KOSZUL COMPLEX OVER SKEW POLYNOMIAL RINGS

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Dedicated to Professor Gennady Lyubeznik on the occasion of his 60th birthday

Abstract. We construct a Koszul complex in the category of left skew polynomial rings associated to a flat endomorphism that provides a finite free resolution of an ideal generated by a Koszul regular sequence.

1. Introduction

Let A be a commutative Noetherian ring of characteristic $p > 0$ and $F : A \longrightarrow A$ the associated Frobenius map for which $F(a) = a^p$ for all $a \in A$. The study of A-modules with an action of the Frobenius map has received a lot of attention over the last decades and is at the core of the celebrated theory of tight closure developed by M. Hochster and C. Huneke in [HH90] and the theory of F-modules introduced by G. Lyubeznik [Lyu97].

To provide an action of the Frobenius on an A-module M is equivalent to give a left $A[\Theta; F]$-module structure on M. Here $A[\Theta; F]$ stands for the *left skew polynomial ring* associated to F, which is an associative, $\mathbb{N}$-graded, not necessarily commutative ring extension of A. More generally, we may also consider the skew polynomial rings $A[\Theta; F^e]$ associated to the e-th iterated Frobenius map and the graded ring $\mathcal{F}^M = \bigoplus_{e \geq 0} \mathcal{F}_e^M$ introduced by G. Lyubeznik and K. E. Smith in [LS01] that collects all the $A[\Theta; F^e]$-module structures on M or equivalently, all possible actions of F^e on M. In the case that $\mathcal{F}^M$ is principally generated then it is isomorphic to $A[\Theta; F]$. We want also to single out here that, under the terminology *skew polynomial ring of Frobenius type*, Y. Yoshino [Yos94] studied $A[\Theta; F]$ and left modules over it, and used this study to obtain some new results about tight closure theory; following the same spirit, R. Y. Sharp [Sha09] also studied the ideal theory of $A[\Theta; F]$. The case when A is a field was studied in detail by F. Enescu [Ene12].

One may also develop a dual notion of *right skew polynomial ring* associated to F that we denote $A[\varepsilon; F]$. A left $A[\varepsilon; F]$-module structure on M is then equivalent to provide an action of a Cartier operator as considered by M. Blickle and G. Böckle in [BB11]. The corresponding ring collecting all the $A[\varepsilon; F^e]$-module structures on M, that we denote $\mathcal{C}^M = \bigoplus_{e \geq 0} \mathcal{C}_e^M$, was developed by K. Schwede [Sch11] and M. Blickle [Bli13] (see [BS13] for a nice survey) and has been a hot topic in recent years because of its role in the theory of *test ideals*.

Although the theory of skew polynomial rings is classical (see [GW04, Chapter 2] and [MR01, Chapter 1, Sections 2 and 6]), the aim of this work is to go back to its basics and develop new tools that should be potentially useful in the study of modules with a Frobenius or Cartier action. Our approach will be in a slightly more general setting as we will consider a Noetherian commutative ring A (not necessarily of positive characteristic) and a ring homomorphism $\varphi : A \longrightarrow A$. Although

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most of the results are mild generalizations of the case of the Frobenius morphism we point out that this general approach has already been fruitful (see [SW07]).

Now, we overview the contents of this paper for the convenience of the reader. In Section 2 we introduce all the basics on left and right skew polynomial rings associated to a morphism φ that will be denoted by $A[\Theta; \varphi]$ and $A[\varepsilon; \varphi]$. More generally, given an A-module M, we will consider the corresponding rings $\mathcal{F}^{M,\varphi}$ and $\mathcal{C}^{M,\varphi}$. The main result of this Section is Theorem 2.11 which states that, whenever φ is a finite morphism, we have an isomorphism between the graded pieces $\mathcal{F}_e^{E_R,\varphi}$ and $\mathcal{C}_e^{R,\varphi}$ induced by Matlis duality. Here, E_R is the injective hull of the residue field of $R = A/I$ with $I \subseteq A := \mathbb{K}[\![x_1, \ldots, x_n]\!]$ being an ideal.

In Section 3 we present the main result of this work. Here we consider a flat morphism φ satisfying some extra condition (which is naturally satisfied in the case of the Frobenius morphism). First we introduce the *φ-Koszul complex* which is a Koszul-type complex associated to $x_1, \ldots, x_n \in A$ in the category of left $A[\Theta; \varphi]$-modules. In Theorem 3.14 we prove that, whenever $x_1, \ldots, x_n$ is an A-Koszul regular sequence (see Definition 3.13), the φ-Koszul complex provides a free resolution of A/I_n in the category of left $A[\Theta; \varphi]$-modules, where I_n is the ideal generated by $x_1, \ldots, x_n$. In the case where A is a regular ring of positive characteristic $p > 0$ and $\varphi = F^e$ is the e-th iteration of the Frobenius, we obtain a free resolution, in the category of left $A[\Theta; F^e]$-modules, of the ideal I_n (see Corollary 3.17).

To the best of our knowledge this is one of the first explicit examples of free resolutions of A-modules as modules over a skew polynomial ring. We hope that a development of this theory of free resolutions would provide, in the case of the Frobenius morphism, more insight in the study of modules with a Frobenius action.

2. Preliminaries

Let A be a commutative Noetherian ring and $\varphi : A \longrightarrow A$ a ring homomorphism. Associated to φ we may consider non necessarily commutative algebra extensions of A that are useful in the sense that, some non finitely generated A-modules become finitely generated when viewed over these extensions. In this section we will collect the basic facts on this theory keeping always an eye on the case of the Frobenius morphism.

First we recall that φ allows to describe a covariant functor, the *pushforward* or *restriction of scalars* functor φ_*, from the category of left A-modules to the category of left A-modules. Namely, given an A-module M, $\varphi_* M$ is the (A, A)-bimodule having the usual A-module structure on the right but the left structure is twisted by φ; that is, if we denote by $\varphi_* m$ $(m \in M)$ an arbitrary element of $\varphi_* M$ then, for any $a \in A$

$$a \cdot \varphi_* m := \varphi_*(\varphi(a)m).$$

Moreover, given a map $g : M \longrightarrow N$ of left A-modules we can define a map $\varphi_* g : \varphi_* M \longrightarrow \varphi_* N$ of left A-modules by setting, for any $m \in M$,

$$\varphi_* g(\varphi_* m) := \varphi_* g(m).$$

2.1. **Left and right skew-polynomial rings.**

The most basic ring extensions given by a ring homomorphism are the so-called *skew-polynomial rings* or *Ore extensions*. This is a classical object of study, and we refer either to [GW04, Chapter 2] or [MR01, Chapter 1, Sections 2 and 6] for further details.

Definition 2.1. Let A be a commutative Noetherian ring and $\varphi : A \longrightarrow A$ a ring homomorphism.

• *Left skew polynomial ring* of A *determined by* φ:

$$A[\Theta;\varphi] := \frac{A\langle\Theta\rangle}{\langle \Theta a - \varphi(a)\Theta \mid \quad a \in A\rangle},$$

that is, $A[\Theta;\varphi]$ is a free left A-module with basis $\{\theta^i\}_{i\in\mathbb{N}_0}$ and $\theta\cdot a = \varphi(a)\theta$ for all $a\in A$.

• *Right skew polynomial ring* of A *determined by* φ:

$$A[\varepsilon;\varphi] := \frac{A\langle\varepsilon\rangle}{\langle a\varepsilon - \varepsilon\varphi(a) \mid \quad a \in A\rangle},$$

that is, $A[\varepsilon;\varphi]$ is a free right A-module with basis $\{\varepsilon^i\}_{i\in\mathbb{N}_0}$ and $a\cdot\varepsilon = \varepsilon\varphi(a)$ for all $a\in A$.

Remark 2.2. *The reader will easily note that* $A[\Theta;\varphi]^{op}\cong A[\varepsilon;\varphi]$ *and* $A[\varepsilon;\varphi]^{op}\cong A[\Theta;\varphi]$*, where* $(-)^{op}$ *denotes the opposite ring.*

Both ring extensions are determined by the following *universal property.*

Proposition 2.3. *Let* $B = A[\Theta;\varphi]$ *be the left (resp.* $B = A[\varepsilon;\varphi]$ *be the right) skew polynomial ring of* A *determined by* φ*. Suppose that we have a ring* T*, a ring homomorphism* $\phi : A \longrightarrow T$ *and an element* $y\in T$ *such that, for each* $a\in A$*,*

$$y\phi(a) = \phi(\varphi(a))y \qquad (\text{resp. } \phi(a)y = y\phi(\varphi(a))).$$

Then, there is a unique ring homomorphism $B\xrightarrow{\psi}T$ *such that* $\psi(\Theta) = y$ *(resp.* $\psi(\varepsilon)=y$*) which makes the triangle*

$$\begin{array}{ccc} A & \hookrightarrow & B \\ & {\scriptstyle\phi}\searrow & \swarrow{\scriptstyle\psi} \\ & T & \end{array}$$

commutative.

This universal property allow us to provide more general examples of skew polynomial rings.

Example 2.4. *Let* $u\in A$*. As* $u\Theta$ *is an element of* $A[\Theta;\varphi]$ *such that, for any* $a\in A$*,*

$$(u\Theta)a = u\Theta a = u\varphi(a)\Theta = \varphi(a)(u\Theta),$$

it follows from the universal property for left skew polynomial rings that there is a unique A*–algebra homomorphism* $A[\Theta';\varphi]\longrightarrow A[\Theta;\varphi]$ *sending* Θ' *to* $u\Theta$*; we shall denote the image of this map by* $A[u\Theta;\varphi]$*. The previous argument shows that* $A[u\Theta;\varphi]$ *can be also regarded as a left skew polynomial ring. Analogously,* $A[\varepsilon u;\varphi]$ *can be also regarded as right skew polynomial ring since we have, for any* $a\in A$*,*

$$a(\varepsilon u) = \varepsilon\varphi(a)u = (\varepsilon u)\varphi(a).$$

2.1.1. *The case of Frobenius homomorphism.* We single out the case where A is a ring of positive characteristic $p>0$ and $\varphi = F$ is the Frobenius homomorphism. The specialization of Definition 2.1 to this case is the following:

• The *Frobenius skew polynomial ring* of A is the non-commutative graded ring $A[\Theta;F]$; that is, the free left A-module with basis $\{\Theta^e\}_{e\in\mathbb{N}_0}$ and right multiplication given by

$$\Theta\cdot a = a^p\Theta.$$

• The *Cartier skew polynomial ring* of A is the non-commutative graded ring $A[\varepsilon;F]$; that is, the free right A-module with basis $\{\varepsilon^e\}_{e\in\mathbb{N}_0}$ and left multiplication given by

$$a\cdot\varepsilon := \varepsilon a^p.$$

It turns out that $A[\Theta;F]$ is rarely left or right Noetherian, as it is explicitly described by Y. Yoshino in [Yos94, Theorem (1.3)]. He also provided effective descriptions of basic examples of left modules over $A[\Theta;F]$. Namely, the ring A, the localizatons A_a at elements $a\in A$ and local cohomology modules $H_I^i(A)$, where I is any ideal of A, have an abstract structure as finitely generated left $A[\Theta;F]$-modules that we collect in the following result. We refer to [Yos94, pp. 2490–2491] for further details.

Proposition 2.5. *Let A be a commutative Noetherian ring of characteristic p, let $a\in A$ be any element and let $J(a)$ denote the left ideal of $A[\Theta;F]$ generated by the infinite set*

$$\{a^{p^e-1}\Theta^e-1 \mid \quad e\in\mathbb{N}\}.$$

Then, the following statements hold.

(i) A has a natural structure as left $A[\Theta;F]$-module given by

$$A\cong A[\Theta;F]/A[\Theta;F]\langle\Theta-1\rangle\cong A[\Theta;F]/J(1).$$

Moreover, if A is a local ring:

(ii) The localization A_a has a natural structure as left $A[\Theta;F]$-module given by

$$A_a\cong A[\Theta;F]/J(a).$$

(iii) The Čech complex of A with respect to $a_1,\dots,a_t$

$$0\longrightarrow A\longrightarrow\bigoplus_{i=1}^{t}A_{a_i}\longrightarrow\bigoplus_{1\leq i<j\leq t}A_{a_ia_j}\longrightarrow\dots\longrightarrow\bigoplus_{i=1}^{t}A_{a_1\cdots\widehat{a_i}\cdots a_t}\longrightarrow A_{a_1\cdots a_t}\longrightarrow 0$$

is a complex of left $A[\Theta;F]$-modules which is isomorphic to the following complex:

$$0\longrightarrow A[\Theta;F]/J(1)\longrightarrow\bigoplus_{i=1}^{t}A[\Theta;F]/J(a_i)\longrightarrow\bigoplus_{1\leq i<j\leq t}A[\Theta;F]/J(a_ia_j)\longrightarrow\dots$$

$$\dots\longrightarrow\bigoplus_{i=1}^{t}A[\Theta;F]/J(a_1\cdots\widehat{a_i}\cdots a_t)\longrightarrow A[\Theta;F]/J(a_1\cdots a_t)\longrightarrow 0.$$

(iv) Any local cohomology module $H_I^i(A)$ has an abstract structure as a finitely generated left $A[\Theta;F]$-module. For example, if $(A,\mathfrak{m},\mathbb{K})$ is a local ring of characteristic p of dimension $d\geq 1$, and $a_1,\dots,a_d$ is a system of parameters for A, then

$$H_{\mathfrak{m}}^d(A)\cong A[\Theta;F]/\left(J(a_1\cdots a_d)+A[\Theta;F]\langle a_1,\dots,a_d\rangle\right).$$

In this section we are going to give a natural generalization of skew polynomial rings associated to a ring homomorphism φ. Before doing so, we briefly recall how to give a module structure over a skew polynomial ring by means of the so-called φ and φ^{-1}-linear maps.

Definition 2.6. Let A be a commutative Noetherian ring, $\varphi: A\longrightarrow A$ a ring homomorphism and M be an A-module. Given $\psi,\phi\in\operatorname{End}_A(M)$:

(i) We say that ψ is *φ-linear* provided $\psi(am)=\varphi(a)\psi(m)$ for any $(a,m)\in A\times M$.
(ii) We say that ϕ is *φ^{-1}-linear* provided $\phi(\varphi(a)m)=a\phi(m)$ for any $(a,m)\in A\times M$.

We denote by $\operatorname{End}_\varphi(M)$ and $\operatorname{End}_{\varphi^{-1}}(M)$ the A-endomorphisms of M which are φ-linear and φ^{-1}-linear respectively. These endomorphisms can be interpreted in terms of the pushforward functor since we have the following bijections:

$$\begin{aligned}\operatorname{End}_\varphi(M)&\longrightarrow\operatorname{Hom}_A(M,\varphi_*M)\ , & \operatorname{End}_{\varphi^{-1}}(M)&\longrightarrow\operatorname{Hom}_A(\varphi_*M,M)\\ \psi&\longmapsto[m\longmapsto\varphi_*(\psi(m))] & \psi&\longmapsto[\varphi_*m\longmapsto\psi(m)]\end{aligned}$$

An $A[\Theta;\varphi]$-module is simply an A-module M together with a suitable action of Θ on M. Actually, one only needs to consider a φ-linear map $\psi : M \longrightarrow M$ and define $\Theta \cdot m := \psi(m)$ for all $m \in M$. Analogously, an $A[\varepsilon;\varphi]$-module is an A-module M together with an action of ε given by a φ^{-1}-linear map $\phi : M \longrightarrow M$; we record all these simple remarks into the following:

Proposition 2.7. *Let M be an A-module.*

(i) There is a bijective correspondence between $\operatorname{End}_{\varphi}(M)$ and the left $A[\Theta;\varphi]$-module structures which can be attached to M.

(ii) There is a bijective correspondence between $\operatorname{End}_{\varphi^{-1}}(M)$ and the left $A[\varepsilon;\varphi]$-module structures which can be attached to M.

More generally, we may consider the e-th powers φ^e of the ring homomorphism φ and define the corresponding notion of φ^e and φ^{-e}-linear maps. We may collect all these morphisms in a suitable algebra that would provide a generalization of the Frobenius algebra introduced by G. Lyubeznik and K. E. Smith in [LS01, Definition 3.5] and the Cartier algebra considered by K. Schwede [Sch11] and generalized by M. Blickle [Bli13] (see also [BS13]).

Definition 2.8. Let A be a commutative Noetherian ring and $\varphi : A \longrightarrow A$ a ring homomorphism. Let M be an A-module. Then we define:

• *Ring of φ-linear operators* on M: Is the associative, $\mathbb{N}$-graded, not necessarily commutative ring

$$\mathcal{F}^{M,\varphi} := \bigoplus_{e\geq 0} \mathcal{F}_e^{M,\varphi},$$

where $\mathcal{F}_e^{M,\varphi} := \operatorname{Hom}_A(M, \varphi_*^e M) = \operatorname{End}_{\varphi^e}(M)$. A product is defined as

$$\psi_{e'} \cdot \psi_e := \varphi_*^e(\psi_{e'}) \circ \psi_e \in \mathcal{F}_{e+e'}^{M,\varphi}.$$

for any given $\psi_e \in \mathcal{F}_e^{M,\varphi}$ and $\psi_{e'} \in \mathcal{F}_{e'}^{M,\varphi}$.

• *Ring of φ^{-1}-linear operators* on M: Is the associative, $\mathbb{N}$-graded, not necessarily commutative ring

$$\mathcal{C}^{M,\varphi} := \bigoplus_{e\geq 0} \mathcal{C}_e^{M,\varphi},$$

where $\mathcal{C}_e^{M,\varphi} := \operatorname{Hom}_A(\varphi_*^e M, M) = \operatorname{End}_{\varphi^{-e}}(M)$.

A product is defined as

$$\phi_{e'} \cdot \phi_e := \phi_{e'} \circ \varphi_*^{e'}(\phi_e) \in \mathcal{C}_{e+e'}^{M,\varphi}$$

for any given $\phi_e \in \mathcal{C}_e^{M,\varphi}$ and $\phi_{e'} \in \mathcal{C}_{e'}^{M,\varphi}$.

Actually, the generalization of Cartier algebra given by M. Blickle in [Bli13] would be interpreted in our context as follows:

Definition 2.9. An *A-Cartier algebra with respect to φ* is an $\mathbb{N}$-graded A-algebra

$$\mathcal{C}^{\varphi} := \bigoplus_{e\geq 0} \mathcal{C}_e^{\varphi}$$

such that, for any $(a, \phi_e) \in A \times \mathcal{C}_e^{\varphi}$, we have that $a \cdot \phi_e = \phi_e \cdot \varphi^e(a)$. The A-algebra structure of $\mathcal{C}^{\varphi}$ is given by the natural map from A to $\mathcal{C}_0^{\varphi}$. We also assume that the structural map $A \longrightarrow \mathcal{C}_0^{\varphi}$ is surjective.

We point out that $\mathcal{C}^{M,\varphi}$ is generally NOT an A-Cartier algebra with respect to φ, since $\mathcal{C}_0^{M,\varphi} = \mathrm{End}_A(M)$ and therefore the natural map $A \longrightarrow \mathrm{End}_A(M)$ is, in general, not surjective. Nevertheless, if $M = A/I$ (where I is any ideal of A) then $\mathrm{End}_A(M) = A/I$ and therefore it follows that $\mathcal{C}^{A/I,\varphi}$ is an A-Cartier algebra.

Whenever the ring of φ-linear (resp. φ^{-1}-linear) operators is principally generated it is isomorphic to a left (resp. right) skew polynomial ring. This is the case of the ring of φ-linear operators of the ring A; notice that, when A is of prime characteristic and $\varphi = F$ is the Frobenius map, this was already observed by G. Lyubeznik and K. E. Smith [LS01, Example 3.6].

Example 2.10. *We have that* $\mathcal{F}^{A,\varphi} \cong A[\Theta;\varphi]$. *Indeed, fix* $e \in \mathbb{N}$ *and let* $\psi_e \in \mathcal{F}_e^{A,\varphi}$. *We point out that, for any* $a \in A$,

$$\psi_e(a) = \psi_e(a \cdot 1) = \varphi^e(a)\psi_e(1) = \psi_e(1)\varphi^e(a).$$

In this way, set

$$\begin{aligned} \mathcal{F}_e^{A,\varphi} &\xrightarrow{b_e} A\Theta^e \\ \psi_e &\longmapsto \psi_e(1)\Theta^e. \end{aligned}$$

The previous straightforward calculation shows the injectivity of this map. In fact, it is a bijective map with inverse

$$\begin{aligned} A\Theta^e &\longrightarrow \mathcal{F}_e^{A,\varphi} \\ a\Theta^e &\longmapsto a\varphi^e. \end{aligned}$$

In this way, setting $\mathcal{F}^{A,\varphi} \xrightarrow{b} A[\Theta;\varphi]$ *as the unique map of rings given in degree* e *by* b_e *it follows that* b *is an isomorphism of graded algebras.*

2.1.2. *Duality between Cartier algebras and Frobenius algebras.* Let $A = \mathbb{K}[\![x_1,\dots,x_d]\!]$ be a formal power series ring with d indeterminates over a field $\mathbb{K}$, $I \subseteq A$ an ideal and $R := A/I$. Let $\varphi : A \longrightarrow A$ be a ring homomorphism such that $\varphi_*^e A$ is finitely generated as A-module. The aim of this subsection is to establish an explicit correspondence, at the level of graded pieces, between $\mathcal{C}^{R,\varphi}$ and $\mathcal{F}^{E_R,\varphi}$ given by Matlis duality. This would extend the correspondence given in the case of the Frobenius map (cf. [BB11, Proposition 5.2] and [SY11, Theorem 1.20 and Corollary 1.21]).

Theorem 2.11. *Let* $\varphi : A \longrightarrow A$ *be a ring homomorphism such that* $\varphi_*^e A$ *is finitely generated as* A*-module. Then we have that*

$$\mathrm{Hom}_A(\varphi_*^e R, R)^\vee \cong \mathrm{Hom}_A(E_R, \varphi_*^e E_R) \quad \textit{and} \quad \mathrm{Hom}_A(E_R, \varphi_*^e E_R)^\vee \cong \mathrm{Hom}_A(\varphi_*^e R, R).$$

Before proving this theorem we have to show a previous statement which we shall need during its proof; albeit the below result was obtained by F. Enescu and M. Hochster in [EH08, Discussion (3.4)] (see also [Yos94, Lemma (3.6)]), we review here their proof for the convenience of the reader.

Lemma 2.12. *Let* $(A,\mathfrak{m},\mathbb{K}) \longrightarrow (B,\mathfrak{n},\mathbb{L})$ *be a local homomorphism of local rings, and suppose that* $\mathfrak{m}B$ *is* $\mathfrak{n}$*-primary and that* $\mathbb{L}$ *is finite algebraic over* $\mathbb{K}$ *(both these conditions hold if* B *is module-finite over* A*). Let* $E := E_A(\mathbb{K})$ *and* $E_B(\mathbb{L})$ *denote choices of injective hulls for* $\mathbb{K}$ *over* A *and for* $\mathbb{L}$ *over* B*, respectively. Then, the functor* $\mathrm{Hom}_A(-,E)$*, on* B*-modules, is isomorphic with the functor* $\mathrm{Hom}_B(-,E_B(\mathbb{L}))$.

Proof. First of all, we underline that $\mathrm{Hom}_A(-,E)$, on B-modules, can be identified via adjunction with

$$\mathrm{Hom}_A((-) \otimes_A B, E) \cong \mathrm{Hom}_B(-, \mathrm{Hom}_A(B,E))$$

and therefore $\operatorname{Hom}_A(B,E)$ is injective as B-module. Moreover, as $\mathfrak{m}B$ is $\mathfrak{n}$-primary any element of $\operatorname{Hom}_A(B,E)$ is killed by a power of $\mathfrak{n}$ and therefore

$$\operatorname{Hom}_A(B,E) \cong E_B(\mathbb{L})^{\oplus l}.$$

In this way, it only remains to check that $l=1$. Indeed, we note that

$$\operatorname{Hom}_A(\mathbb{L},E) \cong \operatorname{Hom}_A(\mathbb{L},\mathbb{K}).$$

However, as A-module, $\operatorname{Hom}_A(\mathbb{L},\mathbb{K})$ is abstractly isomorphic to $\mathbb{L}$ (here we are using the assumption that $\mathbb{L}$ is finite algebraic over $\mathbb{K}$). Thus, all these foregoing facts imply that

$$E_B(\mathbb{L}) \cong \operatorname{Hom}_A(B,E),$$

hence $\operatorname{Hom}_A((-)\otimes_A B, E) \cong \operatorname{Hom}_B(-,E_B(\mathbb{L}))$ and we get the desired conclusion. □

Proof of Theorem 2.11. First of all, we underline that

$$\operatorname{Hom}_A(\varphi_*^e R, R)^\vee \cong \operatorname{Hom}_A(E_R, \varphi_*^e(R)^\vee).$$

Now, let E_* be the injective hull of the residue field of $\varphi_*^e R$. In this way, from Lemma 2.12 we deduce that $\operatorname{Hom}_A(-,E) \cong \operatorname{Hom}_{\varphi_*^e A}(-,E_*)$ as functors of $\varphi_*^e A$-modules. Therefore, combining all these facts joint with the exactness of φ_*^e it follows that

$$\varphi_*^e(R)^\vee \cong \operatorname{Hom}_A(\varphi_*^e R, E) \cong \operatorname{Hom}_{\varphi_*^e A}(\varphi_*^e R, E_*) \cong \varphi_*^e \operatorname{Hom}_A(R,E) \cong \varphi_*^e E_R.$$

Thus, taking into account this last chain of isomorphisms one obtains the first desired conclusion.

On the other hand, using once more Lemma 2.12 it turns out that

$$\varphi_*^e(E_R)^\vee \cong \operatorname{Hom}_A(\varphi_*^e E_R, E) \cong \operatorname{Hom}_{\varphi_*^e A}(\varphi_*^e E_R, \varphi_*^e E) \cong \varphi_*^e(E_R^\vee) \cong \varphi_*^e R.$$

Thus, bearing in mind this last chain of isomorphisms it follows that

$$\operatorname{Hom}_A(E_R, \varphi_*^e E_R)^\vee \cong \operatorname{Hom}_A(\varphi_*^e(E_R)^\vee, E_R^\vee) \cong \operatorname{Hom}_A(\varphi_*^e R, R),$$

just what we finally wanted to show. □

2.1.3. *The case of Frobenius homomorphism.* Once again we single out the case where A is a ring of positive characteristic $p>0$ and $\varphi = F$ is the Frobenius homomorphism. In this case we adopt the terminology of p^e and p^{-e}-linear maps or, following [And00], *Frobenius* and *Cartier* linear maps.

Definition 2.13. Let M be an A-module and $\psi, \phi \in \operatorname{End}_A(M)$.

(i) We say that ψ is p^e-*linear* provided $\psi(am) = a^{p^e}\psi(m)$ for any $(a,m) \in A \times M$. Equivalently, $\psi \in \operatorname{Hom}_A(M, F_*^e M)$.

(ii) We say that ϕ is p^{-e}-*linear* provided $\phi(a^{p^e} m) = a\phi(m)$ for any $(a,m) \in A \times M$. Equivalently, $\phi \in \operatorname{Hom}_A(F_*^e M, M)$.

The corresponding rings of p^e and p^{-e}-linear maps are defined as follows:

Definition 2.14. Let A be a commutative Noetherian ring of prime characteristic p and let M be an A-module.

(i) The *Frobenius algebra* attached to M is the associative, $\mathbb{N}$-graded, not necessarily commutative ring

$$\mathcal{F}^M := \bigoplus_{e \geq 0} \operatorname{Hom}_A(M, F_*^e M).$$

(ii) The *Cartier algebra* attached to M is the associative, $\mathbb{N}$-graded, not necessarily commutative ring

$$\mathcal{C}^M := \bigoplus_{e \geq 0} \operatorname{Hom}_A(F_*^e M, M).$$

In this work we are mainly interested in the case where the Frobenius (resp. Cartier) algebra of a module is principally generated and thus isomorphic to the left (resp. right) skew polynomial ring. G. Lyubeznik and K. E. Smith already carried out such an example in [LS01, Example 3.7].

Example 2.15. *Let $(A, \mathfrak{m}, \mathbb{K})$ be a local ring of characteristic p. Then*

$$\mathcal{F}^{H^{\dim(A)}_{\mathfrak{m}}(A)} \cong S[\Theta; F],$$

where S denotes the S_2-ification of the completion $\widehat{A}$.

Another source of examples is given by the following result.

Proposition 2.16. *Let $(A, \mathfrak{m}, \mathbb{K})$ be a complete F-finite local ring of characteristic p and E_A will stand for a choice of injective hull of $\mathbb{K}$ over A. Then, the following statements hold.*

(i) If A is quasi Gorenstein then $\mathcal{F}^{E_A}$ is principal.
(ii) If A is normal then $\mathcal{F}^{E_A}$ is principal if and only if A is Gorenstein.
(iii) If A is a $\mathbb{Q}$-Gorenstein normal domain then $\mathcal{F}^{E_A}$ is a finitely generated A-algebra if and only if p is relatively prime with the index of A.
(iv) If A is a $\mathbb{Q}$-Gorenstein normal domain then $\mathcal{F}^{E_A}$ is principal if and only if the index of A divides $p-1$.

Proof. If A is quasi Gorenstein then $E_A \cong H^{\dim(A)}_{\mathfrak{m}}(A)$. But we have seen in Example 2.15 that, under our assumptions, $\mathcal{F}^{H^{\dim(A)}_{\mathfrak{m}}(A)} \cong A[\Theta; F]$; indeed, A is complete and any quasi Gorenstein ring is, in particular, S_2. The second part is proved in [Bli13, Example 2.7]. On the other hand, part (iii) follows combining [KSSZ14, Proposition 4.3] and [EY16, Theorem 4.5]; finally, part (iv) also follows from [KSSZ14, Proposition 4.3]. □

We point out that an explicit description of $\mathcal{F}^{E_A}$ can be obtained using the following result due to R. Fedder (cf. [Fed83, pp. 465]).

Theorem 2.17. *Let $A = \mathbb{K}[\![x_1, \ldots, x_d]\!]$ be a formal power series ring over a field $\mathbb{K}$ of prime characteristic p. Let I be an arbitrary ideal of A. Then, one has that*

$$\mathcal{F}^{E_R} \cong \bigoplus_{e \geq 0} \left\{ (I^{[p^e]} :_A I)/I^{[p^e]} \right\} \Theta^e,$$

where E denotes a choice of injective hull of $\mathbb{K}$ over A, $R := A/I$, Θ is the standard Frobenius action on E and $E_R := (0 :_E I)$.

In [ÀMBZ12] we used this result to study Frobenius algebras associated to Stanley-Reisner rings. It turns out that, whenever they are principally generated, they are isomorphic to $A[u\Theta; F]$ with $u = x_1^{p-1} \cdots x_d^{p-1}$.

3. The φ-Koszul chain complex

Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring with coefficients on a commutative ring $\mathbb{K}$, and let $\varphi : S \longrightarrow S$ be a flat map of $\mathbb{K}$–algebras satisfying the extra condition that for any $1 \leq i \leq n$, $\varphi(x_i) \in \langle x_i \rangle$. Thus, there are non–zero elements $s_1, \ldots, s_n$ of S such that $\varphi(x_i) = s_i x_i$, for each $1 \leq i \leq n$.

Remark 3.1. *If $\mathbb{K}$ is a field of positive characteristic $p > 0$ and $\varphi = F^e$ is the iterated Frobenius morphism, then this extra condition is naturally satisfied. Indeed, $F^e(x_i) = x_i^{p^e}$ so $s_i = x_i^{p^e - 1}$. More generally* [SW07, Example 2.2]*, if $\mathbb{K}$ is any field and $t \geq 1$ is an integer, then the $\mathbb{K}$–linear map φ on S sending each x_i to x_i^t also satisfies this condition; indeed, in this case, $s_i = x_i^{t-1}$.*

Our aim is to construct a Koszul complex in the category of $S[\Theta;\varphi]$ -modules that we will denote as φ-Koszul complex. To begin with, let us fix some notations. Let

$$K(x_1,\dots,x_n) :=\ 0 \longrightarrow K_n \xrightarrow{d_n} K_{n-1} \longrightarrow \dots \longrightarrow K_2 \xrightarrow{d_2} K_1 \xrightarrow{d_1} K_0 \longrightarrow 0$$

be the Koszul chain complex of S with respect to $x_1,\dots,x_n$ (regarded as a chain complex in the category of left S-modules) and suppose that each differential d_l is represented by right multiplication by matrix M_l. Moreover, for each $l \geq 0$ $M_l^{[\varphi]}$ denotes the matrix obtained by applying to each entry of M_l the map φ. In particular, for each entry m, the sign of m is equal to the sign of $\varphi(m)$.

Definition 3.2. We define the *φ-Koszul chain complex* with respect to $x_1,\dots,x_n$ as the chain complex

$$\mathrm{FK}_\bullet(x_1,\dots,x_n) :=\ 0 \longrightarrow \mathrm{FK}_{n+1} \xrightarrow{\partial_{n+1}} \mathrm{FK}_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} \mathrm{FK}_0 \longrightarrow 0.$$

Here, for each $0 \leq l \leq n+1$,

$$\mathrm{FK}_l := \bigoplus_{1\leq i_1<\dots<i_l\leq n} S[\Theta;\varphi](\mathbf{e}_{i_1}\wedge\dots\wedge\mathbf{e}_{i_l}) \ \oplus \bigoplus_{1\leq j_1<\dots<j_{l-1}\leq n} S[\Theta;\varphi](\mathbf{e}_{j_1}\wedge\dots\wedge\mathbf{e}_{j_{l-1}}\wedge u),$$

where $\mathbf{e}_1,\dots,\mathbf{e}_n$ corresponds respectively to $x_1,\dots,x_n$ and u corresponds to $\Theta-1$. Here, we are adopting the convention that $\mathrm{FK}_0 := S[\Theta;\varphi]$.

Moreover, one defines $\mathrm{FK}_l \xrightarrow{\partial_l} \mathrm{FK}_{l-1}$ as the unique homomorphism of left $S[\Theta;\varphi]$-modules which, on basic elements, acts in the following manner:

· Given $1 \leq i_1 < \dots < i_l \leq n$, set

$$\partial_l\left(\mathbf{e}_{i_1}\wedge\dots\wedge\mathbf{e}_{i_l}\right) := \sum_{r=1}^{l}(-1)^{r-1}x_{i_r}\left(\mathbf{e}_{i_1}\wedge\dots\wedge\mathbf{e}_{i_{r-1}}\wedge\mathbf{e}_{i_{r+1}}\wedge\mathbf{e}_{i_{r+2}}\wedge\dots\wedge\mathbf{e}_{i_l}\right).$$

· Given $1 \leq j_1 < \dots < j_{l-1} \leq n$, set

$$\partial_l\left(\mathbf{e}_{j_1}\wedge\dots\wedge\mathbf{e}_{j_{l-1}}\wedge u\right) := (-1)^{l-1}\left(\Theta-(s_{j_1}\cdots s_{j_{l-1}})\right)\left(\mathbf{e}_{j_1}\wedge\dots\wedge\mathbf{e}_{j_{l-1}}\right)$$
$$+\sum_{r=1}^{l-1}(-1)^{r-1}\varphi(x_{j_r})\left(\mathbf{e}_{j_1}\wedge\dots\wedge\mathbf{e}_{j_{r-1}}\wedge\mathbf{e}_{j_{r+1}}\wedge\mathbf{e}_{j_{r+2}}\wedge\dots\wedge\mathbf{e}_{j_{l-1}}\wedge u\right).$$

Before going on, we make the following useful:

Discussion 3.3. Given a free, finitely generated left S-module M (whence M is abstractly isomorphic to $S^{\oplus r}$ for some $r\in\mathbb{N}$), we denote by $S[\Theta;\varphi]\otimes_S S^{\oplus r} \xrightarrow{\lambda_M} S[\Theta;\varphi]^{\oplus r}$ the natural isomorphism of left $S[\Theta;\varphi]$-modules given by the assignment $s\otimes m \longmapsto sm$; the reader will easily note that, for each $0\leq l\leq n$, we have the following commutative diagram:

$$\begin{array}{ccc} S[\Theta;\varphi]\otimes_S K_{l+1} & \xrightarrow{\mathbb{1}_{S[\Theta;\varphi]}\otimes d_{l+1}} & S[\Theta;\varphi]\otimes_S K_l \\ \downarrow{\lambda_{K_{l+1}}} & & \downarrow{\lambda_{K_l}} \\ \bigoplus_{1\leq i_1<\dots<i_{l+1}\leq n} S[\Theta;\varphi](\mathbf{e}_{i_1}\wedge\dots\wedge\mathbf{e}_{i_{l+1}}) & \xrightarrow{d'_{l+1}} & \bigoplus_{1\leq i_1<\dots<i_l\leq n} S[\Theta;\varphi](\mathbf{e}_{i_1}\wedge\dots\wedge\mathbf{e}_{i_l}). \end{array}$$

Here, d'_{l+1} denotes the map ∂_{l+1} restricted to the direct summand

$$\bigoplus_{1\leq i_1<\dots<i_{l+1}\leq n} S[\Theta;\varphi](\mathbf{e}_{i_1}\wedge\dots\wedge\mathbf{e}_{i_{l+1}}).$$

As we shall see quickly, this fact turns out to be very useful in what follows.

The first thing we have to check out is that $\mathrm{FK}_\bullet(x_1,\ldots,x_n)$ defines a chain complex in the category of left $S[\Theta;\varphi]$-modules. This fact follows from the next:

Proposition 3.4. *For any $0\leq l\leq n$, one has that $\partial_l\partial_{l+1}=0$.*

Proof. Regarding the very definition of the ∂'s, we only have to distinguish two cases.

· Given $1\leq i_1<\ldots<i_{l+1}\leq n$ one has, keeping in mind Discussion 3.3, that

$$\begin{aligned}&\partial_l\left(\partial_{l+1}\left(\mathbf{e}_{i_1}\wedge\ldots\wedge\mathbf{e}_{i_{l+1}}\right)\right)=d_l'\left(d_{l+1}'\left(\mathbf{e}_{i_1}\wedge\ldots\wedge\mathbf{e}_{i_{l+1}}\right)\right)\\&=\left(\lambda_{K_{l-1}}\circ\left(\mathbb{1}_{S[\Theta;\varphi]}\otimes d_{l-1}\right)\circ\lambda_{K_l}^{-1}\right)\circ\left(\lambda_{K_l}\circ\left(\mathbb{1}_{S[\Theta;\varphi]}\otimes d_l\right)\circ\lambda_{K_{l+1}}^{-1}\right)\left(\mathbf{e}_{i_1}\wedge\ldots\wedge\mathbf{e}_{i_{l+1}}\right)\\&=\left(\lambda_{K_{l-1}}\circ\left(\mathbb{1}_{S[\Theta;\varphi]}\otimes (d_l\circ d_{l+1})\right)\circ\lambda_{K_{l+1}}^{-1}\right)\left(\mathbf{e}_{i_1}\wedge\ldots\wedge\mathbf{e}_{i_{l+1}}\right)\\&=\left(\lambda_{K_{l-1}}\circ\left(\mathbb{1}_{S[\Theta;\varphi]}\otimes 0\right)\circ\lambda_{K_{l+1}}^{-1}\right)\left(\mathbf{e}_{i_1}\wedge\ldots\wedge\mathbf{e}_{i_{l+1}}\right)=0;\end{aligned}$$

indeed, notice that $d_ld_{l+1}=0$ because they are the usual chain differentials in the Koszul chain complex of S with respect to $x_1,\ldots,x_n$.

· Given $1\leq j_1<\ldots<j_l\leq n$, one has that

$$\begin{aligned}&\partial_l\left(\partial_{l+1}\left(\mathbf{e}_{j_1}\wedge\ldots\wedge\mathbf{e}_{j_l}\wedge u\right)\right)=\partial_l\Big((-1)^l\left(\Theta-(s_{j_1}\cdots s_{j_l})\right)\left(\mathbf{e}_{j_1}\wedge\ldots\wedge\mathbf{e}_{j_l}\right)+\\&\sum_{r=1}^{l}(-1)^{r-1}\varphi(x_{j_r})\left(\mathbf{e}_{j_1}\wedge\ldots\wedge\mathbf{e}_{j_{r-1}}\wedge\mathbf{e}_{j_{r+1}}\wedge\mathbf{e}_{j_{r+2}}\wedge\ldots\wedge\mathbf{e}_{j_l}\wedge u\right)\Big)=\\&\sum_{r=1}^{l}(-1)^{r+l-1}\left(\varphi(x_{j_r})\Theta-(s_{j_1}\cdots s_{j_l})x_{j_r}\right)\left(\mathbf{e}_{j_1}\wedge\ldots\wedge\mathbf{e}_{j_{r-1}}\wedge\mathbf{e}_{j_{r+1}}\wedge\mathbf{e}_{j_{r+2}}\wedge\ldots\wedge\mathbf{e}_{j_l}\right)+\\&\sum_{r=1}^{l}\sum_{k=1}^{r-1}(-1)^{r+k-2}\varphi(x_{j_k}x_{j_r})\left(\mathbf{e}_{j_1}\wedge\ldots\wedge\mathbf{e}_{j_{k-1}}\wedge\mathbf{e}_{j_{k+1}}\wedge\mathbf{e}_{j_{k+2}}\wedge\ldots\wedge\mathbf{e}_{j_{r-1}}\wedge\mathbf{e}_{j_{r+1}}\wedge\mathbf{e}_{j_{r+2}}\wedge\ldots\wedge\mathbf{e}_{j_l}\wedge u\right)+\\&\sum_{r=1}^{l}\sum_{k=r}^{l}(-1)^{r+k-2}\varphi(x_{j_k}x_{j_r})\left(\mathbf{e}_{j_1}\wedge\ldots\wedge\mathbf{e}_{j_{r-1}}\wedge\mathbf{e}_{j_{r+1}}\wedge\mathbf{e}_{j_{r+2}}\wedge\ldots\wedge\mathbf{e}_{j_{k-1}}\wedge\mathbf{e}_{j_{k+1}}\wedge\mathbf{e}_{j_{k+2}}\wedge\ldots\wedge\mathbf{e}_{j_l}\wedge u\right)+\\&\sum_{r=1}^{l}(-1)^{l+r-2}\left(\varphi(x_{j_r})\Theta-(s_{j_1}\cdots s_{j_{r-1}}s_{j_{r+1}}s_{j_{r+2}}\cdots s_{j_l})\varphi(x_{j_r})\right)\left(\mathbf{e}_{j_1}\wedge\ldots\wedge\mathbf{e}_{j_{r-1}}\wedge\mathbf{e}_{j_{r+1}}\wedge\mathbf{e}_{j_{r+2}}\wedge\ldots\wedge\mathbf{e}_{j_l}\right).\end{aligned}$$

Starting from the top, the first summand (respectively, the second summand) cancels out the fourth summand (respectively, the third summand) and therefore the whole expression vanishes, just what we finally wanted to check. $\square$

Remark 3.5. *The differentials of the φ-Koszul complex*

$$\mathrm{FK}_\bullet(x_1,\ldots,x_n):=\ 0\longrightarrow\mathrm{FK}_{n+1}\xrightarrow{\partial_{n+1}}\mathrm{FK}_n\xrightarrow{\partial_n}\ldots\xrightarrow{\partial_1}\mathrm{FK}_0\longrightarrow 0.$$

are described as follows:

(1) ∂_{n+1} is represented by right multiplication by matrix $\begin{pmatrix}(-1)^n\left(\Theta-(s_1\cdots s_n)\right) & M_n^{[\varphi]}\end{pmatrix}$.

(2) For each $1\leq l\leq n-1$, ∂_{l+1} is represented by right multiplication by matrix

$$\left(\begin{array}{c|c}M_{l+1} & \mathbf{0}\\\hline (-1)^lD_l & M_l^{[\varphi]}\end{array}\right),$$

where D_l is a diagonal matrix with non-zero entries $\Theta-(s_{i_1}\cdots s_{i_l})$, $1\leq i_1<\ldots<i_l\leq n$.

(3) ∂_1 is represented by right multiplication by matrix $\begin{pmatrix}x_1 & \ldots & x_n & \Theta-1\end{pmatrix}^T$.

For example, when $n=2$, $\mathrm{FK}_\bullet(x_1,x_2)$ boils down to the chain complex

$$0 \longrightarrow S[\Theta;\varphi] \xrightarrow{\partial_3} S[\Theta;\varphi]^{\oplus 3} \xrightarrow{\partial_2} S[\Theta;\varphi]^{\oplus 3} \xrightarrow{\partial_1} S[\Theta;\varphi] \longrightarrow 0,$$

where ∂_3, ∂_2 and ∂_1 are given by right multiplication by matrices

$$\begin{pmatrix} \Theta-(s_1s_2) & -\varphi(x_2) & \varphi(x_1)\end{pmatrix} \quad \left(\begin{array}{cc|c} -x_2 & x_1 & 0 \\ \hline s_1-\Theta & 0 & \varphi(x_1) \\ 0 & s_2-\Theta & \varphi(x_2)\end{array}\right) \quad \begin{pmatrix} x_1 \\ x_2 \\ \Theta-1\end{pmatrix}$$

Now, we want to single out the following technical fact because it will play a key role during the proof of the first main result of this paper (see Theorem 3.10).

Lemma 3.6. *Preserving the notations introduced in Remark 3.5, one has, for any $0\leq l\leq n$, that $M_l^{[\varphi]}D_{l-1}=D_lM_l$.*

Proof. Fix $0\leq l\leq n$. Proposition 3.4 implies that $\partial_l\partial_{l+1}=0$. Then, according to Remark 3.5, this equality corresponds to

$$\left(\begin{array}{c|c} M_{l+1} & \mathbf{0} \\ \hline (-1)^l D_l & M_l^{[\varphi]}\end{array}\right)\left(\begin{array}{c|c} M_{l} & \mathbf{0} \\ \hline (-1)^{l-1} D_{l-1} & M_{l-1}^{[\varphi]}\end{array}\right)=\mathbf{0}.$$

In particular, we must have $(-1)^lD_lM_l+(-1)^{l-1}M_l^{[\varphi]}D_{l-1}=\mathbf{0}$, which is equivalent to say that

$$(-1)^l\left(D_lM_l-M_l^{[\varphi]}D_{l-1}\right)=\mathbf{0}.$$

Whence $M_l^{[\varphi]}D_{l-1}=D_lM_l$, just what we finally wanted to show. □

Before showing our main result, we want to establish a certain technical fact, which is interesting in its own right; namely:

Proposition 3.7. *$S[\Theta;\varphi]$ is a flat right S-module.*

Proof. By the very definition of left skew polynomial rings,

$$S[\Theta;\varphi]=\bigoplus_{e\geq 0} S\Theta^e.$$

Since a direct sum of right S–modules is flat if and only if it is so all its direct summands [Rot09, Proposition 3.46 (ii)], it is enough to check that, for any $e\geq 0$, $S\Theta^e$ is a flat right S-module.

Fix $e\geq 0$. Firstly, albeit the notation $S\Theta^e$ might suggest that it is just a left S-module, this is not the case because $S\Theta^e$ can be identified with $\Theta^e\varphi_*^eS$; from this point of view, it is clear that $S\Theta^e$ may be also regarded as a right S-module. Therefore, keeping in mind the previous identification one has that the map $\Theta^e\varphi_*^eS\longrightarrow\varphi_*^eS$ given by the assignment $\Theta^e\varphi_*^es\longmapsto\varphi_*^es$ defines an abstract isomorphism of right S-modules, whence $S\Theta^e$ is (abstractly) isomorphic to φ_*^eS in the category of right S-modules and then the result follows from the fact that φ_*^eS is a flat right S-module because of the flatness of φ; the proof is therefore completed. □

Remark 3.8. *When S is a commutative Noetherian regular local ring of prime characteristic p and F is the Frobenius map on S, the fact that $S[\Theta;F]$ is a flat right S–module was already observed by Y. Yoshino* [Yos94, Proof of Example (9.2)].

Next result provides some useful properties of $\mathrm{FK}_\bullet(x_1,\ldots,x_n)$.

Proposition 3.9. *$H_0\left(\mathrm{FK}_\bullet(x_1,\ldots,x_n)\right)\cong S/I_n$ as left $S[\Theta;\varphi]$-modules, where $I_n=\langle x_1,\ldots,x_n\rangle$; moreover, $H_{n+1}\left(\mathrm{FK}_\bullet(x_1,\ldots,x_n)\right)=0$.*

Proof. First we notice that, as in the case of the Frobenius morphism studied by Y. Yoshino (see Proposition 2.5), we have $\frac{S[\Theta;\varphi]}{S[\Theta;\varphi](\Theta-1)} \cong S$. Then, using Remark 3.5 it follows that

$$H_0\left(\mathrm{FK}_\bullet(x_1,\dots,x_n)\right) = \frac{S[\Theta;\varphi]}{\mathrm{Im}(\partial_1)} \cong \frac{S[\Theta;\varphi]}{S[\Theta;\varphi]I_n + S[\Theta;\varphi](\Theta-1)} \cong S/I_n.$$

Now, consider the composition

$$S[\Theta;\varphi] \xrightarrow{\partial_{n+1}} S[\Theta;\varphi]\oplus S[\Theta;\varphi]^{\oplus n} \xrightarrow{\pi} S[\Theta;\varphi]^{\oplus n},$$

where π denotes the corresponding projection. In this way, we have that $\pi\partial_{n+1}$ turns out to be, up to isomorphisms, $\mathbb{1}_{S[\Theta;\varphi]}\otimes d_n^{[\varphi]}$ (indeed, this fact follows directly from the commutative square established in Discussion 3.3); regardless, since $d_n^{[\varphi]}$ is an injective homomorphism between free left S-modules, and $S[\Theta;\varphi]$ is a flat right S-module (cf. Proposition 3.7), one has that $\mathbb{1}_{S[\Theta;\varphi]}\otimes d_n^{[\varphi]}$ is an injective homomorphism between free left $S[\Theta;\varphi]$-modules. Therefore, $\pi\partial_{n+1}$ is also an injective homomorphism, whence ∂_{n+1} is so. This fact concludes the proof. □

Now, we state and prove the first main result of this paper, which is the following:

Theorem 3.10. *The φ-Koszul complex $\mathrm{FK}_\bullet(x_1,\dots,x_n)$ provides a free resolution of S/I_n in the category of left $S[\Theta;\varphi]$-modules.*

Proof. By Proposition 3.9, it is enough to check, for any $1\le l\le n$, that $H_l\left(\mathrm{FK}_\bullet(x_1,\dots,x_n)\right)=0$. So, fix $1\le l\le n$. Our goal is to show that $\ker(\partial_l)\subseteq \mathrm{Im}(\partial_{l+1})$; in other words, we have to prove that the chain complex $\mathrm{FK}_{l+1}\xrightarrow{\partial_{l+1}}\mathrm{FK}_l\xrightarrow{\partial_l}\mathrm{FK}_{l-1}$ is midterm exact. First of all, remember that $\mathrm{FK}_l = K_l'\oplus K_l''$, where

$$K_l' := \bigoplus_{1\le i_1<\ldots<i_l\le n} S[\Theta;\varphi](\mathbf{e}_{i_1}\wedge\ldots\wedge\mathbf{e}_{i_l}),\ \text{and}$$
$$K_l'' := \bigoplus_{1\le j_1<\ldots<j_{l-1}\le n} S[\Theta;\varphi](\mathbf{e}_{j_1}\wedge\ldots\wedge\mathbf{e}_{j_{l-1}}\wedge u).$$

Furthermore, Discussion 3.3 implies that the chain complexes

$$K_\bullet':\quad 0\longrightarrow K_n'\xrightarrow{d_n'}K_{n-1}'\longrightarrow\ldots\longrightarrow K_2'\xrightarrow{d_2'}K_1'\xrightarrow{d_1'}K_0'\longrightarrow 0$$

and

$$(K_\bullet')^{[\varphi]}:\quad 0\longrightarrow K_n'\xrightarrow{(d_n')^{[\varphi]}}K_{n-1}'\longrightarrow\ldots\longrightarrow K_2'\xrightarrow{(d_2')^{[\varphi]}}K_1'\xrightarrow{(d_1')^{[\varphi]}}K_0'\longrightarrow 0$$

are respectively canonically isomorphic to $S[\Theta;\varphi]\otimes_S K_\bullet$ and $S[\Theta;\varphi]\otimes_S K_\bullet^{[\varphi]}$; in particular, since $S[\Theta;\varphi]$ is a flat right S-module (cf. Proposition 3.7), $K_\bullet'$ and $(K_\bullet')^{[\varphi]}$ are both acyclic chain complexes in the category of left $S[\Theta;\varphi]$-modules. On the other hand, we also have to keep in mind that d_l' and $(d_l')^{[\varphi]}$ are respectively represented by right multiplication by matrix M_l and $M_l^{[\varphi]}$ (cf. Proposition 3.9 and its corresponding notation).

Now, let $P\in\ker(\partial_l)\subseteq\mathrm{FK}_l$. Since $\mathrm{FK}_l = K_l'\oplus K_l''$, we may write $P=(P',P'')$ for certain $P'\in K_l'$ and $P''\in K_l''$; in this way, as $P\in\ker(\partial_l)$ it follows that

$$\begin{pmatrix}\mathbf{0} & \mathbf{0}\end{pmatrix} = \begin{pmatrix}P' & P''\end{pmatrix}\left(\begin{array}{c|c} M_l & \mathbf{0}\\ \hline (-1)^{l-1}D_{l-1} & M_{l-1}^{[\varphi]}\end{array}\right) = \left(P'M_l + P''(-1)^{l-1}D_{l-1}\quad P''M_{l-1}^{[\varphi]}\right),$$

which leads to the following system of equations:

$$P'M_l + P''(-1)^{l-1}D_{l-1} = \mathbf{0},\ P''M_{l-1}^{[\varphi]} = \mathbf{0}.$$

In particular, since $P''M_{l-1}^{[\varphi]} = \mathbf{0}$ one has that $P'' \in \ker((d'_{l-1})^{[\varphi]}) = \operatorname{Im}((d'_l)^{[\varphi]})$; therefore, there is $Q'' \in K''_l$ such that $Q''M_l^{[\varphi]} = P''$. Using this fact, it follows that

$$P'M_l + Q''(-1)^{l-1}M_l^{[\varphi]}D_{l-1} = 0.$$

Regardless, Lemma 3.6 tells us that $M_l^{[\varphi]}D_{l-1} = D_l M_l$, whence

$$P'M_l + Q''(-1)^{l-1}D_l M_l = 0,$$

which is equivalent to say that $\left(P' + Q''(-1)^{l-1}D_l\right) M_l = 0$. In this way, one has that

$$P' + Q''(-1)^{l-1}D_l \in \ker(d'_l) = \operatorname{Im}(d'_{l+1})$$

and therefore there exists $Q' \in K'_{l+1}$ such that $Q'M_{l+1} = P' + Q''(-1)^{l-1}D_l$.

Summing up, setting $Q := (Q', Q'') \in \mathrm{FK}_{l+1}$, it follows that

$$\begin{aligned}\begin{pmatrix} Q' & Q'' \end{pmatrix} \left(\begin{array}{c|c} M_{l+1} & \mathbf{0} \\ \hline (-1)^l D_l & M_l^{[\varphi]} \end{array}\right) &= \begin{pmatrix} Q'M_{l+1} + Q''(-1)^l D_l & Q''M_l^{[\varphi]} \end{pmatrix} \\ &= \begin{pmatrix} P' + Q''(-1)^{l-1}D_l + Q''(-1)^l D_l & P'' \end{pmatrix} = \begin{pmatrix} P' & P'' \end{pmatrix}\end{aligned}$$

and therefore we can conclude that $P \in \operatorname{Im}(\partial_{l+1})$, which is exactly what we wanted to show. □

The reader will easily note that, during the proof, we have obtained the below result; we specially thank Rishi Vyas to single out to us this fact.

Proposition 3.11. *There is a short exact sequence of chain complexes*

$$0 \to S[\Theta;\varphi]\otimes_S K_\bullet(S;x_1,\ldots,x_n) \to \mathrm{FK}_\bullet(x_1,\ldots,x_n) \to \left(S[\Theta;\varphi]\otimes_S K_\bullet(S;\varphi(x_1),\ldots,\varphi(x_n))\right)[1] \to 0$$

in the category of left $S[\Theta;\varphi]$–modules.

3.1. **The φ-Koszul chain complex in full generality.** Our next aim is to define the φ-Koszul chain complex over a more general setting, and explore some specific situations on which we can ensure that defines a finite free resolution. Let A be a commutative Noetherian ring containing a commutative subring $\mathbb{K}$, and $y_1,\ldots,y_n$ denote arbitrary elements of A; in addition, assume that we fix a flat endomorphism of $\mathbb{K}$–algebras $S := \mathbb{K}[x_1,\ldots,x_n]\xrightarrow{\varphi} S$ satisfying $\varphi(x_i) \in \langle x_i\rangle$ for any $1 \leq i \leq n$. In this section, we regard A as an S-algebra under $S\xrightarrow{\psi} A$, where $S\xrightarrow{\psi} A$ is the natural homomorphism of $\mathbb{K}$-algebras which sends each x_i to y_i. Finally, we suppose that there exists a $\mathbb{K}$–algebra homomorphism $A\xrightarrow{\Phi} A$ making the square

$$\begin{array}{ccc} S & \xrightarrow{\psi} & A \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ S & \xrightarrow[\psi]{} & A \end{array}$$

commutative.

Definition 3.12. We define the *φ-Koszul chain complex* of A with respect to $y_1,\ldots,y_n$ as the chain complex $\mathrm{FK}_\bullet(y_1,\ldots,y_n;A) := A\otimes_S \mathrm{FK}_\bullet(x_1,\ldots,x_n)$.

Under slightly different assumptions, we want to show that $\mathrm{FK}_\bullet(y_1,\ldots,y_n;A)$ still defines a finite free resolution; with this purpose in mind, first of all we review the following notion, which was introduced independently in [Bou07, Definition 2 of page 157] (using a different terminology) and by T. Kabele in [Kab71, Definition 2].

Definition 3.13. Let R be a commutative ring, let $s \in \mathbb{N}$, and let $f_1, \ldots, f_n$ be a sequence of elements in R. It is said that $f_1, \ldots, f_n$ is a *Koszul regular sequence* provided the Koszul chain complex $K_\bullet(f_1, \ldots, f_n; R)$ provides a free resolution of R/I_n, where $I_n = \langle f_1, \ldots, f_n \rangle$.

Next statement may be regarded as a generalization of Theorem 3.10; this is the main result of this section.

Theorem 3.14. *Let A be a commutative Noetherian ring containing a commutative subring $\mathbb{K}$, and let $y_1, \ldots, y_n$ be a sequence of elements in A. Moreover, we assume that Φ is flat and $y_1, \ldots, y_n$ is an A-Koszul regular sequence. Then, $\mathrm{FK}_\bullet(y_1, \ldots, y_n; A)$ defines a finite free resolution of A/I_n in the category of left $A[\Theta; \Phi]$-modules, where $I_n = \langle y_1, \ldots, y_n \rangle$.*

Proof. The proof of this result is, mutatis mutandi, the same as the one of Theorem 3.10 replacing S by A and $x_1, \ldots, x_n$ by $y_1, \ldots, y_n$. Indeed, a simple inspection of the proof of Theorem 3.10 reveals that we only used there the flatness of φ and the fact that the Koszul chain complex $K_\bullet(x_1, \ldots, x_n)$ defines a finite free resolution of $\mathbb{K}$; the proof is therefore completed. $\square$

Remark 3.15. *The global homological dimension of right skew polynomial rings was studied by K. L. Fields in [Fie69, Fie70]. An upper bound for this global dimension is $n+1$ when φ is injective and it is exactly $n+1$ in the case that φ is an automorphism. Passing to the opposite ring (see Remark 2.2) we would get analogous results for left skew polynomial rings. Notice that the length of the φ-Koszul complex $\mathrm{FK}_\bullet(y_1, \ldots, y_n; A)$ is $n+1$.*

3.2. **The case of the Frobenius homomorphism.** For the convenience of the reader, we will specialize the construction of the φ-Koszul complex to the case where A is a commutative Noetherian regular ring of positive characteristic $p > 0$, and $\varphi = F^e$ is an e-th iteration of the Frobenius morphism. In this case, we will denote this complex simply as *Frobenius-Koszul complex.* Namely we have:

$$\mathrm{FK}_\bullet(x_1, \ldots, x_n) := \; 0 \longrightarrow \mathrm{FK}_{n+1} \xrightarrow{\partial_{n+1}} \mathrm{FK}_n \xrightarrow{\partial_n} \ldots \xrightarrow{\partial_1} \mathrm{FK}_0 \longrightarrow 0.$$

where, for each $0 \leq l \leq n+1$,

$$\mathrm{FK}_l := \bigoplus_{1 \leq i_1 < \ldots < i_l \leq n} A[\Theta; F^e](\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_l}) \;\oplus \bigoplus_{1 \leq j_1 < \ldots < j_{l-1} \leq n} A[\Theta; \varphi](\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_{l-1}} \wedge u),$$

and the differentials $\mathrm{FK}_l \xrightarrow{\partial_l} \mathrm{FK}_{l-1}$ are given by:

· For $1 \leq i_1 < \ldots < i_l \leq n$, set

$$\partial_l \left(\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_l}\right) := \sum_{r=1}^{l} (-1)^{r-1} x_{i_r} \left(\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_{r-1}} \wedge \mathbf{e}_{i_{r+1}} \wedge \mathbf{e}_{i_{r+2}} \wedge \ldots \wedge \mathbf{e}_{i_l}\right).$$

· For $1 \leq j_1 < \ldots < j_{l-1} \leq n$, set

$$\partial_l \left(\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_{l-1}} \wedge u\right) := (-1)^{l-1} \left(\Theta - (x_{j_1}^{p^e-1} \cdots x_{j_{l-1}}^{p^e-1})\right) \left(\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_{l-1}}\right)$$
$$+ \sum_{r=1}^{l-1} (-1)^{r-1} x_{j_r}^{p^e} \left(\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_{r-1}} \wedge \mathbf{e}_{j_{r+1}} \wedge \mathbf{e}_{j_{r+2}} \wedge \ldots \wedge \mathbf{e}_{j_{l-1}} \wedge u\right).$$

Remark 3.16. *For $n = 2$, $\mathrm{FK}_\bullet(x_1, x_2)$ is just*

$$0 \longrightarrow A[\Theta; F^e] \xrightarrow{\partial_3} A[\Theta; F^e]^{\oplus 3} \xrightarrow{\partial_2} A[\Theta; F^e]^{\oplus 3} \xrightarrow{\partial_1} A[\Theta; F^e] \longrightarrow 0,$$

where ∂_3, ∂_2 and ∂_1 are given by right multiplication by matrices

$$\begin{pmatrix} \Theta - (x_1^{p^e-1} x_2^{p^e-1}) & -x_2^{p^e} & x_1^{p^e} \end{pmatrix} \quad \left(\begin{array}{cc|c} -x_2 & x_1 & 0 \\ \hline x_1^{p^e-1} - \Theta & 0 & x_1^{p^e} \\ 0 & x_2^{p^e-1} - \Theta & x_2^{p^e} \end{array}\right) \quad \begin{pmatrix} x_1 \\ x_2 \\ \Theta - 1 \end{pmatrix}$$

As a direct consequence of Theorem 3.14 we obtain the below:

Corollary 3.17. *Let A be a commutative Noetherian regular ring of prime characteristic p, and let $y_1, \dots, y_n$ be an A–Koszul regular sequence. Then, $\mathrm{FK}_\bullet(y_1, \dots, y_n; A)$ defines a finite free resolution of A/I_n in the category of left $A[\Theta; F^e]$-modules, where $I_n = \langle y_1, \dots, y_n \rangle$.*

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