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Von Neumann-Morgenstern solution and convex descompositions of TU games

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von Neumann-Morgenstern solution and convex decompositions of TU games

Abstract

We study under which conditions the core of a game involved in a convex decomposition of another game turns out to be a stable set of the decomposed game. Some applications and numerical examples, including the remarkable Lucas' five player game with a unique stable set different from the core, are reckoning and analyzed.

Resum

En l'article s'estudia sota quines condicions el nucli d'un joc cooperatiu amb utilitat transferible que forma part d'una descomposició convexa d'un altre joc és un conjunt estable, en el sentit de von Neumann i Morgenstern, del joc descomposat. També s'analitzen alguns exemples numèrics i aplicacions.

Keywords: Cooperative games, convex games, stable sets

JEL: C71

1 Introduction

The first historical solution concept for cooperative games with transferable utility (games, for short) was proposed and studied in the seminal book of von Neumann and Morgenstern (1944). It is referred to as stable set or von Neumann-Morgenstern solution. There, a stable set of a game is defined to be a set $\mathcal{V}$ of imputations satisfying *internal stability* (no element in $\mathcal{V}$ is dominated by other element in $\mathcal{V}$) and *external stability* (every element outside $\mathcal{V}$ is dominated by some element in $\mathcal{V}$).

Lucas (1968) provides an example that gives a negative answer to the general question about the existence of stable sets for any game. Nevertheless, it is possible to find games with a plethora of stable sets. It is well-known that any stable set contains the core, which is always internally stable. When the core is also externally stable, then it is the unique stable set. However, there are balanced games with a unique stable set different from the core. This situation was showed by Lucas (1992) by means of the following game:[1] let (N, v) be the 5-person game where $N = \{1, 2, 3, 4, 5\}$, $v(N) = 2$, $v(\{1, 2\}) = v(\{3, 4\}) = v(\{1, 3, 5\}) = v(\{2, 4, 5\}) = 1$, and $v(S) = 0$ for all other $S \subset N$. What is specially interesting for our purposes is to see that the superadditive cover of the previous game, $(N, \hat{v})$, can be easily described by decomposing the game as the maximum of a finite set of convex games: $\hat{v} = \max\left\{u_{\{1,2\}} + u_{\{3,4\}}, u_{\{1,3,5\}} + u_{\{2,4,5\}}\right\}$, where u_S, for all $\emptyset \neq S \subseteq N$, denotes the unanimity game associated to coalition S. It is well-known that the superadditive cover process (Gillies, 1959) preserves the imputations set, the core and the stable sets whenever the original game satisfies $v(N) \geq \sum_{C \in \mathcal{P}} v(C)$, for any partition $\mathcal{P} \subseteq 2^N$ of N, which is the case in the above example. As shown by Lucas, the superadditive cover (and so the game itself) of the above game has a unique stable set which is described by $\mathcal{V} = \{(x_1, \ldots, x_5) \in \mathbb{R}^5_+ \mid x_1 + x_2 = 1, x_3 + x_4 = 1, x_5 = 0\}$, being different from its core which is $C(N, v) = \{(\alpha, 1 - \alpha, 1 - \alpha, \alpha, 0) \,|\, 0 \leq \alpha \leq 1\}$. Notice that the unique stable set $\mathcal{V}$ coincides with the core of the first convex game described in the max-convex decomposition of the game $\hat{v}$; i.e. $\mathcal{V} = C\left(N, u_{\{1,2\}} + u_{\{3,4\}}\right)$. A similar situation can be described by using simple monotonic games. For instance,

[1] See Section 2 for formal definitions.

consider the following 3-person simple majority game: $N = \{1, 2, 3\}$, $v(N) = v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 1$ and $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$. This game can be rewritten as $v = \max\left\{u_{\{1,2\}}, u_{\{1,3\}}, u_{\{2,3\}}\right\}$. It is well-known that $C\left(N, u_{\{1,2\}}\right)$, $C\left(N, u_{\{1,3\}}\right)$ and $C\left(N, u_{\{2,3\}}\right)$ are stable sets of this 3-person simple majority unbalanced game. But in this second example there are many others stable sets, one of them being the discrete set $\left\{\left(\frac{1}{2}, \frac{1}{2}, 0\right), \left(0, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, 0, \frac{1}{2}\right)\right\}$, which can not be reached as the core of any related game.

In a previous work, Llerena and Rafels (2006) show that any game can be expressed, in many different ways, as the maximum of a finite collection of convex games. Moreover, if the game is zero monotonic there exists at least a max-convex decomposition where the games involved have the same imputation set. These general decomposition results, together with the above line of argument, open a natural question: which are the relationships between the core of the games involved in a max-convex decomposition and the stable sets of the decomposed game?

With these objectives in mind, the paper is organized as follows. Section 2 contains notation and definitions. Section 3 analyzes the external stability of the intersection between the Weber set and the imputation set of two games related by the usual order. We complete this result stating sufficient conditions to guarantee when the core of a convex game taking part in a max-decomposition happens to be a stable set of the initial game. Section 4 contains examples and applications.

2 Notation and terminology

We denote by $N = \{1, \ldots, n\}$ a finite set of players. A *transferable utility cooperative game (a game)* is a pair (N, v) where $N = \{1, \ldots, n\}$ is the set of players and $v : 2^N \longrightarrow \mathbb{R}$ is the characteristic function with $v(\emptyset) = 0$ and 2^N denotes the set of all subsets (coalitions) of N. For any coalition $S \subseteq N$, $N \backslash S = \{i \in N \mid i \notin S\}$. We use $S \subset T$ to indicate strict inclusion, that is $S \subseteq T$ but $S \neq T$. By $|S|$ we denote the cardinality of the coalition $S \subseteq N$. Given a game (N, v) and a non-empty coalition $S \subseteq N$, we define the *subgame* $(S, v_{|S})$ as $v_{|S}(Q) := v(Q)$, for any $Q \subseteq S$. Given two games (N, v_1), (N, v_2), $v_1 \leq v_2$

means that $v_1(S) \leq v_2(S)$, for all $S \subseteq N$. The *maximum game* (N, v) generated by the set $\{(N, v_t\}_{t=1,\ldots,k}$ is defined as $v(S) := \max\{v_1(S), \ldots, v_k(S)\}$, for all $S \subseteq N$. A *partition* of any set S is a collection of non-empty subsets $\{T_1, \ldots, T_k\}$ such that $T_1 \cup \ldots \cup T_k = S$ and, for all $i, j \in \{1, \ldots, k\}$, $i \neq j$, $T_i \cap T_j = \emptyset$. Let $\mathcal{P}(S)$ denote the set of all partitions of S. Then, the *superadditive cover* of (N, v) is the game $(N, \hat{v})$ defined as $\hat{v}(S) := \max\left\{\sum_{j=1}^{k} v(T_j) \,|\, \{T_1, \ldots, T_k\} \in \mathcal{P}(S)\right\}$, for all $S \subseteq N$. Given a coalition $T \subseteq N$, $T \neq \emptyset$, the *unanimity game* (N, u_T) is defined as

$$u_T(S) := \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The set of unanimity games $\{(N, u_T) \mid \emptyset \neq T \subseteq N\}$ forms a basis of the linear space of the set of N-person games, and the coordinates of a game in this basis are the *unanimity coordinates (or Harsanyi dividends)* of the game. For any (N, v), $v = \sum_{\emptyset \neq T \subseteq N} \lambda_T \cdot u_T$, where $\lambda_T = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S)$, for all $\emptyset \neq T \subseteq N$.

Let $\mathbb{R}^N$ stand for the space of real-valued vectors $x = (x_i)_{i \in N}$ and for all $S \subseteq N$, $x(S) = \sum_{i \in S} x_i$, with the convention $x(\emptyset) = 0$. For each $x \in \mathbb{R}^N$ and $T \subseteq N$, $x_{|T}$ denotes the restriction of x to T: $x_{|T} := (x_i)_{i \in T} \in \mathbb{R}^T$. Given two vectors $x, y \in \mathbb{R}^N$, we denote by $[x, y]$ the closed line segment joining x and y. Formally, $[x, y] := \{\lambda \cdot x + (1 - \lambda) \cdot y \,|\, 0 \leq \lambda \leq 1\}$.

For each game (N, v), the set of *feasible payoff vectors* is defined by $X^* := \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\}$. A *solution* on the set of all games is a mapping σ which associates with each game (N, v) a subset $\sigma(N, v)$ of $X^*(N, v)$. The *pre-imputation set* of a game (N, v) is defined by $X(N, v) := \{x \in \mathbb{R}^N \,|\, x(N) = v(N)\}$, and the set of *imputations* by $I(N, v) := \{x \in \mathbb{R}^N \,|\, x(N) = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N\}$. A game with a non-empty set of imputations is called *essential.* The core of (N, v) is the set of those imputations where each coalition gets at least its worth, that is $C(N, v) = \{x \in I(N, v) \,|\, x(S) \geq v(S) \text{ for all } S \subseteq N\}$. A game with non-empty core is called *balanced.* A game (N, v) is *convex* (Shapley, 1971) if, for all $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$. Each unanimity game (N, u_T) is convex. A game

(N,v) is *monotonic* if $v(S) \leq v(T)$, for all $S \subset T \subseteq N$, and *zero monotonic* if its *zero normalization* (N, v_0), where $v_0(S) = v(S) - \sum_{i \in S} v(\{i\})$, for all $S \subseteq N$, is a monotonic game or, equivalently, $v(S \cup \{i\}) - v(S) \geq v(\{i\})$, for all $i \in N$ and all $S \subseteq N \setminus \{i\}$. A game (N,v) is N-*monotonic* if for all $S \subseteq N$, $v(S) + \sum_{i \in N \setminus S} v(\{i\}) \leq v(N)$.

A *permutation* $\theta \in \Theta_T$ over a non-empty coalition $T \subseteq N$ is a bijection from $\{1, \ldots, |T|\}$ to T. We denote by Θ_T the set of all permutations over T. For any $\emptyset \neq T \subset N$, any $\theta \in \Theta_T$ and any $\theta' \in \Theta_{N \setminus T}$, we define the *appending permutation* $\theta^* = (\theta, \theta') \in \Theta_N$ as follow:

$$\theta^*(i) := \begin{cases} \theta(i) & \text{if } 1 \leq i \leq t, \\ \theta'(i-t) & \text{if } t+1 \leq i \leq n. \end{cases}$$

Given a game (N,v) and a permutation $\theta \in \Theta_N$, the *marginal worth vector* associated to θ, denoted by $m^v_\theta \in \mathbb{R}^N$, is defined as $m^v_{\theta(k)} = v(\{\theta(1), \ldots, \theta(k)\}) - v(\{\theta(1), \ldots, \theta(k-1)\})$, for all $k \in \{2, \ldots, n\}$, and $m^v_{\theta(1)} = v(\{\theta(1)\})$. The convex hull of the marginal worth vectors is called the *Weber set*, $W(N,v) := \textit{convex}\ \{m^v_\theta\}_{\theta \in \Theta_N}$. A game (N,v) is convex if and only if $W(N,v) = C(N,v)$ (Shapley, 1971; Ichiishi, 1981).

Given two pre-imputations $x, y \in X(N,v)$, we say that x dominates y, in short x *dom*^{v} y, if there exists a non-empty coalition $S \subset N$ such that $x_i > y_i$, for all $i \in S$, and $x(S) \leq v(S)$. For $X \subseteq I(N,v)$ we denote by $Dom^v X$ the set of all imputations dominated by some imputation of the set X. Formally, $Dom^v X = \{y \in I(N,v) \mid \exists\, x \in X,\ x\, dom^v\, y\}$. Let (N,v) be an essential game. A set of imputations $\varnothing \neq \mathcal{V} \subseteq I(N,v)$ is a *stable set* for the game (N,v) if it satisfies the next two conditions:

1. $\mathcal{V}$ is v-*internally stable*: no imputation in $\mathcal{V}$ dominates another imputation in $\mathcal{V}$. Formally, $\mathcal{V} \cap Dom^v\, \mathcal{V} = \emptyset$.

2. $\mathcal{V}$ is v-*externally stable*: any imputation outside the set $\mathcal{V}$ is dominated by some imputation in $\mathcal{V}$. Formally, $\mathcal{V} \cup Dom^v\, \mathcal{V} = I(N,v)$.

3 Stable sets and convex decompositions of games

In this section we study under which conditions the core of the games involved in a max-convex decomposition turn out to be stable sets of the decomposed game. We begin stating a general result about the external stability of the intersection between the Weber set and the imputation set of two ordered games with the same efficiency level.

Theorem 1 *Let $(N, v_1), (N, v_2)$ be two games such that $v_1 \leq v_2$, $v_1(N) = v_2(N)$, and*

$$v_1(S) + \sum_{i \in N \setminus S} v_2(\{i\}) \leq v_2(N), \text{ for all } S \subseteq N. \tag{1}$$

Then,

1. *$(C(N, v_1) \cap I(N, v_2)) \cup \; Dom^{v_2} \; (W(N, v_1) \cap I(N, v_2)) = I(N, v_2)$.*

2. *$W(N, v_1) \cap I(N, v_2)$ is a non-empty set and it is v_2-externally stable.*

3. *If $W(N, v_1) \cap I(N, v_2)$ is a stable set for (N, v_2), then $C(N, v_1) \cap I(N, v_2)$ is a stable set for (N, v_2) and both coincide.*

Proof:

1. Note that we only have to see that

$$I(N, v_2) \setminus (C(N, v_1) \cap I(N, v_2)) \subseteq Dom^{v_2} \; (W(N, v_1) \cap I(N, v_2)).$$

Let (N, v_3) be defined as

$$v_3(S) := \max_{R \subseteq S} \left\{ v_1(R) + \sum_{i \in S \setminus R} v_2(\{i\}) \right\}, \text{ for all } S \subseteq N. \tag{2}$$

Clearly, $v_1 \leq v_3$, and from the hypothesis of the theorem it follows $v_1(N) = v_2(N) = v_3(N)$. On the other hand, $v_3(\{i\}) = v_2(\{i\})$ for all $i \in N$, and thus $I(N, v_3) = I(N, v_2)$. It can be easily checked that $C(N, v_1) \cap I(N, v_2) = C(N, v_3)$. Moreover, (N, v_3) is zero monotonic. Indeed, let $i \in N$, $S \subseteq N \setminus \{i\}$ and $R^* \subseteq S$ such that $v_3(S) = v_1(R^*) + \sum_{j \in S \setminus R^*} v_2(\{j\})$. Then, $v_3(S \cup \{i\}) - v_3(S) \geq v_1(R^*) + \sum_{j \in S \cup \{i\} \setminus R^*} v_2(\{j\}) - v_1(R^*) - \sum_{j \in S \setminus R^*} v_2(\{j\}) = v_2(\{i\}) = v_3(\{i\})$.

Let $x \in I(N,v_2)\backslash(C(N,v_1)\cap I(N,v_2))$ and T be a minimal coalition (w.r.t. inclusion) such that $x(T) < v_3(T)$. Let $R \subset T$. Then, $v_3(T) > x(T) = x(R) + x(T\setminus R) \geq v_3(R) + \sum_{i\in T\setminus R} v_2(\{i\}) \geq v_1(R) + \sum_{i\in T\setminus R} v_2(\{i\})$. Now, from the definition of (N,v_3) we get $v_3(T) = v_1(T)$. Let us define the games $(N\backslash T, w_k)$, where $k \in \{1,3\}$, as follow: $w_k(S) := v_k(S\cup T) - v_k(T)$, for all $S \subseteq N\setminus T$. Note that $w_1(S) \leq w_3(S)$ for all $S \subset N\setminus T$ and $w_1(N\setminus T) = w_3(N\setminus T)$, and consequently $W(N\setminus T, w_1)\cap W(N\setminus T, w_3) \neq \emptyset$ (Martínez-de-Albéniz and Rafels, 2004). Let $z \in W(N\setminus T, w_1)\cap W(N\setminus T, w_3)$ and $y \in \mathbb{R}^N$ be defined as follows:

$$y_i := \begin{cases} x_i + \alpha & \text{if } i \in T, \text{ where } \alpha = \frac{v_1(T)-x(T)}{|T|} \\ z_i & \text{if } i \in N\backslash T. \end{cases}$$

Next we show that $y \in W(N,v_1)\cap I(N,v_2)$ and $y\, dom^v\, x$. From the minimality of T, for all $S \subset T$ we have $y(S) = x(S) + \alpha\cdot|S| > x(S) \geq v_3(S) \geq v_1(S)$. The above condition, together with the equality $y(T) = v_3(T) = v_1(T)$, implies $y_{|T} \in C(T, v_{1_{|T}}) = C(T, v_{1_{|T}})\cap C(T, v_{3_{|T}}) \subseteq W(T, v_{1_{|T}})\cap W(T, v_{3_{|T}})$. Thus, $y_{|T} = \sum_{\theta\in\Theta_T} \lambda_\theta \cdot m_\theta^{v_{1|T}}$. Moreover, since $y_{|N\backslash T} = z \in W(N\setminus T, w_1)$, we have $y_{|N\backslash T} = \sum_{\theta'\in\Theta_{N\backslash T}} \mu_{\theta'}\cdot m_{\theta'}^{w_1}$, where $\lambda_\theta \geq 0$, $\mu_{\theta'} \geq 0$, $\sum_{\theta\in\Theta_T}\lambda_\theta = 1$ and $\sum_{\theta'\in\Theta_{N\backslash T}}\mu_{\theta'} = 1$. Now, from the definition of the game $(N\backslash T, w_1)$ we have $y = \sum_{\theta^*=(\theta,\theta')\in\Theta_N} (\lambda_\theta\cdot\mu_{\theta'})\cdot m_{\theta^*}^{v_1}$, where $\theta \in \Theta_T$ and $\theta' \in \Theta_{N\backslash T}$, or equivalently, $y \in W(N,v_1)$. In a similar way we can see that $y \in W(N,v_3)$. From the zero monotonicity of (N,v_3) and the equality $I(N,v_3) = I(N,v_2)$, we have $y \in I(N,v_2)$. Finally, since for all $i \in T$, $y_i > x_i$ and $y(T) = v_1(T) \leq v_2(T)$, we conclude that $y\, dom^{v_2}\, x$ via T.

2. From statement 1, $I(N,v_2) = (C(N,v_1)\cap I(N,v_2))\cup\ Dom^{v_2}\,(W(N,v_1)\cap I(N,v_2)) \subseteq (W(N,v_1)\cap I(N,v_2))\cup\ Dom^{v_2}\,(W(N,v_1)\cap I(N,v_2)) \subseteq I(N,v_2)$, and consequently $(W(N,v_1)\cap I(N,v_2))\cup\ Dom^{v_2}\,(W(N,v_1)\cap I(N,v_2)) = I(N,v_2)$. By condition (1) we have $I(N,v_2) \neq \emptyset$, which implies the non-emptiness of $W(N,v_1)\cap I(N,v_2)$ and the v_2-external stability.

3. By hypothesis, $W(N,v_1)\cap I(N,v_2)$ is a stable set for (N,v_2), and thus $(C(N,v_1)\cap I(N,v_2))\cap\ Dom^{v_2}\,(W(N,v_1)\cap I(N,v_2)) \subseteq (W(N,v_1)\cap I(N,v_2))\cap\ Dom^{v_2}\,(W(N,v_1)\cap$

$I(N, v_2)) = \varnothing$. This fact, together with statement 1, implies $(C(N, v_1) \cap I(N, v_2)) = I(N, v_2) \setminus Dom^{v_2}(W(N, v_1) \cap I(N, v_2)) = W(N, v_1) \cap I(N, v_2)$, and the desired result is reached.

The next example shows that condition (1) is needed to guarantee non-emptiness of the intersection between the Weber set and the imputation set.

Example: Let $v_1 = 3 \cdot u_{\{1,3\}}$ and $v_2 = \max\left\{u_{\{2\}}, 2 \cdot u_{\{1,2\}}, 3 \cdot u_{\{1,3\}}, u_{\{2,3\}}\right\}$. It can be easily checked that $W(N, v_1) = [(3,0,0),(0,0,3)]$ and $W(N, v_1) \cap I(N, v_2) = \emptyset$, but $v_1(\{1,3\}) + v_2(\{2\}) = 4 > v_2(N) = 3$.

Notice that the hypothesis of Theorem 1 holds for two ordered games (N, v_1), (N, v_2), with $v_1 \leq v_2$, such that either **(a)** $v_1(N) = v_2(N)$ and (N, v_2) is N-monotonic, or **(b)** $C(N, v_1) \cap I(N, v_2) \neq \emptyset$. In case **(a)**, condition (1) comes directly from the N-monotonicity of (N, v_2) and the fact that $v_1(N) = v_2(N)$. In case **(b)**, taking $x \in C(N, v_1) \cap I(N, v_2)$ and $S \subset N$, we get $v_1(S) + \sum_{i \in N \setminus S} v_2(\{i\}) \leq x(S) + x(N \setminus S) = v_2(N)$. As a consequence, we obtain the next corollary.

Corollary 1 *Let (N, v_1), (N, v_2) be two games such that $v_1 \leq v_2$. If either*

1. *(N, v_2) is N-monotonic and $v_1(N) = v_2(N)$, or*

2. *$C(N, v_1) \cap I(N, v_2) \neq \emptyset$,*

then $W(N, v_1) \cap I(N, v_2)$ is a non-empty set and it is v_2-externally stable.

A direct and useful consequence for our purposes is the following.

Corollary 2 *Let (N, v_1), (N, v_2) be two games such that $v_1 \leq v_2$, $v_1(N) = v_2(N)$, and condition (1) holds. If (N, v_1) is convex, then $C(N, v_1) \cap I(N, v_2)$ is a non-empty set and it is v_2-externally stable.*

It should be noted that Theorem 1 generalizes some already known results.

Corollary 3 *(Shapley (1971) If (N, v) is a convex game, then $C(N, v)$ is a stable set.*

Proof: Since (N,v) is convex, $C(N,v) = W(N,v)$. By Theorem 1 statement 2, taking $v_1 = v_2 = v$, we have that $C(N,v)$ is v-externally stable. This conclude the proof since $C(N,v)$ is always v-internally stable.

Corollary 4 *(Rafels and Tijs, 1997) A game (N,v) is convex if and only if the Weber set $W(N,v)$ is a stable set.*

Proof: If (N,v) is convex, then $C(N,v) = W(N,v)$ and thus the Weber set is (the unique) stable set. If $W(N,v)$ is a stable set, then $W(N,v) \subseteq I(N,v)$, or equivalently, (N,v) is zero monotonic. Now, from Theorem 1 statement 3, taking $v_1 = v_2 = v$ we get $C(N,v) = W(N,v)$, which implies the convexity of (N,v).

Next, we complete the above result with the analysis of the internal stability for the cores of convex games involved in a max-decomposition.

Definition 1 *Let (N,v) be a game and $\{\lambda_T\}_{\emptyset \neq T \subseteq N}$ its unanimity coordinates. We define $\mathcal{N}_v := \{\varnothing \neq S \subseteq N \mid \lambda_S \neq 0\}$ and $N_v := \bigcup_{S \in \mathcal{N}_v} S$.*

Theorem 2 *Let (N,v) be the maximum game generated by the set of games $\{(N,v_t)\}_{t=1,\ldots,k}$. If for some $t^* \in \{1,\ldots,k\}$ it is satisfied:*

1. *(N,v_{t^*}) is convex and $v_{t^*}(N) = v(N)$,*

2. *$v_{t^*}(S) + \sum_{i \in N \setminus S} v(\{i\}) \leq v(N)$, for all $S \subseteq N$, and*

3. *$R \cap (N \setminus N_{v_t^*}) \neq \varnothing$, for all $R \in \cup_{j=1, j \neq t^*}^{k} \mathcal{N}_{v_j}$,*

then $C(N,v_{t^*}) \cap I(N,v)$ is a stable set for the game (N,v).

Proof: We can take, without loss of generality, $t^* = 1$. External stability comes from corollary 2 applied to (N,v_1) and (N,v). Next we show internal stability. First notice that statement 3 implies $N_{v_1} \neq N$ and, for all $j \in N \setminus N_{v_1}$ and all $x \in C(N,v_1) \cap I(N,v)$, $x_j = 0$ since $0 = v_1(\{j\}) \leq x_j \leq v_1(N) - v_1(N \setminus \{j\}) = 0$. Suppose, on the contrary, there are $x, y \in C(N,v_1) \cap I(N,v)$ such that $y \; dom^v \; x$. Then, there is a coalition $Q \subset N$ such that $y_i > x_i$, for all $i \in Q$, and $y(Q) \leq v(Q)$. We claim that, for any $R \in \mathcal{N}_{v_2} \cup \ldots \cup \mathcal{N}_{v_k}$ and any

$Q' \subseteq Q$, it holds $R \nsubseteq Q'$. Indeed, if $R \subseteq Q'$, by statement 3, there is $j \in R \cap (N \backslash N_{v_1})$, and thus $j \in Q$. Since $x, y \in C(N, v_1) \cap I(N, v)$, we have $x_j = y_j = 0$ and $y_j > x_j$, and reach a contradiction. Therefore, for any $R \in \mathcal{N}_{v_2} \cup \ldots \cup \mathcal{N}_{v_k}$ and any $Q' \subseteq Q$, we have $R \nsubseteq Q'$, which implies $v_2(Q') = \ldots = v_k(Q') = 0$. In addition, since $v = \max\{v_1, \ldots, v_k\}$ we have $v(Q') = \max\{v_1(Q'), 0\}$, for any $Q' \subseteq Q$. In particular, taking $Q' = \{i\}$ we get $v(\{i\}) \geq 0$, for all $i \in Q$. As $y_i > x_i$ for all $i \in Q$ and $x \in I(N, v)$, we have $y_i > 0$ for all $i \in Q$, which implies $y(Q) > 0$. Now, from $0 < y(Q) \leq v(Q) = \max\{v_1(Q), 0\}$, we obtain $v(Q) = v_1(Q)$. But then $x(Q) < y(Q) \leq v_1(Q)$, in contradiction with $x \in C(N, v_1) \cap I(N, v)$. Hence, $C(N, v_1) \cap I(N, v)$ is v-internally stable, which concludes the proof.

In some special cases we can argue that given two ordered games (N, v_1) and (N, v_2), with $v_1 \leq v_2$, the intersection $C(N, v_1) \cap I(N, v_2)$ is the unique stable set for the game (N, v_2). This is what we state in the next theorem.

Theorem 3 *Let $(N, v_1), (N, v_2)$ be two games with $v_1 \leq v_2$, $v_1(N) = v_2(N)$, $v_2(\{i\}) \geq 0$ for all $i \in N$, and such that:*

1. *for all $S, T \in \mathcal{N}_{v_1}$, $S \cap T = \emptyset$, and*
2. *for all $S \in \mathcal{N}_{v_1}$ and for all $x \in C(N, \lambda_S \cdot u_S)$, there is $y \in C(N, v_2)$ such that $y_{|S} = x_{|S}$.*

If $C(N, v_1) \cap I(N, v_2)$ is a stable set for (N, v_2), then it is the unique.

Proof: Suppose there is a stable set $\mathcal{V}$ for (N, v_2) different from $C(N, v_1) \cap I(N, v_2)$ and take $x \in \mathcal{V} \setminus C(N, v_1) \cap I(N, v_2)$. Let T be a minimal coalition (w.r.t. inclusion) such that $x(T) < v_1(T)$. Since $v_2(\{i\}) \geq 0$ for all $i \in N$, we have $v_1(T) > 0$. Thus, there is $S \in \mathcal{N}_{v_1}$ such that $S \subseteq T$. Let $\mathcal{T} := \{S \in \mathcal{N}_{v_1} \mid S \subseteq T\}$ and suppose $S \subset T$, for all $S \in \mathcal{T}$. Then $v_1(T) = \sum_{S \in \mathcal{T}} \lambda_S = \sum_{S \in \mathcal{T}} v_1(S) \leq \sum_{S \in \mathcal{T}} x(S) \leq x(T)$, where the last equality and the two inequalities follow from statement 1, the minimality of T and the fact that $x_i \geq v(\{i\}) \geq 0$ for all $i \in N$. But this is a contradiction, and thus $T = S$ for some $S \in \mathcal{T}$. In fact, from statement 1 we get $\mathcal{T} = \{T\}$. Let $y \in \mathbb{R}^N$ be given by $y_i := x_i + \frac{v_1(T) - x(T)}{|T|}$ if $i \in T$, and $y_i := 0$ otherwise. Clearly $y \in C(N, \lambda_T \cdot u_T)$. Now, from statement 2, there is $z \in C(N, v_2)$ such that $z_{|T} = y_{|T}$. But then, $z\ dom^{v_2}\ x$ via T, in contradiction with the internal stability of $\mathcal{V}$. This concludes the proof.

4 Examples and applications

The applicability of both Theorems 2 and 3 is limited but could be used to find some convex stable sets for a given game. Quoting Aumann (1985), "finding stable sets involve a new tour the force of mathematical reasoning for each game or class of games that is considered. Other than a small number of elementary truisms (e.g. that the core is contained in every stable set), there is no theory, no tools, certainly no algorithms." In this sense, these results are a new tool that not only gives some light on the problem of existence of stable sets but also remarks the fact that a stable set of a given game can be the core of another related game, even if the first game is not balanced. The next examples and applications show some *pragmatism* of our previous results.

Example: Let (N, v) be a game, with $N = \{1, 2, 3, 4, 5, 6\}$ and $v = \max\{v_1, v_2, v_3\}$, where $v_1 = u_{\{1,2\}} + u_{\{1,3\}} + 2 \cdot u_{\{2,3\}} + u_{\{1,3,4\}}$, $v_2 = u_{\{1,5\}} + u_{\{2,6\}} + u_{\{3,5\}} + u_{\{1,2,5\}} + u_{\{1,3,6\}}$ and $v_3 = 3 \cdot u_{\{1,4,6\}} + 2 \cdot u_{\{4,6\}}$. Notice that (N, v_1), (N, v_2) and (N, v_3) have positive unanimity coordinates and thus they are convex games. In addition, $I(N, v_1) = I(N, v_2) = I(N, v_3)$, which implies the N-monotonicity of (N, v). Moreover, $\mathcal{N}_{v_1} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 3, 4\}\}$ and $N \backslash N_{v_1} = \{5, 6\}$; $\mathcal{N}_{v_2} = \{\{1, 5\}, \{2, 6\}, \{3, 5\}, \{1, 2, 5\}, \{1, 3, 6\}\}$ and $N \backslash N_{v_2} = \{4\}$; $\mathcal{N}_{v_3} = \{\{1, 4, 6\}, \{4, 6\}\}$ and $N \backslash N_{v_3} = \{2, 3, 5\}$. Now, from Theorem 2 we obtain that $C(N, v_1)$ and $C(N, v_3)$ are two different stable sets for the initial game (N, v).

Example: (Lucas, 1992) Let (N, v) be the 5-person game where $v(N) = 2$, $v(\{1, 2\}) = v(\{3, 4\}) = v(\{1, 3, 5\}) = v(\{2, 4, 5\}) = 1$ and $v(S) = 0$ for all other $S \subset N$. The core of (N, v) is the segment $[(1, 0, 0, 1, 0), (0, 1, 1, 0, 0)]$. The *superadditive cover* of this game is $\hat{v} = \max\{v_1, v_2\}$, where $v_1 = u_{\{1,2\}} + u_{\{3,4\}}$ and $v_2 = u_{\{1,3,5\}} + u_{\{2,4,5\}}$. Notice that (N, v_1) and (N, v_2) are convex games with the same imputations set, and thus $(N, \hat{v})$ is

N-monotonic. In addition, $\mathcal{N}_{v_1} = \{\{1,2\},\{3,4\}\}$ and $N \backslash N_{v_1} = \{5\}$. Then, from Theorem 2 we obtain that $C(N, v_1)$ is a stable set for the game $(N, \hat{v})$, and consequently for the original game (N, v). To show uniqueness, let $x = (\alpha, 1-\alpha, 0, 0, 0) \in C\left(N, u_{\{1,2\}}\right)$, with $0 \leq \alpha \leq 1$, and $y = (0, 0, \beta, 1-\beta, 0) \in C\left(N, u_{\{3,4\}}\right)$, with $0 \leq \beta \leq 1$. It can be easily checked that $x' = (\alpha, 1-\alpha, 1-\alpha, \alpha, 0)$, $y' = (1-\beta, \beta, \beta, 1-\beta, 0) \in C(N, \hat{v})$. Now, from Theorem 3 we conclude that $C(N, v_1)$ is the unique stable set for the game (N, v).

Example: Let $N = \{1, \ldots, n\}$ and $a_{S_1}, \ldots, a_{S_k}$ be an arbitrary family of strictly positive real numbers associated to the non-empty coalitions $S_1, \ldots, S_k$ of N. Let $v = \max\{a_{S_1} \cdot u_{S_1}, \ldots, a_{S_k} \cdot u_{S_k}\}$ and assume, without loss of generality, $v(N) = a_{S_1}$. Here it is worth to point out that these assumptions are not restrictive in order to analyze existence of stable sets because any monotonic game can be decomposed in this way (see Einy, 1988), and any game is strategically equivalent to a monotonic game (see Peleg and Sudhölter, 2007). In this context, by simple applying Theorem 2, if $1 < |S_r| \leq n$, for all $r = 1, \ldots, k$, and it is satisfied $(N \setminus S_1) \cap S_r \neq \emptyset$, for all $r = 2, \ldots, k$, then $C(N, a_{S_1} \cdot u_{S_1})$ is a stable set for (N, v). This is what happens in the following economic situation.

Example: There are k-different disjoint sets of workers (or types): $N_1, \ldots, N_k$, and one agent who has the capital, denoted by $\mathbf{1}$. A coalition formed by exactly one agent of each type and the owner of the capital, that is $\{\mathbf{1}, i_1, \ldots, i_k\}$ where $i_1 \in N_1, \ldots, i_k \in N_k$, is called a *clique*. The worth of a clique is given by a real positive number $a_{\{\mathbf{1}, i_1, \ldots, i_k\}} > 0$. Let $N = \{\mathbf{1}\} \cup N_1 \cup \ldots \cup N_k$, and suppose that the profit of an arbitrary coalition $S \subseteq N$ is zero if it does not contain any clique, and its profit is the maximum worth generated by the cliques contained in S, otherwise. This situation can be described as a cooperative game (N, v) with $v := \max_{i_1 \in N_1, \ldots, i_k \in N_k} \left\{a_{\{\mathbf{1}, i_1, \ldots, i_k\}} \cdot u_{\{\mathbf{1}, i_1, \ldots, i_k\}}\right\}$. Let $\{\mathbf{1}, i_1^*, \ldots, i_k^*\}$ be a clique such that $v(N) = a_{\{\mathbf{1}, i_1^*, \ldots, i_k^*\}}$. Since $\{\mathbf{1}, j_1, \ldots, j_k\} \cap (N \setminus \{\mathbf{1}, i_1^*, \ldots, i_k^*\}) \neq \emptyset$, for any $j_1 \in N_1, \ldots, j_k \in N_k$, if $(i_1^*, \ldots, i_k^*) \neq (j_1, \ldots, j_k)$, we have that $C\left(N, a_{\{\mathbf{1}, i_1^*, \ldots, i_k^*\}} \cdot u_{\{\mathbf{1}, i_1^*, \ldots, i_k^*\}}\right)$ is a stable set for (N, v). Since $C(N, v) \subset C\left(N, a_{\{\mathbf{1}, i_1^*, \ldots, i_k^*\}} \cdot u_{\{\mathbf{1}, i_1^*, \ldots, i_k^*\}}\right)$, the core $C(N, v)$ is not a stable set for (N, v).

The last example shows that rearranging some factors of an initial max-convex decomposition of a game we can discover other stable sets for the decomposed game.

Example: Let (N, v) be the following game: $N = \{1, 2, 3, 4\}$, and $v(\{i\}) = 0$, for all $i \in N$, $v(\{1, 2\}) = v(\{1, 4\}) = 2$, $v(\{2, 3\}) = v(\{3, 4\}) = 0$, $v(\{1, 3\}) = 3$, $v(\{1, 3, 4\}) = 5$, $v(\{2, 4\}) = v(\{1, 2, 4\}) = v(\{2, 3, 4\}) = v(\{1, 2, 3\}) = v(\{1, 2, 3, 4\}) = 6$. It is easy to check that (N, v) is N-monotonic and it can be decomposed as

$$v = \max\left\{6 \cdot u_{\{1,2,3\}},\, 2 \cdot u_{\{1,2\}},\, 3 \cdot u_{\{1,3\}},\, 2 \cdot u_{\{1,4\}},\, 6 \cdot u_{\{2,4\}},\, 5 \cdot u_{\{1,3,4\}}\right\}.$$

From Theorem 2 we know that $C\left(N, 6 \cdot u_{\{2,4\}}\right)$ is an stable set for (N, v). However, we can rewrite

$$v = \max\left\{2 \cdot u_{\{1,2\}} + 3 \cdot u_{\{1,3\}} + u_{\{1,2,3\}},\, 2 \cdot u_{\{1,4\}},\, 6 \cdot u_{\{2,4\}},\, 5 \cdot u_{\{1,3,4\}}\right\}.$$

Again applying Theorem 2, $C\left(N, 2 \cdot u_{\{1,2\}} + 3 \cdot u_{\{1,3\}} + u_{\{1,2,3\}}\right)$ is another stable set for (N, v).

In future works it could be interesting to analyze conditions in terms of a max-convex decomposition to understand better when a game has no stable sets. Notice that from $n = 5$ to $n = 9$ it is an open problem to get examples without stable sets (if there are any). Perhaps by using these techniques and results we could be able to solve these open problems.

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