

FINITELY GENERATED IDEALS IN THE NEVANLINNA CLASS

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ABSTRACT. In this paper we investigate finitely generated ideals in the Nevanlinna class. We prove analogues to some known results for the algebra of bounded analytic functions H^∞ . We also show that, in contrast to the H^∞ case, the stable rank of the Nevanlinna class is strictly bigger than 1.

1. INTRODUCTION

The aim of this paper is to investigate analogues for the Nevanlinna class of some known results on finitely generated ideals of the algebra H^∞ of bounded analytic functions in the unit disk \mathbb{D} , equipped with the supremum norm $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$.

Let us begin by recalling these results. The first one concerns interpolating sequences. A sequence of points $\Lambda = \{\lambda_n\}_{n \in \mathbb{N}}$ in \mathbb{D} is called *interpolating for H^∞* if for every bounded sequence $\{w_n\}_{n \in \mathbb{N}}$ of complex numbers, there exists a function $f \in H^\infty$ such that $f(\lambda_n) = w_n$, $n \in \mathbb{N}$. By a famous result by Carleson [2] a sequence $\{\lambda_n\}_n$ is interpolating for H^∞ if and only if

$$\inf_{n \in \mathbb{N}} \prod_{k \neq n} \left| \frac{\lambda_k - \lambda_n}{1 - \overline{\lambda_n} \lambda_k} \right| > 0.$$

A Blaschke product with simple zeros is called an *interpolating Blaschke product* if its zeros are an interpolating sequence.

The next important result in the context of this paper is Carleson's corona theorem: every family $\{f_1, \dots, f_m\}$ of functions in H^∞ satisfying

$$\inf_{z \in \mathbb{D}} \sum_{i=1}^m |f_i(z)| > 0$$

generates the whole algebra. See [5] or [21]. More generally, we denote by $I_{H^\infty}(f_1, \dots, f_m)$ the ideal generated by the functions f_1, \dots, f_m in H^∞ . The general structure of these ideals is not well understood (see the references [1], [4], [7]-[11], [18], [19], [24], [25] for more information). As it turns out, in certain situations the ideals can be characterized by growth conditions. More

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precisely, the following ideals have been studied:

$$J_{H^\infty}(f_1, \dots, f_m) = \left\{ f \in H^\infty : \exists c = c(f) > 0, |f(z)| \leq c \sum_{i=1}^m |f_i(z)|, z \in \mathbb{D} \right\}.$$

It is obvious that $I_{H^\infty}(f_1, \dots, f_m) \subset J_{H^\infty}(f_1, \dots, f_m)$. This leads us to the third circle of results we are interested in here. Tolokonnikov [24] proved that the following conditions are equivalent:

- (a) $J_{H^\infty}(f_1, \dots, f_m)$ contains an interpolating Blaschke product,
- (b) $I_{H^\infty}(f_1, \dots, f_m)$ contains an interpolating Blaschke product,
- (c) $\inf_{z \in \mathbb{D}} \sum_{i=1}^m (|f_i(z)| + (1 - |z|^2)|f_i'(z)|) > 0$.

As it turns out, in the special situation of two generators with no common zeros these conditions are equivalent to $I_{H^\infty}(f_1, f_2) = J_{H^\infty}(f_1, f_2)$. In the case of two generators f_1 and f_2 with common zeros, we have $I(f_1, f_2) = J(f_1, f_2)$ if and only if $I(f_1, f_2)$ contains a function of the form BC where B is an interpolating Blaschke product and C is the Blaschke product formed with the common zeros of f_1 and f_2 (see [11]).

Let us now turn to the framework we want to discuss in this paper. We are interested in analogues of the above results for the *Nevanlinna class* N , consisting of the holomorphic functions f on \mathbb{D} such that $\log_+ |f|$ has a positive harmonic majorant on \mathbb{D} . Equivalently, $f \in N$ if and only if f is holomorphic on \mathbb{D} and

$$\lim_{r \rightarrow 1^-} \int_{\partial \mathbb{D}} \log_+ |f(r\zeta)| d\sigma(\zeta) < \infty.$$

Here $d\sigma$ denotes the normalized Lebesgue measure on the unit circle.

As a general rule we shall see that the results for H^∞ translate to the Nevanlinna setting provided that the boundedness of the elements described above is replaced by a control given by a positive harmonic majorant (or minorant). Let $\text{Har}_+(\mathbb{D})$ be the cone of positive harmonic functions in the unit disk \mathbb{D} . Recall that any $H \in \text{Har}_+(\mathbb{D})$ is the Poisson integral of a positive measure μ on the unit circle, that is

$$H(z) = P[\mu](z) = \int_{\partial \mathbb{D}} P(z, \zeta) d\mu(\zeta),$$

where

$$P(z, \zeta) = \text{Re} \left(\frac{\zeta + z}{\zeta - z} \right) = \frac{1 - |z|^2}{|\zeta - z|^2}$$

is the Poisson kernel in \mathbb{D} .

It is a standard fact that functions f in the Nevanlinna class admit non-tangential limits f^* at almost every point of the circle. It is also well-known that any $f \in N$ can be factored as $f = BSE$, where B is a Blaschke product containing the zeros of f , S is a singular inner function and E is the outer function:

$$E(z) = C \exp \left\{ \int_{\partial \mathbb{D}} \frac{\zeta + z}{\zeta - z} \log |f^*(\zeta)| d\sigma(\zeta) \right\},$$

where $|C| = 1$. In particular

$$\log |E(z)| = P[\log |f^*|](z), \quad z \in \mathbb{D}.$$

A function S is singular inner if there exists a positive measure μ on $\partial\mathbb{D}$ singular with respect to the Lebesgue measure such that

$$S(z) = \exp \left\{ - \int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right\}, \quad z \in \mathbb{D}.$$

For the Nevanlinna class R , Mortini observed that a well known result of T. Wolff implies the following corona theorem (see [17] or [16]).

Corona Theorem for N (R. Mortini). *Let $I(f_1, \dots, f_m)$ denote the ideal generated in N by a given family of functions $f_1, \dots, f_m \in N$. Then $I(f_1, \dots, f_m) = N$ if and only if there exists $H \in \text{Har}_+(\mathbb{D})$ such that*

$$\sum_{i=1}^m |f_i(z)| \geq e^{-H(z)}, \quad z \in \mathbb{D}.$$

We need to define the ideal corresponding to J_{H^∞} in N . This will be done in the following way:

$$J(f_1, \dots, f_m) = \left\{ f \in N : \exists H = H(f) \in \text{Har}_+(\mathbb{D}), |f(z)| \leq e^{H(z)} \sum_{i=1}^m |f_i(z)|, z \in \mathbb{D} \right\}.$$

It is clear that $I(f_1, \dots, f_m) \subset J(f_1, \dots, f_m)$. Let us also mention that, by the previous corona theorem, in the case when $J(f_1, \dots, f_m) = N$, then $I(f_1, \dots, f_m) = N$.

Recall that a sequence space is called *ideal* if it is stable with respect to pointwise multiplication by bounded sequences. For the following definition see also [13].

Definition. A sequence of points $\Lambda = \{\lambda_n\}_n$ in \mathbb{D} is called *interpolating for N* (denoted $\Lambda \in \text{Int } N$) if the trace space $N|\Lambda$ is ideal.

Equivalently, $\Lambda \in \text{Int } N$ if for every bounded sequence $\{v_n\}_n$ of complex numbers there exists $f \in N$ such that

$$f(\lambda_n) = v_n, \quad n \in \mathbb{N}.$$

Interpolating sequences for the Nevanlinna class were first investigated by Naftalevitch [20] starting from an *a priori* fixed target space which forces interpolating sequences to be confined in a finite union of Stolz angles.

A rather complete study, based on the above definition, was carried out much later in [13]. In particular, it was proved that a sequence $\{\lambda_n\}_n$ is interpolating for N if and only if there exists a positive harmonic function $H \in \text{Har}_+(\mathbb{D})$ such that

$$(1.1) \quad \prod_{k:k \neq n} \left| \frac{\lambda_k - \lambda_n}{1 - \overline{\lambda_n} \lambda_k} \right| \geq e^{-H(\lambda_n)}, \quad n \in \mathbb{N}.$$

Moreover, it was also shown that if $\Lambda \in \text{Int } N$, then the trace space is given by

$$(1.2) \quad N|\Lambda = \left\{ \{w_n\}_n : \exists H \in \text{Har}_+(\mathbb{D}), \log_+ |w_n| \leq H(\lambda_n) \right\}.$$

It was also noticed that in the previous condition only the factors corresponding to λ_k close to λ_n are relevant. More precisely, fixed any $c \in (0, 1)$, the condition

$$(1.3) \quad \prod_{\substack{k:k \neq n \\ \rho(\lambda_k, \lambda_n) \leq c}} \left| \frac{\lambda_k - \lambda_n}{1 - \bar{\lambda}_n \lambda_k} \right| \geq e^{-H(\lambda_n)}, \quad n \in \mathbb{N}$$

is sufficient for Λ to be interpolating (see [13, Proposition 4.1]).

A Blaschke product the zeros of which forms an interpolating sequence for the Nevanlinna class is called a *Nevanlinna interpolating Blaschke product*.

The analogues of the results mentioned above in the context of H^∞ read as follows.

Theorem 1.1. *Let f_1, \dots, f_m be functions in N . Then the following conditions are equivalent:*

- (a) $I(f_1, \dots, f_m)$ contains a Nevanlinna interpolating Blaschke product,
- (b) $J(f_1, \dots, f_m)$ contains a Nevanlinna interpolating Blaschke product,
- (c) There exists a function $H \in \text{Har}_+(\mathbb{D})$ such that

$$\sum_{i=1}^m (|f_i(z)| + (1 - |z|^2)|f_i'(z)|) \geq e^{-H(z)}, \quad z \in \mathbb{D}.$$

In case $m = 2$, if f_1 and f_2 have no common zeros, the above conditions are equivalent to

- (d) $I(f_1, f_2) = J(f_1, f_2)$.

As in H^∞ each of the conditions (a)-(c) implies $I(f_1, \dots, f_m) = J(f_1, \dots, f_m)$. However, when $m \geq 3$, the converse fails, as will be explained after the proof of the result. Also, like in the H^∞ -situation, if the two generators f_1 and f_2 have common zeros, then $I(f_1, f_2) = J(f_1, f_2)$ if and only if $I(f_1, f_2)$ contains a function of the form BC where B is a Nevanlinna interpolating Blaschke product and C is the Blaschke product formed with the common zeros of f_1 and f_2 .

Our proof of Theorem 1.1 uses some of the ideas from both [24] and [11], but also some specific properties of the Nevanlinna class. In particular we will make use of a new description of Nevanlinna interpolating sequences in terms of harmonic measure, which we discuss now.

Denote by

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{z}w} \right|$$

the pseudohyperbolic distance in \mathbb{D} , and by $D(z, r) = \{w \in \mathbb{D} : \rho(z, w) < r\}$ the corresponding disk of center z and radius $r \in (0, 1)$. Let B denote the Blaschke product with zeros $\Lambda = \{\lambda_n\}_n$ and let

$$b_{\lambda_n}(z) = \frac{\bar{\lambda}_n}{|\lambda_n|} \frac{\lambda_n - z}{1 - \bar{\lambda}_n z}, \quad B_n(z) = \frac{B(z)}{b_{\lambda_n}(z)}.$$

In these terms $B(z) = \prod_n b_{\lambda_n}(z)$ and $|b_{\lambda_n}(z)| = \rho(z, \lambda_n)$. Given $H \in \text{Har}_+(\mathbb{D})$, consider the disks $\mathcal{D}_n^H = D(\lambda_n, e^{-H(\lambda_n)})$ and the domain

$$\Omega_n^H = \mathbb{D} \setminus \bigcup_{\substack{k:k \neq n \\ \rho(\lambda_k, \lambda_n) \leq 1/2}} \mathcal{D}_k^H.$$

It will be clear from the proof of Theorem 1.2 below that the choice of the constant 1/2 in the definition of Ω_n^H is of no relevance; it can be replaced by any $c \in (0, 1)$. Let $\omega(z, E, \Omega)$ denote the harmonic measure at $z \in \Omega$ of the set $E \subset \partial\Omega$ in the domain Ω . The following result collects several new descriptions of Nevanlinna interpolating sequences which will be used in the proof of Theorem 1.1.

Theorem 1.2. *Let $\Lambda = \{\lambda_n\}_n$ be a Blaschke sequence of distinct points in \mathbb{D} and let B be the Blaschke product with zero set Λ . The following statements are equivalent:*

(a) Λ is an interpolating sequence for N , that is, there exists $H \in \text{Har}_+(\mathbb{D})$ such that

$$(1 - |\lambda_n|^2)|B'(\lambda_n)| = |B_n(\lambda_n)| \geq e^{-H(\lambda_n)}, \quad n \in \mathbb{N}.$$

(b) There exists $H \in \text{Har}_+(\mathbb{D})$ such that $|B(z)| \geq e^{-H(z)}\rho(z, \Lambda)$, $z \in \mathbb{D}$,

(c) There exists $H \in \text{Har}_+(\mathbb{D})$ such that $|B(z)| + (1 - |z|^2)|B'(z)| \geq e^{-H(z)}$, $z \in \mathbb{D}$,

(d) There exists $H \in \text{Har}_+(\mathbb{D})$ such that the disks \mathcal{D}_k^H are pairwise disjoint, and

$$\inf_{n \in \mathbb{N}} \omega(\lambda_n, \partial\mathbb{D}, \Omega_n^H) > 0.$$

Statement (d) and its proof are modelled after the corresponding version for H^∞ , proved in [6]. Descriptions of interpolating and sampling sequences in Bergman spaces in terms of harmonic measure can be found in [22]. It will be clear from the proof that (d) can be replaced by a seemingly stronger statement: for every $\varepsilon \in (0, 1)$ there exists $H \in \text{Har}_+(\mathbb{D})$ such that the disks \mathcal{D}_k^H are pairwise disjoint, and

$$\inf_{n \in \mathbb{N}} \omega(\lambda_n, \partial\mathbb{D}, \Omega_n^H) \geq 1 - \varepsilon.$$

The paper is organized as follows. In the next section we shall prove Theorem 1.2 and some Corollaries which will be used later. Section 3 is devoted to the the equivalence of the statements (a), (b) and (c) of Theorem 1.1 and Section 4 to condition (d) in the case $m = 2$. At the end of Section 4 it is also explained that when $m > 2$ then condition (d) does not imply any of the previous ones. The last Section is devoted to present two related open problems. The first one concerns the stable rank of N and the second is a version of the well known f^2 problem of T. Wolff (see [5, p. 319]), solved by S. Treil in the context of H^∞ [27].

A final word about notation. Throughout the paper $A \lesssim B$ will mean that there is an absolute constant C such that $A \leq CB$, and we write $A \asymp B$ if both $A \lesssim B$ and $B \lesssim A$.

It is a pleasure to thank Raymond Mortini for drawing our attention to the Corona Theorem in the Nevanlinna class and to his paper [17].

2. INTERPOLATING SEQUENCES IN THE NEVANLINNA CLASS

We start with an elementary lemma.

Lemma 2.1. *Let $f \in H^\infty$ with $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)| \leq 1$.*

(a) *For all $z, \lambda \in \mathbb{D}$,*

$$|f(w) - f(\lambda)| \leq 2\rho(w, \lambda).$$

(b) *Fix $0 < \delta < 1/5$. If $|f(z)| \leq \delta^4$ and $(1 - |z|^2)|f'(z)| \geq \delta$ for a fixed $z \in \mathbb{D}$, then $|f(w)| \geq \delta^4$ if $\rho(z, w) = \delta^2$.*

(c) If $\rho(z, w) \leq 1/2$, then

$$|(1 - |z|^2)f'(z) - (1 - |w|^2)f'(w)| \leq 6\rho(z, w).$$

Proof of the Lemma 2.1. (a) This is a direct consequence of Schwarz' Lemma:

$$\rho(f(w), f(\lambda)) \leq \rho(w, \lambda) \quad z, \lambda \in \mathbb{D}.$$

(b) Assume first that $z = 0$ and write $f(w) = f(0) + f'(0)w + w^2g(w)$. Since $\|f\|_\infty \leq 1$ and $|f'(0)| \leq 1$, we have $|g(w)| \leq 3$ for every $w \in \mathbb{D}$, and hence,

$$|f(w)| \geq |f'(0)||w| - |f(0)| - 3|w|^2, \quad w \in \mathbb{D}.$$

Since $\delta \leq 1/5$, then for $|w| = \delta^2$ we have $|f(w)| \geq \delta^3 - \delta^4 - 3\delta^4 \geq \delta^4$, as desired.

For arbitrary $z \in \mathbb{D}$ we apply the previous argument to the function $f \circ \phi_z$, where

$$\phi_z(w) = \frac{z - w}{1 - \bar{z}w}$$

is the holomorphic automorphism of \mathbb{D} exchanging 0 and z . Since $|(f \circ \phi_z)'(0)| = (1 - |z|^2)|f'(z)| \geq \delta$ and $|(f \circ \phi_z)(0)| = |f(z)| \leq \delta^4$, taking $\zeta \in \mathbb{D}$ such that $w = \phi_z(\zeta)$, we get $|f(w)| = |(f \circ \phi_z)(\zeta)| \geq \delta^4$ if $\rho(z, w) = |\zeta| = \delta^2$.

(c) Again, assume first that $z = 0$. If $|\zeta| \leq 1/2$ then

$$|f'(0) - (1 - |\zeta|^2)f'(\zeta)| \leq |f'(0) - f'(\zeta)| + |\zeta|^2|f'(\zeta)| \leq |f'(0) - f'(\zeta)| + \frac{|\zeta|^2}{1 - |\zeta|^2}.$$

Let $g(\zeta) = f'(\zeta) - f'(0)$. For $|\zeta| \leq 1/2$ we have

$$|g(\zeta)| = |f'(\zeta) - f'(0)| \leq \frac{1}{1 - |\zeta|^2} + 1 \leq \frac{7}{3}.$$

Applying (a) to $h(z) := 3/7 g(z/2)$ we deduce that

$$|g(\zeta)| \leq \frac{14}{3}|\zeta|, \quad |\zeta| \leq 1/2.$$

Finally, if $|\zeta| \leq 1/2$, from the above estimate we deduce that

$$|f'(0) - (1 - |\zeta|^2)f'(\zeta)| \leq \frac{14}{3}|\zeta| + \frac{|\zeta|^2}{1 - |\zeta|^2} \leq \frac{16}{3}|\zeta| \leq 6|\zeta|,$$

as desired.

For general $z \in \mathbb{D}$ we use the case $z = 0$ and the invariance by automorphisms of $\tilde{\nabla}f(z) = (1 - |z|^2)f'(z)$, that is, $\tilde{\nabla}(f \circ \phi_z)(\zeta) = (\tilde{\nabla}f)(\phi_z(\zeta))$ for any $\zeta, z \in \mathbb{D}$. Then, for $|\zeta| \leq 1/2$,

$$|(1 - |z|^2)f'(z) - (1 - |\zeta|^2)(f \circ \phi_z)'(\zeta)| = |(f \circ \phi_z)'(0) - (1 - |\zeta|^2)(f \circ \phi_z)'(\zeta)| \leq 6|\zeta|.$$

Letting $\zeta = \phi_z(w)$ and using the invariance we see that $(1 - |\zeta|^2)(f \circ \phi_z)'(\zeta) = (1 - |w|^2)f'(w)$ and the result follows. \blacksquare

In the proofs we will repeatedly use the well-known *Harnack inequalities*: for $H \in \text{Har}_+(\mathbb{D})$ and $z, w \in \mathbb{D}$,

$$(2.1) \quad \frac{1 - \rho(z, w)}{1 + \rho(z, w)} \leq \frac{H(z)}{H(w)} \leq \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.$$

In certain parts of this paper, we will need to suppose that z, w are pseudohyperbolically close: $\rho(z, w) < x$ for some $0 < x < 1$, so that $(x - 1)/(x + 1) \leq H(z)/H(w) \leq (x + 1)/(x - 1)$. The constant $(x + 1)/(x - 1)$ will occasionally be called the Harnack constant.

In this section we shall always assume, without loss of generality, that positive harmonic functions $H \in \text{Har}_+(\mathbb{D})$ defining pseudohyperbolic neighborhoods $D(\lambda, e^{-H(\lambda)})$ are big enough so that the corresponding Harnack constant is at most 2. More specifically, let $H \in \text{Har}_+(\mathbb{D})$ be such that $H(z) \geq \ln 3$ for any $z \in \mathbb{D}$; then

$$(2.2) \quad \frac{H(w)}{2} \leq H(z) \leq 2H(w) \quad \text{if} \quad \rho(z, w) \leq e^{-H(z)}.$$

Here is another easy and useful fact.

Lemma 2.2. *There exists a universal constant $C > 0$ such that for any $f \in N$ with $|f(z)| \leq e^{H(z)}$, $z \in \mathbb{D}$, for some $H \in \text{Har}_+(\mathbb{D})$, one has*

- (a) *For every $z \in \mathbb{D}$, $(1 - |z|)|f'(z)| \leq e^{CH(z)}$, $(1 - |z|)^2|f''(z)| \leq e^{CH(z)}$.*
- (b) *For every $z, w \in \mathbb{D}$ with $\rho(z, w) \leq 1/3$, $|f(z) - f(w)| \leq \rho(z, w)e^{CH(z)}$.*

Proof. The estimates in (a) are an easy consequence of Cauchy's formula and Harnack's inequality. The estimate in (b) follows immediately from (a) integrating f' from z to w and using again Harnack's inequality. \blacksquare

Proof of Theorem 1.2. (a) \implies (b). By hypothesis there exists $H_0 \in \text{Har}_+(\mathbb{D})$ satisfying Theorem 1.2(a), and therefore the disks $\mathcal{D}_n^{H_0} = D(\lambda_n, e^{-H_0(\lambda_n)})$ are pairwise disjoint. We will show that condition (b) holds with $H = CH_0$, where C is an absolute constant. Consider the disks $\mathcal{D}_n^{2H_0} = D(\lambda_n, e^{-2H_0(\lambda_n)})$.

- i) Pick $z \in \mathcal{D}_n^{2H_0}$. By construction, λ_n is the closest point of Λ to z and

$$|B(z)| = |B_n(z)||b_{\lambda_n}(z)| = |B_n(z)|\rho(z, \Lambda)$$

Since B_n does not vanish in $\mathcal{D}_n^{2H_0}$, by Harnack's inequalities (2.1) and (2.2), there exists an absolute constant $C > 0$ such that

$$|B_n(z)| \geq |B_n(\lambda_n)|^C \geq e^{-CH_0(\lambda_n)} \geq e^{-2CH_0(z)}.$$

- ii) Let $\Omega := \mathbb{D} \setminus \cup_n \mathcal{D}_n^{2H_0}$. The function B is holomorphic and non-vanishing in Ω . Let F be the holomorphic function with $\text{Re } F = 4CH_0$ on \mathbb{D} . Then $G = Be^F$ is also holomorphic and non-vanishing on Ω . For $z \in \partial\mathcal{D}_n^{2H_0}$, from the preceding case we know that

$$|G(z)| = |B(z)|e^{4CH_0(z)} = |B_n(z)|\rho(z, \Lambda_n)e^{4CH_0(z)} \geq e^{-2CH_0(z) - H_0(\lambda_n) + 4CH_0(z)} \geq 1.$$

For $z \in \partial\mathbb{D}$ we have $|G(z)| = e^{4CH_0(z)} \geq 1$. Hence throughout Ω we have $|G| \geq 1$, that is, $|B(z)| \geq |e^{-F(z)}| = e^{-4CH_0(z)}$ for $z \in \Omega$.

(b) \implies (c). We can assume that the function H in (b) satisfies $\inf\{H(z) : z \in \mathbb{D}\} \geq \ln 3$. Separate into two cases.

i) If $\rho(z, \Lambda) \geq e^{-10H(z)}$ then, by hypothesis, $|B(z)| + (1 - |z|^2)|B'(z)| \geq |B(z)| \geq e^{-11H(z)}$.

ii) If $\rho(z, \Lambda) \leq e^{-10H(z)}$ there exists a unique λ_n such that $\rho(z, \Lambda) = \rho(z, \lambda_n)$. Then by hypothesis

$$\left| (1 - \bar{\lambda}_n z) \frac{B(z)}{z - \lambda_n} \right| \geq e^{-H(z)}, \quad z \neq \lambda_n,$$

and taking the limit as $z \rightarrow \lambda_n$, we deduce that $(1 - |\lambda_n|^2)|B'(\lambda_n)| \geq e^{-H(\lambda_n)}$. Finally, by Lemma 2.1(c) and by Harnack's inequality (2.2)

$$(1 - |z|^2)|B'(z)| \geq e^{-H(\lambda_n)} - 6\rho(z, \lambda_n) \geq e^{-2H(z)} - e^{-8H(z)} \geq \frac{1}{2}e^{-2H(z)} \geq e^{-3H(z)},$$

and therefore

$$|B(z)| + (1 - |z|^2)|B'(z)| \geq e^{-11H(z)}, \quad z \in \mathbb{D}.$$

(c) \implies (a). This implication is immediate taking $z = \lambda_n$.

(a) \implies (d). Let $H \in \text{Har}_+(\mathbb{D})$ such that $|B_n(\lambda_n)| \geq e^{-H(\lambda_n)}$, $n \in \mathbb{N}$, that is,

$$(2.3) \quad \sum_{k:k \neq n} \log \frac{1}{\rho(\lambda_n, \lambda_k)} \leq H(\lambda_n), \quad n \in \mathbb{N}.$$

Again the disks \mathcal{D}_n^H are disjoint, and so will be the smaller disks \mathcal{D}_n^{4H} . By definition

$$\omega(\lambda_n, \partial\mathbb{D}, \Omega_n^H) = 1 - \sum_{\substack{k:k \neq n \\ \rho(\lambda_n, \lambda_k) \leq 1/2}} \omega(\lambda_n, \partial\mathcal{D}_k^H, \Omega_n^H).$$

Since

$$\omega(z, \partial\mathcal{D}_k^{4H}, \mathbb{D} \setminus \mathcal{D}_k^{4H}) = \frac{\log(1/\rho(z, \lambda_k))}{4H(\lambda_k)},$$

estimate (2.3) is equivalent to

$$(2.4) \quad \sup_{n \in \mathbb{N}} \sum_{k:k \neq n} \omega(\lambda_n, \partial\mathcal{D}_k^{4H}, \mathbb{D} \setminus \mathcal{D}_k^{4H}) \frac{4H(\lambda_k)}{H(\lambda_n)} \leq 1.$$

If $\rho(\lambda_n, \lambda_k) \leq 1/2$ Harnack's inequalities (2.1) imply that $1/3 \leq H(\lambda_n)/H(\lambda_k) \leq 3$. Thus, by (2.4),

$$\begin{aligned} \sum_{\substack{k:k \neq n \\ \rho(\lambda_n, \lambda_k) \leq 1/2}} \omega(\lambda_n, \partial\mathcal{D}_k^{4H}, \Omega_n^{4H}) &\leq \sum_{\substack{k:k \neq n \\ \rho(\lambda_n, \lambda_k) \leq 1/2}} \omega(\lambda_n, \partial\mathcal{D}_k^{4H}, \mathbb{D} \setminus \mathcal{D}_k^{4H}) \\ &\leq 3 \sum_{\substack{k:k \neq n \\ \rho(\lambda_n, \lambda_k) \leq 1/2}} \omega(\lambda_n, \partial\mathcal{D}_k^{4H}, \mathbb{D} \setminus \mathcal{D}_k^{4H}) \frac{H(\lambda_k)}{H(\lambda_n)} \leq \frac{3}{4}, \end{aligned}$$

and therefore

$$\omega(\lambda_n, \partial\mathbb{D}, \Omega_n^{4H}) \geq \frac{1}{4}.$$

Observe that by replacing $4H$ by NH in the above reasoning it is possible to get $\omega(\lambda_n, \partial\mathbb{D}, \Omega_n^{NH}) \geq 1 - 3/N$.

(d) \implies (a). For simplicity we drop the superscript H in the notations \mathcal{D}_n^H and Ω_n^H , and let $\delta_n = e^{-H(\lambda_n)}$. Let $\varepsilon = \inf_{n \in \mathbb{N}} \omega(\lambda_n, \partial\mathbb{D}, \Omega_n) > 0$ and consider the bigger domains

$$\tilde{\Omega}_n = \mathbb{D} \setminus \bigcup_{\substack{k: k \neq n \\ \rho(\lambda_n, \lambda_k) \leq 1/4}} \mathcal{D}_k.$$

Notice that then $\omega(\lambda_n, \partial\mathbb{D}, \tilde{\Omega}_n) \geq \omega(\lambda_n, \partial\mathbb{D}, \Omega_n) \geq \varepsilon$. Given $N \geq 1$, to be determined later on, let $\Delta_n = D(\lambda_n, \delta_n^N) \subset D(\lambda_n, \delta_n)$ and

$$\mathcal{V}_n = \mathbb{D} \setminus \bigcup_{\substack{k: k \neq n \\ \rho(\lambda_n, \lambda_k) \leq 1/4}} \Delta_k.$$

Notice that $\Omega_n \subset \tilde{\Omega}_n \subset \mathcal{V}_n$. Define the harmonic functions

$$U_n(z) = \omega(z, \partial\mathbb{D}, \Omega_n) \quad \text{and} \quad u_n(z) = \omega(z, \partial\mathbb{D}, \mathcal{V}_n).$$

Then $u_n(z) \geq U_n(z)$ for $z \in \Omega_n$. In particular $u_n(\lambda_n) \geq \varepsilon > 0$, $n \in \mathbb{N}$. We apply Green's formula to the functions $\Phi(z) = \log(1/\rho(z, \lambda_n))$ and u_n on the domain \mathcal{V}_n :

$$\begin{aligned} u_n(\lambda_n) &= -\frac{1}{2\pi} \int_{\mathcal{V}_n} u_n \Delta \Phi dm \\ &= \frac{1}{2\pi} \int_0^{2\pi} P(\lambda_n, e^{i\theta}) d\theta - \sum_{\substack{k: k \neq n \\ \rho(\lambda_n, \lambda_k) \leq 1/4}} \frac{1}{2\pi} \int_{\partial\Delta_k} \log\left(\frac{1}{\rho(\lambda_n, \zeta)}\right) \frac{\partial u_n}{\partial n}(\zeta) d\sigma(\zeta), \end{aligned}$$

where $\partial/\partial n$ indicates the outer normal derivative. Using the hypothesis and the fact that for $\zeta \in \partial\Delta_k$ one has

$$\log\left(\frac{1}{\rho(\lambda_n, \zeta)}\right) \asymp \log\left(\frac{1}{\rho(\lambda_n, \lambda_k)}\right),$$

we deduce that

$$\begin{aligned} &\sum_{\substack{k: k \neq n \\ \rho(\lambda_n, \lambda_k) \leq 1/4}} \log\left(\frac{1}{\rho(\lambda_n, \lambda_k)}\right) \frac{1}{2\pi} \int_{\partial\Delta_k} \frac{\partial u_n}{\partial n}(\zeta) d\sigma(\zeta) \\ &\lesssim \sum_{\substack{k: k \neq n \\ \rho(\lambda_n, \lambda_k) \leq 1/4}} \frac{1}{2\pi} \int_{\partial\Delta_k} \log\left(\frac{1}{\rho(\lambda_n, \zeta)}\right) \frac{\partial u_n}{\partial n}(\zeta) d\sigma(\zeta) \leq 1 - \varepsilon. \end{aligned}$$

Taking into account (1.3) we will be done as soon as we prove that $\frac{\partial u_n}{\partial n}(\zeta) \geq 0$, $\zeta \in \partial\Delta_k$ and

$$(2.5) \quad \int_{\partial\Delta_k} \frac{\partial u_n}{\partial n}(\zeta) d\sigma(\zeta) \gtrsim \frac{1}{H(\lambda_n)}, \quad k \neq n.$$

Define for $k \neq n$,

$$\begin{aligned} u_{n,k}(z) &= \omega(z, \partial\mathbb{D}, \mathcal{V}_n \cup \Delta_k), \\ v_k(z) &= \omega(z, \partial\Delta_k, \mathbb{D} \setminus \Delta_k) = \frac{\log(1/\rho(z, \lambda_k))}{\log(1/\delta_n^N)} \end{aligned}$$

and notice that, again by the maximum principle,

$$(2.6) \quad u_n \geq u_{n,k} - v_k \quad \text{on } \mathcal{V}_n.$$

For λ_k, λ_j such that $\rho(\lambda_n, \lambda_k), \rho(\lambda_k, \lambda_j) \leq 1/4$ we have $\rho(\lambda_n, \lambda_j) \leq 1/2$ and therefore

$$u_{n,k}(\lambda_k) \geq \omega\left(\lambda_k, \partial\mathbb{D}, \mathbb{D} \setminus \bigcup_{\substack{j \neq k, n \\ \rho(\lambda_j, \lambda_k) \leq 1/2}} \Delta_j\right) \geq \omega(\lambda_k, \partial\mathbb{D}, \Omega_k) \geq \varepsilon > 0.$$

By Harnack's inequalities there exists $\varepsilon' = \varepsilon'(\varepsilon) > 0$ such that $u_{n,k}(z) \geq \varepsilon' > 0$ for $z \in \partial\mathcal{D}_k$. Also, for $z \in \partial\mathcal{D}_k$,

$$v_k(z) = \frac{\log(1/\delta_k)}{\log(1/\delta_k^N)} = \frac{1}{N}$$

and inequality (2.6) yields

$$u_n(z) \geq u_{n,k}(z) - v_k(z) \geq \varepsilon' - \frac{1}{N}, \quad z \in \partial\mathcal{D}_k.$$

Choose N so that $1/N < \varepsilon'/2$. Then $u_n(z) \geq \varepsilon'/2$ for $z \in \partial\mathcal{D}_k, k \neq n$, and by the maximum principle

$$u_n(z) \geq \frac{\varepsilon'}{2} \omega_k(z), \quad z \in \mathcal{D}_k \setminus \Delta_k,$$

where

$$\omega_k(z) = \omega(z, \partial\mathcal{D}_k, \mathcal{D}_k \setminus \Delta_k) = \frac{\log(\rho(z, \lambda_k)/\delta_k^N)}{\log(1/\delta_k^{N-1})}.$$

Since $\log(1/\delta_k) = H(\lambda_k)$, this inequality implies that for $\zeta \in \partial\Delta_k$

$$\frac{\partial u_n}{\partial n}(\zeta) \geq \frac{\varepsilon'}{2} \frac{\partial \omega_k}{\partial n}(\zeta) \gtrsim \frac{1}{H(\lambda_k)} \frac{\partial}{\partial n} \log \rho(\zeta, \lambda_k)$$

and therefore

$$\int_{\partial\Delta_k} \frac{\partial u_n}{\partial n}(\zeta) d\sigma(\zeta) \gtrsim \frac{1}{H(\lambda_k)} \int_{\partial\Delta_k} \frac{\partial}{\partial n} \log \rho(z, \lambda_k) d\sigma(\zeta).$$

Finally, we use Green's formula with $u \equiv 1, v(\zeta) = \log \rho(\zeta, \lambda_k)$ and the domain $\mathbb{D} \setminus \Delta_k$:

$$\int_{\partial\Delta_k} \frac{\partial}{\partial n} \log \rho(\zeta, \lambda_k) d\sigma(\zeta) = \int_{\partial\mathbb{D}} \frac{\partial}{\partial n} \log \rho(\zeta, \lambda_k) d\sigma(\zeta) = \int_{\partial\mathbb{D}} P(\lambda_k, \zeta) d\sigma(\zeta) = 1.$$

■

We end this section with two easy consequences which will be used later. The first one says that Nevanlinna interpolating sequences are stable under convenient pseudohyperbolic perturbations, and will be deduced from Theorem 1.2(d).

Corollary 2.3. *Let $\Lambda = \{\lambda_n\}$ be a Nevanlinna interpolating sequence and let $H \in \text{Har}_+(\mathbb{D})$, satisfying Theorem 1.2(a). If $\Lambda' = \{\lambda'_n\}_n \subset \mathbb{D}$ satisfies*

$$\rho(\lambda_n, \lambda'_n) \leq \frac{1}{4} e^{-H(\lambda_n)}, \quad n \in \mathbb{N},$$

then Λ' is also a Nevanlinna interpolating sequence.

Proof. We shall use the characterization of Nevanlinna interpolating sequences given in Theorem 1.2(d). Consider the domains

$$\Omega_n = \mathbb{D} \setminus \bigcup_{\substack{k:k \neq n \\ \rho(\lambda_k, \lambda_n) \leq 1/2}} D(\lambda_k, e^{-H(\lambda_k)}) \quad , \quad \Omega'_n = \mathbb{D} \setminus \bigcup_{\substack{k:k \neq n \\ \rho(\lambda'_k, \lambda'_n) \leq 1/4}} D(\lambda'_k, e^{-2H(\lambda'_k)})$$

Then $\Omega_n \subset \Omega'_n$, and by Harnack's inequality there exists $c > 0$ such that

$$\omega(\lambda'_n, \partial\mathbb{D}, \Omega'_n) \geq c\omega(\lambda_n, \partial\mathbb{D}, \Omega'_n) \geq c\omega(\lambda_n, \partial\mathbb{D}, \Omega_n).$$

The result follows then from the hypothesis. \blacksquare

Corollary 2.4. *Let Λ be a Nevanlinna interpolating sequence and let $H \in \text{Har}_+(\mathbb{D})$ be such that $\inf\{H(z) : z \in \mathbb{D}\} \geq \ln 3$ and $|B(z)| \geq e^{-H(z)}\rho(z, \Lambda)$, $z \in \mathbb{D}$. Then for every $H_1 \in \text{Har}_+(\mathbb{D})$ with $\inf\{H_1(z) : z \in \mathbb{D}\} \geq \ln 3$, we have*

$$|B(z)| \geq e^{-(2H(z)+2H_1(z))} \quad \text{whenever } z \notin \cup_n \mathcal{D}_n^{H_1}.$$

Proof of Corollary 2.4. Suppose first $z \notin \cup_n D(\lambda_n, 1/2)$. Then $\rho(z, \Lambda) \geq 1/2$ and

$$|B(z)| \geq e^{-H(z)}\rho(z, \Lambda) \geq \frac{1}{2}e^{-H(z)} \geq e^{-2H(z)}.$$

Next, if $z \in \cup_n D(\lambda_n, 1/2)$ picking the closest point $\lambda_0 \in \Lambda$ with $\rho(z, \Lambda) = \rho(z, \lambda_0) \geq e^{-H_1(\lambda_0)}$, Harnack's inequality (2.2) gives

$$|B(z)| \geq e^{-H(z)}\rho(z, \Lambda) = e^{-H(z)}\rho(z, \lambda_0) \geq e^{-H(z)-H_1(\lambda_0)} \geq e^{-H(z)-2H_1(z)}.$$

\blacksquare

3. PROOF OF THEOREM 1.1

Notice first that we can assume throughout the proof that the functions f_i are Blaschke products. For conditions (a), (b) and (d) this is easily seen by considering the Nevanlinna factorization $f_i = B_i e^{g_i}$, where B_i is the Blaschke product with the zeros of f_i and g_i is such that $\text{Re}(g_i) = H_i^+ - H_i^-$, for some $H_i^+, H_i^- \in \text{Har}_+(\mathbb{D})$. Then, since e^{g_i} , $i = 1, \dots, m$, are invertible functions in N , we have $I(f_1, \dots, f_m) = I(B_1, \dots, B_m)$ and $J(f_1, \dots, f_m) = J(B_1, \dots, B_m)$. As for condition (c), let us now see that there exists $H \in \text{Har}_+(\mathbb{D})$ such that

$$(3.1) \quad \sum_{i=1}^m (|B_i(z)| + (1 - |z|^2)|B'_i(z)|) \geq e^{-H(z)}, \quad z \in \mathbb{D}$$

if and only if (c) holds with a suitable, possibly different, $H \in \text{Har}_+(\mathbb{D})$.

Let us first suppose that (3.1) holds. Let $E_i = e^{g_i}$, $i = 1, \dots, m$, and take $H_1 \in \text{Har}_+(\mathbb{D})$ such that

$$|\log |E_i(z)|| = |\text{Re}(g_i(z))| \leq H_1(z), \quad z \in \mathbb{D}, \quad i = 1, \dots, m.$$

Recall from Lemma 2.2 that

$$(3.2) \quad (1 - |z|^2)|E'_i(z)| \leq e^{CH_1(z)}, \quad z \in \mathbb{D},$$

where $C > 0$ is an absolute constant. Fix $z \in \mathbb{D}$. We shall distinguish two cases.

(i) Assume first that $z \in \mathbb{D}$ is such that

$$\sum_{i=1}^m |B_i(z)| \geq \frac{1}{4} e^{-H(z)-(1+C)H_1(z)}.$$

Then

$$\sum_{i=1}^m |f_i(z)| = \sum_{i=1}^m |B_i(z)||E_i(z)| \geq \frac{1}{4} e^{-H(z)-(2+C)H_1(z)}$$

and (c) holds.

(ii) Assume now that

$$\sum_{i=1}^m |B_i(z)| \leq \frac{1}{4} e^{-H(z)-(1+C)H_1(z)},$$

which is in particular bounded by $\frac{1}{4} e^{-H(z)}$. Then by (3.1) we have

$$\sum_{i=1}^m (1 - |z|^2) |B'_i(z)| \geq \frac{3}{4} e^{-H(z)}.$$

Therefore

$$\sum_{i=1}^m (1 - |z|^2) |B'_i(z)||E_i(z)| \geq e^{-H_1(z)} \frac{3}{4} e^{-H(z)},$$

and by (3.2)

$$\sum_{i=1}^m (1 - |z|^2) |B_i(z)||E'_i(z)| \leq \frac{1}{4} e^{-H(z)-H_1(z)}$$

Thus

$$\begin{aligned} \sum_{i=1}^m (1 - |z|^2) |f'_i(z)| &\geq \sum_{i=1}^m (1 - |z|^2) |B'_i(z)||E_i(z)| - \sum_{i=1}^m (1 - |z|^2) |B_i(z)||E'_i(z)| \\ &\geq \frac{3}{4} e^{-H(z)-H_1(z)} - \frac{1}{4} e^{-H(z)-H_1(z)} = \frac{1}{2} e^{-(H(z)+H_1(z))} \end{aligned}$$

and so (c) holds.

The converse is based on exactly the same argument. Observe that we can write $B_i = f_i/E_i = f_i \mathcal{E}_i$ where \mathcal{E}_i is an invertible function in N for which we get similar estimates as for E_i . Now, replacing in the arguments above B_i by f_i and E_i by \mathcal{E}_i , we will reach (3.1) when starting from (c).

Before giving the proof of Theorem 1.1 we shall see that (a) implies

$$I(f_1, \dots, f_m) = J(f_1, \dots, f_m).$$

We only have to show the reverse inclusion. For this, let $g \in J(f_1, \dots, f_m)$ and let $H \in \text{Har}_+(\mathbb{D})$ be such that

$$|g(z)| \leq e^{H(z)} \sum_{i=1}^m |f_i(z)|, \quad z \in \mathbb{D}.$$

Let B be a Nevanlinna interpolating Blaschke product in $I(f_1, \dots, f_m)$ and denote by $\Lambda = \{\lambda_n\}_n$ its zero set. Since for any $i = 1, \dots, m$, we have

$$\frac{|g(\lambda_n)\overline{f_i(\lambda_n)}|}{\sum_{i=1}^m |f_i(\lambda_n)|^2} \leq \frac{e^{H(\lambda_n)}(\sum_{i=1}^m |f_i(\lambda_n)|)^2}{\sum_{i=1}^m |f_i(\lambda_n)|^2} \leq me^{H(\lambda_n)}, \quad n \in \mathbb{N},$$

using the description of the trace space $N|\Lambda$ in (1.2) we see that there exist $h_i \in N$ such that

$$h_i(\lambda_n) = \frac{g(\lambda_n)\overline{f_i(\lambda_n)}}{\sum_{i=1}^m |f_i(\lambda_n)|^2}, \quad n \in \mathbb{N}.$$

Consequently, the function $\sum_{i=1}^m f_i h_i - g$ vanishes on Λ , and therefore there exists $G \in N$ such that

$$\sum_{i=1}^m f_i h_i - g = BG.$$

Since $BG \in I(f_1, \dots, f_m)$, this shows that $g \in I(f_1, \dots, f_m)$ as well.

Let us now move to the proof of Theorem 1.1.

(a) \implies (b) is obvious because $I(f_1, \dots, f_m) \subset J(f_1, \dots, f_m)$.

(b) \implies (c). Assume that $B \in J(f_1, \dots, f_m)$ is a Nevanlinna interpolating Blaschke product and let $\Lambda = \{\lambda_n\}_n$ denote its zero set. By definition and by Theorem 1.2(b) there exist $H, H_1 \in \text{Har}_+(\mathbb{D})$ such that

$$(3.3) \quad \rho(z, \Lambda)e^{-H_1(z)} \leq |B(z)| \leq e^{H(z)} \sum_{i=1}^m |f_i(z)|, \quad z \in \mathbb{D}.$$

Recall from Lemma 2.2 that there exists $H_2 \in \text{Har}_+(\mathbb{D})$ such that

$$|f_i(z)| + (1 - |z|)|f'_i(z)| + (1 - |z|)^2|f''_i(z)| \leq e^{H_2(z)}, \quad z \in \mathbb{D}, \quad i = 1, \dots, m.$$

Now let $H_3 \in \text{Har}_+(\mathbb{D})$, $H_3 \geq H + H_1 + H_2 + \ln 3$ to be chosen later. Observe that the disks $\mathcal{D}_n = \mathcal{D}_n^{H_3} = D(\lambda_n, e^{-H_3(\lambda_n)})$ are disjoint. Observe also that (2.2) holds. By (3.3) and Corollary 2.4, we have

$$\sum_{i=1}^m |f_i(z)| \geq e^{-2(H(z)+H_1(z)+H_3(z))}, \quad z \notin \bigcup_{n \geq 1} \mathcal{D}_n.$$

So, it only remains to discuss the estimate on \mathcal{D}_n . We will prove that

$$(3.4) \quad \sum_{i=1}^m |f_i(z)| + (1 - |z|)|f'_i(z)| \geq e^{-6H_3(z)}, \quad z \in \mathcal{D}_n.$$

We argue by contradiction. Suppose there is a $z \in \mathcal{D}_n$ where this estimate does not hold. Let u be the closest point of $\partial\mathcal{D}_n = \partial\mathcal{D}_n^{H_3}$ to z , that is, $u \in \partial\mathcal{D}_n$ and $\rho(z, u) = \rho(z, \partial\mathcal{D}_n)$. Then using a Taylor expansion at z , as Tolokonnikov did in the H^∞ -case, for every $i = 1, \dots, m$, one has

$$\begin{aligned} |f_i(u)| &= \left| f_i(z) + f'_i(z)(u - z) + \int_z^u (u - t)f''_i(t)dt \right| \\ &\lesssim |f_i(z)| + (1 - |z|)|f'_i(z)|\rho(z, u) + (1 - |z|)^2 \sup_{v \in [z, u]} |f''_i(v)|\rho(z, u)^2 \\ &\lesssim e^{-6H_3(z)} + e^{-6H_3(z) - H_3(\lambda_n)} + e^{H_2(v) - 2H_3(\lambda_n)}, \end{aligned}$$

where v is a suitable point in \mathcal{D}_n . Since $\rho(u, \Lambda) = e^{-H_3(\lambda_n)}$, using (3.3) we deduce

$$e^{-(H(u)+H_1(u))}e^{-H_3(\lambda_n)} \leq \sum_{i=1}^m |f_i(u)| \lesssim m \left(2e^{-6H_3(z)} + e^{H_2(v)-2H_3(\lambda_n)} \right).$$

Harnack's inequality (2.2) gives $H_3(z) \geq H_3(\lambda_n)/2$ and we deduce

$$e^{-(H(u)+H_1(u))}e^{-H_3(\lambda_n)} \leq \sum_{i=1}^m |f_i(u)| \lesssim m \left(2e^{-3H_3(\lambda_n)} + e^{H_2(v)-2H_3(\lambda_n)} \right).$$

Since the functions H , H_1 and H_2 are fixed and H_3 can be taken arbitrarily large, we obtain a contradiction. Hence (3.4) holds and the statement (c) follows.

(c) \implies (a). First of all recall that in condition (c) we can assume that the functions f_i are Blaschke products. We can also assume that the positive harmonic function H appearing in condition (c) satisfies $\inf\{H(z) : z \in \mathbb{D}\} > \ln(3m)$. Then Harnack's inequality (2.1) gives that for any $h \in \text{Har}_+(\mathbb{D})$ one has

$$(3.5) \quad \frac{4}{5} \leq \frac{h(z)}{h(w)} \leq \frac{5}{4} \quad \text{if } \rho(w, z) < e^{-2H(z)}.$$

Now take $C > 1$ big enough to be determined later on, and let

$$E = \left\{ z \in \mathbb{D} : \sum_{i=1}^m |f_i(z)| \leq e^{-CH(z)} \right\} = \cup_n E_n,$$

where E_n are the connected components of E . For every $n \in \mathbb{N}$ choose $\lambda_n \in E_n$, if any, such that

$$\sum_{i=1}^m |f_i(\lambda_n)| \leq e^{-2CH(\lambda_n)},$$

and let $\Lambda = \{\lambda_n\}_n$ (we discard those E_n for which such a λ_n does not exist and keep the indexation with \mathbb{N}). Observe that the sum above is trivially bounded by $e^{-2H(\lambda_n)}$.

Claim 1. Assume $C \geq 24$. Then for every $\lambda_n \in \Lambda$, one has

$$D(\lambda_n, e^{-2CH(\lambda_n)}) \subset E_n \subset D(\lambda_n, e^{-6H(\lambda_n)}).$$

The first inclusion is an immediate consequence of Lemma 2.1(a) and Harnack's inequality (3.5),

$$\begin{aligned} |f_i(z)| &\leq |f_i(\lambda_n)| + 2\rho(z, \lambda_n) \leq e^{-2CH(\lambda_n)} + 2e^{-2CH(\lambda_n)} = 3e^{-2CH(\lambda_n)} \\ &\leq 3e^{-(8/5)CH(z)} \leq e^{-CH(z)}. \end{aligned}$$

In order to see the second inclusion notice that, by hypothesis, on the set E , and so on E_n , the following estimate holds

$$(1 - |z|^2) \sum_{i=1}^m |f'_i(z)| \geq e^{-2H(z)},$$

and in particular there exists i such that $(1 - |\lambda_n|^2)|f'_i(\lambda_n)| \geq e^{-2H(\lambda_n)}/m \geq e^{-3H(\lambda_n)} = \delta$. Thus by Lemma 2.1(b), for every z with $\rho(z, \lambda_n) = e^{-6H(\lambda_n)} = \delta^2$ we have $|f_i(z)| \geq e^{-12H(\lambda_n)}$. By Harnack's inequality (2.2) we get

$$(3.6) \quad |f_i(z)| \geq e^{-24H(z)} \quad \text{if } \rho(z, \lambda_n) = e^{-6H(\lambda_n)}.$$

Thus, taking $C \geq 24$, we get the desired inclusion.

Observe also that $\partial D(\lambda_n, e^{-6H(\lambda_n)}) \cap E = \emptyset$ and in particular

$$\rho(\lambda_n, \lambda_k) \geq \max(e^{-6H(\lambda_n)}, e^{-6H(\lambda_k)}), \quad k \neq n.$$

Lemma 3.1. *The sequence Λ constructed above is interpolating for N .*

Proof. We shall use the characterization given in Theorem 1.2(d). Consider the disks $\mathcal{D}_n^C = D(\lambda_n, e^{-2CH(\lambda_n)})$ and the domains

$$\Omega_n^C = \mathbb{D} \setminus \bigcup_{\substack{k:k \neq n \\ \rho(\lambda_n, \lambda_k) \leq 1/2}} \mathcal{D}_k^C.$$

Since $\mathcal{D}_n^C \subset E_n$, Harnack's inequality (3.5) and the fact that $\|f_i\|_\infty \leq 1$ give that, for every $i = 1, \dots, m$ we have

$$\begin{aligned} \log |f_i(\zeta)| &\leq -CH(\zeta) \leq -\frac{C}{2}H(\lambda_k) \quad \text{if } \zeta \in \partial \mathcal{D}_k^C, \\ \log |f_i(\zeta)| &\leq 0 \quad \text{if } \zeta \in \partial \mathbb{D}. \end{aligned}$$

Hence, by the maximum principle

$$\log |f_i(z)| \leq -\frac{C}{2} \sum_{\substack{k:k \neq n \\ \rho(\lambda_k, \lambda_n) \leq 1/2}} H(\lambda_k) \omega(z, \partial \mathcal{D}_k^C, \Omega_n^C), \quad z \in \Omega_n^C.$$

Notice that, by the separation above, the disk $D(\lambda_n, e^{-6H(\lambda_n)})$ is contained in Ω_n^C . Then, as established in (3.6) there is i such that

$$|f_i(\zeta)| \geq e^{-24H(\zeta)} \quad \text{if } \zeta \in \partial D(\lambda_n, e^{-6H(\lambda_n)}) \subset \Omega_n^C,$$

whence

$$\frac{C}{2} \sum_{\substack{k:k \neq n \\ \rho(\lambda_k, \lambda_n) \leq 1/2}} H(\lambda_k) \omega(\zeta, \partial \mathcal{D}_k^H, \Omega_n^H) \leq 24H(\zeta), \quad \zeta \in \partial D(\lambda_n, e^{-6H(\lambda_n)})$$

By Harnack's inequality applied to both H and $\omega(\cdot, \partial \mathcal{D}_k^C, \Omega_n^C)$, we deduce that

$$\sum_{\substack{k \neq n \\ \rho(\lambda_k, \lambda_n) \leq 1/2}} \omega(\lambda_n, \partial \mathcal{D}_k^H, \Omega_n^H) \leq \frac{192}{C}.$$

Choosing C big enough we finally have

$$\omega(\lambda_n, \partial \mathbb{D}, \Omega_n^H) = 1 - \sum_{\substack{k:k \neq n \\ \rho(\lambda_n, \lambda_k) \leq 1/2}} \omega(\lambda_n, \partial \mathcal{D}_k^H, \Omega_n^H) \geq \frac{1}{2}.$$

■

Notice that, by Theorem 1.2(a) and the proof above, there exists $C_0 > 0$ such that

$$\prod_{k:k \neq n} \rho(\lambda_n, \lambda_k) \geq e^{-C_0 H(\lambda_n)}, \quad n \in \mathbb{N}.$$

Although our choice of $\{\lambda_n\}_n$ depends on C , the constant C_0 is uniform. We indicate to the reader that the Nevanlinna interpolating Blaschke product we are heading for is not constructed with the zero-set Λ but with a sequence close to Λ . This, in view of Lemma 2.3, will guarantee that the new sequence is still interpolating. In the sequel we will need to introduce a new constant $D \gg C \gg C_0$, where $C \geq 24$ is the constant fixed in the preceding discussions. Given a harmonic function G , we denote by \tilde{G} its harmonic conjugate.

Claim 2. For every $n \in \mathbb{N}$ there exists $i \in \{1, \dots, m\}$ such that $g_i := f_i - e^{-12D(H+i\tilde{H})}$ has a unique zero $a_n^{(i)}$ in \mathcal{D}_n^{6H} .

By condition (c) we can assume that for some $i \in \{1, \dots, m\}$ (not necessarily unique) we have $(1 - |\lambda_n|^2)|f'_i(\lambda_n)| \geq e^{-3H(\lambda_n)}$. Since $|f_i(\lambda_n)| \leq e^{-2CH(\lambda_n)}$ and $C \geq 6$, applying again Lemma 2.1(b) we obtain

$$(3.7) \quad |f_i(z)| \geq e^{-12H(z)} \quad \text{for } z \in \partial\mathcal{D}_n^{6H}.$$

We use this and Rouché's theorem to compare the number of zeros of g_i and the function $h_i = f_i - f_i(\lambda_n)$ in \mathcal{D}_n^{6H} . Observe that h_i vanishes at λ_n and $(1 - |\lambda_n|)|h'_i(\lambda_n)| \geq e^{-3H(\lambda_n)}$, so that with Lemma 2.1(b), applied to any $\delta < e^{-6H(\lambda_n)}$, it can be shown that h_i does not vanish at any other point of \mathcal{D}_n^{6H} . Now, for $z \in \partial\mathcal{D}_n^{6H}$, Harnack's inequality (3.5), $D \geq C \geq 24$ and (3.7) give

$$\begin{aligned} |g_i(z) - (f_i(z) - f_i(\lambda_n))| &= |f_i(\lambda_n) - e^{-12D(H+i\tilde{H})(z)}| \leq e^{-2CH(\lambda_n)} + e^{-12DH(z)} \\ &\leq e^{-CH(z)} < |f_i(z) - f_i(\lambda_n)|, \end{aligned}$$

as desired. This proves the Claim. \square

The argument works for every i with $(1 - |\lambda_n|^2)|f'_i(\lambda_n)| \geq e^{-3H(\lambda_n)}$, but we will pick $a_n^{(i)}$ for only one i . We will denote by $i(n)$ the index in $\{1, \dots, m\}$ satisfying Claim 2. The previous argument with Rouché's theorem also allows to show that $\rho(a_n^{(i)}, \lambda_n) \leq e^{-CH(\lambda_n)}$. Since $C \gg C_0$, we deduce from Lemma 2.3 that the sequence $A_i := \{a_n^{(i)}\}_n$ is also interpolating for N . Let I_i denote the Nevanlinna interpolating Blaschke product with zero set A_i .

Claim 3. Assume $C \geq 24$. Then $\sum_{j=1}^m |g_j(z)|/|I_j(z)| \geq e^{-4CH(z)}$ for any $z \in \mathbb{D}$.

To see this consider first $z \notin \cup_n \mathcal{D}_n^{6H}$, so that $\sum_{i=1}^m |f_i(z)| > e^{-CH(z)}$. Hence, there exists f_i such that $|f_i(z)| \geq e^{-2CH(z)}$, and therefore

$$(3.8) \quad \sum_{j=1}^m \left| \frac{g_j(z)}{I_j(z)} \right| \geq |g_i(z)| \geq |f_i(z)| - e^{-12DH(z)} \geq e^{-2CH(z)} - e^{-12DH(z)} \geq e^{-3CH(z)}.$$

Consider now $z \in \mathcal{D}_n^{6H}$. Notice first that for $\zeta \in \partial\mathcal{D}_n^{6H}$ and for $i = i(n)$ by (3.7), we have $|g_i(\zeta)| \geq e^{-3CH(\zeta)}$. Applying the minimum modulus principle to g_i/I_i we deduce that

$$\left| \frac{g_i(z)}{I_i(z)} \right| \geq \inf_{\zeta \in \partial\mathcal{D}_n^{6H}} |g_i(\zeta)| \geq \inf_{\zeta \in \partial\mathcal{D}_n^{6H}} e^{-3CH(\zeta)} \geq e^{-4CH(z)}.$$

This finishes the proof of the Claim. \square

Since $(g_1/I_1, \dots, g_m/I_m)$ is unimodular, by the Corona Theorem for N (see Introduction), there exist $h_i \in N$ such that

$$\sum_{i=1}^m \frac{g_i}{I_i} h_i \equiv 1$$

and

$$\sum_{i=1}^m |h_i(z)| \leq e^{M_0 H(z)}, \quad z \in \mathbb{D}.$$

Here $M_0 = M_0(C) > 0$ is a constant which may depend on C but not D , since the estimate in Claim 3 only depends on C . Since $g_i = f_i - e^{-12D(H+i\tilde{H})}$, we have

$$(3.9) \quad F := \sum_{i=1}^m f_i \left(h_i \frac{\prod_{k=1}^m I_k}{I_i} \right) = \prod_{k=1}^m I_k + e^{-12D(H+i\tilde{H})} \sum_{i=1}^m h_i \frac{\prod_{k=1}^m I_k}{I_i}.$$

Since the function F is obviously in $I(f_1, \dots, f_m)$, we will be done as soon as we show that the zero set of this function is an interpolating sequence for N . In order to consider the zeros of F we will again distinguish two cases.

Observe first that since $\rho(a_n^{(i)}, \lambda_n) \leq e^{-CH(\lambda_n)}$, choosing $C \gg C_0$, and observing that A_i are Nevanlinna interpolating sequences, we will have

$$(3.10) \quad \left| \prod_{k=1}^m I_k(z) \right| \geq e^{-2C_0 H(z)} \quad \text{for } z \in \mathbb{D} \setminus \cup_n \mathcal{D}_n^{6H}.$$

Since

$$\left| e^{-12D(H+i\tilde{H})} \sum_{i=1}^m h_i \frac{\prod_{k=1}^m I_k}{I_i} \right| \leq e^{(-12D+M_0)H},$$

which, choosing D large enough, can be assumed neglectible with respect to $e^{-C_0 H}$, we see that F cannot vanish outside the disks \mathcal{D}_n^{6H} .

To consider the disks \mathcal{D}_n^{6H} , we again use Rouché's theorem to see that F has exactly one zero in such a disk. Since A_i is Nevanlinna interpolating we can then conclude by applying the stability result Lemma 2.3. To apply Rouché's theorem we shall compare the function (3.9) with $\prod_{k=1}^m I_k$. In view of (3.10), for every $n \in \mathbb{N}$ and $z \in \partial \mathcal{D}_n^{6H}$

$$\begin{aligned} & \left| \prod_{k=1}^m I_k(z) + e^{-12D(H+i\tilde{H})(z)} \sum_{i=1}^m h_i(z) \frac{\prod_{k=1}^m I_k(z)}{I_i(z)} - \prod_{k=1}^m I_k(z) \right| \\ & \leq e^{-12DH(z)} \sum_{i=1}^m |h_i(z)| \leq e^{-(12D-M_0)H(z)} \leq e^{-2C_0 H(z)} < \left| \prod_{k=1}^m I_k(z) \right|, \end{aligned}$$

as desired. ■

4. THE CASE OF TWO GENERATORS

In this section we shall assume $m = 2$ and prove the equivalence between condition (d) and (a),(b), (c) in Theorem 1.1. We have already proved that (a) implies that $I(f_1, \dots, f_m) = J(f_1, \dots, f_m)$ for any $m \geq 2$. Hence we only need to prove the sufficiency of condition (d) when $m = 2$. We start with an auxiliary result which allows to reduce the situation to the case where B_1 and B_2 have no common zeros.

Lemma 4.1. *Let \hat{B} be the Blaschke product formed with the common zeros of f_1 and f_2 . Then $I(f_1, f_2) = J(f_1, f_2)$ if and only if $I(f_1/\hat{B}, f_2/\hat{B}) = J(f_1/\hat{B}, f_2/\hat{B})$.*

Proof. If $f \in J(f_1/\hat{B}, f_2/\hat{B})$, then $|f| \leq e^H(|f_1/\hat{B}| + |f_2/\hat{B}|)$ for some $H \in \text{Har}_+(\mathbb{D})$, and so $f\hat{B} \in J(f_1, f_2) = I(f_1, f_2)$ giving $f \in I(f_1/\hat{B}, f_2/\hat{B})$. Conversely, if $f \in J(f_1, f_2)$, then $|f| \leq e^H(|f_1| + |f_2|)$, for some $H \in \text{Har}_+(\mathbb{D})$. In particular \hat{B} divides f . Hence $f/\hat{B} \in J(f_1/\hat{B}, f_2/\hat{B}) = I(f_1/\hat{B}, f_2/\hat{B})$ giving $f \in I(f_1, f_2)$. ■

In order to prove the sufficiency of condition (d) when $m = 2$ we need some more auxiliary results.

Lemma 4.2. *Let $0 < m_1 \leq m_2 \leq \dots \leq m_N \leq 1$, with $N \geq 2$. Assume $\prod_{j=1}^N m_j \leq \eta < 1$ and $\prod_{j=2}^N m_j \leq \eta^{1/2}$. Then there exists an integer k with $1 \leq k < N$ such that $\prod_{j=1}^k m_j \leq \eta^{1/4}$ and $\prod_{j=k+1}^N m_j \leq \eta^{1/2}$ ($\leq \eta^{1/4}$).*

Proof. Let k be the smallest positive integer such that $\prod_{j=1}^k m_j \leq \eta^{1/4}$. Observe that $k < N$, because otherwise $\prod_{j=1}^{N-1} m_j > \eta^{1/4}$ and it would follow that $m_N < \eta^{3/4}$, and then

$$\prod_{j=1}^{N-1} m_j < m_1 \leq m_N < \eta^{3/4} < \eta^{1/4},$$

which is a contradiction. Hence $k < N$. If $k = 1$, the conclusion follows immediately from the assumption $\prod_{j=2}^N m_j \leq \eta^{1/2}$. Next, if $k > 1$, we have $\prod_{j=1}^{k-1} m_j > \eta^{1/4}$ and $\prod_{j=1}^k m_j \leq \eta^{1/4}$. The first estimate gives $m_1 > \eta^{1/4}$ and hence $m_j > \eta^{1/4}$ for any $j = 1, \dots, N$. Then

$$\prod_{j=k+1}^N m_j = \frac{\prod_{j=1}^N m_j}{m_k \prod_{j=1}^{k-1} m_j} \leq \frac{\eta}{\eta^{1/4} \eta^{1/4}} = \eta^{1/2}.$$

■

Lemma 4.3. *Let $f_i, g_i \in N$, $i = 1, 2$, such that $f_1 g_1$ and $f_2 g_2$ have no common zeros. If $I(f_1 g_1, f_2 g_2) = J(f_1 g_1, f_2 g_2)$, then $I(f_1, f_2) = J(f_1, f_2)$.*

Proof. We need to show that $J(f_1, f_2) \subset I(f_1, f_2)$. Let $f \in J(f_1, f_2)$, that is $|f| \leq e^H(|f_1| + |f_2|)$, for some $H \in \text{Har}_+(\mathbb{D})$. Then there exists another $H_1 \in \text{Har}_+(\mathbb{D})$ such that $|f g_1 g_2| \leq e^{H_1}(|f_1 g_1| + |f_2 g_2|)$. By assumption, there exist $h_1, h_2 \in N$, such that $f g_1 g_2 = f_1 g_1 h_1 + f_2 g_2 h_2$. Thus $f_1 g_1 h_1$ vanishes at the zeros of g_2 , and since $f_1 g_1$ and $f_2 g_2$ have no common zeros, so that $f_1 g_1$ and g_2 have no common zeros, it is h_1 vanishing at the zeros of g_2 . We thus may write $h_1 = g_2 h_1^*$ for a suitable $h_1^* \in N$. A similar argument leads to $h_2 = g_1 h_2^*$ for some $h_2^* \in N$. Thus $f = f_1 h_1^* + f_2 h_2^*$. ■

Lemma 4.4. *Let B be a Blaschke product with zero sequence Λ . Let $z \in \mathbb{D}$ be such that $\Lambda \cap D(z, \delta) = \emptyset$ and let ρ_Δ denote the pseudohyperbolic distance in $\Delta = D(z, \delta)$. Then*

$$(a) |B(z)|^{\frac{1+\rho_\Delta(z,w)}{1-\rho_\Delta(z,w)}} \leq |B(w)| \leq |B(z)|^{\frac{1-\rho_\Delta(z,w)}{1+\rho_\Delta(z,w)}}, \quad w \in \Delta,$$

$$(b) (1 - |z|^2)|B'(z)| \leq \frac{|B(z)|}{\delta} \log \frac{1}{|B(z)|^2}.$$

Proof. The estimates in (a) are just Harnack's inequalities rescaled to Δ and applied to the positive harmonic function $u = -\log |B|$. To prove (b) let $\Lambda = \{\lambda_n\}_n$. A direct computation shows that

$$B'(z) = \sum_{n=1}^{\infty} \frac{B(z)}{b_{\lambda_n}(z)} \frac{-\bar{\lambda}_n}{|\lambda_n|} \frac{1 - |\lambda_n|^2}{(1 - \bar{\lambda}_n z)^2}.$$

Hence

$$(1 - |z|^2)|B'(z)| \leq \sum_{n=1}^{\infty} \frac{|B(z)|}{\delta} \frac{(1 - |z|^2)(1 - |\lambda_n|^2)}{|1 - \bar{\lambda}_n z|^2}$$

and we finish by using the estimate $\log(1/x) \geq 1 - x$, $x > 0$, since then

$$\sum_{n=1}^{\infty} \frac{(1 - |z|^2)(1 - |\lambda_n|^2)}{|1 - \bar{\lambda}_n z|^2} = \sum_{n=1}^{\infty} (1 - \rho^2(\lambda_n, z)) \leq 2 \sum_{n=1}^{\infty} \log \frac{1}{\rho(\lambda_n, z)} = \log \frac{1}{|B(z)|^2}.$$

■

Lemma 4.5. *Let $\Lambda = \{\lambda_n\}_n$ be a sequence of distinct points in \mathbb{D} which is the union of two Nevanlinna interpolating sequences. Then the trace of N on Λ is*

$$N|\Lambda = \left\{ \{w_n\}_n : \exists H \in \text{Har}_+(\mathbb{D}) : \sup_{k:k \neq n} \frac{|w_k - w_n|}{\rho(\lambda_k, \lambda_n)} e^{-H(\lambda_n) - H(\lambda_k)} < \infty \right\}.$$

It is also true in general that when Λ is the union of n Nevanlinna interpolating sequences then the trace coincides with the set of sequences such that the pseudohyperbolic divided differences of order $n - 1$ have a positive harmonic majorant (see [12]).

Proof. \squaresubseteq Let $\{w_n\}_n \in N|\Lambda$ and let $f \in N$ with $f(\lambda_n) = w_n$, $n \in \mathbb{N}$. Let H be a positive harmonic majorant of $\log |f|$. Given $\lambda_n, \lambda_k \in \Lambda$, $k \neq n$. Define

$$\Delta f(\lambda_n, \lambda_k) = \frac{f(\lambda_k) - f(\lambda_n)}{b_{\lambda_n}(\lambda_k)}.$$

If $\rho(\lambda_n, \lambda_k) \geq 1/2$ we get

$$|\Delta f(\lambda_n, \lambda_k)| = \left| \frac{f(\lambda_k) - f(\lambda_n)}{b_{\lambda_n}(\lambda_k)} \right| \leq \frac{1}{\rho(\lambda_n, \lambda_k)} (e^{H(\lambda_k)} + e^{H(\lambda_n)}) \leq 2e^{H(\lambda_k) + H(\lambda_n)}.$$

If $\rho(\lambda_n, \lambda_k) < 1/2$ apply the maximum principle to the holomorphic function $z \mapsto \Delta f(\lambda_n, z)$ and use Harnack's inequalities (2.2) to get

$$|\Delta f(\lambda_n, \lambda_k)| \leq \sup_{\zeta: \rho(\lambda_k, \zeta) = 1/2} |\Delta f(\lambda_n, \zeta)| \leq \sup_{\zeta: \rho(\lambda_k, \zeta) = 1/2} 2e^{H(\lambda_n) + H(\zeta)} \leq e^{3H(\lambda_n) + 3H(\lambda_k)}.$$

$\square\supseteq$ Let $\Lambda = \Lambda_1 \cup \Lambda_2$, where $\Lambda_i = \{\lambda_n^{(i)}\}_n$ are Nevanlinna interpolating sequences, $i = 1, 2$, and denote by B_i the corresponding Blaschke products. We will also denote $w_k^i = w_n$ when $\lambda_k^{(i)} = \lambda_n$. A usual technique to interpolate on finite unions of interpolating sequences is to look for an interpolating function of the form $h_0 + B_1 h_1$, where h_0 interpolates on Λ_1 and h_1 interpolates suitable values controlled by the divided differences on Λ_2 . Since by assumption

$\{w_k^{(1)}\}_k$ has a majorant $e^{H(\lambda_k^{(1)})}$, there exists $h_0 \in N$ with $h_0(\lambda_k^{(1)}) = w_k^{(1)}$, $k \in \mathbb{N}$. If we want an interpolating function of the form $h = h_0 + B_1 h_1$, with $h_1 \in N$, then, $h(\lambda_k^{(2)}) = w_k^{(2)}$ reduces to

$$(4.1) \quad h_1(\lambda_k^{(2)}) = \frac{w_k^{(2)} - h_0(\lambda_k^{(2)})}{B_1(\lambda_k^{(2)})}, \quad k \in \mathbb{N}.$$

Since $\Lambda_2 \in \text{Int } N$ we only need to see that the values on the right hand side have a suitable majorant. Given $\lambda_k^{(2)}$ take $\lambda_k^{(1)}$ such that $\rho(\lambda_k^{(2)}, \Lambda_1) = \rho(\lambda_k^{(2)}, \lambda_k^{(1)})$. There is no restriction in assuming that $\rho(\lambda_k^{(2)}, \lambda_k^{(1)}) \leq 1/2$, since otherwise the estimate below is immediate. Since Λ_2 is a Nevanlinna interpolating sequence, by Theorem 1.2(b), there exists $H_1 \in \text{Har}_+(\mathbb{D})$ such that

$$|B_1(\lambda_k^{(2)})| \geq e^{-H_1(\lambda_k^{(2)})} \rho(\lambda_k^{(1)}, \lambda_k^{(2)}), \quad k \in \mathbb{N},$$

and therefore

$$\begin{aligned} \left| \frac{w_k^{(2)} - h_0(\lambda_k^{(2)})}{B_1(\lambda_k^{(2)})} \right| &\leq \left| \frac{w_k^{(2)} - w_k^{(1)}}{B_1(\lambda_k^{(2)})} \right| + \left| \frac{h_0(\lambda_k^{(1)}) - h_0(\lambda_k^{(2)})}{B_1(\lambda_k^{(2)})} \right| \\ &\leq \left(\frac{|w_k^{(2)} - w_k^{(1)}|}{\rho(\lambda_k^{(1)}, \lambda_k^{(2)})} + \frac{|h_0(\lambda_k^{(1)}) - h_0(\lambda_k^{(2)})|}{\rho(\lambda_k^{(1)}, \lambda_k^{(2)})} \right) e^{H_1(\lambda_k^{(2)})}. \end{aligned}$$

By hypothesis the first term between parentheses has a majorant of the form $e^{H(\lambda_k^{(1)})+H(\lambda_k^{(2)})}$. The second term can be assumed to satisfy the same estimate because of the first inclusion and the fact that $h_0 \in N$. Thus, there exists $H_2 \in \text{Har}_+(\mathbb{D})$ such that

$$\left| \frac{w_k^{(2)} - h_0(\lambda_k^{(2)})}{B_1(\lambda_k^{(2)})} \right| \leq 2e^{H_2(\lambda_k^{(1)})+H_2(\lambda_k^{(2)})} e^{H_1(\lambda_k^{(2)})}.$$

By Harnack's inequality this is bounded by $2e^{2H_2(\lambda_k^{(2)})} e^{H_1(\lambda_k^{(2)})}$. Then (1.2) yields the existence of h_1 such that (4.1) holds. \blacksquare

Lemma 4.6. *Let $\Lambda = \{\lambda_n\}_n$ be a separated Blaschke sequence and let $\delta := \inf_{k \neq n} \rho(\lambda_k, \lambda_n) > 0$. Given $0 < \varepsilon_n < \delta/2$ consider the disks $\mathcal{D}_n = D(\lambda_n, \varepsilon_n)$. Let B_1 and B_2 be two Blaschke products without common zeros, having each exactly two zeros in each disk \mathcal{D}_n . Assume $I(B_1, B_2) = J(B_1, B_2)$. Then there exists $H \in \text{Har}_+(\mathbb{D})$ such that*

$$\varepsilon_n > e^{-H(\lambda_n)}, \quad n \in \mathbb{N}.$$

Proof. The assumptions *a priori* allow B_1 and B_2 to have zeros outside $\cup_n \mathcal{D}_n$. In order to get rid of these, let h_i be the Blaschke product vanishing on the zeros of B_i which are not in $\cup_n \mathcal{D}_n$. Setting $B_i^0 = B_i/h_i$, Lemma 4.3 shows that $I(B_1^0, B_2^0) = J(B_1^0, B_2^0)$ (note that B_1 and B_2 are assumed to have no common zeros). Thus we can henceforth assume that the zeros of B_i , $i = 1, 2$, are contained in $\cup_n \mathcal{D}_n$.

Let c_n^i, d_n^i denote the zeros of B_i in \mathcal{D}_n , $n \in \mathbb{N}$, $i = 1, 2$. Pick the largest of the mutual distances $\rho(c_n^1, c_n^2)$, $\rho(c_n^1, d_n^2)$, $\rho(d_n^1, c_n^2)$, $\rho(d_n^1, d_n^2)$, say $\rho(d_n^1, d_n^2)$. Then we have

$$(4.2) \quad 2\varepsilon_n \geq \rho(d_n^1, d_n^2) \geq \max\{\rho(c_n^1, c_n^2), \rho(c_n^1, d_n^2), \rho(d_n^1, c_n^2)\}.$$

For $i = 1, 2$ let D_i be the Blaschke product with zeros $\{d_n^i\}_n$ and let $C_i = B_i/D_i = \prod b_{c_n^i}$. Since Λ is separated and B_i has exactly two zeros on each \mathcal{D}_n we deduce from [13, Corollary

1.9] that C_i and D_i are Nevanlinna interpolating Blaschke products. Hence, taking into account (4.2) there exists $H \in \text{Har}_+(\mathbb{D})$ such that the values

$$\left| \frac{C_1(d_n^2)}{D_1(d_n^2)} \right| = \frac{\rho(c_n^1, d_n^2)}{\rho(d_n^1, d_n^2)} \left| \frac{(C_1/b_{c_n^1})(d_n^2)}{(D_1/b_{d_n^1})(d_n^2)} \right|, \quad \frac{C_2(d_n^1)}{D_2(d_n^1)} = \frac{\rho(c_n^2, d_n^1)}{\rho(d_n^2, d_n^1)} \left| \frac{(C_2/b_{c_n^2})(d_n^1)}{(D_2/b_{d_n^2})(d_n^1)} \right|$$

are bounded by $e^{H(d_n^2)}$ and $e^{H(d_n^1)}$, respectively. Consequently, there exist $h_1, h_2 \in N$ such that

$$h_1(d_n^2) = \frac{C_1(d_n^2)}{D_1(d_n^2)}, \quad h_2(d_n^1) = \frac{C_2(d_n^1)}{D_2(d_n^1)}.$$

Hence, there are $g_1, g_2 \in N$ with $C_1 = D_1 h_1 + D_2 g_1$ and $C_2 = D_2 h_2 + D_1 g_2$. Next we show that $C_1 C_2 \in J(B_1, B_2)$. Indeed, assume (without loss of generality) that $|C_2(z)| \leq |C_1(z)|$. Then

$$|C_1(z)C_2(z)| \leq |(D_1 h_1)(z) + (D_2 g_1)(z)| |C_2(z)| \leq |h_1(z)| |B_1(z)| + |g_1(z)| |B_2(z)|.$$

Hence $C_1 C_2 \in J(B_1, B_2) = I(B_1, B_2)$ so that there exist $f_1, f_2 \in N$ with

$$C_1 C_2 = B_1 f_1 + B_2 f_2 = C_1 D_1 f_1 + C_2 D_2 f_2.$$

Therefore, f_1 vanishes at the zeros of C_2 and f_2 vanishes at the zeros of C_1 , and there exist $f_1^*, f_2^* \in N$ with $f_2 = C_1 f_1^*$ and $f_1 = C_2 f_2^*$. Hence

$$C_1 C_2 = C_1 D_1 C_2 f_1^* + C_2 D_2 C_1 f_2^*$$

and we deduce that $1 = D_1 f_1^* + D_2 f_2^*$. Then there exists $H_1 \in \text{Har}_+(\mathbb{D})$ such that $|D_1| + |D_2| \geq e^{-H_1}$. Consequently, and since $\rho(d_n^1, d_n^2) \leq \varepsilon$ we can use Harnack's inequalities to deduce that

$$\varepsilon_n \geq \rho(d_n^1, d_n^2) \geq |D_1(d_n^2)| \geq e^{-H(d_n^2)} \geq e^{-2H(\lambda_n)}.$$

■

Let us now move to the proof of (d) \implies (c) in Theorem 1.1 in the case $m = 2$. Recall that we can assume that $f_i = B_i$ are Blaschke products. Let Λ_i be the zero set of B_i and denote

$$k(z) = \sum_{i=1}^2 (|B_i(z)| + (1 - |z|^2) |B_i'(z)|), \quad z \in \mathbb{D}.$$

In view of [13, Proposition 4.1], for any $\delta > 0$ there exists $H_\delta \in \text{Har}_+(\mathbb{D})$ such that

$$|B_i(z)| \geq e^{-H_\delta(z)} \quad \text{for } z \text{ with } \rho(z, \Lambda_i) \geq \delta, i = 1, 2.$$

Hence, to prove estimate (c) we can assume that z belongs to a Whitney box $T(I) = \{z = re^{i\theta} \in \mathbb{D} : e^{i\theta} \in I, |I|/2 \leq 1 - r \leq |I|\}$ such that $\rho(T(I), \Lambda_i) \leq 1/2, i = 1, 2$. Here I indicates an arc in $\partial\mathbb{D}$. Let $\{T(I_j)\}_j$ be the collection of Whitney boxes satisfying this condition and pick $\alpha_j \in \overline{T(I_j)}$ such that

$$k(\alpha_j) = \min_{z \in \overline{T(I_j)}} k(z).$$

To prove (c) we need to construct $H \in \text{Har}_+(\mathbb{D})$ such that

$$(4.3) \quad k(\alpha_j) \geq e^{-H(\alpha_j)}, \quad j \in \mathbb{N},$$

since then, by Harnack's inequalities, the inequality propagates to the whole $\overline{T(I_j)}$, that is, if $z \in \overline{T(I_j)}$, we have $k(z) \geq k(\alpha_j) \geq e^{-H(\alpha_j)} \geq e^{-CH(z)}$.

Splitting $\{\alpha_j\}_j$ into finitely many subsequences if necessary, one can assume that the pseudo-hyperbolic disks $\mathcal{D}_j = D(\alpha_j, 1/2)$ are pairwise disjoint.

For $i = 1, 2$ and $j \in \mathbb{N}$ let $B_i(j)$ be the subproduct of B_i formed with the zeros of B_i placed outside \mathcal{D}_j . Then (again using [13, Proposition 4.1], see also (1.3)) there exists $H_0 \in \text{Har}_+(\mathbb{D})$ independent of i and j such that

$$(4.4) \quad |B_i(j)(\alpha_j)| \geq e^{-H_0(\alpha_j)}, \quad j \in \mathbb{N}.$$

We can also assume that each \mathcal{D}_j contains at least two zeros of B_1 and two zeros of B_2 . Indeed, suppose λ is the only zero of B_1 in \mathcal{D}_j (if there is none, then $B_1(j) = B_1$ and $k(\alpha_j) \geq |B_1(\alpha_j)| \geq e^{-H(\alpha_j)}$ so that there is nothing to do). If $\rho(\alpha_j, \lambda) \geq e^{-H_1(\alpha_j)}$ for a suitable fixed H_1 , then since $|B_1| = |B_1(j)||b_\lambda|$ we get (4.3). If $\rho(\alpha_j, \lambda) \leq e^{-H_1(\alpha_j)}$, first observe that $(1 - |\lambda|^2)|B_1'(\lambda)| = |B_1(j)(\lambda)| \geq e^{-2H_0(\lambda)}$. Then, by Lemma 2.2(b) we deduce that, for a sufficiently big H_1 (depending on H_0 only), $(1 - |\alpha_j|)|B_1'(\alpha_j)| \geq e^{-3H_0(\alpha_j)}$, which again yields (4.3).

We can also assume that

$$(4.5) \quad k(\alpha_j) \leq e^{-100H_0(\alpha_j)},$$

since otherwise (4.3) holds.

For $i = 1, 2$ and $j \in \mathbb{N}$ let $\lambda_j^{(i)}$ be a zero of B_i such that $\rho(\alpha_j, \lambda_j^{(i)}) = \rho(\alpha_j, \Lambda_i)$. Denote $B_{i,j} = B_i/b_{\lambda_j^{(i)}}$. We claim that there exists a universal constant $C > 0$ such that

$$(4.6) \quad |B_{i,j}(\alpha_j)| \leq Ck(\alpha_j)^{1/2}, \quad j \in \mathbb{N}, i = 1, 2.$$

To see this notice first that we have $|B_i(\alpha_j)| \leq k(\alpha_j)$. If $\rho(\alpha_j, \lambda_j^{(i)}) \geq k(\alpha_j)^{1/2}$ we obtain (4.6) from

$$|B_{i,j}(\alpha_j)| = \frac{|B_i(\alpha_j)|}{\rho(\alpha_j, \lambda_j^{(i)})} \leq k(\alpha_j)^{1/2}.$$

If $\rho(\alpha_j, \lambda_j^{(i)}) \leq k(\alpha_j)^{1/2}$ we use Lemma 2.1(c) to see that

$$\left| (1 - |\alpha_j|)|B_i'(\alpha_j)| - (1 - |\lambda_j^{(i)}|)|B_i'(\lambda_j^{(i)})| \right| \leq 6\rho(\alpha_j, \lambda_j^{(i)}).$$

Since $(1 - |\alpha_j|)|B_i'(\alpha_j)| \leq k(\alpha_j)$ we deduce that $(1 - |\lambda_j^{(i)}|)|B_i'(\lambda_j^{(i)})| \leq 7k(\alpha_j)^{1/2}$, that is, $|B_{i,j}(\lambda_j^{(i)})| \leq 7k(\alpha_j)^{1/2}$. Since $\rho(\alpha_j, \lambda_j^{(i)}) \leq k(\alpha_j)^{1/2}$, by Schwarz's lemma we deduce that $|B_{i,j}(\alpha_j)| \leq C_1k(\alpha_j)^{1/2}$ for some $C_1 > 0$ and (4.6) holds also in this case.

For $i = 1, 2$ and $j \in \mathbb{N}$ let

$$E_{i,j} = \{z \in \mathbb{D} : B_i(z) = 0 \text{ and } \rho(z, \alpha_j) < 1/2\},$$

and let $b_{i,j}$ be the Blaschke product with zeros in $E_{i,j}$ so that $B_i = b_{i,j}B_i(j)$. Since $|B_i(\alpha_j)| \leq k(\alpha_j)$, estimates (4.4), (4.5) and (4.6) give

$$(4.7) \quad |b_{i,j}(\alpha_j)| \leq k(\alpha_j)e^{H_0(\alpha_j)} \leq k(\alpha_j)^{99/100},$$

$$(4.8) \quad \prod_{\substack{z \in E_{i,j} \\ z \neq \lambda_j^{(i)}}} \rho(z, \alpha_j) = \frac{|B_{i,j}(\alpha_j)|}{|B_i(j)(\alpha_j)|} \leq Ck(\alpha_j)^{1/2}e^{H_0(\alpha_j)} \leq Ck(\alpha_j)^{1/2-1/100}.$$

In order to prove (4.3) we will now split $\{\alpha_j\}_j$ into different pieces and consider different cases according to the number of zeros of B_1 and B_2 in the following neighborhoods of α_j : $U_j = D(\alpha_j, k(\alpha_j)^{1/10})$ and $\tilde{U}_j = D(\alpha_j, k(\alpha_j)^{1/100}) \supset U_j$. Here are the cases we are going to discuss now:

- (i) At least one Blaschke product has at least two zeros in U_j . The set of these α_j will be denoted by A_1 . Splitting possibly A_1 into two subsequences we can assume that B_1 has at least two zeros in U_j (in case B_2 has at least two zeros in U_j while B_1 has not, inverting the rôles of B_1 and B_2 yields the exact same estimate). In this case we will distinguish three subcases.
 - (i)-a. B_2 has at least two zeros in \tilde{U}_j . The set of these α_j will be denoted by A_{11} .
 - (i)-b. B_2 has no zero in \tilde{U}_j . The set of these α_j will be denoted by A_{12} .
 - (i)-c. B_2 has exactly one zero in \tilde{U}_j . The set of these α_j will be denoted by A_{13} .
- (ii) Both Blaschke products have at most one zero in U_j . The set of these α_j will be denoted by A_2 .

We will establish (4.3) in each of these cases.

Case (i)-a. We will start with $\alpha_j \in A_{11}$. For $i = 1, 2$ pick two zeros of $b_{i,j}$ in \tilde{U}_j and let $\tilde{b}_{i,j}$ be the corresponding Blaschke product of degree 2. Consider $\tilde{B}_i = \prod_j \tilde{b}_{i,j}$ where the product is taken over all j such that $\alpha_j \in A_{11}$. Since \tilde{B}_i is a subproduct of B_i , the assumption (d) and Lemma 4.3 give $I(\tilde{B}_1, \tilde{B}_2) = J(\tilde{B}_1, \tilde{B}_2)$. Applying Lemma 4.6 with $\varepsilon_j = k(\alpha_j)^{1/100}$ we obtain $H \in \text{Har}_+(\mathbb{D})$ such that

$$k(\alpha_j)^{1/100} \geq e^{-H(\alpha_j)}, \quad \alpha_j \in A_{11} .$$

This gives the required estimate (4.3) for the points in A_{11} .

Case (i)-b. The idea in this case is to replace B_2 by an appropriate perturbation $B_2 - G\tilde{B}_1$, where \tilde{B}_1 is a sub-product of B_1 vanishing exactly twice in each \tilde{U}_j , in order to generate two zeros (controlled by Rouché's theorem) and then conclude as in Case (ii)-a.

For $\alpha_j \in A_{12}$ the function $b_{2,j}$ has no zero in \tilde{U}_j . For each $\alpha_j \in A_{12}$ pick two zeros of B_1 in U_j and let \tilde{B}_1 be the Blaschke product formed with these zeros as in case (i)-a. Since $U_j \subset \mathcal{D}_j$ and the disks \mathcal{D}_j are disjoint, \tilde{B}_1 is a Blaschke product whose zeros form a union of two Nevanlinna interpolating sequences [13, Corollary 1.9]. Hence there exists $H \in \text{Har}_+(\mathbb{D})$ such that for every zero $\lambda \in U_j$ of \tilde{B}_1 , and z with $\rho(z, \alpha_j) = k(\alpha_j)^{1/30}$,

$$(4.9) \quad |\tilde{B}_1(z)| \geq e^{-H(z)} \rho(z, \lambda) \geq e^{-H(z)} \text{dist}(z, \partial U_j) \geq k(\alpha_j)^{1/15} e^{-H(z)} .$$

Let $G = e^{H+i\tilde{H}}$, where \tilde{H} is the harmonic conjugate of H . By Lemma 4.3, $I(\tilde{B}_1, B_2) = J(\tilde{B}_1, B_2)$. Then, observing that G is invertible in N , one has

$$I(\tilde{B}_1, B_2) = I(\tilde{B}_1, G\tilde{B}_1 - B_2) \subset J(\tilde{B}_1, G\tilde{B}_1 - B_2) \subset J(\tilde{B}_1, B_2) = I(\tilde{B}_1, B_2) ,$$

hence

$$I(\tilde{B}_1, G\tilde{B}_1 - B_2) = J(\tilde{B}_1, G\tilde{B}_1 - B_2) .$$

Now, for points $z \in \mathbb{D}$ such that $\rho(z, \alpha_j) = k(\alpha_j)^{1/30}$ we have, by Lemma 4.4 and the assumption:

$$(4.10) \quad \left| \tilde{B}_1(z)G(z) - (\tilde{B}_1(z)G(z) - B_2(z)) \right| = |B_2(z)| \leq |B_2(\alpha_j)|^{\frac{1-\rho_j}{1+\rho_j}} \leq k(\alpha_j)^{\frac{1-\rho_j}{1+\rho_j}} ,$$

where $\rho_j = \rho_{\tilde{U}_j}(z, \alpha_j)$ and $\rho_{\tilde{U}_j}$ indicates the pseudohyperbolic distance in \tilde{U}_j . Since $\rho(z, \alpha_j) = k(\alpha_j)^{1/30}$ and $\tilde{U}_j = D(\alpha_j, k(\alpha_j)^{1/100}) \supset U_j$, we have

$$\rho_{\tilde{U}_j}(z, \alpha_j) \leq k(\alpha_j)^{1/30-1/100}.$$

Indeed we can assume $\alpha_j = 0$ and let $\phi : \mathbb{D} \rightarrow \tilde{U}_j$ be given by $\phi(w) = k(\alpha_j)^{1/100}w$; then

$$\rho_{\tilde{U}_j}(z, \alpha_j) = \rho\left(\frac{z}{k(\alpha_j)^{1/100}}, 0\right) = \frac{|z|}{k(\alpha_j)^{1/100}} \leq k(\alpha_j)^{1/30-1/100}.$$

Since we can assume that $k(\alpha_j)$ is small, say $k(\alpha_j)^{1/30-1/100} < \varepsilon$, we deduce from (4.10) and (4.9) that

$$\left| \tilde{B}_1(z)G(z) - (\tilde{B}_1(z)G(z) - B_2(z)) \right| \leq k(\alpha_j)^{\frac{1-\varepsilon}{1+\varepsilon}} < k(\alpha_j)^{1/15} < |\tilde{B}_1(z)G(z)|.$$

Then, by Rouché's theorem $\tilde{B}_1G - B_2$ has two zeros in $D(\alpha_j, k(\alpha_j)^{1/30})$. Observe that we can replace $\tilde{B}_1G - B_2$ by the Blaschke product vanishing on the zeros of $\tilde{B}_1G - B_2$, and we can thus argue as we have done for A_{11} (note that $k(\alpha_j)$ now only gives the size of \mathcal{D}_j , U_j and \tilde{U}_j , and it only depends on the fact that the Blaschke products under consideration have zeros in these neighborhoods, but not on the explicit form of these products).

Case (i)-c. Recall that A_{13} is the set of $\alpha_j \in A_1$ such that $b_{2,j}$ has one zero in \tilde{U}_j . If $\alpha_j \in A_{13}$, the zero set of $b_{2,j}$ in \tilde{U}_j must be $\lambda_j^{(2)}$. Recall from (4.6) that

$$|B_{2,j}(\alpha_j)| \leq Ck(\alpha_j)^{1/2}$$

and $B_{2,j}$ has no zeros in \tilde{U}_j . Hence by (4.4) and (4.5), we deduce that

$$\left| \frac{B_2(\alpha_j)}{\prod_{\alpha_k \in A_{13}} b_{\lambda_k^{(2)}}(\alpha_j)} \right| = \left| \frac{B_{2,j}(\alpha_j)}{\prod_{\substack{k \neq j \\ \alpha_k \in A_{13}}} b_{\lambda_k^{(2)}}(\alpha_j)} \right| \leq Ck(\alpha_j)^{1/2} e^{H_0(\alpha_j)} \leq Ck(\alpha_j)^{0.49}.$$

Thus, replacing B_2 by $B_2 / \prod_{\alpha_k \in A_{13}} b_{\lambda_k^{(2)}}$ we can assume that B_2 has no zeros in \tilde{U}_j and we can argue as in the previous case.

Case (ii). For $\alpha_j \in A_2$ and $i = 1, 2$ the function $b_{i,j}$ has at most one zero in U_j . If it has one zero, this must actually be $\lambda_j^{(i)}$. In this case, from (4.4) and (4.6),

$$(4.11) \quad \left| \frac{b_{i,j}(\alpha_j)}{b_{\lambda_j^{(i)}}(\alpha_j)} \right| = \left| \frac{B_{i,j}(\alpha_j)}{B_i(j)(\alpha_j)} \right| \leq Ck(\alpha_j)^{1/2} e^{H_0(\alpha_j)} \leq Ck(\alpha_j)^{0.49}.$$

Hence, replacing $b_{i,j}$ by $b_{i,j}/b_{\lambda_j^{(i)}}$ we can assume that $b_{i,j}$ has no zeros in U_j and satisfies the above estimate (4.11). Observe from this estimate that the initial zero-set $E_{i,j}$ cannot be reduced to the sole point $\lambda_j^{(i)}$. We will henceforth assume that $E_{i,j}$ does not contain any point in U_j . In order to apply Lemma 4.2 write $E_{i,j} = \{a_k(i, j) : k = 1, \dots, N\}$, where the points $a_k(i, j)$ are taken so that the corresponding distances $m_k = m_k(i, j) = \rho(a_k(i, j), \alpha_j)$ satisfy $m_1 \leq m_2 \leq \dots \leq m_N$. In particular $a_1(i, j)$ is the closest point of $E_{i,j}$ to α_j , and it is outside U_j , that is,

$\rho(a_1(i, j), \alpha_j) \geq k(\alpha_j)^{1/10}$. According to (4.11), and setting $\eta = Ck(\alpha_j)^{0.49}$, we have

$$\prod_{k=1}^N m_k = |b_{i,j}(\alpha_j)| \leq Ck(\alpha_j)^{0.49} = \eta.$$

Moreover

$$\prod_{k=2}^N m_k = \frac{|b_{i,j}(\alpha_j)|}{\rho(a_1(i, j), \alpha_j)} \leq Ck(\alpha_j)^{0.39} \leq \eta^{1/2},$$

when $k(\alpha_j)$ is sufficiently small (which we can assume). We are thus in the conditions of Lemma 4.2, which allows to split the product $b_{i,j}$ into two sub-products, denoted by $b_{i,j}^*$, $b_{i,j}^{**}$ each of which is controlled by $\eta^{1/4} = C^{1/4}k(\alpha_j)^{0.49/4}$. More concretely

$$b_{i,j} = b_{i,j}^* b_{i,j}^{**}, \quad j \in \mathbb{N}, \quad i = 1, 2,$$

and

$$(4.12) \quad |b_{i,j}^*(\alpha_j)| \leq |b_{i,j}^{**}(\alpha_j)| \leq C_1 k(\alpha_j)^{0.1225}$$

(if the first inequality does not hold interchange the roles of $b_{i,j}^*$ and $b_{i,j}^{**}$). Let

$$B_i^* = \prod_{j=1}^{\infty} b_{i,j}^*, \quad B_i^{**} = \prod_{j=1}^{\infty} b_{i,j}^{**},$$

where the product is taken over the indices j such that $\alpha_j \in A_2$. For $j \in \mathbb{N}$ and $i = 1, 2$ we have

$$(4.13) \quad |B_i^*(\alpha_j)| + |B_i^{**}(\alpha_j)| \leq 2C_1 k(\alpha_j)^{0.1225}.$$

Moreover, taking into account (4.4), there exists $H_0 \in \text{Har}_+(\mathbb{D})$ such that

$$(4.14) \quad |B_i^*(\alpha_j)| \leq e^{H_0(\alpha_j)} |B_i^{**}(\alpha_j)|.$$

Split A_2 into two sequences $A_2 = A_{21} \cup A_{22}$, where

$$A_{21} = \{\alpha_j : |B_1^{**}(\alpha_j)| \leq |B_2^{**}(\alpha_j)|\}, \quad A_{22} = \{\alpha_j : |B_2^{**}(\alpha_j)| < |B_1^{**}(\alpha_j)|\}.$$

For $i = 1, 2$ we will construct $H_{2i} \in \text{Har}_+(\mathbb{D})$ such that $k(\alpha_j) \geq e^{-H_{2i}(\alpha_j)}$ for any $\alpha_j \in A_{2i}$. This will give (4.3) also in this case and finish the proof. Let us explain how to construct H_{21} . The same argument applies to H_{22} . For $\alpha_j \in A_{21}$ pick $\alpha_j^* \in \mathbb{D}$ with $\rho(\alpha_j^*, \alpha_j) = |B_1^{**}(\alpha_j)|/4$. Observe that (4.14) yields

$$(4.15) \quad \frac{|B_1(\alpha_j)|}{|B_2^{**}(\alpha_j)|} \leq |B_1^*(\alpha_j)| \frac{|B_1^{**}(\alpha_j)|}{|B_2^{**}(\alpha_j)|} \leq |B_1^*(\alpha_j)| \leq e^{H_0(\alpha_j)} 4\rho(\alpha_j^*, \alpha_j).$$

Since $|B_1^{**}(\alpha_j)| \leq |B_2^{**}(\alpha_j)|$ and $\rho(\alpha_j^*, \alpha_j) = |B_1^{**}(\alpha_j)|/4$, Schwarz' Lemma (see Lemma 2.1(a)) gives $|B_2^{**}(\alpha_j^*)| \geq |B_2^{**}(\alpha_j)|/2$ and $|B_1^{**}(\alpha_j^*)| \leq 3|B_1^{**}(\alpha_j)|/2$. Hence, using again Lemma 2.1(a) and (4.14),

$$(4.16) \quad \begin{aligned} \frac{|B_1(\alpha_j^*)|}{|B_2^{**}(\alpha_j^*)|} &\leq |B_1^*(\alpha_j^*)| \frac{|B_1^{**}(\alpha_j^*)|}{|B_2^{**}(\alpha_j^*)|} \leq 3|B_1^*(\alpha_j^*)| \leq 3(|B_1^*(\alpha_j)| + |B_1^*(\alpha_j^*) - B_1^*(\alpha_j)|) \\ &\leq 3(|B_1^*(\alpha_j)| + 2\rho(\alpha_j^*, \alpha_j)) = 3(|B_1^*(\alpha_j)| + \frac{|B_1^{**}(\alpha_j)|}{2}) \\ &\leq C_2 e^{H_0(\alpha_j)} |B_1^{**}(\alpha_j)| = 4C_2 e^{H_0(\alpha_j)} \rho(\alpha_j^*, \alpha_j), \end{aligned}$$

where $C_2 > 0$ is an absolute constant. From (4.15) and (4.16) we get

$$\left| \frac{B_1(\alpha_j^*)}{B_2^{**}(\alpha_j^*)} - \frac{B_1(\alpha_j)}{B_2^{**}(\alpha_j)} \right| \leq \frac{|B_1(\alpha_j^*)|}{|B_2^{**}(\alpha_j^*)|} + \frac{|B_1(\alpha_j)|}{|B_2^{**}(\alpha_j)|} \leq 4(C_2 + 1)e^{H_0(\alpha_j)}\rho(\alpha_j^*, \alpha_j)$$

Hence the sequence defined by $w(\alpha_j) = B_1(\alpha_j)/B_2^{**}(\alpha_j)$ and $w(\alpha_j^*) = B_1(\alpha_j^*)/B_2^{**}(\alpha_j^*)$ is in the trace space defined on the sequence $\{\alpha_j, \alpha_j^*\}_{\alpha_j \in A_{21}}$, and according to Lemma 4.5 we find $h \in N$ such that

$$h(\alpha_j) = \frac{B_1(\alpha_j)}{B_2^{**}(\alpha_j)}, \quad h(\alpha_j^*) = \frac{B_1(\alpha_j^*)}{B_2^{**}(\alpha_j^*)}, \quad \alpha_j \in A_{21}.$$

Setting b the Blaschke product with zeros $\alpha_j \in A_{21}$ and b^* the Blaschke product with zeros α_j^* , we thus get $g \in N$ such that

$$(4.17) \quad B_1 = B_2^{**}h + bb^*g.$$

Since $I(B_1, B_2) = J(B_1, B_2)$, Lemma 4.3 yields $I(B_1, B_2^{**}) = J(B_1, B_2^{**})$. Now (4.17) gives also $I(B_1, B_2^{**}) = I(bb^*g, B_2^{**})$ and $J(B_1, B_2^{**}) = J(bb^*g, B_2^{**})$. Hence, $I(bb^*g, B_2^{**}) = J(bb^*g, B_2^{**})$, and again by Lemma 4.3, $I(bb^*, B_2^{**}) = J(bb^*, B_2^{**})$ (observe that bb^*g and B_2^{**} — which is a subproduct of B_2 — have no common zeros, since by (4.17) those common zeros would be in common with B_1 , which we excluded).

Now notice that bb^* has two zeros in $D(\alpha_j, |B_2^{**}(\alpha_j^*)|/2)$. Also, from (4.12) we can deduce that $D(\alpha_j, |B_2^{**}(\alpha_j^*)|/2) \subset D(\alpha_j, C_1k(\alpha_j)^{0.1})$. Hence we are in the same situation as we were discussing for A_1 , now applied to (B_2^{**}, bb^*) , and therefore there exists $H_{21} \in \text{Har}_+(\mathbb{D})$ such that

$$(4.18) \quad |B_2^{**}(\alpha_j)| + (1 - |\alpha_j|) \left[|(B_2^{**})'(\alpha_j)| + |(bb^*)'(\alpha_j)| \right] \geq e^{-H_{21}(\alpha_j)}, \quad \alpha_j \in A_{21}.$$

Now, by (4.12), $|B_2^{**}(\alpha_j)| \leq 2C_1k(\alpha_j)^{0.1225}$. Also B_2^{**} has no zeros in U_j , and so Lemma 4.4(b) gives that

$$(1 - |\alpha_j|)|(B_2^{**})'(\alpha_j)| \leq \frac{|B_2^{**}(\alpha_j)|}{k(\alpha_j)^{1/10}} \log |B_2^{**}(\alpha_j)|^{-2} \lesssim k(\alpha_j)^{0.02}, \quad \alpha_j \in A_{21}$$

Moreover, since $(bb^*)'(\alpha_j) = b'(\alpha_j)b^*(\alpha_j)$ and $(1 - |\alpha_j|)|b'(\alpha_j)| \leq 2$, we get

$$(1 - |\alpha_j|)|(bb^*)'(\alpha_j)| \leq 2\rho(\alpha_j, \alpha_j^*) \leq |B_1^{**}(\alpha_j)|/2$$

and, again by (4.12), this expression is controlled by $C_1k(\alpha_j)^{0.1225}$. As a result, there exists an absolute constant $C_3 > 0$ such that the left hand side of (4.18) is upper bounded by $C_3k(\alpha_j)^{0.02}$, and we deduce that

$$C_3k(\alpha_j)^{0.02} \geq e^{-H(\alpha_j)}, \quad \alpha_j \in A_{21},$$

as desired. ■

Finally let us show that when $m \geq 3$, condition (d) does not imply the equivalent conditions (a), (b) or (c) in Theorem 1.1. The example is analogous to the one given in the context of H^∞ in [11]. Let B_1, B_2 be Nevanlinna interpolating Blaschke products with zero sets Λ_1 and Λ_2 . We first claim that

$$(4.19) \quad I(B_1^2, B_2^2, B_1B_2) = J(B_1^2, B_2^2, B_1B_2).$$

To prove this we can assume that B_1 and B_2 have no common zeros. Let $f \in J(B_1^2, B_2^2, B_1B_2)$. Then there exists $H \in \text{Har}_+(\mathbb{D})$ such that

$$(4.20) \quad |f(z)| \leq e^{H(z)}(|B_1(z)|^2 + |B_2(z)|^2 + |B_1(z)B_2(z)|), \quad z \in \mathbb{D}.$$

Then $|f(\lambda)| \leq e^{H(\lambda)}|B_1(\lambda)|^2$ for $\lambda \in \Lambda_2$ so that there exists $g_1 \in N$ with $g_1(\lambda) = f(\lambda)/B_1^2(\lambda)$, $\lambda \in \Lambda_2$. This implies that there is $g_2 \in N$ such that $f = g_1B_1^2 + B_2g_2$. Observe that for every $\lambda \in \Lambda_1$ we have $|g_2(\lambda)|/|B_2(\lambda)| = |f(\lambda)|/|B_2(\lambda)|^2$ which by (4.20) is bounded by $e^{H(\lambda)}$. Hence there exists $g_3 \in N$ with $g_3(\lambda) = g_2(\lambda)/B_2(\lambda)$, $\lambda \in \Lambda_1$. Hence, there exists $g_4 \in N$ with $g_2 = B_2g_3 + B_1g_4$. Finally, $f = B_1^2g_1 + B_2^2g_3 + B_1B_2g_4$ and $f \in I(B_1^2, B_2^2, B_1B_2)$. Hence (4.19) holds. However, if the sequences Λ_1 and Λ_2 are too close, then using condition (c) of Theorem 1.1 it can be seen that the ideal $I(B_1^2, B_2^2, B_1B_2)$ cannot contain a Nevanlinna interpolating Blaschke product.

5. TWO OPEN PROBLEMS

5.1. The stable rank of the Nevanlinna class. The first open problem we discuss concerns the stable rank of the Nevanlinna algebra. Recall that an m -tuple (a_1, \dots, a_m) of elements of a commutative unital algebra A is called *unimodular* if the ideal it generates is the whole algebra, that is, there exists an m -tuple (b_1, \dots, b_m) in A^m such that $\sum_{i=1}^m a_i b_i = 1$. The m -tuple (a_1, \dots, a_m) is called *reducible* if there exists an $(m-1)$ -tuple (x_1, \dots, x_{m-1}) in A^{m-1} such that $(a_1 + x_1 a_m, \dots, a_{m-1} + x_{m-1} a_m)$ is unimodular (so, the ideal generated by (a_1, \dots, a_m) contains a specific $(m-1)$ -tuple that already generates A). The *stable rank* of the algebra is the least m for which every unimodular $m+1$ -tuple is reducible.

It is known that the stable rank of the disk algebra and of H^∞ is equal to one (see [3] or [14] for the disk algebra and [26] for H^∞). The stable rank for the Nevanlinna class is unknown, but the following result shows that it is at least two.

Proposition 5.1. *The stable rank of the Nevanlinna class is at least 2.*

It is worth mentioning that any triple $(f_1, f_2, f_3) \in N^3$ such that for some i the zeros of f_i form a Nevanlinna interpolating sequence, can be reduced. The argument uses Theorem 1.2, but it is lengthy and we do not include the details here.

Open problem: Is the stable rank of N equal to 2?

Proof of Proposition 5.1. Suppose to the contrary that the stable rank of N is one and let us reach a contradiction. For any unimodular pair of Blaschke products, there will then exist $\Phi_1 \in N$ such that $B_1 + \Phi_1 B_2$ is invertible in N , i.e.

$$(5.1) \quad B_1 + \Phi_1 B_2 = e^f,$$

where $\text{Re}(f) = H_+ - H_-$, for some $H_+, H_- \in \text{Har}_+(\mathbb{D})$. We will show that this is not possible in general. To this end, let $\Lambda_1 = \{\lambda_n\}_n := \{1 - 2^{-n}\}_n$ and B_1 the associated Blaschke product. The sequence Λ_1 is H^∞ -interpolating. Take now $\{\mu_n\}_n \subset (0, 1)$ with $\rho(\lambda_n, \mu_n)$ small enough so that

$$(5.2) \quad |B_1(\mu_n)| = \begin{cases} e^{-\frac{1}{1-|\lambda_n|}} & \text{if } n \text{ even} \\ e^{-\frac{2}{1-|\lambda_n|}} & \text{if } n \text{ odd.} \end{cases}$$

Set $\Lambda_2 = \{\mu_n\}_n$ and B_2 its Blaschke product.

We shall see first that (B_1, B_2) is unimodular, i.e., that there exists $H \in \text{Har}_+(\mathbb{D})$ such that $|B_1| + |B_2| \geq e^{-H}$. Fix a $\delta > 0$ such that the regions $\Omega_n = D(\lambda_n, \delta) \cup D(\mu_n, \delta)$ are mutually disjoint. Since Λ_1 and Λ_2 are H^∞ -interpolating sequences, there exists $\eta > 0$ such that

$$|B_i(z)| \geq \eta, \quad z \in \mathbb{D} \setminus \cup_n \Omega_n, \quad i = 1, 2.$$

Thus we only need to care about the estimate on $D(\lambda, \delta)$, for $\lambda \in \Lambda_1 \cup \Lambda_2$. So suppose $\lambda = \lambda_n$ or $\lambda = \mu_n$. Since $|B_1/b_{\lambda_n}|$ and $|B_2/b_{\mu_n}|$ are bounded below on $D(\lambda, \delta)$ (by Carleson's condition), we only need to take care of $|b_{\lambda_n}(z)| + |b_{\mu_n}(z)|$. By (5.2)

$$\rho(\lambda_n, \mu_n) = |b_{\lambda_n}(\mu_n)| \geq e^{-\frac{2}{1-|\lambda_n|}}.$$

By the triangular inequality $|\rho(\lambda_n, z) - \rho(z, \mu_n)| \leq \rho(\lambda_n, \mu_n)$, thus either $\rho(\lambda_n, z)$ or $\rho(\mu_n, z)$ are greater than $(1/2)e^{-\frac{2}{1-|\lambda_n|}}$. Take now c (independent of n) such that for $z \in D(\lambda, \delta)$,

$$(1/2)e^{-\frac{2}{1-|\lambda_n|}} \geq e^{-\frac{c}{1-|z|}}.$$

With this

$$|B_1(z)| + |B_2(z)| \geq e^{-\frac{c}{1-|z|}}, \quad z \in D(\lambda, \delta).$$

Since

$$H_0(z) = \text{Re}\left(\frac{1+z}{1-z}\right) = \frac{1-|z|^2}{|1-z|^2} \in \text{Har}_+(\mathbb{D})$$

and

$$H_0(z) \asymp \frac{1}{1-|z|}, \quad z \in D(\lambda, \delta)$$

this finally implies that (B_1, B_2) is unimodular.

Let us now show that the pair (B_1, B_2) cannot be reduced. Equation (5.1) on μ_n yields

$$\log |B_1(\mu_n)| = H_+(\mu_n) - H_-(\mu_n) = P[\nu](\mu_n), \quad n \in \mathbb{N},$$

where ν is a finite measure on $\partial\mathbb{D}$ such that $\text{Re}(f) = P[\nu]$. Then, since $\{\mu_n\}_n$ tends radially towards 1,

$$\lim_{n \rightarrow \infty} (1 - |\mu_n|^2)P[\nu](\mu_n) = \nu(\{1\}).$$

But from (5.2) we see that $\{(1 - |\mu_n|^2) \log |B_1(\mu_n)|\}_n$ has no limit, so we have reached a contradiction. \blacksquare

5.2. The f^2 problem. In the late seventies T. Wolff presented a problem on ideals of H^∞ , known now as the f^2 problem, which was finally solved by S. Treil in [27]. We now discuss an analogous problem in the Nevanlinna class. Let f_1, \dots, f_n be functions in the Nevanlinna class, and let $f \in N$ be such that there exists $H \in \text{Har}_+(\mathbb{D})$ with

$$(5.3) \quad |f(z)| \leq e^{H(z)}(|f_1(z)| + \dots + |f_n(z)|)^p, \quad z \in \mathbb{D},$$

for some $p \geq 1$. Does it follow that $f \in I(f_1, \dots, f_n)$?

As in the H^∞ case, when $p > 2$, the $\bar{\delta}$ estimates by T. Wolff show that the answer is affirmative. When $p < 2$ the answer is in general negative, as the following example shows. Let N be an integer such that $N + 1 > 2Np$, $f = B_1^N B_2^N$, $f_1 = B_1^{N+1}$ and $f_2 = B_2^{N+1}$. Then (5.3) holds but $f \notin I(f_1, f_2)$ if (B_1, B_2) is not unimodular in N .

Open problem: What happens in the case $p = 2$?

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