# ON CARDINAL SEQUENCES OF LENGTH $< \omega_3$

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Abstract. We prove the following consistency result for cardinal sequences of length $< \omega_3$: if GCH holds and $\lambda \geq \omega_2$ is a regular cardinal, then in some cardinal-preserving generic extension $2^\omega = \lambda$ and for every ordinal $\eta < \omega_3$ and every sequence $f = \langle \kappa_\alpha : \alpha < \eta \rangle$ of infinite cardinals with $\kappa_\alpha \leq \lambda$ for $\alpha < \eta$ and $\kappa_\alpha = \omega$ if $\mathrm{cf}(\alpha) = \omega_2$, we have that $f$ is the cardinal sequence of some LCS space.

Also, we prove that for every specific uncountable cardinal $\lambda$ it is relatively consistent with ZFC that for every $\alpha, \beta < \omega_3$ with $\mathrm{cf}(\alpha) < \omega_2$ there is an LCS space $Z$ such that $\mathrm{CS}(Z) = \langle \omega \rangle_\alpha {}^\frown \langle \lambda \rangle_\beta$.

## 1. Introduction

By an *LCS space* we mean a locally compact, Hausdorff and scattered space. Recall that for an LCS space $X$ and an ordinal $\alpha$, the $\alpha^{th}$- *Cantor-Bendixson level* of $X$ is defined by $I_\alpha(X)$ = the set of isolated points of $X \setminus \bigcup \{I_\beta(X) : \beta < \alpha\}$. We define *the height of* $X$ as $ht(X)$ = the least ordinal $\delta$ such that $I_\delta(X) = \emptyset$, and we define its *reduced height* as $ht(X)^-$ = the least ordinal $\delta$ such that $I_\delta(X)$ is finite. Clearly, one has $\mathrm{ht}^-(X) \leq \mathrm{ht}(X) \leq \mathrm{ht}^-(X) + 1$. We define the *width* of $X$ by $\mathrm{wd}(X) = \sup\{|I_\alpha(X)| : \alpha < \mathrm{ht}(X)\}$. And we define the *cardinal sequence* of $X$ as $\mathrm{CS}(X) = \langle |I_\alpha(X)| : \alpha < \mathrm{ht}^-(X) \rangle$. If $\kappa$ is an infinite cardinal and $\alpha$ is an ordinal, we denote by $\langle \kappa \rangle_\alpha$ the cardinal sequence $\langle \kappa_\beta : \beta < \alpha \rangle$ where $\kappa_\beta = \kappa$ for $\beta < \alpha$. If $f$ and $g$ are sequences of cardinals, we denote by $f {}^\frown g$ the concatenation of $f$ with $g$.

Many authors have studied the possible sequences of infinite cardinals that can arise as the cardinal sequence of an LCS space (or, equivalently, of a superatomic Boolen algebra). We refer the reader to the survey papers [1] and [8] for a wide list of results on cardinal sequences of LCS spaces as well as examples and basic facts. It was proved by Juhász and Weiss that if $f = \langle \kappa_\alpha : \alpha < \omega_1 \rangle$ is a sequence of infinite cardinals, then $f$ is the cardinal sequence of an LCS space iff $\kappa_\beta \leq \kappa_\alpha^\omega$ for every $\alpha < \beta < \omega_1$ (see [5, Theorem 5]). However, this result can not be extended to cardinal sequences of length $\omega_1 + 1$, since it was shown by Baumgarter in [2] that in the Mitchell Model there is no LCS space $X$ with $\mathrm{CS}(X) = \langle \omega_1 \rangle_{\omega_1} {}^\frown \langle \omega_2 \rangle$. Also,

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a characterization under GCH for cardinal sequences of length $< \omega_2$ was obtained by Juhász, Soukup and Weiss in [4]. However, no characterization is known for cardinal sequences of length $\omega_2$. Nevertheless, it was shown by Baumgartner and Shelah in [2] that it is relatively consistent with ZFC that there is an LCS space of width $\omega$ and height $\omega_2$. This result was improved by Soukup in [11], where it was shown that if GCH holds and $\lambda \geq \omega_2$ is a regular cardinal, then in some cardinal-preserving generic extension $2^\omega = \lambda$ and every sequence $f = \langle \kappa_\alpha : \alpha < \omega_2 \rangle$ of infinite cardinals with $\kappa_\alpha \leq \lambda$ is the cardinal sequence of some LCS space. However, the following basic proposition shows that Soukup's theorem can not be extended to cardinal sequences of length $< \omega_3$.

**Proposition 1.1.** *Assume that $\kappa$ is a regular cardinal, $\eta > \kappa^{++}$ is an ordinal and $f = \langle \kappa_\xi : \xi < \eta \rangle$ is the cardinal sequence of some LCS space. Assume that $\alpha < \eta$ is an ordinal with $cf(\alpha) > \kappa^+$ such that there is a strictly increasing sequence of ordinals $\langle \alpha_\xi : \xi < cf(\alpha) \rangle$ converging to $\alpha$ in such a way that $\kappa_{\alpha_\xi} \leq \kappa$ for $\xi < cf(\alpha)$. Then, $\kappa_\alpha \leq \kappa$.*

**Proof**. Assume on the contrary that $\kappa_\alpha > \kappa$ . Let $X$ be an LCS space such that $\mathrm{CS}(X) = f$. So, $|I_\alpha(X)| = \kappa_\alpha$. Let $Y$ be a subset of $I_\alpha(X)$ such that $|Y| = \kappa^+$. For every $x \in Y$ let $U_x$ be a compact open neighbourhood of $x$ such that $U_x \cap \bigcup\{I_\beta(X) : \alpha \leq \beta < \eta\} = \{x\}$. Clearly, for every $x, y \in Y$ with $x \neq y$, we have that $U_x \cap U_y \subset \bigcup\{I_\gamma(X) : \gamma < \beta\}$ for some $\beta < \alpha$. Then, we define the function $F : [Y]^2 \longrightarrow \{\alpha_\xi : \xi < \mathrm{cf}(\alpha)\}$ as follows. If $\{x, y\} \in [Y]^2$ we put

$$F\{x, y\} = \text{ the least ordinal } \zeta < \mathrm{cf}(\alpha) \text{ such that } U_x \cap U_y \subset \bigcup\{I_\gamma(X) : \gamma < \alpha_\zeta\}.$$

Since $\mathrm{cf}(\alpha) > \kappa^+$, there is a $\gamma < \mathrm{cf}(\alpha)$ such that for every $\{x, y\} \in [Y]^2$ we have $F\{x, y\} < \gamma$. But then $|I_{\alpha_\gamma}(X)| = \kappa^+$, which contradicts the assumption that $\kappa_{\alpha_\xi} \leq \kappa$ for $\xi < \mathrm{cf}(\alpha)$. □

In particular, there is no LCS space $X$ with $\mathrm{CS}(X) = \langle \omega \rangle_{\omega_2} {}^\frown \langle \omega_1 \rangle$.

Then, we will show in Theorem 2.1 the following consistency result for cardinal sequences of length $< \omega_3$: if GCH holds and $\lambda \geq \omega_2$ is a regular cardinal, then in some cardinal-preserving generic extension $2^\omega = \lambda$ and for every ordinal $\eta < \omega_3$ and every sequence $f = \langle \kappa_\alpha : \alpha < \eta \rangle$ of infinite cardinals with $\kappa_\alpha \leq \lambda$ for $\alpha < \eta$ and $\kappa_\alpha = \omega$ if $\mathrm{cf}(\alpha) = \omega_2$, we have that $f$ is the cardinal sequence of some LCS space.

Also, we will prove in Theorem 3.1 that for every specific uncountable cardinal $\lambda$ it is relatively consistent with ZFC that for every $\alpha, \beta < \omega_3$ with $\mathrm{cf}(\alpha) < \omega_2$ there is an LCS space $Z$ such that $\mathrm{CS}(Z) = \langle \omega \rangle_\alpha {}^\frown \langle \lambda \rangle_\beta$. This theorem improves the results shown in [10] and [9, Section 2].

If $T = \bigcup\{\{\alpha\} \times A_\alpha : \alpha < \eta\}$ where $\eta$ is a non-zero ordinal and each $A_\alpha$ is a non-empty set of ordinals, then for every $s = \langle\alpha, \zeta\rangle \in T$ we write $\pi(s) = \alpha$ and $\rho(s) = \zeta$.

The following notion, which permits us to construct in a direct way LCS spaces from partial orders, will be used in our constructions.

**Definition 1.2.** We say that $\mathcal{T} = \langle T, \preceq, i\rangle$ is an *LCS poset*, if the following conditions hold:

(1) $\langle T, \preceq\rangle$ is a partial order with $T = \bigcup\{T_\alpha : \alpha < \eta\}$ for some non-zero ordinal $\eta$ such that each $T_\alpha = \{\alpha\} \times A_\alpha$ where $A_\alpha$ is a non-empty set of ordinals.
(2) If $s \prec t$ then $\pi(s) < \pi(t)$.
(3) If $\alpha < \beta < \eta$ and $t \in T_\beta$, then $\{s \in T_\alpha : s \prec t\}$ is infinite.
(4) $i : [T]^2 \to [T]^{<\omega}$ such that for every $\{s, t\} \in [T]^2$ the following holds:
  (a) If $v \in i\{s, t\}$, then $v \preceq s, t$.
  (b) If $u \preceq s, t$, then there is a $v \in i\{s, t\}$ such that $u \preceq v$.

If $\mathcal{T} = \langle T, \preceq, i\rangle$ is an LCS poset with $T = \bigcup\{T_\alpha : \alpha < \eta\}$ , we define its *associated LCS space* $X = X(\mathcal{T})$ as follows. The underlying set of $X(\mathcal{T})$ is $T$. If $x \in T$, we write $C(x) = \{y \in T : y \preceq x\}$. Then, for every $x \in T$ we define a basic neighbourhood of $x$ in X as a set of the form $C(x) \setminus (C(x_1) \cup \cdots \cup C(x_n))$ where $n < \omega$ and $x_1, \ldots, x_n \prec x$. It can be checked that $X$ is a locally compact, Hausdorff, scattered space of height $\eta$ such that $I_\alpha(X) = T_\alpha$ for every $\alpha < \eta$ (see [1] for a proof). Then, we will say that $\langle|T_\alpha| : \alpha < \eta\rangle$ is the *cardinal sequence* of $\mathcal{T}$.

If $\mathcal{T} = \langle T, \preceq, i\rangle$ is an LCS poset and $S \subset T$ such that $i\{s, t\} \subset S$ for all $\{s, t\} \in [S]^2$, we define the *restriction of* $\mathcal{T}$ to $S$ as $\mathcal{T} \restriction S = \langle S, \preceq\restriction (S \times S), i \restriction [S]^2\rangle$.

The following notion, which is a refinement of the notion of a skeleton given in [3, Definition 1.5], will also be needed to show our results.

**Definition 1.3.** Assume that $\mathcal{T} = \langle T, \preceq, i\rangle$ is an LCS poset with $T = \bigcup\{\{\alpha\} \times A_\alpha : \alpha < \eta\}$. Let $f$ be the cardinal sequence of $\mathcal{T}$. Then, we say that $\mathcal{T}$ is an *f-skeleton*, if for every $\alpha < \eta$ there is a countable subset $O_\alpha \in [A_\alpha]^\omega$ such that $s \prec t$ and $\pi(s) = \alpha$ implies $\rho(s) \in O_\alpha$, and in such a way that the following two conditions hold:

(1) If $\alpha < \eta$ and $s, t \in \{\alpha\} \times O_\alpha$ with $\rho(s) \neq \rho(t)$, then $i\{s, t\} = \emptyset$.
(2) If $\alpha+1 < \eta$, $t \in \{\alpha+1\}\times A_{\alpha+1}$ and $s \prec t$, then there is a $u \in \{\alpha\}\times O_\alpha$ such that $s \preceq u \prec t$.

## 2. A general consistency result

In this section, our aim is to prove the following result.

**Theorem 2.1.** *If GCH holds and $\lambda \geq \omega_2$ is a regular cardinal, then in some cardinal-preserving generic extension $2^\omega = \lambda$ and for every ordinal $\eta < \omega_3$*

*and every sequence $f = \langle \kappa_\alpha : \alpha < \eta \rangle$ of infinite cardinals with $\kappa_\alpha \leq \lambda$ for $\alpha < \eta$ and $\kappa_\alpha = \omega$ if $cf(\alpha) = \omega_2$, we have that $f$ is the cardinal sequence of some LCS space.*

In order to prove Theorem 2.1, we will use the following refinement of the notion of Shelah's $\Delta$-function due to Soukup.

**Definition 2.2.** A function $f : [\omega_2 \times \lambda]^2 \to [\omega_2]^{<\omega}$ is a $\Delta(\omega_2 \times \lambda)$-*function*, if $f\{x, y\} \subset \min\{\pi(x), \pi(y)\}$ for each $\{x, y\} \in [\omega_2 \times \lambda]^2$ and for every uncountable subset $\{d_\alpha : \alpha < \omega_1\}$ of $[\omega_2 \times \lambda]^{<\omega}$ there are ordinals $\alpha < \beta < \omega_1$ such that for every $x \in d_\alpha \setminus d_\beta$, $y \in d_\beta \setminus d_\alpha$ and $z \in d_\alpha \cap d_\beta \cap (\omega_2 \times \omega)$ the following conditions hold:

(1) if $\pi(z) < \pi(x), \pi(y)$ then $\pi(z) \in f\{x, y\}$,
(2) if $\pi(z) < \pi(y)$ then $f\{x, z\} \subset f\{x, y\}$,
(3) if $\pi(z) < \pi(x)$ then $f\{y, z\} \subset f\{x, y\}$.

The following result was shown in [11].

**Theorem 2.3.** *If GCH holds and $\lambda \geq \omega_2$ is a regular cardinal, then in some cardinal-preserving generic extension $\lambda$ is a regular cardinal with $\lambda^{\omega_1} = \lambda$ and there is a $\Delta(\omega_2 \times \lambda)$-function.*

The following result can be proved by means of a slight refinement of the arguments given in [3, Theorem 1.12] and [11, Theorem 4.2].

**Theorem 2.4.** *Assume that $\lambda \geq \omega_2$ is a regular cardinal such that $\lambda^\omega = \lambda$. Assume that there is a $\Delta(\omega_2 \times \lambda)$-function. Then, in some c.c.c. generic extension $2^\omega = \lambda$ and there is a $\langle \lambda \rangle_{\omega_2}$–skeleton.*

**Theorem 2.5.** *Assume that $\lambda \geq \omega_2$ is a regular cardinal with $2^\omega = \lambda$, $\eta < \omega_3$ is an ordinal and there is a $\langle \lambda \rangle_{\omega_2}$-skeleton. Then, if $f = \langle \kappa_\alpha : \alpha < \eta \rangle$ is a sequence of infinite cardinals with $\kappa_\alpha \leq \lambda$ for $\alpha < \eta$ and $\kappa_\alpha = \omega$ if $cf(\alpha) = \omega_2$, we have that $f$ is the cardinal sequence of some LCS space.*

Note that Theorem 2.1 follows immediately from Theorems 2.3, 2.4 and 2.5.

**Proof of Theorem 2.5**. Assume that $\lambda \geq \omega_2$ is a regular cardinal such that $2^\omega = \lambda$ and that there is a $\langle \lambda \rangle_{\omega_2}$-skeleton $\mathcal{T} = \langle T, \preceq, i \rangle$. We may assume that $O_\alpha = \omega$ for every $\alpha < \omega_2$. Proceeding by transfinite induction on $\eta < \omega_3$, we show that if $f = \langle \kappa_\alpha : \alpha < \eta \rangle$ is a sequence of infinite cardinals with $\kappa_\alpha \leq \lambda$ for $\alpha < \eta$ and $\kappa_\alpha = \omega$ if $\mathrm{cf}(\alpha) = \omega_2$, then there is an LCS space whose cardinal sequence is $f$. If $\eta \leq \omega_2$, we are done by the existence of the $\langle \lambda \rangle_{\omega_2}$-skeleton $\mathcal{T}$. So, assume that $\omega_2 < \eta < \omega_3$ and $f$ is as above.

First, suppose that $\eta$ is a limit ordinal. We assume that $\mathrm{cf}(\eta) = \omega_2$. Otherwise, the argument is similar. We distinguish two cases. First, suppose that there is a strictly increasing sequence of ordinals $\langle \gamma_\xi : \xi < \omega_2 \rangle$ converging to $\eta$ such that $\mathrm{cf}(\gamma_\xi) = \omega_2$ for every $\xi < \omega_2$. Note that if $\xi < \omega_2$ is a limit ordinal, then $\gamma_\xi > \sup\{\gamma_\mu : \mu < \xi\}$. Now, we define the sequence of ordinals $\langle \alpha_\xi : \xi < \omega_2 \rangle$ as follows. We put $\alpha_0 = 0$. If $\xi$ is a successor ordinal, we put $\alpha_\xi = \gamma_\xi$. And if $\xi$ is a limit ordinal, we define $\alpha_\xi = \sup\{\alpha_\mu : \mu < \xi\}$.

Now, for every $\xi < \omega_2$, we define $S_\xi$ as follows. We put $S_0 = \{0\} \times \kappa_0$. If $\xi$ is a successor ordinal, we have $\kappa_{\alpha_\xi} = \omega$ and then we put $S_\xi = \{\xi\} \times \omega$. And if $\xi$ is a limit ordinal, we define $S_\xi = \{\xi\} \times \kappa_{\alpha_\xi}$. We put $S = \bigcup\{S_\xi : \xi < \omega_2\}$. Note that $i\{s,t\} \subset \omega_2 \times \omega \subset S$ for all $\{s,t\} \in [S]^2$. Then, we define $\mathcal{S} = \mathcal{T} \restriction S = \langle S, \preceq \restriction (S \times S), i \restriction [S]^2 \rangle$. Clearly, $\mathcal{S}$ is an $h$-skeleton for $h = \langle \kappa_{\alpha_\xi} : \xi < \omega_2 \rangle$.

In order to carry out the construction, for every ordinal $\xi < \omega_2$ we will insert an adequate LCS space between the $\xi$-th and the $(\xi+1)$-th level of the space associated with $\mathcal{S}$. For every successor ordinal $\xi = \mu + 1 < \omega_2$, we define $\delta_\xi = \mathrm{o.t.}(\alpha_\xi \setminus \alpha_\mu)$. By the induction hypothesis, for every successor ordinal $\xi = \mu + 1 < \omega_2$ there is an LCS space whose cardinal sequence is $f_\xi = \langle \omega \rangle ^\frown \langle \kappa_{\alpha_\mu + \zeta} : 0 < \zeta < \delta_\xi \rangle$.

Let $S' = \bigcup\{\{\xi\} \times \omega : \xi < \omega_2 \text{ is a successor ordinal}\}$. For every successor ordinal $\xi = \mu + 1 < \omega_2$ and every element $x \in \{\xi\} \times \omega$, we put $C_x = \{y \in \{\mu\} \times \omega : y \prec x\}$. Note that since $\mathcal{T}$ is a skeleton, $C_x \cap C_y = \emptyset$ if $\pi(x) = \pi(y)$ and $x \neq y$. Then, for every successor ordinal $\xi < \omega_2$ and every element $x \in \{\xi\} \times \omega$, we consider a compact Hausdorff scattered space $Z_x$ of height $\delta_\xi + 1$ such that $\mathrm{CS}(Z_x) = f_\xi$ and in such a way that $I_0(Z_x) = C_x$ and $I_{\delta_\xi}(Z_x) = \{x\}$. Also, we assume that for every $x \in S'$ we have $Z_x \cap S = C_x \cup \{x\}$ and that for every $x, y \in S'$ with $x \neq y$ we have $(Z_x \setminus \{x\}) \cap (Z_y \setminus \{y\}) = \emptyset$.

Now, we define the required space $Z$ as follows. Its underlying set is $S \cup \bigcup\{Z_x : x \in S'\}$. For every $z \in S$ we put

$$W_z = \{y \in S : y \preceq z\} \cup \bigcup\{Z_y : y \preceq z \text{ and } y \in S'\}.$$

Clearly, $x \prec z$ implies $W_x \subset W_z$. Also, if $z \in Z_x$ for some $x \in S'$ and $U$ is an open neighbourhood of $z$ in $Z_x$, we define

$$U^* = U \cup \bigcup\{W_y : y \in C_x \cap U\}.$$

Note that as $\mathcal{T}$ is a skeleton, $W_x = Z_x^*$ for every $x \in S'$.

Assume that $v \in S \cup \bigcup\{Z_x : x \in S'\}$. Then, if $v \in S \setminus S'$ we define a basic neighbourhood of $v$ in $Z$ as a set of the form $W_v \setminus (W_{v_1} \cup \cdots \cup W_{v_n})$ where $n < \omega$ and $v_1, \ldots, v_n \prec v$. If $v \in S'$, a basic neighbourhood of $v$ in $Z$ is a set $U^*$ where $U$ is a compact open neighbourhood of $v$ in $Z_v$. Otherwise, we have that $v \in Z_x$ for a unique $x \in S'$, and then we define a basic neighbourhood of $v$ in $Z$ as a set $U_v^*$ where $U_v$ is a compact open neighbourhood of $v$ in $Z_x$. In order to show that this collection of sets is in fact a base, suppose that $x \in W_y \setminus (W_{y_1} \cup \cdots \cup W_{y_n})$ where $y_1, \ldots, y_n \prec y \in S \setminus S'$ and $x \in U_z^*$ where $z \in Z_w$ for some $w \in S'$ and $U_z$ is a compact open neighbourhood of $z$ in $Z_w$. The other cases are easier to verify. Without loss of generality, we may assume that $x \in U_z \setminus S$. Since $x \in (W_y \setminus (W_{y_1} \cup \dots \cup W_{y_n})) \cap U_z$, we have that $w \preceq y$ and $w \not\preceq y_i$ for $i \in \{1, \dots, n\}$. Let $\{v_1, \dots, v_k\} = i\{w, y_1\} \cup \dots \cup i\{w, y_n\}$. Since $\mathcal{T}$ is a skeleton, there are $u_1, \dots, u_k \in C_w$ such that $v_i \preceq u_i$ for $i \in \{1, \dots, k\}$. Let $U_x$ be a compact open neighbourhood of $x$ in $Z_w$ such that $U_x \subset U_z$ and $U_x \cap \{u_1, \dots, u_k\} = \emptyset$. Then, $U_x^*$ is as required. Also, it is easy to check that each element of this base is a clopen set.

We show that $Z$ is Hausdorff. Assume that $\{s, t\} \in [Z]^2$. We may assume that $s, t \notin S$. Otherwise, the argument is easier. Let $x, y \in S'$ be such that $s \in Z_x$ and $t \in Z_y$. If $x = y$, the case is obvious. So, assume that $x \neq y$. Note that if $\pi(x) = \pi(y)$, $U_s$ is an open neighbourhood of $s$ in $Z_x$ and $U_t$ is an open neighbourhood of $t$ in $Z_y$, then as $\mathcal{T}$ is a skeleton we have $U_s^* \cap U_t^* = \emptyset$. So, suppose that $\pi(x) < \pi(y)$. If $x \prec y$, then $W_x \cap (W_y \setminus W_x) = \emptyset$ and clearly $W_x$ is a neighbourhood of $s$ and $W_y \setminus W_x$ is a neighbourhood of $t$. Now, assume that $x, y$ are $\preceq$-incomparable. Put $i\{x, y\} = \{v_1, \dots, v_n\}$. We show that $W_x \cap W_y \subset W_{v_1} \cup \dots \cup W_{v_n}$. For this, suppose that $u \in W_x \cap W_y$. Without loss of generality, we may assume that $u \in Z \setminus S$. Let $w \in S'$ be such that $u \in Z_w$. Since $u \in W_x \cap W_y$, it follows that $w \prec x, y$, hence $w \preceq v_i$ for some $i \in \{1, \dots, n\}$, and so $u \in W_{v_i}$. Therefore, $W_x \cap W_y \subset W_{v_1} \cup \dots \cup W_{v_n}$, and so we are done because $s \in W_x$ and $t \in W_y \setminus (W_{v_1} \cup \dots \cup W_{v_n})$.

Also, proceeding by transfinite induction on $\pi(x)$, we can verify that $W_x$ is a compact neighbourhood of $x$ in $Z$ for every $x \in S$, and that if $z \in Z_x \setminus S$ for some $x \in S'$ and $U_z$ is a compact open neighbourhood of $z$ in $Z_x$ then $U_z^*$ is a compact neighbourhood of $z$ in $Z$. Therefore, $Z$ is locally compact.

On the other hand, it is easy to see that $Z$ is a scattered space with $\mathrm{CS}(Z) = f$.

Next, assume that there is no strictly increasing sequence of $\omega_2$ many ordinals of cofinality $\omega_2$ converging to $\eta$. Let $\gamma = \sup\{\xi + 1 : \xi < \eta, \mathrm{cf}(\xi) = \omega_2\}$. Put $g = \langle \lambda_\xi : \xi < \eta \rangle$ where $\lambda_\xi = \omega$ for $\xi \leq \gamma$ and $\lambda_\xi = \kappa_\xi$ for $\gamma < \xi < \eta$. First, we construct an LCS space $Y$ with $\mathrm{CS}(Y) = g$. Put $\zeta = \mathrm{o.t.}(\eta \setminus \gamma)$. Since there is a $\langle \lambda \rangle_{\omega_2}$–skeleton and in $\eta \setminus \gamma$ there is no ordinal of cofinality $\omega_2$, it follows that there is an LCS space $X$ such that $\mathrm{CS}(X) = \langle \lambda_\xi : \xi < \zeta \rangle$ where $\lambda_0 = \omega$ and $\lambda_\xi = \kappa_{\gamma+\xi}$ for $0 < \xi < \zeta$. Let $\{x_n : n \in \omega\}$ be an enumeration without repetitions of the elements of $I_0(X)$. By the induction hypothesis, for every $n < \omega$ we can consider a compact Hausdorff scattered space $X_n$ of height $\gamma + 1$ such that $I_\gamma(X_n) = \{x_n\}$ and $\mathrm{CS}(X_n) = \langle \omega \rangle_\gamma$ and in such a way that $X_n \cap X = \{x_n\}$ and $X_n \cap X_m = \emptyset$ for $n \neq m$. Then, the underlying set of $Y$ is $X \cup \bigcup\{X_n : n < \omega\}$. If $x \in X_n$ for some $n < \omega$, a basic neighbourhood of $x$ in $Y$ is an open neighbourhood of $x$ in $X_n$. And if $x \in X \setminus I_0(X)$, then a basic neighbourhood of $x$ in $Y$ is a set of the form $U \cup \bigcup\{X_n : x_n \in U, n < \omega\}$ where $U$ is an open neighbourhood of $x$ in $X$. Clearly, $Y$ is an LCS space with $\mathrm{CS}(Y) = g$. Also, by the induction hypothesis, there is an LCS space $Z$ such that $\mathrm{CS}(Z) = \langle \kappa_\xi : \xi \leq \gamma \rangle$. We may assume that $Y \cap Z = \emptyset$. Then, the topological sum of $Y$ and $Z$ is the required LCS space of cardinal sequence $f$.

Finally, assume that $\eta = \gamma + 1$ is a successor ordinal. First, suppose that $\gamma$ is a limit ordinal. If $\mathrm{cf}(\gamma) = \omega$, then as $\kappa_\gamma \leq \lambda = 2^\omega$, we can carry out the construction of the desired LCS space by using an almost disjoint family of infinite subsets of $\omega$ of size $\kappa_\gamma$. If $\mathrm{cf}(\gamma) = \omega_1$, we can proceed by means of an argument similar to the one given in the case in which $\eta$ is a limit ordinal of cofinality $\omega_2$. And if $\mathrm{cf}(\gamma) = \omega_2$, then $\kappa_\gamma = \omega$ and so the construction is straightforward. Now, suppose that $\gamma = \delta + 1$ is a successor ordinal. Since $\kappa_\gamma \leq 2^\omega$, there is an LCS space $X$ such that $|I_0(X)| = \omega$, $|I_1(X)| = \kappa_\gamma$ and $I_2(X) = \emptyset$. Then, proceeding as above, we can construct an LCS space $Y$ such that $X$ is a closed subspace of $Y$ and $\mathrm{CS}(Y) = \langle \lambda_\xi : \xi < \eta \rangle$ where $\lambda_\xi = \omega$ for $\xi \leq \delta$ and $\lambda_\gamma = \kappa_\gamma$. Also, by the induction hypothesis, there is an LCS space $Z$ such that $\mathrm{CS}(Z) = \langle \kappa_\xi : \xi \leq \delta \rangle$ and $Y \cap Z = \emptyset$. Clearly, the topological sum of $Y$ and $Z$ is the required LCS space. $\square$

## 3. A further construction of LCS spaces with countable levels

In this section, our aim is to prove the following result.

**Theorem 3.1.** *If $V = L$ holds and $\lambda$ is an uncountable cardinal, then in some cardinal-preserving generic extension we have that for every $\alpha, \beta < \omega_3$ with cf($\alpha$) $< \omega_2$ there is an LCS space $Z$ such that CS($Z$) $= \langle \omega \rangle_\alpha {}^\frown \langle \lambda \rangle_\beta$.*

In order to prove Theorem 3.1, we shall use the main result of [6]. First, we need to introduce the following notion of special function due to Koszmider.

**Definition 3.2.** Assume that $\kappa$ and $\lambda$ are infinite cardinals such that $\kappa$ is regular and $\kappa < \lambda$. We say that a function $F : [\lambda]^2 \longrightarrow \kappa^+$ is a $\kappa^+$-*strongly unbounded function on* $\lambda$, if for every ordinal $\delta < \kappa^+$, every cardinal $\nu < \kappa$ and every family $A \subset [\lambda]^\nu$ of pairwise disjoint sets with $|A| = \kappa^+$, there are different $a, b \in A$ such that $F\{\alpha, \beta\} > \delta$ for every $\alpha \in a$ and $\beta \in b$.

The following result was proved in [6].

**Theorem 3.3.** *If $\kappa$ and $\lambda$ are infinite cardinals such that $\kappa^{+++} \leq \lambda$, $\kappa^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$, then in some cardinal-preserving generic extension there is a $\kappa^+$-strongly unbounded function on $\lambda$.*

In order to prove Theorem 3.1, we need some preparation. Assume that $\lambda$ is an uncountable cardinal. If $\gamma$ is an ordinal, we put

$$Y_\gamma = \gamma \times \omega \cup (\{\gamma, \gamma + 1\} \times \lambda).$$

Let

$$\mathbb{B} = \{S\} \cup \lambda.$$

Let

$$\mathbb{B}_S^{(\gamma)} = \gamma \times \omega$$

and

$$\mathbb{B}^{(\gamma)}_{\zeta} = \{\gamma\} \times [\omega \cdot \zeta, \omega \cdot \zeta + \omega) \cup \{\langle \gamma + 1, \zeta \rangle\}$$

for $\zeta \in \lambda$.

Clearly, $\{\mathbb{B}^{(\gamma)}_t : t \in \mathbb{B}\}$ is a partition of $Y_\gamma$. We define

$$\pi_B : X \longrightarrow \mathbb{B} \text{ by the formula } x \in \mathbb{B}^{(\gamma)}_{\pi_B(x)}.$$

Then, the following notion will be used in the proof of Theorem 3.1.

**Definition 3.4.** If $\gamma$ is an ordinal, we say that $\mathcal{T} = \langle T, \preceq, i \rangle$ is an *adequate $\gamma$-poset*, if $\mathcal{T}$ is an LCS poset with $T = Y_\gamma$ such that for every $\zeta \in \lambda$ the following conditions hold:

(1) If $x \in \mathbb{B}^{(\gamma)}_{\zeta}$ with $\pi(x) = \gamma$, then $x \prec \langle \gamma + 1, \zeta \rangle$ and $x \not\prec \langle \gamma + 1, \xi \rangle$ for $\xi \neq \zeta$.

(2) The restriction of $\mathcal{T}$ to $\mathbb{B}^{(\gamma)}_{S} \cup \mathbb{B}^{(\gamma)}_{\zeta}$ is an $\langle \omega \rangle_{\gamma+1} \frown \langle 1 \rangle$-skeleton.

**Proposition 3.5.** *If there is an adequate $\omega_1$-poset, then there are an adequate $\omega$-poset and an adequate $1$-poset.*

**Proof**. Assume that there is an adequate $\omega_1$-poset. Since its cardinal sequence is $\langle \omega \rangle_{\omega_1} \frown \langle \lambda \rangle_2$, we have that $2^\omega \geq \lambda$. Then, we construct an adequate $\omega$-poset $\mathcal{T} = \langle Y_\omega, \preceq, i \rangle$ as follows. First, for every $n < \omega$, we consider an $\langle \omega \rangle_n \frown \langle 1 \rangle$-skeleton $\mathcal{T}_n = \langle T_n, \preceq_n, i_n \rangle$ such that $\{T_n : n < \omega\}$ is a partition of $\omega \times \omega$. Also, we assume that for each $n < \omega$, there is a top point $v_n$ in $T_n$ such that $u <_n v_n$ for every $u \in T_n \setminus \{v_n\}$. Now, let $Y = \{\omega\} \times \lambda$ and $Y' = \{\omega + 1\} \times \lambda$. For $\xi < \lambda$, put $y'_\xi = \langle \omega + 1, \xi \rangle$. Since $2^\omega \geq \lambda$, there is a pairwise disjoint family $\{a_\xi : \xi < \lambda\}$ of infinite subsets of $\omega$. Then, for every $\{x, y\} \in [(\omega \times \omega) \cup Y']^2$, we put $x \prec' y$ iff either $x \prec_n y$ for some $n < \omega$ or $x \in T_n$, $n \in a_\zeta$ and $y = y'_\zeta$.

Now, for $\nu < \lambda$ and $n < \omega$, we put $z_{\nu,n} = \langle \omega, \omega \cdot \nu + n \rangle$. For $\xi < \lambda$, let $\{a_{\xi,n} : n < \omega\}$ be a partition of $a_\xi$ into infinite subsets. Then, for every $\{x, y\} \in [Y_\omega]^2$, we put $x \prec y$ iff one of the following conditions holds:

(a) $x \prec' y$,
(b) for some $\nu < \lambda$, $x \in T_n$ for some $n \in a_{\nu,n}$ and $y = z_{\nu,n}$,
(c) for some $\nu < \lambda$ and $n < \omega$, $x = z_{\nu,n}$ and $y = y'_\nu$

Now, we define the infimum function $i$ as follows. Assume that $x, y \in Y_\omega$ are $\preceq$-incomparable. If $x, y \in T_n$ for some $n \in \omega$, we put $i\{x, y\} = i_n\{x, y\}$. If $x, y \in Y \cup Y'$, we define $i\{x, y\} = \{v : v = v_n \text{ for some } n < \omega, v \prec x$ and $v \prec y\}$, which is a finite set because $\{a_\xi : \xi < \lambda\}$ is pairwise disjoint. And we put $i\{x, y\} = \emptyset$ otherwise. It is easy to check that $\langle Y_\omega, \preceq, i \rangle$ is as required. And by means of a simpler argument, one can show that there is an adequate 1-poset. $\square$

In order to prove Theorem 3.1, we need to show the following lemma.

**Lemma 3.6.** *Assume that $\square_{\omega_1}$ holds and there is an adequate $\omega_1$-poset. Then, in some cardinal-preserving generic extension we have that for every $\alpha, \beta < \omega_3$ with cf$(\alpha) < \omega_2$ there is an LCS space $Z$ such that CS$(Z) = \langle\omega\rangle_\alpha {}^\frown \langle\lambda\rangle_\beta$.*

**Proof**. Since $\square_{\omega_1}$ holds, there is a cardinal-preserving generic extension $N$ where there is an LCS poset of width $\omega$ and height $\eta$ for every ordinal $\eta < \omega_3$ (see [3] or [7]). By using Proposition 3.5, we have that in $N$ there is an adequate $\gamma$-poset for $\gamma \in \{1, \omega, \omega_1\}$. Then, we prove that in $N$ there is an LCS space $Z$ such that $\mathrm{CS}(Z) = \langle\omega\rangle_\alpha {}^\frown \langle\lambda\rangle_\beta$ for every $\alpha, \beta < \omega_3$ with $\mathrm{cf}(\alpha) < \omega_2$ . Without loss of generality, we may assume that $\beta$ is an infinite successor ordinal $\beta' + 1$. Assume that $\mathrm{cf}(\alpha) = \omega_1$. Let $\mathcal{T} = \langle T, \preceq, i\rangle$ be an adequate $\omega_1$-poset. Let $\{\alpha_\xi : \xi < \omega_1\}$ be a club subset of $\alpha$ with $\alpha_0 = 0$ and $\alpha_\mu < \alpha_\xi$ for $\mu < \xi < \omega_1$. For every countable successor ordinal $\xi = \mu + 1$, we define $\delta_\xi = \mathrm{o.t.}(\alpha_\xi \setminus \alpha_\mu)$. And we write $t_\zeta = \langle\omega_1 + 1, \zeta\rangle$ for $\zeta < \lambda$. Let

$$T' = \bigcup\{\{\xi\} \times \omega : \xi \text{ is a countable successor ordinal}\} \cup \{t_\zeta : \zeta \in \lambda\}.$$

If $\xi = \mu + 1$ is a successor ordinal with $\mu \leq \omega_1$, for every $x \in \{\xi\} \times \omega$ we put $C_x = \{y \in \{\mu\} \times \omega : y \prec x\}$. Then, for every countable successor ordinal $\xi = \mu + 1$ and every $x \in \{\xi\} \times \omega$, we consider a compact Hausdorff scattered space $Z_x$ of height $\delta_\xi + 1$ such that $\mathrm{CS}(Z_x) = \langle\omega\rangle_{\delta_\xi}$ and in such a way that $I_0(Z_x) = C_x$ and $I_{\delta_\xi}(Z_x) = \{x\}$. And for every ordinal $\zeta \in \lambda$ we consider a compact Hausdorff scattered space $Z_{t_\zeta}$ of height $\beta$ such that $\mathrm{CS}(Z_{t_\zeta}) = \langle\omega\rangle_{\beta'}$ and in such a way that $I_0(Z_{t_\zeta}) = C_{t_\zeta}$ and $I_{\beta'}(Z_{t_\zeta}) = \{t_\zeta\}$. Also, we assume that for every $x \in T'$ we have $Z_x \cap T = C_x \cup \{x\}$ and that for every $x, y \in T'$ with $x \neq y$ we have $(Z_x \setminus \{x\}) \cap (Z_y \setminus \{y\}) = \emptyset$. Then, proceeding as in the proof of Theorem 2.5 (replacing in that proof $S$ with $T$ and $S'$ with $T'$), we can construct an LCS space $Z$ whose underlying set is $T \cup \bigcup\{Z_x : x \in T'\}$ such that $\mathrm{CS}(Z) = \langle\omega\rangle_\alpha {}^\frown \langle\lambda\rangle_\beta$.

If $\mathrm{cf}(\alpha) = \omega$, by using an adequate $\omega$-poset we can proceed by means of an argument similar to the one given in the preceding paragraph. And if $\alpha$ is a successor ordinal, we obtain the required LCS space by means of an adequate 1-poset. □

Now, in order to complete the proof of Theorem 3.1, suppose that $V = L$ holds and $\lambda$ is an uncountable cardinal. Without loss of generality, we may assume that $\lambda \geq \omega_3$. We need to prove the following lemma.

**Lemma 3.7.** *In some cardinal-preserving generic extension, there is an adequate $\omega_1$-poset.*

So, we will obtain the conclusion of Theorem 3.1 as an immediate consequence of Lemmas 3.6 and 3.7. Now, in order to show Lemma 3.7, note that it follows from Theorem 3.3 that there is a cardinal-preserving generic extension $N$ where there is an $\omega_1$-strongly unbounded function $F : [\lambda]^2 \longrightarrow \omega_1$.

Then, we construct an adequate $\omega_1$-poset by means of a c.c.c. notion of forcing defined in $N$.

We put $T = Y_{\omega_1}$, $T_\alpha = \{\alpha\} \times \omega$ for $\alpha < \omega_1$, $T_{\omega_1} = \{\omega_1\} \times \lambda$ and $T_{\omega_1+1} = \{\omega_1 + 1\} \times \lambda$. As above, we write $t_\xi = \langle \omega_1 + 1, \xi \rangle$ for $\xi < \lambda$. Also, we put $\mathbb{B}_t = \mathbb{B}_t^{(\omega_1)}$ for $t \in \mathbb{B}$. Then, we define in $N$ the following poset $\mathcal{P} = \langle P, \leq \rangle$. We say that $p = \langle X, \preceq, i \rangle \in P$ iff the following conditions hold:

(P1) $X \in [T]^{<\omega}$.
(P2) $\preceq$ is a partial order on $X$ such that $x \prec y$ implies $\pi(x) < \pi(y)$.
(P3) If $X \cap \mathbb{B}_\zeta \neq \emptyset$ for $\zeta \in \lambda$, then $t_\zeta \in X$.
(P4) If $x \in X \cap T_{\omega_1} \cap \mathbb{B}_\zeta$ for some $\zeta \in \lambda$, then $x \prec t_\zeta$ and $x \not\prec t_\xi$ for $\xi \neq \zeta$.
(P5) If $s, t \in \mathbb{B}_t$ for some $t \in \mathbb{B}$ with $\pi(s) = \pi(t)$ and $s \neq t$, then $i\{s, t\} = \emptyset$.
(P6) If $t \in T_{\alpha+1}$ for $\alpha \leq \omega_1$ and $s \prec t$, then there is a $v \in T_\alpha$ such that $s \preceq v \prec t$.
(P7) $\mathrm{i} : [X]^2 \longrightarrow [X]^{<\omega}$ with $i\{x, y\} = \{x\}$ if $x \prec y$, and such that if $x, y \in X$ are $\preceq$-incomparable then the following conditions hold:
(a)

$$\forall u \in X([u \preceq x \wedge u \preceq y] \text{ iff } u \preceq v \text{ for some } v \in i\{x, y\}).$$

(b) If $x, y \in T_{\omega_1} \cup T_{\omega_1+1}$ with $\pi_B(x) \neq \pi_B(y)$, then $\pi[i\{x, y\}] \subset F\{\pi_B(x), \pi_B(y)\}$.

The order on $P$ is the extension: $\langle X', \preceq', \mathrm{i}' \rangle \leq \langle X, \preceq, \mathrm{i} \rangle$ iff $X \subset X'$, $\preceq = \preceq' \cap (X \times X)$ and $\mathrm{i} \subset \mathrm{i}'$.

**Lemma 3.8.** *(a) If $p = \langle X, \preceq, i \rangle \in P$ and $t \in T \setminus X$, then there is a $p' = \langle X', \preceq', i' \rangle \in P$ with $p' \leq p$ and $t \in X'$.*

*(b) Assume that $p = \langle X, \preceq, i \rangle \in P$, $t \in X$, $\alpha < min\{\pi(t), \omega_1\}$ and $n < \omega$. Then, there is a $p' = \langle X', \preceq', i' \rangle \in P$ with $p' \leq p$ and there is an $s \in X' \setminus X$ with $\pi(s) = \alpha$ and $\rho(s) > n$ such that, for every $x \in X$, $s \preceq' x$ iff $t \preceq x$.*

**Proof**. First, we prove $(a)$. Without loss of generality, we may assume that $t \in T_{\omega_1}$. Let $\zeta \in \lambda$ be such that $t \in \mathbb{B}_\zeta$. Assume that $X \cap \mathbb{B}_\zeta = \emptyset$. Otherwise, the argument is easier. We put $p' = \langle X', \preceq', i' \rangle$ where $X' = X \cup \{t, t_\zeta\}$, $\prec' = \prec \cup \{\langle s, t_\zeta \rangle : s \preceq v \text{ for some } v \in T_{\omega_1} \cap \mathbb{B}_\zeta\} \cup \{\langle t, t_\zeta \rangle\}$, $i'\{x, y\} = i\{x, y\}$ if $\{x, y\} \in [X]^2$, $i'\{x, t_\zeta\} = \{x\}$ if $x \preceq v$ for some $v \in T_{\omega_1} \cap \mathbb{B}_\zeta$, $i'\{x, t_\zeta\} = \bigcup\{i\{x, v\} : v \in X \cap T_{\omega_1} \cap \mathbb{B}_\zeta\}$ if $x \in X$ and there is no $v \in T_{\omega_1} \cap \mathbb{B}_\zeta$ such that $x \preceq v$, $i'\{x, t\} = \emptyset$ for every $x \in X$ and $i'\{t, t_\zeta\} = \{t\}$. It is easy to check that $p'$ is as required.

Now, we prove $(b)$. Let

$$L = \{\alpha\} \cup \{\beta : \alpha < \beta < \pi(t) \wedge \exists j < \omega \ \beta + j = \pi(t)\}.$$

Let $\alpha = \alpha_0, \ldots, \alpha_\ell$ be the increasing enumeration of $L$. Since $X$ is finite, for $j \leq l$ we can pick an $s_j \in T_{\alpha_j} \setminus X$ such that $\rho(s_j) > n$ if $\alpha_j < \omega_1$ and $\pi_B(s_j) = \pi_B(t)$ if $\alpha_j = \omega_1$ and $t \in T_{\omega_1+1}$. Let $X' = X \cup \{s_j : j \leq \ell\}$ and let

$$\prec' = \prec \cup \{\langle s_j, y \rangle : t \preceq y\} \cup \{\langle s_j, s_k \rangle : j < k \leq \ell\}.$$

Now, we put $i'\{x,y\} = i\{x,y\}$ if $x, y \in [X]^2$, $i'\{s_j, s_k\} = \{s_j\}$ for $j < k \leq l$, $i'\{s_j, y\} = \{s_j\}$ if $t \preceq y$ and $i'\{s_j, y\} = \emptyset$ otherwise. It is easy to verify that $\langle X', \preceq', i'\rangle$ is as required. □

Assume that $\mathcal{P} = \langle P, \leq\rangle$ preserves cardinals. Let $G$ be a $\mathcal{P}$-generic filter. Put $p = \langle X_p, \preceq_p, i_p\rangle$ for $p \in G$. By using Lemma 3.8, it is easy to see that $T = \bigcup\{x_p : p \in G\}$ and if we put $\preceq = \bigcup\{\preceq_p : p \in G\}$ and $i = \bigcup\{i_p : p \in G\}$, then $\mathcal{T} = \langle T, \preceq, i\rangle$ is an adequate $\omega_1$-poset. Then, in order to complete the proof of Lemma 3.7 we show the following lemma.

**Lemma 3.9.** *$\mathcal{P}$ is c.c.c.*

**Proof**. Assume that $R = \langle r_\nu : \nu < \omega_1\rangle \subset P$ with $r_\nu \neq r_\mu$ for $\nu < \mu < \omega_1$. For $\nu < \omega_1$, we write $r_\nu = \langle X_\nu, \preceq_\nu, \mathrm{i}_\nu\rangle$. By thinning out $\langle r_\nu : \nu < \omega_1\rangle$ by standard combinatorial arguments, we can assume the following:

(A) (a) $\{X_\nu : \nu < \omega_1\}$ forms a $\Delta$-system with kernel $X$,
 (b) $|X_\nu| = |X_\mu|$ for $\nu < \mu < \omega_1$,
 (c) $\pi[X \cap \mathbb{B}_S]$ is an initial segment of $\pi[X_\nu]$ for each $\nu < \omega_1$,
 (d) for each $\alpha < \omega_1$, either $X_\nu \cap (\{\alpha\} \times \omega) = X \cap (\{\alpha\} \times \omega)$ for every $\nu < \omega_1$ or there is at most one $\nu < \omega_1$ such that $X_\nu \cap (\{\alpha\} \times \omega) \neq \emptyset$.

(B) If $H_\nu = \{t \in \mathbb{B} : X_\nu \cap \mathbb{B}_t \neq \emptyset\}$ for each $\nu < \omega_1$, then $\{H_\nu : \nu < \omega_1\}$ forms a $\Delta$-system such that the following conditions hold:
 (a) $|H_\nu| = |H_\mu|$ for $\nu < \mu < \omega_1$,
 (b) for every $\zeta \in \lambda$, if $X \cap \mathbb{B}_\zeta \neq \emptyset$ then $X_\nu \cap \mathbb{B}_\zeta = X \cap \mathbb{B}_\zeta$ for each $\nu < \omega_1$.

(C) For each $\nu < \mu < \omega_1$ there is an isomorphism $h = h_{\nu,\mu} : \langle X_\nu, \preceq_\nu, \mathrm{i}_\nu\rangle \longrightarrow \langle X_\mu, \preceq_\mu, \mathrm{i}_\mu\rangle$ such that:
 (a) $h \restriction X = \mathrm{id}$,
 (b) $\pi_B(x) = S$ iff $\pi_B(h(x)) = S$,
 (c) $\pi(x) = \omega_1$ iff $\pi(h(x)) = \omega_1$,
 (d) $\pi(x) = \omega_1 + 1$ iff $\pi(h(x)) = \omega_1 + 1$,
 (e) $\pi_B(x) = \pi_B(y)$ iff $\pi_B(h(x)) = \pi_B(h(y))$,
 (f) $\mathrm{i}_\nu\{x,y\} = \mathrm{i}_\mu\{x,y\}$ for all $\{x,y\} \in [X]^2$.

To verify condition $(C)(f)$, assume that $x, y \in X$ are $\preceq_\nu$-incomparable for $\nu < \omega_1$. If either $x \in \mathbb{B}_S$ or $y \in \mathbb{B}_S$, the case is obvious. So, assume that $x, y \in T_{\omega_1} \cup T_{\omega_1+1}$. If $\pi_B(x) = \pi_B(y)$, by conditions $(P4)$ and $(P5)$, we deduce that $i_\nu(x,y\} = \emptyset$ for $\nu < \omega_1$. And if $\pi_B(x) \neq \pi_B(y)$, we apply condition $(P7)(b)$ to obtain $i_\nu\{x,y\} = i_\mu\{x,y\}$ for $\nu < \mu < \omega_1$.

Let $\delta = \max(\pi[X \cap \mathbb{B}_S])$. Since $F : [\lambda]^2 \longrightarrow \omega_1$ is an $\omega_1$-strongly unbounded function, we deduce from condition $(B)$ that there are ordinals $\nu < \mu < \omega_1$ such that if we put $D = \{\xi \in \lambda : (X_\nu \setminus X) \cap \mathbb{B}_\xi \neq \emptyset\}$ and $E = \{\xi \in \lambda : (X_\mu \setminus X) \cap \mathbb{B}_\xi \neq \emptyset\}$, then $F\{\xi, \zeta\} > \delta$ for every $\xi \in D$ and every $\zeta \in E$. We show that $r_\nu$ and $r_\mu$ are compatible in $\mathcal{P}$. We put $p = r_\nu$ and $q = r_\mu$. And we write $p = \langle X_p, \preceq_p, i_p\rangle$ and $q = \langle X_q, \preceq_q, i_q\rangle$. Then, we define the extension $r = \langle X_r, \preceq_r, i_r\rangle$ of $p$ and $q$ as follows. We put $X_r = X_p \cup X_q$ and $\leq_r = \leq_p \cup \leq_q$. Clearly, $\leq_r$ is a partial order on $X_r$. So, we define the infimum function $i_r$. We put $i_r\{s,t\} = i_p\{s,t\}$ if $\{s,t\} \in [X_p]^2$, and $i_r\{s,t\} = i_q\{s,t\}$ if $\{s,t\} \in [X_q]^2$. By condition $(C)(f)$, $i_r \restriction [X]^2$ is well-defined. Now, assume that $s,t \in X_r$ with $s \in X_p \setminus X_q$ and $t \in X_q \setminus X_p$. Note that $s,t$ are not comparable in $\langle X_r, \leq_r \rangle$ and there is no $u \in (X_p \cup X_q) \setminus X$ such that $u \preceq_r s,t$. Then, we define $i_r\{s,t\} = \{u \in X \cap \mathbb{B}_S : u \prec_r s,t\}$. It is easy to check that $r \in P$, and so $r \leq p,q$. $\square$

This concludes the proofs of Lemma 3.9, Lemma 3.7 and Theorem 3.1.

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