The adiabatic theorem and the Landau-Zener model

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Abstract: The adiabatic theorem is exposed and illustrated with an example. Concerning non-adiabatic processes, two different proofs of the Landau-Zener (LZ) formula are studied, and an analogue to the formula is provided for a specific 3×3 case. Lastly, an approximate bound to non-standard interaction (NSI) coefficients of solar neutrino oscillations is derived from the LZ formula.

I. INTRODUCTION

When studying a quantum system, one of the most interesting questions is the computation of transition probabilities. This problem is easily overcome when it is possible to solve the Schrödinger equation analytically or when the hamiltonian is time-independent, but several approximations can be used otherwise.

A first case to consider is that in which the process is adiabatic, i.e., the hamiltonian evolves slowly. In other words, the typical evolution time T_i is far smaller than the time scale T_e of the hamiltonian adiabatic variation.

The adiabatic theorem states, quoting [1], that "a physical system remains in its instantaneous eigenstate if a given perturbation is acting on it slowly enough and if there is a gap between the eigenvalue and the rest of the hamiltonian's spectrum". In fact, if we label $|\psi_n(t)\rangle$, $E_n(t)$ the instantaneous time-dependent eigenstate and eigenvalue, we will find that the state of the system is

$$|\psi(t)\rangle = e^{i(\theta_n(t)+\gamma_n(t))}\,|\psi_n(t)\rangle\,, \tag{1}$$

where $\theta_n(t)$ and $\gamma_n(t)$ are

$$\begin{aligned}\theta_n(t) &\equiv -\frac{1}{\hbar}\int_{t_0}^{t} E_n(\tilde{t})d\tilde{t},\\ \gamma_n(t) &\equiv i\int_{t_0}^{t}\langle\psi_n(\tilde{t})|\partial_{\tilde{t}}\psi_n(\tilde{t})\rangle\,d\tilde{t}.\end{aligned} \tag{2}$$

$\gamma_n(t)$ gains particular interest when the hamiltonian returns to its original form at the end of the process, as in this case it depends on the path taken in the parameter space $\vec{R}(t)$ but not the time used to run through it. [2] In this scenario, it is known as Berry's phase and it becomes

$$\gamma_n = i\oint \langle\psi_n \mid \vec{\nabla}_R\psi_n\rangle\cdot d\vec{R}. \tag{3}$$

It is interesting to provide an example for which the veracity of the theorem can be checked directly.

II. EXAMPLE: A SPIN UNDER A SLOWLY-CHANGING MAGNETIC FIELD

We will now consider the case of an electron spin under a magnetic field that follows a small circle on a sphere,

$$\vec{B}(t) = B_0[\sin\alpha\cos(\omega t)\hat{x} + \sin\alpha\sin(\omega t)\hat{y} + \cos\alpha\hat{z}], \tag{4}$$

where α is the colatitude of said small circle and B_0 is the radius of the sphere. Here, the hamiltonian is

$$\hat{H} = \mu_\mathrm{B}\vec{B}\cdot\vec{\sigma} = \mu_\mathrm{B}B_0\begin{pmatrix}\cos\alpha & e^{-i\omega t}\sin\alpha\\ e^{i\omega t}\sin\alpha & -\cos\alpha\end{pmatrix}, \tag{5}$$

where μ_B is the Bohr magneton. Its instantaneous eigenstates are the up and down spinors in the direction of the field. If we suppose the starting state is $|\psi(0)\rangle = |\uparrow(0)\rangle$, parallel to the initial magnetic field, the solution to the Schrödinger equation can be found in [2] and is, in the basis of the instantaneous eigenvectors,

$$|\psi(t)\rangle = \begin{pmatrix}\left[\cos\frac{\lambda t}{2\hbar} - i\frac{2\mu_\mathrm{B}B_0-\hbar\omega\cos\alpha}{\lambda}\sin\frac{\lambda t}{2\hbar}\right]e^{-i\omega t/2}\\ i\frac{\hbar\omega}{\lambda}\sin\alpha\sin\frac{\lambda t}{2\hbar}e^{i\omega t/2}\end{pmatrix}, \tag{6}$$

with $\lambda \equiv \sqrt{(\hbar\omega)^2 + (2\mu_\mathrm{B}B_0)^2 - 4\hbar\omega\mu_\mathrm{B}B_0\cos\alpha}$. Therefore, the transition probability to spin down is

$$P = |\,\langle\psi(t)\,|\downarrow(t)\rangle\,|^2 = \left[\frac{\hbar\omega}{\lambda}\sin\alpha\sin\frac{\lambda t}{2\hbar}\right]^2, \tag{7}$$

which vanishes for $\omega \ll 2\mu_B B_0/\hbar$, the case of a slow transition, as predicted by the adiabatic theorem. That is, the spin expectation value closely follows the precession of the field.

III. THE LANDAU-ZENER MODEL

Another case in which the transition probabilities are easy to compute is that in which the hamiltonian variation is linear and across $t \in (-\infty, \infty)$. To study it, we now consider the same electron but under magnetic field $\vec{B}(t) = t\vec{\lambda} + \vec{\beta}$. We will then pick the $\hat{z}$ axis to point along $\vec{\lambda}$ and the $\hat{x}$ axes to point in the direction of the normal component to the linear variation. Then, we have $\vec{B}(t) = (\beta_x, 0, \lambda t + \beta_z)$ (taking $\lambda \equiv \vec{\lambda}\cdot\hat{z}$, $\beta_x \equiv \vec{\beta}\cdot\hat{x}$ and $\beta_z \equiv \vec{\beta}\cdot\hat{z}$). Then, we have

$$\hat{H} = \mu_\mathrm{B}\vec{B}\cdot\vec{\sigma} = \mu_\mathrm{B}\begin{pmatrix}\lambda t+\beta_z & \beta_x\\ \beta_x & -\lambda t-\beta_z\end{pmatrix}. \tag{8}$$

Now, we will perform a translation of the origin of t as $\tilde{t} = t + \frac{\beta_z}{\lambda}$, which gets the simplification

$$\hat{H} = \mu_\mathrm{B}\begin{pmatrix}\lambda\tilde{t} & \beta_x\\ \beta_x & -\lambda\tilde{t}\end{pmatrix} \equiv \frac{1}{2}\begin{pmatrix}r\tilde{t} & s\\ s & -r\tilde{t}\end{pmatrix}. \tag{9}$$

where we have defined r and s accordingly. We will assume (without loss of generality) that we have taken the sense of the axes so that both r and s are positive and will omit the tilde in $\tilde{t}$.

This form of the hamiltonian, which is known as the Landau-Zener hamiltonian, can be obtained in much the same way for any 2 dimensional linearly-varying system, and can be used as an approximation for some other cases which will be discussed later.

What makes this form of the hamiltonian useful is the fact that the probability of transition from one eigenstate to the other, that is, the probability of a non-adiabatic transition, is easily expressed as

$$P_{\mathrm{LZ}} = e^{-\pi s^2/2r\hbar}, \tag{10}$$

which is known as the Landau-Zener formula. The fact that time travels through the real line makes the time translation we have performed irrelevant.

The formula returns to the adiabatic case (that is, $P_{\mathrm{LZ}} \to 0$) when $s/r \gg \hbar/s$. Those are , for this hamiltonian, the time scales earlier defined: $T_{\mathrm{i}} = \hbar/s$, $T_{\mathrm{e}} = s/r$.

Since the contribution of s can be neglected for large $|t|$, this is a good model for systems where the interaction is approximately linear when the eigenvalues are closer. Examples of these systems include collisions of molecules or some cases of neutrino oscillations in matter.

A. Analysis of the hamiltonian and Schrödinger equation

Before proving the formula, it is useful to perform a brief discussion on the hamiltonian. Its eigenvalues are $E_\pm = \pm\frac{1}{2}\sqrt{r^2t^2 + s^2}$, so they have asymptotes $\pm rt/2$ and minimal separation s. This situation is usually referred to as an avoided crossing.

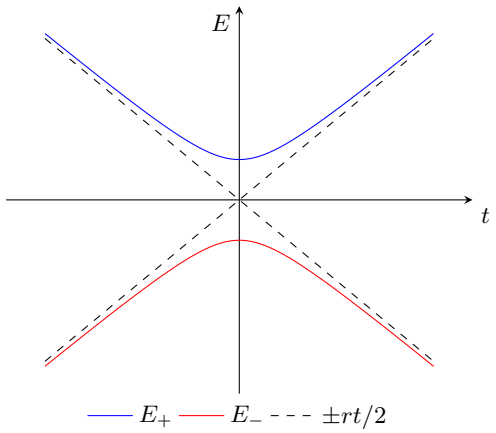

FIG. 1: Spectrum of the hamiltonian with respect to time.

However, it is crucial to notice that the eigenvectors corresponding to these eigenvalues are swapped along t: we find that, for $t \to -\infty$, $|E_+\rangle = \binom{0}{1}$ and $|E_-\rangle = \binom{1}{0}$; while, for $t \to \infty$, $|E_+\rangle = \binom{1}{0}$ and $|E_-\rangle = \binom{0}{1}$. Hence, the adiabatic paths are those that go from $\binom{0}{1}$ to $\binom{1}{0}$ or viceversa, and the non-adiabatic paths are those which go from $\binom{0}{1}$ to $\binom{1}{0}$ and $\binom{1}{0}$ to $\binom{1}{0}$.

Going back to the problem, we write the solution to the Schrödinger equation as $|\psi(t)\rangle = \binom{a(t)}{b(t)}$. Then, the equation becomes

$$i\hbar \begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} rt & s \\ s & -rt \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \tag{11}$$

a set of coupled equations which we can decouple to get

$$4\hbar^2\ddot{a} + (2i\hbar r + s^2 + r^2t^2)a = 0. \tag{12}$$

From here, we will follow two different routes towards the desired result. It is interesting to study both of them as the first one will yield the LZ formula directly whereas the second one is indirect but much more transparent.

B. Proof of the formula by direct solution

The first one, which goes after [3], is a direct attack on the equation, as its solutions can be expressed in terms of Whittaker functions [4] as a linear combination of

$$\begin{aligned} a_1(t) &= t^{-1/2} W_{(2r\hbar - is^2)/8r\hbar, -1/4}(irt^2/2\hbar), \\ a_2(t) &= t^{-1/2} W_{(-2r\hbar + is^2)/8r\hbar, -1/4}(-irt^2/2\hbar). \end{aligned} \tag{13}$$

These have limit behaviour, for $|rt^2/2\hbar| \gg 1$,

$$\begin{aligned} a_1(t) &\propto t^{-is^2/4r\hbar} e^{-irt^2/4\hbar}, \\ a_2(t) &\propto t^{-1+is^2/4r\hbar} e^{irt^2/4\hbar}. \end{aligned} \tag{14}$$

By writing $a = Aa_1 + Ba_2$, for some $A, B \in \mathbb{C}$, we can use the differential equation to obtain, by observing the limit, $b = Ba_1 + Aa_2$. Then, we impose the initial condition $\lim_{t\to\infty} a(t) = 1$, $\lim_{t\to\infty} b(t) = 0$ and, from the second clause, we get $B = 0$.

To finish the proof, we put $t = e^{-i\pi}|t|$ for negative time (it should be noted that the usage of any other branch of the argument yields the same result). Hence, we have $\lim_{t\to-\infty} a(t) = Ae^{\pi s^2/4r\hbar}$ and $\lim_{t\to\infty} a(t) = A$ from where, imposing the first clause of the initial condition, we get $A = e^{-\pi s^2/4r\hbar}$. Thus follows the desired formula,

$$P_{\mathrm{LZ}} = \lim_{t\to\infty} |a(t)|^2 = |A|^2 = e^{-\pi s^2/2r\hbar}. \tag{15}$$

As for the case in which the initial state is $\binom{0}{1}$, the procedure is completely analogous.

C. Proof of the formula by contour integration

The second route consists in a contour integral approach following [5]. The first step is to obtain the behaviour of $\dot{a}/a$ as $t \to \infty$. In this limit, as we saw earlier,

the vectors $\binom{1}{0}$ and $\binom{0}{1}$ become eigenvectors of $\hat{H}$, hence $|a|$ becomes constant. Therefore, for large t, we may write $a(t) \simeq |a|e^{-i\phi(t)}$, where $\phi(t) \in \mathbb{R}$ is the phase. The differential equation (12) then becomes, in this limit,

$$4\hbar^2\dot{\phi}^2 - s^2 - r^2t^2 + i(4\hbar^2\ddot{\phi} - 2\hbar r) = 0. \tag{16}$$

Then, separating the real and imaginary parts, we get

$$\begin{cases} \dot{\phi} = \pm\frac{\sqrt{s^2+r^2t^2}}{2\hbar} = \pm\frac{r|t|}{2\hbar}\sqrt{1+\left(\frac{s}{rt}\right)^2}, \\ \ddot{\phi} = \frac{r}{2\hbar}. \end{cases} \tag{17}$$

Since the second derivative is positive, though, we know we have to take the sign for $\dot{\phi}$ which makes it increase in time, that is, the positive branch.

Now, in this limit, the approximation we have used yields $\dot{a}/a \simeq= -i\dot{\phi}(t)$, and by getting the first-order Taylor of the root for large t we get

$$\frac{\dot{a}(t)}{a(t)} \simeq \frac{rt}{2\hbar} + \frac{s^2}{4\hbar rt}. \tag{18}$$

The last step in the proof is to integrate $\dot{a}/a$ along the real line using the residue theorem, assuming $\dot{a}/a$ extends analytically to the complex plane. We choose a sunrise circuit which gets, writing R for its radius,

$$\int_{-R}^{R} \frac{\dot{a}(x)}{a(x)}dx + iR\int_0^\pi \frac{\dot{a}(Re^{i\theta})}{a(Re^{i\theta})}e^{i\theta}d\theta = 0. \tag{19}$$

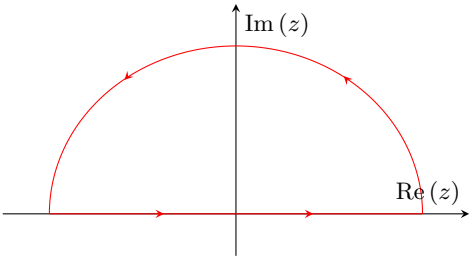

FIG. 2: The sunrise circuit we are using.

The first one of the integrals is precisely $\ln\frac{a(R)}{a(-R)}$, whereas for the second one we will use (18) in the limit case,

$$\lim_{R\to\infty} iR\int_0^\pi \frac{\dot{a}(Re^{i\theta})}{a(Re^{i\theta})}e^{i\theta}d\theta = \lim_{R\to\infty}\int_0^\pi \left(\frac{rR^2e^{i2\theta}}{2\hbar} + \frac{s^2}{4\hbar r}\right)d\theta = \frac{\pi s^2}{4\hbar r}. \tag{20}$$

Therefore, the residue theorem is written, in the limit case and given that $a(t)$ is finite along the real line,

$$\ln\frac{\lim_{t\to\infty} a(t)}{\lim_{t\to-\infty} a(t)} = \frac{-\pi s^2}{4\hbar r}. \tag{21}$$

From here, using the initial condition $\lim_{t\to-\infty} a(t) = 1$, we will get back the desired result,

$$P_{\mathrm{LZ}} = \lim_{t\to\infty}|a(t)|^2 = \lim_{t\to-\infty} e^{-\pi s^2/2r\hbar}|a(t)|^2 = e^{-\pi s^2/2r\hbar}. \tag{22}$$

D. Critique of the LZ model

While the Landau-Zener model has been used as an approximation for a wide variety of systems, its usage must be taken with caution, as noticed in [6], since in some limits the numerical results for the transition probabilities do not agree with the model.

The analysis in said source implies that the Landau-Zener model for the collision of two atomic systems yields good results for low coupling strengths in the limit of low velocity of the nuclei, but does not in other regimes.

IV. THE 3×3 SIMPLE CASE

One interesting extension of the Landau-Zener formula is the 3-level case. This, however, proves very challenging for the general 3×3 linear hamiltonian, which prompts us to study only a particular case.

We will start with the same system we used for the 2-level case: the electron under a magnetic field linear in time. This time, though, we will replace the electron with a charged spin 1 particle. We then take a field $\vec{B}(t) = t\vec{\lambda} + \vec{\beta}$ and perform the same axis election and variable as earlier. Finally, we define parameters r and s analogously to the 2-level case but scaled as to get

$$\hat{H} = \begin{pmatrix} rt & s/\sqrt{2} & 0 \\ s/\sqrt{2} & 0 & s/\sqrt{2} \\ 0 & s/\sqrt{2} & -rt \end{pmatrix}. \tag{23}$$

The reason for this choice will become apparent later on.

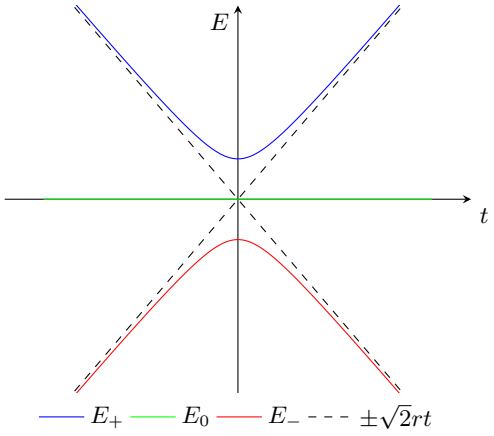

FIG. 3: Spectrum of the 3-level hamiltonian with respect to time.

In this hamiltonian, we find eigenvalues $E_0 = 0$ and $E_\pm = \pm\sqrt{(4r^2t^2 + s^2)/2}$. The latter have assimptotes $\pm\sqrt{2}rt$ and we observe a minimal separation of the eigenvalues of $E_+(0) - E_0(0) = E_0(0) - E_-(0) = s/\sqrt{2}$.

Hence, we have the same avoided crossing situation of the 2-level hamiltonian, with an added third-level in between. The same swap of the eigenvectors is observed for the E_+ and E_- levels.

Again, we write the solution to the Schrödinger equation as $|\psi(t)\rangle = \begin{pmatrix} a(t) & b(t) & c(t) \end{pmatrix}^{\mathrm{T}}$, in this base. Then, the equation becomes

$$i\hbar \begin{pmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} rt & s/\sqrt{2} & 0 \\ s/\sqrt{2} & 0 & s/\sqrt{2} \\ 0 & s/\sqrt{2} & -rt \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{24}$$

This time, we get a system of three coupled linear equations, which we can uncouple to get

$$\begin{cases} \hbar^2 \dddot{a} + (s^2 + 2i\hbar r + r^2t^2)\dot{a} + r^2 t a = 0, \\ \hbar^2 \dddot{b} + (s^2 + r^2t^2)\ddot{b} + 3r^2 t\dot{b} = 0, \\ \hbar^2 \dddot{c} + (s^2 - 2i\hbar r + r^2t^2)\dot{c} + r^2 t c = 0, \end{cases} \tag{25}$$

which are, unfortunately, not easy to solve analytically.

A. Physical reinterpretation

Instead of trying to solve the equations directly, we will again try an indirect approach. In particular, we will consider the coupling of two 1/2 spins under the Landau-Zener 2-level hamiltonian, which will get the same 3-level hamiltonian we are working with. Thus, we consider a hamiltonian

$$\hat{H}' = \hat{h}^{(1)} \otimes \hat{\mathbb{I}}^{(2)} + \hat{\mathbb{I}}^{(1)} \otimes \hat{h}^{(2)}, \tag{26}$$

where $\hat{\mathbb{I}}$ is the identity and $\hat{h}$ is the regular Landau-Zener hamiltonian.

Applying $\hat{H}'$ to the elements of the coupled base, we find

$$\begin{aligned} \hat{H}'\,|11\rangle &= rt\,|11\rangle + \frac{s}{\sqrt{2}}\,|10\rangle\,, \\ \hat{H}'\,|10\rangle &= \frac{s}{\sqrt{2}}\,|11\rangle + \frac{s}{\sqrt{2}}\,|1-1\rangle\,, \\ \hat{H}'\,|1-1\rangle &= -rt\,|1-1\rangle + \frac{s}{\sqrt{2}}\,|10\rangle\,, \\ \hat{H}'\,|00\rangle &= 0. \end{aligned} \tag{27}$$

Since the $|00\rangle$ yields no product and doesn't appear as a product of the other elements of the base, we can omit it and write the hamiltonian in the coupled base as

$$\hat{H}' = \begin{pmatrix} rt & s/\sqrt{2} & 0 \\ s/\sqrt{2} & 0 & s/\sqrt{2} \\ 0 & s/\sqrt{2} & -rt \end{pmatrix}, \tag{28}$$

which is the same 3×3 hamiltonian we were using. This is the reason behind choosing the r and s parameters in the 3-level case as we did.

Using this result, we can just understand the transitions of the 3-level system as transitions in the coupled base of two 2-level ones, and therefore calculate the probabilities as products of the original P_{LZ}.

B. Probability products

We begin with the case with initial state $|E_-\rangle = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^{\mathrm{T}}$, that is $|10\rangle = |\uparrow\uparrow\rangle$. The probability of it jumping to $|E_0\rangle$ is that of one electron jumping and one electron not jumping, that is

$$\begin{aligned} P_{-\to 0} &= P_{\mathrm{LZ}}^{(1)}(1 - P_{\mathrm{LZ}}^{(2)}) + (1 - P_{\mathrm{LZ}}^{(1)})P_{\mathrm{LZ}}^{(2)} \\ &= 2P_{\mathrm{LZ}}(1 - P_{\mathrm{LZ}}) = 2e^{-\pi s^2/2r\hbar}(1 - e^{-\pi s^2/2r\hbar}), \end{aligned} \tag{29}$$

whereas the probability of it jumping to $|E_+\rangle$ is that of no electron jumping, that is

$$P_{-\to +} = P_{\mathrm{LZ}}^{(1)} P_{\mathrm{LZ}}^{(2)} = {P_{\mathrm{LZ}}}^2 = e^{-\pi s^2/r\hbar}. \tag{30}$$

Finally, the probability of remaining in $|E_-\rangle$ (the only adiabatic case) is that of both electrons jumping, that is

$$\begin{aligned} P_{-\to -} &= (1 - P_{\mathrm{LZ}}^{(1)})(1 - P_{\mathrm{LZ}}^{(2)}) \\ &= (1 - P_{\mathrm{LZ}})^2 = (1 - e^{-\pi s^2/2r\hbar})^2. \end{aligned} \tag{31}$$

The case with initial state $|E_+\rangle = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^{\mathrm{T}}$ can be done in a completely analogous manner. Finally, the case with initial state $|E_0\rangle = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^{\mathrm{T}} = |10\rangle$ is slightly more complicated. We will calculate the probability of a jump to $|E_+\rangle$ directly as

$$\begin{aligned} P_{0\to +} &= |\,\langle 10_{\mathrm{i}}|11_{\mathrm{f}}\rangle\,|^2 = \frac{|\,\langle\uparrow\downarrow_{\mathrm{i}}\,|\,\uparrow\uparrow_{\mathrm{f}}\rangle + \langle\downarrow\uparrow_{\mathrm{i}}\,|\,\uparrow\uparrow_{\mathrm{f}}\rangle\,|^2}{2} \\ &= \frac{|\,\langle\uparrow_{\mathrm{i}}^{(1)}\,|\,\uparrow_{\mathrm{f}}^{(1)}\rangle\,\langle\downarrow_{\mathrm{i}}^{(2)}\,|\,\uparrow_{\mathrm{f}}^{(2)}\rangle + \langle\downarrow_{\mathrm{i}}^{(1)}\,|\,\uparrow_{\mathrm{f}}^{(1)}\rangle\,\langle\uparrow_{\mathrm{i}}^{(2)}\,|\,\uparrow_{\mathrm{f}}^{(2)}\rangle\,|^2}{2} \\ &= 2|\,\langle\uparrow_{\mathrm{i}}\,|\,\uparrow_{\mathrm{f}}\rangle\,\langle\downarrow_{\mathrm{i}}\,|\,\uparrow_{\mathrm{f}}\rangle\,|^2 = 2P_{\mathrm{LZ}}(1 - P_{\mathrm{LZ}}) \\ &= 2e^{-\pi s^2/2r\hbar}(1 - e^{-\pi s^2/2r\hbar}), \end{aligned} \tag{32}$$

where the subindices $_{\mathrm{i}}$ and $_{\mathrm{f}}$ indicate initial and final, respectively, and we have used the fact that the relative phase that appears in the products is the same in both particles, as their wavefunctions are identical.

The jump to $|E_-\rangle$ can be computed analogously, and the probability of remaining in $|E_0\rangle$ may be calculated by subtracting the others to one. Thus, we have obtained all the probabilities. These results also coincide with the ones given by [7].

	+	0	−
+	$(1-P_{\text{LZ}})^2$	$2P_{\text{LZ}}(1-P_{\text{LZ}})$	${P_{\text{LZ}}}^2$
0	$2P_{\text{LZ}}(1-P_{\text{LZ}})$	$(1-2P_{\text{LZ}})^2$	$2P_{\text{LZ}}(1-P_{\text{LZ}})$
−	${P_{\text{LZ}}}^2$	$2P_{\text{LZ}}(1-P_{\text{LZ}})$	$(1-P_{\text{LZ}})^2$

TABLE I: Table with the transition probabilities from the state indicated by the row to the one indicated by the column, as we have calculated them.

V. APPLICATION OF THE LZ FORMULA TO SOLAR NEUTRINO NSI

Non-standard interactions (NSI) are a phenomenological model used to study the non-standard phenomena on neutrinos. We will, lastly, study the application of the Landau-Zener formula to the NSI of a neutrino travelling from the center of the sun.

We take a neutrino and consider the oscillations between the e and μ states. In their base, the system has a hamiltonian (as given by, for example, [8])

$$\hat{H} = \frac{1}{2E} U_{\text{PMNS}} \begin{pmatrix} 0 & 0 \\ 0 & \delta m^2 \end{pmatrix} U_{\text{PMNS}}^\dagger + \frac{a(t)}{2E} \begin{pmatrix} 1+\varepsilon_{ee} & \varepsilon_{e\mu} \\ \varepsilon_{e\mu}^* & \varepsilon_{\mu\mu} \end{pmatrix}, \tag{33}$$

where E is the energy of the neutrino, δm is the mass difference between the states, U_{PMNS} is the Pontecorvo-Maki-Nakagawa-Sakata matrix and $a(t) \equiv 2\sqrt{2} G_F E n_e(\vec{r}(t))$, with G_F the Fermi constant and n_e the electron density. We have also set $\hbar = 1$. For a neutrino travelling outwards from the center of the sun, we may consider $\vec{r}(t) \simeq ct\hat{r}$.

ε_{ee}, $\varepsilon_{e\mu}$ and $\varepsilon_{\mu\mu}$ are the phenomenological NSI parameters, for which we are going to try and find a bound. With this goal in mind, we find the eigenvalues of $\hat{H}$,

$$E_\pm(t) = \frac{(1+\varepsilon_{ee}+\varepsilon_{\mu\mu})a(t)}{2E} \pm \frac{1}{2}\sqrt{ua(t)^2 + va(t) + \delta m^2}, \tag{34}$$

where, writing θ for the mixture angle,

$$\begin{aligned} u &\equiv (1+\varepsilon_{ee}-\varepsilon_{\mu\mu})^2 + 4|\varepsilon_{e\mu}|^2, \\ v &\equiv 2[(\varepsilon_{\mu\mu}-1-\varepsilon_{ee})\cos 2\theta + 2\,\text{Re}(\varepsilon_{e\mu})\sin 2\theta]\delta m^2, \end{aligned} \tag{35}$$

Now, if we approximate n_e by a linear function, we get an also linear $a(t) = \beta - \alpha t$. Then, we can approximate the transition probability with the Landau-Zener formula. Since we know that the transition is essentially adiabatic, we can see that, with the parameters used for the LZ formula, $s^2/r \gg 1$.

Then, we only have to find the s and r parameters in this case to get a bound. We recall that r is the asymptotic ratio of separation of the levels, whereas s is the minimal separation of the levels. By analyzing the eigenvalues, then, we find

$$\begin{aligned} r &= \sqrt{u}\alpha, \\ s^2 &= \frac{-3v^2}{4u} + \delta m^4. \end{aligned} \tag{36}$$

Hence, the NSI parameters must fulfill the inequality $4\delta m^4 u - 3v^2 \gg 4u^{3/2}\alpha$.

VI. CONCLUSIONS

We have exposed the adiabatic theorem and given an analytical example in which it applies.

Two different proofs of the Landau-Zener formula have been studied, which yield the desired result in a transparent manner.

An analogue to the LZ formula in a specific 3×3 case has been found and proven using the known 2×2 formula.

An approximate bound to the NSI coefficients of solar neutrino oscillations has been derived from the LZ formula.

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