CONSTRUCTIONS OF LINDELÖF SCATTERED P-SPACES

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ABSTRACT. We construct locally Lindelöf scattered P-spaces (LLSP spaces, in short) with prescribed widths and heights under different set-theoretic assumptions.

We prove that there is an LLSP space of width ω_1 and height ω_2 and that it is relatively consistent with ZFC that there is an LLSP space of width ω_1 and height ω_3 . Also, we prove a stepping up theorem that, for every cardinal $\lambda \geq \omega_2$, permits us to construct from an LLSP space of width ω_1 and height λ satisfying certain additional properties an LLSP space of width ω_1 and height α for every ordinal $\alpha < \lambda^+$. Then, we obtain as consequences of the above results the following theorems:

(1) For every ordinal $\alpha < \omega_3$ there is an LLSP space of width ω_1 and height α .

(2) It is relatively consistent with ZFC that there is an LLSP space of width ω_1 and height α for every ordinal $\alpha < \omega_4$.

1. INTRODUCTION

The cardinal sequence of a scattered space is the sequence of the cardinalities of its Cantor-Bendixson levels. The investigation of the cardinal sequences of different classes of topological spaces is a classical problem of set theoretic topology. Many important results were proved in connection with the cardinal sequences of locally compact scattered (LCS, in short) spaces, see e.g. [1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16]. In [5] a complete characterization of the cardinal sequences of the 0-dimensional, of the regular, and of the Hausdorff spaces was given.

Recall that a topological space X is a *P*-space, if the intersection of every countable family of open sets in X is open in X. The aim of this paper is to start the systematic investigation of cardinal sequences of locally Lindelöf scattered P-spaces. We will see that several methods applied to LCS spaces can be applied here, but typically we should face more serious technical problems.

If X is a topological space and α is an ordinal, we denote by X^{α} the α -th Cantor-Bendixson derivative of X. Then, X is *scattered* if $X^{\alpha} = \emptyset$ for some

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ordinal α . Assume that X is a scattered space. We define the *height* of X by

$$ht(X) =$$
 the least ordinal α such that $X^{\alpha} = \emptyset$.

For $\alpha < \operatorname{ht}(X)$, we write $I_{\alpha}(X) = X^{\alpha} \setminus X^{\alpha+1}$. If $x \in I_{\alpha}(X)$, we say that α is the *level* of x and we write $\rho(x, X) = \alpha$, or simply $\rho(x) = \alpha$ if no confusion can occur. Note that $\rho(x) = \alpha$ means that x is an accumulation point of $I_{\beta}(X)$ for $\beta < \alpha$ but x is not an accumulation point of $X^{\alpha} = \bigcup \{I_{\beta}(X) : \beta \geq \alpha\}$. We define the *width* of X as

$$wd(X) = \sup\{|I_{\alpha}(X)| : \alpha < ht(X)\}.$$

If X is a scattered space, $x \in X$ and U is a neighbourhood of x, we say that U is a *cone on* x, if x is the only point in U of level $\geq \rho(x, X)$.

By an *LLSP space*, we mean a locally Lindelöf, scattered, Hausdorff P-space.

Proposition 1.1. An LLSP space is 0-dimensional.

Proof. By [13, Proposition 4.2(b)], a Lindelöf Hausdorff P-space X is normal, so a locally Lindelöf Hausdorff P-space is regular. Thus, by [13, Corollary 3.3], X is 0-dimensional. \Box

So, by Proposition 1.1 above, if X is an LLSP space, $x \in X$ and \mathbb{B}_x is a neighbourhood basis of x, we may assume that every $U \in \mathbb{B}_x$ is a Lindelöf clopen cone on x.

It was proved by Juhász and Weiss in [6] that for every ordinal $\alpha < \omega_2$ there is an LCS space of height α and width ω . Then, we will transfer this theorem to the setting of LLSP spaces, showing that for every ordinal $\alpha < \omega_3$ there is an LLSP space of height α and width ω_1 .

To obtain an LCS space of height ω_1 and width ω , in [6] Juhász and Weiss, using transfinite recursion, constructed a sequence $\langle X_{\alpha} : \alpha \leq \omega_1 \rangle$ of LCS spaces such that X_{α} had height α and width ω , and for $\alpha < \beta$, the space X_{α} was just the first α Cantor-Bendixson levels of X_{β} .

Since X_{α} is dense in $X_{\alpha+1}$, Juhász and Weiss had to guarantee that X_{α} is not compact. But it was automatic, because if $\alpha = \gamma + 1$, then X_{α} had a top infinite Cantor-Bendixson level, so X_{α} was not compact. If α is a limit ordinal, then the open cover $\{X_{\xi} : \xi < \alpha\}$ witnessed that X_{α} is not compact.

What happens if we try to adopt that approach for LLSP spaces? To obtain an LLSP space of height ω_2 and width ω_1 , we can try, using transfinite recursion, to construct a sequence $\langle X_{\alpha} : \alpha \leq \omega_2 \rangle$ of LLSP spaces such that X_{α} has height α and width ω_1 , and for $\alpha < \beta$, the space X_{α} is just the first α levels of X_{β} .

Since X_{α} is dense in $X_{\alpha+1}$, we have to guarantee that X_{α} is not closed in $X_{\alpha+1}$, in particular, X_{α} is not Lindelöf. (Since in a P-space, Lindelöf subspaces are closed.) However, in our case it is not automatic in limit steps, because the increasing countable union of open non-Lindelöf subspaces can be Lindelöf.

So some extra effort is needed to guarantee the non-Lindelöfness in limit steps.

Assume that κ is an uncountable cardinal and α is a non-zero ordinal. If X is an LLSP space such that $ht(X) = \alpha$ and $wd(X) = \kappa$, we say that X is a (κ, α) -LLSP space.

Then, we will also transfer the results proved in [3] and [11] on thin-tall spaces to the context of locally Lindelöf P-spaces, showing that Con(ZFC) implies Con(ZFC + "there is an (ω_1, α) -LLSP space for every ordinal $\alpha < \omega_4$ ").

2. Construction of an LLSP space of width ω_1 and height ω_2

By a *decomposition* of a set A of size ω_1 , we mean a partition of A into subsets of size ω_1 . In this section we will prove the following result.

Theorem 2.1. There is an (ω_1, ω_2) -LLSP space.

Proof. We construct an (ω_1, ω_2) -LLSP space whose underlying set is ω_2 . For every $\alpha < \omega_2$, we put $I_{\alpha} = (\omega_1 \cdot (\alpha + 1)) \setminus (\omega_1 \cdot \alpha)$, and for every ordinal $\xi < \omega_1$, we define the "column" $N_{\xi} = \{\omega_1 \cdot \mu + \xi : \mu < \omega_2\}$. Write $\xi \in N_{n(\xi)}$. Our aim is to construct, by transfinite induction on $\alpha < \omega_2$ an LLSP space X_{α} satisfying the following:

(1) X_{α} is an $(\omega_1, \alpha + 1)$ -LLSP space such that $I_{\beta}(X_{\alpha}) = I_{\beta}$ for every $\beta \leq \alpha$.

(2) For every $\xi < \omega_1, N_{\xi} \cap X_{\alpha}$ is a closed discrete subset of X_{α} .

(3) If $\beta < \alpha$ and $x \in X_{\beta}$, then a neighbourhood basis of x in X_{β} is also a neighbourhood basis of x in X_{α} .

For every $\alpha < \omega_2$ and $x \in I_{\alpha}$, in order to define the required neighbourhood basis \mathbb{B}_x of x in X_{α} , we will also fix a Lindelöf cone V_x of x in X_{α} such that the following holds:

- (4) $V_x \cap I_\alpha = \{x\}.$
- (5) $V_x = \bigcup \mathbb{B}_x$.

(6) There is a club subset C_x of ω_1 such that $\omega_1 \setminus C_x$ is unbounded in ω_1 and $V_x \cap \bigcup \{N_\nu : \nu \in C_x\} = \emptyset$.

We define X_0 as the set $I_0 = \omega_1$ with the discrete topology, and for $x \in I_0$ we put $V_x = \{x\}$ and $C_x = \{y \in \omega_1 : y \text{ is a limit ordinal } > x\}$. So, assume that $\alpha > 0$. If $\alpha = \beta + 1$ is a successor ordinal, we put $Z = X_\beta$. And if α is a limit ordinal, we define Z as the direct union of $\{X_\beta : \beta < \alpha\}$. So, the underlying set of the required space X_α is $Z \cup I_\alpha$. If $x \in Z$, then a basic neighbourhood of x in X_α is a neighbourhood of x in Z. Our purpose is to define a neighbourhood basis of each element of I_α . Let $\{x_\nu : \nu < \omega_1\}$ be an enumeration without repetitions of Z. By the induction hypothesis, for every $\xi < \omega_1$ there is a club subset C_{ξ} of ω_1 such that $\omega_1 \setminus C_{\xi}$ is unbounded in ω_1 and $V_{x_{\xi}} \cap \bigcup \{N_{\nu} : \nu \in C_{\xi}\} = \emptyset$. Let $C = \Delta \{C_{\xi} : \xi < \omega_1\}$, the diagonal intersection of the family $\{C_{\xi} : \xi < \omega_1\}$. As $V_{x_{\xi}} \cap \bigcup \{N_{\nu} : \nu \in C_{\xi}\} = \emptyset$, by the definition of C, for every $\xi < \omega_1$, $V_{x_{\xi}} \cap \bigcup \{N_{\nu} : \nu \in C\} \subset \bigcup \{N_{\nu} : \nu \leq \xi\}$, and clearly $\omega_1 \setminus C$ is unbounded in ω_1 . Then, we will define for every element $y \in I_{\alpha}$ a neighbourhood basis of y from a set V_y in such a way that for some final segment C' of C we will have that $V_y \cap \bigcup \{N_{\nu} : \nu \in C'\} = \emptyset$. We distinguish the following three cases:

Case 1. $\alpha = \beta + 1$ is a successor ordinal.

For each $\xi < \omega_1$ we take a Lindelöf clopen cone U_{ξ} on some u_{ξ} in Zas follows. We take $U_0 \subset V_{x_0}$ as a Lindelöf clopen cone on x_0 such that $(U_0 \setminus \{x_0\}) \cap N_0 = \emptyset$. Suppose that $\xi > 0$. Let u_{ξ} be the first element x_η in the enumeration $\{x_{\nu} : \nu < \omega_1\}$ of Z such that $u_{\xi} \notin \bigcup \{U_{\mu} : \mu < \xi\}$. Since $I_{\beta} \cap \bigcup \{U_{\mu} : \mu < \xi\} \subset \{u_{\mu} : \mu < \xi\}$, the element u_{ξ} is defined. Then, we choose $U_{\xi} \subset V_{x_{\eta}}$ as a Lindelöf clopen cone on u_{ξ} such that $U_{\xi} \cap \bigcup \{U_{\mu} : \mu < \xi\}$ $\xi\} = \emptyset$ and $(U_{\xi} \setminus \{u_{\xi}\}) \cap \bigcup \{N_{\nu} : \nu \leq \eta\} = \emptyset$. So, as $V_{x_{\eta}} \cap \bigcup \{N_{\nu} : \nu \in C\} \subset$ $\bigcup \{N_{\nu} : \nu \leq \eta\}$, we deduce that $(U_{\xi} \setminus \{u_{\xi}\}) \cap \bigcup \{N_{\nu} : \nu \in C\} = \emptyset$. And clearly, $\{U_{\xi} : \xi < \omega_1\}$ is a partition of Z. Let

$$A = \{\xi \in \omega_1 : u_{\xi} \in I_{\beta} \cap N_{\rho} \text{ for some } \rho \in \omega_1 \setminus C\}.$$

Since $I_{\beta} \subset \{u_{\xi} : \xi < \omega_1\}$, we have $|A| = \omega_1$. Let $\{A_{\xi} : \xi < \omega_1\}$ be a decomposition of A. Fix $\xi < \omega_1$. Let $y_{\xi} = \omega_1 \cdot \alpha + \xi$. Then, we define

$$V_{y_{\xi}} = \{y_{\xi}\} \cup \bigcup \{U_{\nu} : \nu \in A_{\xi}\}.$$

Note that since $\bigcup \{U_{\nu} : \nu \in A_{\xi}\} \cap \bigcup \{N_{\nu} : \nu \in C\} = \emptyset$, we infer that $V_{y_{\xi}} \cap \bigcup \{N_{\nu} : \nu \in C \text{ and } \nu > \xi\} = \emptyset$. Now, we define a basic neighbourhood of y_{ξ} in X_{α} as a set of the form

$$\{y_{\xi}\} \cup \bigcup \{U_{\nu} : \nu \in A_{\xi}, \nu \ge \zeta\}$$

where $\zeta < \omega_1$. Then, it is easy to check that conditions (1) - (6) hold.

Case 2. α is a limit ordinal of cofinality ω_1 .

Let $\langle \alpha_{\nu} : \nu < \omega_1 \rangle$ be a strictly increasing sequence of ordinals cofinal in α . For every $\xi < \omega_1$, we choose a Lindelöf clopen cone U_{ξ} on some point u_{ξ} in Z as follows. If ξ is not a limit ordinal, let u_{ξ} be the first element x_{η} in the enumeration $\{x_{\nu} : \nu < \omega_1\}$ of Z such that $u_{\xi} \notin \bigcup \{U_{\mu} : \mu < \xi\}$ and let $U_{\xi} \subset V_{x_{\eta}}$ be a Lindelöf clopen cone on u_{ξ} such that $U_{\xi} \cap \bigcup \{U_{\mu} : \mu < \xi\}$ and let $U_{\xi} \subset V_{x_{\eta}}$ be a Lindelöf clopen cone on u_{ξ} such that $U_{\xi} \cap \bigcup \{U_{\mu} : \mu < \xi\}$ and let $u_{\xi} < \xi\} = \emptyset$. Now, assume that ξ is a limit ordinal. Let $\nu < \omega_1$ be such that $\alpha_{\nu} > \sup\{\rho(u_{\mu}, Z) : \mu < \xi\}$. Then, we pick u_{ξ} as the first element x_{η} in the enumeration $\{x_{\nu} : \nu < \omega_1\}$ of Z such that $u_{\xi} \in I_{\alpha_{\nu}}(Z) \cap N_{\delta}$ for some $\delta \in \omega_1 \setminus C$ with $\delta > \xi$. Note that by the election of α_{ν} , we have that

 $u_{\xi} \notin \bigcup \{U_{\mu} : \mu < \xi\}$. Then, we choose $U_{\xi} \subset V_{x_{\eta}}$ as a Lindelöf clopen cone on u_{ξ} such that

$$U_{\xi} \cap \bigcup \{U_{\mu} : \mu < \xi\} = \emptyset$$
 and

$$(U_{\xi} \setminus \{u_{\xi}\}) \cap \bigcup \{N_{\nu} : \nu \leq \eta\} = \emptyset.$$

Then since $V_{x_{\eta}} \cap \bigcup \{N_{\nu} : \nu \in C\} \subset \bigcup \{N_{\nu} : \nu \leq \eta\}$ and $\delta \notin C$, we infer that $U_{\xi} \cap \bigcup \{N_{\nu} : \nu \in C\} = \emptyset$.

Now, let $\{A_{\xi} : \xi < \omega_1\}$ be a decomposition of the set of limit ordinals of ω_1 . Fix $\xi < \omega_1$. Let $y_{\xi} = \omega_1 \cdot \alpha + \xi$. Then, we define

$$V_{y_{\xi}} = \{y_{\xi}\} \cup \bigcup \{U_{\mu} : \mu \in A_{\xi}\}.$$

Clearly, $V_{y_{\xi}} \cap \bigcup \{N_{\nu} : \nu \in C, \nu > \xi\} = \emptyset$. Now, we define a basic neighbourhood of y_{ξ} in X_{α} as a set of the form

$$V_{y_{\xi}} \setminus \bigcup \{ U_{\nu} : \nu \in A_{\xi}, \nu < \zeta \}$$

where $\zeta < \omega_1$.

Note that the condition that $\delta > \xi$ in the election of u_{ξ} for ξ a limit ordinal is needed to assure that $N_{\xi} \cap X_{\alpha}$ is a closed discrete subset of X_{α} for $\xi < \omega_1$. So, conditions (1) – (6) hold.

Case 3. α is a limit ordinal of cofinality ω .

Let $\langle \alpha_n : n < \omega \rangle$ be a strictly increasing sequence of ordinals converging to α . Proceeding by transfinite induction on $\xi < \omega_1$, we construct a sequence $\langle u_n^{\xi} : n < \omega \rangle$ of points in Z and a sequence $\langle U_n^{\xi} : n < \omega \rangle$ such that each $U_n^{\xi} \subset V_{u_n^{\xi}}$ is a Lindelöf clopen cone on u_n^{ξ} as follows. Fix $\xi < \omega_1$, and assume that for $\mu < \xi$ the sequences $\langle u_n^{\mu} : n < \omega \rangle$ and $\langle U_n^{\mu} : n < \omega \rangle$ have been constructed. Let $C^* = \bigcap \{ C_{u_n^{\mu}} : \mu < \xi, n < \omega \}$. Note that C^* is a club subset of ω_1 , because it is a countable intersection of club subsets of ω_1 . Now since for every $\mu < \xi$ and $n < \omega$, we have that $V_{u_n^{\mu}} \cap \bigcup \{ N_{\nu} : \nu \in C_{u_n^{\mu}} \} = \emptyset$, we infer that

$$\bigcup \{ V_{u_n^{\mu}} : \mu < \xi, n < \omega \} \cap \bigcup \{ N_{\nu} : \nu \in C^* \} = \emptyset.$$

Hence, for every ordinal $\beta < \alpha$,

$$|I_{\beta} \setminus \bigcup \{V_{u_n^{\mu}} : \mu < \xi, n < \omega\}| = \omega_1.$$

Now, we construct the sequences $\langle u_n^{\xi} : n < \omega \rangle$ and $\langle U_n^{\xi} : n < \omega \rangle$ by induction on n. If n is even, let u_n^{ξ} be the first element x_η in the enumeration $\{x_\nu : \nu < \omega_1\}$ of Z such that $u_n^{\xi} \notin \bigcup \{U_k^{\mu} : \mu < \xi, k < \omega\} \cup \bigcup \{U_k^{\xi} : k < n\}$, and let $U_n^{\xi} \subset V_{x_n}$ be a Lindelöf clopen cone on u_n^{ξ} such that

$$U_n^{\xi} \cap (\bigcup \{U_k^{\mu} : \mu < \xi, k < \omega\} \cup \bigcup \{U_k^{\xi} : k < n\}) = \emptyset.$$

Now, suppose that n is odd. Let $k \in \omega$ be such that $\alpha_k > \sup\{\rho(u_m^{\xi}, Z) : m < n\}$. First, we pick \tilde{u}_n^{ξ} as the first element x_η in the enumeration $\{x_\nu : \nu < \omega_1\}$ of Z such that $\tilde{u}_n^{\xi} \in I_{\alpha_k+1}(Z) \cap N_{\zeta^*}$ for some $\zeta^* \in C^*$. So, $\tilde{u}_n^{\xi} \notin \bigcup\{U_m^{\mu} : \mu < \xi, m < \omega\} \cup \bigcup\{U_m^{\xi} : m < n\}$. Now, we choose $\tilde{U}_n^{\xi} \subset V_{x_\eta}$ as a Lindelöf clopen cone on \tilde{u}_n^{ξ} such that

$$\tilde{U}_n^{\xi} \cap \left(\bigcup \{ U_m^{\mu} : \mu < \xi, m < \omega \} \cup \bigcup \{ U_m^{\xi} : m < n \} \right) = \emptyset.$$

and

$$(\tilde{U}_n^{\xi} \setminus \{\tilde{u}_n^{\xi}\}) \cap \bigcup \{N_{\nu} : \nu \leq \eta\} = \emptyset.$$

Then as $\tilde{u}_n^{\xi} = x_\eta$ and $V_{x_\eta} \cap \bigcup \{N_\nu : \nu \in C\} \subset \bigcup \{N_\nu : \nu \leq \eta\}$, we infer that $(\tilde{U}_n^{\xi} \setminus \{\tilde{u}_n^{\xi}\}) \cap \bigcup \{N_\nu : \nu \in C\} = \emptyset$. However, note that if ζ is the ordinal such that $\tilde{u}_n^{\xi} \in N_{\zeta}$, it may happen that $\zeta \in C$. Then, we pick u_n^{ξ} as the first element x_ρ in the enumeration $\{x_\nu : \nu < \omega_1\}$ of Z such that $u_n^{\xi} \in \tilde{U}_n^{\xi} \cap I_{\alpha_k}(Z) \cap N_{\delta}$ for some $\delta > \xi$. Note that $\delta \notin C$, because $(\tilde{U}_n^{\xi} \setminus \{\tilde{u}_n^{\xi}\}) \cap \bigcup \{N_\nu : \nu \in C\} = \emptyset$. Now, we choose $U_n^{\xi} \subset \tilde{U}_n^{\xi} \cap V_{x_\rho}$ as a Lindelöf clopen cone on u_n^{ξ} such that

$$(U_n^{\xi} \setminus \{u_n^{\xi}\}) \cap \bigcup \{N_{\nu} : \nu \leq \rho\} = \emptyset.$$

Hence as $V_{x_{\rho}} \cap \bigcup \{N_{\nu} : \nu \in C\} \subset \bigcup \{N_{\nu} : \nu \leq \rho\}$ and $\delta \notin C$, we infer that $U_n^{\xi} \cap \bigcup \{N_{\nu} : \nu \in C\} = \emptyset$.

Now, let $\{A_{\xi} : \xi < \omega_1\}$ be a decomposition of ω_1 . Fix $\xi < \omega_1$. Let $y_{\xi} = \omega_1 \cdot \alpha + \xi$. Then, we define

$$V_{y_{\xi}} = \{y_{\xi}\} \cup \bigcup \{U_n^{\mu} : \mu \in A_{\xi}, n \text{ odd}\}.$$

As $\bigcup \{U_n^{\mu} : \mu \in A_{\xi}, n \text{ odd}\} \cap \bigcup \{N_{\nu} : \nu \in C\} = \emptyset$, we deduce that $V_{y_{\xi}} \cap \bigcup \{N_{\nu} : \nu \in C \text{ and } \nu > \xi\} = \emptyset$. Then, we define a basic neighbourhood of y_{ξ} in X_{α} as a set of the form

$$\{y_{\xi}\} \cup \bigcup \{U_n^{\mu} : \mu \in A_{\xi}, \mu \ge \zeta, n \text{ odd}\}$$

where $\zeta < \omega_1$. Now, it is easy to see that conditions (1) - (6) hold.

Then, we define the desired space X as the direct union of the spaces X_{α} for $\alpha < \omega_2$.

Remark 2.2. Note that by the construction carried out in the proof of Theorem 2.1, we have that

if $U \subset X$ is Lindelöf then $\{\xi : N_{\xi} \cap U \neq \emptyset\} \in NS(\omega_1)$.

3. A stepping up theorem

In this section, for every cardinal $\lambda \geq \omega_2$ we will construct from an (ω_1, λ) -LLSP space satisfying certain additional properties an (ω_1, α) -LLSP space for every ordinal $\alpha < \lambda^+$. As a consequence of this construction, we will be able to extend Theorem 2.1 from ω_2 to any ordinal $\alpha < \omega_3$. We need some preparation.

Definitions 3.1. (a) Assume that X is an LLSP space, $\beta + 1 < ht(X)$, $x \in I_{\beta+1}(X)$ and \mathbb{B}_x is a neighbourhood basis for x. We say that \mathbb{B}_x is *admissible*, if there is a pairwise disjoint family $\{U_{\nu} : \nu < \omega_1\}$ such that for every $\nu < \omega_1, U_{\nu}$ is a Lindelöf clopen cone on some point $x_{\nu} \in I_{\beta}(X)$ in such a way that \mathbb{B}_x is the collection of sets of the form

$$\{x\} \cup \bigcup \{U_{\nu} : \nu \ge \xi\},\$$

where $\xi < \omega_1$. Then, we will say that \mathbb{B}_x is the *admissible basis for x given* by $\{U_{\nu} : \nu < \omega_1\}$.

(b) Now, we say that X is an *admissible space* if for every $x \in X$ there is a neighbourhood basis \mathbb{B}_x such that for every successor ordinal $\beta + 1 < \operatorname{ht}(X)$ the following holds:

- (1) \mathbb{B}_x is an admissible basis for every point $x \in I_{\beta+1}(X)$,
- (2) if $x, y \in I_{\beta+1}(X)$ with $x \neq y$ and $\rho(x) = \rho(y)$, \mathbb{B}_x is given by $\{U_{\nu} : \nu < \omega_1\}$ and \mathbb{B}_y is given by $\{U'_{\nu} : \nu < \omega_1\}$, then for every $\nu, \mu < \omega_1$ we have $U_{\nu} \cap U'_{\mu} = \emptyset$.

Note that the space X constructed in the proof of Theorem 2.1 is admissible.

Definition 3.2. We say that an LLSP space X is *good*, if for every ordinal $\alpha < \operatorname{ht}(X)$ and every set $\{U_n : n \in \omega\}$ of Lindelöf clopen cones on points of X, the set $I_{\alpha}(X) \setminus \bigcup \{U_n : n \in \omega\}$ is uncountable.

Note that the space X constructed in the proof of Theorem 2.1 is good.

Assume that X is a good LLSP space. Then, we define the space X^* as follows. Its underlying set is $X \cup \{z\}$ where $z \notin X$. If $x \in X$, a basic neighbourhood of x in X^* is a neighbourhood of x in X. And a basic neighbourhood of z in X^* is a set of the form

$$X^* \setminus \bigcup \{ U_n : n \in \omega \}$$

where each U_n is a Lindelöf clopen cone on some point of X. Clearly, X^* is a Lindelöf scattered Hausdorff P-space with $ht(X^*) = ht(X) + 1$.

Theorem 3.3. Let $\lambda \geq \omega_2$ be a cardinal. Assume that there is a good (ω_1, λ) -LLSP space that is admissible. Then, for every ordinal $\alpha < \lambda^+$ there is a good (ω_1, α) -LLSP space.

So, we obtain the following consequence of Theorems 2.1 and 3.3.

Corollary 3.4. For every ordinal $\alpha < \omega_3$ there is a good (ω_1, α) -LLSP space.

Proof of Theorem 3.3. We may assume that $\lambda \leq \alpha < \lambda^+$. We proceed by transfinite induction on α . If $\alpha = \lambda$, the case is obvious. Assume that $\alpha = \beta + 1$ is a successor ordinal. Let Y be a good (ω_1, β) -LLSP space. For every $\nu < \omega_1$ let Y_{ν} be a P-space homeomorphic to Y^* in such a way that $Y_{\nu} \cap Y_{\mu} = \emptyset$ for $\nu < \mu < \omega_1$. Clearly, the topological sum of the spaces Y_{ν} $(\nu < \omega_1)$ is a good (ω_1, α) -LLSP space.

Now, assume that $\alpha > \lambda$ is a limit ordinal. Let $\theta = cf(\alpha)$. Note that since there is a good admissible (ω_1, λ) -LLSP space and $\theta \leq \lambda$, there is a good admissible LLSP space T of width ω_1 and height θ . Let $\{\alpha_{\xi} : \xi < \theta\}$ be a closed strictly increasing sequence of ordinals cofinal in α with $\alpha_0 = 0$. For every ordinal $\xi < \theta$, we put $J_{\xi} = \{\alpha_{\xi}\} \times \omega_1$. We may assume that the underlying set of T is $\bigcup \{J_{\xi} : \xi < \theta\}, I_{\xi}(T) = J_{\xi}$ for every $\xi < \theta$ and $I_{\theta}(T) = \emptyset$.

Fix a system of neighbourhood bases, $\{\mathbb{B}_x : x \in T\}$, which witnesses that T is admissible. Write $V_s = \bigcup \mathbb{B}_s$ for $s \in T$.

So, writing

 $T' = \{ s \in T : \rho(s, T) \text{ is a successor ordinal} \},\$

for each $s \in T$ with $\rho(s,T) = \xi + 1$, there is $D_s = \{d_{\zeta}^s : \zeta < \omega_1\} \in [I_{\xi}(T)]^{\omega_1}$ and for each $d \in D_s$ there is a Lindelöf cone U_d on d such that

$$\mathbb{B}_s = \{\{s\} \cup \bigcup_{\eta \le \zeta} U_{d^s_{\zeta}} : \eta < \omega_1\}.$$

In order to carry out the desired construction, we will insert an adequate LLSP space between $I_{\xi}(T)$ and $I_{\xi+1}(T)$ for every $\xi < \theta$. If $\xi < \theta$, we define $\delta_{\xi} = \text{o.t.}(\alpha_{\xi+1} \setminus \alpha_{\xi})$. We put $y_{\nu}^{\xi+1} = \langle \alpha_{\xi+1}, \nu \rangle$ for $\xi < \theta$ and $\nu < \omega_1$, and we put $D_{\nu}^{\xi} = \{x \in T : \rho(x,T) = \xi \text{ and } x \in V_{y_{\nu}^{\xi+1}}\} = D_{y_{\nu}^{\xi+1}}$. Since T is admissible, $D_{\nu}^{\xi} \cap D_{\mu}^{\xi} = \emptyset$ for $\nu \neq \mu$.

Now, by the induction hypothesis, for every point $y = y_{\nu}^{\xi+1}$ where $\xi < \theta$ and $\nu < \omega_1$ there is a Lindelöf scattered Hausdorff P-space Z_y of height $\delta_{\xi} + 1$ such that $I_0(Z_y) = D_{\nu}^{\xi}$, $|I_{\nu}(Z_y)| = \omega_1$ for $\nu < \delta_{\xi}$, $I_{\delta_{\xi}}(Z_y) = \{y\}$ and $Z_y \cap T = D_{\nu}^{\xi} \cup \{y\}$. Also, we assume that $Z_{y_{\nu}^{\xi+1}} \cap Z_{y_{\mu}^{\xi+1}} = \emptyset$ for $\nu \neq \mu$ and $(Z_{y_{\nu}^{\xi+1}} \setminus \{y_{\nu}^{\xi+1}\}) \cap (Z_{y_{\mu}^{\eta+1}} \setminus \{y_{\mu}^{\eta+1}\}) = \emptyset$ for $\xi \neq \eta$ and $\nu, \mu < \omega_1$.

Now, our aim is to define the desired (ω_1, α) -LLSP space Z. Its underlying set is

$$Z = T \cup \bigcup \{Z_y : y \in T'\}.$$

If V is a Lindelöf clopen cone on a point $z \in T$, we define

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$$V^* = V \cup \bigcup \{ (Z_y \setminus T) : y \in V \cap T' \}.$$

Observe that if $y \in V \cap T'$, then $Z_y \setminus V^* = D_y \setminus V$ and $D_y \setminus V$ is countable because T is admissible. So $Z_y \cap V^*$ is open in Z_y because Z_y is a P-space.

Now, assume that $x \in Z_s$ for some $s \in T'$. Then, if U is a Lindelöf clopen cone on x in Z_s , we define

$$U^{\sim} = U \cup \bigcup \{ (U_y)^* : y \in D_s \cap U \}.$$

Note that for every $s \in T'$ we have $(V_s)^* = (Z_s)^{\sim}$.

After that preparation we can define the bases of the points of Z. Suppose that $x \in Z = T \cup \bigcup \{Z_y : y \in T'\}$.

If $x \in T \setminus T'$, then let

 $\mathbb{B}_x^Z = \{ V^* : V \text{ is a Lindelöf clopen cone on } x \text{ in } T \}.$

If $x \in (Z \setminus T) \cup T'$, then pick first the unique $s \in T'$ such that $x \in Z_s \setminus I_0(Z_s)$, and let

 $\mathbb{B}_x^Z = \{ U^{\sim} : U \text{ is a Lindelöf clopen cone on } x \text{ in } Z_s \}.$

Claim 1. $\{\mathbb{B}_x^Z : x \in Z\}$ is a system of neighbourhood bases of a topology τ_Z .

Proof. Assume that $y \in W \in \mathbb{B}_x^Z$. We should show that $\mathbb{B}_y^Z \cap \mathcal{P}(W) \neq \emptyset$.

Assume first that $x \in T \setminus T'$, and so $W = V^*$ for some Lindelöf clopen clone V on x in T.

If $y \in T \setminus T'$, then $y \in V$ and so $S \subset V$ for some Lindelöf clopen clone S on y in T. Thus $y \in S^* \subset V^*$ and $S^* \in \mathbb{B}_y^Z$.

If $y \in (Z \setminus T) \cup T'$ then pick first the unique $s \in T'$ such that $y \in Z_s \setminus I_0(Z_s)$. Then $s \in V$ because otherwise $y \in V^*$ is not possible. So as we observed, $V^* \cap Z_s$ is open in Z_s . So let S be a Lindelöf clopen cone on y in Z_s with $S \subset V^* \cap Z_s$. Then $y \in S^{\sim} \subset V^*$ and $S^{\sim} \in \mathbb{B}_y^Z$.

Assume now that $x \in (Z \setminus T) \cup T'$, then pick first the unique $s \in T'$ such that $x \in Z_s \setminus I_0(Z_s)$. Then $W = U^{\sim}$ for some Lindelöf clopen cone U on x in Z_s .

If $y \in Z_s \setminus I_0(Z_s)$, then $S \subset U$ for some Lindelöf clopen clone S on y in Z_s , and so $S^{\sim} \in \mathbb{B}_y^Z$ and $S^{\sim} \subset U^{\sim}$.

If $y \notin Z_s \setminus I_0(Z_s)$, then $y \in (U_d)^*$ for some $d \in I_0(Z_s) \cap U$, and so there is $S \in \mathbb{B}_y^Z$ with $S \subset (U_d)^*$ using what we proved so far. Thus $S \subset U^\sim$ as well.

Claim 2. τ_Z is Hausdorff.

Proof. Assume that $\{x, y\} \in [Z]^2$. Let s and t be elements of T such that $x \in Z_s \setminus I_0(Z_s)$ if $x \notin T \setminus T'$ and s = x otherwise, and $y \in Z_t \setminus I_0(Z_t)$ if $y \notin T \setminus T'$ and t = y otherwise.

If $s \neq t$, consider disjoint Lindelöf clopen cones U and V on s and t in T respectively. Note that if $w \in U \cap T'$, then $Z_w \setminus T \subset U^*$ because $w \in U$, but $(Z_w \setminus T) \cap V^* = \emptyset$ because $w \notin V$, and analogously if $w \in V \cap T'$ then $Z_w \setminus T \subset V^*$ but $(Z_w \setminus T) \cap U^* = \emptyset$. So, U^* and V^* are disjoint open sets containing x and y respectively.

If s = t, then there are disjoint cones in Z_s on x and y, U and V, respectively. Then U^* and V^* are disjoint open sets containing x and y, respectively.

It is trivial from the definition that Z is a P-space because T is a P-space and the Z_s are P-spaces.

By transfinite induction on $\delta < \alpha$ it is easy to check that

$$I_{\delta}(Z) = \begin{cases} J_{\xi} & \text{if } \delta = \alpha_{\xi}, \\ \\ \bigcup \{I_{\eta}(Z_s\} : s \in I_{\xi+1}(T)\} & \text{if } \alpha_{\xi} < \delta = \alpha_{\xi} + \eta < \alpha_{\xi+1}, \end{cases}$$

so Z is scattered with height α and width ω_1 .

Claim 3. Z is locally Lindelöf.

Proof. Note that if $x \in T \setminus T'$ and $U^* \in \mathbb{B}_x^Z$, then for every $V^* \in \mathbb{B}_x^Z$ with $V^* \subset U^*$ we have that $U^* \setminus V^* = \bigcup \{W_n^* : n \in \nu\}$ where $\nu \leq \omega$ in such a way that each W_n is a Lindelöf clopen cone on some point $v_n \in T \cap U$ in T with $\rho(v_n, T) < \rho(x, T)$.

Also, if $x \in T' \cup (Z \setminus T)$ and $U^{\sim} \in \mathbb{B}_x^Z$ then for every $V^{\sim} \in \mathbb{B}_x^Z$ with $V^{\sim} \subset U^{\sim}$, if s is the element of T' with $x \in Z_s \setminus I_0(Z_s)$, we have that $U^{\sim} \setminus V^{\sim} = \bigcup \{U'_n : n \in \nu\}$ where $\nu \leq \omega$ in such a way that for every $n \in \nu$, either $U'_n = U^{\sim}_n$ where U_n is a Lindelöf clopen cone on some point $u_n \in Z_s \cap U$ in Z_s with $0 < \rho(u_n, Z_s) < \rho(x, Z_s)$ or $U'_n = U^*_n$ where U_n is a Lindelöf clopen cone on some point $u_n \in D_s \cap U$ in T.

Now, proceeding by transfinite induction on $\rho(x, Z)$, we can verify that if $x \in T \setminus T'$ and U is a Lindelöf clopen cone on x in T, then U^* is a Lindelöf clopen cone on x in Z, and that if $x \in Z_s \setminus I_0(Z_s)$ for some $s \in T'$ and U is a Lindelöf clopen cone on x in Z_s , then U^\sim is a Lindelöf clopen cone on x in Z. Therefore, Z is locally Lindelöf.

Claim 4. Z is good.

Proof. Let $\delta < \alpha = ht(Z)$ and let $\{W_n : n \in \omega\}$ be a family of Lindelöf cones in Z. Since every W_n is covered by countably many Lindelöf cones from the basis, we can assume that $W_n \in \mathbb{B}_{x_n}^Z$ for some $x_n \in Z$ for each $n \in \omega$. For each n pick $y_n \in T$ such that $y_n = x_n$ if $x_n \in T$ and $x_n \in Z_{y_n}$ otherwise.

Then $W_n \subset W'_n$ for some $W'_n \in \mathbb{B}^Z_{y_n}$, so we can assume that $\{x_n : n \in \omega\} \subset T$. We can also assume that if $x_n \in T'$, then W_n is as large as possible, i.e. $W_n = Z_{x_n}^{\sim} = (V_{x_n})^*$.

If $x_n \in T \setminus T'$, then $W_n = S_n^*$ for some Lindelöf cone S_n on x_n in T.

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If $\delta = \alpha_{\xi}$ for some ξ , then $I_{\delta}(Z) \cap W_n = I_{\delta}(Z) \cap V_{x_n}$ if $x_n \in T'$ and $I_{\delta}(Z) \cap W_n = I_{\delta}(Z) \cap S_n$ if $x_n \in T \setminus T'$. So $I_{\delta}(Z) \setminus \bigcup_{n \in \omega} W_n$ is uncountable because T is good. Assume that $\alpha_{\xi} < \delta < \alpha_{\xi+1}$ and let $\delta = \alpha_{\xi} + \eta$. Pick $s \in I_{\alpha_{\xi+1}}(Z) \setminus \bigcup_{n \in \omega} W_n$. Then $Z_s \setminus \bigcup_{n \in \omega} W_n \supset I_{\eta}(Z_s)$, and so $I_{\delta}(Z) \setminus \bigcup_{n \in \omega} W_n \supset I_{\eta}(Z_s)$, and hence $I_{\delta}(Z) \setminus \bigcup_{n \in \omega} W_n$ is uncountable.

Thus, the space Z is as required.

4. Cardinal sequences of length $< \omega_4$

In this section, we will show the following result.

Theorem 4.1. If V=L, then there is a cardinal-preserving partial order \mathbb{P} such that in $V^{\mathbb{P}}$ there is an (ω_1, α) -LLSP space for every ordinal $\alpha < \omega_4$.

If $S = \bigcup \{ \{\alpha\} \times A_{\alpha} : \alpha < \eta \}$ where η is a non-zero ordinal and each A_{α} is a non-empty set of ordinals, then for every $s = \langle \alpha, \xi \rangle \in S$ we write $\pi(s) = \alpha$ and $\zeta(s) = \xi$.

The following notion is a refinement of a notion used implicitly in [3].

Definition 4.2. We say that $\mathbb{S} = \langle S, \preceq, i \rangle$ is an *LLSP poset*, if the following conditions hold:

- (P1) $\langle S, \preceq \rangle$ is a partial order with $S = \bigcup \{S_{\alpha} : \alpha < \eta\}$ for some non-zero ordinal η such that each $S_{\alpha} = \{\alpha\} \times A_{\alpha}$ where A_{α} is a non-empty set of ordinals.
- (P2) If $s \prec t$ then $\pi(s) < \pi(t)$.
- (P3) If $\alpha < \beta < \eta$ and $t \in S_{\beta}$, then $\{s \in S_{\alpha} : s \prec t\}$ is uncountable.
- (P4) If $\gamma < \eta$ with $cf(\gamma) = \omega$, $t \in S_{\gamma}$ and $\langle t_n : n \in \omega \rangle$ is a sequence of elements of S such that $t_n \prec t$ for every $n \in \omega$, then for every ordinal $\beta < \gamma$ the set $\{s \in S_{\beta} : s \prec t \text{ and } s \not\preceq t_n \text{ for } n \in \omega\}$ is uncountable.
- (P5) $i: [S]^2 \to [S]^{\leq \omega}$ such that for every $\{s, t\} \in [S]^2$ the following holds: (a) If $v \in i\{s, t\}$ then $v \preceq s, t$.
 - (b) If $u \leq s, t$, then there is $v \in i\{s, t\}$ such that $u \leq v$.

If there is an uncountable cardinal λ such that $|S_{\alpha}| = \lambda$ for $\alpha < \eta$, we will say that $\langle S, \leq, i \rangle$ is a (λ, η) -LLSP poset.

If $S = \langle S, \leq, i \rangle$ is an LLSP poset with $S = \bigcup \{S_{\alpha} : \alpha < \eta\}$, we define its associated LLSP space X = X(S) as follows. The underlying set of X(S) is S. If $x \in S$ we write $U(x) = \{y \in S : y \leq x\}$. Then, for every $x \in S$ we define a basic neighbourhood of x in X as a set of the form $U(x) \setminus \bigcup \{U(x_n) : n \in \omega\}$ where each $x_n \prec x$. It is easy to check that X is a locally Lindelöf scattered Hausdorff P-space (see [1] for a parallel proof). And by conditions (P3) and (P4) in Definition 4.2, we infer that $ht(X) = \eta$ and $I_{\alpha}(X) = S_{\alpha}$ for every $\alpha < \eta$.

In order to prove Theorem 4.1, first we will construct an (ω_1, ω_3) -LLSP space X in a generic extension by means of an ω_1 -closed ω_2 -c.c. forcing, by using an argument similar to the one given by Baumgartner and Shelah in [3].

Recall that a function $F : [\omega_3]^2 \to [\omega_3]^{\leq \omega_1}$ has property Δ , if $F\{\alpha, \beta\} \subset \min\{\alpha, \beta\}$ for every $\{\alpha, \beta\} \in [\omega_3]^2$ and for every set D of countable subsets of ω_3 with $|D| = \omega_2$ there are $a, b \in D$ with $a \neq b$ such that for every $\alpha \in a \setminus b, \beta \in b \setminus a$ and $\tau \in a \cap b$ the following holds:

- (a) if $\tau < \alpha, \beta$ then $\tau \in F\{\alpha, \beta\}$,
- (b) if $\tau < \beta$ then $F\{\alpha, \tau\} \subset F\{\alpha, \beta\}$,
- (c) if $\tau < \alpha$ then $F\{\tau, \beta\} \subset F\{\alpha, \beta\}$.

By a result due to Velickovic, it is known that \Box_{ω_2} implies the existence of a function $F : [\omega_3]^2 \to [\omega_3]^{\leq \omega_1}$ satisfying property Δ (see [17, Chapter 7 and Lemma 7.4.9.], for a proof).

Proof of Theorem 4.1. Let $F : [\omega_3]^2 \to [\omega_3]^{\leq \omega_1}$ be a function with property Δ . First, we construct by forcing an (ω_1, ω_3) -LLSP poset. Let $S = \bigcup \{S_\alpha : \alpha < \omega_3\}$ where $S_\alpha = \{\alpha\} \times \omega_1$ for each $\alpha < \omega_3$. S will be the underlying set of the required poset. We define P as the set of all $p = \langle x_p, \preceq_p, i_p \rangle$ satisfying the following conditions:

- (1) x_p is a countable subset of S.
- (2) \leq_p is a partial order on x_p such that:
 - (a) if $s \prec_p t$ then $\pi(s) < \pi(t)$,
 - (b) if $s \prec_p t$ and $\pi(t)$ is a successor ordinal $\beta + 1$, then there is $v \in S_{\beta}$ such that $s \preceq_p v \prec_p t$.
- (3) $i_p: [x_p]^2 \to [x_p]^{\leq \omega}$ satisfying the following conditions:
 - (a) if $s \prec_p t$ then $i_p\{s,t\} = \{s\},\$
 - (b) if $s \not\leq_p t$ and $\pi(s) < \pi(t)$, then $i_p\{s,t\} \subset \bigcup \{S_\alpha : \alpha \in F\{\pi(s), \pi(t)\}\},\$
 - (c) if $s, t \in x_p$ with $s \neq t$ and $\pi(s) = \pi(t)$ then $i_p\{s, t\} = \emptyset$,
 - (d) $v \leq_p s, t$ for all $v \in i_p\{s, t\}$,
 - (e) for every $u \leq_p s, t$ there is $v \in i_p\{s, t\}$ such that $u \leq_p v$.

If $p, q \in P$, we write $p \leq q$ iff $x_q \subset x_p$, $\preceq_p \upharpoonright x_q = \preceq_q$ and $i_p \upharpoonright [x_q]^2 = i_q$. We put $\mathbb{P} = \langle P, \leq \rangle$.

Clearly, \mathbb{P} is ω_1 -closed. And since the function F has property Δ , it is easy to check that \mathbb{P} has the ω_2 -c.c., and so \mathbb{P} preserves cardinals.

Now, let G be a P-generic filter. We write $\leq = \bigcup \{ \leq_p : p \in G \}$ and $i = \bigcup \{ i_p : p \in G \}$. It is easy to see that $S = \bigcup \{ x_p : p \in G \}$ and $\leq i$ is a partial order on S. Then, we have that $\langle S, \leq, i \rangle$ is an (ω_1, ω_3) -LLSP poset. For this, note that conditions (P1), (P2), (P5) in Definition 4.2 are obvious, and condition (P3) follows from a basic density argument. So, we verify condition (P4). For every $t \in S$ such that $\gamma = \pi(t)$ has cofinality ω , for every sequence $\langle t_n : n \in \omega \rangle$ of elements of S, for every ordinal $\beta < \gamma$ and for every ordinal $\xi < \omega_1$ let

 $D_{t,\{t_n:n\in\omega\},\beta,\xi} = \{q \in P : \{t\} \cup \{t_n : n \in \omega\} \subset x_q \text{ and either } (t_n \not\prec_q t \text{ for some } n \in \omega) \text{ or } (t_n \prec_q t \text{ for every } n \in \omega \text{ and there is } y \in S_\beta \cap x_q \text{ with } \zeta(y) > \xi \text{ such that } y \prec_q t \text{ and } y \not\preceq_q t_n \text{ for every } n \in \omega) \}.$

Since \mathbb{P} is ω_1 -closed, we have that $D_{t,\{t_n:n\in\omega\},\beta,\xi} \in V$. Then, consider $p = \langle x_p, \preceq_p, i_p \rangle \in P$. We define a $q \in D_{t,\{t_n:n\in\omega\},\beta,\xi}$ such that $q \leq p$. Without loss of generality, we may assume that $t \in x_p$. We distinguish the following cases.

Case 1. $t_n \notin x_p$ for some $n \in \omega$.

We define $q = \langle x_q, \preceq_q, i_q \rangle$ as follows: (a) $x_q = x_p \cup \{t_n : n \in \omega\}$, (b) $\prec_q = \prec_p$, (c) $i_q\{x, y\} = i_p\{x, y\}$ if $\{x, y\} \in [x_p]^2$, $i_q\{x, y\} = \emptyset$ otherwise.

Case 2. $t_n \in x_p$ for every $n \in \omega$.

If $t_n \not\prec_p t$ for some $n \in \omega$, we put q = p. So, assume that $t_n \prec_p t$ for all $n \in \omega$. Let $u \in S_\beta \setminus x_p$ be such that $\zeta(u) > \xi$. We define $q = \langle x_q, \preceq_q, i_q \rangle$ as follows:

- (a) $x_q = x_p \cup \{u\},\$
- (b) $\prec_q = \prec_p \cup \{ \langle u, v \rangle : t \preceq_p v \},$

(c) $i_q\{x, y\} = i_p\{x, y\}$ if $\{x, y\} \in [x_p]^2$, $i_q\{x, y\} = \{x\}$ if $x \prec_q y$, $i_q\{x, y\} = \{y\}$ if $y \prec_q x$, $i_q\{x, y\} = \emptyset$ otherwise.

So, $D_{t,\{t_n:n\in\omega\},\beta,\xi}$ is dense in \mathbb{P} , and hence condition (P4) holds. Let $X = X(\langle S, \leq, i \rangle)$. For every $x \in S$, we write $U(x) = \{y \in S : y \leq x\}$. By conditions (2)(b) and (3)(c) in the definition of P, we see that if $x \in S_{\beta+1}$ for some $\beta < \omega_3$, then x has an admissible basis in X given by $\{U(y) : y \prec x, \pi(y) = \beta\}$. Thus, X is an admissible space. And clearly, X is good. So, by Theorem 3.3, we can construct from the space X an (ω_1, α) -LLSP space for every ordinal $\omega_3 \leq \alpha < \omega_4$.

Now, assume that κ is an uncountable regular cardinal. Recall that a topological space X is a P_{κ} -space, if the intersection of any family of less than κ open subsets of X is open in X. And we say that X is κ -compact, if every open cover of X has a subcover of size less $< \kappa$. By an SP_{κ} space we mean a scattered Hausdorff P_{κ} -space. Then, we want to remark that by using arguments that are parallel to the ones given in the proofs of the above theorems, we can show the following more general results:

(1) For every uncountable regular cardinal κ and every ordinal $\alpha < \kappa^{++}$, there is a locally κ -compact SP_{κ} space X such that $\operatorname{ht}(X) = \alpha$ and $\operatorname{wd}(X) = \kappa$.

(2) If V=L and κ is an uncountable regular cardinal, then there is a cardinal-preserving partial order \mathbb{P} such that in $V^{\mathbb{P}}$ we have that for every

ordinal $\alpha < \kappa^{+++}$ there is a locally κ -compact SP_{κ} space X such that $ht(X) = \alpha$ and $wd(X) = \kappa$.

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