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PROPORTIONAL CLEARING MECHANISMS IN FINANCIAL SYSTEMS

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JEL Codes: C71, G10

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Proportional clearing mechanisms in financial systems: an axiomatic approach

Pedro Calleja and Francesc Llerena

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Abstract

When a financial network collapses, how should mutual obligations among all agents be cleared? We study this problem taking an axiomatic approach and provide the first characterization of the family of rules based on the principle of proportionality in the entire domain of financial systems. A previous attempt to address this issue was done by Csóka and Herings (2021), but in a tight context where all agents dispose of strictly positive initial endowments. We show that their properties, when accommodated to the full domain of financial systems, no longer characterize the set of proportional rules. To overcome this drawback, we formulate new properties emphasizing the value of equity of the firms in the network. In particular, we show that a clearing mechanism satisfies compatilibity, limited liability, absolute priority, equity continuity, and non-manipulability by clones if and only if each agent is paid proportionally to the value of its claims. Remarkably, our result also holds in the framework studied by Csóka and Herings (2021).

1 Introduction

A financial system is represented by a set of agents or firms (banks, individuals inverstors, hedge funds, etc.), all of them distinguished by their endowments and their obligations towards other agents. Unlike the standard bankruptcy problem, in which one single firm defaults, in this context agents can play the role of debtors and creditors simultaneously, so that the bankruptcy of one firm can lead to the default of others and therefore compromising the stability of the system. An illustrative example of this domino effect of insolvencies is the failure of Lehman Brothers in 2008 and the subsequent crisis on financial markets. Since then, the literature on financial contagion has increased considerably, being the work of Eisenberg and Noe (2001) the reference for further studies. For a detailed reviews of this topic, we refer readers to Glasserman and Young (2016), Caccioli et al. (2018), and Jackson and Pernoud (2021).

When a financial network collapses, a central question is how to settle the mutual obligations between firms. This problem is tackled by means of financial rules that recommend, for each financial network, a set of clearing payment matrices, suggesting the monetary transfer from each node in the network to any other node. To address this problem, in this paper we adopt the axiomatic approach. From a normative point of view, this approach sheds some light on the choice of appropriate mechanisms to clear unstable financial networks and, moreover, they can be easily justified. Thus, the axiomatic motivation of a rule is a prior step and provides a counterpoint to the literature that focuses on the computational challenges of the model, resulting from the complexity in determining the assets value of the entities in the economic network at risk, as they endogenously depend on the extent to which the liabilities of others are satisfied. In this regard, and in line with the evidence that the principle of proportionality is significant in practice,[1] it is worth looking into what normative foundations distinguishes *proportional financial rules*. That is, the family of clearing mechanisms satisfying standard conditions in most insolvency laws such as payments bounded by liabilities (*claim boundedness*), limited liability of equity (*limited liability*), priority of creditors over stockholders (*absolute priority*), and proportional repayments of liabilities. To our knowledge, Csóka and Herings (2021) is the only attempt at investigating what properties identify the set of proportional payment matrices, but focusing on a specific subdomain of financial systems where agents have strictly positive initial endowments or cash flows. This assumption, however, is not harmless and excludes many real economic scenarios in which some of the entities taking part in the system have a very low or zero cash flow. Think, for instance, in the unanticipated economic shock provoked by the recent pandemic due to COVID-19 in which many sectors were damage, turning several companies into zombies with virtually no cash flow.[2] Different scenarios could be those in which one (or more) agent play only the role of debt holder with no obligations and without any initial cash flow, or others in which one (or more) firm defaults once it has exhausted its funds by paying some of its obligations.

As showed by Eisenberg and Noe (2001), for the aforementioned domain of financial systems a unique clearing matrix allows to guarantee claim boundedness, limited liability, absolute priority, and proportional repayments. As a result, Csóka and Herings (2021) restrict their axiomatic study to single-valued solutions. Although this route simplifies the analysis to a certain extent, if the endowments of some agents are allowed to be zero then several clearing payment matrices can be supported by the principle of proportionality. Consequently, as we will show, the accommodation of their properties to multi-valued solution concepts no longer characterize all rules relying on this principle. To overcome this gap, here we present the first axiomatic ground for the family of proportional rules in the full domain of financial networks. Remarkably, our characterization remains valid in the framework considered by Csóka and Herings (2021).

Together with the basic requirements of claim boundedness, limited liability, and absolute priority, they employ *continuity*, *impartiality*, *invariance to mitosis*, and, implicitly, *single-valuedness*. Continuity simple says that small variations in both, the endowments and the liabilities, imply small changes in the payment matrix. Impartiality imposes that two agents with the same claim to another one must receive the same amount from the latter. Invariance to mitosis is the crucial property in Csóka and Herings' result. This requirement extends, in a weak form, *additivity of claims* (Curiel et al., 1987) or *strong non-manipulability* (de Frutos, 1999; Moreno-Ternero, 2006) from the context of bankruptcy problems to the financial systems environment. Roughly speaking, in the first context a rule is non manipulable if it is immune to the strategic behavior of the agents by merging or splitting their claims, and it has been widely used to characterize the proportional bankruptcy rule (O'Neill, 1982; Chun, 1988; de Frutos, 1999; Ju et al., 2007). In financial networks, a way to generalize the notion of non-manipulability is to require that the split of an agent into multiple agents or the merge of a group of agents should not affect the clearing payment matrices. Csóka and Herings (2021) show that, under claim boundedness, limited liability, absolute priority, and for single-valued solutions, there is no financial rule satisfying non-manipulability. To deal with this incompatibility, they weaken this property introducing invariance to mitosis which requires that the division of a firm into a number of identical firms, that is, with the same endowments, claims, and liabilities, or the fusion of a group of identical firms should not affect the final outcome. The idea of imposing restrictions on coalition formation when merges or spin-offs occur is not fresh and also appears in the literature on allocation rules when characterizing extensions of the proportional rule to bankruptcy problems with multiple

[1]See, for instance, Regulation (EU) 2015/848 of the European Parliament and of the Council of 20 May 2015 on Insolvency Proceedings.

[2]See Favara et al. (2021).

types of assets (Ju et al. 2007; Ju, 2013) or in axiomatizing priority rules in the the context of standard insolvencies (Flores-Szwagrzak et al., 2019), to mention some instances. Recently, Calleja and Llerena (2022) restrict fusions and splits to agents that are or become symmetric and convey new axiomatizations of the proportional rule for classical bankruptcy problems. Some of the results of that paper will be essential in the current work.

Except for continuity, the rest of the properties can be naturally accommodated to multi-valued solutions by imposing the same conditions for every or, in a softer manner, some payment matrices. Regarding continuity, there are two different generalizations for correspondences: *lower hemicontinuity* and *upper hemicontinuity*. In words, lower hemicontinuity (upper hemicontinuity) requires that small changes in a financial system does not make the set of recommended payment matrices suddenly implode (explode). As we will see, in the general domain of financial systems some rules built on the principle of proportionality do not satisfy either invariance to mitosis or lower hemicontinuity or upper hemicontinuity, not even if they are single-valued. To handle with this drawback, we relax lower hemicontinuity and invariance to mitosis into *equity continuity* and *non-manipulability by clones*, respectively. Since utility maximization governs the incentives of decision-makers, these properties are formulated in terms of equity values rather than payment matrices. Accordingly, equity continuity says that a small impact in both, the initial endowments of the agents and the liabilities matrix, do not imply large variations in their final equity. Non-manipulability by clones restricts attention to identical firms or *clones* as invariance to mitosis does but, on the contrary, it imposes that splitting a company into several clones or merging some clone firms should have no effect on utilities. This weak form of immunity is in accordance with real-life bankruptcy situations where merging or splitting operations involve firms that are or will become balanced in the sense that an insolvent firm is not allowed to transfer all its liabilities to a spin-off and keeps the endowments and claims for itself. Our last property is *compatibility*, which requires a financial rule to be supported by an inventory of bankruptcy rules establishing the basis driving the clearing process of each defaulting firm. Most of the contributions in the field focus on these type of financial rules as Eisenberg and Noe (2001), Groote et al. (2018), Csóska and Herings (2018), Stutzer (2018), and Ketelaars and Borm (2021), among others. Instances of financial rules supported by the proportionality principle but not directly derived from bankruptcy rules can be found in Demange (2022) and Csóka and Herings (2023). From an axiomatic perspective, Groote et al. (2018) study the extension of the *Talmudic* rule (Aumann and Maschler, 1985) for claims problems to financial systems, and Ketelaars and Borm (2021) accomodate the joint axiomatization of the *proportional*, *constrained equal awards*, and *constrained equal losses* rules for classical bankruptcy problems proposed by Moulin (2000) to the setting of financial systems. These two papers, however, take a different approach considering financial rules as recommendations over the final distribution of equity values rather than on the clearing payment matrices. Therefore, they characterize the resulting equity values of the agents derived from financial rules compatible with their respective bankruptcy counterpart. An interesting explanation of why proportionality is preferred in current bankruptcy laws over the principles of equal awards or equal losses in terms of both egalitarian and utilitarian social welfare can be found in Kibris and Kibris (2013). Stutzer (2018) shows that the strategic justification, coming from bargaining theory, of the constrained equal awards rule for a standard claims problem does not hold in financial networks. Demange (2022) and Csóka and Herings (2023) axiomatize respectively the *constrained-proportional rule* and the *pairwise netting proportional rule*.

The main result of the paper concludes that compatibility, limited liability, absolute priority, equity continuity, and non-manipulability by clones characterize the family of financial rules supported by the principle of proportionality. Moreover, our characterization also holds in the context analyzed by Csóka and Herings (2021). In relation to their result, and under compatibility, we require neither impartiality nor claim boudedness nor single-valuedness (assumed, implicitly, by these authors), while continuity and invariance to mitosis are replaced by the weaker properties of equity continuity and non-manipulability by clones, respectively.

The rest of the paper is organized as follows. Section 2 introduces the model. In Section 3, we connect financial rules and bankruptcy rules. Section 4 contains the axioms. Section 5 provides the characterization result and the logical independence of the axioms. Section 6 concludes. The proofs of the results in each section are relegated to the corresponding appendix.

2 The model

Before describing the model of financial systems, we first provide some basic definitions and introduce well-known insides from the bankruptcy literature.

2.1 Preliminaries

Let $\mathbb{N} = \{1, 2, \ldots\}$ (the set of natural numbers) represent the set of all potential agents and let $\mathcal{N}$ be the collection of all non-empty finite subsets of $\mathbb{N}$. An element $N \in \mathcal{N}$ describes a finite set of agents. For each $x \in \mathbb{R}^N$ and $T \subseteq N$, x_T denotes the restriction of x to T: $x_T = (x_i)_{i \in T} \in \mathbb{R}^T$. For $N \in \mathcal{N}$, we denote by $\mathcal{M}(N)$ the set of all non-negative real $N \times N$ matrices $M = (M_{ij})_{i,j \in N}$ with a zero diagonal, and $\mathcal{M} = \bigcup_{N \in \mathcal{N}} \mathcal{M}(N)$. For $M \in \mathcal{M}(N)$ and $i \in N$, $M_i = (M_{ij})_{j \in N} \in \mathbb{R}^N_+$ denotes the row i of M being $\bar{M}_i = \sum_{j \in N} M_{ij}$. By $\mathbb{Q}_+ = \{a/b \mid a, b \in \mathbb{N}\}$ we denote the set of positive rational numbers.

An important tool in our analysis is Tarski's fixed-point theorem on lattices (Tarski, 1955). Roughly speaking, a lattice is a partially ordered set A in which any two elements $x, y \in A$ have a supremum (a minimum upper bound) and an infimum (a maximum lower bound) in A. A lattice A is complete if every nonempty subset of A has a supremum and an infimum in A. The Tarski's theorem says that the set of all fixed-points of a monotone function f on a complete lattice A (i.e., the set of elements $x \in A$ such that $x = f(x)$) is a complete lattice. In order not to overload the reading of the paper, Appendix A contains the formal statement of this result.

2.2 Bankruptcy problems

A *bankruptcy problem* is a problem of adjudicating claims in which a single firm defaults and its available resources are not enough to satisfy its obligations with creditors. This distributive justice problem has been widely studied from O'Neill (1982) and probably the most complete survey is provided by Thomson (2019). Formally, a bankruptcy problem is a triple (N, E, c) where $N \in \mathcal{N}$ represents the set of creditors to the firm going bankrupt; $c \in \mathbb{R}^N_+$ is the vector of claims, being c_i the claim or the liability of the firm to creditor $i \in N$; and $E \geq 0$ is the net worth or estate of the firm to satisfy its obligations. Additionally, we assume that the issue on how to clear the debts of the firm with the available resources is not trivial, i.e., $\sum_{i \in N} c_i \geq E$. By $\mathcal{B}$ we denote the set of all bankruptcy problems. A *bankruptcy rule* is a function $\beta : \mathcal{B} \longrightarrow \bigcup_{N \in \mathcal{N}} \mathbb{R}^N_+$ that provides for every $(N, E, c) \in \mathcal{B}$ a unique vector or recommendation $\beta(N, E, c) \in \mathbb{R}^N_+$ for the problem satisfying $\sum\limits_{i \in N} \beta_i(N, E, c) = E$ (*budget balance* (BB)) and $\beta_i(N, E, c) \leq c_i$ for all $i \in N$ (*claim boundedness* (CB)). BB imposes that the sum of the payments should be equal to the estate, requiring implicitly that the equity value of the firm after the clearing process neither can be positive, which would ignore the priority of debt claim, nor negative, that would overlook limited liability. CB establishes that no creditor receives more than her claim.

In the following, we introduce a number of properties for bankruptcy rules that will play a central role in the paper. A bankruptcy rule β satisfies:

- *resource monotonicity* (RM) if for all $(N,E,c),(N,E',c) \in \mathcal{B}$ with $E' > E$, $\beta_i(N,E',c) \geq \beta_i(N,E,c)$ for all $i \in N$;

- *equal treatment of equals* (ETE) if for all $(N,E,c) \in \mathcal{B}$ and all $i,j \in N$, if $c_i = c_j$ then $\beta_i(N,E,c) = \beta_j(N,E,c)$;

- *continuity* (CONT) if for all $(N,E,c) \in \mathcal{B}$ and all sequence $\left\{(N,E^n,c^n)\right\}_{n\in\mathbb{N}}$ of bankruptcy problems converging to (N,E,c), the sequence $\left\{\beta(N,E^n,c^n)\right\}_{n\in\mathbb{N}}$ converges to $\beta(N,E,c)$;

- *weak continuity* (WCONT) if for all $(N,E,c) \in \mathcal{B}$ and all sequence $\left\{(N,E^n,c^n)\right\}_{n\in\mathbb{N}}$ of bankruptcy problems converging to (N,E,c), there exists a subsequence $\left\{(N,E^{n_k},c^{n_k})\right\}_{n_k\in\mathbb{N}}$ such that $\left\{\beta(N,E^{n_k},c^{n_k})\right\}_{n_k\in\mathbb{N}}$ converges to $\beta(N,E,c)$;

- *claims continuity* (CCONT) if for all sequence of bankruptcy problems $\left\{(N,E,c^n)\right\}_{n\in\mathbb{N}}$ converging to (N,E,c), the sequence $\left\{\beta(N,E,c^n)\right\}_{n\in\mathbb{N}}$ converges to $\beta(N,E,c)$.

RM says that no one should be worse off when the firm's assets increase. ETE is a weak impartiality requirement meaning that agents with the same claim have to be rewarded equally. CONT imposes that small variations in both, the estate and the claims, imply small variations in the resulting allocation vector; WCONT is a weak version of CONT. Clearly, CONT implies CCONT, that only considers variations in the claims.

Instances of well studied bankruptcy rules that satisfy these properties are the *proportional* (PR), the *constrained equal awards* (CEA), and the *constrained equal losses* (CEL) rules. The PR rule makes awards proportional to the claims and it is probably the most commonly used rule in practice when a firm goes into bankruptcy. Formally, for all $(N,E,c) \in \mathcal{B}$ and all $i \in N$, $PR_i(N,E,c) = \lambda\, c_i$ where $\lambda \in \mathbb{R}_+$ is such that $\sum_{j\in N} \lambda\, c_j = E$. The CEA rule rewards all claimants equally subject to no one receiving more than her claim. Formally, for all $(N,E,c) \in \mathcal{B}$ and all $i \in N$, $CEA_i(N,E,c) = \min\{c_i, \lambda\}$ where $\lambda \in \mathbb{R}_+$ is such that $\sum_{j\in N} \min\{c_j,\lambda\} = E$. In contrast, the CEL rule equalizes the losses of claimants subject to no one receiving a negative amount. That is, for all $(N,E,c) \in \mathcal{B}$ and all $i \in N$, $CEL_i(N,E,c) = \max\{c_i - \lambda, 0\}$ where $\lambda \in \mathbb{R}_+$ is such that $\sum_{j\in N} \max\{c_j - \lambda, 0\} = E$.[3]

Another important property in our analysis is non-manipulability by clones (Calleja and Llerena, 2022), that weakens the classical non-manipulability property for bankruptcy rules (Curiel et al. 1987; de Frutos, 1999) since only symmetric agents (i.e., with the same claim) are allowed to split and merge. Formally, a bankruptcy rule β satisfies

- *non-manipulability by clones* (NMC) if for all (N,E,c), $(N',E,c') \in \mathcal{B}$, if $N' \subset N$ and there is $m \in N'$ such that $c_i = \frac{c'_m}{|N\setminus N'|+1}$ for all $i \in N \setminus N' \cup \{m\}$ and $c'_i = c_i$ for all $i \in N' \setminus \{m\}$, then $\beta_i(N',E,c') = \beta_i(N,E,c)$ for all $i \in N' \setminus \{m\}$.

A rule accomplishes NMC if it does not provide identical agents incentives to merge, neither an agent incentives to split into equal copies. Theorem 3 in Calleja and Llerena (2022) states that NMC together with CCONT characterize the PR rule. Although WCONT does not imply (it is not implied) by CCONT, it can be easily verified along the lines in the proof of the aforementioned theorem that CCONT can be replaced by WCONT.

Theorem 1. *A bankruptcy rule satisfies* WCONT *and* NMC *if and only if it is the proportional rule.*

The above characterization will be important later on in our axiomatic analysis.

[3] For a detailed analysis of bankruptcy rules we refer to Thomson (2019).

2.3 Financial system

A financial system is a non trivial generalization of a bankruptcy problem where agents are connected to each other in a network of contracts that entail mutual obligations. Thus, the default of an agent may provoke the default of others, leading to some systemic risk.[4] Following Eisenberg and Noe (2001), a financial system is described by a triple $\varepsilon = (N, L, e)$ being $N \in \mathcal{N}$ the set of economic entities in the system; the matrix $L \in \mathcal{M}(N)$ represents the structure of *liabilities*, where L_{ij} stands for the liability of firm $i \in N$ to firm $j \in N$ or, equivalently, the claim of firm j against firm i; and the vector $e \in \mathbb{R}^N_+$ indicates the *initial operating cash flows* (or *endowments*) of the agents, that is, its exogenous funds obtained from sources outside the financial system.[5] The vector of total obligations in the system is denoted by $\bar{L} = (\bar{L}_i)_{i \in N} \in \mathbb{R}^N_+$. By $\mathcal{F}$ we represent the set of all financial systems. From a bankruptcy perspective, agents play the role of firms and claimants simultaneously.

A bankruptcy problem $(N, E, c) \in \mathcal{B}$ can be translated into a financial system $(\bar{N}, L, e)$ being $\bar{N} = N \cup \{i\}$ the set of agents, where $i \in \mathbb{N} \setminus N$ represents the firm going bankrupt; the matrix of liabilities L is given by $L_{jk} = 0$ for all $j, k \in N$, $L_{ij} = c_j$, and $L_{ji} = 0$ for all $j \in N$; and the initial endowments $e \in \mathbb{R}^N_+$ by $e_i = E$ and $e_j = 0$ for all $j \in N$.

For each financial system (N, L, e), a *payment matrix* $P \in \mathcal{M}(N)$ specifies a recommendation on what monetary transfer P_{ij} should be paid by any agent $i \in N$ to any other agent $j \in N$. Associated to a payment matrix P and an endowment vector e, the *asset value* of agent $i \in N$ is determined endogenously as the amount of resources of i to clear its debts, that is, by the sum of its endowment and the payments received from other agents,

$$a_i(P, e) = e_i + \sum_{k \in N} P_{ki}. \tag{1}$$

The entities participating in the system will make evaluations on different payment matrices depending on their associated *value of equity*, or *utility*. Given a payment matrix P and an endowment vector e, the equity value of agent $i \in N$ is defined by

$$E_i(P, e) = a_i(P, e) - \bar{P}_i, \tag{2}$$

where $\bar{P}_i$ is the total payment of agent i according to P. By $E(P, e) \in \mathbb{R}^N$ we denote the vector of equity values of the agents. Observe that, indeed, $\sum_{i \in N} E_i(P, e) = \sum_{i \in N} e_i$. Hence, the choice of a particular payment matrix is, in terms of utility or net worth, a recommendation on the distribution of the total initial endowments in the system.

A financial rule associates to each financial system a non-empty set of payment matrices.

Definition 1. *A* financial rule *σ is a correspondence that assigns a non-empty subset $\sigma(N, L, e)$ of $\mathcal{M}(N)$ to each $(N, L, e) \in \mathcal{F}$.*

If a financial rule σ always recommends a unique matrix, then we say that σ is *single-valued* (**SIVA**); in a formal manner, if for all $(N, L, e) \in \mathcal{F}$, $|\sigma(N, L, e)| = 1$.

In line with Eisenberg and Noe (2001), we are interested in financial rules fulfilling three basic criteria: claim boundedness, which imposes that the payment of a firm to any other firm is bounded from above by the liability to it; limited liability of equity, requiring that the payments of the firm to others are limited to its asset value; and absolute priority of debt over equity, demanding that stockholders of each firm can not receive a positive value unless all obligations have been completely paid. Formally, a financial rule σ satisfies

[4] This point is addressed, among others, in Chen at al. (2013) and Demange (2018) which focus on measuring the systemic risk of a financial network.

[5] For a discussion of why the condition of nonnegative operating cash flow is made without a loss of generality we refer readers to Eisenberg and Noe (2001).

- *claims boundedness* (**CB**) if, for all $(N, L, e) \in \mathcal{F}$, all $P \in \sigma(N, L, e)$, and all $i, j \in N$, $P_{ij} \leq L_{ij}$;
- *limited liability* (**LL**) if, for all $(N, L, e) \in \mathcal{F}$, all $P \in \sigma(N, L, e)$, and all $i \in N$, $E_i(P, e) \geq 0$;
- *absolute priority* (**AP**) if, for all $(N, L, e) \in \mathcal{F}$, all $P \in \sigma(N, L, e)$, and all $i \in N$, if $\bar{P}_i < \bar{L}_i$ then $E_i(P, e) = 0$.

In fact, these three basic conditions ensure that the financial rule recommendations clear the debts in the system in a feasible way. The next lemma expresses that, in the presence of **CB**, the combination of **LL** and **AP** is equivalent to require that every firm pays the minimum between its asset value and its total debt obligations. The proof is relegated to Appendix B.

Lemma 1. *Let σ be a financial rule satisfying* **CB**. *Then, the following statements are equivalent:*

1. *σ satisfies* **LL** *and* **AP**.
2. *For all $(N, L, e) \in \mathcal{F}$, all $P \in \sigma(N, L, e)$, and all $i \in N$,*

$$\bar{P}_i = \min \left\{ e_i + \sum_{k \in N} P_{ki}, \bar{L}_i \right\}. \tag{3}$$

3 Financial rules compatible with bankruptcy rules

Since the entities in the system may have different tax addresses, one may ask whether the recommendation proposed by a financial rule is compatible with the recommendations of the insolvency laws of each court or administration taking part. Intuitively, clearing payment matrices should be consistent with the legal rules (bankruptcy solution concept) allocating the value of the estate of a defaulting firm among its debt holders. Obviously, these principles or rules may vary from one court to another, which makes the compatibility issue relevant. Formally,

Definition 2. *A financial rule σ is compatible with an inventory of bankruptcy rules $\beta = (\beta^i)_{i \in \mathbb{N}}$ if for all $(N, L, e) \in \mathcal{F}$, all $P \in \sigma(N, L, e)$, and all $j \in N$, $P_{jk} = \beta^j_k(N \setminus \{j\}, E, c)$ for all $k \in N \setminus \{j\}$, where $(N \setminus \{j\}, E, c)$ is the bankruptcy problem faced by agent $j \in N$ being $E = \bar{P}_j$ and $c \in \mathbb{R}_+^{N \setminus \{j\}}$ with $c_k = L_{jk}$ for all $k \in N \setminus \{j\}$.*

If no confusion arises, we will denote by $\mathbb{PR} \equiv (PR^i)_{i \in \mathbb{N}}$, $\mathbb{CEA} \equiv (CEA^i)_{i \in \mathbb{N}}$, and $\mathbb{CEL} \equiv (CEL^i)_{i \in \mathbb{N}}$ the inventories of bankruptcy rules consisting of all agents applying the *PR*, *CEA*, and *CEL* bankruptcy rule, respectively. The next axiom describes financial rules compatible with bankruptcy rules. A financial rule σ satisfies

- *compatibility* (**C**) if there exists a collection of bankruptcy rules $\beta = (\beta^i)_{i \in \mathbb{N}}$ such that σ is *compatible* with β.

Any compatible financial rule accomplish **CB** as a consequence of the fact that any bankruptcy rule satisfies CB. Moreover, for any given payment matrix P, the value of the estate of any firm $i \in N$ is endogenously determined and defined to be exactly the amount payed to debt holders by the firm according to P which ensures, by CB, that the bankruptcy problem faced by $i \in N$ is well defined (independently if the firm defaults or not). Actually,

$$E = \bar{P}_i \underset{\text{BB}}{=} \sum_{k \in N \setminus \{i\}} \beta^i_k \left(N \setminus \{i\}, \bar{P}_i, (L_{ij})_{j \in N \setminus \{i\}} \right) \underset{\text{CB}}{\leq} \sum_{k \in N \setminus \{i\}} L_{ik} = \sum_{k \in N \setminus \{i\}} c_k.$$

Given an inventory of bankruptcy rules $\beta = (\beta^i)_{i \in \mathbb{N}}$, and regarding the existence of non-empty financial rules compatible with β that additionally meet **LL** and **AP**, the approach in Groote Schaasrsberg et al. (2018) to show existence when all bankruptcy rules are the same can be extended to the general setting in which different bankruptcy rules apply (see Csóka and Herings, 2018). In our analysis, we adopt the methodology of Eisenberg and Noe (2001) that makes use of Tarski's fixed-point theorem (see Appendix A) to prove non-emptiness for the case of all agents applying the proportional rule, exclusively. To do it, let us introduce the following instrumental function.

Definition 3. *Given a collection of bankruptcy rules $\beta = (\beta^i)_{i \in \mathbb{N}}$ and a financial system $\varepsilon = (N, L, e)$, define the function $\Phi^{\varepsilon,\beta} : [\mathbf{0}, \bar{L}] \longrightarrow [\mathbf{0}, \bar{L}]$ as follows:*

$$\Phi_i^{\varepsilon,\beta}(\mathbf{t}) = \min \left\{ e_i + \sum_{k \in N \setminus \{i\}} \beta_i^k \left(N \setminus \{k\}, \mathbf{t}_k, (L_{kj})_{j \in N \setminus \{k\}}\right), \bar{L}_i \right\},$$

for all $i \in N$ and all $\mathbf{t} = (\mathbf{t}_1, ..., \mathbf{t}_n) \in [\mathbf{0}, \bar{L}]$, being $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^N$.

Under **LL** and **AP**, an interpretation of $\Phi^{\varepsilon,\beta}$ is that, for each firm $i \in N$, $\Phi_i^{\varepsilon,\beta}(\mathbf{t})$ represents the total funds it will employ to satisfy obligations assuming that such a firm will receive, from the other firms in the system, inflows specified by the rules in β applied over the vector of payments $\mathbf{t} = (\mathbf{t}_1, ..., \mathbf{t}_n)$. If $FIX(\Phi^{\varepsilon,\beta})$ denotes the set of fixed-points of $\Phi^{\varepsilon,\beta}$, a direct implication of Lemma 1 is the following corollary.

Corollary 1. *Let σ be a financial rule compatible with a collection of bankruptcy rules $\beta = (\beta^i)_{i \in \mathbb{N}}$. Then, the following statements are equivalent:*

1. *σ satisfies* **LL** *and* **AP**.
2. *For all $\varepsilon = (N, L, e) \in \mathcal{F}$ and all $P \in \sigma(\varepsilon)$, $\bar{P} = (\bar{P}_i)_{i \in N} \in FIX(\Phi^{\varepsilon,\beta})$*

Note that, indeed, to obtain a financial rule compatible with β that additionally fulfills **LL** and **AP** is enough to select, for each financial system ε, a vector of payments $\mathbf{t} = (\mathbf{t}_1, ..., \mathbf{t}_n) \in FIX(\Phi^{\varepsilon,\beta})$ and later apply for each agent i the corresponding bankruptcy rule β^i on $\mathbf{t}_i$ to produce a payment matrix. Remark 1 formally contains this observation.

Remark 1. *Given an inventory of bankruptcy rules $\beta = (\beta^i)_{i \in \mathbb{N}}$ and an arbitrary non-empty subset of fixed-points $\mathcal{V}_\varepsilon \subseteq FIX(\Phi^{\varepsilon,\beta})$ for every $\varepsilon = (N, L, e) \in \mathcal{F}$, we can define a financial rule σ compatible with β and satisfying* **LL** *and* **AP** *as follows: for each $\varepsilon = (N, L, e)$ and each $\mathbf{t} \in \mathcal{V}_\varepsilon$, define the matrix $P^{\mathbf{t}}$ as $P_{ij}^{\mathbf{t}} = \beta_j^i(N \setminus \{i\}, \mathbf{t}_i, (L_{ij})_{j \in N \setminus \{i\}})$, for all $i, j \in N$, and then set $\sigma(\varepsilon) = \{P^{\mathbf{t}} \mid \mathbf{t} \in \mathcal{V}_\varepsilon\}$. Note that, for all $\mathbf{t} \in \mathcal{V}_\varepsilon$ and all $i \in N$, by* BB *of β^i, we have that $\bar{P}_i^{\mathbf{t}} = \mathbf{t}_i$ and thus $\bar{P}^{\mathbf{t}} \in FIX(\Phi^{\varepsilon,\beta})$.*

Hence, the problem of combining **C**, **LL**, and **AP** reduces to the existence of fixed-points of $\Phi^{\varepsilon,\beta}$. A way to guarantee that the set of fixed-points is non-empty is requiring RM on the bankruptcy rules contained in β, which implies the monotonicity of the function $\Phi^{\varepsilon,\beta}$. These statements are summarized in Remark 2.

Remark 2. *Given an inventory of bankruptcy rules $\beta = (\beta^i)_{i \in \mathbb{N}}$ satisfying* RM*, there exist financial rules compatible with β satisfying* **LL**, **AP**, *and* **CB**. *This is a direct consequence of the application of Tarski's fixed-point theorem to the non decreasing function $\Phi^{\varepsilon,\beta}$ for each $\varepsilon \in \mathcal{F}$, which ensures that the set $FIX(\Phi^{\varepsilon,\beta})$ is non-empty and forms a complete lattice.*

In view of Remark 1 and Remark 2, the richer structure of the set of fixed-points of the instrumental function Φ allows us to introduce three very special financial rules associated to any inventory of RM bankruptcy rules.

Definition 4. *Let β be a collection of bankruptcy rules satisfying* RM *and let $\mathbf{t}_\varepsilon^+$, $\mathbf{t}_\varepsilon^-$ denote the supremum and the infimum of the set of fixed-points $FIX(\Phi^{\varepsilon,\beta})$ for all $\varepsilon \in \mathcal{F}$, respectively. Define the greatest, σ_+^β, the least, σ_-^β, and the maximal, σ_{max}^β, financial rules compatible with β by setting:*

1. $\sigma_+^\beta(\varepsilon) = \{P^{\mathbf{t}_\varepsilon^+}\}$ *for all* $\varepsilon \in \mathcal{F}$;
2. $\sigma_-^\beta(\varepsilon) = \{P^{\mathbf{t}_\varepsilon^-}\}$ *for all* $\varepsilon \in \mathcal{F}$;
3. $\sigma_{max}^\beta(\varepsilon) = \{P^{\mathbf{t}} \mid \mathbf{t} \in FIX(\Phi^{\varepsilon,\beta})\}$.

Observe that while σ_+^β and σ_-^β satisfy **SIVA**, σ_{max}^β is multi-valued.

Next, we introduce the family of financial rules based on the principles of proportionality, equal awards, and equal losses, respectively.

Definition 5. *A financial rule σ is*

1. *a proportional rule if for all $(N, L, e) \in \mathcal{F}$, all $P \in \sigma(N, L, e)$, and all $i, j \in N$, $P_{ij} = \lambda_i L_{ij}$ where $\lambda_i \in \mathbb{R}_+$ satisfies $\bar{P}_i = \min\{e_i + \sum_{k\in N} \lambda_k L_{ki}, \bar{L}_i\}$;*
2. *a constrained equal awards rule if for all $(N, L, e) \in \mathcal{F}$, all $P \in \sigma(N, L, e)$, and all $i, j \in N$, $P_{ij} = \min\{L_{ij}, \lambda_i\}$ where $\lambda_i \in \mathbb{R}_+$ satisfies $\bar{P}_i = \min\{e_i + \sum_{k\in N} \min\{L_{ki}, \lambda_k\}, \bar{L}_i\}$;*
3. *a constrained equal losses rule if for all $(N, L, e) \in \mathcal{F}$, all $P \in \sigma(N, L, e)$, and all $i, j \in N$, $P_{ij} = \max\{0, L_{ij} - \lambda_i\}$ where $\lambda_i \in \mathbb{R}_+$ satisfies $\bar{P}_i = \min\{e_i + \sum_{k\in N} \max\{0, L_{ki} - \lambda_k\}, \bar{L}_i\}$.*

Note that, from Corollary 1 and Remark 2, proportional, constrained equal awards or constrained equal losses financial rules are compatible with all agents applying their counterpart bankruptcy rule (all of them being RM) satisfying, additionally, **LL** and **AP**, and vice versa.

Remark 3. *It is worth noting that there exist financial rules compatible with an arbitrary inventory of resource monotonic bankruptcy rules β that do not fulfill the requirements of* **LL** *and* **AP**. *Think, for instance, in the following financial rules: $\sigma_1(N, L, e) = \{\mathbf{0}\}$ where $\mathbf{0} \in \mathcal{M}(N)$ denotes the zero matrix and $\sigma_2(N, L, e) = \{L\}$ for all $(N, L, e) \in \mathcal{F}$. Clearly, both, σ_1 and σ_2, are compatible with β since any bankruptcy rule distributing an estate of value zero equals the zero vector, and any other allocating exactly the total debts obligations equal the vector of claims (or liabilities). However, neither σ_1 satisfies* **AP**, *nor σ_2 satisfies* **LL**.

The following examples illustrate two important facts: first, Example 1 shows that proportional, constrained equal awards, and constrained equal losses financial rules need not be **SIVA**; second, Example 2 points out the existence of set-valued financial rules in which different clearing payment matrices may induce different equity values.

Example 1. *(Eisenberg and Noe, 2001) Let $\varepsilon = (N, L, e) \in \mathcal{F}$ with set of players $N = \{1, 2\}$, initial operating cash flows $e = (0, 0)$, and matrix of liabilities*

$$L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now, let σ be an arbitrary financial rule satisfying **CB**, **LL**, **AP**, *and $P \in \sigma(\varepsilon)$. By* **CB**, *$0 \leq P_{12} \leq 1$ and $0 \leq P_{21} \leq 1$. If $E_1(P, e) = P_{21} - P_{12} > 0$ then $E_2(P, e) = P_{12} - P_{21} < 0$, in contradiction with* **LL**. *Thus, $E_1(P, e) = 0$ which implies that $P_{12} = P_{21}$ and*

$$P = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix},$$

where $\lambda \in [0,1]$. Hence,

$$\sigma(\varepsilon) \subseteq \left\{ \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \mid \lambda \in [0,1] \right\}. \tag{4}$$

Clearly, σ is compatible with any inventory of bankruptcy rules β. Thus, in this particular case, the family of proportional, constrained equal awards, and constrained equal losses rules coincide and contain multi-valued solutions.

Example 2. *For all $\varepsilon = (N, L, e) \in \mathcal{F}$, define the financial rule $\sigma(\varepsilon) = \{P_1, P_2\}$ where $P_1 = \sigma_+^{\mathbb{PR}}(\varepsilon)$ and $P_2 = \sigma_-^{\mathbb{CEA}}(\varepsilon)$. Since $\bar{P}_1 \in FIX(\Phi^{\varepsilon,\mathbb{PR}})$ and $\bar{P}_2 \in FIX(\Phi^{\varepsilon,\mathbb{CEA}})$, by Lemma 1 it follows that σ satisfies* **CB**, **LL**, *and* **AP**. *Observe, however, that it does not meet* **C** *since P_1 and P_2 are generated by different bankruptcy rules.*

To see that P_1 and P_2 may induce different equity values for the firms, take $\varepsilon \in \mathcal{F}$ with set of players $N = \{1,2,3\}$, initial operating cash flows $e = (1,0,0)$, and matrix of liabilities

$$L = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Easy calculations lead to

$$P_1 = \begin{pmatrix} 0 & 1/3 & 2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, $E_2(P_1, e) = 1/3 \neq 1/2 = E_2(P_2, e)$ and $E_3(P_1, e) = 2/3 \neq 1/2 = E_3(P_2, e)$.

Nevertheless, Lemma 2 bellow (the proof is relegated to Appendix C) highlights that uniqueness in terms of the equity values (utilities) is guaranteed for set-valued financial rules compatible with resource monotonic bankruptcy rules, regardless of the selected clearing matrices. This is a consequence of the lattice structure of the set of fixed-points of the instrumental function Φ (see Definition 3).

Lemma 2. *Let $\beta = (\beta^i)_{i\in\mathbb{N}}$ be a inventory of resource monotonic bankruptcy rules, $\varepsilon = (N, L, e) \in \mathcal{F}$, and $\mathbf{t}, \mathbf{t}' \in FIX(\Phi^{\varepsilon,\beta})$. Then, $E(P^t, e) = E(P^{t'}, e)$, where the payment matrices P^t and $P^{t'}$ are defined as in Remark 1.*

Two direct consequences of Lemma 2 are specified in the next Remark 4.

Remark 4. *From Lemma 2 it comes that:*

1. *If σ is a financial rule compatible with β meeting* **LL** *and* **AP** *then, for all $(N, L, e) \in \mathcal{F}$ and all $P, P' \in \sigma(N, L, e)$, it holds that $E(P, e) = E(P', e)$.*
2. *Moreover, if σ and σ' are two different financial rules compatible with β meeting* **LL** *and* **AP** *then, for all $(N, L, e) \in \mathcal{F}$, all $P \in \sigma(N, L, e)$, and all $P' \in \sigma'(N, L, e)$, it holds that $E(P, e) = E(P', e)$.*

Lemma 2 is especially important if solutions are multi-valued. In insolvency proceedings involving different courts, that might apply distinct principles in the process of clearing the system, there might be a multiplicity of recommendations compatible with these principles that fulfill **LL** and **AP**. From Eisenberg and Noe (2001), that restrict attention to proportionality, to Csóoka and Herings (2019) that allow for agent-specific bankruptcy rules, through Groote et al. (2018) that assume all agents apply the same bankruptcy rule that need not be the proportional one, the problem of finding a particular clearing payment matrix has been central in the literature of financial networks. Among others, the aforementioned papers study different algorithms and or mechanisms to compute one of such matrices. Recently, Calafiore et al. (2022) offer an iterative procedure to find the whole set of proportional clearing matrices and, in fact, to test uniqueness. Outstandingly, Lemma 2 stresses the fact that, with respect to the value of equity, the agents in the system are indifferent on the chosen clearing payment matrix. This invariance property will be essential in our axiomatic approach.

4 Axioms

In this section, we describe the axioms that we will employ to characterize the set of rules that are supported by the principle of proportionality in the complete domain of financial systems. A first axiomatization was given by Csóka and Herings (2021) in the subdomain of financial systems where all agents dispose of a strictly positive initial endowment. This assumption, however, is not innocuous and discards many economic settings as, for example, those where some agents are exclusively debt holders or some firms dispose of zero cash flows. Moreover, as shown in Eisenberg and Noe (2001) for the more general domain of *regular financial systems*,[6] there is a unique payment matrix compatible with all agents applying the proportional bankruptcy rule. As a result of this particularity, in the context considered by Csóka and Herings (2021) and for the proportional financial rule, there is no difference in working with multi-valued or single-valued solutions concepts. Unfortunately, although this coincidence simplifies their axiomatic analysis, it is not appropriate in the general framework of financial systems.

Together with **CB**, **LL**, and **AP**, Csóka and Herings (2021) impose three additional axioms: continuity, impartiality, and invariance to mitosis. Before stating our axioms, we first emphasize that the accommodation of these properties to multi-valued solution concepts is no longer suitable on the whole domain of financial systems to characterize the family of proportional rules. To overcome this issue, instead of defining the axioms in accordance with the payoff matrices, and considering that utility drives agents' decision-making, we take the novel approach to express them in terms of the value of equity. The proofs of this section are collected in Appendix D.

4.1 Equity continuity

Since Csóka and Herings (2021) implicitly demand **SIVA**, that is, define a financial rule as a function that associates to each financial system a unique clearing payment matrix, they use the classical notion of continuity for functions. For multi-valued solutions, there are two possible generalizations of continuity: lower hemicontinuity and upper hemicontinuity. Formally, a financial rule σ satisfies

- *lower hemicontinuity* (**LHC**) if for all $(N, L, e) \in \mathcal{F}$, all sequence of financial systems $\left\{(N, L^n, e^n)\right\}_{n\in\mathbb{N}}$ converging to (N, L, e), and all clearing payment matrix $P \in \sigma(N, L, e)$, there exists a sequence $\left\{P_n \in \sigma(N, L^n, e^n)\right\}_{n\in\mathbb{N}}$ converging to P;
- *upper hemicontinuity* (**UHC**) if for all $(N, L, e) \in \mathcal{F}$, all sequence of financial systems $\left\{(N, L^n, e^n)\right\}_{n\in\mathbb{N}}$ converging to (N, L, e), and all sequence of clearing payment matrices $\left\{P_n \in \sigma(N, L^n, e^n)\right\}_{n\in\mathbb{N}}$ converging to the matrix P it holds that $P \in \sigma(N, L, e)$;
- *continuity* (**CONT**) if it satisfies simultaneously **LHC** and **UHC**.

For functions, both **LHC** and **UHC** are equivalent to continuity. Informally speaking, these continuity properties require that small changes in the financial system imply small changes in the payment

[6] See Eisenberg and Noe (2001) for a formal definition of regular financial systems.

matrices. Unfortunately, as Example 3 below shows, in the full domain of financial systems, and regardless of the chosen collection of resource monotonic bankruptcy rules β, there might be neither **LHC** nor **UHC** financial rules compatible with β that also satisfy **LL** and **AP**. In particular, there exist proportional financial rules that do not meet any of these continuity properties.

Example 3. *(Eisenberg and Noe, 2001) Let $\varepsilon = (N, L, e) \in \mathcal{F}$ be the financial system described in Example 1. Now, consider the sequence of financial systems $\{\varepsilon^n = (N, L^n, e^n)\}_{n\in\mathbb{N}}$ with set of players $N = \{1, 2\}$, initial operating cash flows $e^n = (0, 0)$, and matrices of liabilities*

$$L^n = \begin{pmatrix} 0 & 1+\frac{1}{n} \\ 1+\frac{1}{n} & 0 \end{pmatrix},$$

for all $n \in \mathbb{N}$.

Let $\sigma_+^{\mathbb{PR}}$ be the greatest, $\sigma_-^{\mathbb{PR}}$ the least, and $\sigma_{max}^{\mathbb{PR}}$ the maximal proportional financial rules (see Definition 4). From Example 1 it follows that

$$\sigma_{max}^{\mathbb{PR}}(\varepsilon) = \left\{ \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \mid \lambda \in [0, 1] \right\}.$$

Consequently,

$$\sigma_+^{\mathbb{PR}}(\varepsilon) = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \text{ and } \sigma_-^{\mathbb{PR}}(\varepsilon) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Following similar arguments, we can additionally set $\sigma_-^{\mathbb{PR}}(\varepsilon^n) = \{\mathbf{0}\}$ and $\sigma_+^{\mathbb{PR}}(\varepsilon^n) = \{L^n\}$ for all $n \in \mathbb{N}$. Now, define the proportional financial rule σ as follows: for all $\varepsilon' \in \mathcal{F}$,

$$\sigma(\varepsilon') = \begin{cases} \sigma_+^{\mathbb{PR}}(\varepsilon) & \text{if} \quad \varepsilon' = \varepsilon, \\ \sigma_-^{\mathbb{PR}}(\varepsilon') & \text{if} \quad \varepsilon' \neq \varepsilon. \end{cases} \tag{5}$$

So, while the sequence of financial systems $\{\varepsilon^n\}_{n\in\mathbb{N}}$ converges to ε when $n \to \infty$, there is not a sequence of matrices $\{P_n \in \sigma(\varepsilon^n)\}_{n\in\mathbb{N}}$ converging to $P \in \sigma(\varepsilon) = \sigma_+^{\mathbb{PR}}(\varepsilon)$, showing that σ fails to satisfy **LHC**. *To see that it neither satisfies* **UHC**, *it is enough to observe that the sequence $\{P_n \in \sigma(\varepsilon^n) = \sigma_-^{\mathbb{PR}}(\varepsilon^n)\}_{n\in\mathbb{N}}$ converges to the zero matrix that is not contained in $\sigma(\varepsilon) = \sigma_+^{\mathbb{PR}}(\varepsilon)$. Furthermore, σ is* **SIVA**, *pointing out that eventhough we restrict ourselves (as in Csóka and Herings, 2021) to financial rules that are functions, in the richer domain of all financial systems continuity of proportional financial rules may fail.*

It is worth noting that the arguments used in Example 3 hold if we replace the proportional bankruptcy rule by any arbitrary collection of resource monotonic bankruptcy rules like, for instance, the constrained equal awards or the constrained equal losses. These bankruptcy rules satisfy CONT but, however, they don't need to produce continuous financial rules. Thus, on the full domain of financial systems, there are financial rules compatible with continuous bankruptcy rules satisfying additionally **LL** and **AP** that do not fulfill either **LHC** or **UHC**. A way to solve this lack of continuity is to formulate it in terms of the utility of the participating agents, that is, according to the value of equity. Formally, a financial rule σ satisfies

- *Equity-continuity* (**E-CONT**) if for all $(N, L, e) \in \mathcal{F}$, all sequence of financial systems $\{(N, L^n, e^n)\}_{n\in\mathbb{N}}$ converging to (N, L, e) and all clearing payment matrix $P \in \sigma(N, L, e)$, there exists a sequence $\{P^n \in \sigma(N, L^n, e^n)\}_{n\in\mathbb{N}}$ with a subsequence of clearing payment matrices $\{P^{n_k} \in \sigma(N, L^{n_k}, e^{n_k})\}_{n_k\in\mathbb{N}}$ such that the associated sequence of equity values $\{E(P^{n_k}, e^{n_k})\}_{n_k\in\mathbb{N}}$ converges to $E(P, e)$.

E-CONT simply says that small changes in the structure of liabilities and in the initial endowments should not lead to large changes in the value of equity. That is, as long as we approach to a financial system (N, L, e), and for any clearing payment matrix P in the solution, there exists a path to approach to the equity values of the agents according to P. Next, we show that **E-CONT** is weaker than **LHC.**

Proposition 1. **LHC** *implies* **E-CONT**.

Interestingly, although there are proportional financial rules that do not satisfy neither **LHC** nor **UHC** (see Example 3 above), all of them meet **E-CONT**.

Lemma 3. *Let σ be a proportional financial rule. Then, σ satisfies* **E-CONT**.

4.2 Equal treatment of equals

The second axiom imposed in Csóka and Herings (2021) is impartiality, which requires that two agents j and k with the same claim on agent i should receive the same payment from i. We now accommodate impartiality for set-valued financial rules. A financial rule σ satisfies

- *impartiality* (**I**) if, for all $(N, L, e) \in \mathcal{F}$ and all $i, j, k \in N$ such that $L_{ij} = L_{ik}$ then, for all $P \in \sigma(N, L, e)$, it holds that $P_{ij} = P_{ik}$.

Impartiality applies only to payments made by agent i to agents j and k, but the repayment capacity of these two agents is not taken into account. Even though impartiality appears to be a mild condition, it applies to pairs of agents that need not be symmetric or identical. Here, and to preserve the ideal that equals should be treated equally, we understand that two agents are *symmetric* if they have the same initial operating cash flow, the same mutual obligations to each other, as well as the same claims and debts to the rest of agents. We interpret that symmetric agents should be treated equally, that is, they should end up with the same utility. Formally, a financial rule σ satisfies

- *equal treatment of equals* (**ETE**) if for all $(N, L, e) \in \mathcal{F}$ and all $i, j \in N$ such that $e_i = e_j$, $L_{ij} = L_{ji}$, $L_{ik} = L_{jk}$, and $L_{ki} = L_{kj}$ for all $k \in N \setminus \{i, j\}$ then, for all $P \in \sigma(N, L, e)$ it holds that $E_i(P, e) = E_j(P, e)$.

ETE ensures that symmetric agents should get the same value of equity. Under the basic requirements of **CB**, **LL**, and **AP**, the next result establishes that **ETE** is weaker than **I**.

Proposition 2. *Under* **CB**, **LL**, *and* **AP**; **I** *implies* **ETE**.

Outstandingly, if we impose the stronger property of **C**, rather than **CB**, we obtain that **I** and **ETE** are equivalent. The underlying reason is that ETE of the bankruptcy rules supporting a financial rule connects both properties.

Proposition 3. *Let σ be a financial rule compatible with a collection of bankruptcy rules $\beta = (\beta^i)_{i \in \mathbb{N}}$, satisfying* **LL** *and* **AP**. *Then, the following statements are equivalent:*

1. *σ satisfies* **I**.
2. *σ satisfies* **ETE**.
3. *β^i satisfies* ETE *for all $i \in \mathbb{N}$.*

A straightforward consequence of Proposition 3 is the following.

Corollary 2. *Let σ be a proportional financial rule. Then, σ satisfies* **I** *and* **ETE**.

To conclude, the next example stresses that, in general, **I** and **ETE** do not imply one another.

Example 4. *We first show that* **ETE** *does not imply* **I***, neither under* **CB**, **LL** *and* **AP***. Consider the financial system $\varepsilon' = (N', L', e')$ being $N' = \{1, 2, 3\}$, initial operating cash flows $e' = (0, 0, 0)$, and matrix of liabilities*

$$L' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix}.$$

Now define the financial rule σ_1 as follows:

$$\sigma_1(\varepsilon) = \begin{cases} \sigma_+^{\mathbb{PR}}(\varepsilon) & \text{if} \quad \varepsilon \neq \varepsilon', \\ \left\{ P' = \begin{pmatrix} 0 & 1/2 & 1 \\ 0 & 0 & 1 \\ 3/2 & 1/2 & 0 \end{pmatrix} \right\} & \text{if} \quad \varepsilon = \varepsilon'. \end{cases}$$

To see that σ_1 satisfies **CB**, **LL**, *and* **AP***, it is enough to observe that, by definition, $\sigma_+^{\mathbb{PR}}$ fulfills the properties and, in the financial system ε', we have $E_1(P', e') = E_2(P', e') = E_3(P', e') = 0$. To check that it also satisfies* **ETE***, we distinguish two cases. If agents i and j are symmetric in $\varepsilon \neq \varepsilon'$, then* **ETE** *follows since proportional financial rules satisfy the property. Otherwise, the only symmetric players in ε' are 1 and 2, which receive the same equity value according to P'. However, σ_1 does not meet* **I** *since $L'_{12} = L'_{13}$ but $P'_{12} \neq P'_{13}$.*

To see that **I** *does not imply* **ETE***, define the financial rule σ_2 as follows:*

$$\sigma_2(\varepsilon) = \begin{cases} \sigma_+^{\mathbb{PR}}(\varepsilon) & \text{if} \quad \varepsilon \neq \varepsilon', \\ \left\{ P'' = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 2 & 0 \end{pmatrix} \right\} & \text{if} \quad \varepsilon = \varepsilon'. \end{cases}$$

Clearly, σ_2 satisfies **I** *as, in case that $\varepsilon \neq \varepsilon'$, $\sigma_+^{\mathbb{PR}}$ meet the property. Otherwise, if $\varepsilon = \varepsilon'$, $L'_{12} = L'_{13}$ and $P''_{12} = P''_{13} = 1$; $L'_{21} = L'_{23}$ and $P''_{21} = P''_{23} = 0$; $L'_{31} = L'_{32}$ and $P''_{31} = P''_{32} = 2$. However, σ_2 fails to satisfy* **ETE** *because players 1 and 2 are symmetric in ε' but they obtain a different equity value according to P'. Indeed, $E_1(P'', e') = 0$ and $E_2(P'', e') = 3$.*

4.3 Non-manipulability by clones

On the setup of bankruptcy problems, O'Neill (1982) paved the route to characterize the proportional rule on the basis of non-manipulability, requiring that agents should not have incentives to merge or split their claims as they will. O'Neill's result was refined in different ways by Chun (1988), de Frutos (1999), and Ju et al. (2007). An important result in our investigation is Theorem 3 in Calleja and Llerena (2022) that states a new characterization for this focal rule using NMC as definced in Subsection 2.2, a weak form of non-manipulability that entitles agents to merge or split only when they are or become symmetric, together with a standard axiom referring continuity on claims.

In the setting of financial networks, Csóka and Herings (2021) interpret non-manipulability as some invariance conditions on the clearing payment matrices enforcing invariance not only on payments made by and received from the merging or splitting agents, but also on payments between agents that are not involved in the merger or split, in the spirit of additivity of claims (Curiel et al., 1987) or strong non-manipulability for bankruptcy problems (Moreno-Ternero, 2006). Contrary to bankruptcy problems, in financial systems no rule is immune to manipulability when combined with the basic requirements of **CB**, **LL**, and **AP**. Therefore, Csóka and Herings (2021) weaken non-manipulability into invariance to mitosis restricting splits and mergers to situations involving identical agents. A natural extension of this property to multi-valued solutions can be derived as follows. A financial rule σ satisfies

- *invariance to mitosis* (**IM**) if for all (N,L,e), $(N',L',e') \in \mathcal{F}$, if $N' \subset N$ and there is $m \in N'$ such that

$$\begin{aligned} e_i &= \tfrac{e'_m}{|N\setminus N'|+1} \text{ for all } i \in N \setminus N' \cup \{m\} \\ e_i &= e'_i \text{ for all } i \in N' \setminus \{m\} \\ L_{kl} &= 0 \text{ for all } k,l \in N \setminus N' \cup \{m\} \\ L_{ij} &= L'_{ij} \text{ for all } i,j \in N' \setminus \{m\} \\ L_{ki} &= \tfrac{L'_{mi}}{|N\setminus N'|+1} \text{ for all } k \in N \setminus N' \cup \{m\}, i \in N' \setminus \{m\} \\ L_{ik} &= \tfrac{L'_{im}}{|N\setminus N'|+1} \text{ for all } k \in N \setminus N' \cup \{m\}, i \in N' \setminus \{m\} \end{aligned} \tag{6}$$

then,

(a) for each $P \in \sigma(N,L,e)$ there exists $P' \in \sigma(N',L',e')$ and

(b) for each $P' \in \sigma(N',L',e')$ there exists $P \in \sigma(N,L,e)$

such that

$$\begin{aligned} P'_{mi} &= P_{mi} + \textstyle\sum_{k\in N\setminus N'} P_{ki} \text{ for all } i \in N' \setminus \{m\}; \\ P'_{im} &= P_{im} + \textstyle\sum_{k\in N\setminus N'} P_{ik} \text{ for all } i \in N' \setminus \{m\}; \\ P'_{kl} &= P_{kl} \text{ for all } k,l \in N' \setminus \{m\}. \end{aligned} \tag{7}$$

Conditions listed in (6), concerning the characteristics of the financial systems for which some invariance is required, are the same as in Csóka and Herings (2021). Observe that only equal agents, as introduced in defining **ETE**, are allowed to split or merge. On the contrary, since financial rules may select a number of payment matrices, we demand that for every payment recommendation made by the rule $P \in \sigma(N,L,e)$ there exists $P' \in \sigma(N',L',e')$ satisfying all conditions in (7). While equations in (7) are exactly as in the aforementioned paper, we want to ensure that for any recommendation in (N,L,e) there is another one in (N',L',e') that do not provide incentives to merge since payments remain invariant. On the other hand, imposing that any $P' \in \sigma(N',L',e')$ can be assigned to a $P \in \sigma(N,L,e)$ for which all equalities in (7) hold guarantees that the rule do not provide incentives to split neither. Observe that, indeed, **IM** requires payments made by and received from agents not involved in the split or the merge remains constant as well.

Unexpectedly, as the next example points out, there are proportional financial rules that do not meet **IM**.

Example 5. *Let $\varepsilon = (N,L,e)$ be the financial system as defined in Example 1, that is, $N = \{1,2\}$, $e = (0,0)$, and matrix of liabilities*

$$L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now consider $\varepsilon' = (N',L',e')$ where agent 2 splits into agents 2 and 3, being $N' = \{1,2,3\}$, $e' = (0,0,0)$, and

$$L' = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}.$$

As in Example 3, we take the proportional financial rule σ defined as follows: for all $\varepsilon'' \in \mathcal{F}$,

$$\sigma(\varepsilon'') = \begin{cases} \sigma_+^{\mathbb{PR}}(\varepsilon) & \text{if} \quad \varepsilon'' = \varepsilon, \\ \sigma_-^{\mathbb{PR}}(\varepsilon'') & \text{if} \quad \varepsilon'' \neq \varepsilon. \end{cases} \tag{8}$$

Recall that,

$$\sigma_{+}^{\mathbb{PR}}(\varepsilon) = \left\{ P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Some calculations lead to

$$\sigma(\varepsilon') = \sigma_{-}^{\mathbb{PR}}(\varepsilon') = \left\{ P' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Observe that, $1 = P_{21} \neq P'_{21} + P'_{31} = 0$, *which prove that* σ *does not meet* **IM**. *However,* $E_2(P, e) = 0 = E_2(P', e) + E_3(P', e)$.

Note that the financial rule in Example 5 is single-valued. Moreover, by means of comparing the equity value of the agents, it turns out that agent 2 does not have incentives to split into 2 and 3, neither agents 2 and 3 have incentives to merge into agent 2, although payments among agents are not invariant. In the same fashion of continuity, a way to solve this issue is to reformulate the property in terms of the equity value of the agents. A financial rule σ satisfies

- *Non-manipulability by clones* (**NMC**) if for all (N, L, e), $(N', L', e') \in \mathcal{F}$, if $N' \subset N$ and there is $m \in N'$ such that all conditions listed in (6) hold, then

  (a) for each $P \in \sigma(N, L, e)$ there exists $P' \in \sigma(N', L', e')$ and

  (b) for each $P' \in \sigma(N', L', e')$ there exists $P \in \sigma(N, L, e)$

  such that, for all $i \in N' \setminus \{m\}$,
  $$E_i(P', e') = E_i(P, e).$$

NMC says that the split of an agent into identical agents or the merge of a group of identical agents should not affect the utility of the remaining agents and, as a consequence, neither the utility of the agents merging or splitting. Observe that,

$$\sum_{i \in N} E_i(P, e) = \sum_{i \in N} e_i = \sum_{i \in N'} e'_i = \sum_{i \in N'} E_i(P', e'),$$

which automatically implies that,

$$E_m(P', e') = E_m(P, e) + \sum_{k \in N \setminus N'} E_k(P, e).$$

The next proposition states that **NMC** is a weak version of **IM**.

Proposition 4. **IM** *implies* **NMC**.

Outstandingly, although proportional financial rules may fail to satisfy **IM** as illustrated in Example 5, they hold **NMC**

Lemma 4. *Let* σ *be a proportional financial rule. Then,* σ *satisfies* **NMC**.

5 Axiomatic characterization

In this section, we provide an axiomatic foundation for the family of proportional financial rules. Remarkably, as a particular case we obtain a new characterization of the (unique) proportional financial rule in the restrictive domain of regular financial systems considered by Csóka and Herings (2021) where agents initially have a positive amount of cash at their disposal. With respect to Csóka and Herings' result, **CB** is strengthened into **C**, **CONT** and **IM** are weakened into **E-CONT** and **NMC**, respectively, while **SIVA** and **I** are no longer required.

Theorem 2. *A financial rule satisfies* **C**, **LL**, **AP**, **E-CONT**, *and* **NMC** *if and only if it is a proportional financial rule.*

Proof. The *only if part* follows from the definition of a proportional financial rule that ensures **C**, **LL**, **AP**, together with Lemma 3 and Lemma 4 that guarantee **E-CONT** and **NMC**, respectively.

To prove the *if part*, let σ be a financial rule compatible with a collection of bankruptcy rules $\beta = (\beta^i)_{i\in\mathbb{N}}$ and satisfying **LL**, **AP**, **E-CONT**, and **NMC**.

- **Claim 1**: For all $i \in \mathbb{N}$, β^i satisfies WCONT.

  To show **Claim 1**, take $i \in \mathbb{N}$. Let $(N, E, c) \in \mathcal{B}$ and $\{(N, E^n, c^n)\}_{n\in\mathbb{N}}$ be a sequence of bankruptcy problems converging to (N, E, c) with $i \notin N$. Let $\varepsilon = (\bar{N}, L, e) \in \mathcal{F}$ and $\varepsilon^n = (\bar{N}, L^n, e^n) \in \mathcal{F}$ be the corresponding associated financial systems with $\bar{N} = N \cup \{i\}$ as defined in Subsection 2.3. Next, we see that $\sigma(\varepsilon)$ and $\sigma(\varepsilon^n)$ select a unique payment matrix. Indeed, let $P \in \sigma(\varepsilon)$. As σ satisfies **CB** (received from **C**), $P_{jl} = 0$ for all $j \in N, l \in \bar{N}$. Moreover, since σ is compatible with β, $P_{ij} = \beta^i_j(N, E, c)$ for all $j \in N$, which is unique by definition of β^i. Hence, $\sigma(\varepsilon) = \{P\}$. In a similar way, we obtain $\sigma(\varepsilon^n) = \{P^n\}$ for all $n \in \mathbb{N}$, being $P^n_{jl} = 0$ and $P^n_{ij} = \beta^i_j(N, E^n, c^n)$ for all $j \in N$, $l \in \bar{N}$. Note that $E = \bar{P}_i$ and $E^n = \bar{P}^n_i$.

  Clearly, the sequence of financial systems $\{\varepsilon^n\}_{n\in\mathbb{N}}$ converges to ε and, by **E-CONT**, there exists a subsequence of clearing payment matrices $\{P^{n_k}\}_{n_k\in\mathbb{N}}$ such that the associated sequence of equity values $\{E(P^{n_k}, e^{n_k})\}_{n_k\in\mathbb{N}}$ converges to $E(P, e)$. Let $j \in N$ and $n_k \in \mathbb{N}$. Then, we have

  $$E_j(P^{n_k}, e^{n_k}) = e^{n_k}_j + \sum_{l\in\bar{N}} P^{n_k}_{lj} - \sum_{l\in\bar{N}} P^{n_k}_{jl} = P^{n_k}_{ij} \underset{\mathbf{C}}{=} \beta^i_j(N, E^{n_k}, c^{n_k})$$

  and

  $$E_j(P, e) = e_j + \sum_{l\in\bar{N}} P_{lj} - \sum_{l\in\bar{N}} P_{jl} = P_{ij} \underset{\mathbf{C}}{=} \beta^i_j(N, E, c).$$

  Hence, for all $j \in N$, the sequence $\{\beta^i_j(N, E^{n_k}, c^{n_k})\}_{n_k\in\mathbb{N}}$ converges to $\beta^i_j(N, E, c)$, which means that β^i meets WCONT.

- **Claim 2**: For all $i \in \mathbb{N}$, β^i satisfies NMC.

  To show **Claim 2**, take $i \in \mathbb{N}$. Let $(N, E, c), (N'E, c') \in \mathcal{B}$ where $N' \subset N$, $i \notin N$, and there is $m \in N'$ such that $c_j = \frac{c'_m}{|N\setminus N'|+1}$ for all $j \in N \setminus N' \cup \{m\}$ and $c'_j = c_j$ for all $j \in N' \setminus \{m\}$. Let $\varepsilon = (\bar{N}, L, e)$ and $\varepsilon' = (\bar{N}', L', e')$ be the associated financial systems being $\bar{N} = N \cup \{i\}$ and $\bar{N}' = N' \cup \{i\}$ as defined in Subsection 2.3. It can easily checked that ε and ε' satisfy all the conditions in (6). Moreover, following the same arguments as in the proof of **Claim 1** we have that $\sigma(\varepsilon) = \{P\}$ where $P_{ij} = \beta^i_j(N, E, c)$ and $P_{jk} = 0$ for all $j \in N, k \in \bar{N}$; and $\sigma(\varepsilon') = \{P'\}$ where $P'_{ij} = \beta^i_j(N', E, c')$ and $P'_{jk} = 0$ for all $j \in N', k \in \bar{N}'$. By **NMC**, for all $j \in \bar{N}' \setminus \{m\}$, we have

  $$E_j(P', e') = E_j(P, e).$$

  In particular, if $j \neq i$, we obtain

  $$E_j(P', e') = e'_j + \sum_{l\in\bar{N}'} P'_{lj} - \sum_{l\in\bar{N}'} P'_{jl} = P'_{ij} \underset{\mathbf{C}}{=} \beta^i_j(N', E, c')$$

  and

  $$E_j(P, e) = e_j + \sum_{l\in\bar{N}} P_{lj} - \sum_{l\in\bar{N}} P_{jl} = P_{ij} \underset{\mathbf{C}}{=} \beta^i_j(N, E, c),$$

  which implies that β^i satisfies NMC.

Hence, from Claims 1 and 2, all bankruptcy rules in β satisfy WCONT and NMC which imply, by Theorem 1, that $\beta^i = PR$, for all $i \in \mathbb{N}$. Finally, since σ is compatible with $\mathbb{PR}$ and, additionally, meets **LL** and **AP**, we conclude that it is a proportional financial rule. $\square$

To finish, we show that the axioms in Theorem 2 are logically independent:

- (All except **C**): for all $(N, L, e) \in \mathcal{F}$, let $\sigma^1(N, L, e) = \sigma_+^{\mathbb{PR}}(N, 2L, e)$. Clearly, σ^1 does not meet **CB** and thus neither **C**. Since $\sigma_+^{\mathbb{PR}}$ satisfies **LL**, **AP**, **E-CONT**, and **NMC**, it follows that σ^1 inherits these properties.

- (All except **LL**): for all $(N, L, e) \in \mathcal{F}$, let $\sigma^2(N, L, e) = \{L\}$. Obviously, σ^2 satisfies **C**, since any bankruptcy rule distributing an estate equal to its total liabilities equals, by CB and BB, the vector of its liabilities. Clearly, it satisfies **AP**, **E-CONT** comes from σ^2 satisfying **LHC**, while **NMC** holds from σ^2 satisfying **SIVA** and **IM**. However, σ^2 does not meet **LL** since there might exist firms with insufficient resources to cover all its liabilities and, consequently, ending up with a negative equity value.

- (All except **AP**): for all $(N, L, e) \in \mathcal{F}$, let $\sigma^3(N, L, e) = \{\mathbf{0}\}$ where $\mathbf{0} \in \mathcal{M}(N)$ denotes the zero matrix. Note that σ^3 satisfies **C** since any bankruptcy rule distributing an estate of zero equals the zero vector. Clearly, σ^3 satisfies **LL**, **E-CONT** and **NMC** comes from **LHC**, **SIVA**, and **IM**. However, it does not meet **AP** since the equity value of each firm coincides with its initial endowment but it could be positive.

- (All except **NMC**): for all $(N, L, e) \in \mathcal{F}$, let $\sigma^4(N, L, e) = \sigma_-^{\mathbb{CEA}}(N, L, e)$. Clearly, σ^4 meets **C**, **LL**, and **AP**. Obviously, σ^4 is a constrained equal awards financial rule. Moreover, the proof of Lemma 3 can be followed almost step by step, if we take a constrained equal awards rule, instead. Thus, σ^4 satisfies **E-CONT**. Finally, to see that it fails to satisfy **NMC**, consider the financial system ε as defined in Example 2. Then,

$$\sigma^4(\varepsilon) = \left\{ P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

  Suppose now that agent 3 splits into clones 3 and 4, defining the corresponding four agents financial system ε'. Some easy algebra yields to

$$\sigma^4(\varepsilon') = \left\{ P' = \begin{pmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

  Observe that $E_3(P, e) = 1/2 < 2/3 = E_3(P', e') + E_4(P', e')$, showing that constrained equal awards financial rules provide incentives to split.

- (All except **E-CONT**): Define first the bankruptcy rule β^* that gives priority to positive non-rational claims, i.e, belonging to the set $\mathbb{R} \setminus \mathbb{Q}_+$, over claims in $\mathbb{Q}_+$, and distributing any amount proportionally in each group. Formally, let $(N, E, c) \in \mathcal{B}$ and $N_{\mathbb{Q}_+}$ be the set of agents with a positive rational claim:

  – If $\sum_{k \in N \setminus N_{\mathbb{Q}_+}} c_k \geq E$, then

$$\beta_i^*(N, E, c) = PR_i(N \setminus N_{\mathbb{Q}_+}, E, c_{N \setminus N_{\mathbb{Q}_+}}) \text{ for all } i \in N \setminus N_{\mathbb{Q}_+}$$

and

$$\beta_i^*(N,E,c) = 0 \text{ for all } i \in N_{\mathbb{Q}_+}.$$

– If $\sum_{k\in N\setminus N_{\mathbb{Q}_+}} c_k < E$, then

$$\beta_i^*(N,E,c) = c_i \text{ for all } i \in N \setminus N_{\mathbb{Q}_+}$$

and

$$\beta_i^*(N,E,c) = PR_i\left(N_{\mathbb{Q}_+}, E - \sum_{k\in N\setminus N_{\mathbb{Q}_+}} c_k, c_{N_{\mathbb{Q}_+}}\right) \text{ for all } i \in N_{\mathbb{Q}_+}.$$

Note that β^* is RM but not CCONT. So, the financial rule $\sigma^5(N,L,e) = \sigma_+^{\beta^*}(N,L,e)$ is well defined. Obviously, σ^5 satisfies **C** (with respect to β^*). This rule was first introduced in Csóka and Herings (2021) and, as they point out, it also meets **LL**, **AP**, and **IM**. Thus, from Proposition 4, it also satisfies **NMC**. To see that is does not satisfy **E-CONT**, consider the financial $\varepsilon = (N,L,e) \in \mathcal{F}$ with set of players $N = \{1,2,3\}$, initial operating cash flows $e = (1,0,0)$, and matrix of liabilities

$$L = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, consider the sequence of financial systems $\{\varepsilon^n = (N,L^n,e^n)\}_{n\in\mathbb{N}}$ with set of players $N = \{1,2,3\}$, initial operating cash flows $e^n = (1,0,0)$, and matrices of liabilities

$$L^n = \begin{pmatrix} 0 & 1+\frac{\sqrt{2}}{n} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for all $n \in \mathbb{N}$. Clearly, $\{\varepsilon^n\}_{n\in\mathbb{N}}$ converges to ε. It is not difficult to check that

$$\sigma^5(\varepsilon) = \left\{ P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \text{ and } \sigma^5(\varepsilon^n) = \left\{ P^n = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \text{ for all } n \in \mathbb{N}.$$

Note that $\{E(P^n,e^n)\}_{n\in\mathbb{N}}$ converges to $(0,1,0)$ while $E(P,e) = (0,1/2,1/2)$. Hence, σ^5 does not meet **E-CONT**.

Table 1 bellow collects the financial rules and the axioms they satisfy.

Table 1: Solutions and Properties

	σ^1	σ^2	σ^3	σ^4	σ^5
Compatibility	No	Yes	Yes	Yes	Yes
Limited **L**iability	Yes	No	Yes	Yes	Yes
Absolute **P**riority	Yes	Yes	No	Yes	Yes
Non **M**anipulability by **C**lones	Yes	Yes	Yes	No	Yes
Equity-**CONT**inuity	Yes	Yes	Yes	Yes	No

6 Final comments

In this paper, we provide an axiomatic ground for the family of proportional financial rules in the whole domain of financial systems. Assuming that all agents dispose of a strictly positive operating cash flow, Csóka and Herings (2021) identify a set of axioms that characterizes the unique proportional clearing mechanism. However, allowing some agents to initially have zero cash flow may result in a multiplicity of proportional payoff matrices, which requires a different set of axioms, putting the focus on the equity value of the entities rather than on the clearing matrices themselves. Outstandingly, our axiomatic characterization (Theorem 2) is also valid in the restricted subdomain of financial networks considered by Csóka and Herings (2021). We impose weaker non-manipulability and continuity axioms and get rid of single-valuedness, claim boundedness, and impartiality to the price of restricting solutions to be compatible with bankruptcy rules. This characterization establishes a parallelism with the axiomatiation of the proportional bankruptcy rule by means of weak continuity and non-manipulability by clones (Theorem 1). Recently, Calleja and Llerena (2022) show that claims monotonicity can replace weak continuity. In this sense, and given that monotonicity principles are widely accepted, an interesting open question is whether or not some suitable monotonicity requirements on liabilities could be used to provide new characterizations of proportional financial rules.

An important result in the literature of bankruptcy problems is owing to Young (1987), who characterizes the so-called *parametric rules* by means of symmetry (or equal treatment of equals), resource continuity, and consistency, a classical invariant principle with respect to variations of population (see Thomson, 2012). A possibility for future research could be to introduce parametric rules in the context of financial networks and extend Young's result to this setup. As noted by Csóka and Herings (2021), the main issue in applying the principle of consistency is that the reduced problem may be outside of the original domain. A natural way to address this drawback is using conditional consistency, a weak form of consistency imposing that the initials payments must be reconfirmed in the reduced problem only when it is a financial network.

A closely related result to Young's can be found in Ju (2002), who characterizes the set of parametric rules that are not manipulable via (pairwise) merging or splitting. Although non-manipulability via splitting is incompatible with the basic requirements of claims boundedness, limited liability, and absolute priority, Calleja et al. (2021) identify a broad class of financial rules immune to manipulations via merging and compatible with these properties. Therefore, an interesting line of research could be the identification of the class of financial rules that fulfill non-manipulability via (pairwise) merging.

Appendix A: Tarski's fixed-point theorem

A *lattice* is a pair $(A, \leq)$ formed by a non-empty set A and a transitive and antisymmetric binary relation $\leq$ on A that determines a partial order on A such that, for any two elements $x, y \in A$, there is a *supremum (join)*, denoted by $x \vee y$, and an *infimum (meet)*, denoted by $x \wedge y$. We write $x < y$ if $x \leq y$ but $x \neq y$. The *supremum* $x \vee y$ is the unique element of A such that $x, y \leq x \vee y$ and if $z \in A$ is such that $z \geq x, y$, then $z \geq x \vee y$. The *infimum* $x \wedge y$ is the unique element of A such that $x, y \geq x \wedge y$ and if $z \in A$ is such that $z \leq x, y$, then $z \leq x \wedge y$. The lattice $(A, \leq)$ is called *complete* if every non-empty subset $B \subseteq A$ has a *supremum* and an *infimum*. Given two elements $x, y \in A$ with $x \leq y$, we denote by $[x, y]$ the interval with the endpoints x and y, i.e., $[x, y] = \{z \in A \mid x \leq z \leq y\}$. Clearly, $([x, y], \leq)$ is a lattice, and it is a complete lattice if $(A, \leq)$ is complete. We shall consider functions $f : B \to C$, where $B, C \subseteq A$. Such a function f is called *non-decreasing* if, for any pair of elements $x, y \in B$, $x \leq y$ implies $f(x) \leq f(y)$. A *fixed point* of f is an element x of B such that $x = f(x)$. Let $FIX(f)$ denote the set of fixed-points of f. The Tarski's fixed-point theorem states that if $(A, \leq)$ is a complete lattice and $f : B \to C$ is a non-decreasing function, then $(\mathrm{FIX}(f), \leq)$ is a complete lattice.

Appendix B: Proofs of Section 2

Proof. (Lemma 1) Let σ be a financial rule satisfying **CB**, **LL**, and **AP**, $(N, L, e) \in \mathcal{F}$, and $P \in \sigma(N, L, e)$. By **LL**, $E_i(P, e) \geq 0$ for all $i \in N$. If $E_i(P, e) = 0$, then $e_i + \sum_{k \in N} P_{ki} = \bar{P}_i \leq \bar{L}_i$, where the inequality comes from **CB**. If $E_i(P, e) > 0$, by **AP** and **CB**, $\bar{P}_i = \bar{L}_i$ and thus $e_i + \sum_{k \in N} P_{ki} > \bar{P}_i = \bar{L}_i$. Hence, $\bar{P}_i = \min\left\{e_i + \sum_{k \in N} P_{ki}, \bar{L}_i\right\}$. To see the reverse implication, let σ be a financial rule fulfilling **CB**, $(N, L, e) \in \mathcal{F}$, and $P \in \sigma(N, L, e)$. If $\bar{P}_i = \min\left\{e_i + \sum_{k \in N} P_{ki}, \bar{L}_i\right\}$, for all $i \in N$, then $E_i(P, e) = e_i + \sum_{k \in N} P_{ki} - \bar{P}_i \geq 0$, which proves **LL**. To check **AP**, select $i \in N$ and suppose that $E_i(P, e) > 0$. Then, $e_i + \sum_{k \in N} P_{ki} > \bar{P}_i$ and thus $\bar{P}_i = \bar{L}_i$. □

Appendix C: Proofs of Section 3

Proof. (Lemma 2)

Let $\beta = (\beta^i)_{i \in \mathbb{N}}$ be a inventory of resource monotonic bankruptcy rules and $\varepsilon = (N, L, e) \in \mathcal{F}$. Since, for all $i \in N$, β^i satisfies RM, by Tarski's theorem the set of fixed-points $\mathit{FIX}(\Phi^{\varepsilon,\beta})$ is non-empty and forms a complete lattice. Let $\mathbf{t} \in [\mathbf{0}, \bar{L}]$ be an arbitrary element of $\mathit{FIX}(\Phi^{\varepsilon,\beta})$ and $P^{\mathbf{t}} \in \mathcal{M}(N)$ defined by $P^{\mathbf{t}}_{ij} = \beta^i_j\left(N \setminus \{i\}, \mathbf{t}_i, (L_{ij})_{j \in N \setminus \{i\}}\right)$ for all $i, j \in N$. As $\bar{P}^{\mathbf{t}} = \mathbf{t}$, for all $i \in N$ we have that

$$
\begin{aligned}
E_i(P^{\mathbf{t}}, e) &= e_i + \sum_{k \in N} P^{\mathbf{t}}_{ki} - \bar{P}^{\mathbf{t}}{}_i \\
&= e_i + \sum_{k \in N} P^{\mathbf{t}}_{ki} - \min\left\{e_i + \sum_{k \in N} P^{\mathbf{t}}_{ki}, \bar{L}_i\right\} \\
&= \max\left\{0, e_i + \sum_{k \in N} P^{\mathbf{t}}_{ki} - \bar{L}_i\right\}.
\end{aligned}
\tag{9}
$$

Let $\mathbf{t}^+$ be the supremum of $\mathit{FIX}(\Phi^{\varepsilon,\beta})$ and $P^{\mathbf{t}^+} \in \mathcal{M}(N)$ the corresponding matrix. Since $\mathbf{t}^+ \geq \mathbf{t}$, by RM of β^i for all $i \in N$, we have that $P^{\mathbf{t}^+} \geq P^{\mathbf{t}}$ and thus, from (9), $E_i(P^{\mathbf{t}^+}, e) \geq E_i(P^{\mathbf{t}}, e)$. If there is $i \in N$ such that $E_i(P^{\mathbf{t}^+}, e) > E_i(P^{\mathbf{t}}, e)$, then $\sum_{i \in N} e_i = \sum_{i \in N} E_i(P^{\mathbf{t}^+}, e) > \sum_{i \in N} E_i(P^{\mathbf{t}}, e) = \sum_{i \in N} e_i$ getting a contradiction. Thus, $E(P^{\mathbf{t}^+}, e) = E(P^{\mathbf{t}}, e)$, which finishes the proof. □

Appendix D: Proofs of Section 4

Proof. (Proposition 1)

Let $\varepsilon = (N, L, e) \in \mathcal{F}$ and $\left\{\varepsilon^n = (N, L^n, e^n)\right\}_{n \in \mathbb{N}}$ be a sequence of financial systems converging to ε. Let σ be a financial rule satisfying **LHC** and $P \in \sigma(\varepsilon)$. By **LHC**, there exists a sequence of clearing payment matrices $\left\{P^n \in \sigma(\varepsilon^n)\right\}_{n \in \mathbb{N}}$ converging to P. Then, for the associated sequence of equity values $\{E(P^n, e^n)\}_{n \in \mathbb{N}}$ we have, for all $i \in N$,

$$
\begin{aligned}
\lim_{n \to \infty} E_i\left(P^n, e^n\right) &= \lim_{n \to \infty} \left(e^n_i + \sum_{k \in N} P^n_{ki} - \sum_{k \in N} P^n_{ik}\right) \\
&= e_i + \sum_{k \in N} P_{ki} - \sum_{k \in N} P_{ik} \\
&= E_i(P, e)
\end{aligned}
$$

which proves **E-CONT** of σ. □

Proof. (Lemma 3)

Let σ be a proportional financial rule. Hence, σ satisfies **CB**, **LL**, and **AP**. Let $\left\{\varepsilon^n = (N, L^n, e^n)\right\}_{n\in\mathbb{N}}$ be a sequence of financial systems converging to $\varepsilon = (N, L, e)$, $P \in \sigma(\varepsilon)$, and $\left\{P^n \in \sigma(\varepsilon^n)\right\}_{n\in\mathbb{N}}$ be a sequence of clearing payment matrices. By **CB**, $0 \le P^n \le L^n$ for all $n \in \mathbb{N}$. Therefore, by the Bolzano-Weierstrass theorem,[7] we can suppose, w.l.o.g., that the sequence $\left\{P^n \in \sigma(\varepsilon^n)\right\}_{n\in\mathbb{N}}$ converges to P^*. Let $\left\{E(P^n, e^n)\right\}_{n\in\mathbb{N}}$ be the associated sequence of equity values. We claim that $\left\{E(P^n, e^n)\right\}_{n\in\mathbb{N}}$ converges to $E(P, e)$.

To prove it, we first see that

$$E(P^*, e) = E(P, e). \tag{10}$$

By **LL**, **AP**, and Corollary 1, $\bar{P^n} \in FIX(\Phi^{\varepsilon^n,\mathbb{PR}})$ for all $n \in \mathbb{N}$. Taking the limit when $n \to \infty$ we have that

$$\begin{aligned}
\bar{P}_i^* &= \lim_{n\to\infty} \bar{P}_i^n \\
&= \lim_{n\to\infty} \min\left\{ e_i^n + \sum_{k\in N} PR_i^k \left(N \setminus \{k\}, \bar{P}_k^n, (L_{kj}^n)_{j\in N\setminus\{k\}}\right), \bar{L}_i^n \right\} \\
&\underset{\text{CONT of } PR}{=} \min\left\{ e_i + \sum_{k\in N} PR_i^k \left(N \setminus \{k\}, \bar{P}_k^*, (L_{kj})_{j\in N\setminus\{k\}}\right), \bar{L}_i \right\},
\end{aligned}$$

for all $i \in N$, where the last equality follows from the continuity of the proportional bankruptcy rule. Thus, $\bar{P}^* \in FIX(\Phi^{\varepsilon,\mathbb{PR}})$ and, consequently, $E(P^*, e) = E(P, e)$, which follows from Lemma 2 and the observation that since $P \in \sigma(\varepsilon)$, by **LL**, **AP**, and Corollary 1, $\bar{P} \in FIX(\Phi^{\varepsilon,\mathbb{PR}})$.

Finally, for all $i \in N$, we obtain

$$\begin{aligned}
\lim_{n\to\infty} E_i(P^n, e^n) &= \lim_{n\to\infty} \left(e_i^n + \sum_{k\in N} PR_i^k \left(N \setminus \{k\}, \bar{P}_k^n, (L_{kj}^n)_{j\in N\setminus\{k\}}\right) \right. \\
&\qquad \left. - \sum_{k\in N} PR_k^i \left(N \setminus \{i\}, \bar{P}_i^n, (L_{ij}^n)_{j\in N\setminus\{i\}}\right) \right) \\
&\underset{\text{CONT of } PR}{=} e_i + \sum_{k\in N} PR_i^k \left(N \setminus \{k\}, \bar{P}_k^*, (L_{kj})_{j\in N\setminus\{k\}}\right) \\
&\qquad - \sum_{k\in N} PR_k^i \left(N \setminus \{i\}, \bar{P}_i^*, (L_{ij})_{j\in N\setminus\{i\}}\right) \\
&= E_i(P^*, e) \\
&\underset{(10)}{=} E_i(P, e),
\end{aligned}$$

which concludes the proof. □

Proof. (Proposition 2)

Let σ be a financial rule satisfying **CB**, **LL**, **AP**, and **I**. Let $(N, L, e) \in \mathcal{F}$, $P \in \sigma(N, L, e)$, and $i, j \in N$ such that $e_i = e_j$, $L_{ij} = L_{ji}$, $L_{ik} = L_{jk}$, and $L_{ki} = L_{kj}$ for all $k \in N \setminus \{i, j\}$. For all $k \in N \setminus \{ij\}$, since $L_{ki} = L_{kj}$, by **I** we have that

$$P_{ki} = P_{kj}. \tag{11}$$

By **LL**, $E_i(P, e) \ge 0$ and $E_j(P, e) \ge 0$. If $E_i(P, e) = E_j(P, e)$ we are done. If not, it is sufficient to consider two cases: **(a)** $E_i(P, e) > 0$ and $E_j(P, e) > 0$; **(b)** $E_i(P, e) > 0$ and $E_j(P, e) = 0$.

In case **(a)**, by **AP** and **CB**, $P_{ik} = L_{ik}$ and $P_{jk} = L_{jk}$ for all $k \in N$. In particular, $P_{ij} = L_{ij} = L_{ji} = P_{ji}$. Hence, since $L_{ik} = L_{jk}$ for all $k \in N \setminus \{ij\}$, we have $\bar{P}_i = \bar{L}_i = \bar{L}_j = \bar{P}_j$. Moreover, by (11) and $P_{ij} = P_{ji}$, we obtain

$$E_i(P, e) = e_i + \sum_{k\in N\setminus\{i\}} P_{ki} - \bar{L}_i = e_j + \sum_{k\in N\setminus\{j\}} P_{kj} - \bar{L}_j = E_j(P, e).$$

[7] In real analysis, this result states that every bounded sequence in the finite-dimensional Euclidean space $\mathbb{R}^n$ has a convergent subsequence.

In case **(b)**, by **AP** and **CB**, $P_{ik} = L_{ik}$ for all $k \in N$ and thus

$$\bar{P}_i = \bar{L}_i \tag{12}$$

Then,

$$\begin{aligned}
E_j(P,e) &= e_j + \sum_{k \in N \setminus \{i,j\}} P_{kj} + P_{ij} - \bar{P}_j \\
&= e_j + \sum_{k \in N \setminus \{i,j\}} P_{kj} + L_{ij} - \bar{P}_j \\
&\underset{(11)}{=} e_i + \sum_{k \in N \setminus \{i,j\}} P_{ki} + L_{ji} - \bar{P}_j \\
&\underset{\mathbf{CB}}{\geq} e_i + \sum_{k \in N} P_{ki} - \bar{L}_j \\
&= e_i + \sum_{k \in N} P_{ki} - \bar{L}_i \\
&\underset{(12)}{=} e_i + \sum_{k \in N} P_{ki} - \bar{P}_i \\
&= E_i(P,e),
\end{aligned}$$

in contradiction with $E_i(P,e) > E_j(P,e) = 0$.

Hence, in both cases, $E_i(P,e) = E_j(P,e)$, which implies **ETE**. □

Proof. (Proposition 3)

First, we show that [1] $\implies$ [2]. Let σ be a financial rule compatible with $\beta = (\beta^i)_{i \in \mathbb{N}}$ fulfilling **LL** and **AP**. Then, from CB of all β^i, σ satisfies **CB**. Hence, by Proposition 2, if σ satisfies **I** then also **ETE**.

Secondly, we show that [2] $\implies$ [3]. Suppose that σ satisfies **ETE**, we prove that each bankruptcy rule in β fulfills ETE. Indeed, select an arbitrary $i \in \mathbb{N}$ and let $(N,E,c) \in \mathcal{B}$ with $i \in \mathbb{N} \setminus N$ and $j,k \in N$ such that $c_j = c_k$. Define the associated financial system $(\bar{N},L,e) \in \mathcal{F}$, as in Subsection 2.3, being $\bar{N} = N \cup \{i\}$; $L_{lh} = 0$ for all $l,h \in N$, $L_{il} = c_l$ and $L_{li} = 0$ for all $l \in N$; $e_i = E$ and $e_l = 0$ for all $l \in N$. Next, we see that $\sigma(\bar{N},L,e)$ selects a unique payment matrix P. As σ satisfies **CB** (received from CB of all β^i), $P_{lh} = 0$ for all $l,h \in N$ and $P_{li} = 0$ for all $l \in N$. Moreover, since σ is compatible with β, $P_{il} = \beta^i_l(N,E,c)$ for all $l \in N$, which is unique by definition of β^i. Thus, $E_j(P,e) = e_j + \sum_{l \in \bar{N}} P_{lj} - \sum_{l \in \bar{N}} P_{jl} = P_{ij} = \beta^i_j(N,E,c)$ and, analogously, $E_k(P,e) = P_{ik} = \beta^i_k(N,E,c)$. To finish, observe that since $e_j = e_k = 0$, $L_{jl} = L_{kl} = 0$ for all $l \in \bar{N}$, $L_{lj} = L_{lk} = 0$ for all $l \in N$ and $L_{ij} = c_j = c_k = L_{ik}$, players j and k are symmetric in $(\bar{N},L,e)$ and then, by **ETE**, $\beta^i_j(N,E,c) = E_j(P,e) = E_k(P,e) = \beta^i_k(N,E,c)$, which proves ETE of β^i.

Finally, we show that [3] $\implies$ [1]. Let (N,L,e) be a financial system with $i,j,k \in N$ such that $L_{ij} = L_{ik}$, and let $P \in \sigma(N,L,e)$. Then, as σ is compatible with β, by ETE of β^i it holds that $P_{ij} = \beta^i_j(N \setminus \{i\}, \bar{P}_i, (L_{il})_{l \in N \setminus \{i\}}) = \beta^i_k(N \setminus \{i\}, \bar{P}_i, (L_{il})_{l \in N \setminus \{i\}}) = P_{ik}$, which shows that σ satisfies **I**. □

Proof. (Proposition 4)

Let σ be a financial rule satisfying **IM**. Let $(N,L,e),(N',L',e')$ be two related financial systems as described in (6). Let $P \in \sigma(N,L,e)$ then, by **IM**, there exist $P' \in \sigma(N',L',e')$ satisfying the conditions in (7). From the relation between P and P', for all $i \in N' \setminus \{m\}$, it follows that

$$\begin{array}{rcl} E_i(P',e') & = & e'_i + \sum_{k\in N'} P'_{ki} - \sum_{k\in N'} P'_{ik} \\ & = & e'_i + \sum_{k\in N'\setminus\{m\}} P'_{ki} + P'_{mi} - \sum_{k\in N'\setminus\{m\}} P'_{ik} - P'_{im} \\ & = & e_i + \sum_{k\in N'\setminus\{m\}} P_{ki} + P_{mi} + \sum_{k\in N\setminus N'} P_{ki} - \sum_{k\in N'\setminus\{m\}} P_{ik} - P_{im} - \sum_{k\in N\setminus N'} P_{ik} \\ & = & e_i + \sum_{k\in N} P_{ki} - \sum_{k\in N} P_{ik} \\ & = & E_i(P,e). \end{array}$$

Following parallel arguments, by **IM**, for all $P' \in \sigma(N', L', e')$ there exist $P \in \sigma(N, L, e)$ satisfying the conditions in (7) and, consequently, for all $i \in N' \setminus \{m\}$ we also obtain $E_i(P', e') = E_i(P, e)$. Thus, σ satisfies **NMC**. □

Proof. (Lemma 4)

Let σ be a proportional financial rule and $\varepsilon = (N, L, e)$, $\varepsilon' = (N', L', e')$ two financial systems as described in (6).

First, we prove item (a). Let $P \in \sigma(\varepsilon)$ and define $P' \in \mathcal{M}(N')$ as in (7) from P. Observe that P' is well defined and unique. We are going to prove that $P' \in \sigma^{\mathbb{PR}}_{max}(\varepsilon')$. Suppose, w.l.o.g., $\bar{L'}_i \neq 0$ for all $i \in N'$. Then, for all $i, j \in N'$, we claim that

$$P'_{ij} = \frac{L'_{ij}}{\bar{L'}_i}\bar{P'}_i. \tag{13}$$

First, let us note that if $i \in N' \setminus \{m\}$, then

$$\begin{array}{rcl} \bar{P'}_i & = & \sum_{j\in N'} P'_{ij} \\ & = & \sum_{j\in N'\setminus\{m\}} P'_{ij} + P'_{im} \\ & = & \sum_{j\in N'\setminus\{m\}} P_{ij} + P_{im} + \sum_{k\in N\setminus N'} P_{ik} \\ & = & \bar{P}_i. \end{array} \tag{14}$$

Otherwise, if $i = m$ we have that

$$\begin{array}{rcl} \bar{P'}_m & = & \sum_{j\in N'} P'_{mj} \\ & = & \sum_{j\in N'} \left(P_{mj} + \sum_{k\in N\setminus N'} P_{kj}\right) \\ & \underset{P_{kj}=P_{mj}}{=} & \sum_{j\in N'} \left(P_{mj} + \sum_{k\in N\setminus N'} P_{mj}\right) \\ & = & \sum_{j\in N'} (P_{mj} + (|N\setminus N'|)P_{mj}) \\ & = & (|N\setminus N'|+1)\sum_{j\in N'} P_{mj} \\ & = & (|N\setminus N'|+1)\bar{P}_m, \end{array} \tag{15}$$

where the last equality comes from the fact that $P_{mj} = 0$ for all $j \in N \setminus N'$.

Now, to prove (13) we distinguish three cases:

Case 1: $i, j \in N' \setminus \{m\}$. In this situation,

$$P'_{ij} = P_{ij} = \frac{L_{ij}}{\bar{L}_i}\bar{P}_i \underset{(14)}{=} \frac{L'_{ij}}{\bar{L'}_i}\bar{P'}_i.$$

Case 2: $i \in N' \setminus \{m\}$ and $j = m$. In this situation,

$$\begin{array}{rcl} P'_{im} & = & P_{im} + \sum_{k\in N\setminus N'} P_{ik} \\ & = & \left(L_{im} + \sum_{k\in N\setminus N'} L_{ik}\right) \frac{\bar{P}_i}{\bar{L}_i} \\ & = & L'_{im} \frac{\bar{P}_i}{\bar{L'}_i} \\ & \underset{(14)}{=} & \frac{L'_{im}}{\bar{L'}_i} \bar{P'}_i. \end{array}$$

Case 3: $i = m$ and $j \in N' \setminus \{m\}$. In this situation,

$$\begin{array}{rcl} P'_{mj} & = & P_{mj} + \sum_{k\in N\setminus N'} P_{kj} \\ & \underset{P_{kj}=P_{mj}}{=} & P_{mj} + \sum_{k\in N\setminus N'} P_{mj} \\ & = & \frac{L_{mj}}{\bar{L}_m} \bar{P}_m(|N\setminus N'|+1) \\ & \underset{(15)}{=} & \dfrac{L_{mj}}{\bar{L}_m} \bar{P'}_m \\ & = & \dfrac{L'_{mj}/(|N\setminus N'|+1)}{(\sum_{k\in N'\setminus\{m\}} L'_{mk})/(|N\setminus N'|+1)} \bar{P'}_m \\ & = & \dfrac{L'_{mj}}{\bar{L'}_m} \bar{P'}_m. \end{array}$$

Thus, (13) holds.

Next, we show that $\bar{P'} \in FIX(\Phi^{\varepsilon',\mathbb{PR}})$. Indeed, if $i \in N' \setminus \{m\}$, then

$$\begin{array}{rcl} \bar{P}'_i & = & \bar{P}_i \\ & = & \min\left\{ e_i + \displaystyle\sum_{k\in N'\setminus\{m\}} P_{ki} + P_{mi} + \sum_{k\in N\setminus N'} P_{ki}, \bar{L}_i \right\} \\ & = & \min\left\{ e'_i + \displaystyle\sum_{k\in N'\setminus\{m\}} P'_{ki} + P'_{mi}, \bar{L'}_i \right\} \\ & = & \min\left\{ e'_i + \displaystyle\sum_{k\in N'} P'_{ki}, \bar{L'}_i \right\} \\ & \underset{(13)}{=} & \min\left\{ e'_i + \displaystyle\sum_{k\in N'} \frac{L'_{ki}}{\bar{L'}_k} \bar{P'}_k, \bar{L'}_i \right\} \\ & = & \min\left\{ e'_i + \displaystyle\sum_{k\in N'} PR^k_i \left(N'\setminus\{k\}, \bar{P'}_k, (L'_{kj})_{j\in N'\setminus\{k\}}\right), \bar{L'}_i \right\}. \end{array}$$

In a similar way, and taking into account that $\bar{P'}_m = (|N \setminus N'| + 1)\bar{P}_m$, we obtain

$$\bar{P'}_m = \min\left\{ e'_m + \sum_{k\in N'} PR^k_m \left(N'\setminus\{k\}, \bar{P'}_k, (L'_{kj})_{j\in N'\setminus\{k\}}\right), \bar{L'}_m \right\}.$$

Hence, $\bar{P'} \in FIX(\Phi^{\varepsilon',\mathbb{PR}})$. Moreover, since σ satisfies **CB**, **LL**, and **AP**, for all $P'' \in \sigma(\varepsilon')$ it holds that $\bar{P''} \in FIX(\Phi^{\varepsilon',\mathbb{PR}})$. Finally, making use of Lemma 2, we have that $E(P', e') = E(P'', e')$. But then, for all $i \in N' \setminus \{m\}$, we obtain

$$\begin{aligned}
E_i(P'', e') &= E_i(P', e') \\
&= e'_i + \sum_{k \in N'} P'_{ki} - \sum_{k \in N'} P'_{ik} \\
&= e_i + \sum_{k \in N' \setminus \{m\}} P'_{ki} + P'_{mi} - \sum_{k \in N' \setminus \{m\}} P'_{ik} - P'_{im} \\
&= e_i + \sum_{k \in N' \setminus \{m\}} P_{ki} + P_{mi} + \sum_{k \in N \setminus N'} P_{ki} - \sum_{k \in N' \setminus \{m\}} P_{ik} - P_{im} - \sum_{k \in N \setminus N'} P_{ik} \\
&= e_i + \sum_{k \in N} P_{ki} - \sum_{k \in N} P_{ik} \\
&= E_i(P, e).
\end{aligned}$$

To conclude, we prove item (b). Let $P' \in \sigma(\varepsilon')$ and define $P \in \mathcal{M}(N)$ as follows: for all $i, j \in N' \setminus \{m\}, P_{ij} = P'_{ij}$; for all $i \in N' \setminus \{m\}$ and all $j \in N \setminus N' \cup \{m\}, P_{ij} = P'_{im}/(|N \setminus N'| + 1)$ and $P_{ji} = P'_{mi}/(|N \setminus N'| + 1)$; and for all $i, j \in N \setminus N' \cup \{m\}, P_{ij} = 0$. Note that $\bar{P}_i = \bar{P}'_i$ for all $i \in N' \setminus \{m\}$; $\bar{P}_j = \bar{P}'_m/(|N \setminus N'| + 1)$ for all $j \in N \setminus N' \cup \{m\}$. From this point, the same arguments as before lead to $\bar{P} \in FIX(\Phi^{\varepsilon, \mathbb{PR}})$ and that $E_i(P, e) = E_i(P', e')$ for all $i \in N' \setminus \{m\}$. This concludes the proof. $\square$

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