very large decrease in the premium of agent 1. The reason is that the probability that agent 1 will die first is decreasing, which increases the probability that the premium he spends will not result in the household receiving an insurance payment.

Table 3.6: The Effect of Which Agent Dies First.

λ_1	ρ_1	c_1, c_2	q_1	q_2	ϵ	$E(dx)$	$V_1^{BA}(x)$	$V_2^{BA}(x)$
0.2	0.6	0.51351	2.87649	1.59683	0.625	-5.45158	-5.52617	-5.59913
0.1	0.7	0.51072	1.58793	1.58793	0.625	-4.14856	-5.59913	-5.59913
0.05	0.75	0.505108	0.863607	1.57004	0.625	-3.39512	-5.67386	-5.67386

3.6 Conclusion

In this chapter, we have solved a life insurance model for a household with two agents. The new feature of our model is that we allow the agents to have different constant discount rates, and we derive a cooperative solution. Thus, our model combines the treatment of inconsistent time preferences and the presence of a household with more than one agent, something which seems to not have been done previously in life insurance models. In solving the model, we assumed that for each agent, the time of death is exponentially distributed, so that each agent has constant discount and mortality rates. This assumption proved to be useful for reason of tractability.

Expansions of the model might go in several directions.

- More than two agents. A model with an arbitrary number of agents in the household was constructed in Bruhn and Steffensen (2011), but without taking the issue of inconsistent time preferences into consideration.

- Nonconstant mortality rates. For example, one might assume that the mortality rates follow Gompertz's Law, as has often been the case in the literature.

- Nonconstant discount rates. This would imply not only that the household as a whole has inconsistent time preferences, but also that each agent on his own has inconsistent time preferences.

3.7 Appendix

3.7.1 Proof of Proposition 1

First, we acknowledge that for $s > t$ and $i \in \{1, 2\}$, the expectation of $1_{\tau_i > s}$ conditional on $\tau_i > t$ can be rewritten as

$$E\left[1_{\tau_i > s} \mid \tau_i > t\right] = P(\tau_i > s \mid \tau_i > t) = \frac{P(\tau_i > s\ ,\ \tau_i > t)}{P(\tau_i > t)} = \frac{P(\tau_i > s)}{P(\tau_i > t)} = e^{-\lambda_i (s-t)}. \qquad (3.13)$$

Then, to rewrite (3.5) into (3.6) we apply the following steps.

$$J^B_{i,t}(x_t; c_{i,s}, \epsilon_s) = E_t\left[\int_t^{\tau_i} e^{-\rho_i(s-t)}\frac{c_{i,s}^{1-\gamma}}{1-\gamma}ds \mid \tau_i > t, x_t\right] \tag{3.14}$$

$$= E_t\left[\int_t^{\infty} e^{-\rho_i(s-t)}\frac{c_{i,s}^{1-\gamma}}{1-\gamma}1_{\tau_i>s}ds \mid \tau_i > t, x_t\right]$$

$$= E_t\left[\int_t^{\infty} e^{-\rho_i(s-t)}\frac{c_{i,s}^{1-\gamma}}{1-\gamma}E\left[1_{\tau_i>s} \mid \tau_i > t\right]ds \mid x_t\right]$$

$$= E_t\left[\int_t^{\infty} e^{-(\rho_i+\lambda_i)(s-t)}\frac{c_{i,s}^{1-\gamma}}{1-\gamma}ds \mid x_t\right] .$$

In the second equality, we change the limit of integration from τ_i to infinity and insert the random variable $1_{\tau_i>s}$. The third equality follows because τ_i is independent of the Brownian motion z_t. The fourth equality follows from (3.13).

3.7.2 Proof of Proposition 2

First, we acknowledge that for $s > t$ and $i \in \{1, 2\}$, the expectation of $1_{\bar{\tau}>s}$ conditional on $\bar{\tau} > t$ can be rewritten as

$$E\left[1_{\bar{\tau}>s} \mid \bar{\tau} > t\right] = P(\bar{\tau} > s \mid \bar{\tau} > t) = P(\tau_1 > s, \tau_2 > s \mid \tau_1 > t, \tau_2 > t) \tag{3.15}$$

$$= P(\tau_1 > s \mid \tau_1 > t)P(\tau_2 > s \mid \tau_2 > t) = \frac{e^{-\lambda_1 s}}{e^{-\lambda_1 t}}\frac{e^{-\lambda_2 s}}{e^{-\lambda_2 t}} = e^{-(\rho_1+\lambda_1+\lambda_2)(s-t)} .$$

Then, to rewrite (3.2) into (3.9), first note that the expectation in (3.2) contains four terms which can be treated separately. For the first term, we have

$$E_t\left[\int_t^{\bar{\tau}} e^{-\rho_1(s-t)}\frac{c_{1,s}^{1-\gamma}}{1-\gamma}ds \mid \bar{\tau} > t, x_t\right] \tag{3.16}$$

$$= E_t\left[\int_t^{\infty} e^{-\rho_1(s-t)}\frac{c_{1,s}^{1-\gamma}}{1-\gamma}1_{\bar{\tau}>s}ds \mid \bar{\tau} > t, x_t\right]$$

$$= E_t\left[\int_t^{\infty} e^{-\rho_1(s-t)}\frac{c_{1,s}^{1-\gamma}}{1-\gamma}E\left[1_{\bar{\tau}>s} \mid \bar{\tau} > t\right]ds \mid x_t\right]$$

$$= E_t\left[\int_t^{\infty} e^{-(\rho_1+\lambda_1+\lambda_2)(s-t)}\frac{c_{1,s}^{1-\gamma}}{1-\gamma}ds \mid x_t\right] ,$$

where the third follows from (3.15). For the second term in (3.2) we have

$$E_t\left[e^{-\rho_1(\bar{\tau}-t)}V_1^B(x_{\bar{\tau}} + q_{2,\bar{\tau}}/\eta_2)1_{\tau_2=\bar{\tau}} \mid \bar{\tau} > t, x_t\right]$$

$$= E_t \left[e^{-\rho_1(\tau_2 - t)} V_1^B(x_{\tau_2} + q_{2,\tau_2}/\eta_2) 1_{\tau_2 < \tau_1} \mid \tau_1 > t, \tau_2 > t, x_t \right]$$

$$= E_t \left[\int_t^\infty \lambda_2 e^{-\lambda_2(s-t)} e^{-\rho_1(s-t)} V_1^B(x_s + q_{2,s}/\eta_2) 1_{s<\tau_1} ds | \tau_1 > t, x_t \right]$$

$$= E_t \left[\int_t^\infty \lambda_2 e^{-\lambda_2(s-t)} e^{-\rho_1(s-t)} V_1^B(x_s + q_{2,s}/\eta_2) E\left[1_{s<\tau_1} \mid \tau_1 > t\right] ds \mid x_t \right] .$$

Recognizing that $E\left[1_{s<\tau_1} \mid \tau_1 > t\right] = e^{-\lambda_1(s-t)}$ and, from (3.7), that

$$V_1^B(x_t + q_{2,t}/\eta_2)$$

$$= \left(\frac{\lambda_1 + \rho_1 - (1-\gamma)r}{\gamma} - \frac{1-\gamma}{2} \frac{(\mu - r)^2}{(\sigma\gamma)^2} \right)^{-\gamma} \frac{(x_t + q_{2,t}/\eta_2 + I_1/r)^{1-\gamma}}{1-\gamma}$$

$$= C_1^{-\gamma} \frac{(x_t + q_{2,t}/\eta_2 + I_1/r)^{1-\gamma}}{1-\gamma} ,$$

the second term in (3.2) becomes

$$E_t \left[\int_t^\infty \lambda_2 e^{-(\rho_1+\lambda_1+\lambda_2)(s-t)} C_1^{-\gamma} \frac{(x_t + q_{2,t}/\eta_2 + I_1/r)^{1-\gamma}}{1-\gamma} ds \mid x_t \right] . \tag{3.17}$$

Combining (3.16) and (3.17) we obtain

$$E_t \left[\int_t^\infty e^{-(\rho_1+\lambda_1+\lambda_2)(s-t)} \frac{c_{1,s}^{1-\gamma} + \lambda_2 C_1^{-\gamma}(x_s + I_1/r + q_{2,s}/\eta_2)^{1-\gamma}}{1-\gamma} ds \mid x_t \right] . \tag{3.18}$$

In a similar way, we can combine the third and fourth terms in (3.2) to obtain

$$E_t \left[\int_t^\infty e^{-(\rho_2+\lambda_1+\lambda_2)(s-t)} \frac{c_{2,s}^{1-\gamma} + \lambda_1 C_2^{-\gamma}(x_s + I_2/r + q_{1,s}/\eta_2)^{1-\gamma}}{1-\gamma} ds \,\middle|\, x_t \right] . \tag{3.19}$$

Combining (3.18) and (3.19) we obtain (3.9), the transformed intertemporal utility function.

3.7.3 Proof of Proposition 3

In short, we need to derive the dynamic programming for an intertemporal equilibrium solution to a problem with stochastic dynamics, in which the intertemporal utility function is the sum of two sums of utility, each with its own constant discount rate. In a deterministic setting and with an arbitrary number of agents, this was done in de Paz *et al* (2013). Here, we present the corresponding proof for the case of two agents and with stochastic dynamics. As was mentioned in the introduction to the thesis, our definition of an equilibrium solution is due to Ekeland and Pirvu (2008) and Ekeland and Lazrak (2010), and we will construct the dynamic programming equation in such a way that it satisfies this definition.

For notational simplicity, we will define

$$U_i(c_{i,s}, q_{-i,s}, x_s) = \frac{c_{i,s}^{1-\gamma} + \lambda_{-i} C_i^{-\gamma}(x_s + I_i/r + q_{-i,s}/\eta_{-i})^{1-\gamma}}{1-\gamma} ,$$

for $i \in \{1, 2\}$. First, we recognize that for each $t > 0$, we have

$$J_t^A(x_t, c_1^*, c_2^*, q_1^*, q_2^*, \epsilon^*) = V_1^A(x_t) + V_2^A(x_t) \ .$$

That is, the value of the intertemporal utility function when we apply the decision rule $\{c_1^*, c_2^*, q_1^*, q_2^*, \epsilon^*\}$ is equal to the sum of the value functions. Next, for a given t, we consider the decision rules that are given by $\{\bar{c}_1, \bar{c}_2, \bar{q}_1, \bar{q}_2, \bar{\epsilon}\}$ if $s \in [t, t + \Delta t]$ and $\{c_1^*, c_2^*, q_1^*, q_2^*, \epsilon^*\}$ if $s > t + \Delta t$, and we let $\bar{x}_s$ be the trajectory of the wealth process when the household follows $\{\bar{c}_1, \bar{c}_2, \bar{q}_1, \bar{q}_2, \bar{\epsilon}\}$. The intertemporal utility function when this decision rule is applied is

$$\begin{aligned} J^{BA}(x_t; \bar{c}_1, \bar{c}_2, \bar{q}_1, \bar{q}_2, \bar{\epsilon}) = \\ = E_t \Bigg[\sum_{i=1}^{2} \int_t^{t+\Delta t} e^{-(\rho_i + \lambda_1 + \lambda_2)(s-t)} U_i(\bar{c}_{i,s}, \bar{q}_{-i,s}, \bar{x}_s) ds \\ + \sum_{i=1}^{2} \int_{t+\Delta t}^{\infty} e^{-(\rho_i + \lambda_1 + \lambda_2)(s-t)} U_i(c_{i,s}^*, q_{-i,s}^*, x_s^*) ds \mid x_t \Bigg] \ . \end{aligned}$$

The idea behind this decision rule is that $[t, t + \Delta t]$ is the period in which the household is able to commit its future selves to a certain behavior. We can refer to this a the "period of commitment". Hence, the household at t is in control of behavior over a nonzero interval of time, as in the discrete time case, and can therefore influence the intertemporal utility function. Next, suppose that for any $\{\bar{c}_1, \bar{c}_2, \bar{q}_1, \bar{q}_2, \bar{\epsilon}\}$, it is the case that

$$\lim_{\Delta t \to 0^+} \frac{J_t^A(x_t, c_1^*, c_2^*, q_1^*, q_2^*, \epsilon^*) - J_t^A(x_t, \bar{c}_1, \bar{c}_2, \bar{q}_1, \bar{q}_2, \bar{\epsilon})}{\Delta t} \geq 0 \ .$$

If this condition is satisfied, we say that $\{c_1^*, c_2^*, q_1^*, q_2^*, \epsilon^*\}$ is the equilibrium decision rule. This is the definition of an equilibrium solution due to Ekeland and Lazrak (2010).

By taking the limit of Δt, the period of commitment vanishes, and the numerator approaches zero. But the expression is scaled up by the denominator, so that the behavior within the period of commitment remains influential. The intuition is that if $\{c_1^*, c_2^*, q_1^*, q_2^*, \epsilon^*\}$ is indeed the equilibrium decision rule, the expression must be nonnegative because no other decision rule can result in a larger value of the intertemporal utility function. Of course, if we set $\{\bar{c}_1, \bar{c}_2, \bar{q}_1, \bar{q}_2, \bar{\epsilon}\} = \{c_1^*, c_2^*, q_1^*, q_2^*, \epsilon^*\}$, the expression evaluates to zero. Hence, we can also define the equilibrium decision rule to be any $\{\bar{c}_1, \bar{c}_2, \bar{q}_1, \bar{q}_2, \bar{\epsilon}\}$ that solves

$$0 = \max_{\bar{c}_1, \bar{c}_2, \bar{q}_1, \bar{q}_2, \bar{\epsilon}} \left\{ \lim_{\Delta t \to 0^+} \frac{J_t^A(x_t, \bar{c}_1, \bar{c}_2, \bar{q}_1, \bar{q}_2, \bar{\epsilon}) - J_t^A(x_t, c_1^*, c_2^*, q_1^*, q_2^*, \epsilon^*)}{\Delta t} \right\} \ , \qquad (3.20)$$

which provides us with a dynamic programming equation for finding the equilibrium solution. To make the dynamic programming equation workable, we need to rewrite it into something more familiar. To this end, we notice that

$$J_t^A(x_t, \bar{c}_1, \bar{c}_2, \bar{q}_1, \bar{q}_2, \bar{\epsilon}) - J_t^A(x_t, c_1^*, c_2^*, q_1^*, q_2^*, \epsilon^*) = \qquad (3.21)$$

$$= E_t \left[\sum_{i=1}^{2} \int_t^{t+\Delta t} e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)} (U_i(\bar{c}_{i,s}, \bar{q}_{-i,s}, \bar{x}_s) - U_i(c^*_{i,s}, q^*_{-i,s}, x^*_s))ds + \right.$$

$$\left. + \sum_{i=1}^{2} \int_{t+\Delta t}^{\infty} e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)} (U_i(c^*_{i,s}, q^*_{-i,s}, \bar{x}_s) - U_i(c^*_{i,s}, q^*_{-i,s}, x^*_s))ds \mid x_t \right] .$$

Moreover,

$$E_t \left[\int_{t+\Delta t}^{\infty} e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)} U_i(c^*_{i,s}, q^*_{-i,s}, \bar{x}_s)ds \mid x_t \right] = \tag{3.22}$$

$$+E_t \left[V_i^A(\bar{x}_{t+\Delta t}) + \int_{t+\Delta t}^{\infty} (e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)} - e^{-(\rho_i+\lambda_1+\lambda_2)(s-t-\Delta t)}) U_i(c^*_{i,s}, q^*_{-i,s}, \bar{x}_s)ds \mid x_t \right]$$

and

$$E_t \left[\int_{t+\Delta t}^{\infty} e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)} U_i(c^*_{i,s}, q^*_{-i,s}, x^*_s)ds \mid x_t \right] = \tag{3.23}$$

$$+E_t \left[V_i^A(x^*_{t+\Delta t}) + \int_{t+\Delta t}^{\infty} (e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)} - e^{-(\rho_i+\lambda_1+\lambda_2)(s-t-\Delta t)}) U_i(c^*_{i,s}, q^*_{-i,s}, x^*_s)ds \mid x_t \right] .$$

Combining (3.22) and (3.23) we have

$$E_t \left[\int_{t+\Delta t}^{\infty} e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)} (U_i(c^*_{i,s}, q^*_{-i,s}, \bar{x}_s) - U_i(c^*_{i,s}, q^*_{-i,s}, x^*_s))ds \mid x_t \right] = \tag{3.24}$$

$$E_t \left[V_i^A \bar{x}_{t+\Delta t}) - V_i^A(x^*_{t+\Delta t}) + \right.$$

$$\left. (o(\Delta) - (\rho_i + \lambda_1 + \lambda_2)\Delta t) \int_{t+\Delta t}^{\infty} e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)} (U_i(c^*_{i,s}, q^*_{-i,s}, \bar{x}_s) - U_i(c^*_{i,s}, q^*_{-i,s}, x^*_s))ds \mid x_t \right] ,$$

where we have used the Taylor expansion $e^{(\rho_i+\lambda_1+\lambda_2)\Delta t} = 1 + (\rho_i + \lambda_1 + \lambda_2)\Delta t + o(\Delta t)$, and $o(\Delta t)$ is terms that converge to zero at a faster rate than Δt. Combining this with (3.21) and dividing by Δt, we have

$$\frac{J_t^A(x_t, \bar{c}_1, \bar{c}_2, \bar{q}_1, \bar{q}_2, \bar{\epsilon}) - J_t^A(x_t, c^*_1, c^*_2, q^*_1, q^*_2, \epsilon^*)}{\Delta t} =$$

$$= E_t \left[\sum_{i=1}^{2} \frac{1}{\Delta t} \int_t^{t+\Delta t} e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)} (U_i(\bar{c}_{i,s}, \bar{q}_{-i,s}, \bar{x}_s) - U_i(c^*_{i,s}, q^*_{-i,s}, x^*_s))ds + \right.$$

$$\sum_{i=1}^{2} \left(\frac{V_i^A(\bar{x}_{t+\Delta t}) - V_i^A(x_t)}{\Delta t} - \frac{V_i^A(x^*_{t+\Delta t}) - V_i^A(x_t)}{\Delta t} \right) +$$

$$\left. \sum_{i=1}^{2} (o(\Delta)/\Delta t - \rho_i - \lambda_1 - \lambda_2) \int_{t+\Delta t}^{\infty} e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)} (U_i(c^*_{i,s}, q^*_{-i,s}, \bar{x}_s) - U_i(c^*_{i,s}, q^*_{-i,s}, x^*_s))ds \mid x_t \right] .$$

When we take the limit $\Delta t \to 0^*$, we can treat each of the terms in the above expression separately. We have

$$\lim_{\Delta t \to 0^+} E_t \left[\frac{1}{\Delta t} \int_t^{t+\Delta t} e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)} (U_i(\bar{c}_{i,s}, \bar{q}_{-i,s}, \bar{x}_s) - U_i(c^*_{i,s}, q^*_{-i,s}, x^*_s))ds \mid x_t \right] =$$

$$= U_i(\bar{c}_{i,t}, \bar{q}_{-i,t}, \bar{x}_t) - U_i(c^*_{i,t}, q^*_{-i,t}, x^*_t)\,,$$

which follows from L'Hopital's Rule. We have

$$\lim_{\Delta t \to 0^+} E_t \left[\frac{V_i^A(\bar{x}_{t+\Delta t}) - V_i^A(x_t)}{\Delta t} \mid x_t \right] = -(\rho_i + \lambda_1 + \lambda_2) V_i(x_t) +$$

$$+ \frac{dV_i^A}{dx_t}(x_t)(rx_t + \bar{\epsilon}_t(\mu - r)x_t - \bar{c}_{1,t} - \bar{c}_{2,t} - \bar{q}_{1,t} - \bar{q}_{2,t} + I_1 + I_2) + \frac{d^2 V_1^A}{dx^2}(x)\frac{(\bar{\epsilon}_t \sigma x_t)^2}{2}$$

and

$$\lim_{\Delta t \to 0^+} E_t \left[\frac{V_i^A(x^*_{t+\Delta t}) - V_i^A(x_t)}{\Delta t} = -(\rho_i + \lambda_1 + \lambda_2) V_i^A(x_t) \mid x_t \right] +$$

$$+ \frac{dV_i^A}{dx_t}(x_t)(rx_t + \epsilon^*_t(\mu - r)x_t - c^*_{1,t} - c^*_{2,t} - q^*_{1,t} - q^*_{2,t} + I_1 + I_2) + \frac{d^2 V_1^A}{dx^2}(x)\frac{(\epsilon^*_t \sigma x_t)^2}{2}\,.$$

For the last terms, we have

$$0 = \lim_{\Delta t \to 0^+} E_t \left[(o(\Delta)/\Delta t - \rho_i - \lambda_1 - \lambda_2) \right.$$

$$\left. \int_{t+\Delta t}^{\infty} e^{-(\rho_i + \lambda_1 + \lambda_2)(s-t)} (U_i(c^*_{i,s}, q^*_{-i,s}, \bar{x}_s) - U_i(c^*_{i,s}, q^*_{-i,s}, x^*_s)) ds \mid x_t \right]$$

because the utility functions converge. Hence, putting everything together we obtain

$$\lim_{\Delta t \to 0^+} \frac{J_t^A(x_t, \bar{c}_1, \bar{c}_2, \bar{q}_1, \bar{q}_2, \bar{\epsilon}) - J_t^A(x_t, c_1^*, c_2^*, q_1^*, q_2^*, \epsilon^*)}{\Delta t} \tag{3.25}$$

$$= \sum_{i=1}^{2} (U_i(\bar{c}_{i,t}, \bar{q}_{-i,t}, \bar{x}_t) - U_i(c^*_{i,t}, q^*_{-i,t}, x^*_t))$$

$$+ \sum_{i=1}^{2} \left(\frac{dV_i^A}{dx_t}(x_t)(rx_t + \bar{\epsilon}_t(\mu - r)x_t - \bar{c}_{1,t} - \bar{c}_{2,t} - \bar{q}_{1,t} - \bar{q}_{2,t} + I_1 + I_2) + \frac{d^2 V_1^A}{dx^2}(x)\frac{(\bar{\epsilon}_t \sigma x_t)^2}{2} \right)$$

$$- \sum_{i=1}^{2} \left(\frac{dV_i^A}{dx_t}(x_t)(rx_t + \epsilon^*_t(\mu - r)x_t - c^*_{1,t} - c^*_{2,t} - q^*_{1,t} - q^*_{2,t} + I_1 + I_2) + \frac{d^2 V_1^A}{dx^2}(x)\frac{(\epsilon^*_t \sigma x_t)^2}{2} \right)\,.$$

Finally, by recognizing that

$$\frac{dV_i^A}{dx_t}(x_t)(rx_t + \epsilon^*_t(\mu - r)x_t - c^*_{1,t} - c^*_{2,t} - q^*_{1,t} - q^*_{2,t} + I_1 + I_2) + \frac{d^2 V_1^A}{dx^2}(x)\frac{(\epsilon^*_t \sigma x_t)^2}{2}$$

$$+ U_i(c^*_{i,t}, q^*_{-i,t}, x^*_t) = E_t \left[\frac{dV_i^A}{dt}(x_t) \mid x_t \right] + (\rho_i + \lambda_1 + \lambda_2) V_i^A(x_t)$$

we obtain Proposition 3.

3.7.4 In Search of a Solution when there is Wage Income

In order to find the constants $\{A_1, A_2, B_1, B_2\}$, we attempt the following method. We first create a system of two equations by taking the time derivative of (3.10) for both agents. With $\{c_1^*, c_2^*, q_1^*, q_2^*, \epsilon^*\}$ as we derived for the case when there is wage income, the system becomes

$$
\begin{aligned}
0 = &(rx + I_1 + I_2 + (\eta_1 + \eta_2)x) \\
&\frac{((A_1(x+B_1))^{-\gamma} + (A_2(x+B_2))^{-\gamma})^{\frac{1-\gamma}{\gamma}}}{1-\gamma}(A_i(x+B_i))^{-\gamma} - (\rho_i + \lambda_1 + \lambda_2)\frac{x+B_i}{1-\gamma} \\
&+\frac{(\mu - r)^2}{\gamma\sigma^2}\frac{(A_1(x+B_1))^{-\gamma} + (A_2(x+B_2))^{-\gamma}}{(A_1(x+B_1))^{-\gamma-1} + (A_2(x+B_2))^{-\gamma-1}} - 2((A_1(x+B_1))^{-\gamma} + (A_2(x+B_2))^{-\gamma})^{-1/\gamma} \\
&-(x+B_i)\frac{(\mu-r)^2}{2\gamma\sigma^2}\left(\frac{(A_1(x+B_1))^{-\gamma} + (A_2(x+B_2))^{-\gamma}}{(A_1(x+B_1))^{-\gamma-1} + (A_2(x+B_2))^{-\gamma-1}}\right)^2 \\
&+\eta_1\left(\frac{\lambda_1}{\eta_1}\right)^{1/\gamma}\left(\left(\frac{C_1}{A_1(x+B_1)}\right)^{\gamma} + \left(\frac{C_1}{A_2(x+B_2)}\right)^{\gamma}\right)^{-1/\gamma} \\
&+\eta_2\left(\frac{\lambda_2}{\eta_2}\right)^{1/\gamma}\left(\left(\frac{C_2}{A_1(x+B_1)}\right)^{\gamma} + \left(\frac{C_2}{A_2(x+B_2)}\right)^{\gamma}\right)^{-1/\gamma}
\end{aligned}
$$

for $i \in \{1, 2\}$. At this point, what we would like to do is to separate each equation in the system into two equations, based on which terms are multiplied by x and which are not. We would then obtain the four equations that we need to solve for the four parameters in $\{A_1, A_2, B_1, B_2\}$. However, there is no way to separate the terms in the equations based on powers of x. The reason is the presence of the terms B_1 and B_2, which are part of our conjecture for the value functions. It is clear that the system if solvable if $B_1 = B_2$, because in that case one can divide by $x + B_i$ in order to remove the complicating factors. But attempting to prove conditions under which $B_1 = B_2$ is not straightforward.

Chapter 4

Fish War Models with Asymmetric Agents and Quasi-Hyperbolic Time Preferences

4.1 Introduction

There exists a large literature dealing with the economics of international fisheries agreements. The need for such research is due to the fact that, at the international political level, the management of territories of water and the harvest (and preservation) of their fish stocks is an important issue. It's long been understood that the management of fish stock suffers from a prisoners' dilemma situation; in the absence of enforced quotas for the harvest of fish, a problem of overfishing and, in the long run, depletion of the stock, tends to arise.

Many authors have used coalition games to model fishing agreements, as such games allow for the investigation of incentives for coordination contra non-coordination among the agents in the model. Kronback *et al* (2015), in surveying the literature on such fishing games, divide these papers into the three categories of characteristic function games, partition function games and dynamic games. The application of dynamic games to fishing agreements can be traced back to Mirman (1979) and Levhari and Mirman (1980). Their model is known as the Great Fish War, and has been expanded upon by many authors. Much of the literature is in discrete time.[1]

In the standard dynamic fish war game, the agents are assumed to derive utility (or profit) from the harvest of fish. The dynamics, usually referred to as the game's growth function, are assumed to be nonlinear. Since the harvest in the present period decreases the total fish stock that remains for harvest in future periods, there arises a trade-off between extraction in the present and in the future. Moreover, since each agent is assumed to derive utility only from

[1]For a model in continuous time, see Clark (1990).

his own harvest, and not from that of the other agents, each agent has an incentive to overfish in the present, else he risks losing utility to the other agents. For this reason, fish war models have often been solved for both noncooperative and cooperative solutions, as this allows for measuring whether there is incentive to deviate from cooperation. In the original paper by Levhari and Mirman (1980), it was shown that (under a set of particular assumptions) the noncooperative solution had a smaller steady state stock of fish than the cooperative solution. This result extends to many other papers in the literature.

Most papers on the fish war assume symmetric agents with consistent time preferences. That is, all agents are assumed to have the same discount and utility functions, and the discount function is assumed to be exponential, so that the discount rate is constant. This has the advantage of simplifying the mathematical treatment necessary for finding solutions. The disadvantage is lack of realism. There is no reason to assume that the economic agents that engage in fishing are “copies” of one another and react to external circumstances in exactly the same way. However, going beyond any of these assumptions bring complications in finding solutions, as we have seen in previous chapters. A coalition in which the members have different discount rates will have a nonconstant discount rate. Hence, the aggregated time preferences of the coalition are inconsistent. Moreover, if the discount functions are nonexponential, the individual agents too will have inconsistent time preferences. The first paper to use asymmetric (but constant) discount rates in a fish war model seems to be Breton and Keoula (2014). They were able to derive analytic solutions under both noncooperation and cooperation, under the assumption that the agents have logarithmic utility functions.

In this chapter, we go a step further and allow the agents to have asymmetric quasi-hyperbolic discount functions. We will study this type of time preference in three fish war models. In the first, we maintain the assumption of logarithmic utility functions and the standard growth function for fish war models. Hence, our first model is identical to that of Breton and Keoula (2014) except for the discount functions.

In the second model, we introduce stochastics with autocorrelated random events into the analysis, and we use general power utility functions. For reasons of tractability, power utility functions have rarely been used in fish war models. When power utility functions are combined with the nonlinear growth function that is standard in a fish war model, the result is that explicit solutions are very difficult (maybe impossible) to find. In Antoniadou *et al* (2013), this problem was circumvented by imposing a particular relationship between the parameter of relative risk aversion of the utility function and a parameter in the growth function. Although this is completely unmotivated from an economic point of view, it at least enhances tractability. In our model we will use a different method, namely that of a linear growth function, i.e. exponential growth of the fish stock. This too is done partially for reasons of tractability, although an economic motivation can be provided: for low levels of the fish stock, it is not inconceivable that the growth is approximately exponential. (For higher levels, one has to imagine that the growth levels off, so that the growth path forms an s-shaped curve.)

In the third model, we reintroduce the nonlinear growth function that is standard, and we also allow the agents to have power utility functions with different coefficients. Hence, this model allows for asymmetry in both the utility and discount functions. Due to the difficulties in using analytic methods, we rely heavily on numeric methods in this part of the chapter. This third model is solved in finite horizon and only for the cooperative solution, and the first two models are solved in infinite horizon for both cooperation and noncooperation.

In order to solve these models, we first develop a general theoretical framework within which our models appear as particular cases. We derive discrete time dynamic programming equations for finding noncooperative and cooperative intertemporal equilibria in dynamic games in which the agents have different and general utility and discount functions. We also allow the dynamics of the game to be stochastic and the random events to be autocorrelated, and we allow for both a finite and an infinite planning horizon. Our first two fish war models are solved using this theory. The third model is solved using a form of backward induction which is not based on dynamic programming equations and value functions. This method is described in the appendix to this chapter.

As mentioned, all of our models assume quasi-hyperbolic discount functions. This is perhaps the simplest type of discount function that preserve the property of inconsistent time preferences. It was introduced by Phelps and Pollak (1968) and has been extensively studied within the field of behavioral economics. According to this model, if $\delta \in (0, 1)$ is the standard geometric discount factor, the intertemporal utility function for the agent at time t is defined as

$$J_t = u_t + \beta(\delta u_{t+1} + \delta^2 u_{t+2} + \delta^3 u_{t+3} + \cdots) ,$$

where $\beta \in (0, 1]$ andu_t is the utility obtained at time t. That is, the discount function for agent i is $\theta_{i,t} = \beta_i \delta_i^t$, for $t \geq 1$, and $\theta_{i,0} = 1$. Therefore, the agent at time t applies not only the fixed discount factor δ, but also a "future discount factor" $\beta > 0$ to all future periods. Clearly, if $\beta \neq 1$, the agent has inconsistent time preferences because the discount rate is not constant. A decrease in β increases the agent's bias toward the immediate present, as opposed to any other period. It doesn't make the agent less patient in general, as a decrease in δ would do. It changes the ratio of the discount function between utility in the present and utility in the any future period, but not between two future periods. Hence, a quasi-hyperbolic discount function differs both from a simple geometric discount function and from more complicated discount functions, such as sums of exponentials with different coefficients.

The main motivation for using power utility functions, which is nonstandard in fish war models, is that it allows for analyzing the relationship between risk and risk aversion. It's a peculiar property of logarithmic utility functions that they make agents indifferent to risk (at least in the context of the standard fish war models). For fish war models this is especially unfortunate, since logarithmic utility functions are so important for tractability. It should not come as a surprise, then, that the survey paper Kronback *et al* (2015) identifies risk in fishing games as an "underexplored area".

The rest of this chapter is organized as follows. Section 2 is devoted to developing the theoretical framework that we make use of in later sections. We derive the dynamic programming equations first for the general case and then for the particular case of quasi-hyperbolic discounting, for which the theory simplifies in an intuitive way. In Section 3 we apply the theory to solving a standard fish war model with asymmetric quasi-hyperbolic discounting. The section ends with some numeric results, in which we investigate how the bias toward the present in the discount functions affect coalitional stability and the steady state fish stock and harvest. In Section 4 we solve the second model, as described above. Again, we end the section with numeric results, looking in particular at the role of the discount functions. In Section 5 we solve the third model and investigate how the behavior of the coalition is affected by risk, asymmetry in preferences for harvest smoothing, and finally the discount functions. Section 6 concludes the chapter.

4.2 The Theory

We shall develop a theoretical framework for solving a stochastic game with N agents discounting the future at different and nonconstant rates. First we will develop the general theory, and later on we will look at the particular case of quasi-hyperbolic discounting. We will look at both finite and infinite planning horizon games.

Fujii and Karp (2008) derived a dynamic programming equation for the general problem of an agent whose preferences at time t are described by

$$J_t = u_t + \theta_1 u_{t+1} + \theta_2 u_{t+2} + \theta_3 u_{t+3} + \cdots \ ,$$

which we interpret as an infinite sum of utilities discounted at potentially nonconstant rates. They assumed that the dynamics were deterministic and applied their theory to finding an intertemporal equilibrium solution to a model on climate change. The theory that we will develop in this section is an extension that generalizes the dynamic programming equation in Fujii and Karp (2008) to a stochastic setting and several agents, both in a finite and an infinite planning horizon. A first informal derivation of such a stochastic dynamic programming equation in the case of just one decision maker was given in Marín-Solano and Navas (2010). In that paper, the probability distributions of future random events were implicitly assumed to be independent of past and present random events, as well as of past and present state and control variables.

The main elements of the game are the following:

- N agents, indexed by $i = 1, \ldots, N$.

- The planning horizon T, that can be finite or infinite. Hence, time is represented by $t \in \{0, \dots, T\}$.

- The state variables, $X_t \in \mathbf{X} \subseteq \mathbf{R}^n$, representing the state of the system at every t, which evolves in accordance with the dynamics $X_{t+1} = f(X_t, c_t(X_t, Z_t), Z_{t+1})$.

- The decision rules, $c_{i,t} \in U_i \subseteq \mathbf{R}^{m_i}$, representing the behavior of agent i in period t. Let us denote $U = U_1 \times \cdots \times U_N$. Without loss of generality, and to simplify the notation, we will take $m_1 = \cdots = m_N = 1$. Hence, we will write $c_s = (c_{1,s}, c_{2,s}, \cdots, c_{N,s}) \subseteq \mathbf{R} \times \cdots \times \mathbf{R}$. The extension to the multidimensional case is straightforward.

- The random variables $Z_t \in \mathcal{Z}$, where $\mathcal{Z}$ is a finite set, representing the uncertainties in the evolution of the state variables. We will assume that the probability that $Z_{t+1} = z \in \mathcal{Z}$, may depend on the outcome z_t at time t, on the state and decision variables X_t and c_t as well as explicitly on time t. Thus, we will write $P_{t+1}(Z_{t+1}|\, z_t, x_t, c_t)$.

- The function $u_i(x_t, c_{i,t}, c_{-i,t})$ gives the utility obtained by agent i at time t if his control variable is $c_{i,t}$ and the control variables of the other agents are $c_{-i,t}$.

We assume that functions u_i, $i = 1, \dots, N$, and f are continuous in the state and control variables.

4.2.1 Noncooperative solution in finite horizon

In a noncooperative setting, each t-agent has to maximize, at every time t, his expected intertemporal utility function (i.e. the sum of expected future utilities discounted back to t), provided that the other agents follow the equilibrium decision rule for all $s \geq t$ and that he himself follows the equilibrium decision rule for $s > t$. Hence, for player i, if $c^*_{i,t} = \phi_{i,t}(x_t)$, is the decision rule of the noncooperative solution, each t-agent i will set $c_{i,t}$ so as to maximize

$$J_{i,t}(x_t, z_t, c_{i,t}, \phi_t(X_s, Z_s)) = \tag{4.1}$$

$$u_i(x_t, c_{i,t}, \phi_{-i,t}(x_t, z_t)) + E\left[\sum_{s=t+1}^{T} \theta_{i,s-t} u_i(X_s, \phi_{i,s}, \phi_{-i,s}(X_s, Z_s))|\, x_t, z_t\right]$$

subject to

$$X_{s+1} = f(X_s, c_{i,s}, \phi_{-i,s}(X_s, Z_s), Z_{s+1})\ , \quad X_t = x_t\ , \quad Z_t = z_t\ , \tag{4.2}$$

where $\theta_{i,t}$ is the discount factor of agent i and $s = t+1, \dots, T-1$. If $\theta_{i,t} = \delta_i^t$, for every t, we recover the problem in which each agent has a constant discount rate. Otherwise, we have a problem with nonconstant discount rates. (For the case of one agent, see Björk *et al* (2021) and references therein.)

First, let us assume that the probability that $Z_{t+1} = z \in \mathcal{Z}$, may depend on the outcome z_t at time t, as well as explicitly on time t, but is independent on the state and control variables x_t and c_t. That is, the probability that $Z_t = z$ can be written as $P_t(z|\, z_t)$. We search for a set of decision rules $c^*_{i,t} = \phi_{i,t}(x_t, z_t)$, characterized by the property that no agent wants to deviate from it in any time period. The value function for the t-agent i is given by

$$V_{i,t}(x_t, z_t) \tag{4.3}$$

$$\sup_{\{c_{i,t}\}} E\left[u_i(x_t, c_{i,t}, \phi_{-i,t}(x_t, z_t) + \sum_{s=t+1}^{T} \theta_{i,s-t} u_i(X_s, \phi_{i,s}(X_s, Z_s), \phi_{-i,s}(X_s, Z_s)|\, x_t, z_t)\right] .$$

The computation of the expectation in (4.3) is based on conditional probabilities of the form $p^*(z_{t+1}, z_{t+2}, \ldots, z_T|\, z_t) = P_T(z_T|\, z_{T-1}) \cdot P_{T-1}(z_{T-1}|\, z_{T-2}) \cdots P_{t+1}(z_{t+1}|\, z_t)$, which denotes the probability at time t of the series of events $\{z_{t+1}, z_{t+2}, \ldots, z_T\}$ conditional on z_t.

In the final period T, the value function of agent i is

$$V_{i,T}(x_T) = \sup_{\{c_{i,T}\}} \{u_i(x_T, c_{i,T}, \phi_{-i,T}(x_T)\} \ , \tag{4.4}$$

as usual. The decision rule $\phi_{i,T}(x_T)$ is the maximizer of the right hand term of the above equation. Note that the value function at time T depends on the state x_T, but not on the event z_T. The reason is that since T is the last period, there is no future event Z_{T+1} which may depend on Z_T. Hence, Z_T is only relevant to the decision rule $c^*_{i,T}$ in so far as it contributes to X_T.

At time $T-1$,

$$V_{i,T-1}(x_{T-1}, z_{T-1}) = \sup_{\{c_{i,T-1}\}} E\left[u_i(x_{T-1}, c_{i,T-1}, \phi_{-i,T-1}(x_{T-1}, z_{T-1}))\right.$$

$$\left. +\theta_{i,1} u_i(X_T, \phi_{i,T}(X_T), \phi_{-i,T}(X_T))|\, x_{T-1}, z_{T-1}\right] \ ,$$

where the expectation is calculated over Z_T given z_{T-1}, since $u_i(X_T, \phi_{i,T}(X_T), \phi_{-i,T}(X_T))$ depends on $X_T = f(x_{T-1}, c_{i,T-1}, \phi_{-i,T-1}(x_{T-1}, z_{T-1}), Z_T)$ which depends on Z_T. We can write

$$V_{i,T-1}(x_{T-1}, z_{T-1}) = u_i(x_{T-1}, \phi_{i,T-1}(x_{T-1}, z_{T-1}), \phi_{-i,T-1}(x_{T-1}, z_{T-1}))$$

$$+\theta_{i,1} E\left[u_i(X_T, \phi_{i,T}(X_T), \phi_{-i,T}(X_T))|\, x_{T-1}, z_{T-1}\right]$$

$$= u_i(x_{T-1}, \phi_{i,T-1}(x_{T-1}, z_{T-1}), \phi_{-i,T-1}(x_{T-1}, z_{T-1})) + \theta_{i,1} L_T^{i,T-1} \ ,$$

where we define $L_T^{i,T-1} = E\left[u_i(X_T, \phi_{i,T}(X_T), \phi_{-i,T}(X_T))|\, x_{T-1}, z_{T-1}\right]$. More generally, for $\tau > s$, we may define

$$L^s_{i,\tau} = E[\cdots[E[E[u_i(X_\tau, \phi_{i,\tau}(X_\tau, Z_\tau), \phi_{-i,\tau}(X_\tau, Z_\tau))|\, X_{\tau-1}, Z_{\tau-1}]|\, X_{\tau-2}, Z_{\tau-2}]\cdots]|\, x_s, z_s] \ .$$

That is, $L_{\tau}^{i,s}$ is the undiscounted utility that agent i expects at time s to receive at time τ when the equilibrium decision rule is employed. It follows that the value function of agent i at time t can be written as

$$V_{i,t}(x_t, z_t) = \sup_{\{c_{i,t}\}} \left\{ u_i(x_t, c_{i,t}, \phi_{-i,t}(x_t, z_t)) + \sum_{s=t+1}^{T-1} \theta_{i,s-t} L_{i,s}^t + \theta_{i,T-t} L_{i,T}^t \right\} . \tag{4.5}$$

In a similar way,

$$V_{i,t+1}(x_{t+1}, z_{t+1}) = \sum_{s=t+1}^{T-1} \theta_{i,s-t-1} L_{i,s}^{t+1} + \theta_{i,T-t-1} L_{i,T}^{t+1}$$

and therefore

$$E\left[V_{i,t+1}(X_{t+1}, Z_{t+1}) \middle| \, x_t, z_t\right] = \sum_{s=t+1}^{T-1} \theta_{i,s-t-1} L_{i,s}^{t+1} + \theta_{i,T-t-1} L_{i,T}^{t+1} \ . \tag{4.6}$$

By solving for $L_{i,T}^{t+1}$ in (4.6) and substituting into (4.5), we obtain

$$\theta_{i,T-t-1} V_{i,t}(x_t, z_t) = \sup_{\{c_{i,t}\}} \{ \theta_{i,T-t-1} u_i(x_t, c_{i,t}, \phi_{-i,t}(x_t, z_t)) \tag{4.7}$$

$$\left. + \sum_{s=t+1}^{T-1} (\theta_{i,T-t-1}\theta_{i,s-t} - \theta_{i,T-t}\theta_{i,s-t-1}) L_{i,s}^t + \theta_{i,T-t} E[V_{i,t+1}(X_{t+1,}, Z_{t+1}) | \, x_t, z_t] \right\} .$$

Equations (4.4) and (4.7), for every $i \in \{1, \dots, N\}$, together with (4.2), constitute the dynamic programming equation for the problem (4.1)-(4.2), and the corresponding decision rules are the noncooperative solution to the game.

The previous result can easily be extended to the case when the probabilities $P_{t+1}[Z_{t+1} = z] = P_{t+1}(z \mid x_t, c_t, z_t)$ depend, not only on time t and the previous outcome z_t, but also on the state and control variables x_t and c_t. Given the decision rules $\phi_{i,0}(x_0, z_0), \dots, \phi_{i,T}(x_T, z_T)$, for $i = 1, \dots, N$, the state X_t depends on the outcomes of $Z_0, \dots, Z_t$, i.e., $X_t = X_t(Z_0, \dots, Z_t)$. Hence, we can write $p^*(z_{t+1}, z_{t+2}, \dots, z_T | \, z_t, x_t, \phi_t(x_t, z_t)) = P_T(z_T | \, z_{T-1}, x_{T-1}, \phi_{T-1}(x_{T-1}, z_{T-1})) \cdot P_{T-1}(z_{T-1} | \, z_{T-2}, x_{T-2}, \phi_{T-2}(x_{T-2}, z_{T-2})) \cdot \dots \cdot P_{t+1}(z_{t+1} | \, z_t, x_t, \phi_t(x_t, z_t))$, and the expectation in (4.1) becomes

$$\sum_{z_{t+1}, \dots, z_T} \sum_{s=t}^{T} \theta_{i,s-t} u_i(X_s, c_{i,s}, \phi_{-i,s}(X_s, Z_s)) p^*(z_{t+1}, \dots, z_T | \, z_t, x_t, \phi_t(x_t, z_t)) \ .$$

In other words, for the decision rule $\phi_t(x_t, z_t)$, we sum over all potential outcomes of the random events that may result from that decision rule, and weight the intertemporal utility function of each outcome by its probability. Since the probabilities p^*, and hence the expected values, depend on the decision rules chosen, we will denote the above expectation as $E_{c_{i,t}, \dots, c_{i,T}}$. We define the value function of the t-agent i as

$$V_{i,t}(x_t, z_t) = \sup_{\{c_{i,t}\}} E_{c_{i,t}, c_{i,t+1}^* \dots, c_{i,T}^*} \left[u_i(x_t, c_{i,t}, \phi_{-i,t}(x_t, z_t)) + \right.$$

$$\left. \sum_{s=t+1}^{T} \theta_{i,s-t} u_i(X_s, \phi_{i,s}(X_s, Z_s), \phi_{-i,s}(X_s, Z_s)) | \, x_t, z_t) \right] ,$$

where the supremum is computed with respect to the decision rule $c_{i,t}$, provided that the future s-agents follow the equilibrium decision rule $\phi_s(x_s, z_s)$, for $s = t+1, \ldots, T$. In the final period T, the value function of agent i, $V_{i,T}(x_T, z_T) = V_{i,T}(x_T)$, is given by (4.4). For $s \geq \tau$, we define

$$L^{\tau}_{i,s} \tag{4.8}$$

$$= \sum_{z_{t+1}, z_{t+2}, \ldots, z_T} p^*(z_{t+1}, z_{t+2}, \ldots, z_T | \, z_t, x_t, \phi_t(x_t, z_t)) u_i(X_s, \phi_i(s, X_s, Z_s), \phi_{-i}(s, X_s, Z_s)) \; .$$

This implies that $L^{i,s}_{\tau}$ is the undiscounted utility that agent i expects at time s to receive at time τ, given that he follows the equilibrium decision rule $\phi_{i,s}(x_s, z_s)$ at times $s = t+1, \ldots, T$ and that all other agents follow $\phi_{-i,s}(x_s, z_s)$ at times $s = t, \ldots, T$.

Proposition 4 *For every initial state x_0, the value function V_i, $i = 1, \ldots, N$, of problem* (4.1)-(4.2) *can be obtained as the solution to the dynamic programming algorithm*

$$\theta_{i,T-t-1} V_{i,t}(x_t, z_t) = \sup_{\{c_{i,t}\}} \{ \theta_{i,T-t-1} u_i(x_t, c_{i,t}, \phi_{-i,t}(x_t, z_t)) + \tag{4.9}$$

$$\left. \sum_{s=t+1}^{T-1} (\theta_{i,T-t-1}\theta_{i,s-t} - \theta_{i,T-t}\theta_{i,s-t-1}) L^t_{i,s} + \theta_{i,T-t} E_{c_{i,t}} \left[V_{i,t+1}(X_{t+1,}, Z_{t+1}) | \, x_t, z_t \right] \right\} ,$$

for $t = 0, \ldots, T-1$, and $V_{i,T}(x_T) = \sup_{\{c_{i,T}\}} \{u_i(x_T, c_{i,T}, \phi_{-i,T}(x_T)\}$. For every $i = 1, \ldots, N$, equations (4.9) *and* (4.2) *are the dynamic programming equations, and the corresponding decision rules $c^*_{i,s} = \phi_{i,s}(X_s, Z_s)$ are the noncooperative solution solving the problem* (4.1)-(4.2).

Proof: Let $c^*_{i,T} = \phi_{i,T}(x_T)$ be the solution to $V_{i,T}(x_T) = \sup_{\{c_{i,T}\}} \{u_i(x_T, c_{i,T}, \phi_{-i,T}(x_T)\}$. Then $V_{i,T}(x_T) = u_i(x_T, \phi_{i,T}(x_T), \phi_{-i,T}(x_T))$. For $t = T-1$,

$$V_{i,T-1}(x_{T-1}, z_{T-1}) = \sup_{\{c_{i,T-1}\}} E_{c_{i,T-1}} \left[u_i(x_{T-1}, c_{i,T-1}, \phi_{-i,T-1}(x_{T-1}, z_{T-1}) \right.$$

$$\left. + \theta_{i,1} u_{i,T}(X_T, \phi_{i,T}(X_T), \phi_{-i,T}(X_T)) | \, x_{T-1}, z_{T-1} \right] = \sup_{\{c_{i,T-1}\}} \{ u_i(x_{T-1}, c_{i,T-1}, \phi_{-i,T-1}(x_{T-1}, z_{T-1})$$

$$+ E_{c_{i,T-1}} \left[\theta_{i,1} u_i(X_T, \phi_{i,T}(X_T), \phi_{-i,T}(X_T)) | \, x_{T-1}, z_{T-1} \right] \} \; .$$

Let $\phi_{i,T-1}(x_{T-1}, z_{T-1})$ be the optimum. Since

$$E_{c_{i,T-1}} \left[u_i(X_T, \phi_{i,T}(X_T), \phi_{-i,T}(X_T)) | \, x_{T-1}, z_{T-1} \right]$$

$$= \sum_{z_T} P_T(\, z | \, x_{T-1}, c_{i,T-1}, z_{T-1}) u_i(X_T, \phi_{i,T}(X_T), \phi_{-i,T}(X_T)) \; ,$$

then, using (4.8),

$$V_{i,T-1}(x_{T-1}, z_{T-1}) = u_i(X_{T-1}, \phi_{i,T-1}(x_{T-1}, z_{T-1}) \phi_{-i,T-1}(x_{T-1}, z_{T-1}))$$

$$+\theta_{i,1}\sum_{z_T} P_{T-1}(\, z|\, x_{T-1}, c_{i,T-1}, z_{T-1})u_i(X_T, \phi_{i,T}(X_T), \phi_{-i,T}(X_T))$$

$$= u_i(X_{T-1}, \phi_{i,T-1}(x_{T-1}, z_{T-1})\phi_{-i,T-1}(x_{T-1}, z_{T-1})) + \theta_{i,1}L^{T-1}_{i,T}\ .$$

In general,

$$V_{i,t+1}(x_{t+1}, z_{t+1}) = \sum_{s=t+1}^{T} \theta_{i,s-t-1}L^{t+1}_{i,s} \tag{4.10}$$

and

$$V_{i,t}(x_t, z_t) = \sup_{\{c_{i,t}\}} \left\{ u_i(x_t, c_{i,t}, \phi_{-i,t}(x_t, z_t)) + \sum_{s=t+1}^{T} \theta_{i,s-t}E_{c_{i,t}}\left[L^{t+1}_{i,s}|\, x_t, z_t \right] \right\}\ . \tag{4.11}$$

By taking the expectation of $V_{i,t+1}(X_{t+1}, Z_{t+1})$ conditioned to x_t, z_t in (4.10), solving $L^t_{i,T}$, substituting in (4.11), and using (4.2), we obtain (4.9). □

It may be useful to provide some intuition for the dynamic programming equation presented in Proposition 4. In comparison with a standard dynamic programming equation (with a constant discount rate), the main new feature is the sum of the L-terms, each of which contains the expected utility of some future period when the equilibrium decision rule is employed. These terms correct for the fact that the discount rates may be nonconstant, and ensures that the solution derived is an intertemporal equilibrium. If they are removed, the decision rule for time t will be based on the assumption that the discount rate is constant and equal to $\theta_{i,T-1}/\theta_{i,T-t-1}$ for all future periods. Indeed, if the discount rates are in fact constant, the L-terms vanish, and the dynamic programming equation becomes more similar to that of the standard case.

4.2.2 Noncooperative solution in infinite horizon

In the case when the planning horizon is infinite, let us restrict our attention to stationary problems, in which the probabilities $P_{t+1}(Z_{t+1} \mid x_t, c_t, z_t)$ are independent of t. This implies that the intertemporal utility functions are also independent of t. The problem for the t-agent i is to set $c_{i,t}$ so as to maximize

$$J_{i,t}(x_t, z_t, c_{i,t}, \phi_s(x_s, z_s)) = \tag{4.12}$$

$$E\left[u_i(x_t, c_{i,t}, \phi_{-i,t}(x_t, z_t)) + \sum_{s=t+1}^{\infty} \theta_{i,s-t}u_i(X_s, c_{i,s}, \phi_{-i,s}(X_s, Z_s))\, |\, x_t\ ,\ z_t \right]$$

subject to (4.2) and with the future decision rule of agent i and the decision rules of the remaining agents given.

It is well-known that, in the case of standard exponential discounting, $\theta_{i,\tau} = \delta_i^\tau$, equilibrium decision rules are stationary, in the sense that they do not depend explicitly on time. Hence, we can write the value functions and the decision rules as $V_i(x, z)$ and $\phi_i(x, z)$, as they are time-independent for every $i = 1, \dots, N$. In Carmon and Schwartz (2009) this result was extended, in problems with one agent, to the class of exponentially representable discount functions $\theta_\tau = \sum_{k=1}^{\infty} c_k \beta_k^\tau$, where the sum converges absolutely, and the sequence $\{\beta_k\}_{k=1}^{\infty}$ is positive, strictly decreasing and $\beta_1 < 1$. For this class of discount functions, it was proved that there exist equilibrium decision rules that are stationary from some time t onward (t-stationary decision rules), and an algorithm for their computation was provided. However, outside this class, optimal t-stationary policies do not exist, in general. In order to avoid this problem, we will assume that there exists a finite time $\bar{T}$ such that, for every $t \geq \bar{T}$, $\theta_{i,\tau} = \theta_{i,\bar{T}} \delta_i^{\tau-\bar{T}}$, for $\delta < 1$. Roughly speaking, this implies that the discount function becomes exponential after a certain amount of time has passed. This condition is also important in the construction of the dynamic programming algorithm (see Karp (2007) and Fujii and Karp (2008)). Using that $\theta_{i,\tau}/\theta_{i,\tau-1} = \delta_i$ for $\tau > \bar{T}$, from Proposition 4 (see also the proof of Proposition 1 in Fujii and Karp (2008) in a deterministic setting), we can write the dynamic programming equation 4.9 as

$$V_i(x,z) = \sup_{\{c_i\}} \{u_i(x, c_i, \phi_{-i}(x,z)) +$$

$$\left. \sum_{s=1}^{\infty} \left(\theta_{i,s} - \frac{\theta_{i,T}}{\theta_{i,T-1}} \theta_{i,s-1} \right) L_{i,s}^0 + \delta_i E_{c_i} \left[V_i(f(x, c_i, \phi_{-i}(x,z), Z), Z) | \, x, z \right] \right\} .$$

Next, observe that

$$\sum_{s=1}^{\infty} \left(\theta_{i,s} - \frac{\theta_{i,T}}{\theta_{i,T-1}} \theta_{i,s-1} \right) L_{i,s}^0 = \sum_{s=1}^{\bar{T}} \left(\theta_{i,s} - \delta \theta_{i,s-1} \right) L_{i,s}^0 .$$

Hence, the dynamic programming equation becomes

$$V_i(x,z) = \sup_{\{c_i\}} \{u_i(x, c_i, \phi_{-i}(x,z)) + \tag{4.13}$$

$$\left. \sum_{s=1}^{\bar{T}} \left(\theta_{i,s} - \delta_i \theta_{i,s-1} \right) L_{i,s}^0(x_s, z_s) + \delta_i E_{c_i} \left[V(f(x, c_i, \phi_{-i}(x,z), Z), Z) | \, x, z \right] \right\} .$$

Note also that our condition on the discount function simplifies the discussion concerning the existence of a well-defined bounded value function (4.12): since the discount rate becomes constant after a finite time $\bar{T}$, the well-known conditions for a problem with a constant discount rate of time preference apply. Equation (4.13) generalizes the dynamic programming equation derived, for the case of one agent in a deterministic setting, in Fujii and Karp (2008).

4.2.3 Cooperative solution in finite horizon

Under cooperation, intertemporal equilibrium decision rules can be obtained by combining the previous derivations and results. For simplicity, we constrain ourselves to the case in which probabilities are independent of the state and control variables. The extension to the more general case proceeds in a similar way to that in the previous sections.

The problem to be solved is the following. The intertemporal utility function of the coalition is

$$J_t(x_t, z_t, c_s) = \sum_{i=1}^{N} \lambda_i J_{i,t}(x_t, z_t, c_s) = \sum_{i=1}^{N} E\left[\sum_{s=t}^{T} \theta_{i,s-t} u_i(X_s, c_s) \middle|\, x_t, z_t)\right]$$

and the dynamics are

$$X_{s+1} = f(X_s, c_s, Z_{s+1}) \ , \quad X_t = x_t \ , \quad Z_t = z_t \ .$$

We search for equilibrium decision rules, $c_t^* = (c_{1,t}^*, \ldots, c_{N,t}^*) = (\phi_{1,t}(x_t, z_t), \ldots, \phi_N(x_t, z_t))$, characterized by the property that at no time period does the coalition have incentive to deviate from it. The value function of the coalition at time t is given by

$$V_t(x_t, z_t) = \sum_{i=1}^{N} \lambda_i J_{i,t}(x_t, z_t, \phi_s(x_s, z_s)) = \sum_{i=1}^{N} \lambda_i E\left[\sum_{s=t}^{T} \theta_{i,s-t} u_i(X_s, \phi_s(X_s, Z_s)) \,| x_t, z_t\right] \ ,$$

with $X_{s+1} = f(X_s, \phi_s(X_s, Z_s), Z_{s+1})$, for $s = t, \ldots, T-1$ and $X_t = x_t$, $Z_t = z_t$ given. Alternatively, we can write

$$V_t(x_t, z_t) = \sum_{i=1}^{N} \lambda_i V_{i,t}(x_t, z_t) \ ,$$

where

$$V_{i,t}(x_t, z_t) = J_{i,t}(x_t, z_t, \phi_s(x_s, z_s)) = E\left[\sum_{s=t}^{T} \theta_{i,s-t} u_i(X_s, \phi_s(X_s, Z_s)) \,| x_t, z_t\right] \ ,$$

We proceed backward in time. In the final period, for $t = T$,

$$V_T(x_T, z_T) = \sup_{\{c_T\}} \sum_{i=1}^{N} \lambda_i u_i(x_T, c_T) \ .$$

Let $c_T = \phi_T(x_T)$ be the solution to the above problem.

For $t = T-1$,

$$V_{T-1}(x_{T-1}, z_{T-1}) = \sup_{\{c_{T-1}\}} \left\{ \sum_{i=1}^{N} \lambda_i E\left[u_i(x_{T-1}, c_{T-1}) + \theta_{i,1} u_i(X_T, \phi_T(X_T) \,| x_{T-1}, z_{T-1}\right] \right\} \ ,$$

with $X_T = f(x_{T-1}, c_{T-1}, Z_T)$, for x_{T-1}, z_{T-1} given. Let $c_{T-1} = \phi_{T-1}(x_{T-1}, z_{T-1})$ be the solution to this maximization problem. Then,

$$V_{T-1}(x_{T-1}, z_{T-1}) = \sum_{i=1}^{N} \lambda_i V_{i,T-1}(x_{T-1}, z_{T-1}) \ ,$$

where

$$V_{i,T-1}(x_{T-1},z_{T-1}) = [u_i(x_{T-1},\phi_{T-1}(x_{T-1},z_{T-1})) + \theta_{i,1}E\left[u_i(X_T,\phi_T(x_T))\,|\,x_{T-1},z_{T-1}\right]\ .$$

As in the noncooperative solution, we define $L_{i,T}^{T-1} = E\left[u_i(X_T,\phi_T(X_T))\,|\,x_{T-1},z_{T-1}\right]$, so that

$$V_{i,T-1}(x_{T-1},z_{T-1}) = u_i(x_{T-1},\phi_{T-1}(x_{T-1},z_{T-1})) + \theta_{i,1}L_{i,T}^{T-1}\ .$$

Similarly, if we define

$$L_{i,\tau}^{s} = E[\cdots[E[E[u_i(X_\tau,\phi_\tau(X_\tau,Z_\tau))\,|\,X_{\tau-1},Z_{\tau-1}]\,|\,X_{\tau-2},Z_{\tau-2}]\cdots]\,|\,x_s,z_s]$$

for $\tau = s+1,\dots,T$, then

$$V_t(x_t,z_t) = \sup_{\{c_t\}}\left\{\sum_{i=1}^{N}\left(\lambda_i u_i(x_t,c_t) + \sum_{s=t+1}^{T-1}\theta_{i,s-t}\lambda_i L_{i,s}^t + \theta_{i,T-t}\lambda_i L_{i,T}^t\right)\right\}\ . \tag{4.14}$$

Next, note that

$$E\left[V_{i,t+1}(X_{t+1},Z_{t+1})\,|\,x_t,z_t)\right] = E\left[\sum_{s=t+1}^{T}\theta_{i,s-t-1}L_{i,s}^{t+1}\,|\,x_t,z_t)\right]$$

$$= \sum_{s=t+1}^{T}\theta_{i,s-t-1}L_{i,s}^t = \sum_{s=t+1}^{T-1}\theta_{i,s-t-1}L_{i,s}^t + \theta_{i,T-t-1}L_{i,T}^t\ .$$

Therefore,

$$L_{i,T}^t = \frac{1}{\theta_{i,T-t-1}}\left(E\left[V_{i,t+1}(X_{t+1},Z_{t+1})\,|\,x_t,z_t)\right] - \sum_{s=t+1}^{T-1}\theta_{i,s-t-1}L_{i,s}^t\right)\ ,$$

and, by substituting in (4.14), we obtain

$$V_t(x_t,z_t) = \sup_{\{c_t\}}\left\{\sum_{i=1}^{N}\lambda_i\left(u_i(x_t,c_t) + \frac{\theta_{i,T-t}}{\theta_{i,T-t-1}}E\left[V_{i,t+1}(X_{t+1},Z_{t+1})\,|\,x_t,z_t)\right]\right.\right.$$
$$\left.\left. + \sum_{s=t+1}^{T-1}\left(\theta_{i,s-t} - \frac{\theta_{i,T-t}}{\theta_{i,T-t-1}}\theta_{i,s-t-1}\right)L_{i,s}^t\right)\right\}\ .$$

Therefore, cooperative decision rules if agents discount the future at different and nonconstant discount rates satisfy the dynamic programming equation

$$\phi_t(x_t,z_t) = \arg\sup_{\{c_t\}}\left\{\sum_{i=1}^{N}\lambda_i\left(u_i(x_t,c_t) + \right.\right. \tag{4.15}$$
$$\left.\left.\sum_{s=t+1}^{T-1}\left(\theta_{i,s-t} - \frac{\theta_{i,T-t}}{\theta_{i,T-t-1}}\theta_{i,s-t-1}\right)L_{i,s}^t + \frac{\theta_{i,T-t}}{\theta_{i,T-t-1}}E\left[V_{i,t+1}(f(x_t,c_t,Z_{t+1}),Z_{t+1})\,|\,x_t,z_t)\right]\right)\right\}$$

for $t = 0,\dots,T-1$, where

$$L_{i,s}^t = E[\cdots[E[E[u_i(X_s,\phi_s(X_s,Z_s))\,|\,X_{s-1},Z_{s-1}]\,|\,X_{s-2},Z_{s-2}]\cdots]\,|\,x_t,z_t]\ , \tag{4.16}$$

and

$$V_T(x_T,z_T) = \sup_{\{c_T\}}\sum_{i=1}^{N}\lambda_i u_i(x_T,c_T)\ . \tag{4.17}$$

4.2.4 Cooperative solution in infinite horizon

If the planning horizon is infinite, we consider the case in which the probabilities $P_{t+1}(z_{t+1}|\, x_t, c_t, z_t)$ are independent of t. In addition, as in the case of the noncooperative solution, we assume that there exists a finite time $\bar{T}$ such that, for every $t \geq \bar{T}$, $\theta_{i,\tau} = \theta_{i,\bar{T}}\delta_i^{\tau-\bar{T}}$, for $\delta < 1$. Therefore, $\dfrac{\theta_{i,\tau}}{\theta_{i,\tau-1}} = \delta_i$, for $\tau > \bar{T}$. In addition, note that

$$\sum_{s=1}^{\infty}\left(\theta_{i,s} - \frac{\theta_{i,T}}{\theta_{i,T-1}}\theta_{i,s-1}\right)L_{i,s}^0 = \sum_{s=1}^{\bar{T}}\left(\theta_{i,s} - \delta\theta_{i,s-1}\right)L_{i,s}^0 \ .$$

Hence, in the search of stationary decision rules in an autonomous problem in infinite horizon, equation (4.15) becomes

$$\phi(x,z) = \arg\sup_{\{c\}}\left\{\sum_{i=1}^{N}\lambda_i\left(u_i(x,c)+\right.\right. \tag{4.18}$$

$$\left.\left.\sum_{s=1}^{\bar{T}}\left(\theta_{i,s} - \delta\theta_{i,s-1}\right)L_{i,s}^0 + \delta_i E\left[V_i(f(x,c,Z),Z)\,|\,x\,,\,z\right]\right)\right\} \ .$$

4.2.5 Quasi-hyperbolic asymmetric discounting

In the models in Section 3 and Section 4 of this chapter, we shall apply the theory that we have developed here to problems in which the planning horizon is infinite and the discount functions are quasi-hyperbolic. This is a special case of the more general model for which we have already derived dynamic programming equations. Starting from the more general discount function that we studied above, a quasi-hyperbolic discount function is obtained by setting $\bar{T} = 1$. In this subsection, we will look at how this simplifies the theory, and describe in detail the algorithms which we will use in the applications.

First, we look at the noncooperative solution. With the planning horizon being infinite and the problem stationary, the decision rules solve

$$V_i(x,z) = \sup_{\{c_i\}}\left\{u_i(x,c_i,\phi_{-i}(x,z)) + \delta_i(\beta_i - 1)L_{i,1}^0(x,z)\right. \tag{4.19}$$

$$\left.+\delta_i E_{c_i}\left[V_i(f(x,c_i,\phi_{-i}(x,z),Z),Z)|\, x,z\right]\right\} \ ,$$

i.e.,

$$V_i(x,z) = \sup_{\{c_i\}}\{u_i(x,c_i,\phi_{-i}(x,z)) + E_{c_i}\left[\delta_i(\beta_i - 1)u_i(f(x,\phi(x,z),Z),\phi(f(x,\phi(x,z),Z))\right.$$

$$\left.+\delta_i V_i(f(x,c_i,\phi_{-i}(x,z),Z),Z)|\, x,z\right]\} \ . \tag{4.20}$$

The main thing to notice here is that in (4.19) the sum of the L-terms have been simplified considerably compared to the case of a general discount function that we derived previously.

Instead of a sum of terms, we have only the one term $\delta_i(\beta_i - 1)L^0_{i,1}$. The reason is that after one time period has passed, the discount rate is constant at $\delta_i - 1$. Hence, there is no need for additional L-terms to correct for a nonconstant discount rate.

It is easy to check that the dynamic programming equation (4.20) can be rewritten as

$$V_i(x,z) = \sup_{\{c_i\}} \left\{ u_i(x, c_i, \phi_{-i}(x,z)) + \beta_i\delta_i E_{c_i}\left[\bar{V}_i(f(x, c_i, \phi_{-i}(x,z), Z), Z)|\, x, z\right] \right\} , \qquad (4.21)$$

where $\bar{V}_i$, for $i = 1, \ldots, N$, are such that

$$\bar{V}_i(x,z) = u_i(x, \phi(x,z)) + \delta_i E\left[\bar{V}_i(f(x, \phi(x,z), Z), Z)|\, x, z\right] . \qquad (4.22)$$

The functions $\bar{V}_i$ will be important. We will refer to them as the auxiliary functions of the model. In order to find the infinite horizon noncooperative decision rules $\phi_i(x,z)$ $(i = 1, \ldots, N)$, we may proceed with the following algorithm.

1. Use (4.21) to find the decision rules in terms of the auxiliary functions.
2. Given the results in step 1, use (4.22) to solve for the auxiliary functions.
3. Having found the auxiliary functions, use the results from step 1 to find the decision rules.
4. Having found the decision rules and auxiliary functions, use (4.21) to find the value functions.

Next, we look at the cooperative solution. For the case of an infinite planning horizon and a stationary problem, the decision rules satisfy the dynamic programming equation

$$\sum_{i=1}^{N} \lambda_i V_i(x,z) = \qquad (4.23)$$

$$\sup_{\{c\}} \left\{ \sum_{i=1}^{N} \lambda_i \left(u_i(x,c) + \delta_i(\beta_i - 1)L^0_{i,1} + \delta_i E\left[V_i(f(x,c,Z),Z)\,|\, x, z)\right]\right) \right\} ,$$

where

$$V_i(x,z) = u_i(x, \phi(x,z)) + \beta_i\delta_i E\left[\bar{V}_i(f(x, \phi(x,z), Z), Z)|\, x, z\right] \qquad (4.24)$$

and

$$\bar{V}_i(x,z) = u_i(x, \phi(x,z)) + \delta_i E\left[\bar{V}_i(f(x, \phi(x,z), Z), Z)|\, x, z\right] . \qquad (4.25)$$

When solving for infinite horizon cooperative equilibrium decision rules $\phi_{i,t}(x_t, z_t)$ $(i = 1, \ldots, N)$, we may proceed with the following algorithm.

1. Use the dynamic programming equation (4.23) to derive the decision rules as functions of the value functions.

2. Given those decision rules, use (4.24) to solve for the auxiliary functions in terms of the value functions.

3. Given the results derived in the previous two steps, use (4.25) to solve for the auxiliary functions.

4. Having found the auxiliary functions, use the results from step 2 to solve for the value functions.

5. Having found the value functions, use the results from step 1 to solve for the equilibrium decision rules.

4.3 The Standard Fish War Model

In this section we will apply the theory developed in the previous section to solve for noncooperative and cooperative solutions in an infinite horizon fish war model with two agents and deterministic dynamics. We will assume that the utility function for agent i is

$$u_i(c_{i,t}) = \ln c_{i,t} \ ,$$

where $c_{i,t}$ is the harvest of agent i in period t. The growth function of the stock of fish is

$$x_{t+1} = b(x_t - c_{1,t} - c_{2,t})^{\alpha} \ ,$$

where x_t is the stock at the beginning of time period t. The logic of the growth function is that we first substract the period's total harvest from the stock of fish, and then raise what remains of the fish stock to the power of α and multiply by b. We impose the parameter constraints $\alpha \in (0,1)$ and $b > 0$. This ensures that, in the absence of any harvest, the stock converges to $b^{\frac{1}{1-\alpha}}$, which is called the saturation level. In addition, we require all variables to be nonnegative and the initial condition $x_0 > 0$.

As we mentioned in the introduction, this model has been studied in several papers, notably by Breton and Keoula (2014), where the model was solved under the assumption of asymmetric and constant discount rates. In the present study, we maintain the assumption of asymmetric time preferences, but also allow both agents to have quasi-hyperbolic time preferences. Hence, the discount function for agent i is $\theta_{i,t} = \beta_i \delta_i^t$, for $t \geq 1$, and $\theta_{i,t} = 1$ for $t = 0$, where all parameters are in the unit interval. If $\beta_1 = \beta_2 = 1$, our model simplifies to the one with asymmetric but constant discount rates that was solved in Breton and Keoula (2014). If in addition $\delta_1 = \delta_2$, the model simplifies to the standard one in which the agents have the same constant discount rate. If we impose that $\delta_1 = \delta_2$ and that one of the agents have a constant discount rate (i.e. that $\beta_1 = 1$ or $\beta_2 = 1$), we obtain a model studied in Turan (2019). Our model can be viewed as a generalization of all of the above.

4.3.1 The Noncooperative Solution

To find the noncooperative solution, we will make use of the dynamic programming equations (4.21)-(4.22), which for this model become[2]

$$V_i(x) = \sup_{\{c_i\}} \left\{ \ln c_i + \beta_i \delta_i \bar{V}_i(b(x - c_i - \phi_{-i}(x))^\alpha) \right\}$$

and

$$\bar{V}_i(x) = \ln \phi_i(x) + \delta_i \bar{V}_i(b(x - \phi_1(x) - \phi_2(x))^\alpha) \ ,$$

where $i \in \{1, 2\}$. We are interested in finding the decision rules $\phi_i(x)$ and the value functions $V_i(x)$. Following standard procedures, we will assume that $V_i(x) = A_i \ln x + B_i$ and $\bar{V}_i(x) = \bar{A}_i \ln x + \bar{B}_i$. The dynamic programming equations then become

$$A_i \ln x + B_i = \sup_{\{c_i\}} \left\{ \ln c_i + \beta_i \delta_i \bar{A}_i \ln b + \beta_i \delta_i \bar{A}_i \alpha \ln(x - c_i - \phi_{-i}(x)) + \beta_i \delta_i \bar{B}_i \right\} \quad (4.26)$$

and

$$\bar{A}_i \ln x + \bar{B}_i = \ln \phi_i(x) + \delta_i \bar{A}_i \ln b + \delta_i \bar{A}_i \alpha \ln(x - \phi_1(x) - \phi_2(x)) + \delta_i \bar{B}_i \ . \quad (4.27)$$

The maximization problem implicit in (4.26) yields the first order conditions

$$\frac{1}{c_i} = \frac{\beta_i \delta_i \alpha \bar{A}_i}{x - c_i - \phi_{-i}(x)} \ . \quad (4.28)$$

Obviously $c_i = \phi_i(x)$. Therefore, we can combine the first order conditions to obtain the linear decision rules

$$\phi_i(x) = \frac{\beta_i \delta_i \alpha \bar{A}_i}{(1 + \beta_1 \delta_1 \alpha \bar{A}_1)(1 + \beta_2 \delta_2 \alpha \bar{A}_2) - 1} x \, . \quad (4.29)$$

Next, we want to find the parameters of the auxiliary functions $\bar{V}_i(x)$. To this end, we substitute the decision rules into (4.27) to obtain

$$\bar{A}_i \ln x + \bar{B}_i = \ln \left(\frac{\beta_i \delta_i \alpha \bar{A}_i}{(1 + \beta_1 \delta_1 \alpha \bar{A}_1)(1 + \beta_2 \delta_2 \alpha \bar{A}_2) - 1} x \right) + \delta_i \bar{A}_i \ln b + \delta_i \bar{B}_i \quad (4.30)$$

$$+ \delta_i \bar{A}_i \alpha \ln \left(x - \frac{\beta_1 \delta_1 \alpha \bar{A}_1}{(1 + \beta_1 \delta_1 \alpha \bar{A}_1)(1 + \beta_2 \delta_2 \alpha \bar{A}_2) - 1} x - \frac{\beta_2 \delta_2 \alpha \bar{A}_2}{(1 + \beta_1 \delta_1 \alpha \bar{A}_1)(1 + \beta_2 \delta_2 \alpha \bar{A}_2) - 1} x \right) \ .$$

By gathering terms that are multiples of $\ln x$, it is easy to show that

$$\bar{A}_i = \frac{1}{1 - \delta_i \alpha} \ .$$

Then, the noncooperative decision rules become

$$\phi_i(x) = \frac{\frac{\beta_i \delta_i \alpha}{1 - \delta_i \alpha}}{\frac{1 - \delta_1 \alpha (1 - \beta_1)}{1 - \delta_1 \alpha} \frac{1 - \delta_2 \alpha (1 - \beta_2)}{1 - \delta_2 \alpha} - 1} x \ .$$

[2] Note that since our growth function is assumed to be deterministic, we can neglect to write out the random variables and the expectation operator that are present in (4.21)-(4.22).

Moreover, by gathering terms that are *not* multiples of $\ln x$, (i.e. the constant terms in ((4.30))), we obtain

$$(1-\delta_i)\bar{B}_i = \ln\left(\frac{\beta_i\delta_i\alpha\bar{A}_i}{(1+\beta_1\delta_1\alpha\bar{A}_1)(1+\beta_2\delta_2\alpha\bar{A}_2)-1}\right) + \delta_i\bar{A}_i\ln b \tag{4.31}$$

$$+\delta_i\bar{A}_i\alpha\ln\left(1-\frac{\beta_1\delta_1\alpha\bar{A}_1}{(1+\beta_1\delta_1\alpha\bar{A}_1)(1+\beta_2\delta_2\alpha\bar{A}_2)-1}-\frac{\beta_1\delta_1\alpha\bar{A}_1}{(1+\beta_1\delta_1\alpha\bar{A}_1)(1+\beta_2\delta_2\alpha\bar{A}_2)-1}\right).$$

which allows us to solve for $\bar{B}_1$ and $\bar{B}_2$ in terms of $\bar{A}_1$ and $\bar{A}_2$. Hence, we have found the auxiliary functions. It remains to find expressions for the value functions $V_i(x)$. This is necessary for investigating coalitional stability. The dynamic programming equation (4.26) can be written as

$$A_i\ln x + B_i = \ln\phi_i(x) + \beta_i\delta_i\bar{A}_i\ln b + \beta_i\delta_i\bar{A}_i\alpha\ln(x-\phi_1(x)-\phi_2(x)) + \beta_i\delta_i\bar{B}_i\ ,$$

which gives $V_i(x)$ as a function of $\bar{A}_i$ and $\bar{B}_i$. By gathering terms that are multiples of $\ln x$, we obtain

$$A_i = 1+\beta_i\delta_i\alpha\bar{A}_i = 1+\frac{\beta_i\delta_i\alpha}{1-\delta_i\alpha}\ .$$

Finally, by gathering terms that are *not* multiples of $\ln x$, we obtain a long, and not very informative, expression for B_i, which we will not provide here.

It is noteworthy that in order to find the decision rules, it is only necessary to solve for $\bar{A}_i$, which, as we have seen, is relatively straightforward. The more complicated part of solving for $\bar{B}_i$ and then B_i is only necessary for finding the value functions in order to make welfare comparisons.

It is also noteworthy that a complete solution can be obtained analytically. In no step of the process are we forced to resort to numeric solutions. However, the expressions that we obtain are quite complicated, and it is therefore difficult to see how specific parameters affect the solution without numeric tests.

4.3.2 The Cooperative Solution

To find the cooperative solution, we will make use of the dynamic programming equations (4.23)-(4.24). For this model, the dynamic programming equations become

$$\sum_{i=1}^{2}\lambda_i V_i(x)$$

$$=\sup_{\{c_i\}}\left\{\sum_{i=1}^{2}\lambda_i\Big(\ln c_i + \delta_i(\beta_i-1)\ln(\phi_i(b(x-c_i-\phi_{-i}(x))^\alpha)) + \delta_i V_i(b(x-c_i-\phi_{-i}(x))^\alpha)\Big)\right\}\ ,$$

and

$$V_i(x) = \ln(\phi(x)) + \beta_i\delta_i\bar{V}_i(b(x-c_i-\phi_{-i}(x))^\alpha)\ ,$$

where $i \in \{1,2\}$. Again, we are interested in finding the decision rules $\phi_i(x)$ and the value functions $V_i(x)$. As in the case of noncooperation, we may assume that $V_i(x) = A_i \ln x + B_i$ and $\bar{V}_i(x) = \bar{A}_i \ln x + \bar{B}_i$. We also guess in advance that $\phi_i(x) = a_i x$, i.e. that harvest is linear in the fish stock.[3] The above equations then become

$$\sum_{i=1}^{N} \lambda_i (A_i \ln x + B_i) \tag{4.32}$$

$$= \sup_{\{c_i\}} \left\{ \sum_{i=1}^{N} \lambda_i \Big(\ln c_i + \delta_i(\beta_i - 1)\ln b a_i + \delta_i A_i \alpha \ln b + (\beta_i - 1 + A_i)\delta_i \alpha \ln(x - c_i - a_{-i}x) + \delta_i B_i \Big) \right\}$$

and

$$A_i \ln x + B_i = \ln a_i x + \beta_i \delta_i \bar{A}_i \ln b + \beta_i \delta_i \bar{A}_i \alpha \ln(x - a_1 x - a_2 x) + \beta_i \delta_i \bar{B}_i \ . \tag{4.33}$$

The maximization problem implicit in (4.32) yields the first order conditions

$$\frac{\lambda_i}{c_i} = \frac{\lambda_1 \delta_1 \alpha(\beta_1 - 1 + A_1) + \lambda_2 \delta_2 \alpha(\beta_2 - 1 + A_2)}{x - c_i - a_{-i}x} \ .$$

Obviously $c_i = a_i x$. Therefore, we can combine the first order conditions to obtain

$$a_i = \frac{\lambda_i(\lambda_{-i} + D) - \lambda_{-i}}{(\lambda_1 + D)(\lambda_2 + D) - 1} \ ,$$

which implies the linear decision rules

$$\phi_i(x) = \frac{\lambda_i(\lambda_{-i} + D) - \lambda_{-i}}{(\lambda_1 + D)(\lambda_2 + D) - 1} x \ ,$$

where, for notational simplicity, we have introduced the parameter $D = \lambda_1 \delta_1 \alpha(\beta_1 - 1 + A_1) + \lambda_2 \delta_2 \alpha(\beta_2 - 1 + A_2)$. Next, we want to find the parameters of the value functions $V_i(x)$. To this end, we substitute the decision rules into (4.33) to obtain expressions that relate the value functions to the auxiliary functions. By gathering terms that are multiples of $\ln x$, we obtain

$$A_i = 1 + \beta_i \delta_i \alpha \bar{A}_i \ ,$$

which gives us A_i in terms of $\bar{A}_i$. Moreover, by gathering terms that are *not* multiples of $\ln x$, (i.e. the constant terms in (4.33)), we obtain

$$B_i = \ln a_i + \beta_i \delta_i (\bar{B}_i + \bar{A}_i \ln b + \bar{A}_i \alpha \ln(1 - a_1 - a_2)) \ , \tag{4.34}$$

which, given that a_i is as derived above, allows us to solve for B_i in terms of $\bar{A}_1$ and $\bar{A}_2$. Hence, to find the value functions, it suffices to find $\bar{A}_i$ and $\bar{B}_i$ for both i. To this end, we make use of equation (4.25), which for this model is

$$\bar{A}_i \ln x + \bar{B}_i = \ln a_i x + \delta_i (\bar{A}_i \ln b + \bar{A}_i \alpha \ln(x - a_1 x - a_2 x) + \bar{B}_i) \ .$$

[3]The reason for guessing linear decision rules in advance is that it allows for separating terms in the dynamic programming equations, in such a way that the procedure of solving the model is greatly simplified.

By gathering terms that are multiples of $\ln x$, it is easy to show that

$$\bar{A}_i = \frac{1}{1-\delta_i\alpha} \ ,$$

which implies that

$$A_i = 1 + \frac{\beta_i\delta_i\alpha}{1-\delta_i\alpha} \ ,$$

and hence

$$D = \lambda_1\delta_1\alpha(\beta_1 + \beta_1\delta_1\alpha(1-\delta_1\alpha)^{-1}) + \lambda_2\delta_2\alpha(\beta_2 + \beta_2\delta_2\alpha(1-\delta_2\alpha)^{-1}) \ .$$

Thus, having found D, we have also found a complete analytic expression for the decision rules. Moreover, by gathering terms that are *not* multiples of $\ln x$, we obtain

$$\bar{B}_i = \ln a_i + \delta_i(\bar{A}_i \ln b + \bar{A}_i\alpha\ln(1-a_1-a_2) + \bar{B}_i) \ , \tag{4.35}$$

which gives us $\bar{B}_i$ in terms of $\bar{A}_i$. To obtain expressions for the value functions, we can use equation (4.35) to obtain $\bar{B}_i$ and then equation (4.34) to obtain B_i. Together with A_i, for which we have a simple analytic expression, this is sufficient for deriving a complete analytic expression for the value functions. However, as in the case of the noncooperative solution, the expressions are too complicated to be very informative.

4.3.3 Numeric Solutions

The central feature of our model is the inclusion of quasi-hyperbolic time preferences, which are captured by the parameters β_1 and β_2. Hence, we are interested in the effect of those variables on the behavior of the agents.

We will analyze two aspects of the model. The first is the steady state level of the fish stock and the corresponding levels of harvest, in both the noncooperative and cooperative solutions. The steady state level can be obtained by solving the equation $x = b(x - \phi_1(x) - \phi_2(x))^\alpha$. (Of course, the solution must be greater than zero and smaller than the saturation level.)

The second aspect is the issue of coalitional stability of the cooperative solution. The coalition of the cooperative solution is stable if, for each agent, the intertemporal utility function is greater than or equal to what it would be if they switched to the noncooperative solution. If this condition holds, no agent has incentive to leave the coalition. If we let the superscripts C and NC denote cooperative and noncooperative, the criteria for coalitional stability can be written as

$$V_i^C(x) = A_i^C \ln x + B_i^C \geq A_i^{NC} \ln x + B_i^{NC} = V_i^{NC}(x)$$

for both i. It's been shown above that $A_i^C = A_i^{NC}$. Hence, stability depends in fact only on B_i^C and B_i^{NC} and is independent of the fish stock. This result is also obtained in Breton and Keoula (2014), where stability is defined in the same way as here.

It should be mentioned that our criteria for stability is not an obvious choice, because it assumes, unrealistically, that if one agent deviates from cooperation then the other one immediately deviates too, in the same period. However, if one agent deviates from cooperation, the other agent should be able to respond only in the next period. This increases the incentive to deviate from cooperation. Following standard procedures in fish war models, we will ignore this complication.

The parameter values are as follows. Following Breton and Keoula (2014) we set $\alpha = 0.7$ and $\delta_1 = \delta_2 = 0.8$. We also set $b = 5$, which implies a saturation level of around 213. Regarding the weights of the cooperative solution we set $\lambda_1 = \lambda_2 = 1$. Finally, we set $\beta_2 = 1$. By varying β_1 we will trace out the effect of quasi-hyperbolic discounting.

In Table 4.1 we display the steady state fish stock and harvest levels for various values of β_1 (keeping all other parameters constant), for the noncooperative and cooperative solutions. As above, the superscripts C and NC are used to denote whether the solution is cooperative or noncooperative. In Table 4.2 we display the value functions for various values of β_1 and determine whether the coalition of the cooperative solution is stable.

Table 4.1: Steady State Fish Stock and Harvest Levels

β_1	x^{NC}	x^C	$\phi_1^{NC}(x^{NC})$	$\phi_1^C(x^C)$	$\phi_2^{NC}(x^{NC})$	$\phi_2^C(x^C)$
1.0	9.176	30.94	2.804	6.807	2.804	6.807
0.9	8.21	28.67	2.426	6.488	2.696	6.488
0.8	7.179	26.39	2.038	6.15	2.548	6.15
0.7	6.088	24.11	1.645	5.792	2.35	5.792
0.6	4.946	21.85	1.256	5.413	2.093	5.413
0.5	3.773	19.61	0.883	5.016	1.766	5.016

Table 4.2: Steady State Value Function Levels

β_1	$V_1^{NC}(x^{NC})$	$V_1^C(x^C)$	$V_1^{NC}(x^C)$	$V_2^{NC}(x^{NC})$	$V_2^C(x^C)$	$V_2^{NC}(x^C)$	Stable
1.0	7.731	11.17	10.49	7.731	11.17	10.49	Yes
0.9	6.477	10.08	9.16	7.625	10.99	10.47	Yes
0.8	5.212	8.998	7.839	7.452	10.79	10.41	Yes
0.7	3.929	7.924	6.532	7.183	10.57	10.31	Yes
0.6	2.623	6.862	5.243	6.773	10.31	10.15	Yes
0.5	1.278	5.815	3.975	6.147	10.02	9.892	Yes

In the first row, where $\beta_1 = 1$, the agents have the same harvest as each other in both the noncooperative solution and cooperative solution. This is in fact the standard model: the agents are symmetric and have consistent time preferences. We see that in this case the coalition is stable, a result which is also obtained in Breton and Keoula (2012).

When β_1 decreases, the harvest decreases for both agents and in both solutions. In the cooperative solution, the reason for this is that when agent 1 becomes less patient, the coalition as a whole becomes less patient. This prompts the coalition to increase the fraction of the stock that is harvested in each period, which in turn decreases the steady state stock. The net effect of this is a decrease in the amount that is harvested. Notice also that in the cooperative solution, the agents have the same harvest, regardless of the level of β_1. Indeed, from the analytic section, we already know that the distribution of harvest across the agents in the cooperative solution is independent of time preferences. That is, a_1 and a_2 are affected by the discount functions in exactly the same way. The decrease in harvest also means that the value functions decrease. The decrease is much larger for agent 1 (compare $V_1^C(x^C)$ to $V_2^C(x^C)$), since in his case there's also a decrease in the valuation of future harvest.

In the noncooperative solution, there is also an increase in the fraction of the fish stock that is harvested when β_1 decreases. However, this time the agents react in different ways to the change in the time preferences of agent 1. The harvest drops faster for agent 1 than for agent 2. Indeed, for agent 1 there's actually a *decrease* in the fraction of the stock that he himself harvests. That is, when agent 1 becomes more biased towards the present, he reacts by harvesting a smaller amount of the stock. (The total fraction still increases because agent 2 increases his own fraction.) This is a counter-intuitive result.

The steady state stock is much larger in the cooperative solution than in the noncooperative solution, for all values of β_1. Moreover, the coalition is always stable (compare $V_i^C(x^C)$ to $V_i^{NC}(x^C)$ for both i), meaning that the greater ecological welfare associated with a larger steady state stock can be maintained. This suggests that the property of coalitional stability is not very sensitive to inconsistent time preferences in one agent.

Finally, we also note that although both agents gain from cooperation, the impatient agent gains much more (compare $V_i^C(x^C)$ to $V_i^{NC}(x^{NC})$ for both i). The more impatient agent 1 becomes, the greater is the steady state stock under cooperation compared to noncooperation. When $\beta_1 = 0.5$, the steady state stock increases by a factor of about five under cooperation, while it only increases by a factor of about three when $\beta_1 = 1.0$. Hence, the benefits of cooperation increase to the extent that one agent has a greater time preference bias than the other agent. In other words, the benefits of cooperation are greater when there's a large asymmetry in time preference bias.

4.4 A Fish War Model with a Linear Autocorrelated Stochastic Growth Function and Power Utility Functions

In this section we generalize the model of the previous section by using the general power utility function

$$u_i(c_{i,t}) = \frac{(c_{i,t})^{1-\gamma}}{1-\gamma} \ .$$

We will also change the growth function to

$$X_{t+1} = (1 + g + \sigma Z_{t+1})(x_t - c_{1,t} - c_{2,t}) \tag{4.36}$$

where Z_t is a stochastic variable and σ is a nonnegative volatility parameter. Starting from the growth function of the previous section, (4.36) can be obtained by setting $b = 1 + g + \sigma Z_{t+1}$ and $\alpha = 1$. Hence, we are both generalizing the growth function by introducing stochastics and restricting it by imposing linearity.

The logic of the growth function (4.36) is that the harvest is first subtracted from the fish stock, and then the random variable is applied to the fish stock that remains. Linearity has the implication that there is no saturation level. If $E[Z_t] = 0$ for any t and Z_t and there is no harvest, the fish stock will, in expectation, grow exponentially forever at the rate g. This is, no doubt, an undesirable aspect of the model. The reason for the restriction is tractability; it allows us to assume linear strategies.

The presence of the stochastic variable Z_t introduces several issues. One issue is its distribution. We need to truncate the distribution in such a way that for any σ and g, it is the case that $1 + g + \sigma Z_t \geq 0$ almost surely. If this condition does not hold, the fish stock can become negative which is nonsensical. We also impose that the unconditional distribution of Z_t is the same for every t.

A second issue is dependence of Z_{t+1} on Z_t. If there is no such dependence, then knowing the outcome of past and present random events provides no information which is relevant for forecasting future random events, and there is no autocorrelation in the stochastic process X_t. This has the convenient consequence that the Z_t are identically and independently distributed. The value functions and the decision rules will depend on the fish stock only, as was the case in the deterministic model of the previous section. That is, we obtain $V_i(x)$ and $\phi_i(x)$ rather than $V_i(x, z)$ and $\phi_i(x, z)$, which simplifies the mathematical derivations.

However, the theory we that we developed in Section 2 allows for Z_{t+1} to be dependent on Z_t. We should make use of this result. The key issue, though, is that when the value functions are $V_i(x, z)$, we have two state variables. Coming up with valid conjectures for the value functions is not straightforward in this case. One way to get around this problem is to assume that the outcome space of Z_t is finite. In that case, we can solve the model separately for each potential outcome of Z_t, and then compare the solutions. In each solution, Z_t is treated as a parameter rather than a variable. For simplicity, we will assume that $Z_t \in \{-1, 1\}$ for all t. Hence, in solving the model we will write the value functions as $V_i(x)$, but with the implicit understanding that the functions depend on a parameter z which is equal to one or minus one. Although this makes for a very simplistic stochastic variable, it is sufficient for analyzing the effects of dependence.

We will model the autocorrelation in the growth function as follows. Let p denote the probability that $Z_{t+1} = Z_t$. We then have the probabilities $P\{Z_{t+1} = 1 | \, Z_t = 1\} = P\{Z_{t+1} = -1 | \, Z_t = -1\} = p$ and $P\{Z_{t+1} = 1 | \, Z_t = -1\} = P\{Z_{t+1} = -1 | \, Z_t = 1\} = 1 - p$. It follows that the conditional expectations are $E\left[Z_{t+1} | \, Z_t = 1\right] = 2p - 1$ and $E\left[Z_{t+1} | \, Z_t = -1\right] = 1 - 2p$. The unconditional expectation of Z_t (for any value of p) is zero. If $p = 0.5$, there is no dependence of Z_{t+1} on Z_t. If $p > 0$, Z_{t+1} on Z_t are positively correlated, and if $p < 0.5$ they are negatively correlated. If p is equal to one or zero, there is no uncertainty in the model because autocorrelation is perfect. Hence, we can view p as a measure of autocorrelation.

It should be noted that the extent to which the future is uncertain is determined by both σ and p. It is clear σ scales the 'jump' that the fish stock takes at the end of each period, and thus it affects the variance of the fish stock. On the other hand, for a given level of σ, the variance of the next "jump" varies with p. Hence, there is no obvious way of measuring uncertainty in a single number. If $p = 0.5$, all random events are mutually independent, and all uncertainty depends on σ. If $\sigma = 0$, uncertainty is removed so p is irrelevant and the model is deterministic.

4.4.1 The Noncooperative Solution

To find the noncooperative solution, we will make use of the dynamic programming equations (4.21)-(4.22), which for this model become

$$V_i(x) = \sup_{\{c_i\}} \left\{ \frac{(c_i)^{1-\gamma}}{1-\gamma} + \beta_i \delta_i E\left[\bar{V}_i((1+g+\sigma Z)(x - c_i - \phi_{-i}(x)))\right] \right\} ,$$

and

$$\bar{V}_i(x) = \frac{(\phi_i(x))^{1-\gamma}}{1-\gamma} + \delta_i E\left[\bar{V}_i((1+g+\sigma Z)(x - \phi_i(x) - \phi_{-i}(x)))\right] ,$$

where $i \in \{1, 2\}$. We are interested in finding the decision rules $\phi_i(x)$ and the value functions $V_i(x)$. Following standard procedures, we will assume that $V_i(x) = A_i^{-\gamma} x^{1-\gamma}/(1-\gamma)$ and $\bar{V}_i(x) = \bar{A}_i^{-\gamma} x^{1-\gamma}/(1-\gamma)$. We then obtain the dynamic programming equations

$$\frac{A_i^{-\gamma} x^{1-\gamma}}{1-\gamma} = \sup_{\{c_i\}} \left\{ \frac{(c_i)^{1-\gamma}}{1-\gamma} + \beta_i \delta_i \frac{\bar{A}_i^{-\gamma}}{1-\gamma} E\left[(1+g+\sigma Z)^{1-\gamma}\right] (x - c_i - \phi_{-i}(x))^{1-\gamma} \right\} \tag{4.37}$$

and

$$\frac{\bar{A}_i^{-\gamma} x^{1-\gamma}}{1-\gamma} = \frac{(\phi_i(x))^{1-\gamma}}{1-\gamma} + \delta_i \frac{\bar{A}_i^{-\gamma}}{1-\gamma} E\left[(1+g+\sigma Z)^{1-\gamma}\right] (x - c_i - \phi_{-i}(x))^{1-\gamma} . \tag{4.38}$$

The maximization problem implicit in (4.37) yields the first order conditions

$$c_i^{-\gamma} = \beta_i \delta_i \bar{A}_i^{-\gamma} E\left[(1+g+\sigma Z)^{1-\gamma}\right](x - c_i - \phi_{-i}(x))^{-\gamma} \ . \tag{4.39}$$

Obviously $c_i = \phi_i(x)$. Therefore, we can combine the first order conditions to obtain the linear decision rules

$$\phi_i(x) = \frac{D_i x}{(1+D_1)(1+D_2)-1} \tag{4.40}$$

where $D_i = \left(\beta_i \delta_i \bar{A}_i^{-\gamma} E\left[(1+g+\sigma Z)^{1-\gamma}\right]\right)^{-1/\gamma}$. Next, we want to find the parameters of the auxiliary functions $\bar{V}_i(x)$. To this end, we substitute the decision rules into (4.38) to obtain

$$\bar{A}_i^{-\gamma} = \left(\frac{D_i}{(1+D_1)(1+D_2)-1}\right)^{1-\gamma}$$
$$+\delta_i \bar{A}_i^{-\gamma} E\left[(1+g+\sigma Z)^{1-\gamma}\right]\left(1 - \frac{D_1+D_2}{(1+D_1)(1+D_2)-1}\right)^{1-\gamma} \ .$$

This provides a system of two equations which can be solved numerically for $\bar{A}_1$ and $\bar{A}_2$. We can then substitute the solutions of the system into (4.40) to obtain numeric solutions for the decision rules. To obtain a numeric solution for the value functions, we substitute the solutions to the system into (4.37) to obtain

$$(1-\gamma)V_i(x) = \left(\frac{D_i x}{(1+D_1)(1+D_2)-1}\right)^{1-\gamma}$$
$$+\beta_i \delta_i \bar{A}_i^{-\gamma} E\left[(1+g+\sigma Z)^{1-\gamma}\right]\left(x - \frac{(D_1+D_2)x}{(1+D_1)(1+D_2)-1}\right)^{1-\gamma} \ .$$

which gives us $V_i(x)$ in terms of $\bar{A}_i$. The value functions depend on the fish stock, which is a random variable. Since the decision rules are linear, the stock will in expectation remain steady if, and only if,

$$G \equiv (1+g)\left(1 - \frac{D_1+D_2}{(1+D_1)(1+D_2)-1}\right) = 1 \ .$$

If G is greater (smaller) than one, the stock will in expectation explode to infinity (converge to zero).

The numeric solutions will depend on the expectation $E\left[(1+g+\sigma Z)^{1-\gamma}\right]$. This is where the parameter z comes in. If $z = 1$ we have

$$E\left[(1+g+\sigma Z)^{1-\gamma}\right] = (1+g+\sigma)^{1-\gamma}p + (1+g-\sigma)^{1-\gamma}(1-p) \ .$$

If $z = -1$ we have

$$E\left[(1+g+\sigma Z)^{1-\gamma}\right] = (1+g+\sigma)^{1-\gamma}(1-p) + (1+g-\sigma)^{1-\gamma}p \ .$$

This is the channel through which the correlation parameter p affects the decision rules and value functions.

4.4.2 The Cooperative Solution

To find the cooperative solution, we will make use of the equations (4.23)-(4.25). For this model, the dynamic programming equations become

$$\sum_{i=1}^{2}\lambda_i V_i(x) = \sup_{\{c_i\}}\left\{\sum_{i=1}^{2}\lambda_i\left(\frac{(c_i)^{1-\gamma}+\delta_i(\beta_i-1)E\left[\phi_i((1+g+\sigma Z)(x-c_i-\phi_{-i}(x)))^{1-\gamma}\right]}{1-\gamma}\right.\right.$$
$$\left.\left.+\delta_i E\left[V_i((1+g+\sigma Z)(x-c_i-\phi_{-i}(x)))\right]\right)\right\}\ ,$$
$$V_i(x) = \frac{\phi(x)^{1-\gamma}}{1-\gamma}+\beta_i\delta_i E\left[\bar{V}_i((1+g+\sigma Z)(x-c_i-\phi_{-i}(x)))\right]\ ,$$

and

$$\bar{V}_i(x) = \frac{\phi(x)^{1-\gamma}}{1-\gamma}+\delta_i E\left[\bar{V}_i((1+g+\sigma Z)(x-c_i-\phi_{-i}(x)))\right]\ .$$

where $i \in \{1,2\}$. Again, we are interested in finding the decision rules $\phi_i(x)$ and the value functions $V_i(x)$. As in the case of noncooperation, we may assume that $V_i(x) = A_i^{-\gamma}x^{1-\gamma}/(1-\gamma)$ and $\bar{V}_i(x) = \bar{A}_i^{-\gamma}x^{1-\gamma}/(1-\gamma)$. We also impose in advance that $\phi_i(x) = a_i x$, i.e. that harvest is linear in the fish stock. The above equations then become

$$\sum_{i=1}^{2}\lambda_i\frac{A_i^{-\gamma}x^{1-\gamma}}{1-\gamma} \tag{4.41}$$
$$= \sup_{\{c_i\}}\left\{\sum_{i=1}^{2}\lambda_i\left(\frac{(c_i)^{1-\gamma}}{1-\gamma}+((\beta_i-1)a_i^{1-\gamma}+A_i^{-\gamma})\delta_i E\left[(1+g+\sigma Z)^{1-\gamma}\right]\frac{(x-c_i-a_{-i}x)^{1-\gamma}}{1-\gamma}\right)\right\}\ ,$$
$$\frac{A_i^{-\gamma}x^{1-\gamma}}{1-\gamma} = \frac{(a_i x)^{1-\gamma}}{1-\gamma}+\beta_i\delta_i\bar{A}_i^{-\gamma}E\left[(1+g+\sigma Z)^{1-\gamma}\right]\frac{(x-a_i x-a_{-i}x)^{1-\gamma}}{1-\gamma} \tag{4.42}$$

and

$$\frac{\bar{A}_i^{-\gamma}x^{1-\gamma}}{1-\gamma} = \frac{\phi(x)^{1-\gamma}}{1-\gamma}+\delta_i\bar{A}_i^{-\gamma}E\left[(1+g+\sigma Z)^{1-\gamma}\right]\frac{(x-a_i x-a_{-i}x)^{1-\gamma}}{1-\gamma}\ . \tag{4.43}$$

The maximization problem implicit in (4.41) yields the first order conditions

$$\lambda_i c_i^{-\gamma}$$
$$= (\lambda_1\delta_1((\beta_1-1)a_1^{1-\gamma}+A_1^{-\gamma})+\lambda_2\delta_2((\beta_2-1)a_2^{1-\gamma}+A_2^{-\gamma}))E\left[(1+g+\sigma Z)^{1-\gamma}\right](x-c_i-a_{-i}x)^{-\gamma}\ .$$

Obviously $c_i = a_i x$. Therefore, the first order conditions are

$$\lambda_i a_i^{-\gamma}$$
$$= (\lambda_1\delta_1((\beta_1-1)a_1^{1-\gamma}+A_1^{-\gamma})+\lambda_2\delta_2((\beta_2-1)a_2^{1-\gamma}+A_2^{-\gamma}))E\left[(1+g+\sigma Z)^{1-\gamma}\right](1-a_1-a_2)^{-\gamma}\ .$$

For given A_1 and A_2, these first order conditions form a system of two equations in a_1 and a_2, which we can solve numerically. Since we don't know the values of A_1 and A_2, a solution to the system will be a correspondence which maps (A_1, A_2) onto (a_1, a_2). Fortunately, it is clear from the first order conditions that $\lambda_1 a_1^{-\gamma} = \lambda_2 a_2^{-\gamma}$. This implies that, when solving the system, if we can find just one of the functions $a_1(A_1, A_2)$ and $a_2(A_1, A_2)$, the other function follows immediately. It also implies that

$$\phi_1(x) = (\lambda_1/\lambda_2)^{1/\gamma}\phi_2(x) \ ,$$

which tells us how the behavior of one agent relates to that of the other agent.

By using the correspondence derived from (4.42) we obtain

$$A_i^{-\gamma} = (a_i(A_1, A_2))^{1-\gamma} + \beta_i\delta_i\bar{A}_i^{-\gamma}E\left[(1+g+\sigma Z)^{1-\gamma}\right](1 - a_1(A_1, A_2) - a_2(A_1, A_2))^{1-\gamma} \ ,$$

which allows us to solve for a second correspondence which maps (A_1, A_2) to $(\bar{A}_1, \bar{A}_2)$. Then, using both correspondences we can write (4.43) as

$$(\bar{A}_i(A_1, A_2))^{-\gamma}$$
$$= (a_i(A_1, A_2))^{1-\gamma} + \delta_i(\bar{A}_i(A_1, A_2))^{-\gamma}E\left[(1+g+\sigma Z)^{1-\gamma}\right](1 - a_1(A_1, A_2) - a_2(A_1, A_2))^{1-\gamma} \ ,$$

which is a system in A_1 and A_2. By solving the system, we obtain the parameters of the value functions. The decision rules can then be obtained by using the first correspondence and set $\phi_i(x) = a_i(A_1, A_2)x$. Then, by setting up the expression

$$G \equiv (1+g)(1 - a_1(A_1, A_2) - a_2(A_1, A_2)) \ ,$$

we can check whether the fish stock will in expectation explode to infinity or converge to zero.

As in the noncooperative solution, we will solve the model twice, one time for $z = 1$ and one time for $z = -1$, with the purpose of measuring the effect of the correlation parameter p.

4.4.3 Numeric Solutions

As in the previous section, we are interested in the effects of β_1 and β_2 on the behavior of the agents. We are also interested in the effects of autocorrelation, determined by p.

The parameter values are as follows. In order to test the effects of quasi-hyperbolic preferences, we set $\gamma = 0.6$, $\delta_1 = \delta_2 = 0.8$, $g = 0.01$, $\sigma = 0.05$, $p = 0.5$, $\lambda_1 = \lambda_2 = 1$ and $\beta_2 = 1$. As in the previous section, we will vary β_1. Since there is no unique steady state level of the fish stock, we set $x = 1$. This implies that the fraction of the current stock that is harvested is equal to the amount that is harvested.

In Table 4.3 we see how the agents react to a decrease in β_1. In both solutions, and for all values of β_1, the growth coefficient G is smaller than one, implying that the fish stock converges to zero. In the cooperative solution, the agents have the same harvest. The harvest increases when β_1 decreases and, as a consequence, the growth coefficient decreases. We can interpret this as meaning that when the coalition as a whole becomes less patient, it decides to exhaust the fish stock faster.

Table 4.3: Fish Stock and Harvest Levels

β_1	G^{NC}	G^C	$\phi_1^{NC}(x)$	$\phi_1^C(x)$	$\phi_2^{NC}(x)$	$\phi_2^C(x)$
1.0	0.31	0.36	0.34	0.32	0.34	0.32
0.8	0.34	0.32	0.36	0.34	0.29	0.34
0.6	0.39	0.28	0.38	0.36	0.22	0.36

Table 4.4: Value Function Levels

β_1	$V_1^{NC}(x)$	$V_1^C(x)$	$V_2^{NC}(x)$	$V_2^C(x)$	Stable
1.0	3.29	3.83	3.29	3.83	Yes
0.8	3.13	3.33	3.20	3.83	Yes
0.6	2.95	2.81	3.08	3.78	No

In the noncooperative solution, on the other hand, the growth coefficient moves in the opposite direction, and the stock is exhausted at a slower rate. The reason is a sharp drop in the harvest of agent 2. It is interesting to note that although it is the time preference of agent 1 that is changing, it is agent 2 who is adjusting his behavior the most.

We can also see, looking at Table (4.4), that stability cannot be maintained when β_1 decreases. At some point, agent 1 has incentive to break the coalition. The reason is that in the noncooperative solution, the decrease in β_1 results in both an increase in the growth coefficient and an increase in the fraction of the stock that is harvested by agent 1. Both of these factors are beneficial for agent 1, and they are absent in the cooperative solution. The result in that although the value function of agent 1 decreases in the both solutions (due to a lower valuation of future harvest), it decreases slower in the noncooperative solution. Hence, eventually $V_1^C(x)$ is overtaken by $V_1^{NC}(x)$.

We will now look at the effects of autocorrelation. Recall that when $p > 0.5$, it is probable that $Z_{t+1} = Z_t$. In the following, we set $\beta_1 = 1.0$, so that the agents are symmetric. We also increase the volatility of the fish stock by setting $\sigma = 0.15$, which raises the usefulness of reliable forecasting. In each of the following three tables, we set a specific value for p, and we provide the value functions for $z = 1$ and for $z = -1$.

In Table 4.5 there is no autocorrelation, and knowing the value of Z_t is irrelevant for forecasting Z_{t+1}. Hence, the value functions are not affected by z. In Table 4.6 the autocorrelation is positive. Here we see that when $z = 1$, the value functions are higher than they were without autocorrelation. This is because the next random event is likely to be beneficial from the point of view of the agents. Likewise, when $z = -1$ the value functions are lower than they were without autocorrelation. The same pattern can be observed in Table 4.7. The stronger the

Table 4.5: Value Function Levels When $p = 0.5$

z	$V_1^{NC}(x)$	$V_1^{C}(x)$	$V_2^{NC}(x)$	$V_2^{C}(x)$	Stable
1	3.29	3.83	3.29	3.83	Yes
-1	3.29	3.83	3.29	3.83	Yes

Table 4.6: Value Function Levels When $p = 0.7$

z	$V_1^{NC}(x)$	$V_1^{C}(x)$	$V_2^{NC}(x)$	$V_2^{C}(x)$	Stable
1	3.33	4.03	3.33	4.03	Yes
-1	3.23	3.65	3.23	3.65	Yes

Table 4.7: Value Function Levels When $p = 0.9$

z	$V_1^{NC}(x)$	$V_1^{C}(x)$	$V_2^{NC}(x)$	$V_2^{C}(x)$	Stable
1	3.37	4.33	3.37	4.33	Yes
-1	3.18	3.46	3.18	3.46	Yes

autocorrelation, the more do the value functions react to the value of the most recent random event.

The most striking result, however, is that as p increases, the change in the value functions is much larger in the cooperative solution than in the noncooperative solution. For example, for $z = 1$, the value of $V_1^{NC}(x)$ changes from 3.29 to 3.33 and then to 3.37, while the value of $V_1^{C}(x)$ changes from 3.83 to 4.03 and then to 4.33. When $z = -1$, we see a similar pattern. There's no obvious economic interpretation to this observation, but one interesting guess is the following. When agents cooperate, they are better disposed at taking advantage of forecasts. That is, cooperation allow them to better use the ability to forecast the future to make accurate predictions of future utilities. Conversely, when they don't cooperate, they are unable to take full advantage of the ability to forecast the future.

4.5 A Fish War Model with Power Utility Functions and Nonlinear Growth Function

In this section, we return to the nonlinear growth function that is standard in the fish war model, but maintain the stochastic element of Section 3. We also keep the assumption of general power utility functions, but add the additional feature of asymmetric utility functions.

The utility functions of agent i is now

$$u(c_{i,t}) = \frac{(c_{i,t})^{1-\gamma_i}}{1-\gamma_i} ,$$

where γ_1 may differ from γ_2. The growth function is

$$X_{t+1} = b((1+\sigma Z_{t+1})(x_t - c_{1,t} - c_{2,t}))^{\alpha} ,$$

where we impose the same parameter restrictions as previously. The logic of this growth function is that we first subtract the total harvest from the fish stock, then apply the random event, and finally raise the remaining stock to the power of α and multiply by b.

When it is not the case that $\gamma_1 = \gamma_2 = 1$ (which would imply logarithmic utility functions), is it difficult to find valid conjectures for the value functions of this model. Hence, the algorithm which we employed in the previous sections will not suffice. Instead, we will solve the model using fully numeric backward induction, without imposing any particular functional form on the value functions. The algorithm which we employ for this purpose is described in the appendix.

Throughout this section, we will only solve for the cooperative solution. We will also assume that the random event Z_t follows the same binary distribution as in the previous section, but without autocorrelation, so that $Z_t \in \{-1, 1\}$ and $P\{Z_t = 1\} = 0.5$ for all t. Moreover, we impose that the model has a finite planning horizon and that some given level of the fish stock must remain in the sea when the horizon has been reached. Hence, we specify a terminal time T and a minimum terminal fish stock $\bar{x}_T$. Intuitively, we can imagine that the agents have been contracted the right to freely harvest over the period $[0, T]$, conditional on that a minimum level of fish remains when the period has ended. Thus, when time T has been reached, the agents are free to harvest until the point where the fish stock reaches $\bar{x}_T$, at which point they must stop.

4.5.1 Attempting an Analytic Solution

Before solving the model numerically, let us first make an attempt at an analytic solution using backward induction. This will clarify why finding a valid conjecture on the value functions is difficult.

Supposing $\bar{x}_T = 0$, the coalition at time T solves the problem

$$\max_{c_{1,T}, c_{2,T}} \left\{ \lambda_1 \frac{c_{1,T}^{1-\gamma_1}}{1-\gamma_1} + \lambda_2 \frac{c_{2,T}^{1-\gamma_2}}{1-\gamma_2} \right\}$$

subject to

$$c_{1,T} + c_{2,T} \leq x_T .$$

Clearly, we can solve for $c_{1,T}$ and $c_{2,T}$ explicitly if, and only if, we impose $\gamma_1 = \gamma_2 = \gamma$, i.e. the agents have the same utility function, in which case we obtain

$$c_{i,T} = \frac{\lambda_{-i}^{-1/\gamma}}{\lambda_1^{-1/\gamma} + \lambda_2^{-1/\gamma}} x_T .$$

From this, it follows that

$$V_{i,T}(x_T) = \left(\frac{\lambda_{-i}^{-1/\gamma}}{\lambda_1^{-1/\gamma} + \lambda_2^{-1/\gamma}} \right)^{1-\gamma} \frac{x_T^{1-\gamma}}{1-\gamma} \ .$$

With this solution, the problem to be solved at time $T-1$ is

$$\max_{c_{1,T-1}, c_{2,T-1}} \left\{ \lambda_1 \frac{c_{1,T-1}^{1-\gamma}}{1-\gamma} + \lambda_2 \frac{c_{2,T-1}^{1-\gamma}}{1-\gamma} \right.$$

$$\left. + \frac{\lambda_1\beta_1\delta_1\lambda_2^{(\gamma-1)/\gamma} + \lambda_2\beta_2\delta_2\lambda_1^{(\gamma-1)/\gamma}}{\lambda_1^{(\gamma-1)/\gamma} + \lambda_2^{(\gamma-1)/\gamma}} E\left[(1+\sigma Z_T)^{\alpha(1-\gamma)}\right] \frac{(x_{T-1} - c_{1,T-1} - c_{2,T-1})^{\alpha(1-\gamma)}}{1-\gamma} \right\}$$

This can be solved for $c_{1,T-1}$ and $c_{2,T-1}$ if we impose that $\alpha(1-\gamma) = 1$.[4] That is, we must assume a particular relationship between a parameter that determines the growth of the fish stock and a parameter that determines the preferences of the agents, as was done in Antoniaduo *et al* (2013). This is a restrictive option.

Without knowing the functional form for the decision rules, we cannot know the functional forms for the value functions. Hence, without imposing restrictions, we cannot proceed further with analytic methods. This motivates the numeric solutions.

4.5.2 Risk and Risk Aversion under Nonlinear Growth Functions

In this subsection, we set $\gamma_1 = \gamma_2 = \gamma$ and $\alpha(1-\gamma) \neq 1$, so that numeric solutions are necessary. We are interested in how the presence of risk interacts with the utility functions to produce behavior. These factors depend on the parameters γ and σ.

First, let's recall an important aspect of power utility functions. The parameter γ determines not only risk aversion, but also the preference for *harvest smoothing*. That is, it determines to what extent the agent prefers to have the same amount of harvest in every period, as opposed to concentrating harvest to a single period. The smaller is γ, the smaller is the preference for smoothing. In the extreme case when $\gamma = 0$, the marginal utility is constant, so the agent is indifferent between, on the one hand, harvesting a given amount of fish in a single period and, on the other hand, spreading the same amount evenly over all time periods. Now, in our model the agents are impatient (the discount functions decline over time) and prefer, all other things equal, harvest in the current period over harvest in any later period. Hence, if the preference for harvest smoothing is low, it follows that harvest will be concentrated mostly to the initial period. If the preference for harvest smoothing is high, extraction will be spread more evenly over the whole interval from 0 to T.

Given the above, how will an increase in risk (i.e. an increase in σ) affect harvest? Risk implies that future planned harvest is uncertain. Only present harvest is free of risk. By moving

[4] Imposing $\alpha = 1$ would also be sufficient. Then we obtain a linear model, as in the previous section.

harvest from the future to the present, the coalition reduces risk, which is beneficial from the point of view of risk aversion. But this implies concentrating harvest in a single period, which is undesirable for an agent for whom smoothing is important. If the preference for smoothing is high, preserving a "risk buffer" for future periods becomes important. The risk buffer ensures that even if some of the stock is lost due to random events, there will still be enough fish in the last few periods to smooth effectively. If the preference for smoothing is low, however, the cost of reducing risk by moving harvest to the present is low.

The point is that, although the agents are risk averse for any $\gamma > 0$, they will not always react to more risk by harvesting more, as one would expect based on risk aversion alone. When γ increases, both risk aversion and the preference for harvest smoothing goes up, and they push the behavior of the agents in different directions. The net effect on behavior will depend on which one of these preferences dominate.

With this in mind, let's look at Table 4.8 and Table 4.9. As in section 3, we set $\alpha = 0.7$, $b = 5$ and $\lambda_1 = \lambda_2 = 1$. We also set $x_0 = 200$, $\bar{x}_T = 0$ and $T = 4$. Regarding the discount functions, we set $\delta_1 = \delta_2 = 0.8$ and $\beta_1 = \beta_2 = 1.0$, implying symmetric and constant discount functions. (Time preferences will not be important in this part of our analysis.)

In each table we provide the development of harvest and the fish stock for every time period and for three levels of σ. Since these variables are random, we provide the expected values (which, in actuality, will not be realized since $Z_t \in \{-1, 1\}$ and $E[Z_t] = 0$). Table 4.8 is for low risk aversions, and therefore also low preference for harvest smoothing, and Table 4.9 is for high risk aversions. We see that when risk aversion is low, the coalition reacts to risk by increasing harvest and thus depleting the stock faster. When risk aversion is high, the opposite occurs. That is, the coalition acts contrary to how one would expect based solely on the concept of risk aversion. We see that as σ grows, the increased preference for harvest smoothing dominates the increased risk aversion, and the net effect is that the coalition acts as one might expect they would if risk aversion had decreased. If we were to set $\gamma = 1$, the logarithmic case, the two tendencies would exactly cancel out. This is why, in the standard fish war model, risk becomes irrelevant. When $\gamma \in [0, 1)$, harvest in the present is increased as a response to risk. When $\gamma > 1$, harvest in the present is decreased as a response to risk.

What we see in Table 4.8 and Table 4.9 is, in fact, an illustration of a general problem in a discounted utility model using power utility functions. Namely: two separate aspects of an agent's subjective preferences, risk aversion and smoothing preference, are determined by the same parameter. We may recall that for power utility functions, the parameter γ is both the coefficient of relative risk aversion and the inverse of the intertemporal elasticity of substitution.

The result obtained here is similar to one in Antoniadou *et al* (2013). They solve a more restricted model, and only for a noncooperative solution, but reach a similar conclusion:

> If the coefficient of relative risk aversion is higher than unity, then, for a given number of symmetric players, each player will conserve the resource once uncertainty

(or an increase in risk) is introduced.

However, they do not characterize this behavior as the net effect of preferences over risk and smoothing, as we have done here.

Table 4.8: The Effect of Risk under Low Risk Aversion: $\gamma = 0.4$

Sigma	**Variable**	**t = 0**	**t = 1**	**t = 2**	**t = 3**	**t = 4**
0.2	$c_{i,t}$	84	21	10	7	7
	$c_{i,t}/x_t$	0.42	0.38	0.36	0.37	0.47
	x_t	200	55	28	19	15
0.4	$c_{i,t}$	87	19	9	6	5
	$c_{i,t}/x_t$	0.43	0.40	0.39	0.40	0.45
	x_t	200	47	23	15	11
0.6	$c_{i,t}$	91	15	7	4	4
	$c_{i,t}/x_t$	0.46	0.42	0.41	0.40	0.50
	x_t	200	36	17	10	8

Table 4.9: The Effect of Risk under High Risk Aversion: $\gamma = 2.0$

Sigma	**Variable**	**t = 0**	**t = 1**	**t = 2**	**t = 3**	**t = 4**
0.2	$c_{i,t}$	48	34	26	21	19
	$c_{i,t}/x_t$	0.24	0.26	0.30	0.34	0.49
	x_t	200	129	88	61	39
0.4	$c_{i,t}$	44	32	26	22	22
	$c_{i,t}/x_t$	0.22	0.24	0.27	0.32	0.50
	x_t	200	134	95	68	44
0.6	$c_{i,t}$	38	30	25	24	25
	$c_{i,t}/x_t$	0.19	0.21	0.24	0.31	0.49
	x_t	200	140	103	77	51

4.5.3 Asymmetric Utility Functions

Next, let's look at the how asymmetric utility functions affect the allocation of harvest. To this end, we set $\gamma_1 = 1.1$ and $\gamma_2 = 0.9$, which means that agent 1 has a slightly higher preference for harvest smoothing than agent 2. We also set $\sigma = 0.0$, so that there is no randomness in the model, and $\beta_1 = \beta_2 = 0.8$ so that the agents have symmetric quasi-hyperbolic time preferences.

The fact that there is no risk implies that γ_1 and γ_2 only influence behavior through their effect on the preference for harvest smoothing, and not through risk aversion.

Table 4.10: Asymmetric Utility Functions: $\gamma_1 = 1.1$ and $\gamma_2 = 0.9$

Variable	**t = 0**	**t = 1**	**t = 2**	**t = 3**	**t = 4**	**t = 5**	**t = 6**	**t = 7**
$c_{1,t}$	33	21	15	13	12	11	11	12
$c_{2,t}$	72	42	27	22	20	20	19	20
$c_{1,t}/x_t$	0.17	0.17	0.17	0.18	0.20	0.21	0.25	0.38
$c_{2,t}/x_t$	0.36	0.35	0.31	0.31	0.33	0.38	0.43	0.63
x_t	200	121	86	71	61	53	44	32

The results are given in Table 4.10. We see that the agent with the lower preference for smoothing (agent 2) has a higher harvest in every period. That is, $c_{2,t} > c_{1,t}$ for all t. However, we also see that the relative size of the harvest of agent 2 decreases. That is, $c_1(t)/c_2(t)$ increases as we approach the end of the planning horizon. We can interpret this in the following way. The presence of time preference means that the agents will want to allocate harvest in the present as opposed to in the future. This is counterbalanced by the preference for smoothing. The agent with the higher preference for smoothing will want to keep a lower present harvest so that the difference between harvests in the present and in the future will be lower. Since this must hold in every "present" (every t) except for in the last period, the result is that the agent with the lower preference for smoothing has a higher harvest throughout.

From a purely mathematical point of view, we can explain this in another way. If the agents have different utility functions, then, for a given level of harvest, one utility function will in general be higher than the other. Thus, equating the marginal utility of harvest for agent 1 with that of agent 2 implies that they will have different levels of harvest.

4.5.4 Time Preferences

We will now investigate the role of the discount functions in the general model with power utility functions and nonlinear growth function. We return to the case of identical agents, and set $\gamma_1 = \gamma_2 = 1.2$. We will vary δ_1 and δ_2, to change the degree to which the coalition is "generally impatient". We will vary β_1 and β_2, to change the degree to which the coalition has quasi-hyperbolic time preferences, and compare the two cases.

Let's first look at Table 4.11, in which we gradually lower δ_1 and δ_2 in order to make the coalition less patient. In this table, it is most instructive to look at the fraction of the fish stock that an agent harvests, which is equal to $c_{i,t}/x_t$. We see that, within most time periods, the fraction increases as δ_i decreases. The exception is the final period, in which each agent always consumes half the remaining stock. As a consequence of this, an impatient coalition will harvest more in the early periods and less in the later periods, compared to a patient coalition. This makes sense from an economic perspective.

Table 4.11: The Effect of General Impatience (δ_i)

δ_i	**Variable**	**t = 0**	**t = 1**	**t = 2**	**t = 3**	**t = 4**	**t = 5**	**t = 6**	**t = 7**
0.9	$c_{i,t}$	36	28	23	21	19	19	20	23
	$c_{i,t}/x_t$	0.18	0.19	0.19	0.21	0.22	0.25	0.31	0.50
	x_t	200	149	119	101	87	76	64	46
0.8	$c_{i,t}$	42	30	23	20	18	17	17	19
	$c_{i,t}/x_t$	0.21	0.22	0.22	0.23	0.24	0.27	0.31	0.49
	x_t	200	139	106	87	74	64	54	39
0.7	$c_{i,t}$	48	32	23	19	16	15	15	16
	$c_{i,t}/x_t$	0.24	0.25	0.25	0.26	0.27	0.29	0.34	0.50
	x_t	200	129	93	73	60	52	44	32
0.6	$c_{i,t}$	54	33	22	17	14	13	11	12
	$c_{i,t}/x_t$	0.27	0.28	0.28	0.29	0.29	0.32	0.33	0.48
	x_t	200	118	79	59	48	41	33	25
0.5	$c_{i,t}$	61	33	21	15	11	9	8	9
	$c_{i,t}/x_t$	0.30	0.31	0.32	0.33	0.31	0.32	0.35	0.50
	x_t	200	106	66	46	35	28	23	18

Next, let's look at Table 4.12, in which we gradually lower β_1 and β_2 in order to make the coalition's time preferences more quasi-hyperbolic. In contrast to in Table 4.11, the fraction $c_{i,t}/x_t$ does not develop in an intuitive way. It sometimes increases, sometimes decreases, in ways which are not easy to interpret. The mathematical reason for this is likely that β_i interacts with γ, α and t in complicated ways. An economic interpretation is difficult to provide.

4.6 Conclusion

The theoretical framework that we developed in Section 2 has many potential applications. Here, we used it to solve two fish war models. The first was the standard one, with the addition of asymmetric quasi-hyperbolic discount functions, and the second was nonstandard due to it having power utility functions and a linear growth function. The first one generalizes the fish war model of Breton and Keoula (2014) by adding quasi-hyperbolic discount functions. The natural next step, then, is to solve the model for a more general discount function. In principle, this can be done with the framework of Section 2.

Table 4.12: The Effect of Quasi-hyperbolic Time Preferences (β_i)

β_i	**Variable**	**t = 0**	**t = 1**	**t = 2**	**t = 3**	**t = 4**	**t = 5**	**t = 6**	**t = 7**
	$c_{i,t}$	42	30	23	20	18	17	17	19
1.0	$c_{i,t}/x_t$	0.21	0.22	0.22	0.23	0.24	0.27	0.31	0.49
	x_t	200	139	106	87	74	64	54	39
	$c_{i,t}$	43	31	24	19	17	17	17	17
0.9	$c_{i,t}/x_t$	0.21	0.23	0.23	0.23	0.24	0.28	0.34	0.49
	x_t	200	137	103	83	71	61	50	35
	$c_{i,t}$	43	30	24	20	18	17	16	16
0.8	$c_{i,t}/x_t$	0.21	0.22	0.23	0.24	0.25	0.29	0.33	0.48
	x_t	200	138	105	85	71	59	48	33
	$c_{i,t}$	41	30	24	21	19	17	16	15
0.7	$c_{i,t}/x_t$	0.20	0.21	0.22	0.24	0.27	0.29	0.36	0.50
	x_t	200	141	108	87	71	58	45	30
	$c_{i,t}$	56	29	20	18	16	15	14	12
0.6	$c_{i,t}/x_t$	0.28	0.25	0.24	0.25	0.27	0.31	0.37	0.48
	x_t	200	114	84	71	59	49	38	25
	$c_{i,t}$	48	29	25	20	17	15	13	10
0.5	$c_{i,t}/x_t$	0.24	0.23	0.26	0.27	0.29	0.33	0.37	0.48
	x_t	200	128	98	75	59	46	35	21

However, it should be pointed out that a more interesting expansion of the standard fish war model lies in generalizing utility functions rather than discount functions. In particular, finding more tractable ways of going beyond the restrictive assumption of logarithmic utilities should be useful. For this purpose, we provided the numeric solutions in the third fish war application of this chapter. The numeric algorithm that we used there can also be used in further research, since it can be employed to (in principle) any type of utility, discount and growth functions. However, a downside of the algorithm is that it requires a finite planning horizon, which is nonstandard in fish war models.

4.7 Appendix

The algorithm that we use in Section 6 was written and compiled in the C++ programming language. It consists of the following steps.

1. Create a matrix in which the number of columns is equal to $T + 1$ (the number of

time periods for which to choose harvest) and the number of rows is equal to 1 plus the maximum amount of the resource stock that can be obtained by the coalition. The maximum amount of stock is the one obtained in the last period if there is no extraction and all random events are positive, i.e. $Z_t = 1$ and $c_{1,t} = c_{2,t} = 0$ for all t. The matrix then contains every combination of time and stock that can potentially be realized. The reason for adding 1 to the maximum amount of the stock is that the stock may be equal to zero.

2. For period T (the last period in which extraction is performed), loop through all the potential states, and record for each state the pair $\{c_{1,T}, c_{2,T}\}$ that maximizes the sum of utilities of period T. All data are recorded as integers. That is, the variables are discretized in such a way that the size of a step is equal to 1.

3. For period $T-1$, loop through all the potential states, and for each state find the pair $\{c_{1,T-1}, c_{2,T-1}\}$ that maximizes the sum of utilities of $T-1$ and T, discounted back to $T-1$.[5] In doing so, we need to calculate the value of x_T that results from each particular $\{c_{1,T-1}, c_{2,T-1}\}$, and then find the $\{c_{1,T}, c_{2,T}\}$ that is optimal given the obtained x_T (which we recorded in the previous step). We also need to take into account that for each $\{c_{1,T-1}, c_{2,T-1}\}$ there are two potential x_T that may be realized: one if $Z_T = 1$ and one if $Z_T = -1$.

4. Perform the equivalent procedure for every period from $T-2$ down to 0. In each period t, and for each potential state x_t, we get the optimal $\{c_{1,t}, c_{2,t}\}$ given that all future harvests have already been determined as functions of the future stock. In doing so, we need to take into account all the different series of shocks that may be realized from $t+1$ to T and weight the utilities of each series by the probability of it being realized. Note that given the assumptions on Z_t that we make in Section 6, every series will have the same probability. This simplifies calculations.

As the algorithm proceeds back in time from T, each new period requires a larger amount of calculations than the previous one. This is because the number of utilities that must be added increases by one for each new period, and because the number of potential series of random events increases exponentially. In period $T-1$, there is only one random event remaining, so the number of potential series is two. In period $T-2$, there are four potential series, and in period $T-k$ there are 2^k potential series. Each one must be considered in order to obtain the decision rule. This implies that as T increases, the time required to run the algorithm can quickly become very long.

[5] If instead we want to find the commitment solution, we would discount every utility back to period 0. This can be done with a minor change in the code.

Chapter 5

Concluding Remarks

5.1 Main Themes of the Thesis

In summarizing the content of this thesis, we will briefly mention some of the main themes that we have encountered throughout the three main chapters.

- In each of the main chapters, we've dealt with the issue of aggregation of preferences across time and across individuals. In solving for noncooperative solutions, we defined the intertemporal utility function of an agent to be equal the sum of all his present and future utilities, discounted back to the present. This is aggregation across time. In solving for cooperative solutions, we defined the intertemporal utility function of a group of agents to be a weighted sum of the intertemporal utility functions of the individual agents. This is aggregation of preferences across agents.

- In most of the models treated in the thesis, we have dealt with agents that have inconsistent time preferences. In some cases, these "agents" have been interpreted as coalitions of agents. This was the case in the cooperative solutions of Chapter 2 and Chapter 3. In Chapter 4, we extended the property of inconsistent time preferences down to the level of the individual agent. In order to find solutions in continuous time, we made use of the definition of an intertemporal equilibrium solution due to Ekeland and Lazrak (2008).

- Most of our models were solved using dynamic programming equations, in which the value functions were the intertemporal utility functions when the equilibrium decision rule is applied. The only exception to this was the numeric solution of the third model in Chapter 4. In Chapter 3 and Chapter 4 we derived new dynamic programming equations, while in Chapter 2 we employed a dynamic programming equation developed in de Paz *et al* (2013).

There is certainly room for extensions and improvements of the models that we have covered here. We've mentioned some of them in the conclusion sections in each chapter. The chapter which has the most potential for improvement is probably Chapter 2, in which the model can be greatly expanded by including additional Macroeconomic aggregates. In this regard, there are plenty of examples in the literature to take inspiration from. In Chapter 3 and Chapter 4, there are the possibilities of experimenting with more complex forms of utility and discount functions, with the purpose of finding interesting results.

5.2 The Philosophical Aspects of a Theory of Inconsistent Time Preferences

We've explained that an intertemporal equilibrium solution is obtained by finding a decision rule such that, at each point in time, it is optimal for the agent to follow this decision rule given that he follows it at all later points in time. This is what makes an intertemporal equilibrium solution different from a commitment solution. The agent at time t is in full control of his actions at t, but can control his action at $t+1, ..., T$ only indirectly. The control is indirect because the agent at time $t+1$ will do what he considers to be optimal given the state that he finds him himself in, and this state is partially a result of the actions of the agent at time t. This is in contrast with the commitment solution, in which the agent at time 0 has full control over the agents at $1, ..., T$.

We recall that it is customary to refer to the agent at time t as the "t-agent". In this thesis, we have occasionally used this terminology. We can think of the t-agent as having a will of his own which, if time preferences are inconsistent, differs from that of the "$(t+1)$-agent". Indeed, given the mathematics of an intertemporal equilibrium, it is perfectly reasonable to think of each t-agent as being a separate individual all together. This individual cares first and foremost about his own utility, but he is also interested in the utility of the agents that follow him, although this altruism decreases with distance in time. Hence, we can view the game as consisting of a series of agents that act one at a time, rather than a single agent who acts repeatedly.

There's an interesting philosophical lesson in this interpretation of the mathematical theory. It reminds us that, ultimately, there's no such thing as a "self" that survives over time. The person who leaves his home for work in the morning is not the same as the one who returns in the evening. They are two separate individuals with different personalities, and with desires and interests that don't necessarily coincide. But they have a peculiar relationship with one another. The present self is interested in the actions and utility of the future self, and he is frustrated because he knows that his influence and control over the future self is limited. The future self, on the other hand, is completely unable to influence the state that the present self finds himself in, but he is not bothered by this fact. He does not care about the utility or disutility of the present self because "the past is in the past". Given this, we may say that in a theory of inconsistent time preferences, there's no such thing as progression of the self through time. There is the present self, and there is the future self. The former does not become the latter.

We close the thesis with the following quote, supposedly due to the 13th century Buddhist priest Dogen Zenji. They are appropriate for a theory of inconsistent time preferences, for reasons which, by now, should be clear to the reader.

> The spring does not become the summer. The summer does not become the autumn. There is spring. Then there is summer. When you burn wood there are ashes. But the wood does not become the ashes. There is wood. And then there are ashes.

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