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Optimal Control Strategies for the Premium Policy of an Insurance Firm with Jump Diffusion Assets and Stochastic Interest Rate

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Abstract: In this paper, we present a stochastic optimal control model to optimize an insurance firm problem in the case where its cash-balance process is assumed to be described by a stochastic differential equation driven by Teugels martingales. Noticing that the insurance firm is able to control its cash-balance dynamics by regulating the underlying premium rate, the aim of the policy maker is to select an appropriate premium in order to minimize the total deviation of the state process to some pre-set target level. As a part of stochastic maximum principle approach, a verification theorem is used to fulfill this achievement.

Keywords: forward-backward stochastic differential equations; teugels martingales; lévy processes; optimal premium policies

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1. Introduction

An insurance is a contract, represented by a policy, used as a method of protection against losses, whether big or small. This means that the insured can receive or reimburse some financial amount to offset his or her losses from an insurance company. Due the fact that an insurance premium is the amount of money that a person or a company ought to pay for an insurance policy, one can perceive it in two different ways. In one side, it can be considered as an income by the insurance company. On the other side, it can also be considered as a liability in that the insurer must provide coverage for claims being made against the policy. In the present paper, we focus on the first case, where the policy maker looks forward to maximize the terminal wealth of its firm's cash-balance under a demand law.

The main problem in optimal control theory is to characterize an optimal control process. There are two main approaches, the Pontryagin's maximum principle and the Bellman's dynamic programming. We note that optimal control theory has been used for example in Hipp and Vogt (2003) in order to determine an optimal dynamic unlimited excess of loss reinsurance strategy to minimize the infinite time ruin probability, see also Højgaard and Taksar (1998), where the authors applied a proportional reinsurance policy for diffusion models in order to find a policy that maximizes a given return function before the time of ruin. In Cairns (2000), Cairns studied the optimization problem of stochastic pension fund models in continuous time. The mean-variance portfolio management for an insurance company was studied by Josa-Fombellida and Rincón-Zapatero (2008) by using dynamic programming techniques and also by Xie et al. (2008) using the general stochastic linear quadratic control technique. In Moore and Young (2006), Moore and Young obtained optimal dynamic consumption, investment, and insurance strategies, using the dynamic programming principle and a Markov chain approximation method. Ngwira and Gerrad in Ngwira and Gerrard (2007) showed that the optimal contribution and asset allocation policies have similar forms as in the pure diffusion case, but with a modification due to the effect of jumps. In Huang et al. (2010), Huang et al. explicitly derived the insurance company's optimal premium strategy and the associated optimal cost function. We refer the reader to a list of recent papers and the references therein for more details in this subject, Asmussen et al. (2019); Li and Young (2021); Lin et al. (2020); Supian and Mamat (2021).

Motivated by the above results, in this paper, we solve an optimal premium policy problem of an insurance firm. The main tool used in proving our main results is the Pontryagin maximum principle. More precisely, the sufficient condition of optimality. Noting that, in pretty much all of the previous papers, the authors dealt with the problem of optimal insurance in continuous-time models, we impose here to work with a quite general semi-martingale framework assuming that the liability process is driven by both a Brownian motion and a family of pairwise orthogonal martingales associated with a Lévy process. This kind of models comes naturally from the fact that in many real cases, the continuity of the trajectories condition cannot be satisfied. Indeed, the empirical distribution of the cash balance process tends to deviate from normal distributions, either due to inspected dusters or huge profits, many successive incidents or even because of the lack of continuity in the real world of applications.

The paper is organized as follows. In Section 2, we introduce the preliminaries and formulate our control problem. In Section 3, we derive a sufficient condition of optimality for stochastic differential equations driven by a Brownian motion and a family of Teugels martingales. In Section 4, we apply our result to study problems of optimal premium rate of an insurance firm with jump diffusion assets with stochastic interest rate.

2. Preliminaries and Problem Statement

2.1. Preliminaries

Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space supporting a one-dimensional standard Brownian motion W and a one-dimensional Lévy process L with triplet (γ, σ^2, ν), defined on $[0, T]$, independent each other. Recall that $\gamma \in \mathbb{R}$, $\sigma^2 > 0$ and ν are Lévy measures. We denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$. Moreover, $\mathbb{F} := \{\mathcal{F}_t, t \in [0, T]\}$ denotes the complete right-continuous natural filtration generated by processes W and L.

Recall briefly the L^2 theory of Lévy processes as it is presented in Nualart and Schoutens (2000). Assume there exists $\alpha > 0$ such that for every $\varepsilon > 0$,

$$\int_{(-\varepsilon,\varepsilon)^c} e^{\alpha|z|}\nu(dz) < \infty.$$

Note that this guarantee that ν has moments of all orders.

Denote by $\Delta L(t) := L(t) - L(t-)$ the jumps of the Lévy process and define the power jump processes $L^{(1)}(t) := L(t)$ and

$$L^{(i)}(t) := \sum_{0<s\leq t} (\Delta L(s))^i,\ i \geq 2.$$

If we define

$$Y^{(i)}(t) := L^{(i)}(t) - \mathbb{E}\Big[L^{(i)}(t)\Big], i \geq 1,$$

then the family of Teugels martingales $\{H^{(i)}(t)\}_{i=1}^{\infty}$ is defined by

$$H^{(i)}(t) := \sum_{j=1}^{i} a_{ij} Y^{(j)}(t)$$

where coefficients a_{ij} correspond to the orthonormalization of polynomials 1, x, x^2, ... with respect to measure $\mu(dx) = x^2\nu(dx) + \sigma^2\delta_0(dx)$. Recall that $\{H^{(i)}(t)\}_{i=1}^{\infty}$ is a family of strongly orthogonal martingales such that for any i and j, $\left\langle H^{(i)}, H^{(j)} \right\rangle_t = \delta_{ij} t$ and $\left[H^{(i)}, H^{(j)}\right] - \left\langle H^{(i)}, H^{(j)} \right\rangle$ is a martingale.

Throughout this article, we will use the following spaces:

- $l^2(\mathbb{R})$: the Hilbert space of real-valued sequences $x = (x_n)_{n \geq 0}$ with norm

$$\left(\sum_{i=1}^{\infty} x_i^2 \right)^{\frac{1}{2}} < \infty.$$

- $\mathcal{P}^2(\mathbb{R})$: the Hilbert space of $l^2(\mathbb{R})$-valued functions $\{f^i\}_{i \geq 0}$ such that

$$\left(\sum_{i=1}^{\infty} \int_0^T |f^i(t)|^2 dt \right)^{\frac{1}{2}} < \infty.$$

- $l^2_{\mathcal{F}}(0, T; \mathbb{R})$: the Hilbert space of $\mathcal{P}^2(\mathbb{R})$-valued $\mathbb{F}$-predictable processes $\{f^i(t)\}_{i \geq 0}$ such that

$$\left(\mathbb{E} \int_0^T \sum_{i=1}^{\infty} |f^i(t)|^2 dt \right)^{\frac{1}{2}} < \infty.$$

- $\mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R})$: the Hilbert space of real-valued $\mathbb{F}$-adapted processes $f(t)$ such that

$$\left(\mathbb{E} \int_0^T |f(t)|^2 dt \right)^{\frac{1}{2}} < \infty.$$

- $\mathcal{S}^2_{\mathcal{F}}(0, T; \mathbb{R})$: the Banach space of real-valued, $\mathbb{F}$-adapted and càdlàg processes $f(t)$ such that

$$\left(\mathbb{E} \sup_{0 \leq t \leq T} |f(t)|^2 \right)^{\frac{1}{2}} < \infty.$$

- $L^2(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R})$: the Hilbert space of real-valued square integrable random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

2.2. Description of the Model

In this paper, we handle a cash management problem of an insurance firm (also called cash-balance problem). The insurer aims to manage the firm's operating cash in order to meet demand. We consider that the insurance firm can constantly modify its premium policy while investing in certain financial accounts. Its concern is to pay due benefits, but at the same time to stabilize the insurance schemes by avoiding a large deviation on the cash balance. We address an optimal premium policy problem of an insurance firm under stochastic interest rate when the liability process, also called the payment function, $B(t)$, is modeled by the stochastic differential equation:

$$-dB(t) = (b(t) + v(t))dt + a(t)dW(t) + \sum_{i=1}^{\infty} \pi^i(t) dH^{(i)}(t), \; B(0) = 0,$$

where b is a locally bounded function that denotes the liability rate, that is, the expected liability per unit time due to premium loading, v is the premium rate (premium policy) and a and $\{\pi^i\}_{i=1}^{\infty}$ are locally bounded functions that denote the volatility rates measuring the liability risks belonging respectively to the Brownian and Teugels martingale components.

For simplicity, from now on, we will write $\pi(t) := \{\pi^i(t)\}_{i=1}^{\infty}$, $H(t) := \left\{H^{(i)}(t)\right\}_{i=1}^{\infty}$ and

$$\pi(t)dH(t) := \sum_{i=1}^{\infty} \pi^i(t)dH^{(i)}(t).$$

Assume moreover that the cash balance process of the insurer $X(\cdot)$ is described by the formula

$$X(t) = e^{\Delta(t)}\left(X(0) - \int_0^t e^{-\Delta(s)}dB(s)\right), \tag{1}$$

where $X(0) = x \geq 0$ represents the initial reserve and $\Delta(\cdot)$ represents the stochastic interest rate, that follows the dynamics

$$d\Delta(t) = \delta(t)dt + \alpha(t)dW(t),\ t \in [0,T],\ \Delta(0) = 0, \tag{2}$$

with δ and α locally bounded functions.

Note that $X(t)$ is the difference between the initial capital and the net expenses up to time t.

Now, the Itô formula applied to the process $X(\cdot)$ leads to the following controlled Stochastic Differential Equation (SDE):

$$\begin{cases} dX(t) = f_1(t, X(t), v(t))dt + \sigma_1(t, X(t), v(t))dW(t) + \pi(t)dH(t), \\ X(0) = x, \end{cases} \tag{3}$$

where

$$f_1(t, X(t), v(t)) = \left(\delta(t) + \frac{1}{2}\alpha^2(t)\right)X(t) + \alpha(t)a(t) + b(t) + v(t),$$

and

$$\sigma_1(t, X(t), v(t)) = X(t)\alpha(t) + a(t).$$

Herein, the process $v(\cdot)$ stands for the control variable. We require that process $v(\cdot)$ is adapted, with càdlàg trajectories and taking values in U, a non-empty convex subset of $\mathbb{R}$, such that the fourth-power condition

$$\mathbb{E}\int_0^T |v(t)|^4 dt < \infty$$

holds. Furthermore, we assume there exists a positive constant c_0 such that the SDE (3) has a unique solution satisfying the terminal constraint

$$\mathbb{E}[X(T)] = c_0. \tag{4}$$

This last equality means that the insurance firm is looking for some regulatory requirement c_0 described by the average value of its cash balance process at the terminal time T.

Herein, a control variable is said to be an admissible control if and only if it satisfies all the above conditions.

Let us point out that the aim of the policy maker is to simultaneously minimize the deviation between the firm's cash-balance process and its dynamic benchmark over the set of all admissible controls, which will be denoted by $\mathcal{U}$, the cost of the premium policy over the whole time interval $[0,T]$, and the terminal variance of the cash-balance process under some given constraint. Therefore, it is quite natural that the cost functional takes the following form:

$$J(v(\cdot)) = \mathbb{E}\left[\int_0^T e^{-\beta t} g_1(t, X(t), v(t))dt + e^{-\beta T}\varphi_1(X(T))\right], \tag{5}$$

with

$$g_1(t,x,v) = \frac{1}{2}\Big(R(t)(x - A(t))^2 + N(t)v^2(t)\Big), \tag{6}$$

and

$$\varphi_1(x) = \frac{1}{2}M(x - c_0)^2. \tag{7}$$

Here, $\beta \geq 0$ is a discounting factor, $A(t)$ is some dynamic pre-set target, representing the dynamic benchmark of X, processes $R(\cdot)$, $N(\cdot)$ and the positive constant M are the weighting factors which make the cost functional (5) more general and flexible to control the preference of the policy-maker. Furthermore, we suppose that process A converges to c_0 as t goes to T.

Now, we can formulate the firm's optimal premium problem as

Problem A: *To find $\hat{v} \in \mathcal{U}$ such that $\hat{v}$ minimizes the cost function (5) subject to (3) and the state constraint (4).*

To deal with this problem, we need to impose the following assumption on the previous coefficients:

Assumption 1. *Functions $R(\cdot) \geq 0$, $N(\cdot) > 0$, $N^{-1}(\cdot)$ and $A(\cdot)$ are all deterministic and bounded on the time interval $[0,T]$.*

Let us now reformulate the above control problem (3)–(5) as a generalized stochastic recursive optimal control problem with state constraint by introducing the following backward stochastic differential equation (BSDE), in term of processes $Y(\cdot)$, $Z(\cdot)$ and $K(\cdot) = \{K^i(\cdot)\}_{i=1}^{\infty}$:

$$\begin{cases} -dY(t) = [g_1(t,X(t),v(t)) - \beta Y(t)]dt - Z(t)dW(t) - K(t)dH(t), \\ Y(T) = \varphi_1(X(T)). \end{cases} \tag{8}$$

Notice that (3) together with (8) form a semi-coupled Forward-Backward Stochastic Differential Equation (FBSDE) driven by both the Teugels martingales and an independent Brownian motion. Lemma 2.1 in Meng and Tang (2009) shows that under (H1), (3) admits a unique solution $X(t)$ for each $v(\cdot) \in \mathcal{U}$. Moreover, we have

$$\sup_{0 \leq t \leq T} \mathbb{E}\Big(X^4(t)\Big) < +\infty.$$

The proof of this estimate is placed in Lemma 1 Section 3.

As a consequence, the terminal condition of (8) is square-integrable. Then, for the foregoing $v(t)$ and $X(t)$, thanks to Theorem 3.1 in Bahlali et al. (2003), the Backward Stochastic Differential Equation (BSDE) (8) admits a unique solution $(Y(\cdot), Z(\cdot), K(\cdot))$ under (H1). That is, for any $v(\cdot) \in \mathcal{U}$, the semi-coupled FBSDE consisting of (3) and (8) admits a unique solution:

$$(X(\cdot), Y(\cdot), Z(\cdot), K(\cdot)).$$

Obviously, by using the dual technique to the BSDE (8) one can get $Y(0) = J(v(\cdot))$. Then, we can reformulate Problem A in the following way:

Problem B: *To find $\hat{v} \in \mathcal{U}$ such that*

$$J(\hat{v}(\cdot)) = \mathbb{E}(Y(0)) \tag{9}$$

subject to (3), (4) and (8).

In the next section, we are going to prove a sufficient condition of optimality for the above problem, but in a more general form by assuming that coefficients are not necessarily linear with respect to the state variables.

3. Sufficient Condition of Optimality

3.1. Problem Formulation

Motivated by the above optimal premium policy of an insurance firm, we are now going to focus on the following control problem where the state process is described by the following controlled stochastic differential equation driven by both a Brownian motion and a family of Teugels martingales,

$$\begin{cases} dX(t) = f(t, X(t), v(t))dt + \sigma(t, X(t), v(t))dW(t) \\ + \sum_{i=1}^{\infty} \pi^i(t, X(t-), v(t))dH^{(i)}(t), \\ X(0) = x, \end{cases} \tag{10}$$

and the general stochastic differential utility is given by the BSDE

$$\begin{cases} -dY(t) = g(t, \mathcal{X}(t), v(t))dt - Z(t)dW(t) - \sum_{i=1}^{\infty} K^i(t)dH^{(i)}(t), \\ Y(T) = \varphi(X(T)). \end{cases} \tag{11}$$

where $\mathcal{X}(t) := (X(t), Y(t), Z(t), K(t))$, and

$$\begin{aligned} &f : [0, T] \times \Omega \times \mathbb{R} \times U \to \mathbb{R}, \\ &\sigma : [0, T] \times \Omega \times \mathbb{R} \times U \to \mathbb{R}, \\ &\pi : [0, T] \times \Omega \times \mathbb{R} \times U \to \mathcal{P}^2(\mathbb{R}), \\ &g : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}^2(\mathbb{R}) \times U \to \mathbb{R}, \\ &\varphi : [0, T] \times \Omega \times \mathbb{R} \to \mathbb{R}, \end{aligned}$$

are progressively measurable functions.

In the sequel, for notational simplicity, we shall use the shorthand notation

$$\pi(t, X(t-), v(t))dH(t) \quad \text{and} \quad K(t)dH(t)$$

instead of

$$\sum_{i=1}^{\infty} \pi^i(t, X(t-), v(t))dH^{(i)}(t) \quad \text{and} \quad \sum_{i=1}^{\infty} K^i(t)dH^{(i)}(t)$$

respectively, where $K(t) = \{K^i(t)\}_{i=1}^{\infty}$ and $\pi(t) = \{\pi^i(t)\}_{i=1}^{\infty}$.

Let us now introduce the following basic assumption on coefficients that will be needed in the sequel:

Assumption 2. *(i)* *The function g is $\mathbb{F}-$progressively measurable for all $(y, z, k) \in \mathbb{R} \times \mathbb{R} \times \mathcal{P}^2(\mathbb{R})$ and for any $v(\cdot) \in \mathcal{U}$,*

$$g(t, \mathcal{X}(t), v(t)) = G(t, Y(t), Z(t), K(t)) + R(t)(X(t) - A(t))^2 + N(t)v^2(t),$$

with

$$\mathbb{E}\left(\int_0^T |g(s, 0, 0, 0, 0, v(t))|^2 ds \right) < \infty.$$

(ii) *For every $x \in \mathbb{R}$, $\varphi \in L^2$, f, σ and π are $\mathbb{F}-$progressively measurable and for any $v(\cdot) \in \mathcal{U}$,*

$$\mathbb{E}\left(\int_0^T \Big(|f(s,0,v(t))|^4 + |\sigma(s,0,v(t))|^4 + \|\pi(s,0,v(t))\|^4_{\mathcal{P}^2(\mathbb{R})}\Big)ds\right) < \infty.$$

(iii) *Functions G, f, σ and π are continuously differentiable with bounded derivatives with respect to x, y, z, k and v.*

Note that Lemma 2.1 in Meng and Tang (2009) shows that under the Assumption (H2) the SDE (10) admits a unique solution that belongs to $\mathcal{S}^2_{\mathcal{F}}(0,T,\mathbb{R})$. On the other hand, since the function g is uniformly Lipschitz with respect to y, z and k, by using the assumptions (H2), one can easily check that the BSDE (11) satisfies all the conditions in Theorem 3.1 in Bahlali et al. (2003), and hence, it has a unique solution that belongs to $\mathcal{S}^2_{\mathcal{F}}(0,T,\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}(0,T,\mathbb{R}) \times l^2_{\mathcal{F}}(0,T,\mathbb{R})$.

The following key lemma gives an L^4 estimate of the solution of the state equation under Assumption (H2). Since Assumption (H1) implies Assumption (H2), the result is still valid under Assumption (H1).

Lemma 1. *Assume that $X(\cdot)$ is a solution of the state Equation (10). Then, under Assumption (H2), the following estimate holds true:*

$$\sup_{0\le t\le T} \mathbb{E}\Big(X^4(t)\Big) < +\infty. \tag{12}$$

Proof. First, we introduce the stopping times

$$R_k = \inf\{t : |X(t)| \ge k\},$$

assuming k great enough to include $X(0) = x$ in $[-k,k]$. Then, applying Itô's formula to $|X(t \wedge R_k)|^4$ we obtain

$$\begin{aligned} X^4(t\wedge R_k) = X^4(0) & \\ &+ 4\int_0^{t\wedge R_k} X^3(s) f(s,X(s),v(s))ds \\ &+ 4\int_0^{t\wedge R_k} X^3(s)\sigma(s,X(s),v(s))dW(s) \\ &+ 4\int_0^{t\wedge R_k} X^3(s-)\pi(s,X(s-),v(s-))dH(s) \\ &+ 6\int_0^{t\wedge R_k} X^2(s)\sigma^2(s,X(s),v(s))ds \\ &+ 6\int_0^{t\wedge R_k} X(s)^2||\pi(s,X(s),v(s))||^2 ds \\ &+ 6\sum_{i,j=1}^{\infty}\int_0^{t\wedge R_k} X^2(s-)(\pi^i\pi^j)(s,X(s-),v(s-))dM_{i,j}(t) \\ &+ I(t\wedge R_k) \end{aligned}$$

where

$$I(t) := \sum_{0<s\le t}\Big\{X^4(s) - X^4(s-) - 4X^3(s-)\Delta X(s) - 12X^2(s-)(\Delta X(s))^2\Big\}.$$

The stochastic integrals with respect to W, H and $M_{i,j}$ in the right hand side of the above equality are in fact martingales with zero expectation. Then, taking expectations and later absolute values, one finds

$$\mathbb{E}|X(t\wedge R_k)|^4 \le \mathbb{E}|X(0)|^4 + 4\mathbb{E}\Big(\int_0^{t\wedge R_k}|X(s)|^3|f(s,X(s),v(s))|ds\Big)$$
$$+6\mathbb{E}\Big(\int_0^{t\wedge R_k}|X(s)|^2|\sigma(s,X(s),v(s))|^2ds\Big)$$
$$+6\mathbb{E}\Big(\int_0^{t\wedge R_k}|X(s)|^2||\pi(s,X(s),v(s))||^2ds\Big) + \mathbb{E}(|I(t\wedge R_k)|).$$

In view of Young's inequality, one can easily check that

$$\mathbb{E}|X(t\wedge R_k)|^4 \le \mathbb{E}|X(0)|^4 + 9\mathbb{E}\int_0^{t\wedge R_k}|X(s)|^4ds + \mathbb{E}(|I(t\wedge R_k)|)$$
$$+\mathbb{E}\int_0^{t\wedge R_k}|f(s,X(s),v(s))|^4ds + 3\mathbb{E}\Big(\int_0^{t\wedge R_k}|\sigma(s,X(s),v(s))|^4ds\Big)$$
$$+3\mathbb{E}\Big(\int_0^{t\wedge R_k}||\pi(s,X(s),v(s))||^4ds\Big).$$

From now on, C will denote a constant that changes from line to line.
Using Lipschitz condition in (H2), we have

$$\mathbb{E}|X(t\wedge R_k)|^4 \le \mathbb{E}|X(0)|^4 + \mathbb{E}|I(t\wedge R_k)| + C\mathbb{E}\int_0^{t\wedge R_k}|X(s)|^4ds$$
$$+C\mathbb{E}\int_0^{t\wedge R_k}\Big(|f(s,0,v(s))|^4 + |\sigma(s,0,v(s))|^4 + ||\pi(s,0,v(s))||^4\Big)ds.$$

We infer, using Gronwall's inequality, that

$$\mathbb{E}|X(t\wedge R_k)|^4 \le e^{C(t\wedge R_k)}\Big(\mathbb{E}|X(0)|^4 + \mathbb{E}|I(t\wedge R_k)| + C\mathbb{E}\int_0^{t\wedge R_k}\Big(|f(s,0,v(s))|^4 +$$
$$|\sigma(s,0,v(s))|^4 + ||\pi(s,0,v(s))||^4\Big)ds\Big).$$

This implies, using Fatou's lemma and taking the supremum over the interval $[0,T]$, that

$$\sup_{0\le t\le T}\mathbb{E}|X(t)|^4 \le C\Big(\mathbb{E}|X(0)|^4 + \mathbb{E}|I(T)| + \mathbb{E}\int_0^T\Big(|f(t,0,v(t))|^4 + |\sigma(t,0,v(t))|^4 + ||\pi(t,0,v(t))||^4\Big)ds\Big), \tag{13}$$

Now, we turn out to estimate $\mathbb{E}(|I(T)|)$. For each k, it is clear that the stopped process $\tilde{X} = X1_{[0,R_k)}$ is bounded by k, and it is a semi-martingale as a product of two semi-martingales.

Note that the estimate

$$\sum_{0<s\le t}\Big\{\tilde{X}^4(s) - \tilde{X}^4(s-) - 4\tilde{X}^3(s)\Delta\tilde{X}(s) - 12\tilde{X}^2(s)\big(\Delta\tilde{X}(s)\big)^2\Big\}$$
$$\le C\big[\tilde{X},\tilde{X}\big]_t$$

holds true $\mathbb{P}-a.s.$ This is a consequence of the fact that $\tilde{X}(\cdot)$ takes values in the compact interval $[-k,k]$ and the fact that for $h(x) = x^4$, it is easy to see that

$$\Big|h(x) - h(y) - (y-x)h'(x) - (y-x)^2h''(x)\Big| \le C(y-x)^2.$$

Thus,

$$\sum_{0<s\le T}|\tilde{X}^4(s) - \tilde{X}^4(s-) - 4\tilde{X}^3(s)\Delta X(s) - 12\tilde{X}^2(s)\big(\Delta\tilde{X}(s)\big)^2|$$
$$\le C\sum_{0<s\le T}\big(\Delta\tilde{X}_s\big)^2 \le C\big[\tilde{X},\tilde{X}\big]_T < \infty, \quad \mathbb{P}-a.s. \tag{14}$$

Since the last inequality is valid for $\tilde{X}$, for each k, it also remains valid for X. Combining (14) with (13), we get the desired result and this completes the proof. □

3.2. Verification Theorem

In this subsection, we study Problem **B** with a more general state process. We establish a sufficient stochastic maximum principle for stochastic control of forward-backward SDEs driven by Brownian motion and Teugels martingales where the control domain is assumed to be convex.

First, by combining (10) and (11), we get the following controlled semi-coupled FBSDE driven by Brownian motion and Teugels martingales:

$$\begin{cases} X(t) = & X(0) + \int_0^t f(s, X(s), v(s))ds + \int_0^t \sigma(s, X(s), v(s))dW(s) \\ & + \int_0^t \pi(s, X(s-), v(s))dH(s), \\ Y(t) = & \varphi(X(T)) + \int_t^T g(s, \mathcal{X}(s), v(s))ds - \int_t^T Z(s)dW(s) - \int_t^T K(s)dH(s), \end{cases} \tag{15}$$

We introduce the following cost functional:

$$J(v(\cdot)) = \mathbb{E}\left[\int_0^T g(s, \mathcal{X}(s), v(s))ds + \varphi(X(T))\right], \tag{16}$$

The optimal control problem is to minimize the cost functional $J(\cdot)$ over the set of all admissible controls.

To deal with the above control problem, we first define the Hamiltonian function

$$\mathcal{H} : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}^2(\mathbb{R}) \times U \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times l^2(\mathbb{R}) \to \mathbb{R}$$

by

$$\begin{aligned} \mathcal{H}(t, x, y, z, k, v, p, q, \lambda, \rho) = g(t, x, y, z, k, v)\lambda + f(t, x, v)p \\ +\sigma(t, x, v)q + \pi(t, x, v)\rho. \end{aligned} \tag{17}$$

We will use the shorthand notation

$$\mathcal{H}_x(t) = \frac{\partial \mathcal{H}}{\partial x}(t, x, y, z, k, v, p, q, \lambda, \rho),$$

and similarly, with $\mathcal{H}_y(t)$, $\mathcal{H}_z(t)$, $\mathcal{H}_k(t)$ and $\mathcal{H}_v(t)$. Similar notations are used for $l(t)$ where $l = f, \sigma, \pi$ and g.

The adjoint equations are described by the following stochastic Hamiltonian system:

$$\begin{cases} d\lambda(t) = \mathcal{H}_y(t)dt + \mathcal{H}_z(t)dW(t) + \mathcal{H}_k(t)dH(t), \\ dp(t) = -\mathcal{H}_x(t)dt + q(t)dW(t) + \rho(t)dH(t), \\ \lambda(0) = 1, \\ p(T) = \lambda(T)\varphi'(X(T)). \end{cases} \tag{18}$$

In addition to (H2), we need the following assumption:

Assumption 3. *If $\hat{v}(\cdot) \in \mathcal{U}$ and*

$$\hat{X}(\cdot), (\hat{Y}(\cdot), \hat{Z}(\cdot), \hat{K}(\cdot)), \hat{p}(\cdot), \hat{q}(\cdot), (\hat{\lambda}(\cdot), \hat{\rho}(\cdot))$$

are the corresponding solutions of (10), (11) and (18), respectively, we assume

(i) Functions $x \to \varphi(x)$ and

$$(t, x, y, z, k, v) \to \mathcal{H}\big(t, x, y, z, k, v, \hat{p}(t), \hat{q}(t), \hat{\lambda}(t), \hat{\rho}(t)\big)$$

are convex.

(ii) Function $\mathcal{H}$ satisfies

$$\begin{aligned} &\mathcal{H}\big(t, \hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t), v(t), \hat{p}(t), \hat{q}(t), \hat{\lambda}(t), \hat{\rho}(t)\big) \\ &\geq \mathcal{H}\big(t, \hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t), \hat{v}(t), \hat{p}(t), \hat{q}(t), \hat{\lambda}(t), \hat{\rho}(t)\big) \end{aligned} \tag{19}$$

for any $v(\cdot) \in \mathcal{U}$ *and for almost all* $(t, w) \in [0, T] \times \Omega$.

Then, we have the following sufficient condition for an optimal control of Problem **B**.

Theorem 1 (Sufficient maximum principle). *Let* $\hat{v}(\cdot) \in \mathcal{U}$ *with the corresponding solutions*

$$\hat{X}(\cdot), (\hat{Y}(\cdot), \hat{Z}(\cdot), \hat{K}(\cdot)), \hat{p}(\cdot), \hat{q}(\cdot), (\hat{\lambda}(\cdot), \hat{\rho}(\cdot))$$

of (10), (11) *and* (18) *respectively. Assume (H3). Then,* $\hat{v}(\cdot)$ *is an optimal control for Problem* ***B***.

Proof. Let $(X^v(\cdot), Y^v(\cdot), Z^v(\cdot), K^v(\cdot), v(\cdot))$ be an admissible solution of (15). It follows from the definition of the cost functional (16) that

$$J(\hat{v}(\cdot)) - J(v(\cdot)) = \mathbb{E}\big[\hat{Y}(0) - Y^v(0)\big].$$

From the forward component of SDE (18), the right hand side of the above equality can be rewritten

$$\mathbb{E}\big[\hat{Y}(0) - Y^v(0)\big] = \mathbb{E}\big[\big(\hat{Y}(0) - Y^v(0)\big)\hat{\lambda}(0)\big].$$

Applying Itô's formula to $(\hat{Y}(t) - Y^v(t))\hat{\lambda}(t)$ from $t = 0$ to $t = T$ and using the fact that $\left\langle H^{(i)}, H^{(j)} \right\rangle_t = \delta_{ij} t$ and $\left[H^{(i)}, H^{(j)}\right] - \left\langle H^{(i)}, H^{(j)} \right\rangle$ is a martingale, we obtain

$$\begin{aligned}
&\mathbb{E}\big[\big(\hat{Y}(0) - Y^v(0)\big)\hat{\lambda}(0)\big] = \mathbb{E}\big[\big(\varphi(\hat{X}(T)) - \varphi(X^v(T))\big)\hat{\lambda}(T)\big] \\
&-\mathbb{E}\Big[\int_0^T \big(\hat{Y}(t) - Y^v(t)\big) d\hat{\lambda}(t)\Big] - \mathbb{E}\Big[\int_0^T \hat{\lambda}(t) d\big(\hat{Y}(t) - Y^v(t)\big)\Big] \\
&-\mathbb{E}\Big[\int_0^T \mathcal{H}_z(t)\big(\hat{Z}(t) - Z^v(t)\big)dt\Big] - \mathbb{E}\Big[\int_0^T \mathcal{H}_k(t)\big(\hat{K}(t) - K^v(t)\big)dt\Big].
\end{aligned}$$

Since φ is convex, one can get

$$\mathbb{E}\big[\big(\varphi(\hat{X}(T)) - \varphi(X^v(T))\big)\hat{\lambda}(T)\big] \leq \mathbb{E}\big[\big(\hat{X}(T) - X^v(T)\big)\varphi'(X^v(T))\hat{\lambda}(T)\big]$$

We remark that $\hat{p}(T) = \varphi'(\hat{X}(T))\hat{\lambda}(T)$, then

$$\begin{aligned}
&\mathbb{E}\big[\big(\hat{Y}(0) - Y^v(0)\big)\hat{\lambda}(0)\big] \leq \mathbb{E}\big[\big(\hat{X}(T) - X^v(T)\big)\hat{p}(T)\big] \\
&-\mathbb{E}\Big[\int_0^T \mathcal{H}_y(t)\big(\hat{Y}(t) - Y^v(t)\big)dt\Big] + \mathbb{E}\Big[\int_0^T (\hat{\lambda}(t)(\hat{g}(t) - g^v(t)))dt\Big] \\
&-\mathbb{E}\Big[\int_0^T \mathcal{H}_z(t)\big(\hat{Z}(t) - Z^v(t)\big)dt\Big] - \mathbb{E}[\int_0^T \mathcal{H}_k(t)\big(\hat{K}(t) - K^v(t)\big)dt].
\end{aligned} \tag{20}$$

On the other hand, Itô's formula applied to $\big(\hat{X}(t) - X(t)\big)\hat{p}(t)$ gives us

$$\begin{aligned}
&\mathbb{E}\big[\big(\hat{X}(T) - X^v(T)\big)\hat{p}(T)\big] = \mathbb{E}\Big[\int_0^T \big(\hat{X}(t) - X^v(t)\big) d\hat{p}(t)\Big] \\
&+\mathbb{E}\Big[\int_0^T \hat{p}(t) d\big(\hat{X}(t) - X^v(t)\big)\Big] + \mathbb{E}\Big[\int_0^T (\hat{\sigma}(t) - \sigma(t))\hat{q}(t)dt\Big] \\
&+\mathbb{E}\Big[\int_0^T (\hat{\pi}(t) - \pi(t))\hat{\rho}(t)dt\Big].
\end{aligned} \tag{21}$$

Substituting (21) into (20), it follows immediately that

$$\begin{aligned}
&\mathbb{E}\big[\big(\hat{Y}(0) - Y^v(0)\big)\big] \leq \mathbb{E}\Big[-\int_0^T \mathcal{H}_x(t)\big(\hat{X}(t) - X^v(t)\big)dt \\
&+\int_0^T \Big\{\Big(\hat{f}(t) - f(t)\Big)\hat{p}(t) + (\hat{\sigma}(t) - \sigma(t))\hat{q}(t) + (\hat{\pi}(t) - \pi(t))\hat{\rho}(t) \\
&+\hat{\lambda}(t)(\hat{g}(t) - g(t)) - \mathcal{H}_y(t)\big(\hat{Y}(t) - Y^v(t)\big) \\
&-\mathcal{H}_z(t)\big(\hat{Z}(t) - Z^v(t)\big) - \mathcal{H}_k(t)\big(\hat{K}(t) - K^v(t)\big)\Big\}dt\Big].
\end{aligned}$$

Then

$$\begin{aligned}
&\mathbb{E}\big[\big(\hat{Y}(0) - Y^v(0)\big)\big] \leq \mathbb{E}\Big[\int_0^T \big\{\big(\hat{\mathcal{H}}(t) - \mathcal{H}(t)\big) - \mathcal{H}_x(t)\big(\hat{X}(t) - X^v(t)\big) \\
&-\mathcal{H}_y(t)\big(\hat{Y}(t) - Y^v(t)\big) - \mathcal{H}_z(t)\big(\hat{Z}(t) - Z^v(t)\big) \\
&-\mathcal{H}_k(t)\big(\hat{K}(t) - K^v(t)\big)\big\}dt\Big].
\end{aligned}$$

By virtue of the convexity property of the Hamiltonian $\mathcal{H}$ with respect to (x, y, z, k) for almost all $(t, w) \in [0, T] \times \Omega$, one can get

$$\begin{aligned}
&\mathbb{E}\big[\big(\hat{Y}(0) - Y^v(0)\big)\big] \leq \mathbb{E}\Big[\textstyle\int_0^T \{\mathcal{H}_x(t)\big(\hat{X}(t) - X^v(t)\big)\\
&+\mathcal{H}_y(t)\big(\hat{Y}(t) - Y^v(t)\big) + \mathcal{H}_z(t)\big(\hat{Z}(t) - Z^v(t)\big)\\
&+\mathcal{H}_k(t)\big(\hat{K}(t) - K^v(t)\big) + \mathcal{H}_v(t)(\hat{v}(t) - v(t))\\
&-\mathcal{H}_x(t)\big(\hat{X}(t) - X^v(t)\big) - \mathcal{H}_y(t)\big(\hat{Y}(t) - Y^v(t)\big)\\
&-\mathcal{H}_z(t)\big(\hat{Z}(t) - Z^v(t)\big) - \mathcal{H}_k\big(\hat{K}(t) - K^v(t)\big)\}dt\big]\\
&= \mathbb{E}\Big[\textstyle\int_0^T \mathcal{H}_v(t)(\hat{v}(t) - v(t))dt\Big].
\end{aligned}$$

By invoking the necessary condition of optimality (19), we conclude that

$$\mathbb{E}\big[\big(\hat{Y}(0) - Y^v(0)\big)\big] \leq 0,$$

which implies,

$$J(\hat{v}(\cdot)) - J(v(\cdot)) \leq 0, \ \ \forall v(\cdot) \in \mathcal{U}.$$

This finishes the proof. □

4. Applications

4.1. Optimal Premium Problem

In this subsection, firstly, we use the Lagrangian method to treat the terminal state constraint, and after that, we apply the sufficient maximum principle to deal with the resulting unconstrained optimization problem. Throughout this subsection, we assume that $\alpha = 0$ in (2), which means that the insurance firm only invests in a money account with compounded interest rate $\delta(t)$, and hence, $\Delta(t) = \int_0^t \delta(s)ds$. We recall that this kind of problem in the continuous Brownian case was solved in Huang et al. (2010).

Then, SDE (3) becomes

$$\begin{cases} dX(t) = (\delta(t)X(t) + b(t) + v(t))dt + a(t)dW(t) + \pi(t)dH(t), \\ X(0) = x. \end{cases} \tag{22}$$

By using the Lagrangian multiplier method, the cost function (5) becomes

$$J(v(\cdot)) = \mathbb{E}\left[\int_0^T e^{-\beta t} g_1(t, X(t), v(t))dt + e^{-\beta T}\varphi_1(X(T)) + \theta((X(T) - c_0))\right] \tag{23}$$

where θ, the Lagrange multiplier, is some constant to be determined. Then, we can reformulate Problem **A**, as the following one: we look forward to find $\hat{v} \in \mathcal{U}$, which minimizes the cost function (23) subject to (22) and (4).

For this end, let us firstly define the Hamiltonian function

$$\mathcal{H} : [0, T] \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}^2(\mathbb{R}) \rightarrow \mathbb{R},$$

by

$$\begin{aligned} \mathcal{H}(t, X, v, p, q, \rho) = {} & (\delta(t)X + b(t) + v)p + a(t)q + \pi(t)\rho \\ & + \frac{1}{2}e^{-\beta t}\Big[R(t)(X - A(t))^2 + N(t)v^2\Big]. \end{aligned} \tag{24}$$

Then, the adjoint equation can be rewritten in a Hamiltonian form as

$$\begin{cases} -dp(t) = & \big[\delta(t)p(t) + (X(t) - A(t))R(t)e^{-\beta t}\big]dt - q(t)dW(t) - \rho(t)dH(t), \\ p(T) = & \theta + Me^{-\beta T}(X(T) - c_0), \end{cases} \tag{25}$$

where $X(\cdot)$ stands for the optimal process under the optimal premium policy $v(\cdot)$ and satisfies the equation (22).

Let ϕ and ψ be the solutions of

$$\begin{cases} \phi'(t) + 2\delta(t)\phi(t) - N^{-1}(t)e^{\beta t}\phi^2(t) + R(t)e^{-\beta t} = 0, \\ \phi(T) = Me^{-\beta T}, \end{cases} \tag{26}$$

and

$$\begin{cases} \psi'(t) + \big(\delta(t) - N^{-1}(t)e^{\beta t}\phi(t)\big)\psi(t) + b(t)\phi(t) - A(t)R(t)e^{-\beta t} = 0, \\ \psi(T) = \theta - c_0 Me^{-\beta T}. \end{cases} \tag{27}$$

We now are in position to state and prove the main result of this subsection.

Theorem 2. *Under Assumption (H1), the optimal premium policy is given by*

$$\hat{v}(t) = -N^{-1}(t)e^{\beta t}(\phi(t)X(t) + \psi(t)),$$

where $X(\cdot)$ satisfies (22) and ϕ and ψ are the solutions of (26) and (27), respectively. Moreover, the optimal cost functional is given by

$$\begin{aligned} J(\hat{v}) = {} & \frac{1}{2}\Big(\int_0^T e^{-\beta t}R(t)A^2(t)dt + Me^{-\beta T}c_0^2\Big) + \frac{1}{2}\phi(0)x^2 + \psi(0)x - c_0\theta \\ & + \frac{1}{2}\int_0^T \big[\phi(t)a^2(t) + \phi(t)\pi^2(t) + \psi(t)\big(2b(t) - N^{-1}(t)e^{\beta t}\psi(t)\big)\big]dt. \end{aligned} \tag{28}$$

Proof. We shall divide the proof into several steps.

Step 1: We start by proving the existence of an optimal premium policy. Since for each t, (5) is quadratic with respect to $X(t), v(t), X(T)$, and the weight $N(t)$ of $v^2(t)$ is larger than 0, then there exists an optimal premium policy $\hat{v}(\cdot)$, which solves Problem **A**. Indeed, from Theorem 6.1 in Meng and Tang (2009), the optimal premium policy $\hat{v}(\cdot)$ satisfies

$$\begin{aligned} 0 &= \frac{\partial \mathcal{H}}{\partial v(t)}(t, X(t), v(t), p(t), q(t), \rho(t)) \\ &= \frac{\partial}{\partial v(t)}\{(\delta(t)X(t) + b(t) + v(t))p(t) + a(t)q(t) + \pi(t)\rho(t) \\ &+ \frac{1}{2}e^{-\beta t}\Big[R(t)(X(t) - A(t))^2 + N(t)v^2(t)\Big]\Big\} \\ &= p(t) + e^{-\beta t}N(t)v(t), \end{aligned}$$

and is given by

$$\hat{v}(t) = -N^{-1}(t)e^{\beta t}p(t). \tag{29}$$

Substituting (29) into (22) and combining it with (25), we obtain the generalized Hamiltonian system

$$\begin{cases} dX(t) = & \big(\delta(t)X(t) - N^{-1}(t)e^{\beta t}p(t) + b(t)\big)dt + a(t)dW(t) + \pi(t)dH(t), \\ -dp(t) = & \big[\delta(t)p(t) + (X(t) - A(t))R(t)e^{-\beta t}\big]dt - q(t)dW(t) - \rho(t)dH(t), \\ X(0) = x, & p(T) = \theta + Me^{-\beta T}(X(T) - c_0), \end{cases} \tag{30}$$

which is a coupled FBSDE system. Using a similar argument to the ones used in Baghery et al. (2014), we can verify that FBSDE (30) admits a unique solution under (H1). Note that the optimal premium policy exists and is unique. Furthermore the relation (29) implies that it is linear with respect to p.

Step 2: In this step, we are going to prove that the optimal premium policy $\hat{v}(\cdot)$ is in fact a linear feedback of the optimal process $X(\cdot)$. First of all, according to the terminal condition of $p(\cdot)$ in (30), it is quite natural to suggest that

$$p(t) = \phi(t)X(t) + \psi(t), \tag{31}$$

with $\phi(T) = Me^{-\beta T}$ and $\psi(T) = \theta - c_0 Me^{-\beta T}$. Then, we apply Itô's formula to $p(\cdot)$ to obtain

$$\begin{aligned} dp(t) = & \big[(\phi'(t) + \delta(t)\phi(t))X(t) + \psi'(t) + \big(b(t) - N^{-1}(t)e^{\beta t}p(t)\big)\phi(t)\big]dt \\ & + \phi(t)\delta(t)dW(t) + \phi(t)\pi(t)dH(t). \end{aligned}$$

Comparing their generator terms with those of the BSDE in (30) we get (26) and (27).

It is well known that (26) is a standard Riccati differential equation that admits a unique solution under (H1), so does (27) and

$$\psi(t) = \int_t^T \Big(b(s)\phi(s) - A(s)R(s)e^{-\beta s}\Big)\Lambda_t(s)ds + \Big(\theta - c_0 Me^{-\beta T}\Big)\Lambda_t(T), \tag{32}$$

where

$$\Lambda_t(s) = \exp\{\int_t^s \Big(\delta(r) - N^{-1}(r)e^{\beta r}\phi(r)\Big)dr\}.$$

Step 3: In this step, we are going to find the value of θ. Combining (29) and (31), we get

$$\left\{ \begin{aligned} dX(t) = & \big[\big(\delta(t) - N^{-1}(t)e^{\beta t}\phi(t)\big)X(t) + b(t) - N^{-1}(t)e^{\beta t}\psi(t)\big]dt \\ & + a(t)dW(t) + \pi(t)dH(t), \\ X(0) = & \; x. \end{aligned} \right.$$

Define $\tilde{X}(t) = \mathbb{E}[X(t)]$, then

$$\left\{ \begin{aligned} \tilde{X}'(t) = & \big(\delta(t) - N^{-1}(t)e^{\beta t}\phi(t)\big)\tilde{X}(t) + b(t) - N^{-1}(t)e^{\beta t}\psi(t), \\ \hat{X}(0) = & \; x \end{aligned} \right. \tag{33}$$

Solving (33) and keeping in mind the terminal constraint (4), we easily derive

$$c_0 = x\Lambda_0(T) + \int_0^T \Big(b(t) - N^{-1}(t)e^{\beta t}\psi(t)\Big)\Lambda_t(T)dt. \tag{34}$$

Note that ϕ in (26) does not depend on θ and ψ in (27) is linear with respect to θ. Inserting (27) into (34), we get the equation

$$\theta = \frac{\int_0^T \{b(t) - N^{-1}(t)[F(t) - Q(t)]\Lambda_t(T)\}dt + x\Lambda_0(T) - c_0}{\int_0^T N^{-1}(t)e^{\beta t}\Lambda_t^2(T)dt}$$

where

$$F(t) = \int_t^T \Big(e^{\beta t}b(s)\phi(s) - A(s)R(s)e^{\beta(t-s)}\Big)\Lambda_t(s)ds$$

and

$$Q(t) = c_0 M \exp\{\int_t^T \Big(\delta(s) - N^{-1}(s)e^{\beta s}\phi(s) - \beta\Big)ds\}.$$

Step 4: We now proceed to determine the optimal cost functional. Substituting (29) into (5) we get

$$\begin{aligned} J(\hat{v}(\cdot)) = \; & \frac{1}{2}\mathbb{E}\Big\{\int_0^T e^{-\beta t}\big[\big(R(t)+N^{-1}(t)e^{2\beta t}\phi^2(t)\big)X^2(t)-2(A(t)R(t) \\ & -N^{-1}(t)e^{2\beta t}\phi(t)\psi(t)\big)X(t)+N^{-1}(t)e^{2\beta t}\psi^2(t)\big]dt \\ & +Me^{-\beta T}X^2(T)-2c_0Me^{-\beta T}X(T)\Big\} \\ & +\frac{1}{2}\Big(\int_0^T e^{-\beta t}R(t)A^2(t)dt+Me^{-\beta T}c_0^2\Big). \end{aligned} \tag{35}$$

From Itô's formula,

$$\begin{aligned} & d\big(\phi(t)X^2(t)+2\psi(t)X(t)\big) = \\ & \big[\phi(t)a^2(t)+2\psi(t)\big(b(t)-N^{-1}(t)e^{\beta t}\psi(t)\big) \\ & +2e^{-\beta t}\big(A(t)R(t)-N^{-1}(t)e^{2\beta t}\phi(t)\psi(t)\big)X(t) \\ & -e^{-\beta t}\big(R(t)+N^{-1}(t)e^{2\beta t}\phi^2(t)\big)X^2(t)\big]dt \\ & +2\sigma(t)(\phi(t)X(t)+\psi(t))dW(t) \\ & +2\pi(t)(\phi(t)X(t)+\psi(t))dH(t)+\phi(t)\sum_{i,j}\pi^i(t)\pi^j(t)d\big[H^i,H^j\big]_t. \end{aligned}$$

Integrating from 0 to T, taking expectations on both sides of the above equality and using the fact that $\big[H^i,H^j\big]_t-\langle H^i,H^j\rangle_t$ is an $\mathbb{F}-$martingale and $\langle H^i,H^j\rangle_t=\delta_{ij}t$, we get

$$\begin{aligned} & \mathbb{E}\big(\phi(T)X^2(T)+2\psi(T)X(T)\big) \\ & =\int_0^T\big[\phi(t)\big(a^2(t)+\pi^2(t)\big)+2\psi(t)\big(b(t)-N^{-1}(t)e^{\beta t}\psi(t)\big)\big]dt \\ & +\phi(0)x^2+2\psi(0)x+2\mathbb{E}\int_0^T e^{-\beta t}\big(A(t)R(t)-N^{-1}(t)e^{2\beta t}\phi(t)\psi(t)\big)X(t)dt \\ & -\mathbb{E}\int_0^T e^{-\beta t}\big(R(t)+N^{-1}(t)e^{2\beta t}\phi^2(t)\big)X^2(t)dt. \end{aligned}$$

On the other hand, it follows from (26) and (27) that

$$\mathbb{E}\Big[\phi(T)X^2(T)+2\psi(T)X(T)\Big]=\mathbb{E}\Big[Me^{-\beta T}X^2(T)-2c_0Me^{-\beta T}X(T)+2\theta X(T)\Big].$$

Inserting the above two equalities into (35) we obtain the optimal cost functional (28). This gives the desired result. □

Specifically, in case we rule out the terminal constraint (4), that is, $\theta=0$, Theorem 2 solves Problem A without the constraint (4).

Note also that ϕ, ψ and θ do not depend on a and π. This leads to the fact that the optimal cost function $J(u)$ increases with respect to σ and π (describing the liability risk). This implies that the more uncertainty in the liability process, the higher the operation costs of our optimal premium policy.

4.2. Optimal Premium Problem under Stochastic Interest Rate

In the current subsection, we want to discuss the optimal premium problem (Problem **B**), in the case where the interest rate is allowed to be stochastic. More precisely, we shall consider two different cases. In the first one, we assume that the payment function and the stochastic interest rate are given by the same Brownian motion and in the second case, we assume that they are given by different and independent Brownian motions.

4.2.1. First Case

In this case, we proceed to solve Problem **B**, assuming that the utility function is that of (9) and the cash-balance process satisfies (3), where the stochastic interest rate is given by (2). We also point out that we solve a mean variance problem in this section as a particular case of Problem **B.**

Since the coefficients in (3), (6) and (7) are linear, we shall assume

$$\begin{cases} dX(t) = & (a_1(t)X(t)+a_2(t)v(t)+a_3(t))dt \\ & +(b_1(t)X(t)+b_2(t)v(t)+b_3(t))dW(t)+\pi(t)dH(t), \\ X(0) = & x, \end{cases} \tag{36}$$

and

$$\begin{cases} dY(t) = & \left(\frac{1}{2}(c_1(t)X^2(t) + c_2(t)X(t) + c_3(t)v^2(t) + c_4(t)) - \beta Y(t)\right)dt \\ & -Z(t)dW(t) - K(t)dH(t) \\ Y(T) = & \frac{1}{2}(X(T) - a)^2 \end{cases} \quad (37)$$

where a is real constant and $a_1, a_2, a_3, b_1, b_2 \neq 0, b_3, c_1, c_2, c_3, c_4$ are deterministic functions satisfying some properties. Note that particular choices of the coefficients of (36) and (37) generate the SDE (10) and the BSDE (11).

Here, the Hamiltonian (17) gets the form

$$\begin{aligned} \mathcal{H} = & \left[\frac{1}{2}(c_1(t)X^2(t) + c_2(t)X(t) + c_3(t)v^2(t) + c_4(t)) - \beta Y(t)\right]\lambda(t) \\ & + (a_1(t)X(t) + a_2(t)v(t) + a_3(t))p(t) \\ & + (b_1(t)X(t) + b_2(t)v(t) + b_3(t))q(t) + \pi(t)\rho(t), \end{aligned}$$

$\lambda(t) = e^{-\beta t}$ and $(p(\cdot), q(\cdot), \rho(\cdot))$ satisfies the adjoint backward equation

$$\begin{cases} dp(t) = & -\left(c_1(t)X(t)\lambda(t) + a_1(t)p(t) + b_1(t)q(t) + \frac{1}{2}c_2(t)\lambda(t)\right)dt \\ & +q(t)dW(t) + \rho(t)dH(t), \\ p(T) = & \theta + \lambda(t)\varphi'(X(T)). \end{cases} \quad (38)$$

By using the sufficient condition (19) given in Section 3, we get

$$\lambda(t)c_3(t)v(t) + a_2(t)p(t) + b_2(t)q(t) = 0. \quad (39)$$

To solve (38), we try a process p of the form

$$p(t) = \phi(t)X(t) + \psi(t), \quad (40)$$

where ϕ and ψ are derivable functions with continuous derivatives.

Applying Itô's formula to (40) and using (36), we get

$$\begin{aligned} dp(t) = & [(\phi(t)a_1(t) + \phi'(t))X(t) + \phi(t)a_2(t)v(t) + \phi(t)a_3(t) + \psi'(t)]dt \\ & + \phi(t)[b_1(t)X(t) + b_2(t)v(t) + b_3(t)]dW(t) + \phi(t)\pi(t)dH(t) \end{aligned} \quad (41)$$

Comparing with (38), we obtain

$$\begin{aligned} \rho(t) &= \phi(t)\pi(t), \\ q(t) &= \phi(t)[b_1(t)X(t) + b_2(t)v(t) + b_3(t)] \end{aligned} \quad (42)$$

and

$$\begin{aligned} & -\left(c_1(t)X(t)\lambda(t) + a_1(t)p(t) + b_1(t)q(t) + \tfrac{1}{2}c_2(t)\lambda(t)\right) \\ & = (\phi(t)a_1(t) + \phi'(t))X(t) + \phi(t)a_2(t)v(t) + \phi(t)a_3(t) + \psi'(t). \end{aligned} \quad (43)$$

Substituting (42) into (39), we obtain

$$\hat{v}(t) = -\frac{(a_2(t) + b_1(t)b_2(t))\phi(t)\hat{X}(t) + a_2(t)\psi(t) + b_2(t)b_3(t)\phi(t)}{\lambda(t)c_3(t) + b_2^2(t)\phi(t)}. \quad (44)$$

On the other hand, from (43) we have

$$\hat{v}(t) = -\frac{\big(c_1(t)\lambda(t) + 2a_1(t)\phi(t) + b_1^2(t)\phi(t) + \phi'(t)\big)\hat{X}(t)}{(a_2(t) + b_1(t)b_2(t))\phi(t)} - \frac{a_1(t)\psi(t) + b_1(t)b_3(t)\phi(t) + \frac{1}{2}c_2(t)\lambda(t) + a_3(t)\phi(t) + \psi'(t)}{(a_2(t) + b_1(t)b_2(t))\phi(t)}. \tag{45}$$

Combining (44) and (45), we obtain

$$\begin{aligned} &\phi^2(t)\Big[(a_2(t) + b_1(t)b_2(t))^2 - b_2^2(t)\big(2a_1(t) + b_1^2(t)\big)\Big] - \phi(t)\phi'(t)b_2^2(t) \\ &-\big[c_3(t)\big(2a_1(t) + b_1^2(t)\big) + c_1(t)b_2^2(t)\big]\phi(t)\lambda(t) \\ &-c_3(t)\phi'(t)\lambda(t) = c_1(t)c_3(t)\lambda^2(t),\ \phi(T) = \lambda(T). \end{aligned} \tag{46}$$

and

$$\begin{aligned} &\psi(t)\big[\phi(t)\big(a_2(t)(a_2(t) + b_1(t)b_2(t)) - a_1(t)b_2^2(t)\big) - c_3(t)a_1(t)\lambda(t)\big] \\ &-\psi'(t)\big(c_3(t)\lambda(t) + \phi(t)b_2^2(t)\big) \\ &= -\phi^2(t)\big[(a_2(t) + b_1(t)b_2(t))b_2(t)b_3(t) - b_2^2(t)(b_1(t)b_3(t) + a_3(t))\big] \\ &+c_3(t)(b_1(t)b_3(t) + a_3(t))\phi(t)\lambda(t) \\ &+\frac{1}{2}\big(c_2(t)b_2^2(t)\phi(t) + c_3(t)c_2(t)\lambda(t)\big)\lambda(t),\ \psi(T) = \theta - \lambda(T)c_0. \end{aligned} \tag{47}$$

Let us emphasize that, under the boundedness of its coefficients, the first order differential equation (46) admits a unique solution. Moreover, a simple computation shows the (47) has an explicit solution, which is given by

$$\psi(t) = (\theta - \lambda(T)c_0)\exp(-\Lambda_t(T)) - \exp(\Lambda_t(T)) \int_t^T e^{-\Lambda_t(s)} G_t(s) ds,$$

where

$$\Lambda_t(T) = \int_t^T \frac{\phi(s)\big[(a_2(s) + b_1(s)b_2(s))a_2(s) - a_1(s)b_2^2(s)\big]}{c_3(s)\lambda(s) + b_2^2(s)\phi(s)} ds - \int_t^T \frac{c_3(s)a_1(s)\lambda(s)}{c_3(s)\lambda(s) + \phi(s)b_2^2(s)} ds,$$

and

$$\begin{aligned} G_t(s) = &- \int_t^T \frac{\phi^2(s)b_2(s)b_3(s)(a_2(s) + b_1(s)b_2(s))}{c_3(s)\lambda(s) + \phi(s)b_2^2(s)} ds \\ &+ \int_t^T \frac{\phi^2(s)b_2^2(s)(b_1(s)b_3(s) + a_3(s))}{c_3(s)\lambda(s) + \phi(s)b_2^2(s)} ds \\ &+ \int_t^T \frac{c_3(s)(b_1(s)b_3(s) + a_3(s))\lambda(s)\phi(s)}{c_3(s)\lambda(s) + \phi(s)b_2^2(s)} ds. \\ &+ \int_t^T \frac{\frac{1}{2}\big(c_2(s)b_2^2(s)\phi(s) + c_3(s)c_2(s)\lambda(s)\big)\lambda(s)}{c_3(s)\lambda(s) + \phi(s)b_2^2(s)} ds \end{aligned}$$

Thus, we have the following theorem:

Theorem 3. *Let $X^v(t)$ be the cash balance satisfying (36). Then, the optimal premium policy of Problem A is*

$$\hat{v}(t) = -\frac{(a_2(t) + b_1(t)b_2(t))\phi(t)\hat{X}(t) + a_2(t)\psi(t) + b_2(t)b_3(t)\phi(t)}{c_3(t)\lambda(t) + b_2^2(t)\phi(t)},$$

where $\lambda(t) = e^{-\beta t}$ and ϕ and ψ are the solutions of (46) and (47), respectively.

This is clearly the unique solution under Assumption (H1). Notice that we can get explicit solutions in some particular cases. For example, if $g_1 = 0$ we can see the problem as a mean-variance one, where our objective is to find $v(t)$ such that it minimizes

$$\mathbb{V}(X(T)) = \mathbb{E}\Big[(X(T) - \mathbb{E}(X(T)))^2\Big],$$

under the terminal constraint condition (4).

By the Lagrangian multiplier method, the problem can be reduced to minimize the following equivalent problem:

$$J(v(\cdot)) = \mathbb{E}\left[\frac{1}{2}(X^v(T) - c_0)^2 + \theta(X^v(T) - c_0)\right]. \tag{48}$$

In this case, the Hamiltonian (17) gets the form

$$\begin{aligned}\mathcal{H} = &(a_1(t)X(t) + a_2(t)v(t) + a_3(t))p(t)\\ &+ (b_1(t)X(t) + b_2(t)v(t) + b_3(t))q(t)\\ &+ \pi(t)\rho(t).\end{aligned}$$

Hence, the adjoint equation takes the following form

$$\begin{cases} dp(t) = -(a_1(t)p(t) + b_1(t)q(t))dt + q(t)dW(t) + \rho(t)dH(t)\\ p(T) = \theta + (X(T) - c_0). \end{cases}$$

Note that in this case $\lambda(t) = 1, \forall t \in [0, T]$.

In the following corollary, we solve Problem (48) assuming that $X^v(t)$ satisfies (36).

Corollary 1. *Let $X^v(t)$ be the cash balance satisfying (36). Then, the optimal premium policy of Problem (48) is*

$$\hat{v}(t) = -\frac{(a_2(t) + b_1(t)b_2(t))\phi(t)\hat{X}(t) + a_2(t)\psi(t) + b_2(t)b_3(t)\phi(t)}{b_2^2(t)\phi(t)},$$

where ϕ and ψ are given, respectively, by

$$\phi(t) = \exp\{-\int_t^T \frac{(a_2(s)+b_1(s)b_2(s))^2 - \big(2a_1(s)+b_1^2(s)\big)b_2^2(s)}{b_2^2(s)}ds\}$$

and

$$\begin{gathered}\psi(t) = (\theta - c_0)\exp(\Lambda_t(T))\\ -\exp(\Lambda_t(T))\int_t^T \exp(-\Lambda_t(s))\phi(s)(a_3(s)b_2(s) - a_2(s)b_3(s))b_2(s)ds,\end{gathered}$$

with

$$\Lambda_t(T) = -\int_t^T \frac{(a_2(s)(a_2(s) + b_1(s)b_2(s))) - a_1(s)b_2^2(s)}{b_2^2(s)}ds.$$

4.2.2. Second Case

In this case, we assume the liability of the surplus process and the interest rate are given by two different and independent Brownian motions. Let $\bar{W} := (\bar{W}(t))_{t\geq 0}$ be another Brownian motion defined in $(\Omega, \mathcal{F}, \mathbb{P})$ and independent of W and L. Assume

$$\mathcal{F}_t = \sigma(W(s), \bar{W}(s), L(s), 0 \leq s \leq t) \vee \mathcal{N}.$$

We consider that the interest rate is given by the stochastic differential equation

$$d\Delta(t) = \delta(t)dt + \alpha(t)d\bar{W}(t),\ t \in [0, T],\ \Delta(0) = 0. \tag{49}$$

Itô formula applied to the process $X(\cdot)$ described by (1) in Section 2, leads to

$$\begin{cases} dX(t) = & \Big(X(t)\Big(\delta(t)+\frac{1}{2}\alpha^2(t)\Big)+b(t)+v(t)\Big)dt+X(t)\alpha(t)d\bar{W}(t) \\ & +a(t)dW(t)+\pi(t)dH(t), \\ X(0) = & x. \end{cases} \tag{50}$$

Define a process V such that

$$X(t)\alpha(t)d\bar{W}(t)+a(t)dW(t)=\Big(X^2(t)\alpha^2(t)+\sigma^2(t)\Big)^{\frac{1}{2}}dV(t),$$

that is,

$$V(t)=\int_0^t\Big(X^2(s)\alpha^2(s)+a^2(s)\Big)^{-\frac{1}{2}}(X(s)\alpha(s)d\bar{W}(s)+a(s)dW(s)).$$

Clearly, the process V is a continuous martingale with quadratic variation $\langle V\rangle_t=t$, and so, it must be a standard Brownian motion, see for example Theorem 6.1 in Chung and Williams (1990). Thus, we can write Equation (50) by

$$\begin{cases} dX(t) = & \Big(X(t)\Big(\delta(t)+\frac{1}{2}\alpha^2(t)\Big)+b(t)+v(t)\Big)dt \\ & +(X^2(t)\alpha^2(t)+a^2(t))^{\frac{1}{2}}dV(t)+\pi(t)dH(t), \\ X(0) = & x. \end{cases} \tag{51}$$

We shall derive the solution of Problem A. Here, the Hamiltonian (17) gets the form

$$\begin{aligned}\mathcal{H} = & \Big(X(t)\Big(\delta(t)+\frac{1}{2}\alpha^2(t)\Big)+b(t)+v(t)\Big)p+(X^2(t)\alpha^2(t)+a^2(t))^{\frac{1}{2}}q \\ & +\pi(t)\rho+\frac{1}{2}e^{-\beta t}\Big[R(t)(X(t)-A(t))^2+N(t)v^2(t)\Big].\end{aligned}$$

The adjoint process $(p(\cdot),q(\cdot),\rho(\cdot))$ satisfies the BSDE

$$\begin{cases} -dp(t) = & (a_4(t)p(t)+a_5(t)+R(t)(X(t)-A(t))e^{-\beta t})dt \\ & -q(t)dV(t)-\rho(t)dH(t), \\ p(T) = & \theta+Me^{-\beta T}(X(T)-c_0). \end{cases}$$

where $a_4(t)=\Big(\delta(t)+\frac{1}{2}\alpha^2(t)\Big)$ and $a_5(t)=\dfrac{\alpha^2(t)X(t)q(t)}{\sqrt{X^2(t)\alpha^2(t)+a^2(t)}}$. Notice that in this case, $\lambda(t)=1, \forall t\in[0,T]$.

By the same technique used in Section 4.1, we infer that the optimal premium policy is given by

$$\hat{v}(t)=-N^{-1}(t)e^{\beta t}(\phi(t)X(t)+\psi(t)),$$

where $X(\cdot)$ satisfies (51) and ϕ and ψ are, respectively, the solutions of the equations

$$\begin{cases} \phi'(t)+2(\delta(t)+\alpha^2(t))\phi(t)-N^{-1}(t)e^{\beta t}\phi^2(t)+R(t)e^{-\beta t}=0 \\ \phi(T)=Me^{-\beta T}, \end{cases}$$

and

$$\begin{cases} \psi'(t)+\Big(\delta(t)+\frac{1}{2}\alpha^2(t)-N^{-1}(t)e^{\beta t}\phi(t)\Big)\psi(t)+b(t)\phi(t)-A(t)R(t)e^{-\beta t}=0, \\ \psi(T)=\theta-c_0Me^{-\beta T}. \end{cases}$$

Moreover, the optimal cost functional is given by

$$J(\hat{v}(\cdot)) = \frac{1}{2}\Big(\int_0^T e^{-\beta t}R(t)A^2(t)dt + Me^{-\beta T}c_0^2\Big) + \frac{1}{2}\phi(0)x^2 + \psi(0)x - c_0\theta$$
$$+\frac{1}{2}\int_0^T \big[\phi(t)a^2(t) + \phi(t)\alpha^2(t) + \psi(t)\big(2b(t) - N^{-1}(t)e^{\beta t}\psi(t)\big)\big]dt.$$

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