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The uncertain ordered weighted averaging adequacy coefficient operator

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ABSTRACT

This article introduces the uncertain ordered weighted averaging adequacy coefficient (UOWAAC) operator. This novel operator uses the ordered weighted averaging (OWA) operator, the adequacy coefficient, and the interval numbers in a single formulation. This article also extends the UOWAAC operator by using order-inducing variables in the reordering process of the input arguments. This new extension is called the uncertain induced ordered weighted averaging adequacy coefficient (UIOWAAC) operator. The article also presents an application of the new approach in a multi-criteria group decision making (MCGDM) problem about international expansion. In addition, a comparative analysis is conducted with the purpose of demonstrating the superiority of the UOWAAC and UIOWAAC aggregation operators in specific situations. Likewise, the use of basic uncertain information (BUI) is discussed. The results show the usefulness of these new aggregation operators in real-life decision making problems under uncertainty, particularly when the decision maker wants to compare different alternatives with an ideal but without giving any penalty or reward in the case that the ideal levels are exceeded.

1. Introduction

Aggregation operators (also referred as aggregation functions) are commonly used in decision making procedures in order to combine several sources of information into a single result. An increasingly popular aggregation operator [1] is the ordered weighted averaging (OWA), which was first presented by Yager [2]. This operator provides a parameterized family of aggregation operators between the minimum and the maximum. Since its introduction, several applications have been studied, including sales forecasting [3,4], portfolio selection [5], retirement planning [6], government transparency [7], agricultural product prices [8], and many others [9].

In the literature we can find a wide range of aggregation operators that extend the OWA operator. Some of the most important are the induced OWA (IOWA) [10] and the IOWA in the expression of weighted averaging (WA) functions [11], the generalized OWA (GOWA) [12], the quasi OWA (QOWA) [13], the probabilistic OWA (POWA) [14], the linguistic OWA (LOWA) [15–17], the fuzzy OWA (FOWA) [18], the uncertain OWA (UOWA) [19], and the OWA distance (OWAD) [20]. Also,

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recently, Jin, Mesiar, and Yager [21] proposed an OWA weight allocation method to deal with convex partially ordered sets (posets).

Furthermore, another extension that received much attention is the OWA adequacy coefficient (OWAAC) operator [20,22], which as its name indicates, uses the OWA operator with the adequacy coefficient [23,24] in a single formulation. This operator is mainly used to compare an ideal set with a real one, but in contrast to other operators, such as the OWAD, it does not penalize the result when the ideal levels are exceeded. However, this aggregation operator only considers exact numbers, which is not always possible, especially when the environment is highly uncertain and complex. In this case, an adequate alternative may be the use of interval numbers [25].

Thus, the aim of this article is to develop a new extension of the OWAAC operator for situations with a high degree of uncertainty. To do so, interval numbers are used instead of exact numbers. It is called the uncertain ordered weighted averaging adequacy coefficient (UOWAAC) operator. Moreover, the UOWAAC operator is extended by using order-inducing variables. As a result, the uncertain induced ordered weighted averaging adequacy coefficient (UIOWAAC) operator is obtained. Lastly, another objective of this article is to demonstrate the utility of these new aggregation operators in real-world situations. To achieve this, the applicability of these novel operators in a multi-criteria group decision making (MCGDM) problem regarding the international expansion of a business is studied.

This article is arranged as follows. Section 2 conducts a review of the OWA operator, the interval numbers, the UOWA operator, the adequacy coefficient, and the OWAAC operator. Section 3 presents the UOWAAC operator, analyze its properties, and discuss its families. Section 4 extends the UOWAAC operator through order-inducing variables. Section 5 provides an illustrative example of the new approach in order to demonstrate its practicability. Section 6 presents a comparison of the developed aggregation operators with the existing ones. Section 7 summarizes the main conclusions of the article and indicates opportunities for future research.

2. Some preliminary concepts

The following section briefly reviews some basic but necessary concepts, which are the OWA operator, the interval numbers, the UOWA operator, the adequacy coefficient, and the OWAAC operator.

2.1. The OWA operator

The OWA operator is an aggregation operator introduced by Yager [2] and it provides a parameterized family of aggregation operators that include among others the minimum, the maximum, and the arithmetic mean decision criteria. A fundamental characteristic of this operator is found in the reordering step of the input arguments in which it is carried out in a descending way. This operator can be defined as follows.

Definition 1. An OWA operator of dimension n is a mapping $\mathrm{OWA}: R^n \rightarrow R$ that has associated a weighting vector $W = (w_1, \ldots, w_n)$, with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\mathrm{OWA}(a_1, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j, \tag{1}$$

where b_j is the jth largest element of the arguments $a_1, \ldots, a_n$.

Additionally, the OWA operator is commutative, monotonic, bounded, and idempotent.

2.2. Interval numbers

In decision making, uncertainty is often an unavoidable problem. A practical way for handling uncertainty is through the use of interval numbers [25]. An interval number can be described as an ordered pair of real numbers or also as a set of the real line R. Mathematically it is defined as follows.

Definition 2. Let $\tilde{a}$ be an interval number. Then $\tilde{a} = [a^L, a^U]$ with $a^L, a^U \in R$ and $a^L \leq a^U$. In the particular case $a^L = a^U$, one can see that $\tilde{a}$ is reduced to a real number, which is known as degenerate interval number.

2.3. The UOWA operator

The UOWA operator was developed by Xu and Da [19] and it is an extension of the OWA operator for uncertain environments where the available information can only be assessed with the use of interval numbers. This aggregation function can be defined as follows.

Definition 3. Let Ω be a set of interval numbers. An UOWA operator of dimension n is a mapping UOWA : $\Omega^n \rightarrow \Omega$ that has associated a weighting vector $W = (w_1, \ldots, w_n)$ with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{UOWA}(\tilde{a}_1, \ldots, \tilde{a}_n) = \sum_{j=1}^{n} w_j \tilde{b}_j, \quad (2)$$

where $\tilde{b}_j$ is the jth largest of the $\tilde{a}_i$, and $\tilde{a}_i$ is the argument variable represented in the form of interval numbers.

The UOWA operator satisfies the mathematical properties of commutativity, monotonicity, boundedness, and idempotency.

2.4. The adequacy coefficient

The adequacy coefficient [23,24] is an index used for calculating the differences between two real numbers in a more effective manner. It can be defined as follows.

Definition 4. Let x and y be two real numbers such that $x, y \in [0, 1]$. Then, the adequacy coefficient between x and y is obtained by applying the following formula:

$$\text{AC}(x, y) = \left[1 \wedge (1 - x + y)\right]. \quad (3)$$

Note that the symbol $\wedge$ is used to indicate the lower value between 1 and $(1 - x + y)$.

Also, it is noteworthy that the adequacy coefficient is similar to the Hamming distance [26] but with the difference that it neutralizes the result when $x < y$.

2.5. The OWAAC operator

In [20,22], the authors presented the OWAAC operator, which uses the adequacy coefficient and the OWA operator in the same formulation. This operator is used for complex comparisons between two sets, normally between an ideal set (X) and a real one (Y).

For two sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$, the OWAAC operator is defined as follows.

Definition 5. An OWAAC operator of dimension n is a mapping OWAAC : $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ that has associated a weighting vector $W = (w_1, \ldots, w_n)$, with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{OWAAC}\big(\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle\big) = \sum_{j=1}^{n} w_j K_j, \quad (4)$$

where K_j is the jth largest of the $[1 \wedge (1 - x_i + y_i)]$, and x_i and y_i are the ith arguments of the sets X and Y.

The OWAAC operator is commutative, monotonic, bounded, idempotent, non-negative and reflexive.

Remark. Some readers may think that an ideal set should always have the highest level in all characteristics. But in the real world, this is not true, in some situations, if the ideal set is fixed at the maximum, when performing the comparison, the decision maker could erroneously exclude the most convenient option. One reason is that a high value in a characteristic may compensate for a low value in another characteristic and consequently obtain an inconsistent result.

For example, imagine that a Spanish cellar wants to offer guided tours for international tourists and therefore needs to hire a professional who can speak English (C_1) and German (C_2), and also who has a general knowledge of wine culture (C_3). Suppose that there are two candidates, Y_1 and Y_2. Assume that the candidates are evaluated for each competence as follows: $Y_1 = \{C_1 = 1, C_2 = 0.3, C_3 = 0.8\}$ and $Y_2 = \{C_1 = 0.7, C_2 = 0.6, C_3 = 0.7\}$, where 0 is the worst evaluation and 1 the best.

If the cellar assumes that the ideal candidate should have the maximum level in all three competences, i.e., $X = \{C_1 = 1, C_2 = 1, C_3 = 1\}$, then, using the OWAAC operator with $W = \{w_1 = 1/3, w_2 = 1/3, w_3 = 1/3\}$, the best candidate would be Y_1.

By contrast, if the cellar establishes the optimal levels of the ideal worker as $X = \{C_1 = 0.6, C_2 = 0.6, C_3 = 0.6\}$, then, the hired worker would be Y_2, which makes more sense. Candidate Y_2 may not be perfect in a specific skill however meets all the requirements. On the other hand, candidate Y_1 is excellent in English but at the same time incapable to do a proper tour in German. Thus, if the hired candidate is Y_1, the cellar probably will have to seek for an additional employee.

3. The UOWAAC operator

The following section first defines the UOWAAC operator, then, analyzes its properties, and lastly, studies its different families.

3.1. Definition of the UOWAAC operator

The UOWAAC operator can be described as an extension of the OWAAC operator that uses interval numbers instead of exact numbers. This new operator is very complete as it offers numerous advantages over traditional aggregation operators. For example, by using interval numbers the decision maker is able to deal with uncertainty. Likewise, as it is built under the OWA operator, it allows to consider the attitudinal character of the decision maker when the information is fused. Furthermore, through the adequacy coefficient, this operator can be used for comparing a set of available alternatives with an ideal, while at the same time establish a threshold from which the results are always the same. Hence, it only penalizes when the optimal levels are not reached.

Definition 6. Let $\tilde{x}$ and $\tilde{y}$ be two interval numbers. An uncertain adequacy coefficient (UAC) is a similarity measure, such that:

$$\text{UAC}(\tilde{x}, \tilde{y}) = \frac{1}{2}\left(\left[1 \wedge \left(1 - x^L + y^L\right)\right] + \left[1 \wedge \left(1 - x^U + y^U\right)\right]\right), \tag{5}$$

where $x^L, x^U \in [0, 1]$ are the lower and upper values of the interval number $\tilde{x}$, and $y^L, y^U \in [0, 1]$ are the lower and upper values of the interval number $\tilde{y}$.

Let $\tilde{X} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ and $\tilde{Y} = (\tilde{y}_1, \ldots, \tilde{y}_n)$ be two sets of interval numbers, then, the uncertain weighted adequacy coefficient (UWAC) operator and the UOWAAC operators can be defined respectively as follows.

Definition 7. Let Ω be a set of interval numbers. An UWAC operator of dimension n is a mapping $\text{UWAC}: \Omega^n \times \Omega^n \to R$ that has associated a weighting vector $W = (w_1, \ldots, w_n)$ with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, in which:

$$\text{UWAC}\left(\langle \tilde{x}_1, \tilde{y}_1 \rangle, \ldots, \langle \tilde{x}_n, \tilde{y}_n \rangle\right) = \sum_{i=1}^{n} w_i \text{UAC}(\tilde{x}_i, \tilde{y}_i), \tag{6}$$

where $\tilde{x}_i$ and $\tilde{y}_i$ are the ith arguments of the sets $\tilde{X}$ and $\tilde{Y}$, and $\text{UAC}(\tilde{x}_i, \tilde{y}_i)$ is the adequacy coefficient between $\tilde{x}_i = [x_i^L, x_i^U]$ and $\tilde{y}_i = [y_i^L, y_i^U]$, with $0 \le x_i^L \le x_i^U \le 1$ and $0 \le y_i^L \le y_i^U \le 1$.

Definition 8. Let Ω be a set of interval numbers. An UOWAAC operator of dimension n is a mapping $\text{UOWAAC}: \Omega^n \times \Omega^n \to R$ that has associated a weighting vector $W = (w_1, \ldots, w_n)$ with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{UOWAAC}\left(\langle \tilde{x}_1, \tilde{y}_1 \rangle, \ldots, \langle \tilde{x}_n, \tilde{y}_n \rangle\right) = \sum_{j=1}^{n} w_j \text{UAC}(\tilde{x}_j, \tilde{y}_j), \tag{7}$$

where $\text{UAC}(\tilde{x}_j, \tilde{y}_j)$ is the jth largest $\text{UAC}(\tilde{x}_i, \tilde{y}_i)$ value of the UOWAAC pair $\langle \tilde{x}_i, \tilde{y}_i \rangle$, and $\text{UAC}(\tilde{x}_i, \tilde{y}_i)$ is the adequacy coefficient between $\tilde{x}_i = [x_i^L, x_i^U]$ and $\tilde{y}_i = [y_i^L, y_i^U]$, with $0 \le x_i^L \le x_i^U \le 1$ and $0 \le y_i^L \le y_i^U \le 1$.

Note that for the previous definitions $\tilde{X}$ represents the ideal set in the comparison, thus, a higher UOWAAC value is preferred.

Next, a simple example will be carried out in order to correctly understand how to calculate the UOWAAC operator.

Example 1. Assume two sets of interval numbers $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) = ([0.7, 0.8], [0.6, 0.7], [0.8, 0.9], [0.5, 0.7])$ and $\tilde{Y} = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4) = ([0.4, 0.6], [0.5, 0.9], [0.7, 0.9], [0.8, 0.9])$. If the weighting vector is $W = (w_1, w_2, w_3, w_4) = (0.5, 0.3, 0.1, 0.1)$, then, the UOWAAC operator is obtained as follows.

First, it is necessary to calculate the UAC for each pair of interval numbers using Eq. (5):

$$\text{UAC}(\tilde{x}_1, \tilde{y}_1) = \frac{1}{2}\left(\left[1 \wedge (1 - 0.7 + 0.4)\right] + \left[1 \wedge (1 - 0.8 + 0.6)\right]\right) = 0.75.$$

Following the same procedure, the remaining outcomes are achieved:

$$\text{UAC}(\tilde{x}_2, \tilde{y}_2) = 0.95, \quad \text{UAC}(\tilde{x}_3, \tilde{y}_3) = 0.95, \quad \text{and} \quad \text{UAC}(\tilde{x}_4, \tilde{y}_4) = 1.$$

Next, with Eq. (7) the aggregation is performed:

$$\text{UOWAAC}(\tilde{X}, \tilde{Y}) = 0.5 \times 1 + 0.3 \times 0.95 + 0.1 \times 0.95 + 0.1 \times 0.75 = 0.955.$$

From a generalized perspective of the reordering step, it is possible to discriminate between the descending UOWAAC (UDOWAAC) operator and the ascending UOWAAC (UAOWAAC) operator. Specifically, the weights of both operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the UDOWAAC (or UOWAAC) operator and w^*_{n-j+1} the jth weight of the UAOWAAC operator.

3.2. Properties of the UOWAAC operator

The UOWAAC operator is commutative, monotonic, bounded, idempotent, non-negative, and reflexive. These properties can be proven with the following theorems:

Theorem 1 *(Commutativity). Assume f is the UOWAAC operator. Then:*

$$f\left(\langle\tilde{x}_1, \tilde{y}_1\rangle, \ldots, \langle\tilde{x}_n, \tilde{y}_n\rangle\right) = f\left(\langle\tilde{x}^*_1, \tilde{y}^*_1\rangle, \ldots, \langle\tilde{x}^*_n, \tilde{y}^*_n\rangle\right), \tag{8}$$

where $(\langle\tilde{x}^*_1, \tilde{y}^*_1\rangle, \ldots, \langle\tilde{x}^*_n, \tilde{y}^*_n\rangle)$ is any permutation of $(\langle\tilde{x}_1, \tilde{y}_1\rangle, \ldots, \langle\tilde{x}_n, \tilde{y}_n\rangle)$.

Proof. Let

$$f\left(\langle\tilde{x}_1, \tilde{y}_1\rangle, \ldots, \langle\tilde{x}_n, \tilde{y}_n\rangle\right) = \sum_{j=1}^{n} w_j \text{UAC}(\tilde{x}_j, \tilde{y}_j),$$

$$f\left(\langle\tilde{x}^*_1, \tilde{y}^*_1\rangle, \ldots, \langle\tilde{x}^*_n, \tilde{y}^*_n\rangle\right) = \sum_{j=1}^{n} w_j \text{UAC}\left(\tilde{x}^*_j, \tilde{y}^*_j\right).$$

As $(\langle\tilde{x}^*_1, \tilde{y}^*_1\rangle, \ldots, \langle\tilde{x}^*_n, \tilde{y}^*_n\rangle)$ is any permutation of $(\langle\tilde{x}_1, \tilde{y}_1\rangle, \ldots, \langle\tilde{x}_n, \tilde{y}_n\rangle)$, we have $\text{UAC}(\tilde{x}_j, \tilde{y}_j) = \text{UAC}(\tilde{x}^*_j, \tilde{y}^*_j)$, for all j, and as a result:

$$f\left(\langle\tilde{x}_1, \tilde{y}_1\rangle, \ldots, \langle\tilde{x}_n, \tilde{y}_n\rangle\right) = f\left(\langle\tilde{x}^*_1, \tilde{y}^*_1\rangle, \ldots, \langle\tilde{x}^*_n, \tilde{y}^*_n\rangle\right). \quad \square$$

Theorem 2 *(Monotonicity). Assume f is the UOWAAC operator. If $\text{UAC}(\tilde{x}_i, \tilde{y}_i) \geq \text{UAC}(\tilde{x}^*_i, \tilde{y}^*_i)$, for all i, then:*

$$f\left(\langle\tilde{x}_1, \tilde{y}_1\rangle, \ldots, \langle\tilde{x}_n, \tilde{y}_n\rangle\right) \geq f\left(\langle\tilde{x}^*_1, \tilde{y}^*_1\rangle, \ldots, \langle\tilde{x}^*_n, \tilde{y}^*_n\rangle\right). \tag{9}$$

Proof. Let

$$f\left(\langle\tilde{x}_1, \tilde{y}_1\rangle, \ldots, \langle\tilde{x}_n, \tilde{y}_n\rangle\right) = \sum_{j=1}^{n} w_j \text{UAC}(\tilde{x}_j, \tilde{y}_j),$$

$$f\left(\langle\tilde{x}^*_1, \tilde{y}^*_1\rangle, \ldots, \langle\tilde{x}^*_n, \tilde{y}^*_n\rangle\right) = \sum_{j=1}^{n} w_j \text{UAC}\left(\tilde{x}^*_j, \tilde{y}^*_j\right).$$

Since $\text{UAC}(\tilde{x}_i, \tilde{y}_i) \geq \text{UAC}(\tilde{x}^*_i, \tilde{y}^*_i)$, for all i, it follows that $\text{UAC}(\tilde{x}_j, \tilde{y}_j) \geq \text{UAC}(\tilde{x}^*_j, \tilde{y}^*_j)$. Therefore:

$$f\left(\langle\tilde{x}_1, \tilde{y}_1\rangle, \ldots, \langle\tilde{x}_n, \tilde{y}_n\rangle\right) \geq f\left(\langle\tilde{x}^*_1, \tilde{y}^*_1\rangle, \ldots, \langle\tilde{x}^*_n, \tilde{y}^*_n\rangle\right). \quad \square$$

Theorem 3 *(Boundedness). Assume f is the UOWAAC operator. Then:*

$$\text{Min}\left\{\text{UAC}(\tilde{x}_i, \tilde{y}_i)\right\} \leq f\left(\langle\tilde{x}_1, \tilde{y}_1\rangle, \ldots, \langle\tilde{x}_n, \tilde{y}_n\rangle\right) \leq \text{Max}\left\{\text{UAC}(\tilde{x}_i, \tilde{y}_i)\right\}. \tag{10}$$

Proof. Consider $\text{Min}\{\text{UAC}(\tilde{x}_i, \tilde{y}_i)\} = z$ and $\text{Max}\{\text{UAC}(\tilde{x}_i, \tilde{y}_i)\} = g$. Subsequently:

$$f\left(\langle\tilde{x}_1, \tilde{y}_1\rangle, \ldots, \langle\tilde{x}_n, \tilde{y}_n\rangle\right) = \sum_{j=1}^{n} w_j \text{UAC}(\tilde{x}_j, \tilde{y}_j) \geq \sum_{j=1}^{n} w_j z = z \sum_{j=1}^{n} w_j,$$

$$f\left(\langle\tilde{x}_1, \tilde{y}_1\rangle, \ldots, \langle\tilde{x}_n, \tilde{y}_n\rangle\right) = \sum_{j=1}^{n} w_j \text{UAC}(\tilde{x}_j, \tilde{y}_j) \leq \sum_{j=1}^{n} w_j g = g \sum_{j=1}^{n} w_j.$$

As $\sum_{j=1}^{n} w_j = 1$, we get:

$$z \leq f\left(\langle \tilde{x}_1, \tilde{y}_1 \rangle, \ldots, \langle \tilde{x}_n, \tilde{y}_n \rangle\right) \leq g.$$

Consequently, we can confirm that the UOWAAC operator is bounded. □

Theorem 4 *(Idempotency). Assume f is the UOWAAC operator. If* $\mathrm{UAC}(\tilde{x}_i, \tilde{y}_i) = \mathrm{UAC}(\tilde{x}, \tilde{y})$*, for all i, then:*

$$f\left(\langle \tilde{x}_1, \tilde{y}_1 \rangle, \ldots, \langle \tilde{x}_n, \tilde{y}_n \rangle\right) = \mathrm{UAC}(\tilde{x}, \tilde{y}). \tag{11}$$

Proof. Since $\frac{1}{2}([1 \wedge (1 - x_i^L + y_i^L)] + [1 \wedge (1 - x_i^U + y_i^U)]) = \frac{1}{2}([1 \wedge (1 - x^L + y^L)] + [1 \wedge (1 - x^U + y^U)])$, for all i, we have:

$$\begin{aligned} f\left(\langle \tilde{x}_1, \tilde{y}_1 \rangle, \ldots, \langle \tilde{x}_n, \tilde{y}_n \rangle\right) &= \sum_{j=1}^{n} w_j \mathrm{UAC}(\tilde{x}_j, \tilde{y}_j) \\ &= \sum_{j=1}^{n} w_j \frac{1}{2}\left(\left[1 \wedge \left(1 - x^L + y^L\right)\right] + \left[1 \wedge \left(1 - x^U + y^U\right)\right]\right) \\ &= \frac{1}{2}\left(\left[1 \wedge \left(1 - x^L + y^L\right)\right] + \left[1 \wedge \left(1 - x^U + y^U\right)\right]\right) \sum_{j=1}^{n} w_j. \end{aligned}$$

Knowing that $\sum_{j=1}^{n} w_j = 1$, we obtain:

$$\begin{aligned} f\left(\langle \tilde{x}_1, \tilde{y}_1 \rangle, \ldots, \langle \tilde{x}_n, \tilde{y}_n \rangle\right) &= \frac{1}{2}\left(\left[1 \wedge \left(1 - x^L + y^L\right)\right] + \left[1 \wedge \left(1 - x^U + y^U\right)\right]\right) \\ &= \mathrm{UAC}(\tilde{x}, \tilde{y}). \quad \square \end{aligned}$$

Theorem 5 *(Nonnegativity). Assume f is the UOWAAC operator. Then:*

$$f\left(\langle \tilde{x}_1, \tilde{y}_1 \rangle, \ldots, \langle \tilde{x}_n, \tilde{y}_n \rangle\right) \geq 0. \tag{12}$$

Proof. Since $0 \leq x_i^L \leq x_i^U \leq 1$ and $0 \leq y_i^L \leq y_i^U \leq 1$, the aggregated value will be always positive. □

Theorem 6 *(Reflexivity). Assume f is the UOWAAC operator. Then:*

$$f\left(\langle \tilde{x}_1, \tilde{x}_1 \rangle, \ldots, \langle \tilde{x}_n, \tilde{x}_n \rangle\right) = 1. \tag{13}$$

Proof. Since $\tilde{x}_i = \tilde{x}_i$, for all i, we have:

$$\frac{1}{2}\left(\left[1 \wedge \left(1 - x_i^L + x_i^L\right)\right] + \left[1 \wedge \left(1 - x_i^U + x_i^U\right)\right]\right) = 1.$$

Thus, we can say that the UOWAAC operator is reflexive.

Additionally, an interesting issue to consider is the measures for characterizing the weighting vector W of the UOWAAC operator. In the following, the measures of attitudinal character [2], entropy of dispersion [2], balance operator [27], and divergence of W [28] are shortly analyzed.

The first measure is the attitudinal character or degree of orness, and it can be defined as follows:

$$\alpha(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right). \tag{14}$$

The second measure is the entropy of dispersion and it shows the amount of information being used. It can be defined as follows:

$$\mathrm{H}(W) = -\sum_{j=1}^{n} w_j \ln(w_j). \tag{15}$$

The balance operator is another interesting measure which evaluates the tendency to the minimum or to the maximum. It can be defined as follows:

$$\mathrm{Bal}(W)=\sum_{j=1}^{n} w_j\left(\frac{n+1-2j}{n-1}\right). \tag{16}$$

Lastly, the divergence of W is a measure that quantifies the divergence of the weights against the attitudinal character measure. It can be defined as follows:

$$\mathrm{Div}(W)=\sum_{j=1}^{n} w_j\left(\frac{n-j}{n-1}-\alpha(W)\right)^2. \quad \square \tag{17}$$

3.3. Families of the UOWAAC operator

One appealing aspect of the UOWAAC operator is that it includes a wide range of particular cases, which can be found by using different manifestations of the weighting vector W. Some interesting particular cases of this operator are the following:

- The minimum UAC (Min-UAC) is found when $w_n=1$ and $w_j=0$, for all $j\neq n$. It corresponds to the pessimistic criteria.
- The maximum UAC (Max-UAC) is found when $w_1=1$ and $w_j=0$, for all $j\neq 1$. It corresponds to the optimistic criteria.
- The median UAC (Med-UAC). If n is an odd number, then, the Med-UAC is obtained when $w_{(n+1)/2}=1$ and $w_j=0$, for all $j\neq(n+1)/2$. If n is an even number, then, the Med-UAC is obtained when $w_{n/2}=w_{(n/2)+1}=0.5$ and $w_j=0$, for all $j\neq n/2,(n/2)+1$.
- The normalized UAC (UNAC) is found when $w_j=1/n$, for all j. It corresponds to the Laplace criteria.
- The Hurwicz-UOWAAC is found when $w_1=\alpha$, $w_n=(1-\alpha)$, and $w_j=0$, for all $j\neq 1,n$.
- The Olympic-UOWAAC is found when $w_1=w_n=0$ and $w_j=1/(n-2)$, for all $j\neq 1,n$.
- The Step-UOWAAC is found when $w_k=1$ and $w_j=0$, for all $j\neq k$.

4. The UIOWAAC operator

The following section studies the UIOWAAC operator, which can be described as an extension of the UOWAAC operator that uses order-inducing variables in the reordering step of the argument variables. Thus, the reordering process does not depend on the values of the argument variables. This feature is very useful as it allows to represent more complex situations.

Let $\tilde{X}=(\tilde{x}_1,\dots,\tilde{x}_n)$ and $\tilde{Y}=(\tilde{y}_1,\dots,\tilde{y}_n)$ be two sets of interval numbers, then, the UIOWAAC operator can be defined as follows.

Definition 9. Let Ω be a set of interval numbers. An UIOWAAC operator of dimension n is a mapping UIOWAAC : $R^n\times\Omega^n\times\Omega^n\to R$ that has associated a weighting vector $W=(w_1,\dots,w_n)$ with $w_j\in[0,1]$ and $\sum_{j=1}^{n} w_j=1$, in which:

$$\mathrm{UIOWAAC}\big(\langle u_1,\tilde{x}_1,\tilde{y}_1\rangle,\dots,\langle u_n,\tilde{x}_n,\tilde{y}_n\rangle\big)=\sum_{j=1}^{n} w_j \mathrm{UAC}(\tilde{x}_j,\tilde{y}_j), \tag{18}$$

where $\mathrm{UAC}(\tilde{x}_j,\tilde{y}_j)$ is the $\mathrm{UAC}(\tilde{x}_i,\tilde{y}_i)$ value of the UIOWAAC triplet $\langle u_i,\tilde{x}_i,\tilde{y}_i\rangle$ having the jth largest u_i value, u_i is the order-inducing variable, and $\mathrm{UAC}(\tilde{x}_i,\tilde{y}_i)$ is the adequacy coefficient between $\tilde{x}_i=[x_i^L,x_i^U]$ and $\tilde{y}_i=[y_i^L,y_i^U]$, with $0\le x_i^L\le x_i^U\le 1$ and $0\le y_i^L\le y_i^U\le 1$.

Like the UOWAAC operator, the UIOWAAC operator is commutative, monotonic, bounded, idempotent, non-negative, and reflexive. The theorems and proofs of the mathematical properties of this operator are omitted as they are quite similar to the ones of the UOWAAC operator and thereby repetitive.

In order to understand numerically the UIOWAAC operator, a simple example is presented below.

Example 2. Assume two sets of interval numbers $\tilde{X}=(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4)=([0.7,0.8],[0.6,0.7],[0.8,0.9],[0.5,0.7])$ and $\tilde{Y}=(\tilde{y}_1,\tilde{y}_2,\tilde{y}_3,\tilde{y}_4)=([0.4,0.6],[0.5,0.9],[0.7,0.9],[0.8,0.9])$. If the weighting vector is $W=(w_1,w_2,w_3,w_4)=(0.5,0.3,0.1,0.1)$ and the inducing vector $U=(u_1,u_2,u_3,u_4)=(9,7,3,5)$, then, the UIOWAAC operator is obtained as follows.

As in Example 1, first of all the UAC for each pair of interval numbers needs to be calculated. By doing this, the following values are obtained:

$$\mathrm{UAC}(\tilde{x}_1,\tilde{y}_1)=0.75,\qquad \mathrm{UAC}(\tilde{x}_2,\tilde{y}_2)=0.95,\qquad \mathrm{UAC}(\tilde{x}_3,\tilde{y}_3)=0.95,\quad \text{and}\quad \mathrm{UAC}(\tilde{x}_4,\tilde{y}_4)=1.$$

Afterwards, with Eq. (18) the aggregation is carried out:

$$\mathrm{UIOWAAC}(\tilde{X},\tilde{Y})=0.5\times 0.75+0.3\times 0.95+0.1\times 1+0.1\times 0.95=0.855.$$

Table 1
Ideal country.

	C_1	C_2	C_3	C_4	C_5
I	[0.9,1]	[0.7,0.8]	[0.7,0.8]	[0.8,0.9]	[0.8,0.9]

Table 2
Assessments of Expert 1.

	C_1	C_2	C_3	C_4	C_5
A_1	[0.7,0.8]	[0.8,0.9]	[0.8,0.9]	[0.9,1]	[0.35,0.45]
A_2	[0.6,0.7]	[0.45,0.55]	[0.7,0.8]	[0.85,0.95]	[0.5,0.6]
A_3	[0.35,0.45]	[0.6,0.7]	[0.45,0.5]	[0.7,0.8]	[0.9,1]
A_4	[0.5,0.6]	[0.5,0.6]	[0.5,0.6]	[0.6,0.7]	[0.8,0.9]
A_5	[0.8,0.9]	[0.85,1]	[0.6,0.7]	[0.8,0.9]	[0.25,0.35]

Table 3
Assessments of Expert 2.

	C_1	C_2	C_3	C_4	C_5
A_1	[0.75,0.85]	[0.7,0.9]	[0.8,0.95]	[0.8,0.95]	[0.35,0.45]
A_2	[0.75,0.85]	[0.5,0.65]	[0.7,0.8]	[0.8,0.95]	[0.4,0.55]
A_3	[0.4,0.5]	[0.6,0.75]	[0.6,0.7]	[0.7,0.85]	[0.85,1]
A_4	[0.6,0.65]	[0.5,0.65]	[0.5,0.65]	[0.6,0.75]	[0.8,0.95]
A_5	[0.75,0.9]	[0.7,0.9]	[0.65,0.75]	[0.8,0.95]	[0.3,0.45]

5. Applications of the UOWAAC operator

The following section presents an illustrative example of the new approach in a MCGDM problem regarding the international expansion of a company. Nevertheless, other applications could be developed, for example in sport management, human resources, asset management, and many others.

Suppose that a company based in Germany and devoted to the production of car components wants to expand internationally in order to increase revenue potential. Therefore, three experts are requested by the company for choosing the most appropriate country to expand in among five options, which are:

- A_1: France.
- A_2: Italy.
- A_3: Portugal.
- A_4: Romania.
- A_5: Spain.

Also, the company considers five different characteristics as key for the assessments, which are:

- C_1: Customer base.
- C_2: Regulatory environment.
- C_3: Economic performance.
- C_4: Skilled labor force.
- C_5: Competitive landscape.

First, the interval numbers of the ideal set I are defined by the company as it is shown in Table 1. Then, each expert evaluates the characteristics of the candidate countries one by one and based on a scale from 0 to 1, where 0 is the worst result and 1 the best. The individual evaluations can be seen in Tables 2, 3, and 4.

In order to obtain a unified payoff matrix, the company aggregates the assessments conducted by the three experts. As the company assumes that the evaluations of each expert are not equally important, the uncertain weighted average (UWA) with $W = (0.4, 0.4, 0.2)$ is used to build this matrix. The results are presented in Table 5.

Next, in order to rank the countries according to the collective assessments, the company decides to use the UNAC, UWAC, UOWAAC, UAOWAAC, and UIOWAAC aggregation operators. To do so, the company has agreed to use the weighting vector $W = (0.4, 0.2, 0.2, 0.1, 0.1)$ and order-inducing vector $U = (10, 5, 6, 7, 8)$. The aggregated results are shown in Table 6. It should be taken into account that the preferred alternative will be the one with the highest value.

As we can see in Table 7, depending on the aggregation operator used, the order of preference may be different. For example, with the UNAC operator, the UOWAAC operator, and the UAOWAAC operator, the best country to expand in is France. However, with the UWAC operator and the UIOWAAC operator, the best country to expand in is Spain. This allows the decision-maker to get a more complete view of the problem and thereby make better decisions.

Table 4
Assessments of Expert 3.

	C_1	C_2	C_3	C_4	C_5
A_1	[0.75,0.9]	[0.7,0.8]	[0.85,0.9]	[0.8,1]	[0.25,0.4]
A_2	[0.65,0.8]	[0.65,0.7]	[0.7,0.8]	[0.7,0.85]	[0.3,0.45]
A_3	[0.3,0.4]	[0.65,0.75]	[0.6,0.7]	[0.75,0.9]	[0.75,0.95]
A_4	[0.45,0.65]	[0.6,0.7]	[0.6,0.7]	[0.7,0.8]	[0.7,0.85]
A_5	[0.8,1]	[0.7,0.85]	[0.7,0.75]	[0.7,0.85]	[0.2,0.4]

Table 5
Collective results.

	C_1	C_2	C_3	C_4	C_5
A_1	[0.73,0.84]	[0.74,0.88]	[0.81,0.92]	[0.84,0.98]	[0.33,0.44]
A_2	[0.67,0.78]	[0.51,0.62]	[0.7,0.8]	[0.8,0.93]	[0.42,0.55]
A_3	[0.36,0.46]	[0.61,0.73]	[0.54,0.62]	[0.71,0.84]	[0.85,0.99]
A_4	[0.53,0.63]	[0.52,0.64]	[0.52,0.64]	[0.62,0.74]	[0.78,0.91]
A_5	[0.78,0.92]	[0.76,0.93]	[0.64,0.73]	[0.78,0.91]	[0.26,0.4]

Table 6
Aggregated results.

	UNAC	UWAC	UOWAAC	UAOWAAC	UIOWAAC
A_1	0.874	0.888	0.937	0.781	0.841
A_2	0.845	0.837	0.904	0.772	0.819
A_3	0.827	0.727	0.898	0.727	0.744
A_4	0.822	0.766	0.874	0.766	0.782
A_5	0.861	0.894	0.923	0.758	0.848

Table 7
Ordering of the countries.

	Ordering
UNAC	$A_1 \succ A_5 \succ A_2 \succ A_3 \succ A_4$
UWAC	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$
UOWAAC	$A_1 \succ A_5 \succ A_2 \succ A_3 \succ A_4$
UAOWAAC	$A_1 \succ A_2 \succ A_4 \succ A_5 \succ A_3$
UIOWAAC	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$

Uncertainty is a major factor that affects the decision making of companies on international expansion [29]. In particular, this uncertainty arises when the information considered in the analysis is incomplete, imprecise, or vague. In this context, the aggregation operators adopted in this practical example provide an effective solution for dealing with uncertainty associated to business internationalization.

6. Comparative analysis

The purpose of this section is to perform a comparative study of the presented aggregation operators with existing aggregation operators. Other uncertainties are also contemplated.

6.1. Existing aggregation operators

In concrete situations the UOWAAC operator and its families are more appropriate than others based on the Hamming distance. To prove this point, a comparative analysis between these two approaches is studied. To do so, the uncertain normalized distance (UND), the uncertain weighted distance (UWD), the uncertain ordered weighted averaging distance (UOWAD) [30], the ascending UOWAD (UAOWAD), and the induced UOWAD (UIOWAD) [31] operators are calculated, considering the same information used in the international expansion MCGDM problem. The comparison results are presented in Table 8. Note that the outcomes of the UNAC, UWAC, UOWAAC, UAOWAAC, and UIOWAAC operators are taken from the example conducted in Section 5.

We can see that the order of preference given by the UOWAAC operator differs from the UOWAD one. The same happens with the other compared aggregation operators, except for the UWAC against the UWD. Obviously, if a candidate country presents better results than the ideal country, it makes no sense to penalize as it happens with the aggregation operators that are built under the Hamming distance. Thus, by using the UOWAAC operators as well as its families and extensions, the decision maker only penalizes when the level of the ideal country is not attained, but he/she neither penalizes nor rewards when the level of the ideal country is exceeded.

Table 8
Comparison with established aggregation operators.

	Ordering		Ordering
UNAC	$A_1 \succ A_5 \succ A_2 \succ A_3 \succ A_4$	UND	$A_2 \succ A_5 \succ A_1 \succ A_4 \succ A_3$
UWAC	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$	UWD	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$
UOWAAC	$A_1 \succ A_5 \succ A_2 \succ A_3 \succ A_4$	UOWAD	$A_2 \succ A_4 \succ A_1 \succ A_5 \succ A_3$
UAOWAAC	$A_1 \succ A_2 \succ A_4 \succ A_5 \succ A_3$	UAOWAD	$A_2 \succ A_5 \succ A_1 \succ A_4 \succ A_3$
UIOWAAC	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$	UIOWAD	$A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$

Table 9
BUI pairs of Expert 1.

	C_1	C_2	C_3	C_4	C_5
A_1	⟨0.75; 0.8⟩	⟨0.85; 0.8⟩	⟨0.85; 0.8⟩	⟨0.95; 0.8⟩	⟨0.4; 0.8⟩
A_2	⟨0.65; 0.8⟩	⟨0.5; 0.8⟩	⟨0.75; 0.8⟩	⟨0.9; 0.8⟩	⟨0.55; 0.8⟩
A_3	⟨0.4; 0.9⟩	⟨0.65; 0.9⟩	⟨0.48; 0.9⟩	⟨0.75; 0.9⟩	⟨0.95; 0.9⟩
A_4	⟨0.55; 0.7⟩	⟨0.55; 0.7⟩	⟨0.55; 0.7⟩	⟨0.65; 0.7⟩	⟨0.85; 0.7⟩
A_5	⟨0.85; 0.9⟩	⟨0.93; 0.9⟩	⟨0.65; 0.9⟩	⟨0.85; 0.9⟩	⟨0.3; 0.9⟩

Table 10
BUI pairs of Expert 2.

	C_1	C_2	C_3	C_4	C_5
A_1	⟨0.8; 0.7⟩	⟨0.8; 0.95⟩	⟨0.88; 0.75⟩	⟨0.88; 0.8⟩	⟨0.4; 0.7⟩
A_2	⟨0.8; 0.7⟩	⟨0.58; 0.95⟩	⟨0.75; 0.75⟩	⟨0.88; 0.8⟩	⟨0.48; 0.7⟩
A_3	⟨0.45; 0.7⟩	⟨0.68; 0.95⟩	⟨0.65; 0.75⟩	⟨0.78; 0.8⟩	⟨0.93; 0.7⟩
A_4	⟨0.63; 0.7⟩	⟨0.58; 0.95⟩	⟨0.58; 0.75⟩	⟨0.68; 0.8⟩	⟨0.88; 0.7⟩
A_5	⟨0.83; 0.7⟩	⟨0.8; 0.95⟩	⟨0.7; 0.75⟩	⟨0.88; 0.8⟩	⟨0.38; 0.7⟩

Table 11
BUI pairs of Expert 3.

	C_1	C_2	C_3	C_4	C_5
A_1	⟨0.83; 0.95⟩	⟨0.75; 0.65⟩	⟨0.88; 0.95⟩	⟨0.9; 0.95⟩	⟨0.33; 0.95⟩
A_2	⟨0.73; 0.85⟩	⟨0.68; 0.6⟩	⟨0.75; 0.85⟩	⟨0.78; 0.85⟩	⟨0.38; 0.85⟩
A_3	⟨0.35; 0.85⟩	⟨0.7; 0.6⟩	⟨0.65; 0.85⟩	⟨0.83; 0.85⟩	⟨0.85; 0.85⟩
A_4	⟨0.55; 0.85⟩	⟨0.65; 0.6⟩	⟨0.65; 0.85⟩	⟨0.75; 0.85⟩	⟨0.78; 0.85⟩
A_5	⟨0.9; 0.85⟩	⟨0.78; 0.6⟩	⟨0.73; 0.85⟩	⟨0.78; 0.85⟩	⟨0.3; 0.85⟩

Table 12
Aggregated results.

	BUI-UNAC	BUI-UWAC	BUI-UOWAAC	BUI-UAOWAAC	BUI-UIOWAAC
A_1	0.859	0.861	0.926	0.772	0.816
A_2	0.810	0.796	0.863	0.748	0.779
A_3	0.807	0.717	0.873	0.715	0.732
A_4	0.784	0.737	0.824	0.734	0.743
A_5	0.838	0.860	0.903	0.745	0.815

6.2. Basic uncertain information

So far, the illustrative example assumed that the assessments are represented directly by interval numbers. However, the experts could also use basic uncertain information (BUI) [32,33] for the evaluations and afterwards transform it into interval numbers.

BUI is a quite recent concept that was introduced to generalize a wide variety of uncertainties. Specifically, it allows to consider the level of certainty that the decision maker has on the input data. A BUI is a real pair $\tilde{x} =< x; c >$, where $x (x \in [0, 1])$ is the input value and $c (c \in [0, 1])$ the certainty degree of x. A BUI can be transformed into a closed interval $[a, b]$, where $a = cx$ and $b = cx + 1 - c$.

In order to enrich this paper, the same example as in Section 5 will be conducted but, in this case, considering BUI assessments. By using BUI, the experts are able to exhibit the amount of confidence that they have in their own assessments. The BUI pair matrix of each expert can be seen in Tables 9, 10, and 11. Likewise, the aggregated results and the ranking of the alternatives can be seen in Tables 12 and 13. Take into account that the weighting vectors and inducing vector used are the same as in Section 5.

Table 13
Ordering of the countries.

	Ordering
BUI-UNAC	$A_1 \succ A_5 \succ A_2 \succ A_3 \succ A_4$
BUI-UWAC	$A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_3$
BUI-UOWAAC	$A_1 \succ A_5 \succ A_3 \succ A_2 \succ A_4$
BUI-UAOWAAC	$A_1 \succ A_2 \succ A_5 \succ A_4 \succ A_3$
BUI-UIOWAAC	$A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_3$

Within each operator, the optimal choice is the alternative with the highest aggregated result. Stated another way, the best option is the one with the closest aggregated result to 1. In all cases it is A_1, i.e., France. Conversely, A_3 and A_4, i.e., Portugal and Romania, are the less preferred options for the company.

7. Conclusions

The UOWAAC operator is a new aggregation operator that uses the adequacy coefficient, the interval numbers, and the OWA operator in the same formulation. As a result, this comprehensive operator presents several advantages. First, it provides a parametrized family of aggregation operators that includes among others the Min-UAC, the Max-UAC, the Med-UAC, the UNAC, the Hurwicz-UOWAAC, the Olympic-UOWAAC, and the Step-UOWAAC. Second, it can aggregate the information according to the attitudinal character of the decision maker. Third, it is practical for dealing with uncertainty, especially when the information cannot be represented with exact numbers, but it is possible to use interval numbers. Fourth, it can be used to compare an ideal with the available alternatives while establishing a threshold from which the results are always the same.

Moreover, the UIOWAAC operator is an extension of the UOWAAC operator. The core difference between these two operators is that the UIOWAAC operator uses order-inducing variables, thus allowing the decision maker to deal with more complex reordering processes of the available information.

An illustrative example of the new approach has been presented in a MCGDM problem regarding the selection of the most suitable country for international business expansion. The adopted aggregation operators permit to consider the judgements of the experts in the form of interval numbers. Moreover, they overcome the limitations of some traditional comparison methods. The numerical results show that depending on the type of aggregation operator used the preference order of the candidate countries may change.

For future research, the proposed operators can be applied to other interesting fields, such as personnel selection, sport management, selection of financial products, and risk management. Also, we suggest investigating new extensions of the UOWAAC operator and the UIOWAAC operator, for example, by including generalized means, quasi arithmetic means, or probabilities.

CRediT authorship contribution statement

Anton Figuerola-Wischke: Conceptualization, Methodology, Writing – original draft, Writing – review & editing. **Anna M. Gil-Lafuente:** Conceptualization, Writing – review & editing. **José M. Merigó:** Conceptualization, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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