TATE MODULE TENSOR DECOMPOSITIONS AND THE SATO–TATE CONJECTURE FOR CERTAIN ABELIAN VARIETIES POTENTIALLY OF GL_2-TYPE

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ABSTRACT. We introduce a tensor decomposition of the ℓ-adic Tate module of an abelian variety A_0 defined over a number field which is geometrically isotypic and potentially of GL_2-type. We use this decomposition as a fundamental tool to describe the Sato–Tate group of A_0 and to prove the Sato–Tate conjecture in certain cases.

CONTENTS

1. INTRODUCTION

Let A_0 be an abelian variety defined over a number field k_0 of dimension $g \geq 1$. Following Ribet, we say that A_0 is of GL_2-type if there exists a number field of degree g that injects into the endomorphism algebra of A_0. In this article, we will make the weaker requirement that A_0 be *potentially of* GL_2*-type*, that is, we will assume that there exists a number field of degree g that injects into the endomorphism algebra of $A_{0,\overline{\mathbb{Q}}} = A_0 \times_{k_0} \overline{\mathbb{Q}}$, the base change of A_0 to an algebraic closure of $\mathbb{Q}$. For the sake of simplicity, assume in this introduction that A_0 does not have potential complex multiplication (CM), that is, there does not exist a number field of degree $2g$ injecting into the endormorphism algebra of $A_{0,\overline{\mathbb{Q}}}$. In this case $A_{0,\overline{\mathbb{Q}}}$ is isogenous to the power of an absolutely simple abelian variety B defined over $\overline{\mathbb{Q}}$. We refer to this property by saying that A_0 is *geometrically isotypic*. The absolutely simple factor B is often referred to as a "building block". It is well known that the endomorphism algebra of B is either isomorphic to a totally real field of degree $\dim(B)$, in which case we say that B has real multiplication (RM), or to a quaternion division algebra over a totally real field of degree $\dim(B)/2$ in

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which case we say that B has quaternionic multiplication (QM). Ribet [Rib92] gave proofs of these facts when A_0 is defined over $\mathbb{Q}$ (see [Gui12, Thm. 3.3, Prop. 3.4] for proofs in the general case).

Ribet extensively studied abelian varieties of GL_2-type and showed that they share many features with elliptic curves. In particular, Ribet [Rib92, §3] showed that one can attach a rank 2 compatible system of ℓ-adic representation to an abelian variety of GL_2-type. This was extended by Wu [Wu18] to abelian varieties potentially of GL_2-type (see Section 3 for a recollection of these results). The main novelty of the present paper is the description of the ℓ-adic Tate module $V_\ell(A_0)$ attached to A_0 as (the induction of) a tensor product of an Artin representation and a rank 2 compatible system of ℓ-adic representations. The next result accounts for the essential statements of Theorem 2.11. Let G_{k_0} denote the absolute Galois group of k_0 and let $m := [M : \mathbb{Q}]$ be the degree of the center M of the endomorphism algebra of B (observe that M is also isomorphic to the center of the endomorphism algebra of $A_{0,\overline{\mathbb{Q}}}$).

Theorem 1.1. *Let A_0 be an abelian variety defined over a number field k_0 of dimension $g \geq 1$. Suppose that A_0 is potentially of GL_2-type and non-CM. Then there exist:*

i) a finite Galois extension k/k_0;
ii) a number field F;
iii) a rank 2 weakly compatible system of ℓ-adic representations $(V_\lambda(B)^{\alpha_B})_\lambda$ of G_k defined over F; and
iv) a finite image representation $V(B,A)^{\alpha_B}$ of G_k realizable over F;

such that for every rational prime ℓ there exists a choice of primes $\lambda_1, \dots, \lambda_r$ of F lying over ℓ, where $r = m/[k : k_0]$, for which there is an isomorphism

$$V_\ell(A_0) \otimes \overline{\mathbb{Q}}_\ell \simeq \mathrm{Ind}_{k_0}^{k} \left(\bigoplus_{i=1}^{r} V_{\lambda_i}(B)^{\alpha_B} \otimes_{\overline{\mathbb{Q}}_\ell} V_{\lambda_i}(B,A)^{\alpha_B} \right)$$

of $\overline{\mathbb{Q}}_\ell[G_{k_0}]$-modules. In the above isomorphism, $V_{\lambda_i}(B,A)^{\alpha_B}$ denotes the tensor product $V(B,A)^{\alpha_B} \otimes_{F,\sigma_i} \overline{\mathbb{Q}}_\ell$ taken with respect to the embedding $\sigma_i \colon F \hookrightarrow \overline{\mathbb{Q}}_\ell$ corresponding to the prime λ_i.

This theorem is proven in the course of Section 2. Along the way we describe the number fields F and k, and construct the Artin representation $V(B,A)^{\alpha_B}$ and the ℓ-adic system $(V_\lambda(B)^{\alpha_B})_\lambda$. The setting of the proof is general enough to show that a similar tensor decomposition holds in the case that A_0 has CM. The representations $V_\lambda(B)^{\alpha_B}$ and $V(B,A)^{\alpha_B}$ arise naturally as projective representations. The obstruction for these projective representations to lift to genuine representations is given by two cohomology classes in $H^2(G_k, M^\times)$ that, by a theorem of Tate, can be trivialized after enlarging the field of coefficients. It lies at the core of the proof of Theorem 1.1 the fact that these two cohomology classes are inverses to each other.

There are at least two known particular cases of this decomposition in the literature. On the one hand, it is known when A_0 is $\overline{\mathbb{Q}}$-isogenous to the power of an elliptic curve B defined over $\overline{\mathbb{Q}}$ which admits a model up to isogeny *defined over* k_0 (this follows from [Fit13, Thm. 3.1] when B does not have CM and from [FS14, (3-8)] when it does). We note that if g is odd, then there does exist a model up to isogeny for B defined over k_0 (see Remark 2.13), but this is not always satisfied when g is even as shown in [FS14, §3D]. On the other hand, an analogous tensor decomposition has been obtained by N. Taylor [Tay19, §3.3] when A_0 is an abelian surface with QM; see also [BCGP18, Prop. 9.2.1]. Taylor's explicit, but intriguing to us, construction of the tensor decomposition in the case of a QM abelian surface was a source of inspiration for the present article. Section 2 is our attempt to give a uniform, general, and more conceptual explanation of this phenomenon.

We find a direct application of the obtained description of the Tate module of A_0 in the context of the Sato–Tate conjecture. In Section 4, we use this description to determine the Sato–Tate group $\mathrm{ST}(A_0)$ of A_0. The Sato–Tate conjecture is an equidistribution statement regarding the Frobenius conjugacy classes acting on $V_\ell(A_0)$. As shown by Serre [Ser89], it can be derived from the analytic behavior of partial Euler products attached to the irreducible representations of $\mathrm{ST}(A_0)$.

In section 5, using recent and deep potential automorphy results (covered by [ACC$^+$18] and [BLGGT14]) relative to the compatible system of ℓ-adic representations $(V_\lambda(B)^{\alpha_B})_\lambda$, we are able to prove the Sato–Tate conjecture for A_0 in certain cases. Let K_0/k_0 denote the minimal extension over which all the endomorphisms of A_0 are defined.

Theorem 1.2. *Suppose that k_0 is a totally real or CM field and that A_0 is an abelian variety defined over k_0 of dimension $g \geq 1$ which is $\overline{\mathbb{Q}}$-isogenous to the power of an abelian variety B which is either:*

i) an elliptic curve; or
ii) an abelian surface with QM; or
iii) an abelian surface with RM; or
iv) an abelian fourfold with QM.

If the field extension k/k_0 from Theorem 1.1 is trivial and K_0/k_0 is solvable, then the Sato–Tate conjecture holds for A_0.

In the above theorem, the condition of B falling in one of the cases $i), \dots, iv)$ amounts to requiring that the center M of the endomorphism algebra of B be a number field of degree $m \leq 2$. This constraint on the degree m ensures the applicability of results of Shahidi [Sha81] on the invertibility of the Rankin-Selberg product of automorphic L-functions, which are essential to the proof.

One can dispense with the hypothesis that K_0/k_0 be solvable when $g \leq 3$. For $g = 2$, the extension K_0/k_0 is in fact known to be always solvable (as a byproduct of the classification in [FKRS12]) and for $g = 3$ it can only fail to be solvable when B is an elliptic with CM (as follows from the upcoming work [FKS19]), in which case the theorem is known to hold as well (see Remark 5.2).

Some particular instances of Theorem 1.2 are known. Indeed, the works of Johansson [Joh17] and N. Taylor [Tay19] altogether imply the theorem when $g = 2$; in other words, when A_0 is $\overline{\mathbb{Q}}$-isogenous to the square of an elliptic curve, to an RM abelian surface, or to a QM abelian surface. Their proof is based on a case-by-case analysis using the classification of Sato–Tate groups of abelian surfaces defined over totally real number fields[1] (as achieved in [FKRS12]). Our proof of Theorem 1.2 is indebted to [Joh17] and [Tay19] in many aspects.

[1]Of the 35 possibilities for the Sato–Tate group of an abelian surface defined over a totally real field, 28 occur only among abelian surfaces which are geometrically isotypic and potentially of GL_2-type. It should be noted that the works of Johansson and N. Taylor yield the Sato–Tate conjecture in 5 non geometrically isotypic cases as well: indeed, they yield the Sato–Tate conjecture in all but 2 of the 35 possible cases.

A second situation where Theorem 1.2 was essentially known is when $g \leq 3$ and B is an elliptic curve with CM that admits a model up isogeny defined over k_0. Indeed, the computation of the moments of the measure governing the equidistribution of the normalized Frobenius traces of A_0 in this situation was carried out in [FS14, §3] (for $g = 2$) and in [FLS18, §2] (for $g = 3$).

Modular abelian varieties are a natural source of geometrically isotypic abelian varieties of GL_2-type defined over $\mathbb{Q}$. Restricted to this setting, the hypotheses on the extensions K_0/k_0 and k/k_0 are automatically satisfied and Theorem 1.2 can be presented in the following way.

Corollary 1.3. *Let $f = \sum a_m q^m \in S_2(\Gamma_1(N))$ be a newform of nebentype ε. Let A_f denote the abelian variety defined over $\mathbb{Q}$ associated to f by the Eichler–Shimura construction. If f is non-CM, suppose that the field $\mathbb{Q}(\{a_m^2/\varepsilon(m)\}_{(m,N)=1})$ has degree at most 2 over $\mathbb{Q}$. Then the Sato–Tate conjecture holds for A_f.*

Conventions and notations. Throughout the article k_0 is a number field and all of its algebraic field extensions are assumed to be contained in a fix algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. For each ℓ, we fix an algebraic closure of $\mathbb{Q}_\ell$ and all finite extensions of $\mathbb{Q}_\ell$ are assumed to be contained in this fixed algebraic closure. We work in the category of abelian varieties up to isogeny. In particular, isogenies become invertible and $\mathrm{Hom}(C, D)$, for a pair abelian varieties C and D defined over k_0, is equipped with a $\mathbb{Q}$-vector space structure. Given a field extension k/k_0, we write C_k to denote the base change $C \times_{k_0} k$ of C from k_0 to k. We refer to nonzero prime ideals of the ring of integers of a number field E simply by primes of E. We denote by I the identity matrix, whose size should always be clear from the context.

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2. A Tate module tensor decomposition

Throughout this section A_0 will denote a simple abelian variety of dimension $g \geq 1$ defined over the number field k_0 such that:

Hypothesis 2.1. *i)* A_0 *is* potentially of GL_2-type*, that is, there exists a number field L of degree $[L : \mathbb{Q}] = \dim(A_0)$ and an injection $L \hookrightarrow \mathrm{End}(A_{0,\overline{\mathbb{Q}}})$.*

ii) A_0 *is* geometrically isotypic*, that is, $A_{0,\overline{\mathbb{Q}}} \sim B^d$, where B is a simple abelian variety defined over $\overline{\mathbb{Q}}$ and $d \geq 1$.*

Remark 2.2. Let C denote the commutant of L in $\mathrm{End}(A_{0,\overline{\mathbb{Q}}})$, and set $t = [C : L]$. Then, either $t = 1$ (non-CM case); or $t = 2$ and C is a CM field of degree $2\dim(A_0)$

(CM case). We note that in the non-CM case, part ii) of the hypothesis is implied by part i) (see [Wu18, Prop. 1.5]).

Remark 2.3. We now make explicit some well-known consequences of Hypothesis 2.1. By the Wedderburn theorem, we know that $\mathrm{End}(B)$ is a central division algebra over a number field M. Let n denote the Schur index of $\mathrm{End}(B)$. Then:

i) If $t = 2$, then $n = 1$, M is a CM field, and $[M : \mathbb{Q}] = 2\dim(B)$.
ii) If $t = 1$, then $n = 1$ or 2, M is totally real, and $n[M : \mathbb{Q}] = \dim(B)$.

We will write $m = [M : \mathbb{Q}]$, so that in both of the above cases we have the equalities

$$t\dim(B) = nm = \frac{\dim_{\mathbb{Q}}\mathrm{End}(B)}{n}\,.$$

Let K_0/k_0 denote the endomorphism field of A_0, that is, the minimal extension of k_0 such that

$$\mathrm{End}(A_{0,K_0}) \simeq \mathrm{End}(A_{0,\overline{\mathbb{Q}}})\,.$$

It is well-known that K_0/k_0 is Galois and finite. Let K/k_0 denote a finite Galois extension containing K_0/k_0. Without loss of generality we may assume that B is defined over K and that

$$\mathrm{Hom}(B_K, A_{0,K}) \simeq \mathrm{Hom}(B_{\overline{\mathbb{Q}}}, A_{0,\overline{\mathbb{Q}}})\,.$$

Definition 2.4. *For a subextension k/k_0 of K/k_0, the abelian variety B is called a k-abelian variety (or k-variety for short) if for every $s \in \mathrm{Gal}(K/k)$ there exists an isogeny $\mu_s\colon {}^sB \to B$ compatible with the endomorphisms of B, that is, such that*

$$\mu_s \circ {}^s\varphi = \varphi \circ \mu_s \text{ for all } \varphi \in \mathrm{End}(B). \tag{2.1}$$

Let k/k_0 be a subextension of K/k_0 such that B is a k-variety (such subextensions obviously exist). From now on, write A for the base change $A_0 \times_{k_0} k$. Fix a system of isogenies $\{\mu_s\}_{s\in\mathrm{Gal}(K/k)}$ compatible with $\mathrm{End}(B)$ in the sense of (2.1). We can, and do, assume that μ_s is the identity for every $s \in G_K$. If we equip $M^\times$ with the trivial action of $\mathrm{Gal}(K/k)$, the map

$$c_B\colon \mathrm{Gal}(K/k) \times \mathrm{Gal}(K/k) \to M^\times\,, \qquad c_B(s,t) = \mu_{st} \circ {}^s\mu_t^{-1} \circ \mu_s^{-1}\,,$$

satisfies the 2-cocycle condition and defines a cohomology class $\gamma_B \in H^2(K/k, M^\times)$. The cocycle c_B (resp. the cohomology class γ_B) gives rise by inflation to a continuous cocycle in $Z^2(G_k, M^\times)$ (resp. a cohomology class in $H^2(G_k, M^\times)$) that we will also denote by c_B (resp. γ_B).

Lemma 2.5. *There is a continuous map*

$$\alpha_B\colon G_k \to \overline{\mathbb{Q}}^\times$$

such that for every $s, t \in G_k$ we have

$$c_B(s,t) = \frac{\alpha_B(s)\alpha_B(t)}{\alpha_B(st)}\,. \tag{2.2}$$

Proof. The lemma is a consequence of a theorem of Tate (see [Rib92, Thm. 6.3]), which states that $H^2(G_k, \overline{\mathbb{Q}}^\times)$ is trivial, where $\overline{\mathbb{Q}}^\times$ is endowed with the trivial action of G_k. □

Let E denote a maximal subfield contained in $\mathrm{End}(B)$. It is well-known that $[E:M]=n$. Since α_B is continuous we may enlarge K so that α_B is trivial when restricted to G_K, and we will do this from now on. Let F denote the compositum of E and the number field generated by the values of α_B.

Fix a rational prime ℓ and an embedding $\sigma\colon F\to\overline{\mathbb{Q}}_\ell$. Denote by $\lambda=\lambda(\sigma)$ the prime of F above ℓ for which σ factors via the natural inclusion of F into its completion F_λ at λ.

Let $V_\ell(A)$ (resp. $V_\ell(B)$) denote the rational Tate module of A (resp. B). For a prime λ of F above ℓ, use the natural E-module structure of $V_\ell(B)$ to define

$$V_\lambda(B)=V_\ell(B)\otimes_{E\otimes\mathbb{Q}_\ell,\sigma}\overline{\mathbb{Q}}_\ell\,. \tag{2.3}$$

Here the tensor product is taken with respect to the map induced by the inclusions $E\subseteq F$ and $\sigma\colon F\hookrightarrow\overline{\mathbb{Q}}_\ell$. The module $V_\lambda(B)$ has dimension

$$\dim_{\overline{\mathbb{Q}}_\ell}(V_\lambda(B))=\frac{2}{t}\,.$$

It is endowed with an action of G_k by the next lemma.

Lemma 2.6. *The map*

$$\varrho_\lambda^{\alpha_B}\colon G_k\to\mathrm{GL}(V_\lambda(B))\,,$$

defined for $s\in G_k$ as the composition

$$\varrho_\lambda^{\alpha_B}(s)\colon V_\lambda(B)\xrightarrow{s\otimes 1}V_\lambda({}^sB)\xrightarrow{\mu_{s,*}}V_\lambda(B)\xrightarrow{1\otimes\sigma(\alpha_B(s))}V_\lambda(B)\,, \tag{2.4}$$

is a continuous representation. Here, $\mu_{s,*}$ denotes the isomorphism of Tate modules induced by the isogeny $\mu_s\colon{}^sB\to B$.

Proof. We first check that the action is indeed $\overline{\mathbb{Q}}_\ell$-linear. This amonunts to note that, in virtue of (2.1), for every $s\in G_k$, $v\in V_\ell(B)$, and $\varphi\in E$ we have

$$\varrho_\lambda^{\alpha_B}(\varphi(v)\otimes 1)=\mu_{s,*}{}^s\varphi_*({}^sv)\otimes\sigma(\alpha_B(s))=\varphi_*\mu_{s,*}({}^sv)\otimes\sigma(\alpha_B(s))=\varphi\varrho_\lambda^{\alpha_B}(v\otimes 1)\,.$$

The proof is then a straightforward computation based on (2.2). Indeed, for every $s,t\in G_k$ and $v\in V_\ell(B)$ we have:

$$\begin{aligned}\varrho_\lambda^{\alpha_B}(st)(v\otimes 1)&=\mu_{st,*}({}^{st}v)\otimes\sigma(\alpha_B(st))\\&=\mu_{s,*}{}^s\mu_{t,*}({}^{st}v)\cdot c_B(s,t)\otimes\sigma(\alpha_B(st))\\&=\mu_{s,*}{}^s\mu_{t,*}({}^{st}v)\otimes\sigma(\alpha_B(s)\alpha_B(t))\\&=\varrho_\lambda^{\alpha_B}(s)(\varrho_\lambda^{\alpha_B}(t)(v\otimes 1))\,.\end{aligned}$$

Since it suffices to verify continuity in a neighborhood of the identity, we are reduced to show that $\varrho_\lambda^{\alpha_B}|_{G_K}$ is continuous. But note that the action of G_K via $\varrho_\lambda^{\alpha_B}$ coincides with the natural action of G_K on $V_\lambda(B)$, which is continuous. □

The map

$$E\to\mathrm{Hom}(B_K,A_K)\,,$$

given by precomposition of maps, equips $\mathrm{Hom}(B_K,A_K)$ with an E-module structure, which we use to define

$$V(B,A)=\mathrm{Hom}(B_K,A_K)\otimes_E F\,.$$

Observe that $V(B,A)$ has dimension

$$\dim_F(V(B,A))=d\frac{\dim_{\mathbb{Q}}\mathrm{End}(B)}{[E:\mathbb{Q}]}=nd\,.$$

We next equip $V(B,A)$ with an action of $\mathrm{Gal}(K/k)$ by means of the following lemma (compare with [FG19, Lemma 2.15]).

Lemma 2.7. *The map*

$$\theta^{\alpha_B}\colon \mathrm{Gal}(K/k) \to \mathrm{GL}(V(B,A))$$

defined for $s \in \mathrm{Gal}(K/k)$ as the composition

$$\theta^{\alpha_B}(s)\colon V(B,A) \xrightarrow{s\otimes 1} V({}^sB,A) \xrightarrow{(\mu_s^{-1})^*} V(B,A) \xrightarrow{1\otimes\sigma(\alpha_B(s)^{-1})} V(B,A) \tag{2.5}$$

is a representation. Here, $(\mu_s^{-1})^*$ is the map obtained by precomposition with μ_s^{-1}.

Proof. We first verify that the action is F-linear. For every $s \in \mathrm{Gal}(K/k)$, $\psi \in \mathrm{Hom}(B_K, A_K)$, and $\varphi \in E$ we have

$$\begin{aligned} \theta^{\alpha_B}(\psi\circ\varphi\otimes 1) &= {}^s\psi\circ{}^s\varphi\circ\mu_s^{-1}\otimes\sigma(\alpha_B(s)^{-1}) \\ &= {}^s\psi\circ\mu_s^{-1}\circ\varphi\otimes\sigma(\alpha_B(s)^{-1}) \\ &= \theta^{\alpha_B}(\psi\otimes 1)\circ\varphi\,. \end{aligned}$$

For every $s,t\in G_k$ and $v \in \mathrm{Hom}(B_K, A_K)$, we have that

$$\begin{aligned} \theta^{c_B}(st)(\psi\otimes 1) &= (\mu_{st}^{-1})^*({}^{st}v)\otimes\sigma(\alpha_B(st)^{-1}) \\ &= ({}^s\mu_t^{-1})^*(\mu_s^{-1})^*({}^{st}v)\cdot c_B(s,t)^{-1}\otimes\sigma(\alpha_B(st)^{-1}) \\ &= ({}^s\mu_t^{-1})^*(\mu_s^{-1})^*({}^{st}v)\otimes\sigma(\alpha_B(s)^{-1}\alpha_B(t)^{-1}) \\ &= \theta^{\alpha_B}(s)(\theta^{\alpha_B}(t)(\psi\otimes 1))\,. \end{aligned}$$

□

For a prime λ of F above ℓ, attached to the embedding $\sigma\colon F \hookrightarrow \overline{\mathbb{Q}}_\ell$, define

$$V_\lambda(B,A) = V(B,A)\otimes_{F,\sigma}\overline{\mathbb{Q}}_\ell\,.$$

Let $\theta_\lambda^{\alpha_B}$ denote the representation on $V_\lambda(B,A)$ obtained by letting $\mathrm{Gal}(K/k)$ act trivially on $\overline{\mathbb{Q}}_\ell$ and via θ^{α_B} on $V(B,A)$. Let us write $V_\lambda(B)^{\alpha_B}$ and $V_\lambda(B,A)^{\alpha_B}$ to denote $V_\lambda(B)$ and $V_\lambda(B,A)$ equipped with actions of G_k via $\varrho_\lambda^{\alpha_B}$ and $\theta_\lambda^{\alpha_B}$, respectively. Define

$$V_\lambda(A) := V_\ell(A)\otimes_{\mathbb{Q}_\ell\otimes M,\sigma}\overline{\mathbb{Q}}_\ell\,,$$

where the tensor product is taken with respect to the map obtained from the iclusions $M \subseteq F$ and $\sigma\colon F\hookrightarrow\overline{\mathbb{Q}}_\ell$. We regard $V_\lambda(A)$ as a $\overline{\mathbb{Q}}_\ell[G_k]$-module by letting G_k act naturally on $V_\ell(A)$ and trivially on $\overline{\mathbb{Q}}_\ell$ (this is well defined by [FG18, Prop. 2.6], for example). Observe that $V_\lambda(A)$ has dimension

$$\dim_{\overline{\mathbb{Q}}_\ell}(V_\lambda(A)) = \frac{2g}{m} = \frac{2nd}{t}$$

Proposition 2.8. *There is an isomorphism of $\overline{\mathbb{Q}}_\ell[G_k]$-modules*

$$V_\lambda(A) \simeq V_\lambda(B)^{\alpha_B}\otimes_{\overline{\mathbb{Q}}_\ell}V_\lambda(B,A)^{\alpha_B}\,. \tag{2.6}$$

Proof. Let us first assume that $V_\lambda(A)$ is an irreducible $\overline{\mathbb{Q}}_\ell[G_k]$-module. Since both $V_\lambda(A)$ and $V_\lambda(B)\otimes V_\lambda(B,A)$ have the same dimension over $\overline{\mathbb{Q}}_\ell$, it will suffice to show that

$$W := \mathrm{Hom}_{G_k}(V_\lambda(A), V_\lambda(B)^{\alpha_B}\otimes V_\lambda(B,A)^{\alpha_B}) \neq 0.$$

Observe that

$$\begin{aligned} W &= \mathrm{Hom}_{G_k}(V_\lambda(A)\otimes (V_\lambda(B)^{\alpha_B})^\vee, V_\lambda(B,A)^{\alpha_B}) \\ &= \mathrm{Hom}_{G_k}(\mathrm{Hom}_{G_k}(V_\lambda(B)^{\alpha_B}, V_\lambda(A)), V_\lambda(B,A)^{\alpha_B})\,. \end{aligned}$$

Thus, to show that $W\neq 0$, it is enough to show that the map

$$\Phi\colon V_\lambda(B,A)^{\alpha_B}\to \mathrm{Hom}_{G_k}(V_\lambda(B)^{\alpha_B}, V_\lambda(A))\,,\qquad \Phi(f):=f_*$$

is G_k-equivariant. But this indeed holds:

$$\begin{aligned} \Phi(\theta_\lambda^{\alpha_B}(s)(f)) &= ({}^s f_*\circ \mu_{s,*}^{-1})\otimes\sigma(\alpha(s)^{-1}) \\ &= {}^s(f_*\circ s^{-1}\circ\mu_{s,*}^{-1})\otimes\sigma(\alpha(s)^{-1}) \\ &= {}^s(f_*\circ \mu_{s^{-1},*})\otimes\sigma(c_B(s,s^{-1})\cdot\alpha(s)^{-1}) \\ &= {}^s(\Phi(f)\circ \varrho_\lambda^{\alpha_B}(s)^{-1})\,, \end{aligned}$$

where we have used that $c_B(s,s^{-1})=\mu_s{}^s\mu_{s^{-1}}$ and $c(s,s^{-1})=\alpha(s)\alpha(s^{-1})$. To conclude, note that if $V_\lambda(A)$ decomposes, then $V_\lambda(B,A)$ does it accordingly, and we can apply the above argument to each of the respective irreducible constituents. □

Note that the proposition implies, in particular, that $V_\lambda(B)^{\alpha_B}\otimes_{\overline{\mathbb{Q}}_\ell} V_\lambda(B,A)^{\alpha_B}$ only depends on the restriction of $\sigma\colon F\to\overline{\mathbb{Q}}_\ell$ to M. Let $\sigma_i\colon F\to\overline{\mathbb{Q}}_\ell$, for $i=1,\dots,m$, denote extensions to F of the distinct embeddings of M into $\overline{\mathbb{Q}}_\ell$. Let λ_i denote the prime of F attached to σ_i.

Proposition 2.9. *There is an isomorphism of $\overline{\mathbb{Q}}_\ell[G_k]$-modules*

$$V_\ell(A)\otimes_{\mathbb{Q}_\ell}\overline{\mathbb{Q}}_\ell \simeq \bigoplus_{i=1}^m V_{\lambda_i}(B)^{\alpha_B}\otimes_{\overline{\mathbb{Q}}_\ell} V_{\lambda_i}(B,A)^{\alpha_B}\,.$$

Proof. This follows from the well-known isomorphism

$$V_\ell(A)\otimes_{\mathbb{Q}_\ell}\overline{\mathbb{Q}}_\ell \simeq \bigoplus_{i=1}^m V_{\lambda_i}(A)\,, \tag{2.7}$$

together with Proposition 2.8. □

Proposition 2.10. *For $i\neq j$, we have that*

$$V_{\lambda_i}(B)^{\alpha_B}\not\simeq V_{\lambda_j}(B)^{\alpha_B}$$

as $\overline{\mathbb{Q}}_\ell[G_{K'}]$-modules for any finite extension K'/k.

Proof. Without loss of generality we may assume that $K\subseteq K'$. On the one hand, we then have

$$\mathrm{End}_{G_{K'}}(V_\ell(A)\otimes\overline{\mathbb{Q}}_\ell)\simeq \mathrm{M}_{nd}(\mathrm{End}(\bigoplus_i V_{\lambda_i}(B)^{\alpha_B}))\,.$$

On the other hand, we have

$$\mathrm{End}(A_{K'})\otimes\overline{\mathbb{Q}}_\ell\simeq \mathrm{M}_d(\mathrm{End}(B)\otimes\overline{\mathbb{Q}}_\ell)\simeq \mathrm{M}_{nd}(M\otimes\overline{\mathbb{Q}}_\ell)\,.$$

By Faltings isogeny theorem [Fal83], we have that $\dim_{\overline{\mathbb{Q}}_\ell}(\mathrm{End}(\oplus_i V_{\lambda_i}(B)^{\alpha_B}))=m$, and the proposition follows. □

So far, the subextension k/k_0 of K/k_0 has only been subject to the constraint that B be a k-variety. We now make a specific choice of k/k_0 that allows for a particularly nice description of the Tate module of A_0 in terms of that of $A = A_0 \times_{k_0} k$.

Theorem 2.11. *Let A_0 be an abelian variety defined over k_0 satisfying Hypothesis 2.1. Let $M_0 = M \cap \mathrm{End}(A_0)$. Then M/M_0 is Galois and there exists a Galois subextension k/k_0 of K_0/k_0 of degree $[M : M_0]$ such that for $A = A_0 \times_{k_0} k$ the following properties hold:*

i) $M \subseteq \mathrm{End}(A)$.
ii) B *is a* k-*variety.*
iii) *For every rational prime* ℓ, *we have*

$$V_\ell(A_0) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell \simeq \bigoplus_\lambda \mathrm{Ind}_{k_0}^k \left(V_\lambda(B)^{\alpha_B} \otimes_{\overline{\mathbb{Q}}_\ell} V_\lambda(A,B)^{\alpha_B} \right),$$

where the sum runs over the primes $\lambda = \lambda(\sigma)$ of F lying over ℓ attached to extensions $\sigma \colon F \to \overline{\mathbb{Q}}_\ell$ of the $[M_0 : \mathbb{Q}]$ distinct embeddings of M_0 into $\overline{\mathbb{Q}}_\ell$.

Proof. The existence of a Galois subextension k/k_0 of K_0/k_0 of degree $[M : M_0]$ such that $M \subseteq \mathrm{End}(A)$ and

$$\bigoplus_{i=1}^{[M:M_0]} V_\ell(A_0) \simeq \mathrm{Ind}_{k_0}^k (V_\ell(A))$$

is [Mil72, Rem. 2, p. 186]. As it is seen in the proof, there is an injection $\mathrm{Gal}(k/k_0) \hookrightarrow \mathrm{Aut}_{M_0}(M)$, which ensures that M/M_0 is Galois. The fact that $M \subseteq \mathrm{End}(A)$ implies that for every $s \in \mathrm{Gal}(K/k)$ we can fix an M-equivariant isogeny $\mu_s \colon {}^sB \to B$ coming from the M-equivariant isogeny

$${}^sB^d \sim {}^sA_K = A_K \sim B^d .$$

The M-equivariant system of isogenies $\{\mu_s\}_{s \in G_k}$ can be modified into a $\mathrm{End}(B)$-equivariant system $\{\lambda_s\}_{s \in G_k}$, so that B becomes a k-variety. Indeed, consider the M-algebra isomorphism

$$\mathrm{End}(B) \to \mathrm{End}(B), \qquad \varphi \mapsto \mu_s \circ {}^s\varphi \circ \mu_s^{-1} .$$

The Skolem–Noether theorem shows the existence of an element $\psi \in \mathrm{End}(B)^\times$ such that $\mu_s \circ {}^s\varphi \circ \mu_s^{-1} = \psi \circ \varphi \circ \psi^{-1}$. Then define $\lambda_s = \psi^{-1} \circ \mu_s$.

We can then apply Proposition 2.9 to $V_\ell(A)$. The result follows from the fact that

$$\mathrm{Ind}_{k_0}^k (V_\lambda(A)^{\alpha_B} \otimes V_\lambda(B,A)^{\alpha_B}) \simeq \mathrm{Ind}_{k_0}^k (V_{\lambda'}(A)^{\alpha_B} \otimes V_{\lambda'}(B,A)^{\alpha_B})$$

if $\lambda = \lambda(\sigma)$, $\lambda' = \lambda'(\sigma')$, and σ and σ' coincide on M_0. Indeed, for $s \in G_k$ we have

$$\mathrm{Tr}\,\mathrm{Ind}_{k_0}^k (V_\lambda(A))(s) = \mathrm{Tr}_{\mathbb{Q}_\ell \otimes \sigma(M)/\mathbb{Q}_\ell \otimes \sigma(M_0)} \mathrm{Tr}(V_\lambda(A))(s) .$$

□

Remark 2.12. We will be later interested in the case that k_0 is totally real. Note that if $[M : M_0]$ is odd, then the injection $\mathrm{Gal}(k/k_0) \hookrightarrow \mathrm{Aut}_{M_0}(M)$ forces k to be totally real as well. In the case that $k_0 = \mathbb{Q}$ and $\mathrm{Aut}_{M_0}(M)$ has a single element of order 2, then k is either totally real or CM (this follows from the fact that all complex conjugations of $\mathrm{Gal}(k/\mathbb{Q})$ are conjugate).

Remark 2.13. Let us review a particular case of Proposition 2.8 implicit in [FG18]. Suppose that A is $\overline{\mathbb{Q}}$-isogenous to the g-th power of a non-CM elliptic curve B and that g is odd. Then, by [FG18, Theorem 2.21], the cohomology class γ_B of c_B in $H^2(G_k, \mathbb{Q}^\times)$ is trivial. By Weil's descend Criterion, if γ_B is trivial, then B admits a model B^* up to isogeny defined over k. If L^*/k denotes the minimal extension such that $\mathrm{Hom}(B_{L^*}, A_{L^*}) \simeq \mathrm{Hom}(B_{\overline{\mathbb{Q}}}, A_{\overline{\mathbb{Q}}})$, then by [Fit13, Thm. 3.1] one has that

$$V_\ell(A) \simeq V_\ell(B^*) \otimes_{\mathbb{Q}_\ell} \mathrm{Hom}(B^*_{L^*}, A_{L^*}),$$

which may be regarded as a particular instance of Proposition 2.8.

3. The weakly compatible system $V_\lambda(B)$

Let A_0 be an abelian variety defined over k_0 satisfying Hypothesis 2.1. In this section we assume further that A_0 is non-CM. Let k/k_0 be as in Theorem 2.11 and write $A = A_0 \times_{k_0} k$. Resume also the notations B, α_B, and F of Section 2.

The goal of this section is to present $\mathcal{R} = (V_\lambda(B))_\lambda$ as a rank 2 weakly compatible system of ℓ-adic representations of G_k defined over F (see [BLGGT14, §5.1] for the definition of weakly compatible system of ℓ-adic representations). This will rely on classical work of Ribet and on the following result of Wu (we note that Wu's result extends work of Ellenberg and Skinner [ES01, Prop. 2.10], who considered the $\dim(B) = 1$ case).

Proposition 3.1 (Cor. 2.1.15, Prop. 2.2.1, [Wu11])**.** *There exists a* GL_2*-type abelian variety* A^{α_B} *defined over* k *satisfying:*

i) $\dim(A^{\alpha_B}) = [F:\mathbb{Q}]$ *and there exists an inclusion* $F \hookrightarrow \mathrm{End}(A^{\alpha_B})$.
ii) There is an isomorphism of $\overline{\mathbb{Q}}_\ell[G_k]$*-modules*

$$V_\lambda(A^{\alpha_B}) \simeq V_\lambda(B)^{\alpha_B},$$

where $V_\lambda(A^{\alpha_B})$ *is the tensor product* $V_\ell(A^{\alpha_B}) \otimes_{\mathbb{Q}_\ell \otimes F, \sigma} \overline{\mathbb{Q}}_\ell$ *taken with respect to the map induced by the embedding* $\sigma\colon F \hookrightarrow \overline{\mathbb{Q}}_\ell$ *attached to the prime* λ.

Proposition 3.2 (Ribet)**.** $\mathcal{R} = (V_\lambda(B)^{\alpha_B})_\lambda$ *is a weakly compatible system of* ℓ*-adic representations of* G_k *defined over* F*, of rank* 2*, and satisfying:*

i) It is pure of weight 1, regular, and with Hodge–Tate weights 0 *and* 1.
ii) Its determinant $\delta_\lambda := \det(V_\lambda(B)^{\alpha_B})$ *is of the form* $\varepsilon_\lambda\chi_\ell$*, where*

$$\varepsilon_\lambda\colon G_k \xrightarrow{\varepsilon} F^\times \overset{\sigma}{\hookrightarrow} \overline{\mathbb{Q}}_\ell^\times$$

is a finite order character and $\chi_\ell\colon G_k \to \mathbb{Q}_\ell^\times$ *is the* ℓ*-adic cyclotomic character.*
iii) It is strongly irreducible and $\mathrm{End}_{G_{K'}}(V_\lambda(B)^{\alpha_B}) \simeq \overline{\mathbb{Q}}_\ell$*, for every finite extension* K'/k.

If k *is totally real, then* $\mathcal{R}$ *is totally odd, in the sense that* $\delta_\lambda(\tau) = -1$ *for every complex conjugation* $\tau \in G_k$.

Proof. By Proposition 3.1, it suffices to prove the corresponding statements for $(V_\lambda(A^{\alpha_B}))_\lambda$. When $k = \mathbb{Q}$, this can be found in the work of Ribet: the Hodge–Tate property and the description of the determinant follow from [Rib92, lem. 3.1], the totally oddness is [Rib92, lem. 3.2], and strong irreducibility amounts to [Rib92, lem. 3.3]. See [Wu11, §2.2] or [Pyl02, §5] for the general statements. ☐

Remark 3.3. We may also regard $(V_\lambda(B,A)^{\alpha_B})_\lambda$ as a compatible system of ℓ-adic representations defined over F. Note that its tensor product $(V_\lambda(A))_\lambda$ with $(V_\lambda(B)^{\alpha_B})_\lambda$ is in fact defined over M. This comes from the fact that $\alpha_B(s)$ appears with inverse exponents in the respective rightmost arrows of (2.4) and (2.5).

We will later make use of the following deep result (which combines [BLGGT14, Thm. 5.4.1] -in the totally real case- and [ACC+18, Thm. 7.1.10] -in the CM case-).

Theorem 3.4. *Suppose that k is a totally real (resp. a CM) field. Then given natural numbers $e_1,\dots,e_r \geq 0$ and a finite extension k^*/k, there exists a totally real (resp. CM) extension k'/k such that:*

i) $\mathrm{Symm}^{e_1}(\mathcal{R}|_{G_{k'}}),\dots,\mathrm{Symm}^{e_r}(\mathcal{R}|_{G_{k'}})$ *are all automorphic;*
ii) k'/k is linearly disjoint from k^* over k; and
iii) $k'/\mathbb{Q}$ is Galois.

4. Sato-Tate groups and Sato–Tate conjecture

Let A_0 be an abelian variety defined over k_0 satisfying Hypothesis 2.1. In this section we assume further that A_0 is non-CM. Let k/k_0 be as in Theorem 2.11 and write $A = A_0 \times_{k_0} k$. Resume also the notations B, α_B, M, M_0, and F of Section 2.

The aim of this section is to describe the Sato–Tate groups of A_0 and A, denoted $\mathrm{ST}(A_0)$ and $\mathrm{ST}(A)$, respectively. We will describe $\mathrm{ST}(A)$ as the Kronecker product of $m = [M:\mathbb{Q}]$ copies of $\mathrm{SU}(2)$ and a finite group H closely related to the image of $\theta_\lambda^{\alpha_B}$. As for $\mathrm{ST}(A_0)$, we will show that it is isomorphic to the semidirect product of the Galois group of the endomorphism field K_0/k_0 by the product of $[M_0:\mathbb{Q}]$ copies of $\mathrm{SU}(2)$. We also state the Sato–Tate conjecture for A_0.

Sato–Tate groups. Let us start by briefly recalling the definition of $\mathrm{ST}(A)$, a compact real Lie subgroup of $\mathrm{USp}(2g)$, only well-defined up to conjugacy (by replacing k with k_0 and A with A_0 in what follows, one obtains the definition of $\mathrm{ST}(A_0)$). Let $G_\ell^{\mathrm{Zar}}(A)$ denote the Zariski closure of the image of the ℓ-adic representation

$$\varrho_\ell\colon G_k \to \mathrm{GL}(V_\ell(A))$$

attached to A. The compatibility of ϱ_ℓ with the Weil pairing, ensures that $G_\ell^{\mathrm{Zar}}(A)$ sits inside $\mathrm{GSp}_{2g}/\mathbb{Q}_\ell$. Let $G_\ell^{\mathrm{Zar},1}(A)$ denote the kernel of the restriction to $G_\ell^{\mathrm{Zar}}(A)$ of the similitude character of GSp_{2g}. Fix an embedding $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ and denote by $G_\iota^1(A)$ the group of $\mathbb{C}$-points of the base change of $G_\ell^{\mathrm{Zar},1}(A)$ from $\mathbb{Q}_\ell$ to $\mathbb{C}$ via ι. The Sato–Tate group $\mathrm{ST}(A)$ is defined as a maximal compact subgroup of $G_\iota^1(A)$ (we refer to [FKRS12, §2.1] for more details).

We next define in a similar way a Sato–Tate group for the system $(V_\lambda(B)^{\alpha_B})_\lambda$ along the lines [BLGG11, §7]. We will formally denote it by $\mathrm{ST}(B^{\alpha_B})$ in order to emphasize that it is not the Sato–Tate group of the abelian variety B defined over K. Let $G_\lambda^{\mathrm{Zar}}(B^{\alpha_B})$ denote the Zariski closure of the image of

$$\varrho_\lambda^{\alpha_B}\colon G_k \to \mathrm{GL}(V_\lambda(B))\,.$$

It is an algebraic group over $\overline{\mathbb{Q}}_\ell$. Let a denote the order of the character ε introduced in Section 3. Let $G_\lambda^{\mathrm{Zar},1}(B^{\alpha_B})$ denote the kernel of the map

$$\det{}^a\colon G_\lambda^{\mathrm{Zar}}(B^{\alpha_B}) \to \mathbb{G}_m\,.$$

Denote by $G^1_\iota(B^{\alpha_B})$ the group of $\mathbb{C}$-points of the base change of $G^{\mathrm{Zar},1}_\lambda(B^{\alpha_B})$ from $\overline{\mathbb{Q}}_\ell$ to $\mathbb{C}$ via ι. The Sato–Tate group $\mathrm{ST}(B^{\alpha_B})$ is defined as a maximal compact subgroup of $G^1_\iota(B^{\alpha_B})$.

Lemma 4.1. *We have* $\mathrm{ST}(B^{\alpha_B}) \simeq \mathrm{SU}(2) \otimes \mu_{2a}$.

Proof. By the definition of $\mathrm{ST}(B^{\alpha_B})$, we clearly have a monomorphism

$$\mathrm{ST}(B^{\alpha_B}) \hookrightarrow \mathrm{U}(2)_a\,,$$

where $\mathrm{U}(2)_a$ is the subgroup of $\mathrm{U}(2)$ consisting of those matrices $g \in \mathrm{U}(2)$ with $\det(g)^a = 1$. We may compose this monomorphism with the inverse of the group isomorphism

$$\mathrm{SU}(2) \otimes \mu_{2a} \to \mathrm{U}(2)_a\,, \qquad A \otimes \zeta \mapsto A\zeta$$

to get a monomorphism φ. Since $-I$ lies in the image of φ it will suffice to show that the induced monomorphism

$$\tilde\varphi\colon \mathrm{ST}(B^{\alpha_B})/\langle -I\rangle \to \mathrm{SU}(2) \otimes \mu_{2a}/\langle -I \otimes 1\rangle$$

is surjective. Consider now the monomorphism

$$\mathrm{ST}(B^{\alpha_B})/\langle -I\rangle \xrightarrow{\tilde\varphi} \mathrm{SU}(2) \otimes \mu_{2a}/\langle -I \otimes 1\rangle \xrightarrow{\pi_1\times\pi_2} \mathrm{SU}(2)/\langle -I\rangle \times \mu_{2a}/\langle -1\rangle\,,$$

where π_i denotes the natural projection map. Let N_i denote the kernel of $\pi_i \circ \tilde\varphi$. By part *iii)* (resp. part *ii)*) of Proposition 3.2, we have that that $\pi_1 \circ \tilde\varphi$ (resp. $\pi_2 \circ \tilde\varphi$) is surjective. Then by Goursat's lemma (as in [Rib76, lem. 5.2.1]) we have that

$$\mathrm{SU}(2)/\tilde\varphi(N_2) \simeq \mu_{2a}/\tilde\varphi(N_1)\,.$$

Since $\mathrm{SU}(2)$ has no proper normal subgroups of finite index, we deduce that $\tilde\varphi(N_2) \simeq \mathrm{SU}(2)/\langle -I\rangle$. This immediately implies that $\tilde\varphi$ is surjective. □

Consider the well-defined group homomorphism

$$\tilde\varepsilon^{1/2}\colon G_k \to F^\times/\langle -1\rangle\,, \qquad \tilde\varepsilon^{1/2}(s) = \sqrt{\varepsilon(s)} \pmod{\langle -1\rangle}\,,$$

where ε is the character appearing in Proposition 3.2. We will denote by k_ε/k the field extension cut out by the homomorphism $\tilde\varepsilon^{1/2}$, which coincides with the field cut out by ε.

Proposition 4.2. *The following field extensions coincide:*

i) The endomorphism field K_0/k_0.

ii) The field extension cut out by the representation $\theta^{\alpha_B}_\lambda \otimes \theta^{\alpha_B,\vee}_\lambda$.

iii) The field extension cut out by the group homomorphism

$$\tilde\varepsilon^{1/2} \otimes \tilde\theta^{\alpha_B}_\lambda\colon G_k \to \mathrm{Aut}(V_\lambda(B,A)/\langle -I\rangle)\,.$$

Proof. By Faltings isogeny theorem, as in the proof of Proposition 2.10, we have that K_0/k is the minimal extension of k such that

$$\mathrm{End}_{G_{K_0}}(V_\lambda(A) \otimes \overline{\mathbb{Q}}_\ell) \simeq \mathrm{M}_{nd}(\overline{\mathbb{Q}}_\ell)\,.$$

Let K'/k be an arbitrary finite extension. By Proposition 2.8, we have

$$\begin{aligned}
\mathrm{End}_{G_{K'}}(V_\lambda(A)) &\simeq \mathrm{End}_{G_{K'}}(V_\lambda(B)^{\alpha_B} \otimes V_\lambda(B,A)^{\alpha_B})\\
&\simeq \mathrm{Hom}_{G_{K'}}(V_\lambda(B)^{\alpha_B} \otimes V_\lambda(B)^{\alpha_B,\vee}, V_\lambda(B,A)^{\alpha_B} \otimes V_\lambda(B,A)^{\alpha_B,\vee})\\
&\simeq (V_\lambda(B,A)^{\alpha_B} \otimes V_\lambda(B,A)^{\alpha_B,\vee})^{G_{K'}}\,,
\end{aligned}$$

where in the last isomorphism we have used that $\mathrm{End}_{G_{K'}}(V_\lambda(B)^{\alpha_B}) \simeq \overline{\mathbb{Q}}_\ell$, as stated in part *ii)* of Proposition 3.2. This shows that the field extensions of *i)* and *ii)* coincide. In fact, we could have alternatively shown the equivalence between *i)* and *ii)*, by establishing the isomorphism

$$V_\lambda(B,A)^{\alpha_B} \otimes V_\lambda(B,A)^{\alpha_B,\vee} \simeq \mathrm{End}(A_{K_0}) \otimes_{M,\sigma} \overline{\mathbb{Q}}_\ell$$

of $\overline{\mathbb{Q}}_\ell[G_k]$-modules (in the same lines as in the proof of Proposition 2.8).

Let L denote the field extension cut out by $\tilde{\varepsilon}^{1/2} \otimes \tilde{\theta}_\lambda^{\alpha_B}$. We first show that $K_0 \subseteq L$. Indeed, for every $s \in G_L$, we have that $\theta_\lambda^{\alpha_B}(s)$ is a scalar diagonal matrix. Thus $\theta_\lambda^{\alpha_B} \otimes \theta_\lambda^{\alpha_B,\vee}(s)$ is trivial, and then by *ii)* we deduce that $s \in G_{K_0}$.

We will give two different proves of the fact that $L \subseteq K_0$. For $s \in G_k$, let $d(\mu_s)$ denote the "degree" of μ_s as defined on [Pyl02, p. 223]. As shown in [Pyl02, Thm. 5.12], for $s \in G_k$, we have that[2]

$$\varepsilon(s) = \frac{\alpha_B(s)^2}{d(s)}\,.$$

Let now $\varphi \in V_\lambda(B,A)$ and $s \in G_{K_0}$. Since μ_s is the identity and $d(s) = 1$, we find that

$$\tilde{\varepsilon}^{1/2} \otimes \tilde{\theta}_\lambda^{\alpha_B}(s)(\varphi) = \alpha_B(s) \cdot {}^s\varphi \circ \mu_s^{-1} \otimes \alpha_B(s)^{-1} = \varphi\,,$$

which gives the first proof of the fact that $G_{K_0} \subseteq G_L$.

As for the second proof, let $s \in G_{K_0}$ so that $\theta_\lambda^{\alpha_B}(s)$ is a scalar matrix. We claim that $\tilde{\theta}_\lambda^{\alpha_B}(s)$ and $\tilde{\varepsilon}^{-1/2}(s)$ coincide as elements in $F^\times/\langle -1\rangle$.

By the Chebotarev density theorem it is enough to show the claim when s is of the form $\mathrm{Frob}_\mathfrak{p}$, for some prime $\mathfrak{p}$ of k of good reduction for A. To shorten notation let us write

$$a_\mathfrak{p} = \mathrm{Tr}(V_\lambda(B)^{\alpha_B}(\mathrm{Frob}_\mathfrak{p})),\ b_\mathfrak{p} = \mathrm{Tr}(V_\lambda(B,A)^{\alpha_B}(\mathrm{Frob}_\mathfrak{p})),\ c_\mathfrak{p} = \mathrm{Tr}(V_\lambda(A))(\mathrm{Frob}_\mathfrak{p})\,.$$

To prove the claim we may even restrict to primes $\mathfrak{p}$ for which $a_\mathfrak{p}$ is nonzero, since the density of those for which $a_\mathfrak{p} = 0$ is zero (this may be seen by applying the argument of [Ser89, Ex. 2, p. IV-13] to $V_\lambda(B)^{\alpha_B}$). Recall that by [Rib92, Thm. 5.3] (see also [Wu11, Prop. 2.2.14]), we have that

$$\frac{a_\mathfrak{p}^2}{\varepsilon_\mathfrak{p}} = a_\mathfrak{p}\overline{a}_\mathfrak{p} \in M\,, \tag{4.1}$$

where $\varepsilon_\mathfrak{p} := \varepsilon(\mathrm{Frob}_\mathfrak{p})$ and $\bar{\cdot}$ denotes the "complex conjugation" in F. By Theorem 2.11, we have that $a_\mathfrak{p} b_\mathfrak{p} = c_\mathfrak{p} \in M$. From this and (4.1), we see that

$$b_\mathfrak{p}^2 \varepsilon_\mathfrak{p} = \frac{c_\mathfrak{p}^2 \varepsilon_\mathfrak{p}}{a_\mathfrak{p}^2} = \frac{c_\mathfrak{p}^2}{a_\mathfrak{p}\overline{a}_\mathfrak{p}}$$

is a totally positve element of the totally real field M. The assumption that $\mathrm{Frob}_\mathfrak{p} \in G_{K_0}$ implies that $b_\mathfrak{p} = nd\zeta_\mathfrak{p}$ for some root of unity $\zeta_\mathfrak{p}$. We deduce that $\zeta_\mathfrak{p}^2\varepsilon_\mathfrak{p} = 1$. This shows that $\tilde{\theta}_{\lambda_i}^{\alpha_B}(\mathrm{Frob}_\mathfrak{p})$ and $\tilde{\varepsilon}^{-1/2}(\mathrm{Frob}_\mathfrak{p})$ coincide as elements in $F^\times/\langle -1\rangle$ and the second proof of the inclusion $L \subseteq K_0$ is complete. □

Proposition 4.3. *The field cut out by the representation*

$$\theta_\lambda^{\alpha_B} \colon G_k \to \mathrm{Aut}(V_\lambda(B,A))$$

is an extension of degree at most 2 of $k_\varepsilon K_0/k$.

[2]Beware that c_B is the inverse of the 2-cocycle chosen by Pyle.

Proof. Let us denote by

$$\tilde{\theta}_\lambda^{\alpha_B}\colon G_k \to \operatorname{Aut}(V_\lambda(B,A)/\langle -I\rangle)$$

the group homomorphism naturally induced by $\theta_\lambda^{\alpha_B}$. It will suffice to show that the field extension L'/k cut out by $\tilde{\theta}_\lambda^{\alpha_B}$ is $k_\varepsilon K_0$. But if we let $\tilde{\varepsilon}^{-1/2}$ stand for the inverse of $\tilde{\varepsilon}^{1/2}$, we have that

$$\tilde{\theta}_\lambda^{\alpha_B} \simeq \tilde{\varepsilon}^{-1/2} \otimes (\tilde{\varepsilon}^{1/2} \otimes \tilde{\theta}_\lambda^{\alpha_B})\,.$$

First note that $K_0 \subseteq L'$. Then, by Proposition 4.2, we have that L'/K_0 is the minimal extension cut out by $\tilde{\varepsilon}^{-1/2}|_{G_{K_0}}$. The proposition now follows from the fact that k_ε is also the field cut out by $\tilde{\varepsilon}^{-1/2}$ □

Definition 4.4. *Let $\tilde{H}$ denote the (isomorphic) image of the Galois group* $\operatorname{Gal}(K_0/k)$ *by the representation $\tilde{\varepsilon}^{1/2} \otimes \tilde{\theta}_\lambda^{\alpha_B}$. We will denote by H the preimage of $\tilde{H}$ by the projection map*

$$\operatorname{Aut}(V_\lambda(B,A)) \to \operatorname{Aut}(V_\lambda(B,A))/\langle -I\rangle\,.$$

Recall the embeddings $\sigma_i\colon F \to \overline{\mathbb{Q}}_\ell$, for $i = 1,\dots,m$, obtained as extensions to F of the distinct embeddings of M into $\overline{\mathbb{Q}}_\ell$. They define primes λ_i of F. Write ε_i for $\iota \circ \varepsilon_{\lambda_i}$ and $\theta_i^{\alpha_B}$ for $\iota \circ \theta_{\lambda_i}^{\alpha_B}$. Let $\varepsilon_i^{1/2}$ denote an arbitrary square root of ε_i. Note that the map $\varepsilon_i^{1/2}$ will not be in general a character. We set

$$\prod_{i=1}^m \mathrm{SU}(2)^{(i)} \otimes H := \left\{ \prod_{i=1}^m g_i \otimes \varepsilon_i^{1/2} \otimes \theta_i^{\alpha_B}(h) \,|\, g_i \in \mathrm{SU}(2),\, h \in \operatorname{Gal}(K_0/k) \right\}.$$

Since $-I$ belongs to SU(2), this definition does not depend on the choice of the square root $\varepsilon_i^{1/2}$.

Proposition 4.5. *Up to conjugacy,* $\mathrm{ST}(A)$ *is the subgroup*

$$\prod_{i=1}^m \mathrm{SU}(2)^{(i)} \otimes H \subseteq \mathrm{USp}(2g)\,.$$

In particular, we have:

i) The identity component $\mathrm{ST}(A)^0$ *of* $\mathrm{ST}(A)$ *satisfies*

$$\mathrm{ST}(A)^0 \simeq \mathrm{ST}(A_{K_0}) \simeq \mathrm{SU}(2)\times \overset{m}{\cdots} \times\mathrm{SU}(2)\,.$$

ii) The group of connected components $\pi_0(\mathrm{ST}(A))$ of $\mathrm{ST}(A)$ is isomorphic to $\operatorname{Gal}(K_0/k)$.

Proof. Proposition 2.9 and Lemma 4.1 imply that there is an injection

$$\varphi\colon \mathrm{ST}(A) \hookrightarrow \prod_{i=1}^m \mathrm{SU}(2)^{(i)} \otimes H\,.$$

That the projection of φ onto the i-th factor $\mathrm{SU}(2)^{(i)} \otimes H$ is surjective is again an application of Goursat's lemma (as in the proof of Lemma 4.1). Since the $V_{\lambda_i}(B)^{\alpha_B}$ are strongly irreducible, the lack of surjectivity of φ would then translate into the existence of an isomorphism $V_{\lambda_i}(B^{\alpha_B}) \simeq V_{\lambda_j}(B^{\alpha_B})$ as $\overline{\mathbb{Q}}_\ell[G_{K'}]$-modules for some $i \neq j$ and some finite extension K'/k. This contradicts Proposition 2.10.

The statement regarding the group of components is an immediate consequence of Proposition 4.2. □

We now define an action of G_{k_0} on the set $\{1,\dots,m\}$. For $s\in G_{k_0}$, set

$$s(j)=i \qquad \text{if } \sigma_i|_M=\sigma_j|_M\circ s^{-1}\,,$$

where we make sense of the last composition via the map

$$G_{k_0}\twoheadrightarrow \mathrm{Gal}(k/k_0)\hookrightarrow \mathrm{Aut}_{M_0}(M)\,.$$

Corollary 4.6. $\mathrm{ST}(A_0)$ *is isomorphic to the semidirect product*

$$(\mathrm{SU}(2)\times \overset{m}{\cdots} \times\mathrm{SU}(2))\rtimes \mathrm{Gal}(K_0/k_0)\,,$$

where $\mathrm{Gal}(K_0/k_0)$ *acts on the first factor according to the rule*

$$s\left(\prod_{i=1}^m g_i\right)=\prod_{i=1}^m g_{s(i)}\,. \tag{4.2}$$

Proof. By Proposition 4.5 and Theorem 2.11, we have that

$$\mathrm{ST}(A_0)^0=\mathrm{SU}(2)\times \overset{m}{\cdots} \times\mathrm{SU}(2)$$

and that $\pi_0(\mathrm{ST}(A_0))\simeq\mathrm{Gal}(K_0/k_0)$. Let $t\in G_{k_0}$ and $s\in G_{K_0}$, so that

$$\varrho_\ell(s)=nd\bigoplus_{i=1}^m \varrho_{\lambda_i}^{\alpha_B}(s)\,.$$

The corollary follows from the computation

$$\varrho_\ell(t)\left(nd\bigoplus_{i=1}^m \varrho_{\lambda_i}^{\alpha_B}(s)\right)\varrho_\ell(t)^{-1}=nd\bigoplus_{i=1}^m \varrho_{t\circ\lambda_i\circ t^{-1}}^{\alpha_B}(s)=nd\bigoplus_{i=1}^m \varrho_{\lambda_{t(i)}}^{\alpha_B}(s)\,.$$

□

Sato–Tate conjecture. Let E/k_0 be a finite Galois extension[3]. It follows from Corollary 4.6 and [FKRS12, Prop. 2.17] that

$$\mathrm{ST}(A_{0,E})\simeq(\mathrm{SU}(2)\times \overset{m}{\cdots} \times\mathrm{SU}(2))\rtimes \mathrm{Gal}(K_0E/E)\,. \tag{4.3}$$

In order to state the Sato–Tate conjecture, we next define Frobenius elements in the set of conjugacy classes of $\mathrm{ST}(A_{0,E})$. For a prime $\mathfrak{p}$ of E, let $\mathrm{Frob}_{\mathfrak{p}}$ denote a Frobenius element at $\mathfrak{p}$ and let $\mathrm{Nm}(\mathfrak{p})$ denote the absolute norm of $\mathfrak{p}$. Let S denote a finite set of primes of E containing those of bad reduction for $A_{0,E}$. As explained in [Ser12, §8.3.3], to each $\mathfrak{p}\notin S$ one can attach an element $x_{\mathfrak{p}}$ in the set of conjugacy classes of $\mathrm{ST}(A_{0,E})$ such that we have an equality of characteristic polynomials

$$\mathrm{Char}(x_{\mathfrak{p}})=\mathrm{Char}\left(\mathrm{Nm}(\mathfrak{p})^{-1/2}\varrho_\ell(\mathrm{Frob}_{\mathfrak{p}})\right)\,.$$

In this specific situation, the general Sato–Tate conjecture (see [FKRS12, §2.1], [Ser12, Chap. 8]) takes the following explicit form.

Conjecture 4.7 (Sato–Tate conjecture for $A_{0,E}$)**.** *The sequence* $\{x_{\mathfrak{p}}\}_{\mathfrak{p}\notin S}$*, where the primes* $\mathfrak{p}$ *are ordered with respect to their absolute norm, is equidistributed on the set of conjugacy classes of* $\mathrm{ST}(A_{0,E})$ *with respect to the projection on this set of the Haar measure of* $\mathrm{ST}(A_{0,E})$.

---

[3]Despite the coinciding notation, E/k_0 should not be confused with the maximal number field of $\mathrm{End}(B)$ introduced in Section 2, which will play no role in the remaining part of the article.

Let ϱ be an irreducible representation of $\mathrm{ST}(A_{0,E})$. For $s \in \mathbb{C}$ with $\Re(s) > 1$, define the partial Euler product

$$L^S(\varrho, A_{0,E}, s) = \prod_{\mathfrak{p}\notin S} \det(1 - \varrho(x_\mathfrak{p})\,\mathrm{Nm}(\mathfrak{p})^{-s})^{-1}\,.$$

By [Ser89, App. to Chap. I], Conjecture 4.7 holds if for every irreducible nontrivial representation ϱ, the partial Euler product $L^S(\varrho, A_{0,E}, s)$ is invertible, that is, it extends to a holomorphic and nonvanishing function on a neighborhood of $\Re(s) \geq 1$.

Frobenius conjugacy classes revisited. In this subsection, we assume $E = k$. In this situation, thanks to Proposition 4.5, we can achieve a more explicit description of the conjugacy classes $x_\mathfrak{p}$. Let $\varrho_i^{\alpha_B}$ stand for $\iota \circ \varrho_{\lambda_i}^{\alpha_B}$. Then we have that $x_\mathfrak{p}$ is the conjugacy class of

$$\prod_{i=1}^{m} g_{i,\mathfrak{p}} \otimes h_{i,\mathfrak{p}}\,,$$

where

$$g_{i,\mathfrak{p}} := \mathrm{Nm}(\mathfrak{p})^{-1/2}\cdot \varepsilon_i^{-1/2} \otimes \varrho_i^{\alpha_B}(\mathrm{Frob}_\mathfrak{p})\,, \qquad h_{i,\mathfrak{p}} := \varepsilon_i^{1/2} \otimes \theta_i^{\alpha_B}(\mathrm{Frob}_\mathfrak{p})\,,$$

for an arbitrary choice of square root $\varepsilon_i(\mathrm{Frob}_\mathfrak{p})^{1/2}$ (note that the Kronecker product $g_{i,\mathfrak{p}} \otimes h_{i,\mathfrak{p}}$ does not depend on this choice). We will simply write $h_\mathfrak{p}$ to denote $h_{1,\mathfrak{p}}$.

Irreducible representations of $\mathrm{ST}(A_0)$**.** In the next section we will prove Conjecture 4.7 in certain cases when $E = k_0 = k$ and $[M : \mathbb{Q}] \leq 2$. Let us describe the type of partial Euler products that one finds in this setting. Let H be as in Definition 4.4.

If $[M : \mathbb{Q}] = 1$, then $k = k_0$, and we see from Proposition 4.5, that the irreducible representations of $\mathrm{ST}(A_0)$ are of the form $\mathrm{Symm}^e \otimes \eta$, where e is an integer ≥ 0, Symm^e is the e-th symmetric power of the standard representation of $\mathrm{SU}(2)$ and η is an irreducible representation of H such that

$$\eta(-I) = (-I)^e. \tag{4.4}$$

We then have

$$L^S(\mathrm{Symm}^e \otimes \eta, A_0, s) = \prod_{\mathfrak{p}\notin S} \det(1 - \mathrm{Symm}^e(g_{1,\mathfrak{p}}) \otimes \eta(h_\mathfrak{p})\,\mathrm{Nm}(\mathfrak{p})^{-s})^{-1}\,. \tag{4.5}$$

Suppose that $[M : \mathbb{Q}] = 2$ and that $k = k_0$. Then the representations of $\mathrm{ST}(A_0)$ are of the form

$$\mathrm{Symm}^{e_1} \otimes \mathrm{Symm}^{e_2} \otimes \eta\,, \tag{4.6}$$

where e_1, e_2 are integers ≥ 0, and η is an irreducible representation of H such that $\eta(-I) = (-I)^{e_1+e_2}$.

For every $e \geq 0$, we next attach to any representation η as above an Artin representation η_e that will be used in §5 to link the partial Euler products described above to the partial Euler products of the compatible systems $(V_\lambda(B)^{\alpha_B})_\lambda$.

Lemma 4.8. *Let $\eta : H \to \mathrm{GL}(V)$ be a complex representation such that $\eta(-I) = (-I)^e$. For every $s \in G_k$, fix a choice $\varepsilon_i^{1/2}(s)$ of a square root of $\varepsilon_i(s)$. Then the map*

$$\eta_e \colon G_k \to \mathrm{Aut}(V), \qquad \eta_e(s) := \varepsilon_i(s)^{-e/2} \otimes \eta(\varepsilon_i^{1/2}(s) \otimes \theta_i^{\alpha_B}(s))$$

is a representation. Moreover, it factors through an extension K_e of degree at most 2 of K_0k_ε.

Proof. For $s,t\in G_k$ define

$$c_\varepsilon(s,t) := \frac{\varepsilon_i^{1/2}(s)\varepsilon_i^{1/2}(t)}{\varepsilon_i^{1/2}(st)} \in \{\pm 1\}.$$

Then

$$\begin{aligned}\eta_e(st) &= \varepsilon_i(st)^{-e/2}\otimes \eta(\varepsilon_i^{1/2}(st)\otimes \theta_i^{\alpha_B}(st))\\ &= c_\varepsilon(s,t)^e\varepsilon_i(s)^{-e/2}\varepsilon_i(t)^{-e/2}\otimes\eta(c_\varepsilon(s,t)\varepsilon_i^{1/2}(s)\varepsilon_i^{1/2}(s)\otimes\theta_i^{\alpha_B}(s)\theta_i^{\alpha_B}(t))\\ &= \eta_e(s)\eta_e(t);\end{aligned}$$

here we have used that $\eta(c_\varepsilon(s,t)) = c_\varepsilon(s,t)^e$, which follows from the hypothesis $\eta(-I)=(-I)^e$.

Let $\tilde\eta_e\colon G_k\to \mathrm{Aut}(V)/\langle -I\rangle$ be the group homomorphism naturally induced by η_e. It factors through K_0k_ε by Proposition 4.2, and therefore η_e factors through an at most quadratic extension of K_0k_ε. □

5. Scenarios of applicability

In this section, we use the description of the Sato–Tate group of an abelian variety A_0 defined over k_0 satisfying Hypothesis 2.1 achieved in §4 to prove the Sato–Tate conjecture in certain cases. The two main theorems of this section generalize [Tay19, Thm 3.4 and Thm 3.6]. The proofs build heavily on those in [Tay19], which in turn are deeply inspired by those in [Joh17]. Many ideas are in fact reminiscent of the seminal works [HSBT10] and [Har09].

Theorem 5.1. *Suppose that k_0 is a totally real or CM field and that A_0 is an abelian variety defined over k_0 of dimension $g\geq 1$ which is $\overline{\mathbb{Q}}$-isogenous to the power of either:*

i) an elliptic curve B without CM; or
ii) an abelian surface B with QM.

Suppose that the endomorphism field K_0 of A_0 is a solvable extension of k_0. Then Conjecture 4.7 holds.

Proof. The setting of the theorem is that of an abelian variety A_0 satisfying Hypothesis 2.1 with $M=\mathbb{Q}$. In particular, we have $k=k_0$. By Theorem 2.11, there is an isomorphism of $\overline{\mathbb{Q}}_\ell[G_{k_0}]$-modules

$$V_\ell(A)\otimes\overline{\mathbb{Q}}_\ell \simeq V_\ell(B)^{\alpha_B}\otimes_{\overline{\mathbb{Q}}_\ell} V_\ell(B,A)^{\alpha_B}\,,$$

where $V_\ell(B,A)^{\alpha_B}$ has dimension g. We want to show that the partial L-function

$$L^S(\mathrm{Symm}^e\otimes\eta, A_0, s) \tag{5.1}$$

as defined in (4.5) is invertible as long as not both $e=0$ and η is trivial. We may assume that $e\geq 1$, since otherwise we have a partial Artin L-function for which the result is well known.

To show invertibility, we will apply the Taylor–Brauer reduction method of [HSBT10] closely following the presentation of [MM09]. We first invoke Theorem 3.4 to obtain a Galois extension[4] k'/k such that

$$\mathcal{R}_e|_{G_{k'}}\,,\qquad \text{where } \mathcal{R}_e := \mathrm{Symm}^e(V_\lambda(B)^{\alpha_B})_\lambda\,,$$

is automorphic. Note that the partial L-function of (5.1) is the normalized partial L-function $L^S(\mathcal{R}_e \otimes \eta_e, s)$ attached to the weakly compatible system of λ-adic representations $\mathcal{R}_e \otimes \eta_e$.

Set $L = k'K_e$, where K_e is the field introduced in Lemma 4.8, and inflate η_e to a representation of $\mathrm{Gal}(L/k_0)$. By Brauer's induction theorem, we may write η_e as a finite sum

$$\eta_e = \bigoplus_i c_i \,\mathrm{Ind}_{k_0}^{E_i}(\chi_i)\,,$$

where c_i is an integer, E_i/k_0 is a subextension of L/k_0 such that $\mathrm{Gal}(L/E_i)$ is solvable, and $\chi_i\colon \mathrm{Gal}(L/E_i) \to \mathbb{C}^\times$ is a character. We therefore have

$$L^S(\mathcal{R}_e \otimes \eta_e, s) = \prod_i L^S(\mathcal{R}_e|_{G_{E_i}} \otimes \chi_i, s)^{c_i}\,,$$

and it suffices to show that the L-functions $L^S(\mathcal{R}_e|_{G_{E_i}} \otimes \chi_i, s)$ are invertible.

By assumption, K_0/k_0 is solvable, and thus so is K_e/k_0. Therefore L/k' is solvable. Then automorphic base change [AC89] implies that $\mathcal{R}_e|_{G_L}$ is automorphic. Since L/E_i is solvable, automorphic descent implies that $\mathcal{R}_e|_{G_{E_i}}$ is automorphic. Via Artin reciprocity, we may interpret χ_i as a Hecke character of E_i, and deduce that

$$\mathcal{R}_e|_{G_{E_i}} \otimes \chi_i \tag{5.2}$$

is automorphic. This implies that $L^S(\mathcal{R}_e|_{G_{E_i}} \otimes \chi_i, s)$ is invertible. □

Remark 5.2. A statement analogous to Theorem 5.1 holds true when B is a CM elliptic curve. In this case, the solvability assumption on K_0/k_0 is not necessary, since the automorphicity of Hecke characters is well known. We will however disregard the CM setting in this section, since it has already been treated in [Joh17, §3].

Theorem 5.3. *Suppose that k_0 is a totally real or CM field and that A_0 is an abelian variety defined over k_0 of dimension $g \geq 1$ which is $\overline{\mathbb{Q}}$-isogenous to the power of either:*

i) an abelian surface B wih RM; or
ii) an abelian fourthfold B with QM[5]*.*

Suppose that the endomorphism field K_0 of A_0 is a solvable extension of k_0 and that the field extension k/k_0 from Theorem 2.11 is trivial. Then Conjecture 4.7 holds.

Proof. The hypotheses of the theorem are a reformulation of the assumption that A_0 satisfies Hypothesis 2.1 and that M is a (real) quadratic field.

Since $k = k_0$, we have that $M = M_0$. By Theorem 2.11, we have an isomorphism of $\overline{\mathbb{Q}}_\ell[G_{k_0}]$-modules

$$V_\ell(A) \otimes \overline{\mathbb{Q}}_\ell \simeq V_\lambda(B)^{\alpha_B} \otimes V_\lambda(B,A)^{\alpha_B} \oplus V_{\overline{\lambda}}(B)^{\alpha_B} \otimes V_{\overline{\lambda}}(B,A)^{\alpha_B}\,,$$

[4]It is not really necessary to assume that k'/k is linearly disjoint from K_0 over k_0.

[5]Recall that in our terminology this means that $\mathrm{End}(B)$ is a quaternion algebra over a quadratic number field.

where $\lambda, \overline{\lambda}$ are attached to extensions to F of the two distinct embeddings of M_0 into $\overline{\mathbb{Q}}_\ell$. Note that $V_\lambda(B,A)^{\alpha_B}$ has dimension $g/2$ as a $\overline{\mathbb{Q}}_\ell$-vector space. It will suffice to show that the partial L-function

$$L^S(\mathrm{Symm}^{e_1} \otimes \mathrm{Symm}^{e_2} \otimes \eta, A_0, s)$$

attached to (4.6) is invertible whenever $e_1 > 0$ or $e_2 > 0$. Invoke Theorem 3.4 to obtain a Galois extension k'/k_0 such that $\mathcal{R}_{e_1}|_{G_{k'}}$ and $\overline{\mathcal{R}}_{e_2}|_{G_{k'}}$ are automorphic, where

$$\mathcal{R}_{e_1} := \mathrm{Symm}^{e_1}(V_\lambda(B)^{\alpha_B})_\lambda\,, \qquad \overline{\mathcal{R}}_{e_2} := \mathrm{Symm}^{e_2}(V_{\overline{\lambda}}(B)^{\alpha_B})_\lambda\,.$$

Set $L = k'K_{e_1+e_2}$ and inflate $\eta_{e_1+e_2}$ to a representation of $\mathrm{Gal}(L/k_0)$. Note that

$$L^S(\mathrm{Symm}^{e_1} \otimes \mathrm{Symm}^{e_2} \otimes \eta, A_0, s) = L^S(\mathcal{R}_{e_1} \otimes \overline{\mathcal{R}}_{e_2} \otimes \eta_{e_1+e_2}, s).$$

As in the proof of Theorem 5.1, by Brauer's induction theorem applied to $\eta_{e_1+e_2}$, there exist integers c_i, subextensions E_i/k_0 of L/k_0 with $\mathrm{Gal}(L/E_i)$ solvable, and characters $\chi_i\colon \mathrm{Gal}(L/E_i) \to \mathbb{C}^\times$ such that

$$L^S(\mathcal{R}_{e_1} \otimes \overline{\mathcal{R}}_{e_2} \otimes \eta_{e_1+e_2}, s) = \prod_i L^S(\mathcal{R}_{e_1}|_{G_{E_i}} \otimes \overline{\mathcal{R}}_{e_2}|_{G_{E_i}} \otimes \chi_i, s)^{c_i}\,. \tag{5.3}$$

As in the proof of Theorem 5.1 we have that L/k' is solvable. Using automorphic base change and automorphic descent, we find that

$$\mathcal{R}_{e_1}|_{G_{E_i}} \text{ and } \overline{\mathcal{R}}_{e_2}|_{G_{E_i}} \otimes \chi_i \tag{5.4}$$

are automorphic. The invertibility of the L-function attached to (5.3) follows from [Sha81]. Note that the discussion in the paragraph preceding [Har09, Thm. 5.3], together with Proposition 2.10, ensures that the systems of (5.4) are not dual to each other.

□

Remark 5.4. When $g \le 3$, the hypothesis that K_0/k_0 be solvable in Theorem 5.1 and Theorem 5.3 is always satisfied. This follows from the classification results achieved in [FKRS12] and [FKS19]. Note also that the hypothesis that k be CM or totally real is trivially satisfied when $k_0 = \mathbb{Q}$.

Examples: modular abelian varieties. A natural source of examples of abelian varieties satisfying Hypothesis 2.1 are the modular abelian varieties associated to modular forms by the Eichler–Shimura construction. Let $f = \sum a_m q^m \in S_2(\Gamma_1(N))$ be a non-CM newform of nebentype ε, and let $F_f = \mathbb{Q}(\{a_m\}_m)$ be the number field generated by its Fourier coefficients. Put $g = [F_f : \mathbb{Q}]$. There exists an abelian variety A_f defined over $\mathbb{Q}$ of dimension g which is uniquely characterized up to isogeny by the equality of L-functions

$$L(A_f, s) = \prod_{\sigma : F_f \hookrightarrow \mathbb{C}} L(f^\sigma, s),$$

and satisfying that $\mathrm{End}(A_f) \simeq F_f$. The variety A_f is simple, but it may not be geometrically simple. The structure of the base change $A_{f,\overline{\mathbb{Q}}}$ was determined by Ribet [Rib92] and Pyle [Pyl02]. They proved that $A_{f,\overline{\mathbb{Q}}} \sim B^d$ for some abelian variety $B/\overline{\mathbb{Q}}$ satisfying that:

- B is a $\mathbb{Q}$-variety, and
- $\mathrm{End}(B)$ is a central division algebra over a totally real field M_f of Schur index $n \le 2$ and $n[M_f : \mathbb{Q}] = \dim B$.

Moreover, the center M_f of $\mathrm{End}(B)$ can be described in terms of f as the field generated by all the numbers $a_m^2/\varepsilon(m)$ with m coprime to N.

Denote by K_f the smallest field of definition of $\mathrm{End}(A_{f,\overline{\mathbb{Q}}})$ (this is the field called K_0 in §2). It is well-known that the extension $K_f/\mathbb{Q}$ is abelian (cf. [GL01, Proposition 2.1]), hence in particular solvable.

All these properties of the varieties A_f give the following consequence of Theorems 5.1 and 5.3.

Corollary 5.5. *Let $f = \sum a_m q^m \in S_2(\Gamma_1(N))$ be a newform of nebentype ε. If f is non-CM, suppose that the number field $M_f = \mathbb{Q}(\{a_m^2/\varepsilon(m)\}_{(m,N)=1})$ has degree at most 2 over $\mathbb{Q}$. Then the Sato–Tate conjecture is true for A_f.*

Examples of these modular forms are certainly abundant even for small levels N. For example, in the tables of [Que09] (the complete tables are available at [Que12]), where levels up to 500 are considered, one finds many examples of modular abelian varieties A_f which are geometrically isogenous to powers of elliptic curves ([Que12, §4.1]), abelian surfaces with RM by a quadratic field M_f ([Que12, §4.2]), abelian surfaces with QM by a quaternion algebra over $\mathbb{Q}$ ([Que12, §5.1]) or abelian fourfolds with QM by a quaternion algebra over a quadratic field M_f ([Que12, §5.2]).

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