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# Exceptional Gegenbauer polynomials via isospectral deformation

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## Abstract

In this paper, we show how to construct exceptional orthogonal polynomials (XOP) using isospectral deformations of classical orthogonal polynomials. The construction is based on confluent Darboux transformations, where repeated factorizations at the same eigenvalue are allowed. These factorizations allow us to construct Sturm–Liouville problems with polynomial eigenfunctions that have an arbitrary number of real-valued parameters. We illustrate this new construction by exhibiting the class of *deformed* Gegenbauer polynomials, which are XOP families that are isospectral deformations of classical Gegenbauer polynomials.

KEYWORDS

confluent Darboux transformations, exceptional polynomials, Gegenbauer polynomials, isospectral deformations, Sturm–Liouville problems

## 1 | INTRODUCTION

Classical orthogonal polynomials (OPs) have been traditionally characterized as the only OP bases of an $L^2$ space that are also eigenfunctions of a Sturm–Liouville problem. Since the mid-19th century until today, classical OPs appear in ubiquitous applications in mathematical physics, numerical analysis, approximation theory or statistics, among other fields.

As it is well known, classical OPs can be classified into three main families, depending on whether they are defined on the whole real line (Hermite), the half-line (Laguerre), or a compact interval (Jacobi). These polynomial families are characterized by two free real parameters in the case of Jacobi, one for Laguerre, and none for Hermite.

This paper tackles the following question: *Is it possible to construct an orthogonal basis of an* $L^2$ *space, which is also formed by polynomial eigenfunctions of a Sturm–Liouville problem, but contains a higher (possibly arbitrary) number of free real parameters?*

The flexibility of deforming classical OPs to contain so many free real parameters and yet mantain many of their defining properties would open the way to many potential applications in all of the fields where classical OPs appear naturally.

In this paper we show how to construct Sturm–Liouville problems with polynomial eigenfunctions that contain an arbitrary number of real parameters, thus providing a positive answer to the previous question. More specifically, let $T$ be a *classical differential operator*, that is, a second-order differential operator whose eigenfunctions are classical OPs,

$$T(z, D_z) := p(z)D_z^2 + q(z)D_z + r(z), \tag{1}$$

and let $\{P_i\}_{i=0}^{\infty}$ be a set of polynomial eigenfunctions of $T$

$$TP_i = \lambda_i P_i, \quad i = 0, 1, 2, \ldots \tag{2}$$

We will show how to construct a new operator $\hat{T}$ and polynomials $\{\hat{P}_i\}_{i=0}^{\infty}$ such that

$$\hat{T}(z, D_z; t_{m_1}, \ldots, t_{m_n}) := p(z)D_z^2 + \hat{q}(z; t_{m_1}, \ldots, t_{m_n})D_z + \hat{r}(z; t_{m_1}, \ldots, t_{m_n}), \tag{3}$$

where $t_{m_1}, \ldots, t_{m_n}$ are $n$ real parameters, $\hat{q}$ and $\hat{r}$ are rational functions of $z$, and

$$\widehat{T}\widehat{P}_i = \lambda_i \widehat{P}_i, \quad i \in \{0, 1, 2, \ldots\}. \tag{4}$$

The transformed eigenvalue problem has the same spectrum $\{\lambda_i\}_{i=0}^{\infty}$ as the original one and the leading coefficient $p(z)$ does not change in the transformation. We speak thus of an *isospectral deformation* because we will also see that

$$\widehat{T}(z, D_z; 0, \ldots, 0) = T(z, D_z), \quad \widehat{P}_i(z; 0, \ldots, 0) = P_i(z). \tag{5}$$

So much extra freedom and flexibility comes at a cost. Although the set of new polynomials is still $L^2$-complete, there is a finite number of exceptional degrees for which no polynomial of that degree exists in the basis. Indeed, in general, we will have that

$$i = \deg P_i \neq \deg \widehat{P}_i, \quad i \in \{0, 1, 2, \ldots\}, \tag{6}$$

despite the fact that both $\hat{P}_i$ and $P_i$ have the same eigenvalue, and both of them have $i$ zeros in the domain of orthogonality. We will refer to operator $\hat{T}$ as a *deformed classical operator*, and to the set of polynomials $\hat{P}_i$ as *deformed classical polynomials*, because:

(a) the deformed classical operator has the same spectrum as the associated classical one,
(b) the deformed classical polynomial family contains an arbitrary family of real parameters,
(c) the deformed classical polynomials become classical polynomials when the deformation parameters are set to zero.

The new polynomial families introduced in this paper fit thus into the definition of *exceptional orthogonal polynomials* (XOPs) originally introduced in Ref. 1. In the past 10 years, the mathematical study of XOPs has been concerned with their classification,[2,3] the properties of their zeroes,[4–6] and their recurrence relations.[7–10] The spectral–theoretic aspects of exceptional operators (the second-order differential operators whose eigenfunctions are the XOPs) have been studied in Refs. 11–15. It is known that every exceptional operator can be related to a classical Bochner operator by a finite number of Darboux transformations (see Ref. [2, Theorem 1.2)], which can be state-adding, state-deleting, or isospectral.

In the context of mathematical physics, XOPs appear as exact solutions to Dirac's equation[16] and as bound states of exactly solvable rational extensions.[17–19] Additionally, they are connected to finite-gap potentials[20] and super-integrable systems.[19,21]

The aim of this paper is:

(a) to draw attention to the existence of XOPs whose construction requires confluent Darboux transformations (CDTs),
(b) to exhibit a different construction of exceptional polynomials based on determinants and matrices, and
(c) to describe the differences between the class of generic exceptional polynomials and the new class of exceptional polynomials obtained via CDTs.

As an illustration of this new construction, we show how to define an isospectral deformation of the classical Gegenbauer operator

$$T^{(\alpha)} := (1-z^2)D_z^2 - (2\alpha+1)zD_z \tag{7}$$

through the application of a finite number of CDTs, also referred to as the "double commutator" method.[22]

Nondegenerate exceptional Jacobi polynomials are indexed by discrete parameters[23,24] and, as a result, cannot be continuously deformed into their classical counterparts. By contrast, every CDT introduces a new deformation parameter. Therefore, by performing a chain of $n$ CDTs on the classical Gegenbauer operator (7) at distinct eigenvalues, we will arrive at an exceptional operator that depends on $n$ discrete parameters and $n$ real parameters. The eigenpolynomials of the resulting exceptional operator may be deformed Gegenbauer polynomials, which also depend on $n$ real parameters, and can be continuously deformed to the classical Gegenbauer polynomials by letting the parameters tend to zero.

The new construction of exceptional polynomials and weights described in this paper can also be understood from the point of view of the theory of inverse scattering,[25-27] and it is conceptually related to the construction of Korteweg–de Vries (KdV) multisolitons. While KdV solitons are obtained by applying a state-adding deformation on the zero potential, the exceptional operators in this paper are related to isospectral deformations of particular instances of the Darboux–Pöschl–Teller (DBT) potential.[28] In spectral–theoretic terms, the consequence of our construction is the continuous modification of the norming constants of a finite number of the bound states. However, because our focus is on OPs, rather than quantum mechanics or evolution equations, our approach is formulated in the gauge and coordinates of the classical differential operator rather than working with Schrödinger operators. The resulting procedure can be easily implemented using a computer algebra system.

For the remainder of the paper, we develop the theory of CDTs in the algebraic gauge and apply it for the construction of deformed Gegenbauer polynomials. The paper is organized as follows: in Section 2 we describe the formal theory of rational multistep Darboux–Crum transformations in the algebraic gauge. In Section 3 we describe CDTs as a two-step Darboux transformation whose seed functions are an eigenfunction and a generalized eigenfunction at the same eigenvalue. In Section 4, we describe the class of exceptional Gegenbauer operators and their factorizations and we provide a recursive construction for the operators and eigenfunctions connected by CDTs. In Section 5 we provide matrix formulas for deformed Gegenbauer polynomials, we prove the equivalence of the matrix and recursive definitions and thereby establish the proofs of our main results concerning the Sturm–Liouville properties and $L^2$-completeness of the deformed Gegenbauer polynomials. Finally, in Section 6, we provide explicit examples of families of deformed Gegenbauer polynomials, for both one and two deformation parameters.

Exceptional polynomials obtained by CDTs were introduced by Grandati and Quesne,[29] who exhibited one-step examples with one deformation parameter. In this paper, we aim to generalize these results and to provide a detailed construction of deformed classical OPs via CDTs. The class of deformed Legendre polynomials was recently introduced by some of the present authors,[30] showing, for the first time, that isospectral deformations of classical polynomials with an arbitrary number of real parameters exist.

Shortly after that, Durán has shown that there is an alternative way to construct deformed OPs, by first perturbing the measure of the discrete Hahn polynomials, dualizing and taking a suitable limit.[31] Durán's work implies that the most general class of XOPs contains not just deformations of classical OPs, but deformations of other XOPs. He did this by exhibiting constructions and examples where setting the deformation parameters to zero recovers exceptional, rather than classical OPs. In other words, there are "mixed" cases that combine the usual Wronskian and the novel deformation constructions where isospectrality with the classical families no longer holds.

Another significant feature of Durán's construction is that the deformation parameters are introduced via degenerate Darboux transformations (DDTs) rather than CDTs. In this regard, his construction generalizes the one presented in Ref. 32, which exhibited some examples with one deformation parameter. The DDT construction relies on the spectral degeneracy of certain classical operators (the so-called para-Jacobi class[33]) that, for certain eigenvalues, possess a two-dimensional polynomial eigenspace. Finally, Ref. 31 is significant because of explicit examples of alternative constructions of deformable Legendre polynomials via DDTs of classical Jacobi operators with negative integer parameters.

Summarizing, there are three rather different constructions leading to the same mathematical objects:

(i) one based in iterating the action of differential operators, leading to Wronskian determinants whose seed functions are generalized eigenfunctions;
(ii) one based in matrix and integral formulas, coming from the inverse scattering method; and

(iii) one based in Durán's DDT construction, where the base of the construction is a para-Jacobi operator and the deformation parameters appear as linear combinations of polynomial seed functions at the same eigenvalue.

In this paper we demonstrate the equivalence of the first two approaches. In a subsequent publication we will classify the class of all XOP that admit real deformation parameters, and in doing so, establish the equivalence of the DDT and the CDT approaches. We will demonstrate that all these phenomena arise *precisely* because of the spectral degeneracy of classical para-Jacobi operators with integer parameters.

### 1.1 | Notation

Throughout the paper we use $\mathbb{N} = \{1, 2, \ldots\}$ to denote the set of natural numbers and $\mathbb{N}_0 = \{0, 1, \ldots\}$ to denote the set of nonnegative integers. We use "half-integer" to refer to an odd integer divided by 2. The set of positive half-integers will be denoted by $\mathbb{N}_0 + \frac{1}{2}$.

We let $D_z$ denote the derivative with respect to $z$. For the sake of notational convenience, we will often drop the explicit dependence on the indeterminate $z$ and write $\phi = \phi(z)$, $\phi' = D_z\phi$, and $D = D_z$.

We call a differential expression of the form $\sum_{k=0}^{n} p_k(z)D_z^k$, where $p_0(z), \ldots, p_n(z)$ are rational functions and $p_n \neq 0$, an $n$th-*order rational operator*. We will call a function $\phi(z)$ quasi-rational if its log-derivative $w(z) = \frac{\phi'(z)}{\phi(z)}$ is rational. We denote by $\mathrm{Wr}[y_1, \ldots, y_k]$ the Wronskian determinant of the functions $y_1, \ldots, y_k$.

We denote matrices by calligraphic symbols, such as $\mathcal{R}$, whereas one-dimensional tuples will be given bold symbols such as $\boldsymbol{m}$, $\boldsymbol{t}$, or $\boldsymbol{Q}$. To access the components of a vector or tensor we will employ square brackets, that is, $[\mathcal{R}]_{k\ell}$ denotes the $(k, \ell)$ entry of $\mathcal{R}$.

We denote an $n$-tuple of integers by $\boldsymbol{m} = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$, and associate to it the $n$-tuple of real parameters $\boldsymbol{t}_{\boldsymbol{m}} = (t_{m_1}, \ldots, t_{m_n}) \in \mathbb{R}^n$. We will separate objects of different natures, such as real parameters and tuples, by semicolons. The concatenation of tuples will be shown using commas, for example, if $i_1, \ldots, i_k \in \mathbb{N}_0$, then $(\boldsymbol{m}, i_1, \ldots, i_k)$ denotes the $(n + k)$-tuple $(m_1, \ldots, m_n, i_1, \ldots, i_k)$. We will frequently omit parentheses when denoting 1-tuples, opting to write $m_1$ instead of $(m_1)$. Occasionally, we will omit the dependence on the parameter $\alpha$, so as not to conflict with other superscript notations.

## 2 | DARBOUX TRANSFORMATIONS AND FACTORIZATION CHAINS

The formal theory of Darboux transformations for Schödinger operators in mathematical physics (also known as supersymmetric quantum mechanics) has been developed in numerous works, mostly with the aim of generating new solvable problems from known ones,[26,34,35] but also in the construction of solutions to Painlevé-type equations.[36–38] It was recently shown that every exceptional polynomial family must be related to a classical family by a sequence of Darboux transformations.[2] We first describe here Darboux transformations with seed functions that have no repeated eigenvalues. This is the class of transformations that leads to the generic XOPs, when applied on the classical polynomial families.[3,7,17,23,24]

In this section we revise some of these well known results, albeit with a little twist: We describe Darboux transformations for the class of general second-order differential operators, which include Schrödinger operators as a particular case. For the purpose of this paper, we focus on second-order differential operators with rational coefficients that have an infinite number of polynomial eigenfunctions, and we restrict to rational Darboux transformations of these operators that preserve this property by construction. However, the results derived in this section can be trivially extended to general second-order operators. All of this section is written at a purely formal level, so in an abuse of notation we speak of differential operators without specifying their domain, or we speak of eigenfunctions as solutions of an eigenvalue problem, without defining a proper spectral–theoretic setting.

**Definition 1.** For $n \in \mathbb{N}_0$, let $T_0, T_1, \dots, T_n$ be second-order rational operators. We say that $T_0 \to T_1 \to \cdots \to T_n$ is an *n-step rational Darboux transformation* if there exist first-order rational operators $A_1, A_2, \dots, A_n$ such that

$$A_k T_{k-1} = T_k A_k, \quad k = 1, \dots, n. \tag{8}$$

The next Proposition states that if $A_k$ is a first-order intertwiner operator between $T_{k-1}$ and $T_k$ as in (8), then its kernel must be spanned by a formal eigenfunction of $T_{k-1}$.

**Proposition 1.** *For $n \in \mathbb{N}_0$, let $T_0 \to T_1 \to \cdots \to T_n$ be an n-step rational Darboux transformation, and $A_1, \dots, A_n$ be the first-order rational operators satisfying the intertwining relation (8). Let $b_k(z)$ and $w_k(z)$ be rational functions such that*

$$A_k = b_k(D - w_k), \tag{9}$$

*and define (up to a constant factor) the quasi-rational functions*

$$\psi_k(z) := \exp\left(\int^z w_k(u)du\right). \tag{10}$$

*We then have that*

$$T_{k-1}\psi_k = \lambda_k \psi_k, \qquad k = 1, \dots, n, \tag{11}$$

*where $\lambda_1, \lambda_2, \dots, \lambda_n$ are constants.*

*Proof.* By (9) and (10) we have that $A_k \psi_k = 0$, so (8) implies that $A_k T_{k-1} \psi_k = 0$. Because $A_k$ is a first-order operator, its kernel is one-dimensional and it is therefore spanned by $\psi_k$. This implies that there exists a constant $\lambda_k$ such that (11) holds. ■

*Remark* 1. Assume that $T_k$ has the form

$$T_k = pD^2 + q_k D + r_k, \quad k = 0, 1, \dots, n. \tag{12}$$

Observe that its coefficients $p, q_k$, and $r_k$ are related to the rational functions $w_1, \dots, w_n$ defined in (9) by the following Ricatti-type equations

$$p(w_k' + w_k^2) + q_{k-1} w_k + r_{k-1} = \lambda_k, \qquad k = 1, \dots, n. \tag{13}$$

For a given $n$-step Darboux transformation, the corresponding $\psi_1, \dots, \psi_n$ are unique, up to a choice of multiplicative constant. In light of this remark and future ones, we introduce the following defintions:

**Definition 2.** We call the quasi-rational functions $\psi_1, \dots, \psi_n$ *factorization eigenfunctions*, the set of numbers $\lambda_1, \dots, \lambda_n$ *factorization eigenvalues*, and the rational functions $b_1, \dots, b_n$ *factorization gauges*.

A chain of Darboux transformations is more often defined as a factorization of each second-order operator $T_k$ for $k = 0, \dots, n$, followed by a permutation of the two factors to yield the next operator $T_{k+1}$. We make precise the notion of a factorization chain in the next definition and establish later that both approaches (intertwining of operators and factorization) coincide.

**Definition 3.** Let $T_0, T_1, \dots, T_n$ be second-order rational operators. We say that $T_0 \to T_1 \to \cdots \to T_n$ is a *factorization chain* if there exist first-order rational operators $A_k, B_k,\ k = 1, \dots, n$ and constants $\lambda_1, \dots, \lambda_n$ such that

$$T_{k-1} = B_k A_k + \lambda_k, T_k = A_k B_k + \lambda_k, \qquad k = 1, \dots, n, \tag{14}$$

We now show that these two formulations of Darboux transformations are equivalent.

**Proposition 2.** *A multistep rational Darboux transformation is necessarily a factorization chain, and vice versa.*

*Proof.* Suppose that (14) holds. Then, the intertwining relations (8) follow by the associativity of operator composition.

Conversely, suppose that (8) holds. Let $A_k$ be as in (9) and set

$$B_k := \hat{b}_k(D - \hat{w}_k), \tag{15}$$

where

$$\widehat{w}_k := -w_k - \frac{q_{k-1}}{p} + \beta_k, \quad \beta_k := \frac{b_k'}{b_k}, \quad \widehat{b}_k := \frac{p}{b_k}. \tag{16}$$

Let $\lambda_1, \dots, \lambda_n$ denote the eigenvalues as per (11). By (13) and a direct calculation, we have

$$\begin{aligned} B_k A_k &= \hat{b}_k(D - \hat{w}_k) b_k (D - w_k) \\ &= p(D - \hat{w}_k + \beta_k)(D - w_k) \\ &= p(D + w_k)(D - w_k) + q_{k-1}(D - w_k) \\ &= pD^2 + q_{k-1} D - p(w_k' + w_k^2) - q_{k-1} w_k \\ &= T_{k-1} - \lambda_k. \end{aligned} \tag{17}$$

Hence, it follows that

$$(T_k - A_kB_k - \lambda_k)A_k = A_k(T_{k-1} - B_kA_k - \lambda_k) = 0. \tag{18}$$

The ring of differential operators does not have zero divisors, so relations (14) follow immediately. ■

*Remark* 2. As a direct consequence we may observe that Darboux transformations are invertible. Indeed, by (14), we have that

$$B_kT_k = T_{k-1}B_k, \quad k = 1, \dots, n, \tag{19}$$

where $B_1, \dots, B_n$ are the rational operators defined by (15) and (16). Thus, the chain $T_n \to T_{n-1} \to \cdots \to T_0$ also satisfies the definition of an $n$-step rational Darboux transformation.

## 2.1 | Seed eigenfunctions and generalized Crum formula

So far we have defined $n$-step Darboux transformations referring to the intertwining or factorization of operators $T_0, T_1, \dots, T_n$ at each step of the chain, which requires an eigenfunction $\psi_k$ for each operator $T_{k-1}$ in the chain. In this section we show how to define an $n$-step Darboux transformation using only eigenfunctions of the first operator $T_0$, which we will call *seed functions*. In the case of Darboux chains for Schrödinger operators, this construction leads to the well-known Darboux–Crum Wronskian formula.[39] Theorem 1 in this section can thus be seen as a generalization of the Crum formula.

Let $T_0$ be a second-order rational operator and let $\phi_1, \phi_2, \dots, \phi_n$ be quasi-rational eigenfunctions of $T_0$. Explicitly, we have

$$T_0\phi_k = \lambda_k\phi_k, \quad k = 1, \dots, n, \tag{20}$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are constants. We refer to $\phi_1, \dots, \phi_n$ as *seed eigenfunctions*, because, as we show below, a Darboux transformation is determined by a choice of seed eigenfunctions and factorization gauges.

Going forward, for a set of indices $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$, we write

$$\phi_{(i_1,i_2,\dots,i_k)} := \mathrm{Wr}[\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_k}]. \tag{21}$$

We now arrive at the key result of this section: explicit formulas for the coefficients of an operator obtained by a multistep Darboux transformation.

**Theorem 1.** *Let $T_0$ be a second-order rational operator as in (12), $\{\phi_1, \dots, \phi_n\}$ be quasi-rational eigenfunctions of $T_0$ with distinct eigenvalues $\lambda_1, \dots, \lambda_n$ and let $\{b_1, \dots, b_n\}$ be a set of nonzero rational functions. Define the operators*

$$T_k := pD^2 + q_kD + r_k, \quad A_k := b_k(D - w_k). \tag{22}$$

*where*

$$\sigma_k := \sum_{j=1}^{k} (\log b_j)', \quad v_k := \frac{\varphi'_{(1,2,\dots,k)}}{\varphi_{(1,2,\dots,k)}}, \quad k = 1, \dots, n; \tag{23}$$

*and*

$$\begin{aligned} q_k &:= q_0 + kp' - 2p\sigma_k, \\ r_k &:= r_0 + kq_0' + \frac{1}{2}k(k-1)p'' + v_k p' - \sigma_k(q_0 + kp') + (\sigma_k^2 - \sigma_k' + 2v_k')p, \end{aligned} \tag{24}$$

$$w_1 := v_1, \quad w_k := \sigma_{k-1} + v_k - v_{k-1}, \quad k = 2, \dots, n. \tag{25}$$

*Then, the sequence of operators $T_0 \to T_1 \to \cdots \to T_n$ is an $n$-step rational Darboux transformation in the sense of Definition 1, that is, (8) holds with the operators $A_k$ defined above.*

Because we assume that the eigenvalues $\lambda_1, \dots, \lambda_n$ are distinct, $\phi_1, \dots, \phi_n$ are linearly independent. Hence, the Wronskians in the denominator of (23) are nonzero, and the functions $v_k$ are well defined.

*Remark* 3. The well-known Crum formula for an $n$-step Darboux transformation of a Schrödinger operator

$$T_n = T_0 - 2(\log \phi_{(1,2,\dots,n)})'' \tag{26}$$

is a special case of Theorem 1 that corresponds to starting from a Schrödinger operator ($p = -1$ and $q_0 = 0$) and choosing the gauge $b_1 = \cdots = b_n = 1$. Thus, this result can be regarded as an extension of the Darboux–Crum formula from Schrödinger operators to general second-order operators. Although for the purpose of the paper we restrict to second-order operators with rational coefficients and polynomial eigenfunctions, the generalized Crum formula in Theorem 1 is valid on a general setting.

The rest of this section is devoted to the proof of Theorem 1. The strategy is to first establish the result for the particular gauge $b_1 = \cdots = b_n = 1$ and then show the transformation rules under a different gauge. Most of the proofs proceed by induction and make use of differential algebra and properties of Wronskian determinants. We will first need to state and prove a number of auxiliary lemmas, where operators with tilde denote the restriction of the same objects in Theorem 1 to the special case $b_1 = \cdots = b_n = 1$.

**Lemma 1.** *Under the same setting as in Theorem 1, let $\tilde{A}_k = D - \tilde{w}_k$ for $k \in \{1, \dots, n\}$ with $\tilde{w}_k = v_k - v_{k-1}$. Then,*

$$(\tilde{A}_{k-1}\tilde{A}_{k-2} \cdots \tilde{A}_1)\phi_j = \frac{\phi_{(1,2,\dots,k-1,j)}}{\phi_{(1,2,\dots,k-1)}}, \quad 2 \le k \le j \le n. \tag{27}$$

*Proof.* For convenience, set

$$\varphi_{k,j} := (\tilde{A}_{k-1}\tilde{A}_{k-2}\cdots\tilde{A}_1)\phi_j. \tag{28}$$

We argue by induction. The case of $k = 2$ follows directly from the definition. Indeed,

$$\varphi_{2,j} = \tilde{A}_1\phi_j = \frac{1}{\phi_1}\,\mathrm{Wr}[\phi_1, \phi_j] = \frac{\phi_{1,j}}{\phi_1}. \tag{29}$$

Now, we suppose that (27) holds for a particular $k < n$. By the inductive hypothesis, in particular we have

$$\varphi_{k,k} = \frac{\phi_{(1,2,\ldots,k)}}{\phi_{(1,2,\ldots,k-1)}}, \qquad \text{so} \quad \tilde{\omega}_k = \frac{\varphi'_{k,k}}{\varphi_{k,k}} \quad \text{and} \quad \tilde{A}_k = D - \frac{\varphi'_{k,k}}{\varphi_{k,k}}. \tag{30}$$

Hence,

$$\begin{aligned}\varphi_{k+1,j} &= (\tilde{A}_k\cdots\tilde{A}_1)\phi_j = \tilde{A}_k\varphi_{k,j}\\ &= \frac{1}{\varphi_{k,k}}\,\mathrm{Wr}\left[\varphi_{k,k}, \varphi_{k,j}\right] = \frac{1}{\varphi_{k,k}}\,\mathrm{Wr}\left[\frac{\phi_{(1,2,\ldots,k-1,k)}}{\phi_{(1,2,\ldots,k-1)}}, \frac{\phi_{(1,2,\ldots,k-1,j)}}{\phi_{(1,2,\ldots,k-1)}}\right].\end{aligned} \tag{31}$$

By well-known properties of the Wronskian operator, we finally have

$$\varphi_{k+1,j} = \frac{1}{\varphi_{k,k}}\frac{\mathrm{Wr}[\phi_{1,2,\ldots,k-1,k}, \phi_{(1,2,\ldots,k-1,j)}]}{\left(\phi_{(1,2,\ldots,k-1)}\right)^2} = \frac{\phi_{(1,2,\ldots,k-1)}\phi_{(1,2,\ldots,k,j)}}{\varphi_{k,k}\left(\phi_{(1,2,\ldots,k-1)}\right)^2} = \frac{\phi_{(1,2,\ldots,k,j)}}{\phi_{(1,2,\ldots,k)}}. \tag{32}$$

■

The following lemma establishes the desired result in the restricted gauge $b_1 = \cdots = b_n = 1$.

**Lemma 2.** *Under the same setting as in Theorem 1, let $\tilde{T}_0 = T_0$ and $\tilde{T}_k := pD^2 + \tilde{q}_k D + \tilde{r}_k$ for $k \in \{1, \ldots n\}$, where*

$$\tilde{q}_k := q_0 + kp' \tag{33}$$

$$\tilde{r}_k := r_0 + kq_0' + \frac{1}{2}k(k-1)p'' + \upsilon_k p' + 2\upsilon_k' p. \tag{34}$$

*Let $\tilde{A}_k$ be as in Lemma 1. Then*

$$\tilde{A}_k\tilde{T}_{k-1} = \tilde{T}_k\tilde{A}_k, \quad k = 1, \ldots, n. \tag{35}$$

*Proof.* By direct calculation, we find that

$$\begin{aligned}\tilde{T}_k\tilde{A}_k &= (pD^2 + \tilde{q}_k D + \tilde{r}_k)(D - \tilde{\omega}_k)\\ &= pD^3 + (\tilde{q}_k - p\tilde{\omega}_k)D^2 + (\tilde{r}_k - \tilde{q}_k\tilde{\omega}_k - 2p\tilde{\omega}_k')D - (\tilde{r}_k\tilde{\omega}_k + \tilde{q}_k\tilde{\omega}_k' + p\tilde{\omega}_k''),\end{aligned} \tag{36}$$

$$\begin{aligned}\tilde{A}_k\tilde{T}_{k-1} &= (D-\tilde{w}_k)(pD^2+\tilde{q}_{k-1}D+\tilde{r}_{k-1})\\ &= pD^3+(\tilde{q}_{k-1}-p\tilde{w}_k+p')D^2+\big(\tilde{r}_{k-1}-\tilde{q}_{k-1}\tilde{w}_k+\tilde{q}'_{k-1}\big)D+\tilde{r}'_{k-1}-\tilde{w}_k\tilde{r}_{k-1}.\end{aligned}\tag{37}$$

Hence, by inspection of the coefficients, the desired intertwining relation is equivalent to the following three relations:

$$\tilde{q}_k = \tilde{q}_{k-1}+p',\tag{38}$$

$$\tilde{r}_k = \tilde{r}_{k-1}+\tilde{q}'_{k-1}+\tilde{w}_k p'+2p\tilde{w}'_k,\tag{39}$$

$$\tilde{r}_k\tilde{w}_k+\tilde{q}_k\tilde{w}'_k+p\tilde{w}''_k = \tilde{r}_{k-1}\tilde{w}_k-\tilde{r}'_{k-1}.\tag{40}$$

By inspection, (33) entails (38). Then, using (33), (34), and $\tilde{w}_k = v_k - v_{k-1}$, we find that

$$\begin{aligned}\tilde{r}_k-\tilde{r}_{k-1}-\tilde{q}'_{k-1} &= q'_0+(k-1)p''+(v_k-v_{k-1})p'+2(v'_k-v'_{k-1})p-(q'_0+(k-1)p'')\\ &= \tilde{w}_k p'+2\tilde{w}'_k p,\end{aligned}\tag{41}$$

which establishes (39). Using (38) and (39), we can rewrite (40) as

$$\begin{aligned}0 &= (\tilde{r}_k-\tilde{r}_{k-1})\tilde{w}_k+\tilde{q}_k\tilde{w}'_k+p\tilde{w}''_k+\tilde{r}'_{k-1}\\ &= (\tilde{q}'_{k-1}+p'\tilde{w}_k+2p\tilde{w}'_k)\tilde{w}_k+(\tilde{q}_{k-1}+p')\tilde{w}'_k+p\tilde{w}''_k+\tilde{r}'_{k-1}\\ &= \big(p(\tilde{w}'_k+\tilde{w}_k^2)+\tilde{q}_{k-1}\tilde{w}_k+\tilde{r}_{k-1}\big)'.\end{aligned}\tag{42}$$

Let $\tilde{\psi}_k$ denote

$$\tilde{\psi}_k = \varphi_{k,k} = \frac{\phi_{(1,\dots,k)}}{\phi_{(1,\dots,k-1)}},\qquad \text{so}\qquad \tilde{w}_k = (\log\tilde{\psi}_k)',\tag{43}$$

and observe that

$$p(\tilde{w}'_k+\tilde{w}_k^2)+\tilde{q}_{k-1}\tilde{w}_k+\tilde{r}_{k-1} = \frac{\tilde{T}_{k-1}\tilde{\psi}_k}{\tilde{\psi}_k}.\tag{44}$$

Then (40) is equivalent to establishing

$$\tilde{T}_{k-1}\tilde{\psi}_k = \lambda_k\tilde{\psi}_k,\qquad k\in\{1,\dots,n\}.\tag{45}$$

The rest of the proof follows by induction. The base case holds because $\tilde{\psi}_1 = \phi_1$ and by assumption, $T_0\phi_1 = \lambda_1\phi_1$. Now, suppose that we have established (35) for $j = 1,\dots,k-1$. By Lemma 1 and the inductive hypothesis,

$$\begin{aligned}\tilde{T}_{k-1}\tilde{\psi}_k &= \tilde{T}_{k-1}\tilde{A}_{k-1}\cdots\tilde{A}_1\phi_k = \tilde{A}_{k-1}\tilde{T}_{k-2}\tilde{A}_{k-2}\cdots\tilde{A}_1\phi_k\\ &= \tilde{A}_{k-1}\cdots\tilde{A}_1T_0\phi_k = \lambda_k\tilde{A}_{k-1}\cdots\tilde{A}_1\phi_k = \lambda_k\tilde{\psi}_k.\end{aligned}\tag{46}$$

■

The next lemma shows the gauge transformation that connects $T_k$ with $\tilde{T}_k$.

**Lemma 3.** *Let $T_k$ be as in Theorem 1 and $\tilde{T}_k$ be as in Lemma 2. Setting $s_k := b_1 \dots b_k$, we have*

$$T_k = s_k \tilde{T}_k s_k^{-1}, \quad k = 1, \dots, n. \tag{47}$$

*Proof.* Observe that $\sigma_k = (\log s_k)'$. By direct calculation,

$$s_k D s_k^{-1} = D - \sigma_k, \tag{48}$$

$$s_k D^2 s_k^{-1} = (s_k D s_k^{-1})^2 = (D - \sigma_k)^2 = D^2 - 2\sigma_k D + \sigma_k^2 - \sigma_k'. \tag{49}$$

Hence,

$$q_k = \tilde{q}_k - 2\sigma_k p, \tag{50}$$

$$r_k = \tilde{r}_k + (\sigma_k^2 - \sigma_k')p - \sigma_k \tilde{q}_k, \tag{51}$$

as was to be shown. ∎

With all the previous elements, proving Theorem 1 for general factorization gauges becomes a straightforward computation

*Proof of Theorem* 1. We observe that $A_k$ is related with $\tilde{A}_k$ in Lemma 1 by $s_k = b_1 \dots b_k$ as follows:

$$s_k \tilde{A}_k s_k^{-1} = D - \tilde{w}_k - \sigma_k = A_k. \tag{52}$$

Hence, by Lemma 3,

$$T_k A_k = s_k \tilde{T}_k \tilde{A}_k s_k^{-1} = s_k \tilde{A}_k \tilde{T}_{k-1} s_k^{-1} = A_k T_{k-1}, \tag{53}$$

as was to be shown. ∎

*Remark* 4. In light of Lemma 3, a different choice of $b_1, \dots, b_n$ results in a gauge transformation of the operators $T_1, \dots, T_n$. It is for this reason that we refer to $b_1, \dots, b_n$ as factorization gauges. Moreover, by (24), the coefficients of $T_n$ are defined directly in terms of $\sigma_n = (\log s_n)'$ and $v_n$. The Wronskian $\phi_{(1,2,\dots,n)}$ is alternating in its indices and its log-derivative $v_n$ is invariant with respect to permutations of the set $\{1, 2, \dots, n\}$. Thus, we see that $T_n$ depends only on the choice of the seed eigenfunctions $\phi_1, \dots, \phi_n$ — irrespective of their order—and on the product of the factorization gauges $s_n = b_1 \cdots b_n$. Fixing the seed eigenfunctions, but choosing a different $s_n$ amounts to a gauge transformation of $T_n$.

## 3 | CDTs

In this section we generalize the concept of Darboux transformations to allow for repeated eigenvalues. Notice that the construction in Section 2 fails if the eigenvalues of the factorization eigenfunctions are not all distinct, because then some of the seed functions may not be linearly independent, which leads to the vanishing of the Wronskians in the denominator of (23). To allow for repeated eigenvalues, we will allow some of our seed eigenfunctions to become generalized eigenfunctions.

**Definition 4.** We say that two second-order rational operators $T_0$ and $T_2$ are connected by a *CDT* if there exists a second-order rational operator $T_1$ such that $T_0 \to T_1 \to T_2$ is a two-step Darboux transformation and the corresponding eigenvalues, as defined in Proposition 1, satisfy $\lambda_1 = \lambda_2$.

As we shall see, the factorization eigenfunction for the second step $T_1 \to T_2$ will not be related to an eigenfunction of $T_0$ but to a generalized eigenfunction, which motivates the following definition.

**Definition 5.** Let $T$ be a linear differential operator. We say that $\phi$ is an *nth-order generalized eigenfunction* of $T$ if $(T-\lambda)^n\phi = 0$, but $(T-\lambda)^{n-1}\phi \neq 0$.

For our purposes it will be sufficient to use only second-order generalized eigenfunctions; however, the more general construction can be found in Ref. 26. We now show that a CDT can be generated by a seed eigenfunction and a corresponding second-order generalized eigenfunction.

To build a CDT, we start with a second-order rational operator $T_0 = pD^2 + q_0D + r_0$, a quasi-rational seed eigenfunction $\phi$ with eigenvalue $\lambda$ and rational factorization gauges $b_1$ and $b_2$. Let $T_0 \to T_1$ be a one-step Darboux transformation with factorization function $\phi$ at factorization eigenvalue $\lambda$. We now wish to perform a second Darboux transformation on $T_1$ using the repeated eigenvalue $\lambda$ and factorization gauge $b_2$. As shown below, this requires that the second seed function be a generalized eigenfunction of $T_0$. We therefore seek to construct a function $\phi^{(1)}$ such that $(T_0 - \lambda)\phi^{(1)} = \phi$. The following lemma shows how to achieve this.

**Lemma 4.** *Let $T_0 = pD^2 + q_0D + r_0$ and let $\phi$ be an eigenfunction of $T_0$ with eigenvalue $\lambda$, that is, $T_0\phi = \lambda\phi$. Then the particular solution of the inhomogeneous equation $(T_0 - \lambda)y = \phi$ is given by*

$$\phi^{(1)}(z) = \phi(z) \int^z \left( \frac{\mu(u)}{\phi(u)^2} \int^u \frac{\phi^2(s)}{p(s)\mu(s)} ds \right) du, \tag{54}$$

*where*

$$\mu(z) := \exp\left( -\int^z \frac{q_0(u)}{p(u)} du \right). \tag{55}$$

*Also, a linearly independent solution of the homogenous equation $(T_0 - \lambda)y = 0$ is given by*

$$\phi^{\perp}(z) := \phi(z) \int^z \frac{\mu(u)}{\phi(u)^2} du. \tag{56}$$

*Remark* 5. Having fixed $\mu$ as in (55), the above definition of $\phi^{(1)}$ incorporates two additional constants of integration which correspond to linear combination of the solutions of the homogeneous equation, $\phi$ and $\phi^{\perp}$. We will fix one of these constants once we consider an explicit form for $p$, by imposing an appropriate lower bound for the integral. The other constant of integration serves as a natural deformation parameter in the CDT construction. This will be explained in more detail later on.

*Proof of Lemma* 4. We first consider a complementary solution $\phi^{\perp}$ to the eigenvalue equation $T_0 y = \lambda y$. We can obtain an explicit formula for $\phi^{\perp}$ via reduction of order. Substituting the form $\phi^{\perp} := f\phi$ into the equation $T_0\phi^{\perp} = \lambda\phi^{\perp}$ yields the equation

$$\frac{f''}{f'} = -\frac{q_0}{p} - 2w, \quad \text{with} \quad w = \frac{\phi'}{\phi}. \tag{57}$$

Noticing that $\frac{\mu'}{\mu} = -\frac{q_0}{p}$, it follows that

$$f(z) = \int^{z} \frac{\mu(u)}{\phi(u)^2} du. \tag{58}$$

The above definitions are purely formal in that we have not specified the lower bound of the integrals. This means that $\phi^{\perp}$ is defined up to a constant multiple of $\phi$, and that $\mu$ is defined up to a choice of positive multiplicative constant.

We now construct $\phi^{(1)}$ using variation of parameters. We set

$$\phi^{(1)} := \hat{\rho}\phi + \rho\phi^{\perp}, \tag{59}$$

where $\hat{\rho}$ and $\rho$ are unknown functions satisfying

$$\hat{\rho}'\phi + \rho'\phi^{\perp} = 0. \tag{60}$$

Because $(T_0 - \lambda)\phi^{(1)} = \phi$, then they also satisfy

$$p\big(\hat{\rho}''\phi + 2\hat{\rho}'\phi' + \rho''\phi^{\perp} + 2\rho'(\phi^{\perp})'\big) = \hat{\rho}\phi + \rho\phi^{\perp}. \tag{61}$$

Solving for the functions $\hat{\rho}$ and $\rho$ satisfying the above system of equations, we find that

$$\hat{\rho}(z) = -\int^{z} \phi(u)\phi^{\perp}(u)W(u)du, \tag{62}$$

$$\rho(z) = \int^{z} \phi^2(u)W(u)du, \tag{63}$$

where

$$W(z) := (p(z)\mu(z))^{-1} = \frac{1}{p(z)} \exp\left(\int^{z} \frac{q_0(u)}{p(u)} du\right). \tag{64}$$

Now integrating by parts it follows that we can express $\phi^{(1)}$ as

$$\begin{aligned} \phi^{(1)}(z) = \phi(z)(f(z)\rho(z) + \hat{\rho}(z)) &= \phi(z)\left(f(z)\rho(z) - \int^{z} f(u)\rho'(u)du\right) \\ &= \phi(z)\int^{z} \rho(u)f'(u)du = \phi(z)\int^{z} \frac{\rho(u)\mu(u)}{\phi(u)^2}du. \end{aligned} \tag{65}$$

■

After the first Darboux transformation $T_0 \to T_1$ at factorization eigenvalue $\lambda$, we define the intertwiner $A_1 = b_1(D - w_1)$, with $w_1 = (\log \phi)'$. The first candidate for factorization function for the second Darboux transformation $T_1 \to T_2$ would be the image of $\phi^\perp$ under $A_1$:

$$\psi^\perp := A_1\phi^\perp = b_1\frac{\mu}{\phi}. \tag{66}$$

Indeed, because $T_1\psi^\perp = \lambda\psi^\perp$, we could employ $\psi^\perp$ as a factorization eigenfunction for a one-step Darboux transformation on $T_1$. However, this choice of eigenfunction produces the *inverse Darboux transformation* $T_1 \to T_0$.

To construct an operator $T_2$ distinct from $T_0$, we need to consider another candidate: the image of the generalized eigenfunction $\phi^{(1)}$ under $A_1$. Indeed, we define the factorization eigenfunction for $T_1$ to be

$$\psi_2 := A_1\phi^{(1)} = \rho\psi^\perp, \tag{67}$$

with $\rho$ as in (63). The second equality is true because

$$\mathrm{Wr}[\phi, \phi^{(1)}] = \mathrm{Wr}\left[\phi, \phi \int^z \frac{\rho\mu}{\phi^2}\right] = \rho\mu \tag{68}$$

and therefore

$$\psi_2 = \frac{b_1}{\phi}\mathrm{Wr}[\phi, \phi^{(1)}] = \frac{b_1}{\phi}\rho\mu = \rho\psi^\perp. \tag{69}$$

The key observation is that although $\phi^{(1)}$ is only a generalized eigenfunction of $T_0$, its image $\psi_2$ is a true eigenfunction of $T_1$ at eigenvalue $\lambda$:

$$(T_1 - \lambda)\psi_2 = (T_1 - \lambda)A_1\phi^{(1)} = A_1(T_0 - \lambda)\phi^{(1)} = A_1\phi_1 = 0, \tag{70}$$

and thus it can be employed for the second Darboux transformation $T_1 \to T_2$. We summarize the construction of a CDT $T_0 \to T_2$ with a generalized eigenfunction in the following proposition.

**Proposition 3.** *Let $T_0 = pD^2 + q_0D + r_0$ be a second-order rational operator, $\phi$ a quasi-rational eigenfunction of $T_0$ with eigenvalue $\lambda$, and $b_1, b_2$ a choice of nonzero rational functions. Let $\mu, \rho$ be defined as per (55) and (63), respectively, and assume that $\rho$ is a quasi-rational function. Let $T_2 = pD^2 + q_2D + r_2$, where $p_2$ and $r_2$ are defined in (24) with*

$$v_1 := (\log \phi)', \quad v_2 := (\log(\rho\mu))'. \tag{71}$$

*Then $T_0$ and $T_2$ are connected by a CDT*

*Proof.* First, we observe that $T_2$ is a second-order rational operator because of the assumption on $\mu$. Then, it remains to show there is $T_1$ such that $T_0 \to T_1 \to T_2$ is a two-step Darboux transforation with $\lambda_1 = \lambda_2$. Defining $T_1, A_1, A_2$ as per (22)–(25), with $b_j, v_j$, $j \in \{1, 2\}$ as in the statement, the result then follows. ∎

*Remark* 6. Observe that the construction of a CDT amounts to applying the extended Crum Wronskian formula derived in Theorem 1 for $k = 2$ with the only modification that one of the seed functions of $T_0$ is a true eigenfunction, but the second one is a generalized eigenfunction, that is, apply the formulas in Theorem 1 with

$$\phi_1 = \phi, \quad \phi_2 = \phi^{(1)}, \quad \lambda_1 = \lambda_2 = \lambda. \tag{72}$$

*Remark* 7. The definition (54) of the second seed function $\phi^{(1)}$ involves two indefinite integrals. The outer integral provides no extra freedom, because it means that $\phi^{(1)}$ is defined up to an additive term $C\phi$ but this term will vanish in the Wronskian $\phi_{(1,2)}$. The lower bound of the inner integral (which is also shown in (63)) however, gives rise to a term $C\phi^{\perp}$, which introduces a free real parameter, a characteristic feature of the CDT. We will see below, that this coefficient of $\phi^{\perp}$ is—roughly speaking—the reciprocal of a deformation parameter $t$. Thus sending $C \to \infty$ (equivalently, setting $t \to 0$) recovers the starting operator.

*Remark* 8. The assumption that $\frac{\rho'}{\rho}$ is rational ensures that $v_2$, as defined above, is rational. This assumption may be restated as the condition that $\phi^2 W$, the integrand of (63), is a rational function with vanishing residues. Verifying the rationality of the CDT is key to ensure that the transformed operator has polynomial eigenfunctions.

*Remark* 9. The confluent aspect of a CDT comes from a conceptual formula for the generalized eigenvalue equation $(T_0 - \lambda)\phi^{(1)} = \phi$. We note that, despite not being made explicit, the seed eigenfunction $\phi$ depends on the eigenvalue $\lambda$. This dependence can be recovered by imposing initial conditions on $\phi$ and $\phi'$. Thus, starting from the eigenvalue equation $(T_0 - \lambda)\phi = 0$, we can differentiate with respect to $\lambda$ to find that

$$(T_0 - \lambda)\left[\frac{\partial \phi}{\partial \lambda}\right] = \phi. \tag{73}$$

This equation implies a rather simple formula for defining $\phi^{(1)}$, which is

$$\phi^{(1)} = \frac{\partial \phi}{\partial \lambda}. \tag{74}$$

This formula is not of much practical use, because the functional dependence on $\lambda$ is typically impossible to express explicitly. However, this expression is of conceptual importance and the derivative can be seen as a limiting case of the ordinary Darboux transformation, where the eigenvalues converge as $\lambda_2 \to \lambda_1$. Hence the name CDT.

In this section we have seen how to build a CDT as a two-step Darboux transformation with a generalized eigenfunction, introducing in the process a free real parameter. This construction can be iterated at different eigenvalues to create chains of CDTs: Perform a CDT on the first operator $T_0$ at eigenvalue $\lambda_1$, which is followed by a CDT on $T_2$ at eigenvalue $\lambda_2 \neq \lambda_1$, which yields the operator $T_4$, etc.

Chains of operators may be constructed through an arbitrary finite number of CDTs in this fashion, thereby leading to an operator with an arbitrary number of free real parameters, which is Darboux connected to the original $T_0$. In the following section we show how to construct a CDT chain starting on the classical Gegenbauer polynomials, and leading to the deformed Gegenbauer polynomials.

# 4 | EXCEPTIONAL GEGENBAUER OPERATORS AND POLYNOMIALS

In this section we apply the theory developed in Sections 2 and 3 to construct a chain of CDTs on the classical Gegenbauer operator.

## 4.1 | Definition of exceptional Gegenbauer operators and polynomials

In Ref. 2 it was shown that an exceptional operator in the Hermite, Laguerre, or Jacobi class must have a very specific form. We define in this section exceptional Gegenbauer polynomials and operators attending to this particular form (as a particular class of exceptional Jacobi operators), postponing for later sections the discussion on how the construction of specific families is achieved.

**Definition 6.** Let $\tau = \tau(z)$ be a nonzero polynomial and $\alpha \in \mathbb{R}$. We say that the differential expression

$$T_\tau^{(\alpha)}(z, D) := (1 - z^2)\left(D^2 - 2\frac{\tau'}{\tau}D + \frac{\tau''}{\tau}\right) - (2\alpha + 1)zD + (2\alpha - 1)z\frac{\tau'}{\tau} \tag{75}$$

is an *exceptional Gegenbauer operator* if $T_\tau^{(\alpha)}$ admits eigenpolynomials $\{\pi_i(z)\}_{i\in\mathbb{N}_0}$ such that the degree sequence $\{\deg \pi_i\}_{i\in\mathbb{N}_0}$ is missing finitely many "exceptional" degrees.

In the above definition, it should be stressed that only very specific polynomials $\tau(z)$ in (75) will lead to $T_\tau^{(\alpha)}$ being an exceptional Gegenbauer operator, that is, having an infinite number of polynomial eigenfunctions. The following sections are devoted to describing a class of polynomials $\tau$ obtained by applying a multistep CDT on $\tau = 1$, which ensures that this is indeed the case. Note that there is no restriction on the parameter $\alpha$ at this stage. In the following section we will see that $\alpha$ must be a half-integer for the CDTs to be rational. This means that for standard Darboux transformations (see Section 2) the parameter $\alpha$ can be real, leading to generic exceptional Gegenbauer polynomials, but for CDTs (see Section 3) the parameter $\alpha \in \mathbb{N}_0 + \frac{1}{2}$, which leads to deformed Gegenbauer polynomials.

Observe also, by contrast to classical OPs, that we *are not* assuming that $\deg \pi_i = i$. Furthermore, without loss of generality, it will be convenient to assume that

$$\deg \pi_i \neq \deg \pi_j \quad \text{if } i \neq j. \tag{76}$$

As usual, we speak of exceptional Gegenbauer polynomials when the eigenpolynomials $\{\pi_i(z)\}_{i\in\mathbb{N}_0}$ define a complete OP system.

**Definition 7.** Let $\tau(z)$ be a polynomial and $\alpha \in \mathbb{R}$ . We say that the set $\{\pi_i(z)\}_{i\in\mathbb{N}_0}$ is a family of *exceptional Gegenbauer polynomials* with weight

$$W_\tau^{(\alpha)}(z) := \frac{(1-z^2)^{\alpha-\frac{1}{2}}}{\tau(z)^2}, \tag{77}$$

if the following conditions hold:

(a) $\tau(z)$ does not vanish on $I = [-1, 1]$;
(b) $\{\pi_i(z)\}_{i\in\mathbb{N}_0}$ are eigenpolynomials of an $X$-Gegenbauer operator (75);
(c) The polynomials $\{\pi_i(z)\}_{i\in\mathbb{N}_0}$ form a complete set in the Hilbert space $\mathrm{L}^2(I, W_\tau^{(\alpha)})$.

Note that there is no need to include an explicit orthogonality assumption in the above definition, because orthogonality of the eigenpolynomials follows from assumptions (a) and (b). Also note that there is no need for supplementary assumptions regarding the corresponding eigenvalues as these are necessarily quadratic functions of the degree sequence. More specifically, we establish these results in the next two lemmas.

**Lemma 5.** *Let $T_\tau^{(\alpha)}$ be an exceptional Gegenbauer operator and let $\{\pi_i\}_{i\in\mathbb{N}_0}$ be its associated eigenpolynomials, that is,*

$$T_\tau^{(\alpha)}\pi_i = \lambda_i\pi_i. \tag{78}$$

*Then, necessarily*

$$\lambda_i = -d_i(2\alpha + d_i), \quad \textit{where } d_i = \deg \pi_i - \deg \tau. \tag{79}$$

*Proof.* Let $\tau(z)$ be a polynomial and $\pi_i(z)$ an eigenpolynomial of $T_\tau^{(\alpha)}$ with eigenvalue $\lambda_i$. Explicitly, by (75), we have

$$(1-z^2)\big(\tau\pi_i'' - 2\tau'\pi_i' + \tau''\pi_i\big) - (2\alpha+1)z\tau\pi_i' + (2\alpha-1)z\tau'\pi_i = \lambda_i\tau\pi_i. \tag{80}$$

Let $n = \deg \pi_i$ and $m = \deg \tau$ and without loss of generality suppose that both $\tau(z), \pi_i(z)$ are monic; that is, $\tau(z) = z^m + \cdots$ and $\pi(z) = z^n + \cdots$. Notice that the highest power of $z$ on either side of (80) is $m + n$. Hence, for the above equation to hold, the two coefficients on $z^{m+n}$ must be equal. Considering only the highest power of $z$ in each term, yields the equation

$$\lambda_i z^{m+n} = (-n(n-1) + 2mn - m(m-1) - (2\alpha+1)n + (2\alpha-1)m)z^{m+n}. \tag{81}$$

Hence

$$\begin{aligned} \lambda_i &= -n(n-1) + 2mn - m(m-1) - (2\alpha+1)n + (2\alpha-1)m \\ &= -n^2 + 2mn - m^2 - 2\alpha n + 2\alpha m \\ &= -(n-m)^2 - 2\alpha(n-m) \\ &= -(n-m)(2\alpha + n - m), \end{aligned} \tag{82}$$

as was to be shown. ■

**Lemma 6.** *A family of exceptional Gegenbauer polynomials $\{\pi_i\}_{i\in\mathbb{N}_0}$ is necessarily orthogonal with respect to the corresponding weight (77):*

$$\int_I \pi_i(z)\pi_j(z)W_\tau^{(\alpha)}(z)dz = 0 \quad \text{if } i \neq j. \tag{83}$$

*Proof.* Multiplying the eigenvalue equation $T_\tau^{(\alpha)}y = \lambda y$ by $W_\tau^{(\alpha)}$ yields a Sturm–Liouville eigenvalue equation

$$(P_\tau^{(\alpha)}y')' + R_\tau^{(\alpha)}y = W_\tau^{(\alpha)}\lambda y, \tag{84}$$

where

$$\begin{aligned} P_\tau^{(\alpha)} &:= (1-z^2)^{\alpha+\frac{1}{2}}\tau^{-2}, \\ R_\tau^{(\alpha)} &:= (1-z^2)^{\alpha+\frac{1}{2}}\tau''\tau^{-3} + (2\alpha-1)z(1-z^2)^{\alpha-\frac{1}{2}}\tau'\tau^{-3}. \end{aligned} \tag{85}$$

Lagrange's identity now gives

$$\int_{-1}^{z}\left(y_1T_\tau^{(\alpha)}y_2 - y_2T_\tau^{(\alpha)}y_1\right)W_\tau^{(\alpha)}du = P_\tau^{(\alpha)}\,\mathrm{Wr}[y_1,y_2]. \tag{86}$$

Because the eigenvalues $\lambda_i$ are distinct, assumption (a) and (86) imply that the eigenpolynomials satisfy orthogonality relations. ∎

The base case of the class of exceptional Gegenbauer operators is the classical Gegenbauer operator $T^{(\alpha)} = T_{\tau_0}^{(\alpha)}$, where $\tau_0(z) = 1$. In this case the general form (75) simplifies to (7) and the eigenpolynomials are the classical Gegenbauer polynomials[40]

$$C_i^{(\alpha)} := \sum_{k=0}^{\lfloor i/2\rfloor}(-1)^k\frac{\Gamma(i-k+\alpha)}{\Gamma(\alpha)k!(i-2k)!}(2z)^{i-2k}. \tag{87}$$

These classical OPs do have $\deg C_i^{(\alpha)} = i$, and they satisfy the eigenvalue relation

$$T^{(\alpha)}C_i^{(\alpha)} = \lambda_i C_i^{(\alpha)}, \quad i \in \mathbb{N}_0, \tag{88}$$

with $\lambda_i = -i(2\alpha+i)$. If $\alpha > -\frac{1}{2}$ the polynomials $\{C_i^{(\alpha)}\}_{i\in\mathbb{N}_0}$ form a complete set in $\mathrm{L}^2(I, W^{(\alpha)})$ and they satisfy the orthogonality relation

$$\int_I C_i^{(\alpha)}(u)C_j^{(\alpha)}(u)W^{(\alpha)}(u)du = \nu_i^{(\alpha)}\delta_{ij}, \quad i,j \in \mathbb{N}_0, \tag{89}$$

where

$$W^{(\alpha)}(z) := W_{\tau_0}^{(\alpha)}(z) = (1-z^2)^{\alpha-\frac{1}{2}}; \tag{90}$$

$$\nu_i^{(\alpha)} := \frac{\pi 2^{1-2\alpha}\Gamma(i+2\alpha)}{i!(i+\alpha)\Gamma(\alpha)^2}, \quad i \in \mathbb{N}_0. \tag{91}$$

Once the class of exceptional Gegenbauer operators and polynomials has been defined, we will describe in the next sections the construction of deformed Gegenbauer polynomials via a sequence of CDTs.

### 4.2 | Factorizations of exceptional Gegenbauer operators

Let us start by describing the factorization of an exceptional Gegenbauer operator (75) to define a one-step Darboux transformation. After that, we will combine two Darboux transformations to define a CDT.

Given rational functions $\tau(z)$ and $\pi(z)$, and a real constant $\alpha$, we define the following two first-order rational operators:

$$\begin{aligned} A_{\tau\pi}(z,D) &:= \tau(z)^{-1}\big(\pi(z)D - \pi'(z)\big), \\ B^{(\alpha)}_{\pi\tau}(z,D) &:= (1-z^2)A_{\pi\tau}(z,D) - (2\alpha+1)z\tau(z)\pi(z)^{-1}. \end{aligned} \tag{92}$$

It will be useful to express the above operators in terms of Wronskian determinants by setting

$$\begin{aligned} \hat{\pi}(z) &:= (1-z^2)^{-\alpha-\frac{3}{2}}\pi(z), \\ \hat{\tau}(z) &:= (1-z^2)^{-\alpha-\frac{1}{2}}\tau(z), \end{aligned} \tag{93}$$

so that

$$A_{\tau\pi}y = \tau^{-1}\,\mathrm{Wr}[\pi, y], \tag{94}$$

$$B^{(\alpha)}_{\pi\tau}y = \hat{\pi}^{-1}\,\mathrm{Wr}[\hat{\tau}, y]. \tag{95}$$

Observe that it would be sufficient to define only $A_{\tau\pi}$ because the two operators are related by

$$B^{(\alpha)}_{\pi\tau} = A_{\hat{\pi}\hat{\tau}}. \tag{96}$$

However, defining both operators separately will make for simpler notation going forward. With these first order operators, we can now describe the factorization of an exceptional Gegenabuer operator (75) in the following proposition.

**Proposition 4.** *Let $\alpha \in \mathbb{R}$, $\tau(z)$ be a polynomial, and $T^{(\alpha)}_{\tau}$ be an exceptional Gegenabuer operator as in (75). Assume that $\pi(z)$ is an eigenpolynomial of $T^{(\alpha)}_{\tau}$ with eigenvalue $\lambda$, that is, $T^{(\alpha)}_{\tau}\pi = \lambda\pi$. We then have the following factorizations:*

$$\begin{aligned} B^{(\alpha)}_{\pi\tau}A_{\tau\pi} &= T^{(\alpha)}_{\tau} - \lambda, \\ A_{\tau\pi}B^{(\alpha)}_{\pi\tau} &= T^{(\alpha+1)}_{\pi} - \hat{\lambda}, \end{aligned} \tag{97}$$

*where*

$$\hat{\lambda} = \lambda + 2\alpha + 1. \tag{98}$$

*Proof.* The proof follows by a straightforward computation. ■

We say that the transformation $T_\tau^{(\alpha)} \to T_\pi^{(\alpha+1)} - (2\alpha + 1)$ is a formally *state-deleting* Darboux transformation, because the second operator no longer has an eigenpolynomial at the eigenvalue $\lambda$. Likewise, we refer to the transformation $T_\pi^{(\alpha+1)} \to T_\tau^{(\alpha)} + (2\alpha + 1)$ as a formally *state-adding* Darboux transformation, because the operator $T_\tau^{(\alpha)}$ gains an extra polynomial eigenfunction $\pi$ at eigenvalue $\lambda$. The following proposition describes the factorization eigenfunction for this dual factorization.

**Proposition 5.** *Let $\alpha \in \mathbb{R}$, $\tau(z)$ and $\pi(z)$ be nonzero polynomials and let $\hat{\tau}$ be as in (93). Suppose that the following eigenvalue equation holds: $T_\pi^{(\alpha+1)}\hat{\tau} = \hat{\lambda}\hat{\tau}$. Then the factorizations (97) also hold.*

*Proof.* By direct calculation, we obtain that

$$\pi T_\pi^{(\alpha+1)}\hat{\tau} = \hat{\tau} T_\tau^{(\alpha)}\pi + (2\alpha + 1)\pi\hat{\tau}. \tag{99}$$

Hence, $T_\tau^{(\alpha)}\pi = \lambda\pi$ is equivalent to $T_\pi^{(\alpha+1)}\hat{\tau} = \hat{\lambda}\hat{\tau}$. This, together with (96), implies the result.

## 4.3 | CDTs of exceptional Gegenbauer operators

In this section, we apply the theory of CDTs developed in Section 3 to exceptional Gegenbauer operators. With the factorizations introduced in the previous section, we will realize a CDT of an exceptional Gegenbauer operator $T_0$ by a state-deleting transformation $T_0 \to T_1$ followed by a one-parameter family of state-adding transformations $T_1 \to T_2$ at the same eigenvalue. Next we derive certain recursive formulas that connect the $\tau$-functions and eigenpolynomials of two exceptional Gegenbauer operators connected by a CDT.

Starting from the classical Gegenbauer operator we will construct a chain of CDTs and a recursive construction of deformed Gegenbauer operators. This recursive construction will be discussed in more detail in the following section.

*Remark* 10. While the definitions of exceptional Gegenbauer operators and their factorizations in the previous section hold for any $\alpha \in \mathbb{R}$, we need to assume that certain integrals like (100) define rational functions, which requires that $\alpha \in \mathbb{N}_0 + \frac{1}{2}$ from here on.

Suppose moreover that $\tau(z)$ is a polynomial and $T_\tau^{(\alpha)}$ is an exceptional Gegenbauer operator, as per (75), with eigenpolynomials $\{\pi_i\}_{i\in\mathbb{N}_0}$. We define the functions

$$\rho_{ij}(z) := \int_{-1}^{z} \pi_i(u)\pi_j(u)W_\tau^{(\alpha)}(u)du, \quad i, j \in \mathbb{N}_0; \tag{100}$$

$$\tau_m(z, t) := \tau(z)(1 + t\rho_{mm}(z)), \quad m \in \mathbb{N}_0; \tag{101}$$

$$\pi_{m;i}(z,t) := (1 + t\rho_{mm}(z))\pi_i(z) - t\rho_{im}(z)\pi_m(z), \quad i, m \in \mathbb{N}_0, \tag{102}$$

where the integral in (100) denotes a formal antiderivative that vanishes at $z = -1$.

*Remark* 11. Throughout this section, we assume that $\tau_m(z)$ and $\pi_{m;i}(z,t)$ are polynomials in $z$ and $\rho_{ij}(z)$ is a rational function of $z$. In principle, this assumption seems a strong requirement when looking at (100) and (77). However, we will see in Section 5 that whenever these quantities are connected recursively to the classical Gegenbauer operator and polynomials, there exist matrix formulas that establish the polynomial character of $\tau_m(z)$ and $\pi_{m;i}(z,t)$ and the rational character of $\rho_{ij}(z)$ by construction.

**Proposition 6.** *For a given $m \in \mathbb{N}_0$, the operators $T_\tau^{(\alpha)}, T_{\tau_m}^{(\alpha)}$ are related by a CDT generated by the seed functions $\{\pi_m, \pi_m^{(1)}\}$.*

*Proof.* Set $T_0 = T_\tau^{(\alpha)}$ and recall that, by assumption, $T_0\pi_m = \lambda_m\pi_m$. We set

$$T_1 := T_{\pi_m}^{(\alpha+1)} - (2\alpha + 1), \quad T_2 := T_{\tau_m}^{(\alpha)}, \tag{103}$$

$$A_1 := A_{\tau,\pi_m}, \quad A_2 := B_{\pi_m,\tau_m}^{(\alpha)}. \tag{104}$$

By Proposition 4, $T_0 \to T_1$ is a state-deleting Darboux transformation. Consequently,

$$A_1T_0 = T_1A_1. \tag{105}$$

We now claim that the transformation $T_1 \to T_2$ is a one-parameter family of state-adding Darboux transformations. By inspection of (75), we find that the relevant coefficient functions of $T_0$ are

$$p := 1 - z^2, \quad q_0 := -2(1 - z^2)\frac{\tau'}{\tau} - (2\alpha + 1)z. \tag{106}$$

Then the function $\mu(z)$ in (55) satisfies

$$\frac{\mu'}{\mu} = -\frac{q_0}{p} = 2\frac{\tau'}{\tau} + \frac{(2\alpha + 1)z}{1 - z^2}, \tag{107}$$

and hence

$$\mu(z) := (1 - z^2)^{-\alpha-\frac{1}{2}}\tau(z)^2. \tag{108}$$

Following (65), we construct a generalized eigenfunction of $T_0$ as shown in Section 3. We define

$$\pi_m^{(1)}(z;t) := \pi_m(z)\int_{-1}^{z}(1 + t\rho_{mm}(s))\hat{\tau}(s)^2W_{\pi_m}^{(\alpha+1)}(s)ds, \tag{109}$$

where

$$\hat{\tau}(z) := (1-z^2)^{-\alpha-\frac{1}{2}}\tau(z), \tag{110}$$

and $W^{(\alpha+1)}_{\pi_m}$ is given in (77). Observe that $W^{(\alpha)}_{\tau} = (p\mu)^{-1}$, in agreement with (64). The definitions of $\rho_{mm}, \pi^{(1)}_m, \hat{\tau}$ agree with the definitions of $\rho, \phi^{(1)}, \psi^{\perp}_1$ in (63), (65), and (66), respectively. Consequently, as shown in Section 3, $\pi^{(1)}_m$ is a first-order generalized eigenfunction of $T_0$, because it satisfies the equation

$$(T_0 - \lambda_m)\pi^{(1)}_m = t\pi_m. \tag{111}$$

A direct calculation shows that

$$A_1\pi^{(1)}_m = \hat{\tau}_m, \tag{112}$$

where

$$\hat{\tau}_m(z,t) := (1-z^2)^{-\alpha-\frac{1}{2}}\tau_m(z,t) = \hat{\tau}(z)(1+t\rho_{mm}(z)). \tag{113}$$

Hence, by (105),

$$T_1\hat{\tau}_m = T_1A_1\pi^{(1)}_m = A_1T_0\pi^{(1)}_m = \lambda_m\hat{\tau}_m, \tag{114}$$

which means that $\hat{\tau}_m$ is a factorization eigenfunction of $T_1$ at $\lambda_m$. By Proposition 5, we have thus

$$\begin{aligned} B^{(\alpha)}_{\tau_m,\pi}A_{\pi,\tau_m} &= T^{(\alpha)}_{\tau_m} - \lambda, \\ A_{\pi,\tau_m}B^{(\alpha)}_{\tau_m,\pi} &= T^{(\alpha+1)}_{\pi} - 2\alpha - 1 - \lambda. \end{aligned} \tag{115}$$

We conclude therefore that $T_2A_2 = A_2T_1$, as was to be shown. ■

**Proposition 7.** *The polynomials $\pi_{m,i}(z;t)$ as defined in (102) are eigenfunctions of $T^{(\alpha)}_{\tau_m}$.*

*Proof.* Combining the intertwining relations in the preceding proof, we have the second-order intertwining relation

$$A_{21}T_0 = T_2A_{21}, \quad \text{where } A_{21} := A_2A_1. \tag{116}$$

Our first claim is that

$$A_{21}\pi_i = (\lambda_i - \lambda_m)\pi_{m;i}, \quad i \neq m. \tag{117}$$

Let $i \in \mathbb{N}_0$ be given. To establish the previous equation, we consider $A_1$ and $A_2$ in terms of their Wronskian formulations:

$$A_1y = A_{\tau\pi_m}y = \tau^{-1}\,\mathrm{Wr}[\pi_m, y], \tag{118}$$

$$A_2 y = B^{(\alpha)}_{\pi_m \tau_m} y = \hat{\pi}_m^{-1} \mathrm{Wr}[\hat{\tau}_m, y], \tag{119}$$

where

$$\hat{\pi}_m(z) = (1-z^2)^{-\alpha-\frac{3}{2}} \pi_m(z). \tag{120}$$

Lagrange's identity (86) implies that

$$\mathrm{Wr}[\pi_m, \pi_i] = (\lambda_i - \lambda_m)\mu\, \rho_{mi}. \tag{121}$$

Hence,

$$A_1 \pi_i = \tau^{-1}\, \mathrm{Wr}[\pi_m, \pi_i] = (\lambda_i - \lambda_m)\hat{\tau}\rho_{mi}. \tag{122}$$

We define $B_1 := B^{(\alpha)}_{\pi_m,\tau}$ and note that by Proposition 4 we have the factorization $T_0 = B_1 A_1 + \lambda_m$. By the linearity of the Wronskian, we also have

$$\begin{aligned} A_2 y &= \hat{\pi}_m^{-1}\, \mathrm{Wr}[\hat{\tau}_m, y] = \hat{\pi}_m^{-1}\, \mathrm{Wr}[\hat{\tau}(1 + t\rho_{mm}), y] \\ &= \hat{\pi}_m^{-1}\, \mathrm{Wr}[\hat{\tau}, y] + \hat{\pi}_m^{-1}\, \mathrm{Wr}[t\hat{\tau}\rho_{mm}, y] \\ &= B_1 y + t\hat{\pi}_m^{-1}\, \mathrm{Wr}[\hat{\tau}\rho_{mm}, y]. \end{aligned} \tag{123}$$

Finally, we have that

$$\begin{aligned} A_{21}\pi_i &= B_1 A_1 \pi_i + t(\lambda_i - \lambda_m)\hat{\pi}_m^{-1}\, \mathrm{Wr}[\hat{\tau}\rho_{mm}, \hat{\tau}\rho_{mi}] \\ &= (T_0 - \lambda_m)\pi_i + (\lambda_i - \lambda_m) t\hat{\tau}^2 \hat{\pi}_m^{-1}\, \mathrm{Wr}[\rho_{mm}, \rho_{mi}] \\ &= (\lambda_i - \lambda_m)(\pi_i + t\hat{\tau}^2\hat{\pi}_m^{-1}(\rho_{mm}\pi_m\pi_i W^{(\alpha)}_\tau - \rho_{mi}\pi_m^2 W^{(\alpha)}_\tau) \\ &= (\lambda_i - \lambda_m)(\pi_i + t(\rho_{mm}\pi_i - \rho_{mi}\pi_m)), \end{aligned} \tag{124}$$

which establishes (117). Next suppose that $i \neq m$. Then, by (116), it follows that

$$T_2 \pi_{m;i} = (\lambda_i - \lambda_m)^{-1} T_2 A_{21}\pi_i = (\lambda_i - \lambda_m)^{-1} A_{21} T_0 \pi_i \lambda_i \pi_{m;i}. \tag{125}$$

Finally, observe that $\pi_{m;m} = \pi_m$ and recall that

$$T_2 = B^{(\alpha)}_{\pi_m,\tau_m} A_{\tau_m,\pi_m} + \lambda_m. \tag{126}$$

Hence, $T_2\pi_m = \lambda_m \pi_m$, as was to be shown. ■

The last proposition of this section shows that the CDT of an exceptional Gegenbauer family falls into the same class provided we impose suitable bounds on the introduced parameter $t$. The form of the norming constants of the new eigenpolynomials relative to the weight $W^{(\alpha)}_{\tau_m}$ follow as a direct corollary to this proposition. We denote the norm of $\pi_i$ by $\nu_i$, and similarly the norm of

$\pi_{m;i}$ is $\nu_{m;i}$. Explicitly, we have

$$\nu_i := \int_{-1}^{1} \pi_i(s)^2 W_\tau^{(\alpha)}(s)ds = \rho_{ii}(1), \quad i \in \mathbb{N}_0; \tag{127}$$

$$\nu_{m;i} := \int_{-1}^{1} \pi_{m;i}(s)^2 W_{\tau_m}^{(\alpha)}(s)ds = \rho_{m;ii}(1), \quad m, i \in \mathbb{N}_0; \tag{128}$$

where

$$\rho_{m;ij}(z) := \int_{-1}^{z} \pi_{m;i}(u;t)\pi_{m;j}(u;t)W_{\tau_m}^{(\alpha)}(u)du, \quad m, i, j \in \mathbb{N}_0. \tag{129}$$

The following proposition exhibits a recursive formulation for the functions $\rho_{m;ij}$. As a consequence, if the $\rho_{ij}(z)$ are rational, then so are $\rho_{m;ij}(z)$, that is, rationality is preserved by a CDT.

**Proposition 8.** *Let $\rho_{ij}, \tau_m, \pi_{m;i}, \rho_{m;ij}$ be defined by (100)–(102) and (129). Then,*

$$\rho_{m;ij}(z,t) = \rho_{ij}(z) - \frac{t\rho_{mi}(z)\rho_{mj}(z)}{1 + t\rho_{mm}(z)}, \quad m, i, j \in \mathbb{N}_0. \tag{130}$$

*Proof.* Observe that

$$\begin{aligned}
\left(\rho_{ij} - \frac{t\rho_{mi}\rho_{mj}}{1+t\rho_{mm}}\right)' &= \rho_{ij}' - \frac{t\rho_{mi}'\rho_{mj}}{1+t\rho_{mm}} - \frac{t\rho_{mi}\rho_{mj}'}{1+t\rho_{mm}} + \frac{t^2\rho_{mi}\rho_{mj}}{(1+t\rho_{mm})^2}\rho_{mm}' \\
&= \left((1+t\rho_{mm})^2\pi_i\pi_j - t\pi_m(1+t\rho_{mm})(\pi_i\rho_{mj} + \pi_j\rho_{mi}) + t^2\rho_{mi}\rho_{mj}\pi_m^2\right)\frac{W_\tau^{(\alpha)}}{(1+t\rho_{mm})^2} \\
&= \left((1+t\rho_{mm})\pi_i - t\rho_{mi}\pi_m\right)\left((1+t\rho_{mm})\pi_j - t\rho_{mj}\pi_m\right)W_{\tau_m}^{(\alpha)} \\
&= \pi_{m;i}\pi_{m;j}W_{\tau_m}^{(\alpha)}.
\end{aligned} \tag{131}$$

The desired result follows then by integration because $\rho_{ij}(-1) = 0$ by definition, and $\rho_{ij}(z)$ is rational by assumption. ■

To demonstrate that the $\pi_{m;i}(z)$ are exceptional Gegenbauer polynomials in the sense of Definition 7, we must first establish the positivity of $\tau_m$ on $[-1,1]$ conditioned on the positivity of $\tau$.

**Proposition 9.** *Suppose that $\tau(z) > 0$ for all $z \in I = [-1,1]$. Then, $\tau_m$ is positive on $I$ if and only if $1 + t\nu_m > 0$.*

*Proof.* By (101), $\tau_m$ is positive on $I$ if and only if the same is true for $1 + t\rho_{mm}(z)$. We know that $\rho_{mm}(z)$ is an increasing function, because

$$\rho_{mm}'(z) = \pi_m^2(z)\frac{(1-z^2)^{\alpha-\frac{1}{2}}}{\tau^2(z)} > 0. \tag{132}$$

Additionally, we notice that $\rho_{mm}$ is differentiable, and hence continuous, on the interval $I$. Fix $t$ and define $g(z) := 1 + t\rho_{mm}(z)$, so that $g'(z) = t\rho'_{mm}(z)$. If $t \geq 0$, then $g$ is positive on $I$, because $\rho_{mm}$ is an increasing function and $g(-1) = 1$. However, if $t < 0$, then $g$ is a decreasing function on $I$. Importantly, $g$ cannot have any local extrema on the interval. Hence, if $g(1) = 1 + t\nu_m > 0$, then $g$ must be positive on all of $I$. As a consequence, $1 + t\rho_{mm}(z)$ is positive on $z \in I$ if and only if $1 + t\nu_m > 0$. ■

**Proposition 10.** *Let $\pi_{m;i}(z, t)$ be defined as in (102), and assume that $1 + t\nu_m > 0$. Then, the norms (127) and (128) are related by*

$$\nu_{m;i}^{-1} = \nu_m^{-1} + \delta_{im} t. \tag{133}$$

*Proof.* By Proposition 8, we have that

$$\nu_{m;i} = \nu_i - \frac{t\rho_{mi}(1)^2}{1 + t\rho_{mm}(1)} = \nu_i, \quad i \neq m, \tag{134}$$

Suppose that $i \neq m$. Then, $\pi_i$ and $\pi_m$ for $i \neq m$ are orthogonal relative to $W_\tau^{(\alpha)}$. Hence, $\rho_{mi}(1) = 0$ and $\nu_{m;m} = \nu_m$. By definition, $\nu_m = \rho_{mm}(1)$. Hence, if $i = m$, we have

$$\nu_{m;m} = \nu_m - \frac{t\nu_m^2}{1 + t\nu_m} = \frac{\nu_m}{1 + t\nu_m} = (t + \nu_m^{-1})^{-1}. \tag{135}$$

■

*Remark* 12. As a consequence of (128), we can recover the deformation parameter as the difference of the following norm reciprocals:

$$t = \nu_{m;m}^{-1} - \nu_m^{-1}. \tag{136}$$

Consequently,

$$\nu_{m;m}^{-1} = \nu_m^{-1}(1 + t\nu_m). \tag{137}$$

Hence, by Proposition 9, the positivity of $\tau_m$ is equivalent to the condition that $\nu_{m;m} > 0$.

**Proposition 11.** *Let $\alpha \in \mathbb{N}_0 + \frac{1}{2}$ and suppose that $\{\pi_i(z)\}_{i \in \mathbb{N}_0}$ are exceptional Gegenbauer polynomials with respect to the weight $W_\tau^{(\alpha)}(z)$. Let $m \in \mathbb{N}_0$ and suppose that $1 + t\nu_m > 0$. Then, the family $\{\pi_{m;i}(z)\}_{i \in \mathbb{N}_0}$ are also exceptional Gegenbauer polynomials with respect to the weight $W_{\tau_m}^{(\alpha)}(z)$.*

*Proof.* Proposition 9 establishes condition (a) of Definition 7. Proposition 6 establishes (b). It remains to prove that the completeness condition (c) also holds. Set

$$\begin{aligned}\tilde{\pi}_i(z) &:= \left(W_\tau^{(\alpha)}(z)\right)^{\frac{1}{2}} \pi_i(z) = (1-z^2)^{\frac{\alpha}{2}-\frac{1}{4}} \frac{\pi(z)}{\tau(z)},\\ \tilde{\pi}_{m;i}(z) &:= \left(W_{\tau_m}^{(\alpha)}(z)\right)^{\frac{1}{2}} \pi_{m;i}(z) = (1-z^2)^{\frac{\alpha}{2}-\frac{1}{4}} \frac{\pi_{m;i}(z)}{\tau_m(z)}.\end{aligned} \tag{138}$$

By assumption, the eigenpolynomials $\{\pi_i(z)\}_{i\in\mathbb{N}_0}$ form a complete basis of $\mathrm{L}^2(I, W_\tau^{(\alpha)}(z)dz)$. Equivalently, $\{\tilde{\pi}_i(z)\}_{i\in\mathbb{N}_0}$ are complete in $\mathrm{L}^2(I, dz)$. We seek to show that $\{\tilde{\pi}_{m;i}(z)\}_{i\in\mathbb{N}_0}$ are complete in $\mathrm{L}^2(I, dz)$ also.

Following an argument adapted from the appendix of Ref. 25, we observe that the completeness of $\{\tilde{\pi}_i\}_{i\in\mathbb{N}_0}$ in $\mathrm{L}^2(I, dz)$ is equivalent to

$$\sum_{i\in\mathbb{N}_0} \nu_i^{-1} \tilde{\pi}_i(z)\tilde{\pi}_i(w) = \delta(z-w), \tag{139}$$

where the equality must be understood in distributional sense on $I \times I$. Let $f$ be piece-wise continuous function in $I$ and set

$$[f]_i = \nu_i^{-1} \int_I \tilde{\pi}_i(u) f(u) du. \tag{140}$$

Relation (139) then entails

$$\sum_{i\in\mathbb{N}_0} [f]_i\, \tilde{\pi}_i(z) = f(z), \quad \text{a.e. } z \in I. \tag{141}$$

Moreover, if $g$ is another piece-wise continuous in $I$ function, then

$$\int_I f(u) g(u) du = \sum_{i\in\mathbb{N}_0} \nu_i [f]_i [g]_i. \tag{142}$$

For a given $w \in I$, and a function $f(z),\ z \in I$, define the truncation operator

$$(\mathcal{T}_w f)(z) := \theta(w-z) f(z), \quad z \in I, \tag{143}$$

where

$$\theta(u) := \begin{cases} 1 & \text{if } u > 0,\\ \frac{1}{2} & \text{if } u = 0,\\ 0 & \text{if } u < 0; \end{cases} \tag{144}$$

denotes the Heaviside step function. We may now rewrite (100) as

$$\rho_{ij}(w) = \int_I (\mathcal{T}_w \tilde{\pi}_j)(u)\, \tilde{\pi}_i(u)\, du = \nu_i [\mathcal{T}_w \tilde{\pi}_j]_i, \quad i, j \in \mathbb{N}_0,\ w \in I. \tag{145}$$

Applying (141) then gives

$$\theta(w - z)\tilde{\pi}_j(z) = (\mathcal{T}_w \tilde{\pi}_j)(z) = \sum_{i \in \mathbb{N}_0} \nu_i^{-1} \rho_{ij}(w) \tilde{\pi}_i(z), \quad z, w \in I. \tag{146}$$

Moreover, (142) implies that

$$\sum_{i \in \mathbb{N}_0} \nu_i^{-1} \rho_{ij}(z) \rho_{ij}(w) = \int_I (\mathcal{T}_w \tilde{\pi}_j)(u)(\mathcal{T}_z \tilde{\pi}_j)(u) du, \quad z, w \in I,\ j \in \mathbb{N}_0. \tag{147}$$

Observe that

$$\theta(z)\theta(w) = \theta(\min(z, w)), \quad z, w \in I. \tag{148}$$

Hence,

$$\int_I (\mathcal{T}_w \tilde{\pi}_j)(u)(\mathcal{T}_z \tilde{\pi}_j)(u) du = \int_I \theta(w - u)\theta(z - u) \tilde{\pi}_j(u) \tilde{\pi}_j(u)\, du = \rho_{jj}(\min(z, w)). \tag{149}$$

Therefore,

$$\sum_{i \in \mathbb{N}_0} \nu_i^{-1} \rho_{ij}(z) \rho_{ij}(w) = \theta(w - z)\rho_{jj}(z) + \theta(z - w)\rho_{jj}(w). \tag{150}$$

By (101), (102), and (138),

$$1 + t\rho_{mm} = \frac{\tau_m}{\tau}; \tag{151}$$

$$\tilde{\pi}_{m;i} = (1 - z^2)^{\frac{\alpha}{2} - \frac{1}{4}} \frac{1}{\tau_m} \left( \frac{\tau_m}{\tau} \pi_i - t\rho_{im} \pi_m \right) = \tilde{\pi}_i - t\rho_{im} \tilde{\pi}_{m;m}; \tag{152}$$

$$\tilde{\pi}_m = (1 + t\rho_{mm}) \tilde{\pi}_{m;m}. \tag{153}$$

By (128), (152), (139), (146), (150), and (153), we have

$$\begin{aligned}
&\sum_{i \in \mathbb{N}_0} \nu_{m;i}^{-1} \tilde{\pi}_{m;i}(z) \tilde{\pi}_{m;i}(w) \\
&= t\tilde{\pi}_{m;m}(z)\tilde{\pi}_{m;m}(w) + \sum_i \nu_i^{-1} \tilde{\pi}_{m;i}(z) \tilde{\pi}_{m;i}(w) \\
&= t\tilde{\pi}_{m;m}(z)\tilde{\pi}_{m;m}(w) + \sum_i \nu_i^{-1} \big(\tilde{\pi}_i(z) - t\rho_{im}(z)\tilde{\pi}_{m;m}(z)\big)\big(\tilde{\pi}_i(w) - t\rho_{im}(w)\tilde{\pi}_{m;m}(w)\big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_i \nu_i^{-1}\tilde{\pi}_i(z)\tilde{\pi}_i(w) + t\tilde{\pi}_{m;m}(z)\tilde{\pi}_{m;m}(w)\left(1 + t\sum_i \nu_i^{-1}\rho_{im}(z)\rho_{im}(w)\right)\\
&\quad - t\,\tilde{\pi}_{m;m}(z)\sum_i \nu_i^{-1}\rho_{im}(z)\tilde{\pi}_i(w) - t\,\tilde{\pi}_{m;m}(w)\sum_i \nu_i^{-1}\rho_{im}(w)\tilde{\pi}_i(z)\\
&= \delta(z-w) + t\tilde{\pi}_{m;m}(z)\tilde{\pi}_{m;m}(w)(1 + \theta(w-z)t\rho_{mm}(z) + \theta(z-w)t\rho_{mm}(w))\\
&\quad - t\,\theta(z-w)\tilde{\pi}_{m;m}(z)\tilde{\pi}_m(w) - t\,\theta(w-z)\tilde{\pi}_m(z)\tilde{\pi}_{m;m}(w)\\
&= \delta(z-w) + t\tilde{\pi}_{m;m}(z)\tilde{\pi}_{m;m}(w)(1 + \theta(w-z)t\rho_{mm}(z) + \theta(z-w)t\rho_{mm}(w)\\
&\quad - \theta(z-w)(1 + t\rho_{mm}(w)) - \theta(w-z)(1 + t\rho_{mm}(z)))\\
&= \delta(z-w) + t\tilde{\pi}_{m;m}(z)\tilde{\pi}_{m;m}(w)(1 - \theta(w-z) - \theta(z-w))\\
&= \delta(z-w).
\end{aligned}
\tag{154}
$$

We conclude that the set $\{\tilde{\pi}_{m;i}\}_{i\in\mathbb{N}_0}$ is complete in $\mathrm{L}^2(I, dz)$, and by virtue of the previously stated equivalence, the set $\{\pi_{m;i}\}_{i\in\mathbb{N}_0}$ is complete in $\mathrm{L}^2(I, W^{(\alpha)}_{\tau_m}(z)dz)$. ∎

Note that in Proposition 11 and in fact, all throughout the current section, we have assumed that $\pi_{m;i}$ are polynomials, which is not guaranteed by their defining expressions (100)–(102) and (77), even if $\pi_i$ are polynomials and $\alpha \in \mathbb{N}_0 + \frac{1}{2}$ (but they are certainly not polynomials if $\alpha$ is not half-integer). In the following section we will show that when these objects are connected to the classical Gegenbauer polynomials by a chain of CDTs, the assumptions on polynomiality are guaranteed at each step of the chain.

## 5 | DEFORMED GEGENBAUER POLYNOMIALS

In this section we provide explicit formulas for the construction of deformed Gegenbauer polynomials and their associated operators in terms of a matrix whose entries involve classical Gegenbauer polynomials. Throughout this section, we also assume that $\alpha \in \mathbb{N}_0 + \frac{1}{2}$ is a positive half-integer. Note that, with this assumption in place, the weight $W^{(\alpha)}(z)$ in (90) becomes a polynomial.

We next define several objects that allow us to describe the exceptional Gegenbauer polynomials and operators below. Set

$$
\rho^{(\alpha)}_{ij}(z) := \int_{-1}^{z} C^{(\alpha)}_i(u)C^{(\alpha)}_j(u)W^{(\alpha)}(u)\,du, \quad i,j\in\mathbb{N}_0 \tag{155}
$$

and observe that the above functions are polynomials precisely because $\alpha$ is a positive half-integer. Given an $n$-tuple $\boldsymbol{m}\in\mathbb{N}_0^n$ and the associated $\boldsymbol{t}_{\boldsymbol{m}}\in\mathbb{R}^n$, we then define $\mathcal{R}^{(\alpha)}_{\boldsymbol{m}} = \mathcal{R}^{(\alpha)}_{\boldsymbol{m}}(z;\boldsymbol{t}_{\boldsymbol{m}})$ as the $n\times n$ matrix with polynomial entries given by

$$
[\mathcal{R}^{(\alpha)}_{\boldsymbol{m}}]_{k\ell} = \delta_{k\ell} + t_{m_\ell}\rho^{(\alpha)}_{m_k m_\ell}(z), \quad k,\ell\in\{1,\dots,n\}. \tag{156}
$$

We denote its determinant by

$$\tau_{\boldsymbol{m}}^{(\alpha)} := \det \mathcal{R}_{\boldsymbol{m}}^{(\alpha)}. \tag{157}$$

Next, define the $n$-tuple of polynomials

$$\left(\boldsymbol{Q}_{\boldsymbol{m}}^{(\alpha)}\right)^T := \tau_{\boldsymbol{m}}^{(\alpha)} \left(\mathcal{R}_{\boldsymbol{m}}^{(\alpha)}\right)^{-1} \left(C_{m_1}^{(\alpha)}, \dots, C_{m_n}^{(\alpha)}\right)^T. \tag{158}$$

We are now ready to define the fundamental objects of this section.

**Definition 8.** Let $\boldsymbol{m} = (m_1, \dots, m_n) \in \mathbb{N}_0^n$ be a tuple of distinct integers, $\boldsymbol{t}_{\boldsymbol{m}} \in \mathbb{R}^n$ and $\boldsymbol{Q}_{\boldsymbol{m}}^{(\alpha)}$ be the $n$-tuple of polynomials defined by (155)–(158). We define the *deformed Gegenbauer polynomials* associated to $\boldsymbol{m}$ as

$$C_{\boldsymbol{m};i}^{(\alpha)} := \left[\boldsymbol{Q}_{(\boldsymbol{m},i)}^{(\alpha)}\right]_{n+1}, \quad i \in \mathbb{N}_0. \tag{159}$$

*Remark* 13. Note that, by construction, $\tau_{\boldsymbol{m}}^{(\alpha)} = \tau_{\boldsymbol{m}}^{(\alpha)}(z; \boldsymbol{t}_{\boldsymbol{m}})$ is invariant with respect to permutations of the indices $\boldsymbol{m} = (m_1, \dots, m_n)$ and that $\boldsymbol{Q}_{\boldsymbol{m}}^{(\alpha)} = \boldsymbol{Q}_{\boldsymbol{m}}^{(\alpha)}(z; \boldsymbol{t}_{\boldsymbol{m}})$ is equivariant with respect to such permutations. In addition, $C_{\boldsymbol{m};i}^{(\alpha)} = C_{\boldsymbol{m};i}^{(\alpha)}(z; \boldsymbol{t}_{\boldsymbol{m}})$ is symmetric in $\boldsymbol{m}$ and does not depend on $t_i$ because $\tau_{(\boldsymbol{m},i)}^{(\alpha)}[(\mathcal{R}_{(\boldsymbol{m},i)}^{(\alpha)})^{-1}]_{n+1,j}$ correspond to the minors of the last column of $\mathcal{R}_{(\boldsymbol{m},i)}^{(\alpha)}$, the only column where $t_i$ appears.

The main result of this section states that the polynomials $\{C_{\boldsymbol{m};i}^{(\alpha)}(z; \boldsymbol{t}_{\boldsymbol{m}})\}_{i \in \mathbb{N}_0}$ defined above are indeed exceptional Gegenbauer polynomials (in the sense of Definition 7), provided the real parameters $\boldsymbol{t}_{\boldsymbol{m}}$ satisfy certain constraints to ensure that $\tau_{\boldsymbol{m}}^{(\alpha)}(z; t_{\boldsymbol{m}})$ is positive on $z \in [-1, 1]$. We first state that inserting the polynomial $\tau_{\boldsymbol{m}}^{(\alpha)}$ in (75) leads to an exceptional Gegenbauer operator in the sense of Definition 6.

**Theorem 2.** *Let $\alpha \in \mathbb{N}_0 + \frac{1}{2}$, $\boldsymbol{m} \in \mathbb{N}_0^n$ and $\boldsymbol{t}_{\boldsymbol{m}} \in \mathbb{R}^n$. Consider the n-parameter family of operators $T_{\boldsymbol{m}}^{(\alpha)} := T_{\tau_{\boldsymbol{m}}^{(\alpha)}}^{(\alpha)}$ given by (75) and (157). For each value of the parameters $\boldsymbol{t}_{\boldsymbol{m}}$, this operator is an exceptional Gegenbauer operator that satisfies*

$$T_{\boldsymbol{m}}^{(\alpha)} C_{\boldsymbol{m};i}^{(\alpha)} = \lambda_i C_{\boldsymbol{m};i}^{(\alpha)}, \quad i \in \mathbb{N}_0, \tag{160}$$

*with $\lambda_i = -i(2\alpha + i)$.*

The following theorem provides necessary and sufficient conditions for the polynomials $\{C_{\boldsymbol{m};i}^{(\alpha)}(z; \boldsymbol{t}_{\boldsymbol{m}})\}_{i \in \mathbb{N}_0}$ to be a family of exceptional Gegenbauer polynomials according to Definition 7. If these conditions hold the family $\{C_{\boldsymbol{m};i}^{(\alpha)}(z; \boldsymbol{t}_{\boldsymbol{m}})\}_{i \in \mathbb{N}_0}$ is orthogonal and complete, like their classical counterparts.

**Theorem 3.** *Let $\alpha \in \mathbb{N}_0 + \frac{1}{2}$, $\boldsymbol{m} \in \mathbb{N}_0^n$ with $m_1, \dots, m_n$ distinct and $\boldsymbol{t_m} \in \mathbb{R}^n$. Then the polynomial $\tau_{\boldsymbol{m}}^{(\alpha)}(z; \boldsymbol{t_m})$ in (157) has no zeros on $z \in [-1, 1]$ if and only if*

$$t_{m_j} > -\left(\nu_{m_j}^{(\alpha)}\right)^{-1}, \quad j = 1, \dots, n, \tag{161}$$

*with $\nu_{m_j}^{(\alpha)}$ as in (91). If the above conditions hold, then $\{C_{\boldsymbol{m};i}^{(\alpha)}(z; \boldsymbol{t_m})\}_{i\in\mathbb{N}_0}$ are exceptional Gegenbauer polynomials with weight $W_{\tau_m}^{(\alpha)}$ and norms given by*

$$\int_I \left(C_{\boldsymbol{m};i}^{(\alpha)}(u)\right)^2 W_{\tau_m}^{(\alpha)}(u) du = \frac{\nu_i^{(\alpha)}}{1 + \delta_{i,\boldsymbol{m}} t_i \nu_i^{(\alpha)}}, \tag{162}$$

*where*

$$\delta_{i,\boldsymbol{m}} := \begin{cases} 1 & \text{if } i \in \{m_1, \dots, m_n\}; \\ 0 & \text{otherwise.} \end{cases} \tag{163}$$

As mentioned above, the degree of the $i$th exceptional Gegenbauer polynomial $C_{\boldsymbol{m};i}^{(\alpha)}$ is not necessarily $i$. The next proposition provides this result. It is also worth noting that, as opposed to the generic exceptional families, the degree sequence of the deformed Gegenbauer polynomials is not an increasing sequence, which is further evidence of the different nature of this new construction.

**Proposition 12.** *Let $\alpha \in \mathbb{N}_0 + \frac{1}{2}$, $\boldsymbol{m} \in \mathbb{N}_0^n$ with $m_1, \dots, m_n$ distinct and $\boldsymbol{t_m} \in \mathbb{R}^n$. Let $\tau_{\boldsymbol{m}}^{(\alpha)}, C_{\boldsymbol{m};i}^{(\alpha)}$ be as defined in (157) and (159). Then,*

$$\deg_z \tau_{\boldsymbol{m}}^{(\alpha)} = 2(m_1 + \cdots + m_n + \alpha n), \tag{164}$$

$$\deg_z C_{\boldsymbol{m};i}^{(\alpha)} = 2(m_1 + \cdots + m_n + \alpha n) + i - 2\delta_{i,\boldsymbol{m}}(i + \alpha), \quad i \in \mathbb{N}_0. \tag{165}$$

*Moreover,*

$$C_{\boldsymbol{m};m_k}^{(\alpha)} = C_{m_1,\dots,\widehat{m_k},\dots,m_n;m_k}^{(\alpha)}, \quad k = 1, \dots, n, \tag{166}$$

*where the hat symbol denotes the omission of the $k$th entry of $\boldsymbol{m}$.*

*Remark* 14. From Proposition 12, we see that the codimension (number of missing degrees) of the exceptional Gegenbauer family indexed by $\boldsymbol{m} = (m_1, \dots, m_n)$ is $2(m_1 + \cdots + m_n + \alpha n)$. As it happens for all exceptional polynomials,[2] this coincides with the degree of $\tau_{\boldsymbol{m}}^{(\alpha)}$.

*Remark* 15. Below we will show that the $n$-parameter family $\{C_{\boldsymbol{m};i}^{(\alpha)}(z; t_{\boldsymbol{m}})\}_{i\in\mathbb{N}_0}$ is the result of applying $n$ CDTs to the classical family $\{C_i^{(\alpha)}(z)\}_{i\in\mathbb{N}_0}$. Identity (162) then tells us that a single CDT leaves invariant all but one of the norming constants. By contrast, identity (166) tells us that after a single CDT there is exactly one polynomial that remains the same but whose norm undergoes a change. It is this phenomenon that accounts for the Kronecker delta term in (165).

*Remark* 16. The degree formulas (164), (165) allow us to relate the eigenvalue formula (79) with the value $\lambda_i = -i(2\alpha + i)$ given in Theorem 2. Indeed if $i \notin \{m_1, \dots, m_n\}$ then $\deg C^{(\alpha)}_{\boldsymbol{m},i} - \deg \tau^{(\alpha)}_{\boldsymbol{m}} = i$ in full agreement with (79). By contrast, if $i = m_k$ for some $k \in \{1, \dots, n\}$ then

$$d_i := \deg C^{(\alpha)}_{\boldsymbol{m},i} - \deg \tau^{(\alpha)}_{\boldsymbol{m}} = -i - 2\alpha \tag{167}$$

and therefore

$$d_i(2\alpha + d_i) = (i + 2\alpha)i. \tag{168}$$

Hence, (79) is correct in this case also.

In Theorem 3 and Proposition 5 we have considered the case when $\boldsymbol{m} = (m_1, \dots, m_n)$ contains distinct indices. Let us now show that this choice entails no loss of generality. Indeed, the repeated application of a CDT at the same eigenvalue only serves to modify the deformation parameter.

**Proposition 13.** *Let $\alpha \in \mathbb{N}_0 + \frac{1}{2}$, $\boldsymbol{m} \in \mathbb{N}_0^n$ and let $\tau^{(\alpha)}_{\boldsymbol{m}}, C^{(\alpha)}_{\boldsymbol{m};i}$ be as defined in (157) and (159). Then, for any $j \in \mathbb{N}_0$, we have*

$$\tau^{(\alpha)}_{(\boldsymbol{m},j,j)}(z; (\boldsymbol{t}_{\boldsymbol{m}}, t_j, t'_j)) = \tau^{(\alpha)}_{(\boldsymbol{m},j)}(z; (\boldsymbol{t}_{\boldsymbol{m}}, t_j + t'_j)), \tag{169}$$

$$C^{(\alpha)}_{(\boldsymbol{m},j,j);i}(z; (\boldsymbol{t}_{\boldsymbol{m}}, t_j, t'_j)) = C^{(\alpha)}_{(\boldsymbol{m},j);i}(z; ((\boldsymbol{t}_{\boldsymbol{m}}, t_j + t'_j)). \tag{170}$$

## 5.1 | Proof of the results

In this section we provide proofs for all of the theorems and propositions in this section. The general strategy is the following:

1. We define polynomials $\tilde{\tau}_j$, $\tilde{\pi}_{0;i}$ and rational functions $\tilde{\rho}_{j;i_1 i_2}$ recursively, starting the recursion at the objects corresponding to the classical Gegenbauer Sturm–Liouville problem.
2. We show that these recursion formulas describe a multistep CDT.
3. We show in Proposition 15 that the recursively defined objects coincide with those obtained via the matrix-based definitions (155)–(159).
4. Because the objects defined by the matrix formulas are polynomial by construction, we can dispense with the rationality and polynomiality assumptions made at the beginning of Section 4.3.
5. Propositions 6–11 then ensure that at each step of the recursion we have an exceptional Gegenbauer Sturm–Liouville problem, provided the parameters are chosen in the right range.

Fix $\boldsymbol{m} = \boldsymbol{m}_n = (m_1, \dots, m_n) \in \mathbb{N}_0^n$ and a positive half-integer $\alpha \in \mathbb{N}_0 + \frac{1}{2}$. For $j = 0, 1, \dots, n$ let $\boldsymbol{m}_j \in \mathbb{N}_0^n$ denote the initial segment of $\boldsymbol{m}$; that is, $\boldsymbol{m}_j = (m_1, \dots, m_j)$. Note that, throughout this section, we are going to omit the explicit dependence on $z$ and $\boldsymbol{t}_{\boldsymbol{m}}$ of the objects, which must be understood from the dependence on $\boldsymbol{m}$, that is, we will write $\mathcal{R}^{(\alpha)}_{\boldsymbol{m}}$ instead of $\mathcal{R}^{(\alpha)}_{\boldsymbol{m}}(z; \boldsymbol{t}_{\boldsymbol{m}})$. To simplify the notation, we will sometimes make the dependence of the various objects on $\alpha$ implicit rather than explicit.

We start the recursion at $j = 0$ by setting

$$\tilde{\rho}_{0;i_1i_2} := \rho^{(\alpha)}_{i_1i_2}, \quad \tilde{\tau}_0 := 1, \quad \tilde{\pi}_{0;i} := C^{(\alpha)}_i, \tag{171}$$

where $\rho^{(\alpha)}_{i_1i_2}(z)$ is given by (155) and $C^{(\alpha)}_i(z)$ are the classical Gegenbauer polynomials (97). For $j = 1, \dots, n$ we then define

$$\tilde{\tau}_j = \left(1 + t_{m_j}\tilde{\rho}_{j-1;m_jm_j}\right)\tilde{\tau}_{j-1}, \tag{172}$$

$$\tilde{\pi}_{j;i} = \left(1 + t_{m_j}\tilde{\rho}_{j-1;m_jm_j}\right)\tilde{\pi}_{j-1;i} - t_{m_j}\tilde{\rho}_{j-1;im_j}\tilde{\pi}_{j-1;m_j}, \quad i \in \mathbb{N}_0; \tag{173}$$

$$\tilde{\rho}_{j;i_1i_2} = \tilde{\rho}_{j-1;i_1i_2} - \frac{t_{m_j}\tilde{\rho}_{j-1;i_1m_j}\tilde{\rho}_{j-1;i_2m_j}}{1 + t_{m_j}\tilde{\rho}_{j-1;m_jm_j}}, \quad i_1, i_2 \in \mathbb{N}_0. \tag{174}$$

These recursive definitions match the formulas (101), (102), and (130). Thus, in effect we are defining the objects associated with an $n$-step CDT applied to classical Gegenbauer operators.

**Proposition 14.** *Let $i_1, i_2, j \in \mathbb{N}_0$ and $\tilde{\rho}_{j;i_1i_2}$, $\tilde{\pi}_{j,i}$ be as in (173), (174). We have that*

$$\tilde{\rho}_{j;i_1i_2}(z) := \int_{-1}^{z} \tilde{\pi}_{j,i_1}(u)\tilde{\pi}_{j,i_2}(u)\tilde{W}_j(u)du, \tag{175}$$

*where*

$$\tilde{W}_j(z) := (1 - z^2)^{\alpha - \frac{1}{2}}\tilde{\tau}_j(z)^{-2}, \tag{176}$$

*where the integral denotes an antiderivative that vanishes at $z = -1$.*

*Proof.* The proof follows directly from (129) and Proposition 8. ■

**Proposition 15.** *Let $\tau^{(\alpha)}_{\boldsymbol{m}}$ and $C^{(\alpha)}_{\boldsymbol{m};i}$ be as in (157), (159) and let $\tilde{\tau}_n$, $\tilde{\pi}_{n;i}$ be as in (172), (173) Then,*

$$\tau^{(\alpha)}_{\boldsymbol{m}} = \tilde{\tau}_n, \tag{177}$$

$$C^{(\alpha)}_{\boldsymbol{m};i} = \tilde{\pi}_{n;i}, \quad i \in \mathbb{N}_0. \tag{178}$$

*Proof.* The proof follows the same argument as the analogous result for exceptional Legendre polynomials, see Ref. [30, Proposition 5]. ■

As a direct consequence of (176) and (177) we see that the recursively defined $\tilde{\tau}_j, \tilde{\pi}_{j;i}$ are polynomials for each $j = 1, \dots, n$. We have also established that the antiderivative in the RHS of (175) describes a rational function. This allows us to dispense with the polynomiality and rationality assumptions used in Section 4.3.

*Proof of Theorem* 2. Starting form the classical Gegenbauer operator, the application of a rational CDT indexed by an integer $m_i$ introduces an extra real parameter $t_{m_i}$. Proposition 15 establishes the equivalence of the objects defined by the CDT recursion (172), (173) and the matrix-based definitions (157), (159). This allows us to apply Proposition 7. The eigenvalue relation (160) follows immediately. ∎

*Proof of Theorem* 3. First of all, it is clear by construction that the objects $C^{(\alpha)}_{\boldsymbol{m};i}$ are polynomials. In virtue of Proposition 15, the results of Section 4 can be exploited. We recursively define for $i \in \mathbb{N}_0$

$$\tilde{\nu}_{0;i} = \nu^{(\alpha)}_i, \quad \left(\tilde{\nu}_{j;i}\right)^{-1} = \left(\tilde{\nu}_{j-1;i}\right)^{-1} + \delta_{im_j} t_{m_j}, \quad j = 1, \dots, n. \tag{179}$$

By Propositions 9 and 10, condition (a) is satisfied, that is, $\tilde{\tau}_j(z) > 0$ for $z \in I$, if and only if $\tilde{\nu}_{j;m_j} > 0$. By the above definition,

$$\tilde{\nu}_{j;m_j} = \frac{\nu^{(\alpha)}_{m_j}}{1 + t_{m_j}\nu^{(\alpha)}_{m_j}}. \tag{180}$$

Therefore, $\tilde{\nu}_{j;m_j} > 0$ for $j = 1, \dots, n$ if and only if (161) holds. Relation (162) also follows by Proposition 10 and a similar induction argument. Finally, the completeness condition (c) follows by induction with Proposition 11 serving as the inductive step. ∎

We conclude this section by proving the remaining results on the degrees of the polynomials and the case of repeated indices.

*Proof of Proposition* 12. By (173), we have

$$\tilde{\pi}_{j;m_j} = \tilde{\pi}_{j-1;m_j}, \quad j = 1, \dots, n. \tag{181}$$

Identity (166) then follows by (177). Thanks to (166) no generality is lost by assuming that $i \notin \{m_1, \dots, m_n\}$. We use induction to show that

$$\deg \tilde{\tau}_j = 2(m_1 + \cdots + m_j + \alpha j), \quad j = 0, 1, \dots, n, \tag{182}$$

$$\deg \tilde{\pi}_{j;i} = \deg \tilde{\tau}_j + i. \tag{183}$$

The desired relations (164), (165) then follow by (176) and (177).

By inspection, (181) and (182) hold for $j = 0$. Assume that these relations hold for a given $j < n$. By (175),

$$\tilde{\rho}_{j,m_{j+1}m_{j+1}}(z) = \int_{-1}^{z} \left(\frac{\tilde{\pi}_{m_{j+1}}(u)}{\tilde{\tau}_j(u)}\right)^2 \tilde{W}_0(u)du. \tag{184}$$

Because $m_{j+1} \notin \{m_1, \dots, m_j\}$, by the inductive hypothesis,

$$\deg_z \tilde{\rho}_{j,m_{j+1}m_{j+1}} = 2m_{j+1} + 2\alpha, \tag{185}$$

where the degree of a rational function is understood as the difference between the degrees of the numerator and the denominator. Hence,

$$\deg_z \tilde{\tau}_{j+1} = 2m_{j+1} + 2\alpha + \deg_z \tilde{\tau}_j, \tag{186}$$

which agrees with (181) for the $j+1$ case.

By (173) and (175), we have

$$\tilde{\pi}_{j+1;i} = \tilde{\pi}_{j;i} + t_{m_{j+1}}\Pi, \tag{187}$$

where

$$\Pi(z) = \tilde{\pi}_{j;i}(z)\int_{-1}^{z}\frac{\tilde{\pi}_{j;m_{j+1}}(u)}{\tilde{\tau}_j(u)}\frac{\tilde{\pi}_{j;m_{j+1}}(u)}{\tilde{\tau}_j(u)}\tilde{W}_0(u)du - \tilde{\pi}_{j;m_{j+1}}(z)\int_{-1}^{z}\frac{\tilde{\pi}_{j;i}(u)}{\tilde{\tau}_j(u)}\frac{\tilde{\pi}_{j;m_{j+1}}(u)}{\tilde{\tau}_j(u)}\tilde{W}_0(u)du. \tag{188}$$

Because $i \neq m_{j+1}$, the leading degree terms in the above difference do not cancel and therefore

$$\deg_z \tilde{\pi}_{j+1;i} = \deg_z \Pi = 2m_{j+1} + 2\alpha + \deg \tilde{\pi}_{j;i}. \tag{189}$$

Hence (182) also holds for $j+1$. ■

*Proof of Proposition* 13. We apply Proposition 15 and the definitions (172), (173) twice to obtain

$$\begin{aligned}
\tau^{(\alpha)}_{(\boldsymbol{m},j,j)}(z;(\boldsymbol{t_m},t_j,t'_j)) &= \left(1+t'_j\rho^{(\alpha)}_{(\boldsymbol{m},j);jj}\right)\tau^{(\alpha)}_{(\boldsymbol{m},j)}\\
&= \left(1+t'_j\rho^{(\alpha)}_{\boldsymbol{m};jj} - \frac{(t'_j)^2\left(\rho^{(\alpha)}_{\boldsymbol{m};jj}\right)^2}{1+t'_j\rho^{(\alpha)}_{\boldsymbol{m};jj}}\right)\left(1+t_j\rho^{(\alpha)}_{\boldsymbol{m};jj}\right)\tau^{(\alpha)}_{\boldsymbol{m}}\\
&= \left(1+(t_j+t'_j)\rho^{(\alpha)}_{\boldsymbol{m};jj}\right)\tau^{(\alpha)}_{\boldsymbol{m}}\\
&= \tau^{(\alpha)}_{(\boldsymbol{m},j)}(z;(\boldsymbol{t_m},t_j+t'_j)),
\end{aligned} \tag{190}$$

$$\begin{aligned}
C^{(\alpha)}_{(\boldsymbol{m},j,j);i}(z;(\boldsymbol{t_m},t_j,t'_j)) &= \left(1+t'_j\rho^{(\alpha)}_{(\boldsymbol{m},j);jj}\right)C^{(\alpha)}_{(\boldsymbol{m},j);i} - t'_j\rho^{(\alpha)}_{(\boldsymbol{m},j);ij}C^{(\alpha)}_{(\boldsymbol{m},j);j}\\
&= \left(1+t'_j\rho^{(\alpha)}_{(\boldsymbol{m},j);jj}\right)\left(\left(1+t_j\rho^{(\alpha)}_{\boldsymbol{m};jj}\right)C^{(\alpha)}_{\boldsymbol{m};i} - t_j\rho^{(\alpha)}_{\boldsymbol{m};ij}C^{(\alpha)}_{\boldsymbol{m};i}\right)\\
&\quad - t'_j\left(\rho^{(\alpha)}_{\boldsymbol{m};ij} - \frac{t_j\rho^{(\alpha)}_{\boldsymbol{m};jj}\rho^{(\alpha)}_{\boldsymbol{m};ij}}{1+t_j\rho^{(\alpha)}_{\boldsymbol{m};jj}}\right)C^{(\alpha)}_{\boldsymbol{m};j}\\
&= \left(1+(t_j+t'_j)\rho^{(\alpha)}_{\boldsymbol{m};jj}\right)C^{(\alpha)}_{\boldsymbol{m};i} - (t_j+t'_j)\rho^{(\alpha)}_{\boldsymbol{m};ij}C^{(\alpha)}_{\boldsymbol{m};j}\\
&= C^{(\alpha)}_{(\boldsymbol{m},j);i}(z;(\boldsymbol{t_m},t_j+t'_j)).
\end{aligned} \tag{191}$$

■

# 6 | EXAMPLES

We conclude by showing some explicit examples of deformed Gegenbauer polynomials, together with their properties. It can be readily checked that these families are an isospectral deformation of the classical Gegenbauer polynomials.

## 6.1 | One-parameter deformed Gegenbauer polynomials

The one-parameter exceptional Gegenbauer polynomials arise after a single CDT on the classical operator. For $m \in \mathbb{N}_0$, we follow definitions (101)–(102) to write

$$\tau_m^{(\alpha)}(z, t_m) = 1 + t_m \rho_{mm}^{(\alpha)}(z), \tag{192}$$

$$C_{m;i}^{(\alpha)}(z, t_m) = \Big(1 + t_m \rho_{mm}^{(\alpha)}(z)\Big) C_i^{(\alpha)}(z) - t_m \rho_{im}^{(\alpha)}(z) C_m^{(\alpha)}(z), \tag{193}$$

with $\rho_{ij}^{(\alpha)}$ as per (55) and $C_i^{(\alpha)}$ the classical Gegenbauer polynomials. Notice that by construction, we have $C_{m;m}^{(\alpha)} = C_m^{(\alpha)}$. The $\{C_{m;i}^{(\alpha)}\}_{i \in \mathbb{N}_0}$ is a family of exceptional Gegenbauer polynomials with weight

$$W_m^{(\alpha)}(z) = \frac{(1 - z^2)^{\alpha - \frac{1}{2}}}{\left(\tau_m^{(\alpha)}(z)\right)^2}, \tag{194}$$

as long as $t_m$ satisfies the inequality

$$t_m > -\left(\nu_m^{(\alpha)}\right)^{-1} = -\frac{2^{2\alpha-1} m!(m+\alpha)\Gamma(\alpha)^2}{\pi \Gamma(m + 2\alpha)}. \tag{195}$$

The orthogonality relations are

$$\int_{-1}^{1} C_{m;i}^{(\alpha)}(z, t_m) C_{m;j}^{(\alpha)}(z, t_m) W_m^{(\alpha)}(z) dz = \delta_{ij} \nu_i^{(\alpha)} = \delta_{ij} \frac{\pi 2^{1-2\alpha} \Gamma(i + 2\alpha)}{i!(i+\alpha)\Gamma(\alpha)^2}, \quad i, j \in \mathbb{N}_0 \backslash \{m\}, \tag{196}$$

$$\int_{-1}^{1} C_{m;m}^{(\alpha)}(z, t_m)^2 W_m^{(\alpha)}(z, t_m) dz = \frac{\nu_m^{(\alpha)}}{1 + t\nu_m^{(\alpha)}} = \frac{\pi 2^{1-2\alpha} \Gamma(m + 2\alpha)}{m!(m+\alpha)\Gamma(\alpha)^2 + t_m \pi 2^{1-2\alpha} \Gamma(m + 2\alpha)}. \tag{197}$$

The function $\tau_m^{(\alpha)}$ is shown below in Figure 1, for a particular choice of the parameters $\alpha, m, t_m$. Clearly, there are no zeroes. Figure 2 shows the polynomial families for $\alpha = 3/2$, $m = 2$, and different values of $t_m$.

The first few polynomials for $m = 4$ and $\alpha = 3/2$ are explicitly given by

$$C_{4;0}^{(3/2)}(z, t_4) = C_0^{(3/2)}(z) + \frac{15}{176} t_4 \big(945 z^{11} - 3080 z^9 + 3630 z^7 - 1848 z^5 + 385 z^3 + 32\big),$$

$$\begin{aligned} C_{4;1}^{(3/2)}(z, t_4) = C_1^{(3/2)}(z) + \frac{45}{5632} t_4 \big(& 19845 z^{12} - 59290 z^{10} + 59455 z^8 \\ & - 20636 z^6 + 275 z^4 + 1430 z^2 + 1024 z - 55\big), \end{aligned} \tag{198}$$

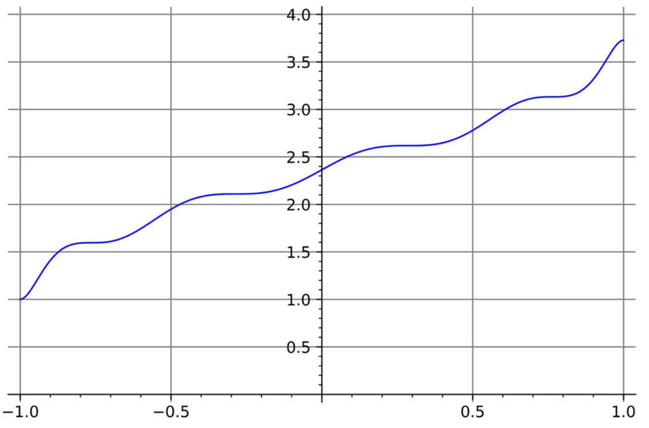


**FIGURE 1** The function $\tau_m^{(\alpha)}(z, t_m)$ for $m = 4$ and $t_m = 0.5$

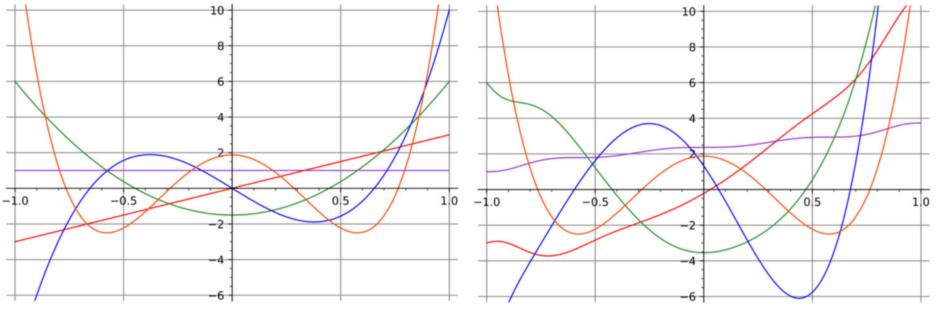


**FIGURE 2** First few deformed Gegenbauer polynomials $C_{m;i}^{(\alpha)}(z; t_m)$ for $m = 4$, with $t_m = 0$ (left) and $t_m = 0.5$ (right)

$$\begin{aligned}C_{4;2}^{(3/2)}(z, t_4) = C_2^{(3/2)}(z) + \frac{45}{704} t_4 \big(&3675z^{13} - 11515z^{11} + 13310z^9 - 7590z^7 \\ &+ 2871z^5 - 495z^3 + 320z^2 - 64\big),\end{aligned} \tag{199}$$

$$\begin{aligned}C_{4;3}^{(3/2)}(z, t_4) = C_3^{(3/2)}(z) + \frac{75}{2816} t_4 \big(&9261z^{14} - 30919z^{12} + 39501z^{10} - 24783z^8 \\ &+ 7007z^6 + 2739z^4 + 1792z^3 - 1881z^2 - 768z + 99\big),\end{aligned} \tag{200}$$

$$C_{4;4}^{(3/2)}(z, t_4) = C_4^{(3/2)}(z). \tag{201}$$

For larger values of the index, we have, For larger values of the index, we have,

$$C_{4;i}^{(3/2)}(z, t_4) = C_4^{(3/2)}(z) + t_4\left(\rho_{44}^{(3/2)}(z) C_i^{(3/2)}(z) - \rho_{4i}^{(3/2)}(z) C_4^{(3/2)}(z)\right), \quad i \geq 5, \tag{202}$$

where, by (63),

$$\deg \rho_{4i}^{(3/2)} = 4 + i + 2 + 1 = 7 + i,$$

and consequently,

$$\deg C^{(3/2)}_{4;i} = i + 11, \qquad i \geq 5.$$

Notice that, as illustrated by Figure 2, the exceptional polynomials are continuous deformations of the corresponding classical polynomials, aside from $C^{(3/2)}_{4;4}$. It is worth observing also that each polynomial $C^{(3/2)}_{4,i}(z)$ has precisely $i$ zeros in the domain of orthogonality $(-1, 1)$ (which evidently follows from the Sturm–Liouville character of the family). Thus, for all polynomials except for $i = 4$, we see that polynomial $C^{(3/2)}_{4,i}(z)$ has 11 *exceptional* zeros that lie outside the support of the measure. Finally, notice one more difference with respect to generic XOPs: When ordered by eigenvalue (or equivalently, when ordered by their number of zeros in the interval of orthogonality), the sequence of degrees is not an increasing sequence. This is a consequence of the fact that not all polynomials in the family have the same number of exceptional zeros, in this example $C^{(3/2)}_{4;4}$ has no exceptional zeros.

## 6.2 | Two-parameter deformed Gegenbauer polynomials

We construct the two-parameter exceptional polynomials by applying the recursive construction to the one-parameter formulas. We start with $\boldsymbol{m} = (m_1, m_2) \in \mathbb{N}_0^2$ and the associated tuple $\boldsymbol{t_m} = (t_{m_1}, t_{m_2})$. Following Equations (100)–(102), and using (130), we find that

$$\tau^{(\alpha)}_{\boldsymbol{m}}(z; \boldsymbol{t_m}) = \tau^{(\alpha)}_{m_1}(z, t_{m_1})\tau^{(\alpha)}_{m_2}(z, t_{m_2}) - t_{m_1}t_{m_2}\rho^{(\alpha)}_{m_1m_2}(z)^2, \tag{203}$$

$$\begin{aligned} C^{(\alpha)}_{\boldsymbol{m};i}(z; \boldsymbol{t_m}) = {} & C^{(\alpha)}_{i}(z)\tau^{(\alpha)}_{\boldsymbol{m}}(z; \boldsymbol{t_m}) - t_{m_1}C^{(\alpha)}_{m_1}(z)\tau^{(\alpha)}_{m_2}(z; t_{m_2})\rho^{(\alpha)}_{m_2;m_1i}(z, t_{m_2}) \\ & - t_{m_2}C^{(\alpha)}_{m_2}(z)\tau^{(\alpha)}_{m_1}(z; t_{m_2})\rho^{(\alpha)}_{m_1;m_2i}(z, t_{m_1}), \end{aligned} \tag{204}$$

where

$$\begin{aligned} \rho^{(\alpha)}_{m;ij}(z, t_m) &= \int_{-1}^{z} C^{(\alpha)}_{m;i}(u, t_m)C^{(\alpha)}_{m;j}(u, t_m)W^{(\alpha)}_{m}(u)du \\ &= \rho^{(\alpha)}_{ij}(z) - \frac{t_m\rho^{(\alpha)}_{mi}(z)\rho^{(\alpha)}_{mj}(z)}{1 + t_m\rho^{(\alpha)}_{mm}(z)}, \qquad m, i, j \in \mathbb{N}_0. \end{aligned} \tag{205}$$

Once again, the polynomials form a complete orthogonal basis relative to the weight

$$W^{(\alpha)}_{\boldsymbol{m}}(z) = (1 - z^2)^{\alpha - \frac{1}{2}}\tau^{(\alpha)}_{m_1m_2}(z; t_{m_1}, t_{m_2})^{-2}, \tag{206}$$

provided

$$t_{m_i} > -\frac{2^{2\alpha-1}m_i!(m_i + \alpha)\Gamma(\alpha)^2}{\pi\Gamma(m_i + 2\alpha)}, \quad i = 1, 2. \tag{207}$$

The orthogonality relations are

$$\int_{-1}^{1} C^{(\alpha)}_{\boldsymbol{m};i}(z,\boldsymbol{t_m})C^{(\alpha)}_{\boldsymbol{m};j}(z,\boldsymbol{t_m})W^{(\alpha)}_{\boldsymbol{m}}(z,\boldsymbol{t_m})dz = \delta_{ij}\frac{\pi 2^{1-2\alpha}\Gamma(i+2\alpha)}{i!(i+\alpha)\Gamma(\alpha)^2}, \quad i,j \in \mathbb{N}_0\backslash\{m_1,m_2\}, \tag{208}$$

$$\int_{-1}^{1} C^{(\alpha)}_{\boldsymbol{m};m_i}(z,\boldsymbol{t_m})^2 W^{(\alpha)}_{\boldsymbol{m}}(z,\boldsymbol{t_m})dz = \frac{\pi 2^{1-2\alpha}\Gamma(m_i+2\alpha)}{m_i!(m_i+\alpha)\Gamma(\alpha)^2 + t_{m_i}\pi 2^{1-2\alpha}\Gamma(m_i+2\alpha)}, \quad i=1,2. \tag{209}$$

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