Normal forms and Sternberg conjugation theorems for infinite dimensional coupled map lattices

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Abstract

In this paper we present local Sternberg conjugation theorems near attracting fixed points for lattice systems. The interactions are spatially decaying and are not restricted to finite distance. The conjugations obtained retain the same spatial decay. In the presence of resonances the conjugations are to a polynomial normal form that also has decaying properties.

1 Introduction

Coupled map lattices are used to model many systems in physics, chemistry and biology. They are formed by sequences of nodes, each one having its own internal dynamics and being influenced by the dynamics of other nodes through some interactions.

Its origin can be found in the first models for the dynamics of chains of particles under the action of a potential, with a nearest neighbours interaction, models which were first considered by Prandtl [29] and Dehlinger [9]. Later, these models were also considered by Frenkel and Kontorova for specific cases in dislocation models of solids in [16] and [17].

Several problems can be studied under the Frenkel-Kontorova model (or some generalization), ranging from chains of coupled pendula, dislocation dynamics and surface physics to DNA and neural dynamics. See [4] for a modern description and many applications of this model.

In statistical mechanics, coupled oscillations were used to study numerically the equirepartion of energy, starting with [11]. See also [19] for a modern treatment of the problem. Mathematically, they appear as models of discretised partial differential equations. Several objects and notions are studied in this setting, such as travelling waves, wave fronts, invariant measures and spatio-temporal chaos, see [25], [5], [2], [18], [8], [12], [23], [26]. For applications to neuroscience and biology see [10], [21], [27], [28].

One can consider higher dimensional lattices with interactions among all particles. In this case, we have to require some decay in the strength of the interaction, because as it is physically natural, the larger the separation between particles is, the smaller the force of interaction should be.

Assuming each node is represented by $\mathbb{R}^n$, in this paper we consider m-dimensional lattices modelled as

$$\ell^\infty = \ell^\infty(\mathbb{R}^n) = \{x : \mathbb{Z}^m \to \mathbb{R}^n \mid \sup_{k\in\mathbb{Z}^m} |x(k)| < \infty\}.$$

We allow each node to interact with every other node, but the strength of the interactions decay with the distance between them with a spatial decay controlled by a function $\Gamma : \mathbb{Z}^m \to [0,\infty)$ satisfying certain properties. To this end, we will use the decay functions introduced in [22] and presented in Section 2.1. We will work with differentiable maps $F : \ell^\infty \to \ell^\infty$ such that the derivative of the component of F corresponding to the i-th node with respect to the variable x_j of the j-th node satisfies

$$\Big|\frac{\partial F_i}{\partial x_j}\Big| \le C\Gamma(i-j).$$

More generally, we will work with spaces $C^r_\Gamma(\ell^\infty, \ell^\infty)$ of C^r functions having decay properties. To be able to work with them, first we have to introduce linear and multilinear maps with decay in ℓ^∞ spaces. See Sections 3 and 4 for the precise definitions and properties. These spaces or similar constructs were introduced in [22], [13], [14], [15].

In this paper we describe normal forms and Sternberg theorems [32] around fixed points in the setting described above, which give differentiable conjugations of the map to their normal forms in neighbourhoods of attracting fixed points. In absence of resonances, the normal forms reduce to the linearisation of the map at the fixed point. For more general fixed points, even in finite dimensions, the study seems to require the use of differentiable bump functions in the ambient space, as it is the case in other settings we are aware of [33], [3], [20], [7]. However, such bump functions do not exist in $\ell^\infty(\mathbb{R}^n)$.

One important consequence of our results is that the normal forms and the obtained conjugations have the same kind of decay as the original map.

From the differentiable conjugation to the linear map we can obtain several invariant manifolds: if we can linearise the map, we can find as many manifolds as linear invariant subspaces the linear map has. Among them, the slow manifolds. These define the motion which converges to the attractor the slowest, and contain the dynamics that can be observed in simulations or physical systems. These manifolds have parameterisations that decay in the aforementioned sense. This property is important in the study of statistical mechanics, see [22].

In the study of normal forms and the linearisation procedure we have to deal with cohomological equations in $\ell^\infty(\mathbb{R}^n)$, in the setting of linear maps with decay. For this, we use the so-called Sylvester operators (see [6]) and we adapt the theory to work in the space of k-linear maps with decay and study their invertibility properties. To that end we introduce the Γ-spectrum in Section 5, a tool enabling us to study these operators in this setting.

We obtain Sternberg theorems for the conjugation of a map to its linear part or to its normal form, in the case that the linear part is a contraction (Poincaré domain). Assuming decay properties for the map we obtain decay properties for the conjugating map. For the results where we allow the existence of resonances, we use a normal form theory with decay which we develop here (analogous to the standard normal form theory around a fixed point) and based on the use of

Sylvester operators in spaces of k-linear maps in $\ell^\infty(\mathbb{R}^n)$, introduced in Section 6.

We study two cases, the first one for maps that are small perturbations of an uncoupled map with equal dynamics in each node. For this class of maps we add conditions on the eigenvalues of the linearisation of the unperturbed map at the fixed point restricted to a node (all maps are the same in each node).

In the absence of resonances among eigenvalues we have the following result that gives differentiable conjugation to the linear part of the map. The norm $\|\cdot\|_\Gamma$ is introduced in Section 3 and the space of C^r functions with decay C^r_Γ is introduced in Section 4.

Theorem 1.1. *Let U be an open set of $\ell^\infty(\mathbb{R}^n)$ such that $0 \in U$. Let $F : U \to \ell^\infty(\mathbb{R}^n)$ be a C^r_Γ map of the form $F = F_0 + F_1$ where F_0 is an uncoupled map and $F_0(0) = F_1(0) = 0$. Let $A = DF_0(0)$, $B = DF_1(0)$ and $M = A + B$. Assume that $A_{ij} = \mathfrak{a}\delta_{ij}$ with $\mathfrak{a} \in L(\mathbb{R}^n, \mathbb{R}^n)$.*

Let $\operatorname{Spec}(\mathfrak{a}) = \{\lambda_1, \dots, \lambda_n\}$, $\alpha = \min_i |\lambda_i|$, $\beta = \max_i |\lambda_i|$, $\nu = \frac{\log \alpha}{\log \beta}$ and $r_0 = [\nu] + 1$. Assume

(H1) $0 < |\lambda_i| < 1, \quad 1 \le i \le n,$

(H2) $\lambda_i \neq \lambda^k, \quad k \in (\mathbb{Z}^+)^n,\ 2 \le |k| \le r_0,\ 1 \le i \le n.$

Then, if $F \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$ with $r \ge r_0$ and $\|B\|_\Gamma$ is small enough, there exists $R \in C^r_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ such that $R(0) = 0$, $DR(0) = \mathrm{Id}$ and

$$R \circ F = MR$$

in some neighborhood $U_1 \subseteq U$ of 0 in $\ell^\infty(\mathbb{R}^n)$.

If we allow for the existence of resonances, i.e. omitting Hypothesis (H2) above, we have an analogous result giving a differentiable conjugation to a polynomial normal form (Theorem 8.6 in Section 8).

The second case we consider is non-perturbative and requires conditions over the Γ-spectrum of the linear part, introduced in Section 5.

Theorem 1.2. *Let U be an open set of $\ell^\infty(\mathbb{R}^n)$ such that $0 \in U$. Let $F \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$ with $F(0) = 0$. Let $A = DF(0)$, $\alpha_\Gamma = \inf\{|\lambda| \mid \lambda \in \operatorname{Spec}_\Gamma(A)\}$, $\beta_\Gamma = \sup\{|\lambda| \mid \lambda \in \operatorname{Spec}_\Gamma(A)\}$, $\nu = \frac{\log \alpha_\Gamma}{\log \beta_\Gamma}$ and $r_0 = [\nu] + 1$. Assume*

(H1) $0 \notin \operatorname{Spec}_\Gamma(A)$ *and* $\operatorname{Spec}_\Gamma(A) \subset \mathbb{D}(0,1),$

(H2) $\operatorname{Spec}_\Gamma(A) \cap (\operatorname{Spec}_\Gamma(A))^j = \emptyset, \qquad 2 \le j \le r_0.$

Then, if $F \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$ with $r \ge r_0$ there exists $R \in C^r_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ such that $R(0) = 0$, $DR(0) = \mathrm{Id}$ and

$$R \circ F = AR$$

in a neighborhood $U_1 \subset U$ of 0.

The paper is structured as follows. Section 2.1 provides the definition of the class of decay functions we will work with and the main examples of them. Section 3 deals with linear and multilinear maps with decay while Section 4 deals with C^k maps with decay. In Section 5 we introduce the Γ-spectrum of linear map with decay. In Section 6 we study some spectral properties of Sylvester operators. Finally, Sections 7 and 8 provide the normal forms and the conjugation results respectively.

2 Lattices, decay functions and dynamical systems

In this work we will consider dynamical systems in the space of bounded sequences of points of $\mathbb{R}^n$ with indices in $\mathbb{Z}^m$. That is, we will work in the infinite product space $(\mathbb{R}^n)^{\mathbb{Z}^m}$, where as usual we will call *node* each individual $\mathbb{R}^n$ in the lattice. Associated with this space we will consider a decay function, which will control the strength of the interactions between different nodes.

The space of bounded sequences in the infinite product space $(\mathbb{R}^n)^{\mathbb{Z}^m}$ is denoted by $\ell^\infty(\mathbb{R}^n)$ and formally defined as

$$\ell^\infty(\mathbb{R}^n) = \left\{ (x_i)_{i\in\mathbb{Z}^m} \mid x_i \in \mathbb{R}^n, \sup_{i\in\mathbb{Z}^m} \|x_i\| < \infty \right\},$$

where $\|\cdot\|$ is a given norm in $\mathbb{R}^n$. We endow $\ell^\infty(\mathbb{R}^n)$ with the norm $\|x\|_\infty = \sup_{i\in\mathbb{Z}^m} \|x_i\|$ as usual. Note that if we change the norm in $\mathbb{R}^n$ we end up with an equivalent norm in $\ell^\infty(\mathbb{R}^n)$. We denote by $\mathrm{proj}_i : \ell^\infty(\mathbb{R}^n) \to \mathbb{R}^n$ the projection onto the i-th component, and the related function, $\mathrm{emb}_i : \mathbb{R}^n \to \ell^\infty(\mathbb{R}^n)$ the i-th embedding. They satisfy $\mathrm{proj}_j(\mathrm{emb}_i(u)) = 0$, $i \neq j$, and $\mathrm{proj}_i(\mathrm{emb}_i(u)) = u$ for every $u \in \mathbb{R}^n$. This embedding is an isometry if the norm in $\ell^\infty(\mathbb{R}^n)$ is induced by the norm considered in $\mathbb{R}^n$.

2.1 Decay functions in lattices

To be able to define meaningful localised perturbations in $\ell^\infty(\mathbb{R}^n)$, we consider an appropriate set of weighted Banach spaces. The main idea is that the coupling term in the perturbed system belongs to a weighted space, which controls the strength of the interaction between nodes. We should note that nearest-neighbour coupling (or any other finite rank coupling) will satisfy these hypotheses.

We will make use of the following decay functions, originally introduced in [22].

Definition 2.1. *We say that a function $\Gamma : \mathbb{Z}^m \to \mathbb{R}^+$ is a decay function when it satisfies:*

1. $\sum_{k\in\mathbb{Z}^m} \Gamma(k) \leq 1$,
2. $\sum_{k\in\mathbb{Z}^m} \Gamma(i-k)\Gamma(k-j) \leq \Gamma(i-j), \qquad i, j \in \mathbb{Z}^m.$

The first property ensures that interaction propagation related to such a decay function is finite, while the second property is akin to a triangular inequality in a discrete lattice. As pointed out by Prof. L. Sadun, the second property can be interpreted as that the sum of the interactions between two nodes through the interactions involving third nodes is dominated by the direct interaction between them.

The following proposition can be found in [22] and provides a family of examples of decay functions satisfying Definition 2.1.

Proposition 2.2. *Given $\alpha > m, \theta \geq 0$, there exists $a > 0$, depending on α, θ, m such that the function defined by*

$$\Gamma(j) = \begin{cases} a, & j = 0, \\ a|j|^{-\alpha} e^{-\theta|j|}, & j \neq 0, \end{cases}$$

is a decay function on $\mathbb{Z}^m$.

Note that the standard exponential function $\Gamma(j) = Ce^{-\theta|j|}$ is not a decay function for any $C, \theta > 0$, as proved in [22].

3 Linear and multilinear maps with decay

To define C^r maps with decay properties in lattices we need to first introduce spaces of linear and multilinear mappings with suitable decay properties. Then we can use these definitions to introduce spaces of C^r maps with these predefined decay properties for its derivatives. In this section we will define linear maps with decay and its related norm $\|\cdot\|_\Gamma$. From now on we will use $\|\cdot\|$ to denote the norm induced in the space of linear or multilinear maps by the same norm in $\ell^\infty(\mathbb{R}^n)$. All decay functions will satisfy Definition 2.1. We reproduce some statements from [22] and [13], and provide some details and some additional results for the convenience of the reader.

3.1 The space of linear maps with decay

The most natural way to define linear maps with decay is to require the components of "infinite matrices" to have decay properties with respect to their indices. This is formalised as follows. Given $\|\cdot\|$ a norm in $\mathbb{R}^n$ we define

$$L_\Gamma = L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)) = \big\{A \in L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)) \,|\, \|A\|_\Gamma < \infty\big\},$$

where

$$\|A\|_\Gamma = \max\{\|A\|, \gamma(A)\},$$

with $\|A\|$ the operator norm of A and

$$\gamma(A) = \sup_{i,k \in \mathbb{Z}^m} \sup_{\substack{\|u\| \le 1, \\ \mathrm{proj}_j u = 0,\ j \ne k}} \|(Au)_i\| \Gamma(i-k)^{-1}.$$

Remark 3.1. *We will use* L_Γ *as an abbreviation for* $L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$. *Although the definition has been written for* $\ell^\infty(\mathbb{R}^n)$*, we can define linear maps with decay among arbitrary vector subspaces* $\mathcal{E}, \mathcal{F}$ *of* $\ell^\infty(\mathbb{R}^n)$ *as*

$$L_\Gamma = L_\Gamma(\mathcal{E}, \mathcal{F}) = \big\{A \in L(\mathcal{E}, \mathcal{F}) \,|\, \|A\|_\Gamma < \infty\big\}.$$

All results stated for $L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *extend in a straightforward way to* $L_\Gamma(\mathcal{E}, \mathcal{F})$.

We can provide an interpretation of $\gamma(A)$ in terms of the elements of the infinite dimensional matrix. If we denote $A_{ij} = \mathrm{proj}_i A \,\mathrm{emb}_j$ then we have

$$\gamma(A) = \sup_{i,j \in \mathbb{Z}^m} \|A_{ij}\| \Gamma(i-j)^{-1}.$$

It is worth emphasizing that the elements A_{ij} do not determine A. We can find a specific counter-example in [13, p.2843]

Remark 3.2. *Observe that given an uncoupled linear map* $A_{ij} = \mathfrak{a}\delta_{ij}$*, with* $\mathfrak{a} \in L(\mathbb{R}^n, \mathbb{R}^n)$ *we have* $A \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$*,* $\gamma(A) = \Gamma(0)^{-1}\|\mathfrak{a}\|$ *and* $\|A\|_\Gamma = \Gamma(0)^{-1}\|\mathfrak{a}\|$.

Several basic properties follow.

Proposition 3.3. *The space* $(L_\Gamma, \|\cdot\|_\Gamma)$ *is a Banach space.*

Proposition 3.4 (Algebra properties)**.** *Let* $A, B \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$*. Then* $AB \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *and*

(1) $\gamma(AB) \leq \gamma(A)\gamma(B)$*,*

(2) $\|AB\|_\Gamma \leq \|A\|_\Gamma \|B\|_\Gamma$*.*

Remark 3.5. *Proposition 3.3 and Proposition 3.4 imply* $L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *is a Banach algebra. It has a unit element* Id *but* $\|\operatorname{Id}\|_\Gamma \neq 1$*. This makes spectral theory in* L_Γ *less straightforward, since the classic results (cf. [30]) require unit elements with norm 1 in most proofs. There is however a standard trick to overcome this difficulty, see Section 5.*

Proposition 3.6. *Let* $M_0 \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *invertible such that*

$$M_0^{-1} \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$$

and $M_1 \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *such that* $\|M_0^{-1}\|_\Gamma \|M_1\|_\Gamma < 1$*. Then* $M = M_0 + M_1$ *is invertible,* $M^{-1} \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *and*

$$\big|\|M^{-1}\|_\Gamma - \|M_0^{-1}\|_\Gamma\big| \leq \|M^{-1} - M_0^{-1}\|_\Gamma = \mathcal{O}(\|M_1\|_\Gamma).$$

Proof. Since M_0 is invertible, we can write

$$M = M_0\big(\operatorname{Id} + M_0^{-1} M_1\big).$$

Since $\|M_0^{-1}\|_\Gamma \|M_1\|_\Gamma < 1$ we can write M^{-1} as a Neumann series as

$$M^{-1} = \sum_{j=0}^{\infty} (-M_0^{-1} M_1)^j M_0^{-1} = M_0^{-1} + \sum_{j=1}^{\infty} (-M_0^{-1} M_1)^j M_0^{-1}$$

which is convergent in $\|\cdot\|_\Gamma$. □

3.2 The space of k-linear maps with decay

To characterise higher order differentiable functions with decay we inductively define multilinear maps with decay. Recall that we can define the space of k-linear maps $L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ via the identification

$$L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)) = L(\ell^\infty(\mathbb{R}^n), L^{k-1}(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))).$$

There are k possible identifications defined by the isomorphisms ι_j as follows. Given the map

$$\iota_j : L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)) \to L(\ell^\infty(\mathbb{R}^n), L^{k-1}(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))), \quad 1 \leq j \leq k,$$

and $A \in L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$, let

$$\iota_j(A)(w)(v_1, \ldots, v_{k-1}) = A(v_1, \ldots, \overbrace{w}^{j}, \ldots, v_{k-1}). \tag{3.1}$$

The maps ι_j, $1 \le j \le k$, are isometries in the corresponding operator norms. We define

$$L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)) = \big\{A \in L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)) \mid \\ \iota_p(A) \in L_\Gamma(\ell^\infty(\mathbb{R}^n), L^{k-1}(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))),\ 1 \le p \le k\big\},$$

with the norm

$$\|A\|_\Gamma = \max\{\|A\|, \gamma(A)\},$$

where

$$\gamma(A) = \max_{1\le p\le k}\{\gamma(\iota_p(A))\}.$$

Remark 3.7. *Note that this definition is consistent because we can identify $L^{k-1}(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ with the ℓ^∞ space $\ell^\infty(L^{k-1}(\ell^\infty(\mathbb{R}^n), \mathbb{R}^n))$.*

With the definition of this norm we can prove that $L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ is a Banach space using the same tools as in the proof of Proposition 3.3.

The following proposition gives bounds to the norm of multilinear contractions. These bounds are fundamental later on, since multilinear contractions appear naturally when differentiating repeatedly invariance equations, a basic step in the study of normal form equations and the parameterisation method.

Proposition 3.8. *Let $A \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$, $k \ge 2$, and $u_1, \dots, u_p \in \ell^\infty(\mathbb{R}^n)$, $1 \le p \le k-1$. Then, for any permutation of k elements $\tau \in S_k$, the map*

$$B_{\tau,u_1,\dots,u_p} : \ell^\infty(\mathbb{R}^n) \times \overset{(k-p)}{\cdots} \times \ell^\infty(\mathbb{R}^n) \to \ell^\infty(\mathbb{R}^n)$$

defined by

$$B_{\tau,u_1,\dots,u_p}(v_1, \dots, v_{k-p}) = A(\tau(v_1, \dots, v_{k-p}, u_1, \dots, u_p))$$

belongs to $L^{k-p}_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$. Moreover

$$\gamma(B_{\tau,u_1,\dots,u_p}) \le \gamma(A)\|u_1\| \cdots \|u_p\|$$

and

$$\|B_{\tau,u_1,\dots,u_p}\|_\Gamma \le \|A\|_\Gamma \|u_1\| \cdots \|u_p\|.$$

Proposition 3.8 can be used to bound Γ-norms of contractions in such a way that decay properties can be ignored except for the bound of just one component, as the next proposition shows.

From Propositions 3.4 and 3.8 we also obtain the following composition property, which will prove crucial for later developments.

Given $A \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$, $B_j \in L^{l_j}_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ for $j = 1, \dots, k$ and $w_{l_j} \in \ell^\infty(\mathbb{R}^n)^{l_j}$, we define the composition $AB_1 \cdots B_k$ by

$$AB_1 \cdots B_k(w_{l_1}, \dots, w_{l_k}) = A(B_1 w_{l_1}, \dots, B_k w_{l_k}).$$

Proposition 3.9. *If $A \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ and $B_j \in L^{l_j}_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$, for $j = 1, \dots, k$, then the composition $AB_1 \cdots B_k \in L^{l_1+\cdots+l_k}_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ and*

$$\gamma(AB_1 \cdots B_k) \le \gamma(A)\|B_1\|_\Gamma \cdots \|B_k\|_\Gamma, \tag{3.2}$$

$$\|AB_1 \cdots B_k\|_\Gamma \le \|A\|_\Gamma \|B_1\|_\Gamma \cdots \|B_k\|_\Gamma. \tag{3.3}$$

A consequence of the proof is that if $B_p = B_q$ for all $q \neq p$ the bounds can be written instead as

$$\begin{aligned}\gamma(AB \cdots B) \leq& \gamma(A)\|B\|_\Gamma \|B\| \cdots \|B\|,\\ \|AB \cdots B\|_\Gamma \leq& \|A\|_\Gamma \|B\|_\Gamma \|B\| \cdots \|B\|.\end{aligned}$$

An important special case of the previous proposition is the following result.

Corollary 3.10. *If $A \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ and $B \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ then $A \cdot B \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ and*

$$\begin{aligned}\gamma(A\,B) &\leq \gamma(A)\gamma(B),\\ \|A\,B\|_\Gamma &\leq \|A\|_\Gamma \|B\|_\Gamma.\end{aligned}$$

4 Spaces of differentiable functions with decay

With the definitions of linear and multilinear applications with decay from the previous section we are prepared to define spaces of differentiable functions with decay whose domain is an open set in $\ell^\infty(\mathbb{R}^n)$.

Definition 4.1. *Let U be an open set of $\ell^\infty(\mathbb{R}^n)$. We define*

$$\begin{aligned}C^1_\Gamma(U, \ell^\infty(\mathbb{R}^n)) = \{F \in C^1(U, \ell^\infty(\mathbb{R}^n)) \mid & \sup_{x \in U} \|F(x)\|_\infty < \infty,\\ & DF(x) \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)), \forall x \in U,\\ & \sup_{x \in U} \|DF(x)\|_\Gamma < \infty\}\end{aligned}$$

with norm

$$\|F\|_{C^1_\Gamma} = \max\big(\|F\|_{C^0}, \sup_{x \in U} \|DF(x)\|_\Gamma\big),$$

where $\|F\|_{C^0} = \sup_{x \in U} \|F(x)\|_\infty$ as usual. We can also define

$$\begin{aligned}C^1_\Gamma(U, L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))) = \big\{F \in C^1(U, L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))) \mid &\\ F(x) \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)),\ \forall x \in U,&\\ \sup_{x \in U} \|F(x)\|_\Gamma < \infty\big\}.&\end{aligned}$$

This definition is consistent since

$$L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)) \sim \ell^\infty(L^k(\ell^\infty(\mathbb{R}^n), \mathbb{R}^n)).$$

Based on the above, we define spaces of C^r_Γ functions as:

$$\begin{aligned}C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n)) = \big\{F \in C^r(U, \ell^\infty(\mathbb{R}^n)) \mid D^k F \in C^1_\Gamma(U, & L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))),\\ & 0 \leq k \leq r-1\big\}\end{aligned}$$

with norm

$$\|F\|_{C^r_\Gamma} = \max\left(\|F\|_{C^0}, \max_{0 \leq k \leq r-1} \sup_{x \in U} \|DD^k F(x)\|_\Gamma\right).$$

Remark 4.2. *The inclusions* $C^i_\Gamma \subset C^{i-1}_\Gamma$, $1 \le i \le r$, *are satisfied.*

It is easy to check that $C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$ is a Banach space. We have the following result concerning the composition of maps, which can be found in [13].

Proposition 4.3. *Let* U, V *be open sets of* $\ell^\infty(\mathbb{R}^n)$, $F \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$ *and* $G \in C^r_\Gamma(V, \ell^\infty(\mathbb{R}^n))$ *such that* $F(U) \subseteq V$. *Then*

(1) $G \circ F \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$,

(2) $\|G \circ F\|_{C^r_\Gamma} \le K(1 + \|F\|^r_{C^r_\Gamma})\|G\|_{C^r_\Gamma}$.

Remark 4.4. *An important particular case appears when* G *is a linear map in* $L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$. *In this case the estimates in the proof are much easier and the bound is*

$$\|A \circ F\|_{C^r_\Gamma} \le \|A\|_\Gamma \|F\|_{C^r_\Gamma}. \tag{4.1}$$

Theorem 4.5 (Inverse Function Theorem)**.** *Let* U *be an open set of* $\ell^\infty(\mathbb{R}^n)$ *and* $F \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$, $r \ge 1$. *Let* $p \in U$ *and* $q = F(p)$. *Assume that* $DF(p)$ *is invertible and* $DF(p)^{-1} \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$. *Then* F *is locally invertible around* p *and* $F^{-1} \in C^r_\Gamma(V, \ell^\infty(\mathbb{R}^n))$, *where* V *is a suitable neighbourhood of* q.

We defer the proof of this result until Section 5.

5 Spectral theory for Γ-coupled linear maps

In this section we recall some results in spectral theory of linear operators and also introduce a new notion, the Γ-spectrum of a linear operator in a lattice, associated to a decay function Γ satisfying the definition introduced in Section 2.1.

Given a Banach space E the space of continuous linear maps $L(E, E)$ is also a Banach space with the standard operator norm. Moreover, it is a Banach algebra with the product given by the composition of maps.

Let $E = \ell^\infty(\mathbb{R}^n)$. Given a decay function Γ as in Definition 2.1, the space $L_\Gamma(E, E)$ introduced in Section 3.1 is a Banach algebra (see Proposition 3.4).

The inclusion $L_\Gamma(E, E) \subset L(E, E)$ holds considering both spaces as sets, but $L_\Gamma(E, E)$ is not a closed subalgebra of $L(E, E)$, hence it is not a Banach subalgebra of $L(E, E)$. Indeed, consider a specific decay function:

$$\Gamma(j) = a|j|^{-\alpha} e^{-\theta|j|}, \qquad j \in \mathbb{Z}^m,$$

with $\alpha > m$, $\theta > 0$ and $a > 0$ small enough.

Consider the sequence of linear maps $\{A^k\}_{k\in\mathbb{N}}$ defined by

$$A^k = \begin{cases} A^k_{i,j} = |i-j|\Gamma(i-j), & |i-j| \le k, \\ A^k_{i,j} = 0, & \text{otherwise.} \end{cases}$$

Clearly $A^k \in L_\Gamma(E, E)$. Next we check that $\{A^k\}_{k\in\mathbb{N}}$ converges to A^∞ in

$L(E,E)$, where $A^\infty_{i,j} = |i-j|\Gamma(i-j)$, $\forall i,j \in \mathbb{Z}^m$. Indeed

$$\begin{aligned}\|A^\infty - A^k\| &= \sup_{\substack{u\in E\\ \|u\|\le 1}} \|(A^\infty - A^k)u\| = \sup_{\|u\|\le 1}\sup_{i\in\mathbb{Z}^m} \|\sum_{|i-j|>k} |i-j|\Gamma(i-j)u_j\| \\ &\le \sum_{|l|>k} |l|\Gamma(l)\end{aligned}$$

which goes to zero as $k\to\infty$ because $\sum_{l\in\mathbb{Z}^m}|l|\Gamma(l)$ is convergent provided either $\theta>0$ or $\theta=0$ and $\alpha>m+1$. However $A^\infty \notin L_\Gamma(E,E)$ because

$$\gamma(A^\infty) = \sup_{i,j\in\mathbb{Z}^m} |A_{i,j}|\Gamma(i-j)^{-1} = \sup_{i,j\in\mathbb{Z}^m} |i-j| = \infty.$$

The space $L_\Gamma(E,E)$ is a Banach algebra with the identity as unit, but $\|\operatorname{Id}\|_\Gamma = \Gamma(0)^{-1} \neq 1$. To be able to apply the general results of Banach algebras with unit, we can introduce an equivalent norm in $L_\Gamma(E,E)$, say $\|\cdot\|'$, such that $\|\operatorname{Id}\|' = 1$. The procedure is standard (see [24]). We define

$$\|A\|' = \sup\left\{\|AC\|_\Gamma,\, C\in L_\Gamma(E,E),\, \|C\|_\Gamma \le 1\right\}.$$

The properties of norm are easily checked from the definition, proving the equivalence requires the following. On one hand,

$$\|A\|' = \sup_{\|C\|_\Gamma\le 1}\|AC\|_\Gamma \le \sup_{\|C\|_\Gamma\le 1}\|A\|_\Gamma\|C\|_\Gamma = \|A\|_\Gamma,$$

on the other hand,

$$\|A\|' \ge \|A\frac{\operatorname{Id}}{\|\operatorname{Id}\|_\Gamma}\|_\Gamma = \frac{1}{\|\operatorname{Id}\|_\Gamma}\|A\|_\Gamma.$$

Finally,

$$\|\operatorname{Id}\|' = \sup_{\|C\|_\Gamma\le 1}\|\operatorname{Id}\cdot C\|_\Gamma = \sup_{\|C\|_\Gamma\le 1}\|C\|_\Gamma = 1.$$

To illustrate some features of $L_\Gamma(E,E)$ we present an example of an invertible linear map in $L_\Gamma(\ell^\infty(\mathbb{C}^n),\ell^\infty(\mathbb{C}^n))$ such that its inverse may not be in $L_\Gamma(\ell^\infty(\mathbb{C}^n),\ell^\infty(\mathbb{C}^n))$ depending on the decay function Γ considered.

Let $\ell^\infty(\mathbb{C})$ be a one dimensional lattice ($m=1$), $r\in\mathbb{N}$, $a_0,\dots,a_r\in\mathbb{C}$ and $A\in L(\ell^\infty(\mathbb{C}^n),\ell^\infty(\mathbb{C}^n))$ determined by

$$\begin{aligned} A_{i,j} &= 0, && \text{if either } j<i \text{ or } j>i+r,\\ A_{i,i+k} &= a_k, && 0\le k\le r,\end{aligned}$$

with $(Ax)_i = \sum_{j=i}^{i+r} A_{ij}x_j$.

Clearly $A\in L_\Gamma(\ell^\infty(\mathbb{C}),\ell^\infty(\mathbb{C}))$ for any decay function Γ, since

$$\gamma(A) = \sup_{i,j}|A_{ij}|\Gamma(i-j)^{-1} = \max\{|a_0|\Gamma(0)^{-1}, |a_1|\Gamma(1)^{-1},\dots,|a_r|\Gamma(r)^{-1}\} < \infty$$

and

$$\|A\| = \sup_{\|x\|\le 1}\|Ax\| = \sup_{\|x\|\le 1}\sup_{i\in\mathbb{Z}}\|(\dots,\sum_{j=i}^{i+r}A_{ij}x_j,\dots)\| = |a_0|+\dots+|a_r|.$$

We look for the inverse B of A assuming *a priori* that the inverse is upper triangular and a band matrix. That is, $B_{ij} = b_{j-i}$ for some $b_k \in \mathbb{C}$, with $b_k = 0$ if $k < 0$.

Imposing the condition $AB = \mathrm{Id}$, or equivalently

$$\sum_{k\in\mathbb{Z}} A_{ik} B_{kj} = \delta_{ij}$$

we get

$$a_0 b_{j-i} + a_1 b_{j-i-1} + \ldots + a_r b_{j-i-r} = \delta_{ij}.$$

When $i = j$ we have $a_0 b_0 = 1$. This condition implies $a_0 \neq 0$. We assume it from now on. Then we proceed by induction and recursively obtain b_j for $j > 0$. Actually b_j satisfies the r-th order linear difference equation

$$b_j = -\frac{a_1}{a_0} b_{j-1} - \frac{a_2}{a_0} b_{j-2} - \ldots - \frac{a_r}{a_0} b_{j-r}, \qquad j \geq 1,$$

with initial conditions $b_0 = 1/a_0$, $b_{-1} = 0, \ldots, b_{-r+1} = 0$.

Using the theory of linear difference equations we can compute b_j in terms of the zeros of the characteristic polynomial of this equation,

$$a_0 x^r + a_1 x^{r-1} + \ldots + a_r = 0.$$

Once we have determined b_j and hence B, we can check that indeed

$$AB = BA = \mathrm{Id}\,.$$

For this to hold we strongly use that $A_{ik} \in L(\mathbb{R}, \mathbb{R}) \sim \mathbb{R}$. It remains to check that B sends $\ell^\infty(\mathbb{R})$ to itself. This will depend on the choice of the values of a_i.

To work with a specific example, assume $r = 2$ and $a_0 = 1$. Hence we can determine the zeros of the characteristic polynomial and write the general solution of the difference equation as

$$b_j = \beta_1 \left(\frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} \right)^j + \beta_2 \left(\frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2} \right)^j, \qquad j \geq 0,$$

for suitable values β_1, β_2.

Now we can choose a_1, a_2 to adjust the growth of the coefficients b_j. For instance, taking $a_1 = -\frac{3}{4}$, $a_2 = \frac{1}{8}$, then $b_j = 2\left(\frac{1}{2}\right)^j - \left(\frac{1}{4}\right)^j$. In this case $B \in L(\ell^\infty(\mathbb{R}), \ell^\infty(\mathbb{R}))$, because $\sum_{j\geq 0} |b_j| < \infty$. With the choice of decay function $\Gamma(j) = a|j|^{-\alpha} e^{-\theta|j|}$ we have that

$$\begin{aligned}\gamma(B) &= \sup_{i,j} |B_{ij}| \Gamma(i-j)^{-1} \\ &= \max\left(\frac{1}{a}, \sup_{j-i\geq 1} \left[2\left(\frac{1}{2}\right)^{j-i} - \left(\frac{1}{4}\right)^{j-i} \right] a^{-1} |j-i|^\alpha e^{\theta|j-i|} \right)\end{aligned}$$

which is finite when $\theta < \log 2$. Hence $B \in L_\Gamma(\ell^\infty(\mathbb{R}), \ell^\infty(\mathbb{R}))$ (with this particular choice of decay function Γ) if and only if $\theta < \log 2$.

5.1 Γ-spectrum of linear maps on lattices

Consider the lattice $\ell^\infty(\mathbb{R}^n)$ and a decay function Γ. Also consider the complexified space

$$\ell^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C} \sim \ell^\infty(\mathbb{R}^n) \oplus i\ell^\infty(\mathbb{R}^n) \sim \ell^\infty(\mathbb{C}^n).$$

Let $\mathcal{E}$ be a linear subspace of $\ell^\infty(\mathbb{C}^n)$. Given $A \in L_\Gamma(\mathcal{E}, \mathcal{E})$ we define:

- Γ-resolvent of A as
$$\rho_\Gamma(A) = \{\lambda \in \mathbb{C} \,|\, A - \lambda \,\mathrm{Id} \text{ is invertible and } (A - \lambda \,\mathrm{Id})^{-1} \in L_\Gamma(\mathcal{E}, \mathcal{E})\},$$
- Γ-spectrum of A as
$$\mathrm{Spec}_\Gamma(A) = \mathbb{C} \backslash \rho_\Gamma(A),$$
- Γ-spectral radius of A as
$$r_\Gamma(A) = \sup\{|\lambda| \,|\, \lambda \in \mathrm{Spec}_\Gamma(A)\}.$$

From the definitions above it is immediate that

$$\rho_\Gamma(A) \subset \rho(A)$$

and therefore

$$\mathrm{Spec}(A) \subset \mathrm{Spec}_\Gamma(A), \qquad r(A) \leq r_\Gamma(A).$$

Also from the definitions, it is clear that $\mathrm{Spec}_\Gamma(A)$ is the spectrum of A as an element of the Banach algebra $L_\Gamma(\mathcal{E}, \mathcal{E})$. Hence all general properties of spectra in unitary Banach algebras apply to Spec_Γ.

5.2 Operational calculus

The results in this section are similar to those for the spectrum of a linear operator, a standard reference is [31]. For the remaining of this section let $\mathcal{E}$ denote a complex Banach space.

Let $A \in L_\Gamma(\mathcal{E}, \mathcal{E})$ and Ω be an open set such that $\mathrm{Spec}_\Gamma(A) \subset \Omega$. Let ω be an open set such that

$$\mathrm{Spec}_\Gamma(A) \subset \omega \subset \overline{\omega} \subset \Omega \tag{5.1}$$

and $\partial\omega$ is a finite union of closed curves.

Then, given $f : \Omega \to \mathbb{C}$ analytic we define

$$f(A) = \frac{1}{2\pi i} \int_{\partial\omega} f(z)(z - A)^{-1} \, dz.$$

This definition is independent of the choice of ω provided it satisfies the conditions above, and we have that

$$f(A) \in L_\Gamma(\mathcal{E}, \mathcal{E}).$$

In the case that f is a polynomial, $f(z) = \sum_{k=0}^m a_k z^k$, the previous definition gives $f(A) = \sum_{k=0}^m a_k A^k$.

The following proposition proves upper semicontinuity of $\mathrm{Spec}_\Gamma(A)$ with respect to A.

Proposition 5.1. *Let $A \in L_\Gamma(\mathcal{E}, \mathcal{E})$ and $\mu \in \rho_\Gamma(A)$. Then if $B \in L_\Gamma(\mathcal{E}, \mathcal{E})$ and $\|B\|_\Gamma$ is small enough, then $\mu \in \rho_\Gamma(A + B)$.*

Moreover, if $f, g : \Omega \to \mathbb{C}$ are analytic functions and $h(z) = f(z)g(z)$, hence

$$h(A) = f(A)g(A). \tag{5.2}$$

As a consequence, under these conditions if we assume $f : \Omega \to \mathbb{C}$ is not zero, thus $f(A)$ is invertible, $f(A)^{-1} \in L_\Gamma(\mathcal{E}, \mathcal{E})$ and

$$[f(A)]^{-1} = \frac{1}{2\pi i} \int_{\partial\omega} \frac{1}{f(z)} (z - A)^{-1} \, dz \tag{5.3}$$

where ω satisfies (5.1).

With these results on spectra of Γ-linear maps we can now prove the inverse function theorem in lattices with spatial decay.

Proof of Theorem 4.5. From the standard inverse function theorem in Banach spaces, F is locally invertible and F^{-1} is defined in a neighbourhood V of q. Moreover, $DF^{-1}(q) = DF(p)^{-1}$ and by the continuity of DF and continuity of Spec_Γ, $DF^{-1}(x) \in L_\Gamma$ for $x \in V$, provided V is small. Since

$$DF^{-1}(x) = \big(DF(F^{-1}(x))\big)^{-1} \tag{5.4}$$

we can obtain the higher order derivatives of F^{-1} by taking derivatives in the right hand side of (5.4). For instance,

$$\begin{aligned} D^2F^{-1}(x) &= -\big(DF(F^{-1}(x))\big)^{-1} D^2F(F^{-1}(x))(DF(F^{-1}(x)))^{-1} \\ &= -DF^{-1}(x)D^2F(F^{-1}(x))DF^{-1}(x). \end{aligned} \tag{5.5}$$

Then, by Proposition 3.9, we have $D^2F^{-1}(x) \in L^2_\Gamma$. Proceeding in the same way for the other derivatives we get that $F^{-1} \in C^r_\Gamma(V, \ell^\infty(\mathbb{R}^n))$. Alternatively, we can use (5.5) to prove inductively that $F^{-1} \in C^i_\Gamma$ assuming $F^{-1} \in C^{i-1}_\Gamma$, for $i \leq r$.

□

5.3 Spectral projections associated to a gap in the Γ-spectrum

We can adapt the spectral projection theorem (see for instance [31]) to the setting of Γ-spectrum. The statements and proofs are very similar to the ones corresponding to $L(\mathcal{E}, \mathcal{E})$ but we give them here for the sake of completeness.

Assume that

$$\mathrm{Spec}_\Gamma(A) = \sigma_1 \cup \sigma_2,$$

with

$$\sigma_i \subset \omega_i \subset \overline{\omega_i} \subset \Omega_i, \qquad i = 1, 2,$$

where Ω_i are disjoint open sets and ω_i are open sets such that $\partial\omega_i$ are finite union of simple closed curves.

We define

$$P = \frac{1}{2\pi i} \int_{\partial\omega_1} (z - A)^{-1} \, dz.$$

Results on integration of elements of L_Γ can be found in [13].

Lemma 5.2. *We have*

(i) $P \in L_\Gamma(\mathcal{E}, \mathcal{E})$,

(ii) $P^2 = P$,

(iii) $P(\mathcal{E})$ *and* $\mathrm{Ker}(P)$ *are closed and invariant.*

Proof. Part (i) follows from the properties of integrals of functions in the Banach algebra L_Γ in [13]. Part (ii) follows from the fact that P can be written as

$$\frac{1}{2\pi i}\int_{\partial\omega} f(z)(z-A)^{-1}\,dz,$$

with $f : \Omega_1 \to \mathbb{C}$, defined by $f(z) = 1$. Since $f(z) = f(z)f(z)$, by (5.2)

$$PP = f(A)f(A) = f(A) = P,$$

proving P is a projection.

For Part (iii), $P(\mathcal{E})$ and $\mathrm{Ker}(P)$ are invariant when P is a projection, and $\mathcal{E} = P(\mathcal{E}) \oplus \mathrm{Ker}(P)$. Moreover since $P(\mathcal{E}) = \mathrm{Ker}(\mathrm{Id} - P)$ and P is continuous, both $P(\mathcal{E})$ and $\mathrm{Ker}(\mathrm{Id} - P)$ are closed.

□

We denote $\mathcal{E}^1 = P(\mathcal{E})$ and $\mathcal{E}^2 = (\mathrm{Id} - P)(\mathcal{E}) = \mathrm{Ker}(P)$ and $A_i = A|_{\mathcal{E}_i}$.

Theorem 5.3. *We have that*

$$\mathrm{Spec}_\Gamma(A_i) = \sigma_i, \qquad i = 1, 2.$$

Proof. Let

$$f(z) = \begin{cases} 1, & \text{if } z \in \Omega_1, \\ 0, & \text{if } z \in \Omega_2. \end{cases}$$

Moreover, let $\lambda \notin \sigma_1$ and $g_1(z) = \frac{f(z)}{\lambda - z}$. The function g_1 is analytic in a neighbourhood U of $\mathrm{Spec}_\Gamma(A)$ and satisfies

$$\begin{aligned} f(z) &= (\lambda - z)g_1(z), \\ f(z)g_1(z) &= g_1(z), \end{aligned}$$

for $z \in U$.

By (5.2),

$$\begin{aligned} f(A) &= (\lambda - A)g_1(A), \\ f(A)g_1(A) &= g_1(A), \end{aligned} \tag{5.6}$$

and hence

$$P = (\lambda - A)g_1(A) = g_1(A)(\lambda - A), \tag{5.7}$$

$$Pg_1(A) = g_1(A)P = g_1(A). \tag{5.8}$$

If $x \in \mathcal{E}^1$,

$$g_1(A)x = Pg_1(A)x = P(g_1(A)x) \in \mathcal{E}^1.$$

Moreover, from (5.6)

$$g_1(A) = (\lambda - A)^{-1}\big|_{\mathcal{E}^1} = (\lambda - A_1)^{-1}$$

which implies that $\lambda \in \rho_\Gamma(A_1)$. Therefore $\mathrm{Spec}_\Gamma(A_1) \subset \sigma_1$. Since $\mathrm{Id} - P$ is a projection onto $\mathcal{E}^2$ a completely analogous argument shows that $\mathrm{Spec}_\Gamma(A_2) \subset \sigma_2$. Indeed, given $\lambda \notin \sigma_2$, let $g_2(z) = \frac{1-f(z)}{\lambda - z}$. Then

$$(\mathrm{Id} - P) = (\lambda - A)g_2(A) = g_2(A)(\lambda - A),$$
$$(\mathrm{Id} - P)g_2(A) = g_2(A)(\mathrm{Id} - P) = g_2(A)$$

and

$$g_2(A) = (\lambda - A)^{-1}\big|_{\mathcal{E}^2} = (\lambda - A_2)^{-1}.$$

Now suppose that $\lambda \in \rho_\Gamma(A_1) \cap \rho_\Gamma(A_2)$. Since $g_1(z) + g_2(z) = \frac{1}{\lambda - z}$, by (5.3) we have

$$\begin{aligned}(\lambda - A)^{-1} &= g_1(A) + g_2(A) = g_1(A)P + g_2(A)(\mathrm{Id} - P)\\ &= (\lambda - A_1)^{-1}P + (\lambda - A_2)^{-1}(\mathrm{Id} - P).\end{aligned}$$

Then $\lambda \in \rho_\Gamma(A)$, which implies that

$$\mathrm{Spec}_\Gamma(A) \subset \mathrm{Spec}_\Gamma(A_1) \cup \mathrm{Spec}_\Gamma(A_2),$$

and therefore

$$\sigma_i \subset \mathrm{Spec}_\Gamma(A_i), \qquad i = 1, 2.$$

□

6 Sylvester operators in L^k_Γ

In this section we will introduce Sylvester operators and prove some results in spaces of k linear maps with decay.

Definition 6.1. *Let $E = \ell^\infty(\mathbb{R}^n)$. Given $A, B \in L_\Gamma(E, E)$ we define the operators*

$$\mathcal{R}_{j,A} : L^k_\Gamma(E, E) \to L^k_\Gamma(E, E), \qquad 1 \le j \le k,$$

by

$$\mathcal{R}_{j,A}(W)(u_1, \dots, u_k) = W(u_1, \dots, Au_j, \dots, u_k),$$

and $\mathcal{L}_B, \mathcal{S}_{B,A} : L^k_\Gamma(E, E) \to L^k_\Gamma(E, E)$ by

$$\mathcal{L}_B(W)(u_1, \dots, u_k) = BW(u_1, \dots, u_k),$$
$$\mathcal{S}_{B,A}(W)(u_1, \dots, u_k) = BW(Au_1, \dots, Au_k),$$

respectively.

Note that by Proposition 3.9, if $W \in L^k_\Gamma(E, E)$ then $\mathcal{R}_{j,A}(W), \mathcal{L}_B(W)$ and $\mathcal{S}_{B,A}(W)$ are in $L^k_\Gamma(E, E)$ so that the operators are well defined.

Given two subsets X, Y of $\mathbb{C}$ we denote by $X \cdot Y$ the set

$$X \cdot Y = \{x \cdot y \,|\, x \in X, \, y \in Y\}.$$

Analogously, we define

$$X^k = X \cdot \overset{k}{\ldots} \cdot X.$$

Proposition 6.2. *We have*

$$\mathrm{Spec}(\mathcal{S}_{B,A}, L^k_\Gamma(E,E)) \subset \mathrm{Spec}_\Gamma(B)\cdot(\mathrm{Spec}_\Gamma(A))^k, \qquad k\in\mathbb{N}.$$

The proof of this proposition is a consequence of the following theorem and the next lemma.

Theorem 6.3. *[Theorem 11.23, [30]] Let $\mathfrak{a}$ and $\mathfrak{b}$ be two commuting elements in a unitary Banach algebra. Then*

$$\mathrm{Spec}(\mathfrak{a}\mathfrak{b}) \subseteq \mathrm{Spec}(\mathfrak{a})\cdot\mathrm{Spec}(\mathfrak{b}).$$

Lemma 6.4. *Given $A, B \in L_\Gamma(E,E), k\in\mathbb{N}, 1\le j\le k$, then*

$$\mathrm{Spec}(\mathcal{R}_{j,A}, L^k_\Gamma(E,E)) \subset \mathrm{Spec}_\Gamma(A),$$
$$\mathrm{Spec}(\mathcal{L}_B, L^k_\Gamma(E,E)) \subset \mathrm{Spec}_\Gamma(B).$$

Proof. Let $\lambda\in\rho_\Gamma(A)$, thus $(A-\lambda\,\mathrm{Id})^{-1}\in L_\Gamma(E,E)$. To study the invertibility of $\mathcal{R}_{j,A}-\lambda\,\mathrm{Id}$ we consider the equation

$$W(u_1,\dots,Au_j,\dots,u_k)-\lambda W(u_1,\dots,u_j,\dots,u_k)=H(u_1,\dots,u_j,\dots,u_k),$$

for W, $H\in L^k_\Gamma(E,E)$, which is equivalent to

$$W(u_1,\dots,(A-\lambda\,\mathrm{Id})u_j,\dots,u_k)=H(u_1,\dots,u_j,\dots,u_k).$$

Formally,

$$W=\mathcal{R}_{j,(A-\lambda\,\mathrm{Id})^{-1}}H$$

and hence $W\in L^k_\Gamma(E,E)$ and $\lambda\in\rho(\mathcal{R}_{j,A})$.

The proof of the result for $\mathcal{L}_B$ is completely analogous.

□

Proof of Proposition 6.2. It follows directly from the fact that

$$\mathcal{S}_{B,A}=\mathcal{L}_B\circ\mathcal{R}_{1,A}\circ\ldots\circ\mathcal{R}_{k,A} \tag{6.1}$$

and the fact that the operators on the r.h.s. of (6.1) commute. Then Theorem 6.3 proves the result.

□

7 Normal forms of maps in lattices

In this section we consider the computation of normal forms around a fixed point of a map in a lattice, assuming the map has decay properties. We estimate the decay properties of the normal form and the transformation leading to it.

To study the decay properties of normal forms we will use Sylvester operators in spaces with decay, introduced in the previous section.

We consider an open set U of $\ell^\infty(\mathbb{R}^n)$ such that $0\in U$ and a map

$$F: U\to\ell^\infty(\mathbb{R}^n)$$

such that $F(0)=0$ and $F\in C^r_\Gamma(U,\ell^\infty(\mathbb{R}^n))$. Let $A=DF(0)$, with $A\in L_\Gamma(\ell^\infty(\mathbb{R}^n),\ell^\infty(\mathbb{R}^n))$, invertible and consider its Γ-spectrum $\mathrm{Spec}_\Gamma(A)$.

Theorem 7.1. *In the previously described setting there exist polynomials* $K \in C^\infty_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *and* $H \in C^\infty_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *of degree at most* r *such that* $K(0) = 0$*,* $DK(0) = \mathrm{Id}$ *and*

$$F \circ K(x) - K \circ H(x) = o(\|x\|^r)$$

and $H(x) = Ax + \sum_{j \in J} H_j x^{\otimes j}$ *with* $H_j \in L^j_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *where*

$$J = \{2 \leq j \leq r \,|\, (\mathrm{Spec}_\Gamma(A))^j \cap \mathrm{Spec}_\Gamma(A) \neq \emptyset\}.$$

Corollary 7.2. *Under the conditions of the previous theorem, if*

$$(\mathrm{Spec}_\Gamma(A))^j \cap \mathrm{Spec}_\Gamma(A) = \emptyset, \qquad 2 \leq j \leq r,$$

then there exists a polynomial $K \in C^\infty_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *such that*

$$F \circ K(x) - K \circ Ax = o(\|x\|^r).$$

Proof of Theorem 7.1. We look for K and H in the form

$$K(x) = \sum_{j=1}^{r} K_j x^{\otimes j},$$
$$H(x) = \sum_{j=1}^{r} H_j x^{\otimes j},$$

where $K_j, H_j \in L^j_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$. Taking derivatives on both sides of

$$F \circ K = K \circ H \tag{7.1}$$

and evaluating at 0 we have

$$A\, K_1 = K_1\, H_1.$$

This equation has the obvious solution $K_1 = \mathrm{Id}$, $H_1 = A$, although other solutions are possible, for instance taking K_1 as any linear map which commutes with A, like $K_1 = \alpha\,\mathrm{Id}$, $\alpha \in \mathbb{R}$ and $H_1 = A$.

Taking k-th order derivatives on both sides of (7.1), using the Faà di Bruno formula,

$$\sum_{j=1}^{k} \sum_{\substack{i_1,\ldots,i_j \geq 1 \\ i_1+\ldots+i_j=k}} C D^j F \circ K(D^{i_1}K \cdots D^{i_j}K)$$
$$= \sum_{j=1}^{k} \sum_{\substack{i_1,\ldots,i_j \geq 1 \\ i_1+\ldots+i_j=k}} C D^j K \circ H(D^{i_1}H \cdots D^{i_j}H)$$

(where for the sake of simplicity we have not written the dependence of C on the indices), and evaluating the derivatives at 0 we can write

$$AK_k + G^1_k = H_k + K_k A^{\otimes r} + G^2_k, \tag{7.2}$$

where G_k^1, G_k^2 are k-linear maps which depend on $D^jF(0)$, $2 \le j \le r$, and K_i, H_i, $2 \le i \le r-1$.

Let $G_k = G_k^1 - G_k^2$. Observe that G_k consists of sums and contractions of multilinear operators.

Using Sylvester operators (introduced in Section 6) we rewrite Equation (7.2) as

$$\left(\mathcal{S}_{A^{-1},A} \; - \mathrm{Id}\right) K_k = A^{-1}(-H_k + G_k). \tag{7.3}$$

Now we proceed inductively from $k = 2$ up to $k = r$. Assume that for j up to the $(k-1)$-th step we have obtained K_j and H_j in $L^j_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ by solving Equation (7.3). From the way G_k is defined, Proposition 3.9 proves that $G_k \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ and hence

$$A^{-1}(-H_k + G_k) \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)).$$

Now by Proposition 6.2, if

$$\mathrm{Spec}_\Gamma(A) \cap (\mathrm{Spec}_\Gamma(A))^k = \emptyset,$$

then $1 \notin \mathrm{Spec}(\mathcal{S}_{A^{-1},A} \;)$, thus $\left(\mathcal{S}_{A^{-1},A} - \mathrm{Id}\right) : L^k_\Gamma \to L^k_\Gamma$ is invertible. This implies we can choose $H_k = 0$ and $K_k = (\mathcal{S}_{A^{-1},A} - \mathrm{Id})^{-1}A^{-1}G_k$.

Obviously, with this choice $H_k, K_k \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$. On the other hand, if

$$\mathrm{Spec}_\Gamma(A) \cap (\mathrm{Spec}_\Gamma(A))^k \neq \emptyset$$

the operator $\mathcal{S}_{A^{-1},A}$ may not be invertible and we set $H_k = G_k$ and $K_k = 0$. This is not the only possible choice, it is only the simplest one and also has $K_k, H_k \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$.

Another standard possibility is to decompose $L^k_\Gamma(E, E) = \mathrm{Im}\,\mathcal{S}_{A^{-1},A} \; \oplus V$, where Im stands for the range of $\mathcal{S}_{A^{-1},A}$ and V is a complementary subspace in $L^k_\Gamma(E, E)$. Then one decomposes $A^{-1}G_k$ according to this splitting of the space as $(A^{-1}G_k)^{\mathrm{Im}} + (A^{-1}G_k)^V$ and chooses K_k such that $(S_{A^{-1},A} \; - \mathrm{Id})K_k = (A^{-1}G_k)^{\mathrm{I}}$ and $H_k = A(A^{-1}G_k)^V$. Of course this choice also depends on the choice of the complementary space V.

By the choices of K_k, H_k, $1 \le k \le r$,

$$D^k \left[F \circ K - K \circ H\right] = 0, \qquad 1 \le k \le r,$$

and hence, by Taylor's theorem, $F \circ K(x) - K \circ H(x) = o(\|x\|^r)$.

□

8 Sternberg theorems in lattices

In this section we will prove several Sternberg conjugation theorems for contractions under several non-resonance hypotheses. All of them are adaptations to our setting of the classical proof in [32], using the normal form theory developed in the previous section.

We will begin by proving Theorem 1.1. First, two remarks from the requirements.

Remark 8.1. *Note that we do not require F_1 to be small but only $B = DF_1(0)$ to be small in the Γ-norm.*

Remark 8.2. *Since r_0 is finite, assumption (H2) involves only a finite set of conditions.*

Before starting the proof we perform a rescaling of F in order to transfer the smallness conditions on the domain of definition to smallness of an auxiliary parameter.

Let $\delta \in \mathbb{R}$, $\delta > 0$, and the rescaling map $T_\delta x = \delta x$. We define

$$F_\delta(x) = T_\delta^{-1} \circ F \circ T_\delta(x) = Mx + N_\delta(x),$$

where $N_\delta(x) = \delta^{-1} N(\delta x)$.

From now on we will not write the dependence on δ of $F_\delta(x)$ and $N_\delta(x)$ and we will assume that F is defined on $B(0,1) \subset \ell^\infty(\mathbb{R}^n)$ and δ is as small as needed. In particular, if F is at least of class C^2, we have that

$$\|N\|_{C^0} = \mathcal{O}(\delta), \quad \|DN\|_{C^0} = \mathcal{O}(\delta) \quad \text{and} \quad \|D^j N\|_{C^0} = \mathcal{O}(\delta^{j-1}), \qquad j \geq 2,$$

and moreover

$$\|N\|_{C^r_\Gamma} = \mathcal{O}(\delta).$$

Given $r, r_0 \in \mathbb{N}$, $r \geq r_0$, we introduce the spaces

$$\chi^{r,r_0} = \left\{ g \in C^r(B(0,1), \ell^\infty(\mathbb{R}^n)) \,|\, D^j g(0) = 0,\, 0 \leq j \leq r_0,\, \|g\|_{C^r} < \infty \right\},$$
$$\chi^{r,r_0}_\Gamma = \left\{ g \in C^r_\Gamma(B(0,1), \ell^\infty(\mathbb{R}^n)) \cap \chi^{r,r_0} \,|\, \|g\|_{C^r_\Gamma} < \infty \right\}.$$

Observe that χ^{r,r_0}_Γ is a closed subspace of C^r_Γ.

Lemma 8.3. *Assume the hypotheses of Theorem 1.1.*

Then, if $r \geq r_0$, $F \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$ and $\|B\|_\Gamma$ and the rescaling parameter δ are small enough, the linear operator $\mathcal{G}_m : \chi^{r,r_0}_\Gamma \to \chi^{r,r_0}_\Gamma$ defined by

$$\mathcal{G}_m(g) = M^{-m} g \circ F^m$$

is well defined and is a contraction in the C^r_Γ-norm.

Proof. First, we fix some quantities to be used throughout the proof. From the definition of r_0 and the fact that $\beta < 1$ we have

$$\alpha^{-1}\beta^{r_0} < 1.$$

Then there exists $m \in \mathbb{N}$ such that

$$\Gamma(0)^{-2} \left(\alpha^{-1}\beta^{r_0}\right)^m < 1$$

and also there exist positive numbers ε_1, ε_2 and ε_3 such that

$$\Gamma(0)^{-1}(\alpha^{-1} + \varepsilon_1)^m \left[\Gamma(0)^{-1}\left((\beta + \varepsilon_1)^m + \varepsilon_2\right)^{r_0} + \varepsilon_3\right] < 1. \qquad (8.1)$$

Note that this condition requires $\beta + \varepsilon_1 < 1$. There exists a norm in $\mathbb{R}^n$ such that

$$\|\mathfrak{a}\| < \beta + \frac{\varepsilon_1}{2}, \qquad \|\mathfrak{a}^{-1}\| < \alpha^{-1} + \frac{\varepsilon_1}{2},$$

where $\|\cdot\|$ is the associated operator norm. Clearly,

$$\|\mathfrak{a}^m\| < \left(\beta + \frac{\varepsilon_1}{2}\right)^m, \qquad \|\mathfrak{a}^{-m}\| < \left(\alpha^{-1} + \frac{\varepsilon_1}{2}\right)^m$$

and

$$\|A^m\|_\Gamma < \Gamma(0)^{-1}\left(\beta + \frac{\varepsilon_1}{2}\right)^m, \qquad \|A^{-m}\|_\Gamma \le \Gamma(0)^{-1}\left(\alpha^{-1} + \frac{\varepsilon_1}{2}\right)^m.$$

Moreover,

$$\begin{aligned}\|M^m\|_\Gamma &= \|(A+B)^m\|_\Gamma \\ &\le \|A^m\|_\Gamma + \mathcal{O}(\|B\|_\Gamma) \le \Gamma(0)^{-1}(\beta + \frac{\varepsilon_1}{2})^m + \Gamma(0)^{-1} m \frac{\varepsilon_1}{2}\beta^{m-1} \\ &< \Gamma(0)^{-1}(\beta+\varepsilon_1)^m\end{aligned}$$

if $\|B\|_\Gamma$ is small enough.

In the same way, now using Proposition 3.6,

$$\begin{aligned}\|M^{-m}\|_\Gamma &\le \|A^{-m}\|_\Gamma + \mathcal{O}(\|B\|_\Gamma) \\ &\le \Gamma(0)^{-1}(\alpha^{-1} + \frac{\varepsilon_1}{2})^m + \Gamma(0)^{-1} m \frac{\varepsilon_1}{2}\alpha^{-(m-1)} < \Gamma(0)^{-1}(\alpha^{-1}+\varepsilon_1)^m.\end{aligned}$$

Analogously,

$$\|M^m\| \le \|M\|^m \le (\beta + \varepsilon_1)^m.$$

Let $g \in \chi_\Gamma^{r,r_0}$. By Remark 4.4 we have

$$\|\mathcal{G}_m(g)\|_{C^r_\Gamma} \le \|M^{-m}\|_\Gamma \|g \circ F^m\|_{C^r_\Gamma}.$$

To estimate $\|g \circ F^m\|_{C^r_\Gamma}$ we will use the Faà di Bruno formula for the p-th derivative of $g \circ F^m$, $1 \le p \le r$,

$$\begin{aligned}D^p(g\circ F^m)(x) =& D^p g(F^m(x))(DF^m(x))^{\otimes p} \\ &+ \sum_{j=1}^{p-1} \sum_{\substack{i_1,\dots,i_j \ge 1 \\ i_1+\dots+i_j = p}} C D^j g(F^m(x)) D^{i_1}F^m(x)\cdots D^{i_j}F^m(x),\end{aligned} \tag{8.2}$$

where C is a combinatorial coefficient which depends on all indices in the sum. From (8.2) it is clear that $D^p(g \circ F^m)(0) = 0$ for $1 \le p \le r_0$, since $F(0) = 0$.

Since $g \in \chi^{r,r_0}$, by Taylor's theorem in integral form (see [1]),

$$g(x) = \frac{1}{(r_0-1)!}\int_0^1 (1-t)^{r_0-1} D^{r_0} g(tx) x^{\otimes r_0}\, dt$$

and also

$$D^j g(x) = \frac{1}{(r_0-j-1)!}\int_0^1 (1-t)^{r_0-j-1} D^{r_0} g(tx) x^{\otimes(r_0-j)}\, dt, \qquad 0 \le j \le r_0 - 1.$$

Using the previous formulas, Proposition 3.8 and usual results about integration in Banach spaces (see [1] for the theory of Cauchy-Bochner integration on Banach spaces) we have

$$\begin{aligned}&\|D^j g(F^m(x))\|_\Gamma \\ &\le \frac{1}{(r_0-j-1)!}\int_0^1 (1-t)^{r_0-j-1}\|D^{r_0} g(tF^m(x))\|_\Gamma \|F^m(x)\|^{r_0-j}\, dt \\ &\le \frac{1}{(r_0-j)!}\|g\|_{C^r_\Gamma}\|F^m(x)\|^{r_0-j}, \qquad 0 \le j \le r_0 - 1\end{aligned}$$

and

$$\|D^j g(F^m(x))\|_\Gamma \le \|g\|_{C^r_\Gamma}, \qquad r_0 \le j \le r.$$

As a consequence of the two previous bounds, we can write the more compact form

$$\|D^j g(F^m(x))\|_\Gamma \le \|g\|_{C^r_\Gamma} \|F^m(x)\|^{(r_0-j)_+}, \qquad 0 \le j \le r,$$

where $(t)_+ = \max(t, 0)$.

Then, using Proposition 3.9

$$\begin{aligned}
&\|D^p(g \circ F^m)(x)\|_\Gamma \\
&\quad \le \|g\|_{C^r_\Gamma} \|F^m(x)\|^{(r_0-p)_+} \|DF^m(x)\|_\Gamma \|DF^m(x)\|^{p-1} \\
&\quad\quad + \sum_{j=1}^{p-1} \sum_{\substack{i_1,\dots,i_j \ge 1 \\ i_1+\dots+i_j = p}} C\|g\|_{C^r_\Gamma} \|F^m(x)\|^{(r_0-j)_+} \|D^{i_1} F^m(x)\| \cdots \|D^{i_j} F^m(x)\|_\Gamma.
\end{aligned}$$

By the rescaling, if $x \in B(0,1)$,

$$\begin{aligned}
\|F^m(x)\| &\le \|M^m x\| + \mathcal{O}(\delta) \le \|M^m\| + \mathcal{O}(\delta), \\
\|DF^m(x)\| &\le \|M^m\| + \mathcal{O}(\delta), \\
\|DF^m(x)\|_\Gamma &\le \|M^m\|_\Gamma + \mathcal{O}(\delta)
\end{aligned}$$

and

$$\|D^j F^m(x)\|_\Gamma = \mathcal{O}(\delta), \qquad j \ge 2.$$

Also note that for $p \ge 0$, we have $(r_0 - p)_+ + p \ge r_0$.

Then

$$\|D^p(g \circ F^m)(x)\|_\Gamma \le \|g\|_{C^r_\Gamma} \left[\big(\|M^m\|_\Gamma + \mathcal{O}(\delta)\big) \big(\|M^m\| + \mathcal{O}(\delta)\big)^{r_0-1} + \mathcal{O}(\delta) \right],$$

for $1 \le p \le r$, and finally,

$$\begin{aligned}
\|\mathcal{G}_m(g)\|_{C^r_\Gamma} &\le \|M^{-m}\|_\Gamma \left[(\|M^m\|_\Gamma + \mathcal{O}(\delta)) (\|M^m\| + \mathcal{O}(\delta))^{r_0-1} + \mathcal{O}(\delta) \right] \|g\|_{C^r_\Gamma} \\
&\le \Gamma(0)^{-1} (\alpha^{-1} + \varepsilon_1)^m \\
&\quad \times \left[\big(\Gamma(0)^{-1} (\beta + \varepsilon_1)^m + \mathcal{O}(\delta)\big) \left((\beta + \varepsilon_1)^m + \mathcal{O}(\delta)\right)^{r_0-1} + \mathcal{O}(\delta) \right] \|g\|_{C^r_\Gamma}.
\end{aligned}$$

Then if δ is small enough, by (8.1) the factor in front of $\|g\|_{C^r_\Gamma}$ is strictly less than 1 and hence $\mathcal{G}$ is a contraction in χ^{r,r_0}_Γ.

□

Now we use the normal form theory in the previous section to find a decay map which linearises our map F up to order r_0. The form of A implies that $\mathrm{Spec}(A) = \{\lambda_1, \dots, \lambda_n\}$ and $\mathrm{Spec}\, A^m = \{\lambda_1^m, \dots, \lambda_n^m\}$. Since A^m is uncoupled, $\mathrm{Spec}_\Gamma(A^m) = \mathrm{Spec}(A^m)$. Moreover the non-resonance condition (H2) implies that

$$\lambda_i^m \ne \lambda_1^{mk_1} \cdots \lambda_n^{mk_n}, \qquad k \in (\mathbb{Z}^+)^n, \quad |k| \ge 2,$$

and therefore

$$(\mathrm{Spec}_\Gamma(A^m))^j \cap \mathrm{Spec}_\Gamma(A^m) = \emptyset, \qquad j \ge 2.$$

Taking $\|B\|_\Gamma$ sufficiently small, since Spec_Γ is continuous, we have

$$(\mathrm{Spec}_\Gamma(M^m))^j \cap \mathrm{Spec}_\Gamma(M^m) = \emptyset, \qquad 2 \le j \le r_0,$$

because we are only dealing with a finite set of conditions.

Hence Corollary 7.2 gives us that there exists a polynomial $K \in C^\infty_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ of degree (at most) r_0 such that $K(0) = 0$, $DK(0) = \mathrm{Id}$ and

$$F^m \circ K(x) - K \circ M^m(x) = o(\|x\|^{r_0}).$$

Let $S_0 = K^{-1}$ be the local inverse. Taking the rescaling parameter δ smaller if necessary we can assume that S_0 is defined in $B(0,1) \subset \ell^\infty(\mathbb{R}^n)$. By Theorem 4.5, we have $S_0 \in C^r_\Gamma$ and satisfies

$$S_0(0) = 0, \qquad DS_0(0) = \mathrm{Id},$$
$$M^{-m} \circ S_0 \circ F^m - S_0 = o(\|x\|^{r_0}).$$

Starting with this approximate conjugation we define the sequence

$$S_n = M^{-m} S_{n-1} \circ F^m = \mathcal{G}_m(S_{n-1}), \qquad n \ge 1.$$

The next lemma proves that S_n converges to a well-defined conjugation in the space C^k_Γ.

Lemma 8.4. *The sequence $\{S_n\}_{n\in\mathbb{N}}$ defined above converges to a function $S \in C^k_\Gamma(B(0,1), \ell^\infty(\mathbb{R}^n))$ satisfying $S(0) = 0$, $DS(0) = \mathrm{Id}$ and*

$$S \circ F^m = M^m S.$$

Proof. Since m is fixed we will drop the dependence of $\mathcal{G}_m$ on m. First, we prove the following relation

$$S_n = S_0 + \sum_{j=0}^{n-1} \mathcal{G}^j (M^{-m} S_0 \circ F^m - S_0), \tag{8.3}$$

where $\mathcal{G}^0 = \mathrm{Id}$ and $\mathcal{G}^j = \mathcal{G} \circ \mathcal{G}^{j-1}$, $j \ge 1$. Observe that $M^{-m} S_0 \circ F^m - S_0 \in \chi^{r,r_0}$, since S_0 solves the conjugation equation formally up to order r_0. Moreover $M^{-m} S_0 \circ F^m - S_0 \in C^k_\Gamma$ since $M \in L_\Gamma$ and $S_0, F \in C^k_\Gamma$. We prove (8.3) by induction. When $n = 1$, we use the definition $S_1 = \mathcal{G}(S_0)$:

$$S_1 = \mathcal{G}(S_0) = M^{-m} S_0 \circ F^m = S_0 + \mathcal{G}^0(M^{-m} S_0 \circ F^m - S_0).$$

Now assume Equation (8.3) is true up to index n, then

$$\begin{aligned}
S_{n+1} &= M^{m(-n-1)} S_0 \circ F^{m(n+1)} \\
&= M^{-mn} S_0 \circ F^{mn} + M^{m(-n-1)} S_0 \circ F^{m(n+1)} - M^{-mn} S_0 \circ F^{mn} \\
&= S_n + M^{-mn} \big(M^{-m} S_0 \circ F^m - S_0\big) \circ F^{mn} \\
&= S_n + \mathcal{G}^n (M^{-m} S_0 \circ F^m - S_0) \\
&= S_0 + \sum_{j=0}^{n} \mathcal{G}^j \big(M^{-m} S_0 \circ F^m - S_0\big).
\end{aligned}$$

By Lemma 8.3, $\mathcal{G}$ is a contraction in χ_Γ^{r,r_0} and therefore the series arising from (8.3) converges and $\lim_{n\to\infty} S_n$ exists and belongs to χ_Γ^{r,r_0}.

Finally, we check the conjugacy property. Indeed,

$$\begin{aligned} S \circ F^m &= \lim_{n\to\infty} S_n \circ F^m = \lim_{n\to\infty} M^{-mn} S_0 \circ F^{mn+m} \\ &= \lim_{n\to\infty} M^m M^{-mn-m} S_0 F^{mn+m} = M^m S. \end{aligned}$$

Also

$$S(0) = \lim_{n\to\infty} M^{-mn} S_0 \circ F^{mn}(0) = 0,$$

and since $DF^{mn}(0) = DF(F^{mn-1}(0)) \cdots DF(0) = M^{mn}$ and $DS_0(0) = \mathrm{Id}$,

$$DS(0) = \lim_{n\to\infty} M^{-mn} DS_0(F^{mn}(0)) DF^{mn}(0) = \lim_{n\to\infty} M^{-mn} \,\mathrm{Id}\, M^{mn} = \mathrm{Id}\,.$$

□

Thus, S conjugates F^m to M^m. The final step is to show that S also conjugates F to M.

By the spectral properties and Corollary 7.2, there exists a polynomial $\widetilde{K} \in C^\infty(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ such that

$$\widetilde{K}(0) = 0, \qquad D\widetilde{K}(0) = \mathrm{Id},$$

and

$$F \circ \widetilde{K}(x) - \widetilde{K} \circ M(x) = o(\|x\|^{r_0}).$$

Let $R_0 = \widetilde{K}^{-1}$, which similarly to S_0, we can assume is defined in $B(0,1) \subset \ell^\infty(\mathbb{R}^n)$. Thus

$$M^{-1} R_0 \circ F(x) = R_0(x) + o(\|x\|^{r_0})$$

and as a consequence

$$M^{-m} R_0 \circ F^m(x) = R_0(x) + o(\|x\|^{r_0}). \tag{8.4}$$

Lemma 8.5. *Under the hypotheses of Theorem 1.1, if $\|B\|$ and the rescaling parameter δ are small enough, the operator*

$$\widetilde{\mathcal{G}} : C^r(B(0,1), \ell^\infty(\mathbb{R}^n)) \to C^r(B(0,1), \ell^\infty(\mathbb{R}^n))$$

defined by

$$\widetilde{\mathcal{G}}(g) = M^{-1} g \circ F$$

is well defined and is a contraction in the C^r-norm.

We omit the proof of this lemma since it is completely analogous to the proof of Lemma 8.3 but the estimates are much simpler, since they do not involve decay functions.

We define the sequence

$$R_n = M^{-1} R_{n-1} \circ F, \qquad n \geq 1.$$

The same arguments as the ones used in the proof of Lemma 8.4 but now in the space C^r instead of C^r_Γ give that there exists $R = \lim_{n\to\infty} R_n$ with $R \in C^r(B(0,1), \ell^\infty(\mathbb{R}^n))$ such that $R \circ F = MR$.

Now consider the iteration S_n introduced in the proof of Lemma 8.4 with $S_0 = R_0 \in C^r_\Gamma$. Since R_0 satisfies (8.4), we see that S_n is a subsequence of R_n, being both sequences convergent in the larger space C^r. Then

$$R = \lim_{n\to\infty} R_n = \lim_{n\to\infty} S_n = S \in C^r_\Gamma$$

and therefore S also conjugates F with M, proving Theorem 1.1.

An improvement of Theorem 1.1 consists of not assuming the non-resonance condition (H2). In such case we obtain a C^r_Γ local conjugation to a normal form of F instead to a conjugation to its linear part.

Theorem 8.6. *Under the conditions and notation of Theorem 1.1 except hypothesis (H2), if $F \in C^r_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ with $r \geq r_0$ and $\|B\|_\Gamma$ is small enough there exists a polynomial $H \in C^\infty_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ of degree not larger than r_0 and $R \in C^r_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ such that*

$$R(0) = 0, \qquad DR(0) = \mathrm{Id}$$

and

$$R \circ F = H \circ R$$

in some neighborhood $U_1 \subset U$ of 0.

Proof. We will only comment on the differences of this proof with the proof of Theorem 1.1. We rescale the map, we consider the spaces χ^{r,r_0} and χ^{r,r_0}_Γ and we use the same integer m as in the proof of that theorem. We take a normal form H provided by Theorem 7.1 and define the operator

$$\overline{\mathcal{G}}_m(g) = H^{-m} \circ g \circ F^m.$$

Now the estimates on $\overline{\mathcal{G}}_m$ become more involved because in this case H is not linear. This implies that $\overline{\mathcal{G}}_m$ is not linear anymore. However, because of the rescaling, H^{-1} is very close to M^{-1} in C^r_Γ (and C^r) norm, a fact which gives similar estimates and thus proves $\mathrm{Lip}\,\overline{\mathcal{G}}_m < 1$. The remaining part of the proof is analogous.

□

The previous theorems assume that the linear part of the maps is close to an uncoupled map with identical dynamics on each node. This gives sufficient conditions for the conjugation in terms of the eigenvalues of the projections to the nodes.

Theorem 1.2 requires instead conditions on the Γ-spectrum of the linear part of the map.

Note that, since $\mathrm{Spec}_\Gamma(A)$ is compact, Hypothesis (H1) in Theorem 1.2 implies that $0 < \alpha_\Gamma \leq \beta_\Gamma < 1$ and $r_0 < \infty$.

Proof of Theorem 1.2. The structure of the proof is very similar to the one of the proof of Theorem 1.1 but it has some technical differences. Let r_0 and r be as in the statement of the theorem. Note that $\beta_\Gamma = r_\Gamma(A)$ and $\alpha_\Gamma^{-1} = r_\Gamma(A^{-1})$. Since $r_0 > \nu$ then $\alpha_\Gamma^{-1}\beta_\Gamma^{r_0} < 1$ and there exists $\varepsilon_1 > 0$ such that

$$(\alpha_\Gamma^{-1} + \varepsilon_1)(\beta_\Gamma + \varepsilon_1)^{r_0} < 1.$$

In any Banach algebra

$$r_\Gamma(A) = \lim_{n\to\infty} (\|A^n\|_\Gamma)^{1/n} = \inf_{n\geq 1} (\|A^n\|_\Gamma)^{\frac{1}{n}}\,, \tag{8.5}$$

thus there exists $m \in \mathbb{N}$ such that

$$\|A^n\|_\Gamma \leq (r_\Gamma(A) + \varepsilon_1)^n\,, \qquad n \geq m,$$

and

$$\|A^{-n}\|_\Gamma \leq (r_\Gamma(A^{-1}) + \varepsilon_1)^n, \qquad n \geq m.$$

Obviously,

$$(\alpha_\Gamma^{-1} + \varepsilon_1)^m (\beta_\Gamma + \varepsilon_1)^{mr_0} < 1$$

and there exists $\varepsilon_2, \varepsilon_3 > 0$ such that

$$(\alpha_\Gamma^{-1} + \varepsilon_1)^m \left[((\beta_\Gamma + \varepsilon_1)^m + \varepsilon_2)^{r_0} + \varepsilon_3\right] < 1.$$

Now we introduce the operator $\mathcal{G}_m : \chi_\Gamma^{r,r_0} \to \chi_\Gamma^{r,r_0}$ defined by

$$\mathcal{G}_m(g) = A^{-m} g \circ F^m.$$

Analogous estimates as in Lemma 8.3 yield that if the rescaling parameter is small enough, $\mathcal{G}_m$ is well defined in χ_Γ^{r,r_0} and is a contraction. Then the proof follows the same lines as the proof of Theorem 1.1.

For the uniqueness arguments needed at the end of the proof, we consider the operator $\widetilde{\mathcal{G}}(g) = A^{-1} g \circ F$ in $C^r(B(0,1), \ell^\infty(\mathbb{R}^n))$. Since $\mathrm{Spec}(A) \subset \mathrm{Spec}_\Gamma(A)$, condition (H2) implies that there are also no resonances among the elements of $\mathrm{Spec}(A)$ and that $r(A) < \beta_\Gamma$ and $r(A^{-1}) < \alpha_\Gamma^{-1}$. Hence we can find a norm in the space $\ell^\infty(\mathbb{R}^n)$, equivalent to the original one, such that

$$\|A^{-1}\| \|A\|^{r_0} < 1, \tag{8.6}$$

where in the previous expression $\|\cdot\|$ stands for the corresponding operator norm. The bound (8.6) allows us to prove the estimates needed to show that $\widetilde{\mathcal{G}}$ is a contraction. With these ingredients we can finish the proof in this setting in the same way as in Theorem 1.1.

□

The analogous version of Theorem 8.6 in this setting is the following.

Theorem 8.7. *Under the conditions of Theorem 1.2, except condition (H2), if $F \in C^r_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ with $r \geq r_0$ there exists a polynomial $H \in C^\infty_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ of degree not larger than r_0 and $R \in C^r_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ such that*

$$R(0) = 0, \quad DR(0) = \mathrm{Id}$$

and

$$R \circ F = H \circ R$$

in some neighborhood $U_1 \subset U$ of 0.

The proof of this theorem is a combination of the arguments in the proofs of Theorem 8.6 and Theorem 1.2.

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