NON-TWIST INVARIANT CIRCLES IN CONFORMALLY SYMPLECTIC SYSTEMS

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Abstract. Dissipative mechanical systems on the torus with a friction that is proportional to the velocity are modeled by conformally symplectic maps on the annulus, which are maps that transport the symplectic form into a multiple of itself (with a conformal factor smaller than 1). It is important to understand the structure and the dynamics on the attractors. It is well-known that, with the aid of parameters, and under suitable non-degeneracy conditions, one can obtain that there is an attractor that is an invariant torus whose internal dynamics is conjugate to a rotation. By analogy with symplectic dynamics, a natural question is establishing appropriate definitions for twist and non-twist invariant tori in conformally symplectic systems.

The main goals of this paper are: (a) to establish proper definitions of twist and non-twist invariant tori in families of conformally symplectic systems; (b) to interpret these definitions in terms of dynamical properties; (c) to derive algorithms to compute twist and non-twist invariant tori; (d) to implement these algorithms in examples; (e) to explore the mechanisms of breakdown of twist and non-twist invariant tori. Hence, the last part of the paper is devoted to implementations of the algorithms, illustrating the definitions presented in this paper, and studying robustness properties of invariant tori.

1. The Introduction

Conformally symplectic systems model some mechanical systems with dissipation, in which the friction is proportional to the velocity. Geometrically, conformally symplectic systems transport a symplectic form into a multiple of itself. When the conformal factor is less than one the systems contract the form and are dissipative. In contrast to symplectic systems, dissipative systems have attractors. Although dissipative systems have less asymptotic behaviors by themselves, one recovers asymptotic behaviors by adding adjusting parameters. There has been a lot of interest in the case these attractors are invariant smooth tori that contain quasi-periodic dynamics (see e.g. [GOY85, BHS96, CLHB05, CC09, CCdlL13a]). Obtaining quasi-periodic dynamics is proved thanks to the presence of parameters in the system and some non-degeneracy condition that is referred to as twist condition in analogy of the common twist condition that appears in symplectic systems, [Mos66, Mos67, BHS96, CCdlL13b, CH17b]. Systems that do not have the twist condition appear in several applications. For example, in Celestial Mechanics the motion of a satellite near an oblate planet violates the twist condition at a critical inclination

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[Kyn68]. Also, conformally symplectic flows arise in certain models of electric field lines in non-neutral plasmas.

In this paper we are interested in developing algorithms for computing quasi-periodic circles when an analogue of the twist condition fails. In fact, a first task is to identify the proper definition for a non-twist circle in this context.

To the best of our knowledge, this paper presents a first attempt for considering non-twist tori in dissipative systems. We will present algorithms for the simplest 2D case. We have not proved here the convergence of the algorithms, but just applied them in several examples. However, we expect that a proof could be done by using standard KAM techniques, see for example [BHS96, dlLGJV05, CCdlL13b, GHdlL14, CH17b].

Organization of the paper. In Section 2 we introduce the setting, and present suitable definitions of twist and non-twist tori in the context of conformally symplectic dynamics. In Section 3 we describe a methodology for the computation of invariant tori in conformally symplectic systems, and, more importantly, for the computation and continuation of non-twist invariant tori in these systems. Section 4 is devoted to implementations of the algorithms to several examples, referred to as dissipative standard non-twist families, illustrating the concepts and algorithms presented in this paper, and to the analysis of the breakdown of non-twist invariant tori.

2. The definitions

In this section, we present and motivate the definition of twist and non-twist invariant circles in families of conformally symplectic systems of the annulus.

2.1. **Conformally symplectic maps and their invariant circles.**

In this paper, the phase space is the annulus $\mathbb{T}\times\mathbb{R}$, endowed with coordinates $z=(x,y)$, being $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ the torus. We will assume also that all maps appearing in this paper are sufficiently smooth.

Definition 2.1. *A conformally symplectic map in $\mathbb{T}\times\mathbb{R}$, with conformal factor $\sigma\in]0,1[$, is a diffeomorphism $F=(F^x,F^y):\mathbb{T}\times\mathbb{R}\to\mathbb{T}\times\mathbb{R}$, homotopic to the identity (i.e., lifting to the covering space, $F^x(x,y)-x$ is a periodic function), such that for all $z\in\mathbb{T}\times\mathbb{R}$, $\det \mathrm{D}F(z)=\sigma$.*

Notice that, by considering the symplectic product given by matrix

$$\Omega=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

the condition $\det \mathrm{D}F(z)=\sigma$ may be written as

$$(\mathrm{D}F(z))^\top\,\Omega\,\mathrm{D}F(z)=\sigma\Omega,$$

motivating the nomenclature. In the limiting case $\sigma=1$, these diffeomorphisms are symplectic.

We are interested in the computation of invariant rotational circles for F, and most particularly, those for which the internal dynamics is quasi-periodic (with a certain fixed Diophantine rotation number ω).

Definition 2.2. *We say that the circle $\mathcal{K}$ parameterized by $K:\mathbb{T}\to\mathbb{T}\times\mathbb{R}$ is an F*-invariant rotational circle *with internal dynamics $f:\mathbb{T}\to\mathbb{T}$, if $\mathcal{K}$ is homotopic to the zero section (so that, $\theta\to K^x(\theta)-\theta$ is 1-periodic) and the couple (K,f) satisfies the invariance equation:*

$$F(K(\theta))-K(f(\theta))=0. \tag{1}$$

Notice that the internal dynamics is homotopic to the identity (hence, the lift of $f(\theta) - \theta$ *is* 1-*periodic).*

A particular case, is when the internal dynamics is (smoothly) conjugate to a rotation by a certain angle $\omega \in \mathbb{R} \setminus \mathbb{Q}$ *and, hence, we can reparametize* $\mathcal{K}$ *so that*

$$F(K(\theta)) - K(\theta + \omega) = 0, \tag{2}$$

that is $f(\theta) = \theta + \omega$. *We will say then that* $\mathcal{K}$ *is a* quasi-periodic F-invariant rotational circle.

Remark 2.3. *The phase of the parameterization* K *of the circle* $\mathcal{K}$ *is* $\langle K^x(\theta) - \theta\rangle$ *where here, and in the following,* $\langle g\rangle$ *denotes the average of a periodic function* g. *Notice that the phase of a reparameterization* K_φ, *where* $\varphi \in \mathbb{R}$ *and* $K_\varphi(\theta) = K(\theta + \varphi)$, *is* $\langle K^x_\varphi(\theta) - \theta\rangle = \varphi + \langle K^x(\theta) - \theta\rangle$. *Hence, one can adjust* φ *so that the phase is zero: if* $\varphi = -\langle K^x(\theta) - \theta\rangle$ *then* $\langle K^x_\varphi(\theta) - \theta\rangle = 0$. *In summary, by a change of variables, we can assume the phase condition*

$$\langle K^x(\theta) - \theta\rangle = 0. \tag{3}$$

Notice also that, if f *is the internal dynamic corresponding to* K, *then* f_φ *given by* $f_\varphi(\theta) = f(\theta + \varphi) - \varphi$ *is the dynamics corresponding to* K_φ.

Remark 2.4. *Conformal symplecticity imposes severe restrictions for the existence of invariant circles. For instance, there can not exist invariant librational circles (those that are homotopically trivial), and no more than one invariant rotational circle. This is due to the fact that a conformally symplectic map "contracts area", that is, any bounded open domain is mapped onto a bounded open domain whose area is the one of the former domain multiplied by the conformal factor.*

In the sequel, we formulate the property of an invariant circle of being normally attracting in a rather computational way, since we are interested here in numerical algorithms and their implementations. The tangent bundle $\mathcal{TK}$ to the circle $\mathcal{K}$ is spanned by the derivative map $K' : \mathbb{T} \to \mathbb{R} \times \mathbb{R}$, where here, and in the following, $'$ denotes the derivative with respect to θ. We can consider a normal bundle $\mathcal{N}_0$ over $\mathcal{K}$ generated by $N_0 : \mathbb{T} \to \mathbb{R} \times \mathbb{R}$, where

$$N_0(\theta) = \Omega\, K'(\theta)\, (K'(\theta)^\top K'(\theta))^{-1}. \tag{4}$$

Notice that, with this choice

$$N_0(\theta)^\top \Omega\, K'(\theta) = 1.$$

The geometrical meaning is that the area of the rectangle generated by $K'(\theta)$ and $N_0(\theta)$ is 1. T2
While the tangent bundle is invariant for the linearized dynamics, and in particular

$$\mathrm{D}F(K(\theta))K'(\theta) = K'(f(\theta))f'(\theta),$$

the normal bundle $\mathcal{N}_0\mathcal{K}$ could be non-invariant, since

$$\mathrm{D}F(K(\theta))N_0(\theta) = K'(f(\theta))\, t_0(\theta) + N_0(f(\theta))\, \frac{\sigma}{f'(\theta)},$$

where

$$t_0(\theta) = N_0(f(\theta))^\top \Omega\, \mathrm{D}F(K(\theta))N_0(\theta). \tag{5}$$

In order to construct an invariant normal bundle $\mathcal{N}^s$ over $\mathcal{K}$, the stable bundle, spanned by a suitable $N : \mathbb{T} \to \mathbb{R} \times \mathbb{R}$, we write

$$N(\theta) = K'(\theta)\, \vartheta(\theta) + N_0(\theta), \tag{6}$$

for which the area of the parallelogram generated by $K'(\theta)$ and $N(\theta)$ is $N(\theta)^\top \Omega\, K'(\theta) = 1$, and realize that

$$\mathrm{D}F(K(\theta))N(\theta) = K'(f(\theta))\, t(\theta) + N(f(\theta))\, \frac{\sigma}{f'(\theta)},$$

where

$$t(\theta) = t_0(\theta) + f'(\theta)\, \vartheta(\theta) - \frac{\sigma}{f'(\theta)}\, \vartheta(f(\theta)).$$

Hence, we make $t(\theta) = 0$ by taking

$$\vartheta(\theta) = -\sum_{k=0}^{\infty} \frac{\sigma^k}{\left(f'(f^{k-1}(\theta)) \dots f'(\theta)\right)^2} \cdot \frac{t_0(f^k(\theta))}{f'(f^k(\theta))}. \tag{7}$$

The convergence of the series is guaranteed by the following property of normal attractivity.

Definition 2.5. *We say that an F-invariant rotational circle $\mathcal{K}$, parameterized by $K : \mathbb{T} \to \mathbb{T}\times\mathbb{R}$ with internal dynamics $f : \mathbb{T} \to \mathbb{T}$, is normally attracting if there exists positive constants C, λ, with $\sigma \le \lambda < 1$ such that, for all $\theta \in \mathbb{T}$, $k \in \mathbb{Z}^+$,*

$$\sigma^k \left(f'(f^{k-1}(\theta)) \dots f'(f(\theta)) f'(\theta)\right)^{-2} \le C\lambda^k, \tag{8}$$

In a nutshell, for a normally attracting invariant circle we have just constructed a frame $P : \mathbb{T} \to \mathbb{R}^{2\times 2}$, defined by juxtaposing K' and N, i.e.

$$P(\theta) = \begin{pmatrix} K'(\theta) & N(\theta) \end{pmatrix}, \tag{9}$$

that satisfies $\det P(\theta) = 1$ (the frame is symplectic) and reduces the linearized dynamics to diagonal form:

$$P(f(\theta))^{-1} \mathrm{D}F_{a_0,\mu_0,\varepsilon_0}(K(\theta))P(\theta) = \begin{pmatrix} f'(\theta) & 0 \\ 0 & \frac{\sigma}{f'(\theta)} \end{pmatrix}.$$

Remark 2.6. *Definition 2.5 is a particularization of the general definition of normally hyperbolic invariant manifold* [Fen72, HPS77] *to the context of the present paper. We have a splitting $T_{\mathcal{K}}(\mathbb{T}\times\mathbb{R}) = T\mathcal{K} \oplus \mathcal{N}^s$ of the tangent bundle to the phase space on the invariant circle $\mathcal{K}$, $T_{\mathcal{K}}(\mathbb{T}\times\mathbb{R})$, as a direct sum of the tangent bundle of $\mathcal{K}$, $T\mathcal{K}$, and the stable bundle $\mathcal{N}^s$. Both bundles in the splitting are invariant, and are characterized by rates of growth, so that the rate of contraction in the stable bundle dominates the rate of contraction in the tangent bundle. More specifically, there exist constants $C_s, C_c > 0$ and $0 < \lambda_s < \lambda_c \le 1$, to be specified below, such that:*

(a) For $\theta \in \mathbb{T}$, $v = N(\theta) \in \mathcal{N}^s$, for all $n \ge 0$:

$$\begin{aligned} |\mathrm{D}F^n(K(\theta))v| &= |\mathrm{D}F^n(K(\theta))N(\theta)| = \sigma^n \left(f'(f^{n-1}(\theta)) \dots f'(f(\theta))f'(\theta)\right)^{-1} |N(f^n(\theta))| \\ &\le C^{1/2} \frac{|N(f^n(\theta))|}{|N(\theta)|} (\lambda\sigma)^{n/2} |N(\theta)| \le C_s \lambda_s^n |v|, \end{aligned}$$

where $C_s = C^{1/2}(\max_{\theta\in\mathbb{T}} |N(\theta)|)/(\min_{\theta\in\mathbb{T}} |N(\theta)|)$ and $\lambda_s = (\lambda\sigma)^{1/2} \le \lambda < 1$.

(b) For $\theta \in \mathbb{T}$, $v = K'(\theta) \in T\mathcal{K}$, for all $n \ge 0$:

$$\begin{aligned} |\mathrm{D}F^{-n}(K(\theta))v| &= |\mathrm{D}F^{-n}(K(\theta))K'(\theta)| = \left(f'(f^{-n}(\theta)) \dots f'(f^{-2}(\theta))f'(f^{-1}(\theta))\right)^{-1} |K'(f^{-n}(\theta))| \\ &\le C^{1/2} \frac{|K'(f^{-n}(\theta))|}{|K'(\theta)|} (\lambda/\sigma)^{n/2} |K'(\theta)| \le C_c \lambda_c^{-n} |v|, \end{aligned}$$

where $C_c = C^{1/2}(\max_{\theta\in\mathbb{T}} |K'(\theta)|)/(\min_{\theta\in\mathbb{T}} |K'(\theta)|)$ *and* $\lambda_c = (\sigma/\lambda)^{1/2}$*, so that* $\lambda_s < \lambda_c \leq 1$*. The results in* [Fen72, HPS77] *imply that, even if in the assumption the invariant circle is* C^1*, then there is a bootstrap in the regularity and, for* $r \geq 1$ *such that* $\lambda_s < \lambda_c^r$*, the invariant circle is* C^r*.*

Remark 2.7. *The rate of contraction* λ *is a dynamical observable of the contracting condition, and has to be* $\lambda < 1$*. Another important observable that measures the quality of the hyperbolicity property is the (minimum) angle between the invariant bundles. In the setting of the present paper, this is given by*

$$\alpha = \min_{\theta\in\mathbb{T}} \left| \arctan\left(\frac{1}{\vartheta(\theta)[K'(\theta)^\top K'(\theta)]} \right) \right|. \tag{10}$$

In the case $\alpha > 0$*, there is a well-defined splitting in tangent and invariant normal bundles.*

Remark 2.8. *We will be mainly interested in the quasi-periodic case, for which* $f(\theta) = \theta + \omega$ *(see* (2)*). Hence,* $f'(\theta) = 1$*, and the rate of contraction is* $\lambda = \sigma < 1$*. In this case, the quality of the hyperbolicity property is essentially given by the positiveness of the angle* α *between bundles defined in Remark 2.7.*

Remark 2.9. *There are several methods available to compute normally hyperbolic invariant manifolds, for instance* [BOV97, BHV07, Hen11, Can14, HCF+16, Gra17, BC19]*. In the computations performed in this paper, we have tailor the algorithm presented in* [Can14, HCF+16] *to obtain an algorithm to compute normally attracting circles in conformally symplectic systems.*

2.2. **Adjustment of parameters and twist condition.** It is well-known the need of parameters in order to adjust the dynamics on an invariant circle to a specific rotation [BHS96, CC10, CCdlL13b, CF12, CH14, BC19], but to do so non-degeneracy conditions are also mandatory. While persistence under perturbations of an invariant circle has to do with the property of normal hyperbolicity [Fen72, HPS77, Mañ78], the adjustment to an specific quasi-periodic dynamics is related to KAM theory [Arn61]. Thus, if F_p is a p-parameter family of conformally symplectic systems, where $p \in P \subset \mathbb{R}^m$ and P is an open set of parameters, and $\mathcal{K}_p$ is a normally attracting F_{p_0}-invariant circle for a certain $p_0 \in P$, then it persists as a normally attracting F_p-invariant circle $\mathcal{K}_p$ for p values in a neighborhood of p_0. Dynamics on the invariant circle also depends on p. When we impose dynamics to be conjugated to a rigid rotation with a fixed rotation number ω we need a one dimensional parameter since ω is 1-dimensional. In the p-dimensional case, there is typically a codimension 1 manifold in the parameter space for which dynamics on the circle is the rigid rotation by ω.

In the context of the present paper, assume we have a 1-parameter family of conformally symplectic maps $a \to F_a$ on the annulus, with a fixed conformal factor σ ($|\sigma| < 1$). Hence, if $\mathcal{K}_{a_0}$ is a normally attracting F_{a_0}-invariant rotational circle for a particular parameter value a_0, then there is an open neighborhood of a_0 for which there is a normally contracting F_a-invariant rotational circle $\mathcal{K}_a$ for each a in such a neighborhood. Hence, we assume there are smooth maps $a \to K_a$ and $a \to f_a$ such that, for each a:

$$F_a(K_a(\theta)) - K_a(f_a(\theta)) = 0.$$

Hence, we can define a rotation number function $a \to \rho(a)$ such that, $\rho(a)$ is the rotation number of the internal circle dynamics f_a. The resonant set is the set of parameters for which the rotation number is rational (and the internal dynamics possesses periodic orbits), and the non-resonant set corresponds to irrational rotation numbers. The regularity of the rotational circles jump from being (typically) finitely differentiable in the resonant set (generically with non-empty interior) to C^∞ (or even real-analytic if the maps are real-analytic) in the non-resonant set (generically with empty interior). The graph of the rotation number function is (typically) a devil staircase, whose steps correspond to resonances. Then, we say that an invariant rotational circle K_{a_0} that is quasi-periodic (say with rotation number ω) satisfies the twist rotation number if the rotation number function is strictly monotone at a_0, otherwise we say it is non-twist with respect to parameter a (or that it is an non-a-twist invariant rotational circle). Notice that for the rotation number ω one can (locally) adjust parameter a to the value a_0 (that is, close to a_0 there is no other parameter for which the corresponding invariant circle has rotation number ω).

In place of previous dynamical definitions of the twist and non-twist properties we will use the following analytical definition, which is more practical (and can be easily generalized to higher dimensions).

Definition 2.10. *Let $\mathcal{K}$ be a quasi-periodic F_{a_0}-invariant rotational circle $\mathcal{K}$ parameterized by $K : \mathbb{T} \to \mathbb{T} \times \mathbb{R}$, with irrational rotation number ω, so that*

$$F_{a_0}(K(\theta)) - K(\theta + \omega) = 0. \tag{11}$$

We define the a-twist (the twist with respect to parameter a) to be the number

$$b_a(K, a_0) = \langle N(\theta + \omega)^\top \Omega \, \mathrm{D}_a F_{a_0}(K(\theta))\rangle, \tag{12}$$

where N is the parameterization of the normal invariant bundle. Then, the circle is a-twist if $b_a(K, a_0) \neq 0$, and non-a-twist if $b_a(K, a_0) = 0$.

The definition of the a-twist condition in Definition 2.10 is motivated by the fact that, for ω Diophantine, KAM techniques can be applied to obtain real-analytic solutions of the invariance equation

$$F_a(K(\theta)) - K(\theta + \omega) = 0, \tag{13}$$

under suitable sufficient conditions including the a-twist condition [CCdlL13b]. The unknowns in (13) are both the parameterization K and the *adjusting parameter a* (since it has to be adjusted to get the quasi-periodic dynamics on the invariant circle with specific rotation number ω).

In this paper we are interested in studying the boundaries of twist property, particularly in developing algorithms for computing rotational invariant circles with internal dynamics given by the rotation by ω when the twist condition, with respect to a parameter, fails. It is usually the case that at the boundaries of the non-degeneracy conditions there appear new phenomena and transitions (think for instance in bifurcations of fixed points, for which degeneracies do appear at bifurcation values of the parameters). In the context of the present paper, it is important to know where theorems such as [CCdlL13b] do not work (in a similar fashion in the symplectic case, where one can be interested in detecting where the classical KAM theorems are not applicable), and then to try to get new results and methodologies to cover degenerate cases. It is also useful to detect where the rotation number of an invariant circle in a one-parameter family of conformally symplectic maps is not a monotone function of the parameter (in the same way in the symplectic case one can be interested in detecting where the frequency map, that gives the rotation number of a rotational invariant curve as a function of its action, is not monotone).

2.3. **Examples: dissipative standard non-twist maps.** The examples we will consider in this paper are families of *dissipative standard non-twist maps*, that are conformally symplectic maps given by 3-parameter maps $F_{a,\mu,\varepsilon} : \mathbb{T}\times\mathbb{R} \to \mathbb{T}\times\mathbb{R}$ given by

$$F_{a,\mu,\varepsilon}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + (\sigma y + \varepsilon p(x) - a)^2 + \mu \\ \sigma y + \varepsilon p(x) \end{pmatrix}, \tag{14}$$

where $p : \mathbb{T} \to \mathbb{R}$ is a 1-periodic function, σ is the conformal factor (which we assume it is fixed), a, μ are adjusting parameters (whose roles will be explained below), and ε is the perturbative parameter.

We start by analyzing (14) for $\varepsilon = 0$, which is integrable. For each a, μ, there is an $F_{a,\mu,0}$-invariant circle parameterized by

$$K_{a,\mu,0}(\theta) = \begin{pmatrix} \theta \\ 0 \end{pmatrix},$$

whose internal dynamics is given explicitly by

$$f_{a,\mu,0}(\theta) = \theta + a^2 + \mu.$$

Moreover, the adapted frame and the corresponding linearized dynamics are

$$P(\theta) = \begin{pmatrix} \mathrm{D}K(\theta) & N(\theta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Lambda(\theta) = \begin{pmatrix} \mathrm{D}f(\theta) & 0 \\ 0 & \Lambda_N(\theta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}.$$

Normal attractivity applies and in fact for ε small enough there is an $F_{a,\mu,\varepsilon}$-invariant circle parameterized by $K_{a,\mu,\varepsilon}$.

If we are looking for an initial invariant tori with fixed quasi-periodic frequency ω, then parameter a, μ, ε are linked. In particular, for $\varepsilon = 0$, a and μ satisfy the equation

$$\omega = a^2 + \mu. \tag{15}$$

That is, $\mu = \omega - a^2$. The a-twist is

$$b_a(K, a, \mu, 0) = \langle N(\theta+\omega)^\top \Omega\, \mathrm{D}_a F_{a,\mu,0}(K(\theta))\rangle = 2a,$$

while the μ-twist is

$$b_\mu(K, a, \mu, 0) = \langle N(\theta+\omega)^\top \Omega\, \mathrm{D}_\mu F_{a,\mu,0}(K(\theta), a, \mu)\rangle = 1.$$

Since the μ-twist is non zero we can isolate μ. Notice however, that the invariant circle with frequency ω is non-a-twist for whenever $a = 0$ (and hence $\mu = \omega$).

Since the μ-twist is non zero, from an implicit function theorem we get that for a, ε close to zero, we can find $\mu = \mu(a, \varepsilon)$ and a circle parameterized by $K_{a,\mu,\varepsilon}$ which is invariant for $F_{a,\mu,\varepsilon}$ and whose internal dynamics is a rotation with frequency ω. So given a, ε, the parameter μ is used to adjust the frequency to ω. By writing $\bar{b}_a(a, \varepsilon) = b_a(K_{a,\mu(a,\varepsilon),\varepsilon}; a, \mu(a, \varepsilon), \varepsilon)$, then the equation we need to solve in order to find a non-a-twist invariant circle is

$$\bar{b}_a(a, \varepsilon) = 0,$$

and to apply the implicit function theorem in order to find a for small enough ε we also need that

$$\frac{\partial \bar{b}_a}{\partial a}(0, 0) \neq 0.$$

In our example, $\frac{\partial \bar{b}_a}{\partial a}(0, 0) = 2$.

In the following we will present algorithms for computing a-twist and non-a-twist invariant circles, and for computing surfaces in 3D parameter space corresponding to invariant circles of dissipative standard non-twist maps with a fixed rotation number ω. Section 4 is devoted to illustrate actual implementations and to explore the dynamics.

3. The algorithms

In this section we present algorithms for computing and continuing with respect to parameters invariant rotational circles of 2-dimensional conformally symplectic systems with a prescribed quasi-periodic dynamics. Algorithms for computing invariant circles with fixed quasi-periodic dynamics under twist conditions with respect to parameters (in particular, with respect to an adjusting parameter a) are presented in [CC10, CF12, CH14, CH17a]. We present here algorithms for computing both a-twist (with a fixed a-twist) and non-a-twist invariant rotational circles (using an unfolding parameter μ), and continuation (with respect to a perturbing parameter ε). The algorithms to solve the invariance equations are based on Newton's method. For the continuation, we use three parameters. Typically in order to have a non-twist circle with respect to a parameter one needs at least two parameters. The third parameter is for continuing the degeneracy. Of course, one could have even more parameters in a particular problem.

In this setting, assume we are given a 3-parameter family of conformally symplectic maps given by a map $F : \mathbb{T} \times \mathbb{R} \times A \times U \times E \to \mathbb{T} \times \mathbb{R}$, where $A, U, E \subset \mathbb{R}$ are open sets, such that, for each $(a, \mu, \varepsilon) \in A \times U \times E$, $F_{a,\mu,\varepsilon} = F(\,\cdot\,; a, \mu, \varepsilon)$ is a conformally symplectic map with conformal factor $\sigma \in]0, 1[$). Geometrically, the goal is obtaining a surface Σ_ω in parameter space $A \times U \times E$ such that for each triple $(a, \mu, \varepsilon) \in \Sigma_\omega$ there is an invariant circle with a fixed (Diophantine) rotation number ω. From normal attractivity, for (a, μ, ε) in a neighborhood of Σ_ω there is also an invariant circle, whose rotation number we denote as $\rho(a, \mu, \varepsilon)$.

Let us fix the parameter ε_0 and the a-twist b_a^0, so we look for solutions (K, a, μ) of the system of equations

$$F(K(\theta), a, \mu, \varepsilon_0) - K(\theta + \omega) = 0, \tag{16}$$
$$\langle K^x(\theta) - \theta \rangle = 0, \tag{17}$$
$$b_a(K, a, \mu, \varepsilon_0) - b_a^0 = 0. \tag{18}$$

Equation (16) is the invariance equation of the circle, with quasi-periodic dynamics with rotation number ω, (17) is the phase equation for the parameterization, and (18) is the equation for the a-twist b_a^0. By moving ε_0 and b_a^0 we will generate the goal parameter surface Σ_ω. Moreover, we will obtain the curve Γ_ω in Σ_ω, corresponding to $b_a^0 = 0$, at which (typically) the rotation number is non-monotonic as a function of parameter a (e.g., for $(a_0, \mu_0, \varepsilon_0) \in \Gamma_\omega$, the function $a \to \rho(a, \mu_0, \varepsilon_0)$ is non-monotonic).

Assume then we have an approximate solution (K, a, μ) of (16), (17), (18). The aim to perform one step of the Newton's method is computing the corrections $(\Delta K, \Delta a, \Delta\mu)$ to obtain a new approximate solution $(\bar{K}, \bar{a}, \bar{\mu})$ which will have an error that is quadratically small with respect to the initial error, even though the linearized equations are solved approximately using appropriate frames. The starting point is a triple (K, a, μ) such that

$$F(K(\theta), a, \mu) - K(\theta + \omega) = E(\theta), \tag{19}$$
$$\langle K^x(\theta) - \theta\rangle = e_p, \tag{20}$$
$$b_a(K, a, \mu) - b_a^0 = e_b, \tag{21}$$

where $E : \mathbb{T} \to \mathbb{R}^2$ and e_p, e_b are error terms. For the moment, we remove the dependence on ε because at this point this parameter plays no role. In the following, we will proceed in two steps: 1) for any Δa, we compute ΔK and $\Delta\mu$ to improve (19) and (20); 2) we adjust Δa (and hence ΔK and $\Delta\mu$) to improve (21).

The first step consists in solving

(22)
$$\mathrm{D}F(K(\theta), a, \mu, \varepsilon_0)\Delta K(\theta) + \tfrac{\partial F}{\partial a}(K(\theta), a, \mu, \varepsilon_0)\Delta a + \tfrac{\partial F}{\partial \mu}(K(\theta), a, \mu, \varepsilon_0)\Delta\mu - \Delta K(\theta + \omega) = -E(\theta),$$

for any Δa. To do so, we first compute, from $L(\theta) = K'(\theta)$, the expressions of $N_0(\theta)$ and $t_0(\theta)$, $\vartheta(\theta)$ and $N(\theta)$, and then the frame $P : \mathbb{T} \to \mathbb{R}^{2\times 2}$ given by

$$P(\theta) = \begin{pmatrix} K'(\theta) & N(\theta) \end{pmatrix},$$

that satisfies $\det P(\theta) = 1$ and

$$\mathrm{D}F(K(\theta), a, \mu)P(\theta) = P(\theta + \omega)\Lambda(\theta) + E_r(\theta), \tag{23}$$

where

$$\Lambda(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}$$

and the reducibility error is

$$E_r(\theta) = \begin{pmatrix} E'(\theta) & E_r^N(\theta) \end{pmatrix},$$

with

$$E_r^N(\theta) = \left(E'(\theta)^\top \Omega\, \mathrm{D}F(K(\theta), a, \mu)N_0(\theta)\right) N_0(\theta + \omega) + E'(\theta)\, \vartheta(\theta).$$

We emphasize that the cohomological equation for ϑ,

$$\vartheta(\theta) - \sigma\, \vartheta(\theta + \omega) = -t_0(\theta). \tag{24}$$

can be solved in Fourier space:

$$\vartheta(\theta) = \sum_{k\in\mathbb{Z}} \frac{-t_{0k}}{1 - \sigma e^{2\pi \mathrm{i} k\omega}} e^{2\pi \mathrm{i} k\theta}.$$

(Notice that, since $|\sigma| < 1$, the divisors are uniformly far from 0.) Then, we write the correction term of the parameterization as $\Delta K(\theta) = P(\theta)\xi(\theta)$, where $\xi : \mathbb{T} \to \mathbb{R}^2$ is a periodic function. By multiplying (22) by $P(\theta+\omega)^{-1}$, using approximate reducibility (23) and neglecting quadratically small terms, we obtain the following cohomological equation

$$\Lambda(\theta)\xi(\theta) - \xi(\theta + \omega) + B_a(\theta)\Delta a + B_\mu(\theta)\Delta\mu = \eta(\theta),$$

where

$$B_a(\theta) = P(\theta + \omega)^{-1} \tfrac{\partial F}{\partial a}(K(\theta), a, \mu),\ B_\mu(\theta) = P(\theta + \omega)^{-1} \tfrac{\partial F}{\partial \mu}(K(\theta), a, \mu)$$

and $\eta(\theta) = -P(\theta+\omega)^{-1}E(\theta)$ is the error of invariance in the adapted frame. Notice that the previous system splits into the diagonal system

$$\xi^L(\theta) - \xi^L(\theta+\omega) + B_a^L(\theta)\Delta a + B_\mu^L(\theta)\Delta\mu = \eta^L(\theta), \tag{25}$$

$$\sigma\xi^N(\theta) - \xi^N(\theta+\omega) + B_a^N(\theta)\Delta a + B_\mu^N(\theta)\Delta\mu = \eta^N(\theta), \tag{26}$$

where

$$\begin{aligned} B_a^L(\theta) &= N(\theta+\omega)^\top\Omega\,\tfrac{\partial F}{\partial a}(K(\theta),a,\mu), \quad & B_\mu^L(\theta) &= N(\theta+\omega)^\top\Omega\,\tfrac{\partial F}{\partial \mu}(K(\theta),a,\mu)\\ B_a^N(\theta) &= -L(\theta+\omega)^\top\Omega\,\tfrac{\partial F}{\partial a}(K(\theta),a,\mu), \quad & B_\mu^N(\theta) &= -L(\theta+\omega)^\top\Omega\,\tfrac{\partial F}{\partial \mu}(K(\theta),a,\mu). \end{aligned}$$

In particular: $b_a(K,a,\mu) = \langle B_a^L(\theta)\rangle$, $b_\mu(K,a,\mu) = \langle B_\mu^L(\theta)\rangle$. It is the moment to face cohomological equations (25) and (26), which are in fact very different, and introduce some notation. We will denote by $\xi = \mathcal{R}_\sigma\eta$ the solution of

$$\sigma\xi(\theta) - \xi(\theta+\omega) = \eta(\theta),$$

that is, in Fourier series:

$$\xi(\theta) = \mathcal{R}_\sigma\eta(\theta) = \sum_{k\in\mathbb{Z}} \frac{\eta_k}{\sigma - e^{2\pi \mathbf{i} k\omega}} e^{2\pi\mathbf{i}k\theta}.$$

Notice again that, since $|\sigma| < 1$, the divisors are uniformly far from 0. We will denote by $\xi = \mathcal{R}\eta$ the solution of

$$\xi(\theta) - \xi(\theta+\omega) = \eta(\theta) - \langle\eta\rangle$$

with zero average, that is, in Fourier series:

$$\xi(\theta) = \mathcal{R}\eta(\theta) = \sum_{k\in\mathbb{Z}^*} \frac{\eta_k}{1 - e^{2\pi \mathbf{i} k\omega}} e^{2\pi\mathbf{i}k\theta}.$$

The solution involves small divisors and it suffices Diophantine conditions on ω to ensure the convergence of the expansions.

With the aid of these operators, we solve (25) and (26) as follows. We compute $\Delta\mu = \Delta\mu[\Delta a]$ by adjusting averages in (25), so that

$$\Delta\mu = \frac{\langle\eta^L\rangle - \langle B_a^L\rangle\Delta a}{\langle B_\mu^L\rangle} \simeq \frac{\langle\eta^L\rangle - b_a^0\Delta a}{b_\mu},$$

where in the last approximation we are skipping second order error terms. Notice that we need a twist condition with respect to parameter μ. We emphasize the dependence of $\Delta\mu$ on Δa (as we will do in the sequel for other objects). With this choice of $\Delta\mu$ we compute $\xi^L = \xi^L[\Delta a]$, $\xi^N = \xi^N[\Delta a]$ as follows:

$$\xi^N(\theta) = \mathcal{R}_\sigma\eta(\theta) - \mathcal{R}_\sigma B_a^N(\theta)\,\Delta a - \mathcal{R}_\sigma B_\mu^N(\theta)\,\Delta\mu$$

for the solution of (26)

$$\hat\xi^L(\theta) = \mathcal{R}\eta(\theta) - \mathcal{R}B_a^N(\theta)\,\Delta a - \mathcal{R}B_\mu^N(\theta)\,\Delta\mu$$

for the zero-average solution of (25),

$$\xi_0^L = -e_p - \langle L^x(\theta)\hat\xi^L(\theta) + N^x(\theta)\xi^N(\theta)\rangle$$

to fix the phase (notice that $\langle L^x\rangle = 1$) and, finally

$$\xi^L(\theta) = \xi_0^L + \hat\xi^L(\theta).$$

Finally, we obtain the correction $\Delta K = \Delta K[\Delta a]$ for improving (19) and (20):

$$\Delta K[\Delta a](\theta) = L(\theta)\xi^L[\Delta a](\theta) + N(\theta)\xi^N[\Delta a](\theta).$$

In summary, from the previous recipe we obtain a univariate function

$$\Delta a \to b_a[\Delta a] = b_a(K + \Delta K[\Delta a], a + \Delta a, \mu + \Delta\mu[\Delta a], \varepsilon_0)$$

for which we have to solve the equation

$$b_a[\Delta a] - b_a^0 = 0. \tag{27}$$

In the implementation of each step of Newton method, instead of solving this equation, we apply one step of Steffensen's method to this equation starting with $\Delta a = 0$. In the implementation, we control the non-degeneracy condition to solve (27).

With the previous Newton method we compute an invariant rotational circle with fixed a-twist for a fixed value of ε_0. In order to implement the continuation with respect to parameter ε one can compute derivatives of (K, a, μ) with respect to ε, at ε_0. The type of equations one has to solve are of the same type as to perform a Newton step. In particular, one has (22) with

$$E(\theta) = \frac{\partial F}{\partial \varepsilon}(K(\theta), a, \mu, \varepsilon_0),$$

and $\Delta K = \frac{\partial K}{\partial \varepsilon}$, $\Delta a = \frac{\partial a}{\partial \varepsilon}$, $\Delta_\mu = \frac{\partial \mu}{\partial \varepsilon}$.

For the implementation of Newton's method and continuation method described here we use Fourier series to represent periodic functions. Thus, we use FFTs to switch from grid representation to Fourier representation. All operations can be done at linear cost in grid or Fourier representations, except the ones switching representations. Hence, the cost of the algorithms is $O(N\log(N))$ where N is the size of the representation (the size of the grid or the number of Fourier modes). See e.g. [CdlL09, CdlL10, HCF+16] for some guidelines.

4. The applications

In this section, we implement the algorithms presented in the previous section to a couple of families of dissipative standard maps (14): (symmetric) $p(x) = \frac{1}{2\pi}\sin(2\pi x)$, and (non-symmetric) $p(x) = \frac{1}{2\pi}(\sin(2\pi x) + \cos(4\pi x))$.

4.1. Continuation of the non-a-twist circle in the symmetric case.

In this section we study the family (14) with $p(x) = \frac{1}{2\pi}\sin(2\pi x)$. Since $p(x-\frac{1}{2}) = -p(x)$ then the involution $S : \mathbb{T}\times\mathbb{R} \to \mathbb{T}\times\mathbb{R}$ defined by

$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - \frac{1}{2} \\ -y \end{pmatrix}$$

is a symmetry of the family with respect to parameter a, meaning that

$$S \circ F_{a,\mu,\varepsilon} \circ S = F_{-a,\mu,\varepsilon}.$$

This symmetry property implies that if $K_{a,\mu,\varepsilon}$ is a parameterization of an invariant circle for $F_{a,\mu,\varepsilon}$ with internal dynamics $f_{a,\mu,\varepsilon}$, then $K_{-a,\mu,\varepsilon} = S \circ K_{a,\mu,\varepsilon}$ is parameterization of an invariant circle for $F_{-a,\mu,\varepsilon}$ with internal dynamics $f_{-a,\mu,\varepsilon} = f_{a,\mu,\varepsilon}$. In particular, for $a = 0$, the invariant circle parameterized by $K_{0,\mu,\varepsilon}$ is S-symmetric, since it is also parameterized by $S \circ K_{0,\mu,\varepsilon}$. In fact, since they is a unique parameterization such that $\langle K^x_{0,\mu,\varepsilon}(\theta) - \theta\rangle = 0$, it satisfies:

$$K_{0,\mu,\varepsilon}(\theta) = S \circ K_{0,\mu,\varepsilon}(\theta + \tfrac{1}{2}).$$

In this case, by selecting μ so that $f_{0,\mu,\varepsilon}(\theta) = \theta + \omega$, we have

$$\bar{b}_a(0, \varepsilon) = b_a(K_{0,\mu,\varepsilon}, 0, \mu, \varepsilon) = 0.$$

Q10: parameter $\sigma =$ 0.8

That is, the tori with $a = 0$ are non-a-twist. This example will be a first test of our algorithms.

In the following, the conformal factor is selected to be $\sigma = 0.8$. First, we continue with respect to parameter ε a non-a-twist circle with rotation number $\omega = \frac{1}{2}(\sqrt{5} - 1)$, and adjust parameters a and μ accordingly. The adjusting parameters are shown in Figure 1. Starting from $\varepsilon = 0$, the continuation goes up to $\varepsilon = 3.658600$, close to breakdown, in which the number of Fourier modes demanded by the algorithm is 262144. Some of the non-a-twist circles are shown in Figure 2, together with the corresponding tangent and stable bundles, represented by their angles with respect to the horizontal axis α. The complex behavior observed in the bundles preludes the breakdown of the invariant circle. Notice that when both bundles collide, the normal hyperbolicity property fails, and this happens even though the contraction factor is far from 1 (it is $\sigma = 0.8$). This collision behavior has been observed in other contexts [CH14, CH17a, FH15, HdlL06, HdlL07], and in [CF12] for a-twist circles in conformally symplectic systems. From these references one conjectures that, even though the behavior is very wild, there is some sort of regularity and the minimum angle between the invariant bundles behaves very smoothly, in fact asymptotically in a linear fashion when approaching the breakdown, as shown in Figure 3. This behavior lets us extrapolate the critical breakdown parameter very consistingly, being $\varepsilon_c \simeq 3.662396$.

Q10: parameter $\sigma =$ 0.8

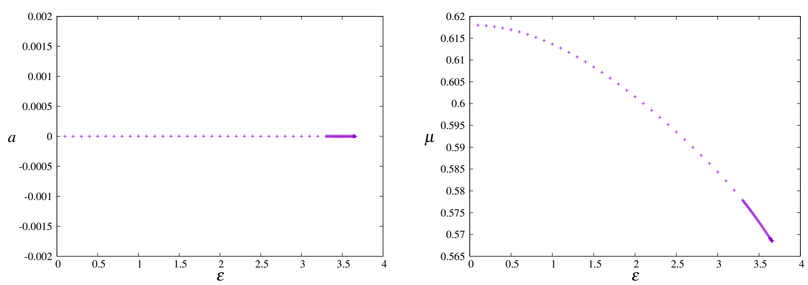

FIGURE 1. Continuation w.r.t. ε of a non-a-twist circle with frequency ω (symmetric case): (left) adjusting parameter a; (right) unfolding parameter μ.

The symmetry properties of the family lead to several features. First, parameter a is always 0, as it is shown in Figure 1. Moreover, the non-a-twist circles and their bundles have also symmetry properties, as it is shown in Figure 2. In particular, We note that the collapse in this symmetric case happens on both sides of the bundles. We expect that when the bundles collapse, there will be no gap between the bundles on either side of the bundles with respect to α. Later in this section, we will see that in the nonsymmetric the bundles collapse leaving a gap between the bundles for all values of θ, but only on one side of the bundles with respect to α.

We also performed some computations to illustrate that the analytic condition that the invariant circle is non-a-twist translates into dynamical properties of the rotation number of the invariant cicle when we move parameters. We have tailored the algorithm in chapter 5 of [HCF+16] to continue invariant tori regardless of the internal dynamics and compute the corresponding rotation number, by starting with a non-a-twist circle from the previous implementation (see

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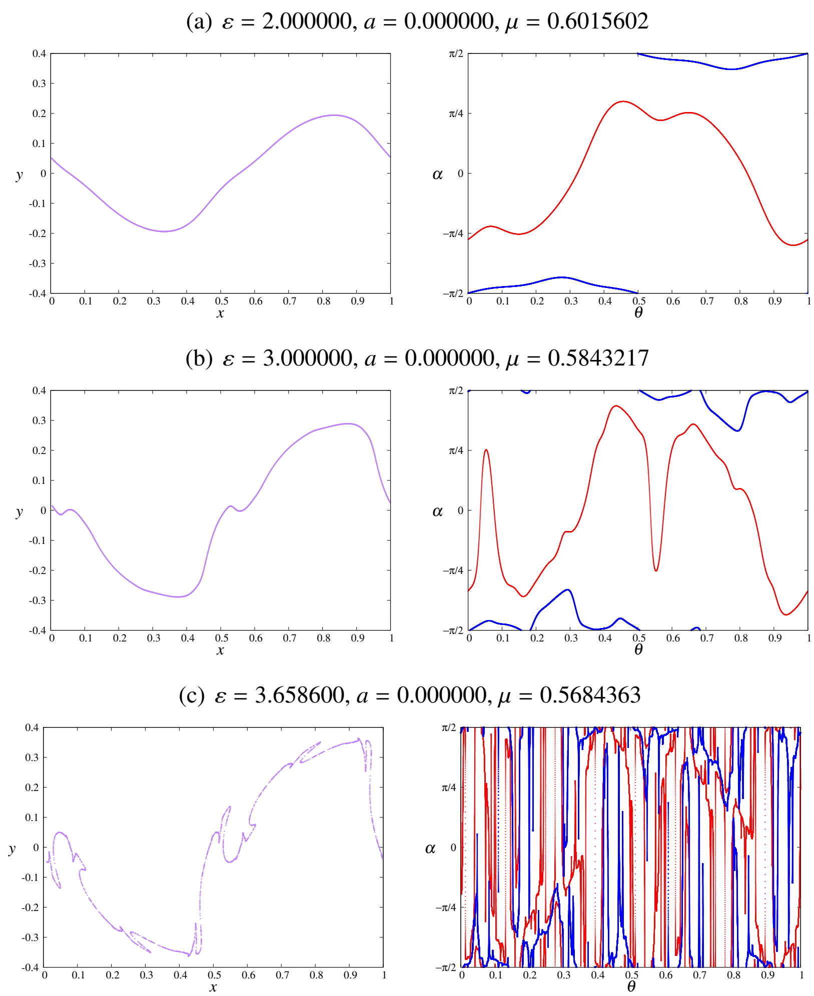

FIGURE 2. Continuation w.r.t. ε of a non-a-twist circle with frequency ω (symmetric case): (left) invariant circle; (right) projectivized tangent bundle (in red) and stable bundle (in blue).

e.g. [BOV97, BHV07, Hen11] for other algorithms of computation of normally hyperbolic invariant manifolds). In particular, we have selected a non-a-twist circle for $\varepsilon = 2.2$, so that $\mu = 1.5984626393$ and $a = 0$. We first perform continuations for μ and ε fixed, increasing and decreasing the parameter a, respectively. The graph of the rotation number of the invariant circle as a function of a is shown in Figure 4 (Left). As expected, the non-a-twist circle corresponds to a critical point of this graph. The graph is symmetric, also as expected from the symmetry properties of the family being studied. Notice also the presence of visible resonances, corresponding to rotation number $5/8$. However, by performing a continuation with respect to μ instead of a (and starting with the same initial torus), we observe that the starting torus does not correspond

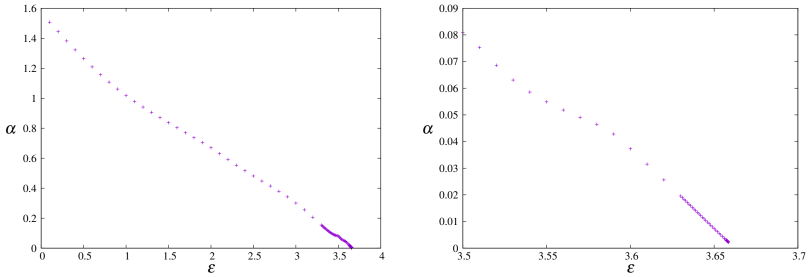

FIGURE 3. Continuation w.r.t. ε of a non-a-twist circle with frequency ω (symmetric case): (left) minimum angle α between the stable and tangent bundles as function of ε; (right) critical behavior. The breakdown of the circle is produced at $\varepsilon_c \simeq 3.662396$.

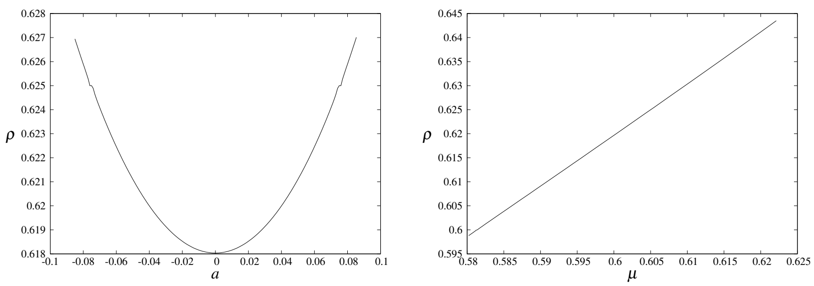

FIGURE 4. Rotation number versus continuation parameter from the non-a-twist circle with $a = 0.000000$, $\mu = 0.5984626$, $\varepsilon = 2.20000$ (symmetric case): (left) continuation w.r.t. a; (right) continuation w.r.t. μ.

to a minimum of the rotation number as a function of μ, as shown in Figure 4 (Right). This is because the non-twist-property is associated to parameter a, and the invariant circle is μ-twist.

4.2. **Continuation of the non-a-twist circle in the nonsymmetric case.** In this section we consider the family (14) with $p(x) = \frac{1}{2\pi}(\sin(2\pi x) + \cos(4\pi x)$, that (apparently) does not have symmetry properties. We again take $\sigma = 0.8$, and $\omega = \frac{1}{2}(\sqrt{5} - 1)$. We have followed the same plan as in previous example.

Q8: parameter $\sigma = 0.8$

First, with the algorithms of Section 3, we continue with respect to parameter ε a non-a-twist circle with rotation number $\omega = \frac{1}{2}(\sqrt{5} - 1)$. The adjusting parameters a and μ as functions of perturbation parameter ε are shown in Figure 5. Unlike the symmetric case, parameter a varies, and remains bounded inside an interval of size 2.6×10^{-3} around zero. The continuation reaches the value $\varepsilon = 1.230340$, in which the invariant circle is approximated with a truncated Fourier series with 524288 modes. The process of breakdown and the collision of the invariant bundles is shown in Figure 6. We notice that in contrast with the bundle collapse in the symmetric version of the dissipative standard non-twist map, the collapse for this example only happens on one side of the bundles, leaving a gap between the bundles. The minimum angle between bundles is also asymptotically linear when close to breakdown, see Figure 7, from which we can extrapolate the critical value $\varepsilon_c \simeq 1.240522$.

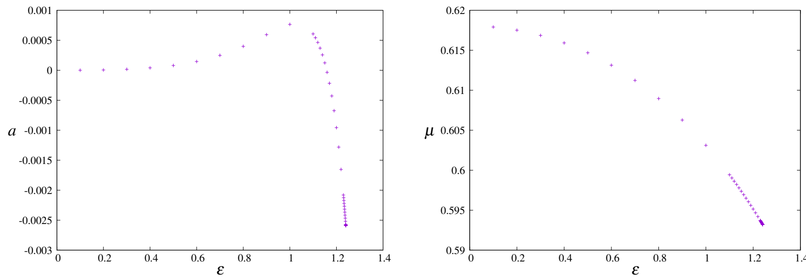

FIGURE 5. Continuation w.r.t. ε of a non-a-twist circle with frequency ω (non-symmetric case): (left) adjusting parameter a; (left) unfolding parameter μ.

As in the first example, in Figure 8 we show the graph of the rotation number of the invariant circle as a function of a parameter of continuation (either a or μ) starting at a non-a-twist circle for $\varepsilon = 1.00000$, $a = 7.646104 \cdot 10^{-4}$, $\mu = 0.6031124$. The figure provides again a dynamical interpretation of the fact that the invariant circle is non-a-twist, but μ-twist.

In Figure 9, we show continuations with respect to ε of invariant circles with fixed frequency ω and different values of the a-twist. That is, we compute the surface Σ_ω of parameter points for which there is an invariant circle with frequency ω. We plotted this surface showing the values of a and μ along the ε continuation. In particular, the continuation curve corresponding to an a-twist b_a starts with $a = \frac{1}{2}b_a$, $\mu = \omega - a^2$ and $\varepsilon = 0$. We have highligted the curve Γ_ω corresponding to zero a-twist. Note that the surface is not symmetric with respect to a and for negative a-twist there is a region where the circles seem to persist for larger values of μ and ε.

5. THE CONCLUSIONS

In this paper we have clarified the property of being non-twist for a circle, in the context of conformally symplectic systems. This non-twist property has to do with the degeneracy condition arising when tuning a particular parameter to fix the dynamics of an invariant circle to a given rotation number. Hence, the non-twist condition is with respect to a particular parameter. As such, the concept can be extended to many other systems in which parameters have to be adjusted to fix the frequency, as in [CH17a]. In symplectic systems, the parameters to adjust are the actions of a torus.

We have also presented several algorithms for computing invariant circles, including non-twist circles and a methodology to compute parametric surfaces in parameter space corresponding to invariant circles with a prescribed (Diophantine) frequency. The key of our methodology is introducing a concept of twist with respect to a parameter, so one can compute continuation curves corresponding to a fix twist. Unlike the symplectic case, non-twist tori in conformally symplectic systems do not seem to be the more robust, meaning they are not the ones that survive for greater values of perturbation parameters.

The algorithms are very efficient, and let us compute invariant circles even with hundreds of thousands of Fourier coefficients, and then explore the regimes at the verge of analyticity breakdown.

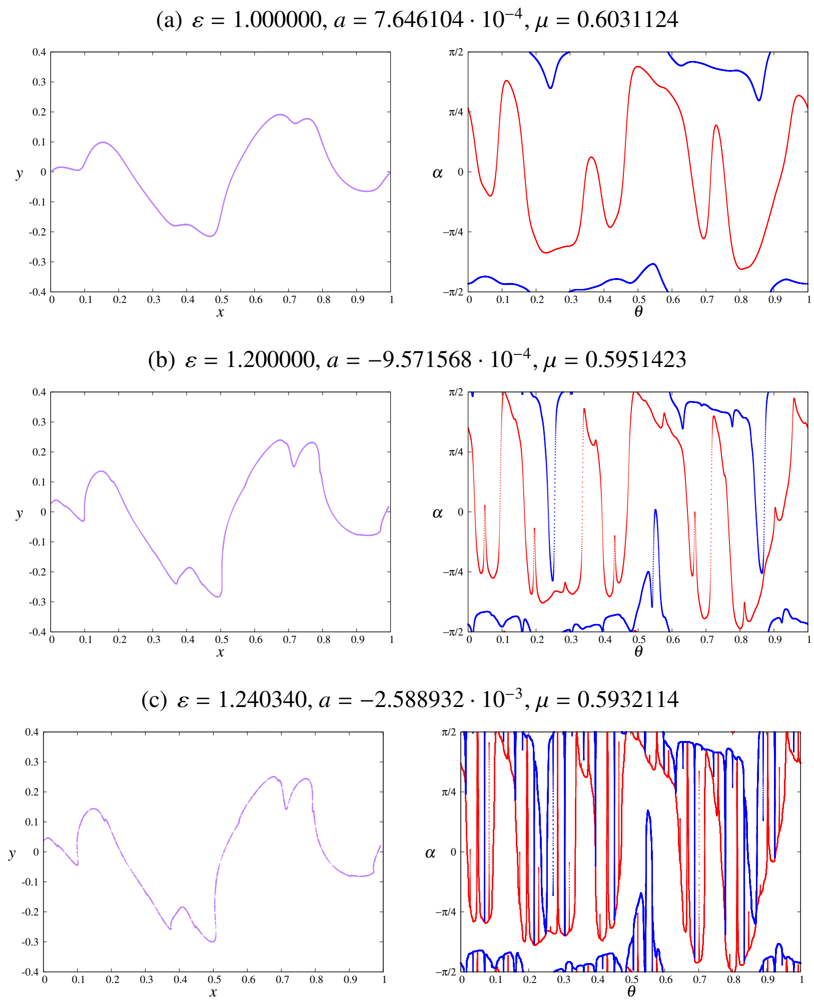

FIGURE 6. Continuation w.r.t. ε of a non-a-twist circle with frequency ω (non-symmetric case): (left) invariant circle; (right) projectivized tangent bundle (in red) and stable bundle (in blue).

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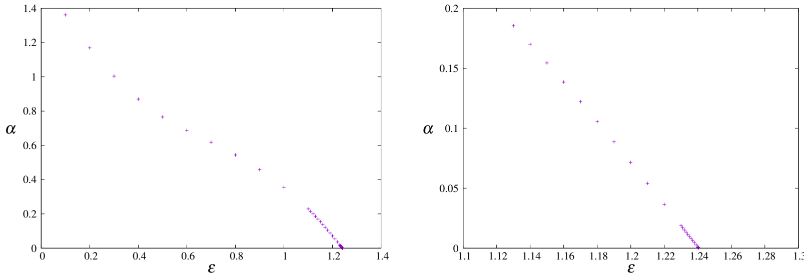

FIGURE 7. Continuation w.r.t. ε of a non-a-twist circle with frequency ω (non-symmetric case): (left) minimum angle α between the stable and tangent bundles; (right) critical behavior. The breakdown of the circle is produced at $\varepsilon_c \simeq 1.240522$.

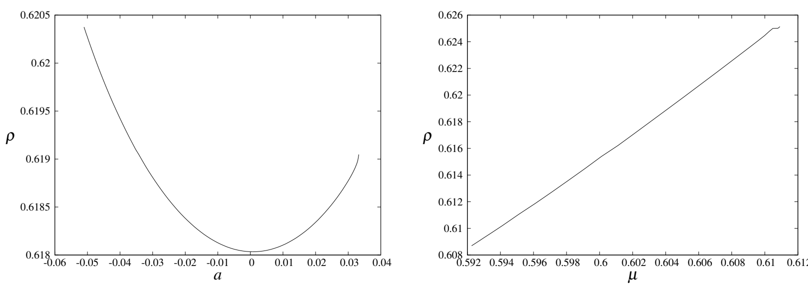

FIGURE 8. Rotation number versus continuation parameter from the non-a-twist circle with $a = 7.646104 \cdot 10^{-4}$, $\mu = 0.6031124$, $\varepsilon = 1.00000$ (non-symmetric case): (left) Continuation w.r.t. a; (right) Continuation w.r.t. μ.

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REFERENCES

[Arn61] V.I. Arnold, *Small denominators. I. Mapping the circle onto itself*, Izv. Akad. Nauk SSSR Ser. Mat. **25** (1961), 21–86.

[BC19] Adrián P. Bustamante and Renato C. Calleja, *Computation of domains of analyticity for the dissipative standard map in the limit of small dissipation*, Phys. D **395** (2019), 15–23. MR 3958298

[BHS96] H.W. Broer, G.B. Huitema, and M.B. Sevryuk, *Quasi-periodic motions in families of dynamical systems. Order amidst chaos*, Lecture Notes in Math., Vol 1645, Springer-Verlag, Berlin, 1996.

[BHV07] H.W. Broer, A. Hagen, and G. Vegter, *Numerical continuation of normally hyperbolic invariant manifolds*, Nonlinearity **20** (2007), no. 6, 1499–1534.

[BOV97] H.W. Broer, H.M. Osinga, and G. Vegter, *Algorithms for computing normally hyperbolic invariant manifolds*, Z. Angew. Math. Phys. **48** (1997), no. 3, 480–524.

[Can14] Marta Canadell, *Computation of normally hyperbolic invariant manifolds*, Ph.D. thesis, Universitat de Barcelona, Barcelona, Spain, June 2014.

[CC09] A. Celletti and L. Chierchia, *Quasi-periodic attactors in Celestial Mechanics*, Arch. Rational Mech. Anal. **191** (2009), no. 2, 311–345.

[CC10] Renato Calleja and Alessandra Celletti, *Breakdown of invariant attractors for the dissipative standard map*, Chaos **20** (2010), no. 1, 013121, 9. MR 2730168

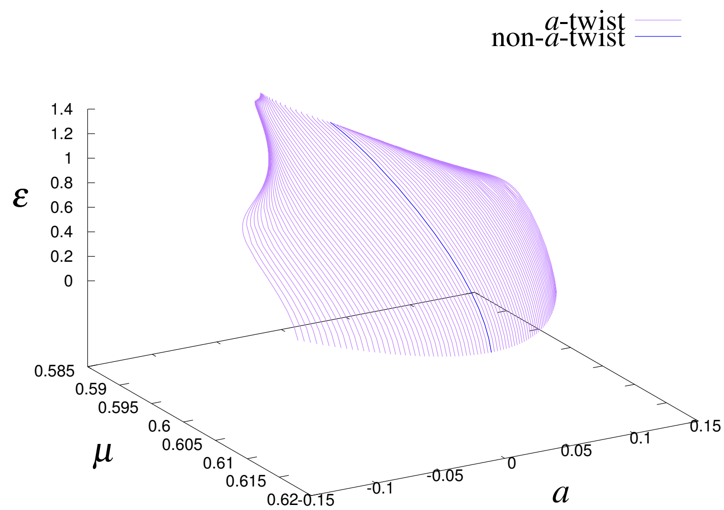

FIGURE 9. Parameter surface Σ_ω generated by continuation w.r.t. ε of invariant circles with frequency ω and fix a-twist (non-symmetric case). In blue, the parameter curve Γ_ω corresponding to non-a-twist.

[CCdlL13a] R. Calleja, A. Celletti, and R. de la Llave, *Construction of response functions in forced strongly dissipative systems*, Discrete Contin. Dyn. Syst. **33** (2013), no. 10, 4411–4433.

[CCdlL13b] ———, *A KAM theory for conformally symplectic systems: efficient algorithms and their validation*, J. Differential Equations **255** (2013), no. 5, 978–1049.

[CdlL09] R. Calleja and R. de la Llave, *Fast numerical computation of quasi-periodic equilibrium states in 1D statistical mechanics, including twist maps*, Nonlinearity **22** (2009), no. 6, 1311–1336.

[CdlL10] Renato Calleja and Rafael de la Llave, *Computation of the breakdown of analyticity in statistical mechanics models: numerical results and a renormalization group explanation*, J. Stat. Phys. **141** (2010), no. 6, 940–951. MR 2740396

[CF12] R. Calleja and J.-Ll. Figueras, *Collision of invariant bundles of quasi-periodic attractors in the dissipative standard map*, Chaos: An Interdisciplinary Journal of Nonlinear Science **22** (2012), no. 3, 033114.

[CH14] M. Canadell and A. Haro, *Parameterization method for computing quasi-periodic reducible normally hyperbolic invariant tori*, F. Casas, V. Martínez (eds.), *Advances in Differential Equations and Applications*, SEMA SIMAI Springer Series, vol. 4, Springer, 2014.

[CH17a] M. Canadell and À. Haro, *Computation of Quasi-Periodic Normally Hyperbolic Invariant Tori: Algorithms, Numerical Explorations and Mechanisms of Breakdown*, J. Nonlinear Sci. **27** (2017), no. 6, 1829–1868. MR 3713932

[CH17b] ———, *Computation of Quasiperiodic Normally Hyperbolic Invariant Tori: Rigorous Results*, J. Nonlinear Sci. **27** (2017), no. 6, 1869–1904. MR 3713933

[CLHB05] Maria-Cristina Ciocci, Anna Litvak-Hinenzon, and Henk Broer, *Survey on dissipative KAM theory including quasi-periodic bifurcation theory*, Geometric mechanics and symmetry, London Math. Soc. Lecture Note Ser., vol. 306, Cambridge Univ. Press, Cambridge, 2005, Based on lectures by Broer, pp. 303–355. MR 2172981

[dlLGJV05] R. de la Llave, A. González, À. Jorba, and J. Villanueva, *KAM theory without action-angle variables*, Nonlinearity **18** (2005), no. 2, 855–895.

[Fen72] N. Fenichel, *Persistence and smoothness of invariant manifolds for flows*, Indiana Univ. Math. J. **21** (1971/1972), 193–226.

[FH15] Jordi-Lluís Figueras and Àlex Haro, *Different scenarios for hyperbolicity breakdown in quasiperiodic area preserving twist maps*, Chaos **25** (2015), no. 12, 123119, 16. MR 3436748

[GHdlL14] A. González, A. Haro, and R. de la Llave, *Singularity theory for non-twist KAM tori*, Mem. Amer. Math. Soc. **227** (2014), no. 1067, vi+115.

[GOY85] Celso Grebogi, Edward Ott, and James A. Yorke, *Attractors on an n-torus: Quasiperiodicity versus chaos*, Physica D: Nonlinear Phenomena **15** (1985), no. 3, 354 – 373.

[Gra17] Albert Granados, *Invariant manifolds and the parameterization method in coupled energy harvesting piezoelectric oscillators*, Phys. D **351/352** (2017), 14–29. MR 3659393

[HCF+16] À. Haro, M. Canadell, J.-Ll. Figueras, A. Luque, and J.-M. Mondelo, *The parameterization method for invariant manifolds*, Applied Mathematical Sciences, vol. 195, Springer, [Cham], 2016, From rigorous results to effective computations. MR 3467671

[HdlL06] A. Haro and R. de la Llave, *Manifolds on the verge of a hyperbolicity breakdown*, Chaos **16** (2006), no. 1, 013120, 8.

[HdlL07] ———, *A parameterization method for the computation of invariant tori and their whiskers in quasi-periodic maps: explorations and mechanisms for the breakdown of hyperbolicity*, SIAM J. Appl. Dyn. Syst. **6** (2007), no. 1, 142–207 (electronic).

[Hen11] M.E. Henderson, *Flow box tiling methods for compact invariant manifolds*, SIAM J. Appl. Dyn. Syst. **10** (2011), no. 3, 1154–1176.

[HPS77] M.W. Hirsch, C.C. Pugh, and M. Shub, *Invariant manifolds*, Lecture Notes in Mathematics, Vol. 583, Springer-Verlag, Berlin, 1977.

[Kyn68] W. T. Kyner, *Rigorous and formal stability of orbits about an oblate planet*, Mem. Amer. Math. Soc. No. 81, Amer. Math. Soc., Providence, R.I., 1968. MR 0229449

[Mañ78] R. Mañé, *Persistent manifolds are normally hyperbolic*, Trans. Amer. Math. Soc. **246** (1978), 261–283.

[Mos66] J. Moser, *A rapidly convergent iteration method and non-linear differential equations. II*, Ann. Scuola Norm. Sup. Pisa (3) **20** (1966), 499–535.

[Mos67] ———, *Convergent series expansions for quasi-periodic motions*, Math. Ann. **169** (1967), 136–176.

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