Kyle-Back's model with a random horizon

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Abstract

The continuous-time version of Kyle (1985)) developed by Back (1992)) is here studied. In Back's model there is asymmetric information in the market in the sense that there is an insider having information on the real value of the asset. We extend this model by assuming that the fundamental value evolves with time and that it is announced at a future random time. First we consider the case when the release time of information is predictable to the insider and then when it is not.

The goal of the paper is to study the structure of equilibrium, which is described by the optimal insider strategy and the competitive market prices given by the market makers. We provide necessary and sufficient conditions for the optimal insider strategy under general dynamics for the asset demands. Moreover, we study the behavior of the price pressure and the market efficiency. In particular we find that when the random time is not predictable, there can be equilibrium without market efficiency. Furthermore, for the two cases of release time and for classes of pricing rules, we provide a characterization of the equilibrium.

Key words: Market microstructure, equilibrium, insider trading, stochastic control, semimartingales, enlargement of filtrations.
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1 Introduction

Models of financial markets with the presence of an insider or informational asymmetries are largely studied in the literature with different approaches and perspectives.

A conspicuous part of this literature proposes models with stock prices fixed exogenously, i.e., the insider does not affect the stock price dynamics and the privileged information is a functional of the stock price process, i.e. the final value, the maximum, etc. The aim of these studies is often to find the optimal strategy of the insider and, in some cases, provide an evaluation on how much better the insider performs in the market using the larger information at his disposal, compared with a trader using only market information. In this direction we find, e.g. Karatzas & Pikovsky (1996), Amendiger et al. (1998), Grorud & Pontier (1998, 2001, 2005), Imkeller et al. (2001), Corcuera et al. (2004), Biagini & Øksendal (2005, 2006), Kohatsu-Higa (2007), Di Nunno et al. (2006, 2011), Draouil & Øksendal (2016), Enrst et al. (2017). In some cases, the impact of the insider strategy affects the stock price dynamics in the sense that these dynamics are dependent on the insider strategy itself; see, e.g., Di Nunno et al. (2008).

As pointed by Danilova (2010), in an equilibrium situation market prices are determined by the demand of the market participants. So, in such a situation, the privileged information cannot be a functional of the stock price process because this implies the knowledge of the future demand and this is unrealistic. Then

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the privileged information is exogenous. This can be the value of the fundamental price, or some signal of it, or the time of the announcement of the fundamental value, which evolves independently of the demand.

The original model is due to Kyle (1985). He considers three kinds of actors in the market: market makers, uninformed traders and one insider who knows the fundamental or liquidation value of an asset at certain fixed released time. In the model, there is also a price function establishing the relation between the market prices and the total demand. Kyle works in the discrete time setting and with noises given by Gaussian random walks. Back (1992) extends the previous work to the continuous time case. These are seminal papers which paved the way to various generalisations and extensions. To mention some, see Wu (1999) extending Back (1992) and also Back & Pedersen (1998), who consider a *dynamic* fundamental price and Gaussian noises with time varying volatility; Cho (2003), who considers pricing functions depending on the path of the demand process and studies what happens when the informed trader is risk-averse; Lasserre (2004), who considers a multivariate setting; Aase et al. (2012a), (2012b) and Campi et al. (2011) who put emphasis in filtering techniques to find the equilibrium problem; Campi & Çetin (2007), who consider a defaultable bond instead of a stock as in the Kyle-Back model and also consider the default time as privileged information; Danilova (2010), who deals with non-regular pricing rules; Corcuera et al. (2010), where the presence of jumps and a drift in the aggregate demand of the liquidity traders is analised; Caldentey & Stacchetti (2010) who take a random release time into account; and Campi et al. (2013), who consider again a defaultable bond, but this time the privileged information is represented by some dynamic signal related with the default time. The here mentioned paper constitute a still incomplete list, even including the references therein, as the field is in simmering activity.

In the present work we propose a unified framework to study equilibrium, able to capture most of the different situations and contexts that have been presented in the literature. Specifically, we consider that the insider has access to an exogenous information flow, which includes the knowledge of some signal related to the fundamental value of the asset. The fundamental value is actually going to be released at a random time, which is a stopping time for the insider. We consider two situations, first when the release time is predictable for the insider and then when this is not.

Our framework is capturing a large number of previous extensions of Kyle (1985) as it is illustrated by the several examples presented. Excluded from our framework are the insider risk-aversion attitudes considered in some previous works and the multivariate setting, as in Lasserre (2004) even thought both aspects can be treated in a technical but direct extension of our framework and methodology . Also our present framework is dealing with a *price pressure* λ that is a deterministic function. This does not allow to include the case of Back and Baruch (2004), where λ depends on the market price of the stock, and Collin-Dufresne and Fos (2016), where λ depends on the random volatility of the noise in the market. Extending our framework to the case of λ stochastic is possible, but is matter of future research.

The main focus of the present paper is to study properties of the equilibrium. Given a set of admissible triplets of insider's strategies, pricing rules, and price pressure, conditions for the equilibrium are given by those admissible triplets for which the insider's strategy is optimal and the pricing rule is rational. We also show how these properties can be used for finding the equilibrium.

The framework presents the interplay of agents having different roles and asymmetric information. The market makers set rational market prices, which are assumed to be a function of time and the aggregate demand for the asset. For such given pricing rule, the insider optimizes his position to maximize his expected wealth. In our work we consider a very general insider's information flow, a random release time of information, and very general dynamics for the aggregate demand, i.e. a predictable semi-martingale. To the best of our knowledge it is the first time that these three features are considered together at this level of generality.

In this framework we study the necessary and sufficient conditions for an insider's strategy to be optimal in terms of the properties that the pricing rule and the information flows should have.

Moreover, we study market efficiency and we can see a role of the insider in the market in this respect. If the insider can predict the release time, then the market is actually efficient. This shows a certain beneficial effect of the presence of an insider in the market. On the other side we can see that in the case when the release time is not predictable for the insider, then the market is not efficient in general and we can also show that an equilibrium is still possible if the sensitivity of the prices decreases in time according to the survival probability of the announcement. In other words, the prices become more stable as the release time approaches.

Finally, in the two cases of predictable and non predictable time of information release and for classes of pricing rules, we can provide necessary and sufficient conditions to characterize the equilibrium for classes of pricing rules.

To conclude we include various examples in which we illustrate how the present framework covers many relevant examples present in the literature and opens for the study of new situations. In the examples we provide specific demand dynamics and value processes. In this cases then the analysis can proceed to a further stage and we show how our results, coupled with the mathematical tools of enlargement of filtrations or filtering techniques, allow to actually find explicitly the insider's optimal strategy. Several of these examples are treated in the literature. Here we show how it is possible to approach the study in a unified framework.

The paper is structured as follows. In the next section we describe the model that gives rise to the stock prices. We discuss the insider's optimal strategies for a given pricing rule and we define the concept of admissibility for pricing rules and insider strategies. In Section 3 we specifically study the case when the release time is predictable for the insider, while in Section 4 the case when the release time is not predictable. In Section 5 we show how to apply the previous results to find the equilibrium in our general framework.

2 The model and equilibrium

We consider a market with two assets, a stock and a bank account with interest rate r equal to zero for the sake of simplicity. With abuse of terminology we will just write "prices" even though they are sometimes "discounted prices". The trading is continuous in time over the period $[0, \infty)$ and it is order driven. There is a (possibly random) release time $\tau < \infty$ a.s., when the fundamental value of the stock is revealed. The fundamental value process represents the actual value of the asset, which would be known only if *all* the information was public. The fundamental value process is denoted by V.

We shall denote the market price of the stock at time t by P_t. This represents the market evaluation of the asset. Just after the revelation time the price of the stock coincides with the fundamental value. Then we consider P_t defined only on $t \le \tau$. It is possible that $P_t \neq V_t$ for $t \le \tau$. We stop our studies at this (random) time of release τ, which is reasonable to assume finite.

We assume that all the random variables and processes mentioned are defined in the same complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{H}, \mathbb{P})$ where the filtration $\mathbb{H}$ is complete and right-continuous.

There are three kinds of traders. A *large* number of liquidity traders, who trade for liquidity or hedging reasons, an informed trader or insider, who observe all the random processes $\mathbb{H}$-adapted, in particular she has information about the firm and can deduce its fundamental value, and the market makers, who set the market price according to the total aggregate demand and clear the market.

2.1 The agents and the equilibrium

As we say above, at time t, the insider information is given by $\mathcal{H}_t$ and $\mathbb{H} = (\mathcal{H}_t)_{t \ge 0}$. The filtration $\mathbb{H}$ is the reference filtration so, if there is no possibility of confusion, we shall omit $\mathbb{H}$ in the notation. If other filtration is used, then this will be specified. In some cases we shall assume that the firm value V is depending on some adapted *signal process* η, but we do not specify any exact functional relationship between V and η and we refer instead to the various examples provided in the sequel.

We assume that V is a càdlàg martingale (if not otherwise specified) such that $\sigma_V^2(t) := \frac{d[V,V]_t^c}{dt}$ is well defined (where $[V, V]^c$ indicates the continuous part of the quadratic variation of V).

Hereafter we describe in detail the three types of agents involved in this market model, namely their role, their demand process and their information.

Let Z be the *aggregate* demand process of the **liquidity traders**. We recall that these are a large number of traders motivated by liquidity or hedging reasons. They are perceived as constituting noise in the market, thus also called *noise* traders. We assume that Z is a continuous martingale, with $Z_0 = 0$, independent of η and V, such that $\sigma_Z^2(t) := \frac{\mathrm{d}[Z,Z]_t}{\mathrm{d}t}$ is well defined. We do not consider the presence of jumps or a drift in Z, this was analysed in Corcuera et al. (2010) and it was shown that there is not equilibrium when we introduce jumps in Z and that the presence of a drift, in the risk neutral case, produces similar equilibrium situations. Therefore the liquidity traders observe the market prices and the release time of information, as any other trader in the market, but their investment or trading attitude is not strategic.

Market makers clear the market giving the market prices. They rely on the information given by the total aggregate demand Y, which they observe. Specifically, $Y := X + Z$, where X denotes the insider demand process. X is naturally a predictable process and we assume that it is a càdlàg semimartingale with $X_0 = 0$. Just like the noise traders, the market makers instantly know about the time of release of information when that occurs. Hence, their information flow is: $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, where $\mathcal{F}_t = \bar{\sigma}(Y_s, \tau \wedge s, 0 \leq s \leq t)$. Here $\bar{\sigma}$ denotes the σ-field corresponding to the usual augmentation of the natural filtration (see Revuz & Yor 1999, Ch. I, Def. 4.13 and the paragraphs following this definition). That is, e.g.,

$$\bar{\sigma}(Y_s, \tau \wedge s, 0 \leq s \leq t) := \bigcap_{r>t} \left(\sigma(Y_s, \tau \wedge s, 0 \leq s \leq r) \cup \mathcal{N}\right), \tag{2.1}$$

where $\mathcal{N}$ is the family of $\mathbb{P}$-null sets in $\mathcal{F}$, and $(\sigma(Y_s, \tau \wedge s, 0 \leq s \leq r))_{r\geq 0}$ is the natural filtration generated by Y and $\tau \wedge s$.

From the economic point of view, due to the competition among market makers, the market prices $(P_t)_{t\geq 0}$ are *rational*, or *competitive*, in the sense that

$$P_t = \mathbb{E}(V_t|\mathcal{F}_t), \quad 0 \leq t \leq \tau. \tag{2.2}$$

In our model, consistent with the original idea of Kyle (1985) and later literature, we suppose that market makers give market prices through a pricing rule, which consists of a formula that here takes the form:

$$P_t = H(t, \xi_t), \quad t \geq 0, \tag{2.3}$$

where the deterministic function H is $C^{1,2}$ and, for all $t \geq 0$, $H(t, \cdot)$ is strictly increasing and where

$$\xi_t := \int_0^t \lambda(s)\mathrm{d}Y_s, \tag{2.4}$$

with the deterministic function λ strictly positive and integrable with respect to Y. In this paper we call λ the *price pressure*. Observe that

$$\mathcal{F}_t = \bar{\sigma}(P_s, \tau \wedge s, 0 \leq s \leq t) \subseteq \mathcal{H}_t, \tag{2.5}$$

for all t and therefore, since V is an $\mathbb{H}$-martingale, equality (2.2) implies that P is an $\mathbb{F}$-martingale. Furthermore, from the assumptions on Y and λ, we observe that ξ is a càdlàg $\mathbb{H}$-semimartingale. Hence, applying the Itô formula to (2.3), we can see that P is also an $\mathbb{H}$-semimartingale.

Definition 2.1 (Pricing rule)**.** *Let $\mathfrak{H}$ denote the class of pairs (H, λ) described above. An element of $\mathfrak{H}$ is called a* pricing rule.

The **insider** or **informed trader** will have *some knowledge* about the (random) release time of information τ, which is in general assumed finite and it is a $\mathbb{H}$-stopping time. We shall consider the two following cases from the insider perspective:

(i) τ is predictable, i.e. there is an increasing sequence of stopping times (τ_n) such that a.s., $\tau_n < \tau$ and $\lim_n \tau_n = \tau$. In this case, we assume τ bounded.

(ii) τ is not predictable. In this case, we assume τ to have probability density with respect to the Lebesgue measure and to be independent of V, Z, and P.

Remark 2.1. In this equilibrium model the random time τ and the processes V and Z are exogenously given. The modelling assumptions above state in what terms the insider relates to these.

The informed trader is assumed risk-neutral and she aims at maximizing her expected final wealth. Let W be the wealth process corresponding to insider's portfolio X. To illustrate the relationship among the processes V, P, X, and W we first consider a multi-period model where trades are made at times $i = 1, 2, \dots N$, and where $\tau = N$ is random. If at time $i-1$, there is an order to buy $X_i - X_{i-1}$ shares, its *cost* will be $P_i(X_i - X_{i-1})$, so, there is a change in the bank account given by $-P_i(X_i - X_{i-1})$. Then the total (cumulated) change at $\tau = N$ is $-\sum_{i=1}^N P_i(X_i - X_{i-1})$, and due to the fact that at the release time $\tau = N$ the price of the asset becomes the fundamental one, there is the extra income: $X_N V_N$. So, the total wealth W_τ at τ is

$$
\begin{aligned}
W_\tau &= -\sum_{i=1}^N P_i(X_i - X_{i-1}) + X_N V_N \\
&= -\sum_{i=1}^N P_{i-1}(X_i - X_{i-1}) - \sum_{i=1}^N (P_i - P_{i-1})(X_i - X_{i-1}) + X_N V_N.
\end{aligned} \tag{2.6}
$$

Consider now the continuous time setting where we have the processes X, P, and V, and we take N trading periods, where N is random and the trading times are: $0 \leq t_1 \leq t_2 \leq ... \leq t_N = \tau$, then we have

$$
W_\tau = -\sum_{i=1}^N P_{t_{i-1}}(X_{t_i} - X_{t_{i-1}}) - \sum_{i=1}^N (P_{t_i} - P_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}) + X_{t_N} V_{t_N}, \tag{2.7}
$$

so, if the time between trades goes to zero, we will have

$$
\begin{aligned}
W_\tau &= X_\tau V_\tau - \int_0^\tau P_{t-} \mathrm{d}X_t - [P, X]_\tau \\
&= \int_0^\tau X_{t-} \mathrm{d}V_t + \int_0^\tau V_{t-} \mathrm{d}X_t + [V, X]_\tau - \int_0^\tau P_{t-} \mathrm{d}X_t - [P, X]_\tau \\
&= \int_0^\tau (V_{t-} - P_{t-}) \, \mathrm{d}X_t + \int_0^\tau X_{t-} \mathrm{d}V_t + [V, X]_\tau - [P, X]_\tau,
\end{aligned} \tag{2.8}
$$

where (here and throughout the whole article) $P_{t-} = \lim_{s \uparrow t} P_s$ a.s and we have $X_0 = 0$. Having assumed that X is a $\mathbb{H}$-*predictable* càdlàg semimartingale we can give meaning to the stochastic integrals above in the framework of Itô stochastic integration.

In the next subsection we discuss the characterization of an insider's optimal strategy in equilibrium. For this we shall consider an insider's demand process X that is optimal in the sense that it maximizes

$$
J(X) := \mathbb{E}(W_\tau) = \mathbb{E}\left(\int_0^\tau (V_{t-} - H(t, \xi_{t-}))\mathrm{d}X_t + \int_0^\tau X_{t-} \mathrm{d}V_t + [V, X]_\tau - [P, X]_\tau \right), \tag{2.9}
$$

for a pricing rule $(H, \lambda) \in \mathfrak{H}$. However for technical and modelling reasons, we require additional properties to the triplet (H, λ, X).

Here and in the sequel $\partial_i H$, $\partial_{ij} H$ denote the first and second derivatives with respect to the i^{th}, i^{th} and j^{jh} variables, respectively.

Definition 2.2 (Admissibility). *We say that* (H, λ, X) *is an* admissible *triplet, if the process* X *(which may* *also be $X \equiv 0$) and the price function $(H, \lambda) \in \mathfrak{H}$ satisfy:*

(A1) $X_t = M_t + A_t + \int_0^t \theta_s \mathrm{d}s$, *for all $t \geq 0$, where M is a continuous $\mathbb{H}$-martingale, A a bounded variation $\mathbb{H}$-predictable process, with $A_t = \sum_{0<s\leq t} (X_s - X_{s-})$, and θ a càdlàg $\mathbb{H}$-adapted process,*

(A2) $\mathbb{E}\left(\left(\int_0^\tau (\partial_2 H(s,\xi_s))^2 + (H(s,\xi_s))^2 + V_s^2\right)\left(\sigma_Z^2(s)\mathrm{d}s + \sigma_M^2(s)\mathrm{d}s\right)\right) < \infty$, *where* $\sigma_M^2(s) := \frac{\mathrm{d}[M,M]_s}{\mathrm{d}s}$,

(A3) $\mathbb{E}\left(\int_0^\tau (\partial_2 H(s,\xi_s) + H(s,\xi_s) + V_s) |\theta_s| \mathrm{d}s\right) < \infty$,

(A4) $\mathbb{E}\left(\sum_0^\tau \partial_2 H(s,\xi_{s-})|\Delta X_s|\right) < \infty, \Delta X_s := X_s - X_{s-}$,

(A5) $\mathbb{E}\left(\int_0^\tau \left(H^{-1}(\tau,\cdot)(V_{s-})\right)^2 + |Z_s|^2 + |X_{s-}|^2)\mathrm{d}[V,V]_s\right) < \infty$,

(A6) $\mathbb{E}\left(\int_0^\tau \lambda(s)|\partial_{22}H(s,\xi_s)|\left(\sigma_M^2(s) + |\sigma_{M,Z}(s)|\right))\mathrm{d}s\right) < \infty$,*where* $\sigma_{M,Z}(s) := \frac{\mathrm{d}[M,Z]_s}{\mathrm{d}s}$.

Remark 2.2. Note that, since X is a càdlàg *predictable* process, given $(A1)$ above, its martingale part is predictable, then it cannot have jumps, see Corollary 2.31 in Jacod and Shiryaev (1987). Similarly, we have chosen Z to be a continuous martingale before.

Definition 2.3 (Optimality)**.** *Let (H, λ, X) be an admissible triplet, the strategy X is called* optimal *with respect to (H, λ) if it maximizes $J(X)$ (2.9).*

Definition 2.4 (Equilibrium)**.** *An admissible triplet (H, λ, X) is an equilibrium if we have both that*

1. *given X, the pricing rule (H, λ) is such that the price process $P_\cdot := H(\cdot, \xi_\cdot)$ is rational* (2.2)
2. *given (H, λ), the strategy X is optimal.*

Remark 2.3. Notice that the processes V, Z and the random variable τ are fixed exogenously in our model, while we find an equilibrium endogeneously. The equilibrium is obtained among admissible triplets (H, λ, X) by first fixing (H, λ) and looking for the optimal X and later choosing (H, λ) such that prices are rational.

2.2 The optimality condition

In this last part of the section we provide necessary conditions for the insider's demand in an admissible triplet (H, λ, X) to be optimal. In our model the insider information advantage can be relevant up to the time τ of information release about the fundamental value of the stock. Then, hereafter, we consider two kinds of stopping times: τ bounded, or τ finite but independent of (V, P, Z). In both cases, by the assumptions that V is a martingale and X a predictable càdlàg semimartingale satisfying (A5), we have that $\mathbb{E}(\int_0^\tau X_t \mathrm{d}V_t) = 0$. In fact, we can argue that, if τ is bounded, we can apply Doob's Optional Sampling Theorem and, if τ is finite but independent of (V, P, Z) (and consequently of X), we have that

$$\mathbb{E}\left(\int_0^\tau X_{t-}\mathrm{d}V_t\right) = \mathbb{E}\left(\mathbb{E}\left(\int_0^\tau X_{s-}\mathrm{d}V_s \middle| \tau\right)\right) = \mathbb{E}\left(\mathbb{E}\left(\int_0^t X_{s-}\mathrm{d}V_s\right)\Big|_{t=\tau}\right) = 0. \tag{2.10}$$

Hence, (2.9) reduces to

$$J(X) := \mathbb{E}(W_\tau) = \mathbb{E}\left(\int_0^\tau (V_{t-} - H(t,\xi_{t-}))\mathrm{d}X_t + [V,X]_\tau - [P,X]_\tau\right). \tag{2.11}$$

We now present a series of observations. First, note that

$$\int_0^{\tau} (V_{t-} - H(t,\xi_{t-}))\mathrm{d}X_t + [V,X]_\tau - [P,X]_\tau = \int_0^{\tau-} (V_{t-} - H(t,\xi_{t-}))\mathrm{d}X_t + [V,X]_{\tau-} - [P,X]_{\tau-} + (V_\tau - H(\tau,\xi_\tau))\,\Delta X_\tau.$$

Then suppose that X is optimal and we modify only the last jump of this strategy by taking $(1+\varepsilon\gamma)\Delta X_\tau$, with γ an $\mathcal{H}_{\tau-}$-measurable and bounded random variable and $\varepsilon > 0$ small enough. We recall that $\mathcal{H}_{\tau-} := \mathcal{H}_0 \vee \sigma(A \cap (\tau > t) : A \in \mathcal{H}_t, t \geq 0)$ (see, e.g., Revuz and Yor (1999), page 46). Denote $X^{(\varepsilon)}$ this new strategy.

Then, since ΔX_τ is bounded (see (A1) in Definition 2.2), we can see that

$$0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\, J(X^{(\varepsilon)})\Big|_{\varepsilon=0} = \mathbb{E}\left(\gamma\left((V_\tau - H(\tau,\xi_\tau))\,\Delta X_\tau - \lambda(\tau)\partial_2 H(\tau,\xi_\tau)\,(\Delta X_\tau)^2\right)\right), \tag{2.12}$$

so we obtain

$$\mathbb{E}\left((V_\tau - H(\tau,\xi_\tau))\,\Delta X_\tau - \lambda(\tau)\partial_2 H(\tau,\xi_\tau)\,(\Delta X_\tau)^2\Big|\,\mathcal{H}_{\tau-}\right) = 0. \tag{2.13}$$

Now we modify the strategy X by taking an $\mathbb{H}$-adapted càdlàg process β such that $X + \varepsilon\int\beta_s\mathrm{d}s$ is admissible, with $\varepsilon > 0$ small enough.

We have

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} J(X + \varepsilon\int\beta_s\mathrm{d}s)\Big|_{\varepsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbb{E}\left(\int_0^\tau (V_{t-} - H(t, \int_0^{t-}\lambda(s)(\mathrm{d}X_s + \varepsilon\beta_s\mathrm{d}s + \mathrm{d}Z_s)))(\mathrm{d}X_t + \varepsilon\beta_t\mathrm{d}t)\right)\Big|_{\varepsilon=0} \\
&- \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbb{E}\left([V, X + \varepsilon\int\beta_s\mathrm{d}s]_\tau - [H(\cdot, \int\lambda(s)(\mathrm{d}X_s + \varepsilon\beta_s\mathrm{d}s + \mathrm{d}Z_s), X + \varepsilon\int\beta_s\mathrm{d}s]_\tau\right)\Big|_{\varepsilon=0} \\
&= \mathbb{E}\left(\int_0^\tau (V_{t-} - H(t,\xi_t))\beta_t\mathrm{d}t\right) - \mathbb{E}\left(\int_0^\tau \partial_2 H(t,\xi_{t-})\left(\int_0^t \lambda(s)\beta(s)\mathrm{d}s\right)\mathrm{d}X_t\right) \\
&- \mathbb{E}\left(\left[\partial_2 H(\cdot,\xi)\left(\int\lambda(s)\beta(s)\mathrm{d}s\right), X\right]_\tau\right) \\
&= \mathbb{E}\left(\int_0^\tau \left((V_t - H(t,\xi_t)) - \lambda(t)\int_{t\wedge\tau}^\tau \partial_2 H(s,\xi_{s-})\mathrm{d}X_s\right)\beta_t\mathrm{d}t\right) \\
&- \mathbb{E}\left(\int_0^\tau \left(\int_0^t \lambda(s)\beta(s)\mathrm{d}s\right)\mathrm{d}\left[\partial_2 H(\cdot,\xi), X\right]_t\right) \\
&= \mathbb{E}\left(\int_0^\tau \left((V_t - H(t,\xi_t)) - \lambda(t)\left(\int_{t\wedge\tau}^\tau \partial_2 H(s,\xi_{s-})\mathrm{d}X_s + [\partial_2 H(\cdot,\xi), X]_t^\tau\right)\right)\beta_t\mathrm{d}t\right),
\end{aligned} \tag{2.14}$$

where $[\cdot,\cdot]_t^\tau := [\cdot,\cdot]_\tau - [\cdot,\cdot]_t$. Since we can take $\beta_t = \alpha_u \mathbf{1}_{(u,u+h]}(t)$, with an $\mathcal{H}_u$-measurable and bounded α_u, we have

$$\mathbb{E}\left(\int_u^{u+h}\left[\mathbb{E}\big(\mathbf{1}_{[0,\tau]}(t)\,(V_t - H(t,\xi_t))\big|\,\mathcal{H}_t\big) - \lambda(t)\mathbb{E}\Big(\int_{t\wedge\tau}^\tau \partial_2 H(s,\xi_{s-})\mathrm{d}X_s + [\partial_2 H(\cdot,\xi), X]_{t\wedge\tau}^\tau\Big|\,\mathcal{H}_t\Big)\right]\mathrm{d}t\,\Big|\,\mathcal{H}_u\right) = 0$$

and this means that the process Ξ_t, $t \geq 0$:

$$\Xi_t := \int_0^t \left[\mathbb{E}\big(\mathbf{1}_{[0,\tau]}V_u|\mathcal{H}_u\big) - \mathbb{E}\big(\mathbf{1}_{[0,\tau]}(u)H(u,\xi_u)|\mathcal{H}_u\big) - \lambda(u)\mathbb{E}\Big(\int_{u\wedge\tau}^\tau \partial_2 H(s,\xi_{s-})\mathrm{d}X_s + [\partial_2 H(\cdot,\xi), X]_{u\wedge\tau}^\tau\,|\mathcal{H}_u\Big)\right]\mathrm{d}u \tag{2.15}$$

is a continuous $\mathbb{H}$-martingale with bounded variation. In particular this implies that, for a.a. $t \geq 0$,

$$\mathbb{E}\left(\mathbf{1}_{[0,\tau]}(t)V_t\big|\,\mathcal{H}_t\right) - \mathbb{E}\left(\mathbf{1}_{[0,\tau]}(t)H(t,\xi_t)\big|\,\mathcal{H}_t\right) - \lambda(t)\mathbb{E}\left(\int_{t\wedge\tau}^\tau \partial_2 H(s,\xi_{s-})\mathrm{d}X_s + [\partial_2 H(\cdot,\xi), X]_{t\wedge\tau}^\tau\,|\mathcal{H}_t\right) = 0, a.s. \tag{2.16}$$

Since τ is an $\mathbb{H}$-stopping time, then for a.a. t and for a.a. $\omega \in \{\tau \geq t\}$, or equivalently a.s. on the stochastic interval $[[0,\tau]]$, we can write

$$V_t - H(t,\xi_t) - \lambda(t)\mathbb{E}\left(\int_t^\tau \partial_2 H(s,\xi_s) \mathrm{d}^- X_s \middle| \mathcal{H}_t \right) = 0, \tag{2.17}$$

where we have used a shorthand notation by means of $\mathrm{d}^- X_s$ as the *backward* integral in the sense of Revuz and Yor (1999) (see page 144), here extended to semimartingales with jumps. As a summary we have the following necessary condition, which is instrumental for identifying insider's optimal strategies.

Theorem 2.1. *An admissible triple (H, λ, X) such that X is optimal for the insider satisfies the equations:*

$$\mathbb{E}\left((V_\tau - H(\tau,\xi_\tau))\,\Delta X_\tau - \lambda(\tau)\partial_2 H(\tau,\xi_\tau)\,(\Delta X_\tau)^2 \middle| \mathcal{H}_{\tau-} \right) = 0. \tag{2.18}$$

$$V_t - H(t,\xi_t) - \lambda(t)\mathbb{E}\left(\int_t^\tau \partial_2 H(s,\xi_s) \mathrm{d}^- X_s \middle| \mathcal{H}_t \right) = 0. \tag{2.19}$$

a.s. on $[[0,\tau]]$.

In the sequel we study two different cases of knowledge of τ from the insider's perspective. First the case in which the insider can predict the time τ of release of information about the firm value, then we study the case when τ is not predictable.

3 Case when τ is predictable to the insider

In this section we consider the case when the insider can predict the release time of information τ. Namely, there is an increasing sequence of stopping times (τ_n) such that a.s., $\tau_n < \tau$ and $\lim_n \tau_n = \tau$. Moreover, we assume that τ is bounded. These are standing assumptions throughout this section.

We observe that a particular case in this section is when τ is known to the insider at time $t = 0$, that is τ is $\mathcal{H}_0$-measurable.

3.1 Necessary conditions for the equilibrium

Our first observation is that optimal strategies lead the market price to the fundamental one, which means that the market is efficient. In fact we have the following proposition.

Proposition 3.1. *If (H, λ, X) is admissible with X optimal, then the optimal strategy X has no jump at τ and the market is efficient, i.e.*

$$V_{\tau-} = H(\tau,\xi_{\tau-}) = H(\tau,\xi_\tau) = P_\tau \quad \text{a.s.} \tag{3.1}$$

Proof. By the assumptions (A1) and (A2) in Definition 2.2, equation (2.19) can be rewritten by using the announcing sequence $(\tau_n)_{n\geq 0}$:

$$\begin{aligned}
&V_{\tau_n} - H(\tau_n, \xi_{\tau_n}) - \lambda(\tau_n)\mathbb{E}\left(\int_{\tau_n\wedge\tau}^{\tau} \partial_2 H(s,\xi_s)\mathrm{d}^- X_s \middle| \mathcal{H}_{\tau_n}\right) \quad (3.2)\\
&= V_{\tau_n} - H(\tau_n, \xi_{\tau_n}) - \lambda(\tau_n)\mathbb{E}\left(\int_{\tau_n}^{\tau} \partial_2 H(s,\xi_s)\mathrm{d}^- X_s \middle| \mathcal{H}_{\tau_n}\right)\\
&= V_{\tau_n} - H(\tau_n, \xi_{\tau_n}) - \lambda(\tau_n)\mathbb{E}\left(\int_{\tau_n}^{\tau} \partial_2 H(s,\xi_s)\theta_s | \mathcal{H}_{\tau_n}\right)\mathrm{d}s\\
&- \lambda(\tau_n)\mathbb{E}\left(\sum_{\tau_n}^{\tau} \partial_2 H(s,\xi_s)\Delta X_s \middle| \mathcal{H}_{\tau_n}\right)\\
&- \lambda(\tau_n)\mathbb{E}\left(\int_{\tau_n}^{\tau} \lambda(s)\partial_{22} H(s,\xi_s)(\sigma_M^2(s) + \sigma_{Z,M}(s))\mathrm{d}s \middle| \mathcal{H}_{\tau_n}\right)\\
&= 0 \quad \text{a.s. on } [[0,\tau]]. \quad (3.3)
\end{aligned}$$

Now by assumption (A3) in Definition 2.2 and Corollary (2.4) in Revuz & Yor (1999), we have that

$$\lim_{\tau_n\uparrow\tau} \mathbb{E}\left(\int_{\tau_n}^{\tau} \partial_2 H(s,\xi_s)|\theta_s|\mathrm{d}s \middle| \mathcal{H}_{\tau_n}\right) = 0. \quad (3.4)$$

Analogously we also have that

$$\lim_{\tau_n\uparrow\tau} \lambda(\tau_n)\mathbb{E}\left(\int_{\tau_n}^{\tau} \lambda(s)\partial_{22} H(s,\xi_s)(\sigma_M^2(s) + \sigma_{Z,M}(s))\mathrm{d}s \middle| \mathcal{H}_{\tau_n}\right) = 0 \quad \text{a.s.}, \quad (3.5)$$

whereas

$$\lim_{\tau_n\uparrow\tau} \lambda(\tau_n)\mathbb{E}\left(\sum_{\tau_n}^{\tau} \partial_2 H(s,\xi_s)\Delta X_s \middle| \mathcal{H}_{\tau_n}\right) = \lambda(\tau)\partial_2 H(\tau,\xi_\tau)\Delta X_\tau. \quad (3.6)$$

Consequently

$$V_{\tau-} - H(\tau,\xi_{\tau-}) - \lambda(\tau)\partial_2 H(\tau,\xi_\tau)\Delta X_\tau = 0 \quad \text{a.s.} \quad (3.7)$$

Now, since V is a martingale and τ is predictable, then $\mathbb{E}(V_\tau|\mathcal{H}_{\tau-}) = V_{\tau-}$ (see Jacod and Shiryaev (1987), Lemma 2.27). (We recall that $\mathcal{F}_{\tau-} := \mathcal{F}_0 \vee \sigma(A\cap(\tau > t) : A \in \mathcal{F}_t, t \geq 0)$, see, e.g., Revuz and Yor (1999), page 46). Moreover, since X is $\mathbb{H}$-predictable, Z is continuous (and consequently ξ is predictable), we have

$$\mathbb{E}\left((H(\tau,\xi_\tau))\,\Delta X_\tau + \lambda(\tau)\partial_2 H(\tau,\xi_\tau)\,(\Delta X_\tau)^2 \middle| \mathcal{H}_{\tau-}\right) = H(\tau,\xi_\tau)\Delta X_\tau + \lambda(\tau)\partial_2 H(\tau,\xi_\tau)\,(\Delta X_\tau)^2. \quad (3.8)$$

Therefore equation (2.18) gives

$$(V_{\tau-} - H(\tau,\xi_\tau))\,\Delta X_\tau - \lambda(\tau)\partial_2 H(\tau,\xi_\tau)\,(\Delta X_\tau)^2 = 0 \quad \text{a.s.} \quad (3.9)$$

If it was $\Delta X_\tau \neq 0$, then we would have that

$$V_{\tau-} - H(\tau,\xi_\tau) - \lambda(\tau)\partial_2 H(\tau,\xi_\tau)\Delta X_\tau = 0. \quad (3.10)$$

However, comparing the above equation with (3.7) we have that $H(\tau,\xi_\tau) = H(\tau,\xi_{\tau-})$, which actually contradicts $\Delta X_\tau \neq 0$, being H strictly increasing in the second variable. Then this shows that an optimal strategy X has no jump at τ and that $V_{\tau-} = H(\tau,\xi_{\tau-}) = H(\tau,\xi_\tau)$, see by (3.7). □

Remark 3.1. In Aase et al. (2012a) it was already observed that market efficiency is a consequence of the optimality of the insider's strategy. Here we obtain an extension of this result for a more general behaviour of the fundamental value and of the demand process of the noise traders.

Remark 3.2. This efficiency situation is also the case in Campi and Çetin (2007). In our notation they have the signal $\eta = \bar\tau$, with $\bar\tau$ known by the insider and representing the default time of a bond with face value 1, the fundamental value $V_t = \mathbf{1}_{\{\bar\tau > 1\}}$, and the release time is $\tau = \bar\tau \wedge 1$. So, τ is $\mathcal{H}_0$-measurable and it is bounded. Then, they obtain

$$\mathbf{1}_{\{\bar\tau>1\}} - H(\bar\tau \wedge 1, \xi_{\bar\tau\wedge 1}) = 0 \quad \text{a.s.}$$

Within this study, the authors also assume that $\bar\tau$ is the first passage time of a standard Brownian motion independent of Z.

Remark 3.3. If we take the fundamental value $V_t \equiv V$ and the deterministic fixed release time $\tau \equiv 1$, then we retrieve Back's framework (1992). There it is shown that market prices converge to V when $t \to 1$.

Hereafter we consider necessary conditions for an admissible triplet (H, λ, X) to be an equilibrium. These conditions show the synergy between the optimal insider strategy and the pricing rule in an equilibrium state. Note that one cannot use these conditions to (uniquely) identify a pricing rule. The choice of pricing rules is not unique. In the next subsection we will provide both necessary and sufficient conditions for the equilibrium in a wide class of pricing rules. Before that we have the following result. Here we assume that the process V is quasi-left continuous.

Proposition 3.2. *Consider an admissible triple (H, λ, X), with $\lambda \in C^1$. If (H, λ, X) is an equilibrium, we have*

(*i*) $H(\tau, \xi_\tau) = V_\tau \quad$ *a.s.,*

(*ii*) $\dfrac{\lambda'(t)}{\lambda^2(t)} V_t - \dfrac{\lambda'(t)}{\lambda^2(t)} H(t, \xi_t) + \dfrac{\partial_1 H(t,\xi_t)}{\lambda(t)} + \dfrac{1}{2}\partial_{22} H(t,\xi_t)\lambda(t)\big(\sigma_Y^2(t) - 2\sigma_{M,Y}(t)\big) = 0 \quad$ *a.s. on* $[[0,\tau))$,

(*iii*) $\partial_1 H(t,\xi_t) + \dfrac{1}{2}\partial_{22}H(t,\xi_t)\lambda^2(t)\mathbb{E}\left(\sigma_Z^2(t) - \sigma_M^2(t) | \mathcal{F}_t\right) = 0 \quad$ *a.s. on* $[[0,\tau))$.

Proof. (*i*) It is just Proposition 3.1 together with the fact that V is quasi-left continuous and that τ is a predictable time. We prove (*ii*) and (*iii*). By using Itô formula on $\frac{H(t,\xi_t)}{\lambda(t)}$, with (A2) in Definition 2.2 applied, we have

$$\begin{aligned}
\mathbb{E}\left(\int_{t\wedge\tau}^{\tau} \frac{1}{\lambda(s)}\partial_2 H(s,\xi_{s-})\mathrm{d}\xi_s \,\middle|\, \mathcal{H}_t\right) &= \mathbb{E}\left(\frac{H(\tau,\xi_\tau)}{\lambda(\tau)}\,\middle|\,\mathcal{H}_t\right) - \frac{H(t\wedge\tau, \xi_{t\wedge\tau})}{\lambda(t\wedge\tau)} \\
&\quad - \mathbb{E}\left(\int_{t\wedge\tau}^{\tau}\left(-\frac{\lambda'(s)}{\lambda^2(s)}H(s,\xi_s) + \frac{\partial_1 H(s,\xi_s)}{\lambda(s)} + \frac{1}{2}\partial_{22}H(s,\xi_s)\lambda(s)\sigma_Y^2(s)\right)\mathrm{d}s\,\middle|\,\mathcal{H}_t\right) \\
&\quad - \mathbb{E}\left(\sum_{t\wedge\tau\le s\le\tau}\left(\frac{\Delta H(s,\xi_s)}{\lambda(s)} - \partial_2 H(s,\xi_{s-})\Delta X_s\right)\,\middle|\,\mathcal{H}_t\right),
\end{aligned}$$

where $\sigma_Y^2(s) := \frac{\mathrm{d}[Y,Y]^c_s}{\mathrm{d}s}$. Since X is optimal given (H,λ), by the equation (2.19) and (*i*) we can write for all $t \ge 0$.

$$\begin{aligned}
0 &= V_{t\wedge\tau} - \lambda(t)\mathbb{E}\left(\frac{V_\tau}{\lambda(\tau)}\,\middle|\,\mathcal{H}_t\right) \\
&\quad + \lambda(t)\,\mathbb{E}\left(\int_{t\wedge\tau}^{\tau}\left(-\frac{\lambda'(s)}{\lambda^2(s)}H(s,\xi_s) + \frac{\partial_1 H(s,\xi_s)}{\lambda(s)} + \frac{1}{2}\partial_{22}H(s,\xi_s)\lambda(s)\sigma_Y^2(s)\right)\mathrm{d}s\,\middle|\,\mathcal{H}_t\right) \\
&\quad + \lambda(t)\mathbb{E}\left(\sum_{t\wedge\tau\le s\le\tau}\left(\frac{\Delta H(s,\xi_s)}{\lambda(s)} - \partial_2 H(s,\xi_s)\Delta X_s\right)\,\middle|\,\mathcal{H}_t\right) \\
&\quad - \lambda(t)\mathbb{E}\left(\int_{t\wedge\tau}^{\tau}\lambda(s)\partial_{22}H(s,\xi_s)(\sigma_M^2(s) + \sigma_{Z,M}(s))\mathrm{d}s\,\middle|\,\mathcal{H}_t\right).
\end{aligned}$$

Hence, we have

$$
\begin{aligned}
0 &= \frac{V_{t\wedge\tau}}{\lambda(t)} - \mathbb{E}\left(\frac{V_\tau}{\lambda(\tau)}\middle|\mathcal{H}_t\right)\\
&+\mathbb{E}\left(\left(\int_{t\wedge\tau}^{\tau} -\frac{\lambda'(s)}{\lambda^2(s)}H(s,\xi_s)+\frac{\partial_1 H(s,\xi_s)}{\lambda(s)}+\frac{1}{2}\partial_{22}H(s,\xi_s)\lambda(s)(\sigma_Y^2(s)-2\sigma_{M,Y}(s))\right)\mathrm{d}s\middle|\mathcal{H}_t\right)\\
&+\mathbb{E}\left(\sum_{t\wedge\tau\le s\le\tau}\left(\frac{\Delta H(s,\xi_s)}{\lambda(s)}-\partial_2 H(s,\xi_s)\Delta X_s\right)\middle|\mathcal{H}_t\right),
\end{aligned}
\tag{3.11}
$$

where $\sigma_{M,Y}(t) := \frac{\mathrm{d}[M,Y]_t}{\mathrm{d}t} = \sigma_M^2(t)+\sigma_{M,Z}(t)$. We study the summands in the previous expression. By taking infinitesimal increments over time, we can identify the bounded variation and the martingale parts. In fact, for the first term we have

$$
\mathrm{d}\left(\frac{V_{t\wedge\tau}}{\lambda(t)} - \mathbb{E}\left(\frac{V_\tau}{\lambda(\tau)}\middle|\mathcal{H}_t\right)\right) = -\frac{\lambda'(t)}{\lambda^2(t)}V_{t\wedge\tau}\mathrm{d}t + \frac{\mathrm{d}V_{t\wedge\tau}}{\lambda(t)} - \mathrm{d}\mathbb{E}\left(\frac{V_\tau}{\lambda(\tau)}\middle|\mathcal{H}_t\right).
$$

If we define,

$$
\mathcal{M}_t := \mathbb{E}\left(\int_0^{\tau}\left(-\frac{\lambda'(s)}{\lambda^2(s)}H(s,\xi_s)+\frac{\partial_1 H(s,\xi_s)}{\lambda(s)}+\frac{1}{2}\partial_{22}H(s,\xi_s)\lambda(s)(\sigma_Y^2(s)-2\sigma_{M,Y}(s))\right)\mathrm{d}s\middle|\mathcal{H}_t\right),\quad t\ge 0,
$$

we can see that $\mathcal{M}$ is an $\mathbb{H}$-martingale and then we have

$$
\begin{aligned}
&\mathrm{d}\mathbb{E}\left(\int_{t\wedge\tau}^{\tau}\left(-\frac{\lambda'(s)}{\lambda^2(s)}H(s,\xi_s)+\frac{\partial_1 H(s,\xi_s)}{\lambda(s)}+\frac{1}{2}\partial_{22}H(s,\xi_s)\lambda(s)(\sigma_Y^2(s)-2\sigma_{M,Y}(s))\right)\mathrm{d}s\middle|\mathcal{H}_t\right)\\
&=\mathbf{1}_{[0,\tau]}(t)\left(\frac{\lambda'(t)}{\lambda^2(t)}H(t,\xi_t)-\frac{\partial_1 H(t,\xi_t)}{\lambda(t)}-\frac{1}{2}\partial_{22}H(t,\xi_t)\lambda(t)(\sigma_Y^2(t)-2\sigma_{M,Y}(t))\right)\mathrm{d}t+\mathrm{d}\mathcal{M}_t
\end{aligned}
$$

for the second term. Analogously the for the third summand we have

$$
\mathrm{d}\mathbb{E}\left(\sum_{t\wedge\tau\le s\le\tau}\left(\frac{\Delta H(s,\xi_s)}{\lambda(s)}-\partial_2 H(s,\xi_s)\Delta X_s\right)\middle|\mathcal{H}_t\right) = -\mathbf{1}_{[0,\tau]}(t)\frac{\Delta H(t,\xi_t)-\partial_2 H(t,\xi_t)\Delta\xi_t}{\lambda(t)}+\mathrm{d}\mathcal{L}_t,
$$

with

$$
\mathcal{L}_t := \mathbb{E}\left(\sum_{0\le s\le\tau}\left(\frac{\Delta H(s,\xi_s)}{\lambda(s)}-\partial_2 H(s,\xi_s)\Delta X_s\right)\middle|\mathcal{H}_t\right).
$$

Then the continuous and jump parts of the bounded variation part of (3.11) will be equal to zero. So

$$
\frac{\Delta H(t,\xi_t)-\partial_2 H(t,\xi_t)\Delta\xi_t}{\lambda(t)} = 0 \quad \text{a.s. on } [[0,\tau)) \tag{3.12}
$$

and

$$
0 = \frac{\lambda'(t)}{\lambda^2(t)}V_t - \frac{\lambda'(t)}{\lambda^2(t)}H(t,\xi_t)+\frac{\partial_1 H(t,\xi_t)}{\lambda(t)}+\frac{1}{2}\partial_{22}H(t,\xi_t)\lambda(t)(\sigma_Y^2(t)-2\sigma_{M,Y}(t)) \quad \text{a.s. on } [[0,\tau)), \tag{3.13}
$$

which gives (ii). Recall that (X,λ,H) is an equilibrium and that the prices are rational given X. So, by taking conditional expectations with respect to $\mathcal{F}_t$ in (3.13), we have

$$
\begin{aligned}
0 &= \frac{\lambda'(t)}{\lambda^2(t)}\left(\mathbb{E}(V_t|\mathcal{F}_t)-\mathbb{E}(H(t,\xi_t)|\mathcal{F}_t)\right)+\frac{\partial_1 H(t,\xi_t)}{\lambda(t)}+\frac{1}{2}\partial_{22}H(t,\xi_t)\lambda(t)\mathbb{E}\left(\sigma_Y^2(t)-2\sigma_{M,Y}(t)|\mathcal{F}_t\right)\\
&=\frac{\partial_1 H(t,\xi_t)}{\lambda(t)}+\frac{1}{2}\partial_{22}H(t,\xi_t)\lambda(t)\left(\sigma_Y^2(t)-2\mathbb{E}\left(\sigma_{M,Y}(t)|\mathcal{F}_t\right)\right) \quad \text{a.s. on } [[0,\tau)),
\end{aligned}
\tag{3.14}
$$

because of the rationality of prices, which gives (iii). □

Proposition 3.3. *Assume that* (X, λ, H) *with* $\lambda \in C^1$ *is an equilibrium. If in addition the pricing rule* $H(t, \cdot)$ *is linear, for all* t*, or the optimal strategy* X *is absolutely continuous, then we have:*
(i) Y *is an* $\mathbb{F}$*-local martingale;*
(ii) If $V_t \neq P_t$ *a.s. (except for a set with* $\mathrm{d}\mathbb{P} \otimes \mathrm{d}t$ *zero measure) on* $[[0, \tau))$*, then* $\lambda(t) = \lambda_0 > 0$.

Proof. (i) From (3.12) and (3.14) we have

$$\mathrm{d}P_t = dH(t, \xi_t) = \lambda(t)\partial_2 H(t, \xi_{t-})\mathrm{d}Y_t,$$

and, since P is an $\mathbb{F}$- martingale and $\lambda(t)\partial_2 H(t, y) > 0$, we have that Y is an $\mathbb{F}$-local martingale.
(ii) From (3.12) and (3.14) we have that

$$\frac{\lambda'(t)}{\lambda^2(t)} V_t - \frac{\lambda'(t)}{\lambda^2(t)} H(t, \xi_t) = 0,$$

then $V_t \neq H(t, \xi_t)$ implies that $\lambda'(t) = 0$. □

Example 3.1. Consider the case $\tau \equiv 1$, $V_t \equiv V$ and such that $\log V \sim N(m, v^2)$, $Z = \sigma B$ where B is a Brownian motion. Assume that the price functions are of the form

$$H(t, u) = \exp\left\{ m + \frac{v^2}{2} + \frac{v}{\lambda}\frac{1}{\sigma(1-\alpha)}u - \frac{1}{2}\frac{1+\alpha}{1-\alpha}v^2 t \right\}, \quad 0 < \alpha < 1.$$

Note that

$$\partial_1 H(t, u) = -\frac{1}{2} H(t, u)\frac{1+\alpha}{1-\alpha}v^2$$

and

$$\partial_{22} H(t, u) = H(t, u)\left(\frac{v}{\lambda}\right)^2 \frac{1}{\sigma^2(1-\alpha)^2}.$$

So we have

$$\partial_1 H(t, u) + \frac{1}{2}\partial_{22} H(t, u)\lambda^2\sigma^2(1-\alpha^2) = 0.$$

We look for optimal strategies of the form

$$\mathrm{d}X_t = \mathrm{d}M_t + \mathrm{d}\theta_t,$$

where M is an $\mathbb{H}$-martingale and such that $[X, Z]_t = -\alpha\sigma^2 t$, $0 \le t \le 1$. Let $\bar{Y}$ be the solution of

$$\bar{Y}_t = \sigma(1-\alpha)B_t + \int_0^t \frac{\bar{Y}_1 - \bar{Y}_s}{1-s}\mathrm{d}s,$$

where we take

$$\bar{Y}_1 = \sigma(1-\alpha)\frac{\log V - m}{v}.$$

Then if we set

$$X_t = -\sigma\alpha B_t + \int_0^t \frac{\bar{Y}_1 - \bar{Y}_s}{1-s}\mathrm{d}s + v\sigma\alpha t, \quad 0 \le t \le 1,$$

we also have that

$$Y_1 = \bar{Y}_1 + v\sigma\alpha,$$

and

$$P_1 = H(1, \lambda Y_1) = \exp\left\{ m + \frac{v}{\sigma(1-\alpha)}\bar{Y}_1 \right\} = V.$$

Then X satisfies the necessary conditions to be an equilibrium in the class of strategies with quadratic variation equal to $\sigma^2\alpha^2 t$.

3.2 Characterization of the equilibrium

In this subsection we shall give necessary and sufficient conditions to guarantee that (H, λ, X) is an equilibrium in the context of pricing rules $(H, \lambda) \in \mathfrak{H}$ satisfying

$$0 = \partial_1 H(t, y) + \frac{1}{2}\partial_{22} H(t, y)\lambda(t)^2\sigma^2(t) \qquad \text{a.a. } t \geq 0,\ y \in \mathbb{R}, \tag{3.15}$$

where σ^2 is a deterministic and càdlàg function and $0 < \sigma^2(t) \leq \sigma_Z^2(t)$ for a.a. t. Condition (3.15) specifies a subclass of pricing rules (Definition 2.1) and thus of admissible strategies (Definition 2.2). Note that condition (3.15) is close to condition (iii) in Proposition 3.2 (with $\sigma^2(t) = \mathbb{E}\left(\sigma_Z^2(t) - \sigma_M^2(t) \middle| \mathcal{F}_t\right)$), which is a necessary condition for the equilibrium. Observe also that the pricing rules (H, λ) are deterministic, by construction. Consequently, if we consider pricing rules satisfying (3.15), except for the linear pricing rules, we need $\mathbb{E}\left(\sigma_Z^2(t) - \sigma_M^2(t) \middle| \mathcal{F}_t\right) = \mathbb{E}\left(\sigma_Z^2(t) - \sigma_M^2(t)\right)$. One possibility is that $\sigma_Z^2(t) - \sigma_M^2(t) = \sigma^2(t)$ is deterministic in equilibrium. This is what we consider here. Note also that we do not enter into the study of the case $\sigma^2 = \sigma^2(t, y)$, which could be object for future research.

Moreover, if we consider pricing rules satisfying (3.15), by (iii) in Proposition 3.2, we obtain that, in the equilibrium,

$$\partial_1 H(t, \xi_t) + \frac{1}{2}\partial_{22} H(t, \xi_t)\lambda^2(t)(\sigma_Z^2(t) - \sigma_M^2(t)) = 0, \tag{3.16}$$

now if $\partial_{22} H(t, y) \neq 0$ and (3.15) holds, we have that $\sigma_Z^2(t) - \sigma_M^2(t) = \sigma^2(t)$. So, we will have equilibrium only in the class of admissible strategies with $\sigma_M^2(t) = \sigma_Z^2(t) - \sigma^2(t)$.

Definition 3.1. *Let C denote the class of those admissible strategies (Definition 2.2) such that*

$$\sigma_M^2(t) = \sigma_Z^2(t) - \sigma^2(t).$$

Theorem 3.1. *Consider an admissible triple (H, λ, X) with (H, λ) satisfying (3.15) with $\partial_{22} H(t, y) \neq 0$ for all $(t, y) \in \mathbb{R}_+ \times \mathbb{R}$, $\lambda(t) = \lambda_0 > 0$, and $\int_0^t \mathbb{E}\Big(\big(\partial_2 H(s, \lambda_0 \int_0^s \sigma(u)\mathrm{d}B_u)\big)^2\Big)\sigma^2(s)\mathrm{d}s < \infty$, for all $t \geq 0$, where B is a Brownian motion independent of τ. Then (H, λ, X) is an equilibrium, in the class C, if and only if the following conditions hold:*

(i) $H(\tau, \xi_\tau) = V_\tau$

(ii) $Y = X + Z$ *has no jumps*

(iii) $Y_t + \lambda_0 \displaystyle\int_0^t \frac{\partial_{22} H(s, \xi_s)}{\partial_2 H(s, \xi_s)}(\sigma_{M,Z}(s) + \sigma_M^2(s))\mathrm{d}s,\ 0 \leq t < ess\sup\tau,$ *is an $\mathbb{F}$-local martingale.*

Proof. Assume $(i) - (iii)$, we show that (H, λ, X) is an equilibrium. Consider a process ς such that

$$\varsigma_t := \lambda_0 \int_0^t \sigma(s)\mathrm{d}B_s.$$

where B is a Brownian motion independent of τ (possibly defined in an extension of $(\Omega, \mathcal{F}, \mathbb{P})$). First if $H(t, y)$ is a solution of (3.15).

$$H(t, \varsigma_t) = H(0, 0) + \lambda_0 \int_0^t \partial_2 H(s, \varsigma_s)\sigma(s)\mathrm{d}B_s,$$

then, by the hypothesis, $(H(t, \varsigma_t))_{t\geq 0}$ is a martingale (w.r.t. its own filtration) and since ς has independent increments and τ is bounded and independent of ς

$$H(t \wedge \tau, y) = \mathbb{E}\left(H(\tau, \varsigma_\tau) \middle| \varsigma_{t\wedge\tau} = y, \tau\right) = \mathbb{E}\left(H(\tau, y + \varsigma_\tau - \varsigma_{t\wedge\tau}) \middle| \tau\right).$$

Set now, for $T \in [0, \infty)$,

$$i(T, y, v) := \int_y^{H^{-1}(T,\cdot)(v)} \frac{v - H(T, x)}{\lambda_0}\mathrm{d}x$$

and define

$$I(t,y,v) := \mathbb{E}\left(i\left(\tau, y+\varsigma_\tau - \varsigma_{t\wedge\tau}, v\right) \middle| \tau\right), \quad t\geq 0.$$

Note that $I(t,y,v)$ is a random-field. We have that

$$\begin{aligned}\partial_2 I(t,y,v) &= \mathbb{E}\left(\partial_2 i(\tau, y+\varsigma_\tau - \varsigma_{t\wedge\tau}, v) \middle| \tau\right) \\ &= \mathbb{E}\left(-\frac{v - H(\tau, y+\varsigma_\tau - \varsigma_{t\wedge\tau})}{\lambda_0} \middle| \tau\right) = \frac{-v + H(t\wedge\tau, y)}{\lambda_0}. \end{aligned} \tag{3.17}$$

We can take the derivative under the expectation sign because $H(\tau(\omega), \cdot)$ is monotone and $\mathbb{E}(H(\tau,\varsigma_\tau)|\tau) < \infty$. Then $I(t,y,v)$ is well defined and

$$\begin{aligned} I(t,y,v) &= \mathbb{E}\left(i\left(\tau, y+\varsigma_\tau - \varsigma_{t\wedge\tau}, v\right) \middle| \tau\right) \\ &= \mathbb{E}(\, i(\tau, \varsigma_\tau, v) | \, \varsigma_{t\wedge\tau} = y, \tau), \end{aligned}$$

then, fixed v, $(I(t,\varsigma_{t\wedge\tau}, v))_{t\geq 0}$ is a martingale (w.r.t. its own filtration), so

$$\partial_1 I(t,\varsigma_t, v) + \frac{1}{2}\partial_{22} I(t,\varsigma_t,v)\lambda_0^2\sigma^2(t) = 0, \; a.s. \text{ on } [[0,\tau]]. \tag{3.18}$$

Now, consider an admissible strategy X, by using Itô-Wentzell's formula (see for instance Bank & Baum (2004)), we have

$$\begin{aligned} I(\tau,\xi_\tau,V_\tau) &= I(0,0,V_0) + \int_0^\tau \partial_3 I(t,\xi_{t-},V_{t-})\mathrm{d}V_t + \int_0^\tau \partial_1 I(t,\xi_t,V_t)\mathrm{d}t \\ &+ \int_0^\tau \partial_2 I(t,\xi_{t-},V_{t-})\mathrm{d}\xi_t + \frac{1}{2}\int_0^\tau \partial_{22} I(t,\xi_t,V_t)\mathrm{d}[\xi^c,\xi^c]_t \\ &+ \int_0^\tau \partial_{23} I(t,\xi_t,V_t)\mathrm{d}[\xi^c,V^c]_t + \frac{1}{2}\int_0^\tau \partial_{33} I(t,\xi_t,V_t)\sigma_V^2(t)\mathrm{d}t \\ &+ \sum_{0\leq t\leq\tau} \left(\Delta I(t,\xi_t,V_t) - \partial_2 I(t,\xi_{t-},V_t)\Delta\xi_t\right) \\ &+ \sum_{0\leq t\leq\tau} \left(\Delta I(t,\xi_t,V_t) - \partial_2 I(t,\xi_t,V_{t-})\Delta V_t\right) \end{aligned}$$

By construction, $\xi_0 = 0$ and $\mathrm{d}\xi_t = \lambda_0 \mathrm{d}Y_t$. Now we have that

$$\mathrm{d}[\xi^c,\xi^c]_t = \lambda_0^2 \mathrm{d}[X^c,X^c]_t + 2\lambda_0^2\mathrm{d}[X^c,Z]_t + \lambda_0^2\sigma_Z^2(t)\mathrm{d}t.$$

Also by (3.17) and the fact that V and Z are independent,

$$\partial_{23} I(t,\xi_t,V_t)\mathrm{d}[\xi^c,V^c]_t = -\frac{1}{\lambda_0}\mathrm{d}[\xi^c,V^c]_t = -\mathrm{d}[X,V]_t,$$

and the last equality is because since X is predictable and V a martingale X and V cannot jump at the same time (see Corollary 2.31 in Jacod & Shiryaev (1987)). Then using (3.17) and (3.18), and the fact that Z has not jumps, we get

$$\begin{aligned} I(\tau,\xi_\tau,V_\tau) &= I(0,0,V_0) + \int_0^\tau \partial_3 I(t,\xi_{t-},V_{t-})\mathrm{d}V_t + \int_0^\tau (P_{t-} - V_{t-})(\mathrm{d}X_t + \mathrm{d}Z_t) \\ &+ \frac{1}{2}\int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda_0^2\mathrm{d}[X^c,X^c]_t - [X,V]_\tau + \frac{1}{2}\int_0^\tau \partial_{33} I(t,\xi_t,V_t)\sigma_V^2(t)\mathrm{d}t \\ &+ \int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda_0^2\mathrm{d}[X^c,Z]_t + \frac{1}{2}\int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda_0^2\left(\sigma_Z^2(t) - \sigma^2(t)\right)\mathrm{d}t \\ &+ \sum_{0\leq t\leq\tau} \left(I(t,\xi_t,V_t) - I(t,\xi_{t-},V_t) - \partial_2 I(t,\xi_{t-},V_t)\lambda_0\Delta X_t\right) \\ &+ \sum_{0\leq t\leq\tau} \left(I(t,\xi_t,V_t) - I(t,\xi_t,V_{t-}) - \partial_2 I(t,\xi_t,V_{t-})\Delta V_t\right) \end{aligned}$$

Subtracting $[P, X]_\tau$ from both sides and rearranging the terms, we obtain

$$\begin{aligned}
&\int_0^\tau (V_{t-} - P_{t-})\mathrm{d}X_t - [P, X]_\tau + [X, V]_\tau - \left(I(0, 0, V_0) + \frac{1}{2}\int_0^\tau \partial_{33} I(t, \xi_t, V_t)\sigma_V^2(t)\mathrm{d}t \right) \\
&= -I(\tau, \xi_\tau, V_\tau) + \int_0^\tau \partial_3 I(t, \xi_{t-}, V_{t-})\mathrm{d}V_t + \int_0^\tau (P_t - V_t)\mathrm{d}Z_t \\
&+ \frac{1}{2}\int_0^\tau \partial_{22} I(t, \xi_t, V_t)\lambda_0^2 \mathrm{d}[X^c, X^c]_t + \int_0^\tau \partial_{22} I(t, \xi_t, V_t)\lambda_0^2 \mathrm{d}[X^c, Z]_t \\
&+ \frac{1}{2}\int_0^\tau \partial_{22} I(t, \xi_t, V_t)\lambda_0^2 \left(\sigma_Z^2(t) - \sigma^2(t)\right)\mathrm{d}t \\
&+ \sum_{0 \le t \le \tau} \left(I(t, \xi_t, V_t) - I(t, \xi_t, V_{t-}) - \partial_3 I(t, \xi_t, V_{t-})\Delta V_t\right) \\
&+ \sum_{0 \le t \le \tau} \left(I(t, \xi_t, V_t) - I(t, \xi_{t-}, V_t) - \partial_2 I(t, \xi_{t-}, V_t)\lambda_0 \Delta X_t\right) - [P, X]_\tau. \qquad (3.19)
\end{aligned}$$

We have that

$$[P, X]_\tau = [P^c, X^c]_\tau + \sum_{0 \le t \le \tau} \Delta P_t \Delta X_t.$$

Then Itô's formula for H shows that the continuous local martingale part of P is $\int \partial_2 H(t, \xi_t)\mathrm{d}\xi_t^c$, so by using (3.17), we obtain

$$\begin{aligned}
[P^c, X^c]_\tau &= \left[\int_0^\cdot \partial_2 H(t, \xi_t)\mathrm{d}\xi_t^c, X^c\right]_\tau = \int_0^\tau \partial_2 H(t, \xi_t)\mathrm{d}\left[\xi^c, X^c\right]_t \\
&= \int_0^\tau \partial_{22} I(t, \xi_t, V_t)\lambda_0^2 \mathrm{d}\left[X^c, X^c\right]_t + \int_0^\tau \partial_{22} I(t, \xi_t, V_t)\lambda_0^2 \mathrm{d}\left[X^c, Z\right]_t,
\end{aligned}$$

and

$$\begin{aligned}
\lambda_0 \partial_2 I(t, \xi_{t-}, V_t)\Delta X_t + \Delta P_t \Delta X_t &= (P_{t-} - V_t)\Delta X_t + \Delta P_t \Delta X_t \\
&= (P_t - V_t)\Delta X_t = \lambda_0 \partial_2 I(t, \xi_t, V_t)\Delta X_t.
\end{aligned}$$

Substituting the above relationships in the right-hand side of the equation (3.19), it becomes

$$\begin{aligned}
&- I(\tau, \xi_\tau, V_\tau) + \int_0^\tau \partial_3 I(t, \xi_{t-}, V_{t-})\mathrm{d}V_t + \int_0^\tau (P_t - V_t)\mathrm{d}Z_t - \frac{1}{2}\int_0^\tau \partial_{22} I(t, \xi_t, V_t)\lambda_0^2 \mathrm{d}[X^c, X^c]_t \\
&+ \frac{1}{2}\int_0^\tau \partial_{22} I(t, \xi_t, V_t)\lambda_0^2 \left(\sigma_Z^2(t) - \sigma^2(t)\right)\mathrm{d}t \\
&+ \sum_{0 \le t \le \tau} \left(I(t, \xi_t, V_t) - I(t, \xi_{t-}, V_t) - \lambda_0 \partial_2 I(t, \xi_t, V_t)\Delta X_t\right) \\
&+ \sum_{0 \le t \le \tau} \left(I(t, \xi_t, V_t) - I(t, \xi_t, V_{t-}) - \partial_3 I(t, \xi_t, V_{t-})\Delta V_t\right) \\
&= -I(\tau, \xi_\tau, V_\tau) + \int_0^{\tau-} \partial_3 I(t, \xi_{t-}, V_{t-})\mathrm{d}V_t + \int_0^\tau (P_t - V_t)\mathrm{d}Z_t \\
&+ \sum_{0 \le t \le \tau} \left(I(t, \xi_t, V_t) - I(t, \xi_t, V_{t-}) - \partial_3 I(t, \xi_t, V_{t-})\Delta V_t\right) \\
&+ \sum_{0 \le t \le \tau} \left(I(t, \xi_t, V_t) - I(t, \xi_{t-}, V_t) - \lambda_0 \partial_2 I(t, \xi_t, V_t)\Delta X_t\right). \qquad (3.20)
\end{aligned}$$

Recall the expected total wealth of an insider's strategy (2.11). Then, taking the expectation in the right-hand side of (3.19), or equivalently of (3.20), we show that the maximum is achieved at X. For this it is important to note that $\partial_{33} I(t, y, v)$ does not depend on y and so $\partial_{33} I(t, \xi_t, V_t)$ does not depend of ξ. Then $I(0, 0, V_0) + \frac{1}{2}\int_0^\tau \partial_{33} I(t, \xi_t, V_t)\sigma_V^2(t)\mathrm{d}t$ has the same value for *any* insider's strategy. The result follows from the following points.

1. (ii) guarantees that $\Delta X_t = 0$.
2. The processes $\int_0^\cdot \partial_3 I(t,\xi_t,V_t)\mathrm{d}V_t$ and $\int_0^\cdot (P_t - V_t)\mathrm{d}Z_t$ are martingales by (A5) and (A2) in Definition 2.2, hence they have null expectation.
3. The term $\sum_{0\le t<\tau} \left(I(t,\xi_t,V_t) - I(t,\xi_t,V_{t-}) - \partial_2 I(t,\xi_t,V_{t-})\Delta V_t\right)$ does not depend on ξ:
$$\begin{aligned}&I(t,\xi_t,V_t) - I(t,\xi_t,V_{t-}) - \partial_2 I(t,\xi_t,V_{t-})\Delta V_t\\ &= \int_{H^{-1}(T,\cdot)(V_{t-})}^{H^{-1}(T,\cdot)(V_t)} \frac{V_t - H(T,x)}{\lambda_0}\mathrm{d}x.\end{aligned}$$
4. We know that $\lambda_0\partial_{22} I(\tau,\xi_\tau,V_\tau) = \partial_2 H(\tau,\xi_\tau) > 0$ and that $\lambda_0\partial_2 I(\tau,\xi_\tau,V_\tau) = -V_\tau + H(\tau,\xi_\tau)$ so by (i) we have a maximum value of $-E[I(\tau,\xi_\tau,V_\tau)]$ for our strategy X.

Assumption (iii) and (i) together with condition (A2) in Definition 2.2 guarantee the rationality of prices, given X. In fact from (3.15)

$$\mathrm{d}P_t = \lambda_0\partial_2 H(t,\xi_t)\mathrm{d}Y_t + \frac{1}{2}\lambda_0^2\left(\sigma_Y^2(t) - \sigma^2(t)\right)\partial_{22} H(t,\xi_t)\mathrm{d}t$$

and by (ii)

$$\begin{aligned}\mathrm{d}P_t &= \lambda_0\partial_2 H(t,\xi_t)\mathrm{d}Y_t + \lambda_0^2\left(\sigma_M^2(t) + \sigma_{M,Z}(t)\right)\partial_{22} H(t,\xi_t)\mathrm{d}t\\ &= \lambda_0\partial_2 H(t,\xi_t)\left(\mathrm{d}Y_t + \lambda_0\left(\sigma_M^2(t) + \sigma_{M,Z}(t)\right)\frac{\partial_{22} H(t,\xi_t)}{\partial_2 H(t,\xi_t)}\mathrm{d}t\right)\end{aligned}$$

so, P is an $\mathbb{F}$-local martingale and, by condition (A2) in Definition 2.2, it is an $\mathbb{F}$-martingale. Then from (i), and on the set $\{t\le\tau\}$ we have

$$\mathbb{E}\left(H(\tau,\xi_\tau)\middle|\mathcal{F}_t\right) = \mathbb{E}\left(V_\tau\middle|\mathcal{F}_t\right) = \mathbb{E}\left(\mathbb{E}\left(V_\tau\middle|\mathcal{H}_t\right)\middle|\mathcal{F}_t\right) = \mathbb{E}\left(V_t\middle|\mathcal{F}_t\right).$$

Conversely, assume that (H,λ,X) is an equilibrium. We show that $(i)-(iii)$ hold true. First note that (i) is a necesssary condition for equilibrium by (i) in Proposition 3.2. Now, from the computations above we can see that $\partial_{22} I = \frac{\partial_2 H}{\lambda_0} > 0$ (convexity) implies that

$$I(t,x+h,v) - I(t,x,v) - \partial_2 I(t,x,v)h \le 0, \quad \text{for any } h.$$

So,

$$\sum_{0\le t\le\tau}\left(I(t,\xi_{t-} + \lambda_0\Delta X_t, V_t) - I(t,\xi_{t-},V_t) - \partial_2 I(t,\xi_t,V_t)\lambda_0\Delta X_t\right) \le 0.$$

Since X is optimal, then $\Delta X_t = 0$. So (ii) is a necessary condition for equilibrium. Finally, from the Itô formula, we have that

$$\mathrm{d}Y_t + \lambda_0\left(\sigma_M^2(t) + \sigma_{M,Z}(t)\right)\frac{\partial_{22} H(t,\xi_t)}{\partial_2 H(t,\xi_t)}\mathrm{d}t = \frac{\mathrm{d}P_t}{\lambda_0\partial_2 H(t,\xi_t)}.$$

Since prices are rational, given X, then we see that (iii) holds true. □

Remark 3.4. Notice that (3.20) is true for strategies in the class C since for strategies in this class we have

$$-\frac{1}{2}\int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda_0^2\mathrm{d}[X^c,X^c]_t + \frac{1}{2}\int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda_0^2\left(\sigma_Z^2(t) - \sigma^2(t)\right)\mathrm{d}t = 0.$$

However given a strategy X in the class C, there is always an absolutely continuous strategy strictly better, in the sense that the final optimal wealth is higher. In fact we can approximate X by an absolutely continuous process, say $\tilde{X}$, and then

$$\begin{aligned}&-\frac{1}{2}\int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda_0^2\mathrm{d}[X^c,X^c]_t + \frac{1}{2}\int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda_0^2\left(\sigma_Z^2(t) - \sigma^2(t)\right)\mathrm{d}t\\ &< \frac{1}{2}\int_0^\tau \partial_{22} I(t,\tilde{\xi}_t,V_t)\lambda_0^2\left(\sigma_Z^2(t) - \sigma^2(t)\right)\mathrm{d}t,\end{aligned}$$

by the continuity $\partial_{22}I$. This proves that if $\sigma_Z^2(t) - \sigma^2(t) > 0$ class C is suboptimal. These results are in agreement with that of Theorem 2 and Lemma 2 in Back (1992). However equation (14) in Theorem 2, Back (1992), is obtained under the restrictive assumption that the quadratic variation of the total aggregate demand is equal to that of the noise traders. This results in forcing the continuous part of the insider's strategies not to have a continuous martingale part, as it is shown in Lemma 2, Back (1992), which uses equation (14) in Theorem 2, Back (1992), in its proof.

Remark 3.5. We have seen that there is not a substantial difference when we introduce jumps in the fundamental value V, so we assume for simplicity that V is continuous in the following.

For the linear pricing rules case or (3.15) with $\sigma_Z^2(t) = \sigma^2(t)$ for $t \geq 0$, we have an equilibrium in the class of all admissible strategies:

Theorem 3.2. *Consider an admissible triple (H, λ, X) with (H, λ) satisfying (3.15) with $\partial_{22}H(t, y) = 0$ for all $(t, y) \in \mathbb{R}_+ \times \mathbb{R}$, or pricing rules satisfying (3.15) with $\sigma_Z^2(t) = \sigma^2(t)$ for $t \geq 0$, and $\lambda(t) = \lambda_0$ for $t \geq 0$. Assume that V is a continuous martingale. Then (H, λ, X) is an equilibrium if and only if the following conditions hold:*

$$
\begin{aligned}
&(i)\ H(\tau, \xi_\tau) = V_\tau \\
&(ii)\ \sigma_M(t) = 0\ , \quad 0 \leq t < ess \sup \tau \\
&(iii)\ Y = X + Z \textit{ has no jumps} \\
&(iv)\ Y_t,\ 0 \leq t \leq ess \sup \tau, \textit{ is an } \mathbb{F}\textit{-local martingale}
\end{aligned}
$$

Proof. Let (H, λ, X) be an equilibrium. First note that if H is linear and $\lambda(t) = \lambda_0$, point (ii) in Proposition 3.2 does not imply any condition on σ_M. Secondly,

$$
\mathrm{d}P_t = \lambda_0 \partial_2 H(t, \xi_t) \mathrm{d}Y_t,
$$

so we have (iv). Thirdly, by (3.17), $\partial_{22}I(t, \xi_t, V_t) = C(t) > 0$ (deterministic). So, in the linear case, the term

$$
\frac{1}{2} \int_0^\tau \partial_{22} I(t, \xi_t, V_t) \lambda_0^2 \left(\sigma_Z^2(t) - \sigma^2(t) \right) \mathrm{d}t,
$$

in the equality (3.19), does not depend on the insider's strategy, and obviously neither when $\sigma_Z^2 = \sigma^2$. Then we can pass the term to the left-hand side of (3.19) and

$$
I(0, 0, V_0) + \frac{1}{2} \int_0^\tau \partial_{33} I(t, \xi_t, V_t) \sigma_V^2 \mathrm{d}t + \frac{1}{2} \int_0^\tau \partial_{22} I(t, \xi_t, V_t) \lambda_0^2 \left(\sigma_Z^2(t) - \sigma^2(t) \right) \mathrm{d}t
$$

becomes a true bound for the insider's wealth. We have that

$$
\begin{aligned}
&\int_0^\tau (V_t - P_{t-}) \mathrm{d}X_t - [P, X]_\tau + [X, V]_\tau \\
&- \left(I(0, 0, V_0) + \frac{1}{2} \int_0^\tau \partial_{33} I(t, \xi_t, V_t) \sigma_V^2 \mathrm{d}t + \frac{1}{2} \int_0^\tau \partial_{22} I(t, \xi_t, V_t) \lambda_0^2 \left(\sigma_Z^2(t) - \sigma^2(t) \right) \mathrm{d}t \right) \\
&= -I(\tau, \xi_\tau, V_\tau) + \int_0^\tau \partial_3 I(t, \xi_{t-}, V_t) \mathrm{d}V_t + \int_0^\tau (P_t - V_t) \mathrm{d}Z_t \\
&- \frac{1}{2} \int_0^\tau \partial_{22} I(t, \xi_t, V_t) \lambda_0^2 \mathrm{d}[X^c, X^c]_t + \sum_{0 \leq t \leq \tau} \left(\Delta I(t, \xi_t, V_t) - \partial_2 I(t, \xi_t, V_t) \lambda_0 \Delta X_t \right).
\end{aligned}
$$

Now, the arguments in the proof of the previous theorem apply and we obtain (i) and (iii) . Finally, since

$$
-\frac{1}{2} \int_0^\tau \partial_{22} I(t, \xi_t, V_t) \lambda_0^2 \mathrm{d}[X^c, X^c]_t \leq 0,
$$

its maximum value is achieved if and only if $[X^c, X^c] \equiv 0$ and we conclude (ii). The converse is directly obtained from the previous theorem. $\square$

4 Case when τ is not predictable to the insider

In this section we consider the case when the insider cannot predict the time τ of release of information. We also assume that the (finite) stopping time τ is independent of the rest of observable random objects $(V, P, Z, ...)$ that is

$$\mathcal{H}_t = \overline{\mathcal{G}_t \vee \sigma(\tau \wedge s, 0 \leq s \leq t)},$$

with $\mathcal{G}_t$ independent of τ, that $\mathbb{P}(\tau > t) > 0$ for all $0 \leq t < T \in \bar{\mathbb{R}}_+$ and that τ has a density with respect to the Lebesgue measure. For the sake of simplicity we also assume that V is a continuous martingale. All these are standing assumptions for this section.

Proposition 4.1. *Consider an admissible triple (H, λ, X) with $\lambda \in C^1$ and $\lim_{\bar{T}\uparrow T} \frac{\mathbb{P}(\tau>\bar{T})}{\lambda(\bar{T})} =: c < \infty$. If (H, λ, X) is an equilibrium, we have:*

(i) $\lim_{\bar{T}\uparrow T} H(\bar{T}, \xi_{\bar{T}}) = \lim_{\bar{T}\uparrow T} V_{\bar{T}}$ *a.s. on* $[[0, \tau))$.

(ii) $\partial_t \left(\frac{\mathbb{P}(\tau > t)}{\lambda(t)}\right)(V_t - H(t, \xi_t)) - \frac{\mathbb{P}(\tau > t)}{\lambda(t)}\partial_1 H(t, \xi_t) - \frac{1}{2}\partial_{22}H(t, \xi_t)\mathbb{P}(\tau > t)\lambda(t)(\sigma_Y^2(t) - 2\sigma_{M,Y}(t)) = 0,$

(iii) $\partial_1 H(t, \xi_t) + \frac{1}{2}\partial_{22}H(t, \xi_t)\lambda^2(t)\mathbb{E}\left(\sigma_Z^2(t) - \sigma_M^2(t)|\mathcal{F}_t\right) = 0$ *a.s. on* $[[0, \tau))$.

Proof. Going back to Theorem 2.1, we can see that equation (2.19) can be written as:

$$\mathbf{1}_{[0,\tau]}(t)\left(V_t - H(t, \xi_t)\right) - \lambda(t)\mathbb{E}\left(\int_t^T \mathbf{1}_{[0,\tau]}(s)\left(\partial_2 H(s, \xi_s)\mathrm{d}^- X_s\right)|\mathcal{H}_t\right) = 0 \quad t \geq 0, \text{ a.s.}$$

We recall that the optimal total demand X for the insider satisfies (A1) - (A6) in Definition 2.2. Then we have

$$\begin{aligned}
0 &= \mathbf{1}_{[0,\tau]}(t)\left(V_t - H(t, \xi_t)\right) - \lambda(t)\mathbb{E}\left(\int_t^T \mathbf{1}_{[0,\tau]}(s)\left(\partial_2 H(s, \xi_s)\mathrm{d}^- X_s\right)\middle| \mathcal{H}_t\right) \\
&= \mathbf{1}_{[0,\tau]}(t)\left(V_t - H(t, \xi_t)\right) - \lambda(t)\mathbb{E}\left(\int_t^T \mathbf{1}_{[0,\tau]}(s)\partial_2 H(s, \xi_s)\theta_s \mathrm{d}s\middle| \mathcal{H}_t\right) \\
&- \lambda(t)\mathbb{E}\left(\sum_t^T \mathbf{1}_{[0,\tau]}(s)\partial_2 H(s, \xi_s)\Delta X_s\middle| \mathcal{H}_t\right) \\
&- \lambda(t)\mathbb{E}\left(\int_t^T \mathbf{1}_{[0,\tau]}(s)\lambda(s)\partial_{22}H(s, \xi_s)(\sigma_M^2(s) + \sigma_{Z,M}(s))\mathrm{d}s\middle| \mathcal{H}_t\right) \\
&= \mathbf{1}_{[0,\tau]}(t)\left(V_t - H(t, \xi_t)\right) - \lambda(t)\mathbb{E}\left(\int_t^T \mathbb{P}(\tau > s|\mathcal{H}_t)\partial_2 H(s, \xi_s)\theta_s \mathrm{d}s\middle| \mathcal{H}_t\right)\mathbf{1}_{[0,\tau]}(t) \\
&- \lambda(t)\mathbb{E}\left(\sum_t^T \mathbb{P}(\tau > s|\mathcal{H}_t)\partial_2 H(s, \xi_s)\Delta X_s\middle| \mathcal{H}_t\right)\mathbf{1}_{[0,\tau]}(t) \\
&- \lambda(t)\mathbb{E}\left(\int_t^T \mathbb{P}(\tau > s|\mathcal{H}_t)\lambda(s)\partial_{22}H(s, \xi_s)(\sigma_M^2(s) + \sigma_{Z,M}(s))\mathrm{d}s\middle| \mathcal{H}_t\right)\mathbf{1}_{[0,\tau]}(t).
\end{aligned}$$

Where in the third equality we apply conditional Fubini (see for instance Theorem 1.1.7 in Applebaum 2004) and the independence of τ with respect to $\mathcal{G}_t$. Observe that, for $s > t$,

$$\mathbb{P}(\tau > s|\mathcal{H}_t) = \mathbb{P}(\tau > s|\tau > t)\mathbf{1}_{[0,\tau]}(t) = \frac{\mathbb{P}(\tau > s)}{\mathbb{P}(\tau > t)}\mathbf{1}_{[0,\tau]}(t). \tag{4.1}$$

Hence, substituting in the previous expression, we have

$$0=\mathbf{1}_{[0,\tau]}(t)\left(V_t-H(t,\xi_t)\right)-\frac{\lambda(t)}{\mathbb{P}(\tau>t)}\mathbb{E}\left(\int_t^T\mathbb{P}(\tau>s)\partial_2H(s,\xi_s)\theta_s\mathrm{d}s\middle|\mathcal{H}_t\right)\mathbf{1}_{[0,\tau]}(t)$$
$$-\frac{\lambda(t)}{\mathbb{P}(\tau>t)}\mathbb{E}\left(\sum_t^T\mathbb{P}(\tau>s)\partial_2H(s,\xi_s)\Delta X_s\middle|\mathcal{H}_t\right)\mathbf{1}_{[0,\tau]}(t)$$
$$-\frac{\lambda(t)}{\mathbb{P}(\tau>t)}\mathbb{E}\left(\int_t^T\lambda(s)\mathbb{P}(\tau>s)\partial_{22}H(s,\xi_s)(\sigma_M^2(s)+\sigma_{Z,M}(s))\mathrm{d}s\middle|\mathcal{H}_t\right)\mathbf{1}_{[0,\tau]}(t)\quad t\geq 0.\text{ a.s.}\tag{4.2}$$

First of all we note that, by assumption (A3) in Definition 2.2, and Corollary (2.4) in Revuz & Yor (1999) we have that

$$\lim_{t\uparrow T}\mathbb{E}\left(\int_t^T\mathbb{P}(\tau>s)\partial_2H(s,\xi_s)|\theta_s|\mathrm{d}s\middle|\mathcal{H}_t\right)=0\quad\text{a.s.}$$

Analogously $\mathbb{E}\Big(\sum_t^T\mathbb{P}(\tau>s)\partial_2H(s,\xi_{s-})|\Delta X_s|\Big|\mathcal{H}_t\Big)$ and $\mathbb{E}\Big(\int_t^T\lambda(s)\mathbb{P}(\tau>s)\partial_{22}H(s,\xi_s)(\sigma_M^2(s)+\sigma_{Z,M}(s))\mathrm{d}s\Big|\mathcal{H}_t\Big)$ vanish for $t\uparrow T$. Then, taking the limit in (4.2), we are left with

$$\lim_{t\uparrow T}\frac{\left(V_t-H(t,\xi_t)\right)\mathbb{P}(\tau>t)}{\lambda(t)}\mathbf{1}_{[0,\tau]}(t)=0\qquad\text{a.s.}\tag{4.3}$$

This leads to (i). Moreover, applying the Itô's formula to $\frac{H(t,\xi_t)\mathbb{P}(\tau>t)}{\lambda(t)}$, $t\leq\bar{T}$, and studying the limit for $\bar{T}\to T$, we obtain

$$\mathbb{E}\left(\int_t^T\mathbb{P}\left(\tau>s\right)\partial_2H(s,\xi_{s-})\mathrm{d}X_s\middle|\mathcal{H}_t\right)$$
$$=\lim_{\bar{T}\uparrow T}\mathbb{E}\left(\frac{H(\bar{T},\xi_{\bar{T}})\mathbb{P}(\tau>\bar{T})}{\lambda(\bar{T})}\middle|\mathcal{H}_t\right)-\frac{H(t,\xi_t)\mathbb{P}(\tau>t)}{\lambda(t)}$$
$$-\mathbb{E}\left(\int_t^T(\partial_s\left(\frac{\mathbb{P}\left(\tau>s\right)}{\lambda(s)}\right)H(s,\xi_s)+\frac{\mathbb{P}\left(\tau>s\right)}{\lambda(s)}\partial_1H(s,\xi_s)\right.$$
$$\left.+\frac{1}{2}\partial_{22}H(s,\xi_s)\mathbb{P}\left(\tau>s\right)\lambda(s)\sigma_Y^2(s))\mathrm{d}s\middle|\mathcal{H}_t\right)$$
$$-\mathbb{E}\left(\sum_t^T\frac{\mathbb{P}(\tau>s)\Delta H(s,\xi_s)}{\lambda(s)}-\mathbb{P}(\tau>s)\partial_2H(s,\xi_{s-})\Delta X_s\middle|\mathcal{H}_t\right).\tag{4.4}$$

Moreover, by (4.3), we have

$$\lim_{\bar{T}\uparrow T}\mathbb{E}\left(\frac{H(\bar{T},\xi_{\bar{T}})\mathbb{P}\left(\tau>\bar{T}\right)}{\lambda(\bar{T})}\middle|\mathcal{H}_t\right)\mathbf{1}_{[0,\tau]}(t)=\lim_{\bar{T}\uparrow T}\mathbb{E}\left(\frac{V_{\bar{T}}\mathbb{P}\left(\tau>\bar{T}\right)}{\lambda(\bar{T})}\middle|\mathcal{H}_t\right)\mathbf{1}_{[0,\tau]}(t)$$
$$=\mathbf{1}_{[0,\tau]}(t)V_t\lim_{\bar{T}\uparrow T}\frac{\mathbb{P}(\tau>\bar{T})}{\lambda(\bar{T})}=c\mathbf{1}_{[0,\tau]}(t)V_t.\tag{4.5}$$

By substituting (4.4) and (4.5) into (4.2), we obtain the equation

$$0=\mathbf{1}_{[0,\tau]}(t)V_t\left(c-\frac{\mathbb{P}\left(\tau>t\right)}{\lambda(t)}\right)-\mathbf{1}_{[0,\tau]}(t)\mathbb{E}\left(\int_t^T\left(\partial_s\left(\frac{\mathbb{P}(\tau>s)}{\lambda(s)}\right)H(s,\xi_s)\right.\right.$$
$$\left.\left.+\frac{\mathbb{P}(\tau>s)}{\lambda(s)}\partial_1H(s,\xi_s)+\frac{1}{2}\partial_{22}H(s,\xi_s)\mathbb{P}\left(\tau>s\right)\lambda(s)(\sigma_Y^2(s)-2\sigma_{M,Y}(s))\right)\mathrm{d}s\middle|\mathcal{H}_t\right)$$
$$-\mathbf{1}_{[0,\tau]}(t)\mathbb{E}\left(\sum_t^T\frac{\mathbb{P}(\tau>s)\Delta H(s,\xi_s)}{\lambda(s)}-\mathbb{P}(\tau>s)\partial_2H(s,\xi_s)\Delta X_s\middle|\mathcal{H}_t\right).\tag{4.6}$$

Proceeding in the same way as in the proof of Proposition 3.2 with the right-hand side of the equation (4.6) and taking into account that $\left(\frac{\mathbf{1}_{[0,\tau]}(t)}{\mathbb{P}(\tau>t)}\right)_{t\geq 0}$ is an $\mathbb{H}$-martingale by (4.1) , we can identify the bounded variation and martingale parts. This yields that

$$\begin{aligned}0 = &\left\{\partial_t\left(\frac{\mathbb{P}(\tau>t)}{\lambda(t)}\right)(V_t - H(t,\xi_t)) + \right.\\ &\left. -\frac{\mathbb{P}(\tau>t)}{\lambda(t)}\partial_1 H(t,\xi_t) - \frac{1}{2}\partial_{22}H(t,\xi_t)\mathbb{P}(\tau>t)\,\lambda(t)(\sigma_Y^2(t) - 2\sigma_{M,Y}(t))\right\}\mathrm{d}t\\ &-\left(\frac{\mathbb{P}(\tau>t)\Delta H(t,\xi_t)}{\lambda(t)} - \frac{\mathbb{P}(\tau>t)}{\lambda(t)}\partial_2 H(t,\xi_t)\Delta X_t\right),\end{aligned}$$

a.s. on $[[0,\tau))$. The above gives us (ii). Now since (H,λ,X) is a local equilibrium, then prices are rational. By taking conditional expectations with respect to $\mathcal{F}_t$, we obtain

$$0 = \frac{\mathbb{P}(\tau>t)}{\lambda(t)}\partial_1 H(t,\xi_t) + \frac{1}{2}\partial_{22}H(t,\xi_t)\mathbb{P}(\tau>t)\,\lambda(t)\left(\sigma_Y^2(t) - 2\mathbb{E}(\sigma_{M,Y}(t)|\mathcal{F}_t)\right)$$

a.s. on $[[0,\tau))$, because of rationality of prices. This leads to (iii). ☐

The following result is then immediate.

Proposition 4.2. *Consider an admissible triple (H,λ,X) with $\lambda\in C^1$. Moreover, let $H(t,\cdot)$ be linear, for all t, or the strategy X be absolutely continuous. Then if (H,λ,X) is an equilibrium we have:*

$$(i)\ Y \text{ is an } \mathbb{F}\text{-local martingale} \tag{4.7}$$

$$(ii)\ \text{If } V_t\neq P_t,\ \text{a.s. on } [[0,\tau)),\ \text{then } \lambda(t) = c\mathbb{P}(\tau>t) \text{ with } c>0\,. \tag{4.8}$$

Proof. The result derives from the proposition before with an argument similar to the one in Proposition 3.3. ☐

Now we study sufficient conditions for having an equilibrium. We obtain a result in line with Theorem 3.1.

Theorem 4.1. *Consider an admissible triple (H,λ,X) with (H,λ) satisfying*

$$\partial_1 H(t,y) + \frac{1}{2}\partial_{22}H(t,y)\lambda^2(t)\sigma^2(t) = 0, \qquad t\geq 0, \tag{4.9}$$

with

(a) σ^2 deterministic and càdlàg and $0<\sigma^2(t)\leq\sigma_Z^2(t)$, for all $t\geq 0$,

(b) $\partial_{22}H(t,y)\neq 0$ for all $(t,y)\in\mathbb{R}_+\times\mathbb{R}$, $\lambda(t) = c\mathbb{P}(\tau>t)$, for all t $(c>0)$,

(c) $\int_0^T \mathbb{E}\left(\left(\partial_2 H(s,\int_0^s\lambda(u)\sigma(u)\mathrm{d}B_u)\right)^2\right)\lambda^2(s)\sigma^2(s)\mathrm{d}s<\infty$, where B is a Brownian motion independent of τ.

The triple (H,λ,X) is an equilibrium, in the class C (Definition 3.1), if

(i) $\lim_{\bar{T}\uparrow T} H(\bar{T},\xi_{\bar{T}}) = \lim_{\bar{T}\uparrow T} V_{\bar{T}}$, a.s.

(ii) $Y = X+Z$ has no jumps,

(iii) $Y_t + \int_0^t \lambda(s)\frac{\partial_{22}H(s,\xi_s)}{\partial_2 H(s,\xi_s)}(\sigma_{M,Z}(s)+\sigma_M^2(s))\mathrm{d}s,\ 0\leq t\leq T,$ is an $\mathbb{F}$-local martingale.

Proof. See Appendix. ☐

Remark 4.1. Note that condition (i) in the previous theorem is stronger than (i) in (4.1).

For linear pricing rules or pricing rules satisfying (4.9) with $\sigma_Z^2(t) = \sigma^2(t)$, $t \geq 0$, we have a result in the class of all admissible strategies in line with Theorem 3.2.

Theorem 4.2. *Consider an admissible triple (H, λ, X) with (H, λ) satisfying (4.9) with $\partial_{22}H(t, y) = 0$ for all $(t, y) \in \mathbb{R}_+ \times \mathbb{R}$ or pricing rules satisfying (4.9) with $\sigma_Z^2(t) = \sigma^2(t)$, $t \geq 0$, and $\lambda(t) = c\mathbb{P}(\tau > t)$, $c > 0$. Then (H, λ, X) is an equilibrium if the following conditions hold:*

$$(i)\ \lim_{\bar{T}\uparrow T} H(\bar{T}, \xi_{\bar{T}}) = \lim_{\bar{T}\uparrow T} V_{\bar{T}} \quad a.s.$$

$$(ii)\ \sigma_M(t) = 0,\ 0 \leq t \leq T,$$

$$(iii)\ Y = X + Z \text{ has no jumps}$$

$$(iv)\ Y_t,\ 0 \leq t \leq T, \text{ is an } \mathbb{F}\text{-local martingale.}$$

Remark 4.2. Here we can draw analogous conclusions to the one in Cho (2003), where the author considers a risk-averse insider (and a deterministic release time). Cho concludes that, in equilibrium, a risk-adverse insider would do most of her trading early to avoid the risk that the prices get closer to the asset value, and consequently the risk of a lower profit, unless the trading conditions become more favourable over time. Similarly in our case, when the (risk-neutral) insider does not know the release time of information, she would trade early in order to use her piece of information before the announcement time comes. This behaviour would continue unless the price pressure (the factor lambda) decreases over time providing more favourable trading conditions also at a later time. A similar conclusion is obtained also by Baruch (2002), who studies the same problem about the effect of risk-aversion for the insider. In his study he assumes that the demand of noise traders follows a Brownian motion with time varying instantaneous variance.

Example 4.1. We can consider the context of Caldentey & Stacchetti (2010) where the authors assume that V and Z are arithmetic Brownian motion with variances σ_V and σ_Z, respectively, and that τ follows an exponential distribution with scale parameter μ, independent of (V, P, Z). Then, by Proposition 4.2, we have that, for a.a. t and a.a. $\omega \in \{t < \tau\}$,

$$V_t - H(t, \xi_t) - \lambda(t)\mathbb{E}\left(\int_t^\infty e^{-\mu(s-t)} \partial_2 H(s, \xi_s) \mathrm{d}X_s \middle| \mathcal{H}_t \right) = 0.$$

And to have an equilibrium, provided that $V_t - H(t, \xi_t) \neq 0$, we need $\lambda(t) = \lambda_0 e^{-\mu t}$.

5 Explicit insider's optimal strategies

In this section we shall apply our results to explicitly find the insider's optimal strategy in equilibrium. We will show how our general framework serves different models known in the literature, which are presented as different extensions of the Kyle-Back model, and opens for new more. In order to perform the explicit computations we will use techniques both of enlargements of filtrations and of filtering.

To explain how enlargement of filtrations enters the topic we consider a total demand $Y = Z + X$ in equilibrium given by:

$$Y_t = Z_t + \int_0^t \theta(V_u, Y_u, 0 \leq u \leq s) \mathrm{d}s, \quad 0 \leq t \leq T. \tag{5.1}$$

Here X is an absolutely continuous process with respect to the Lebesgue measure. We recall that Z is a martingale independent of V. Let $\mathbb{F}^{Y,V} = (\mathcal{F}_t^{Y,V})_{t\geq 0}$ be the filtration $\mathcal{F}_t^{Y,V} := \bar{\sigma}(Y_s, V_s, 0 \leq s \leq t)$. Since $\mathcal{F}_t^{Y,V} \subseteq \mathcal{H}_t$, for all t, and Z is adapted to $\mathbb{F}^{Y,V}$, we see that Z is also an $\mathbb{F}^{Y,V}$-martingale. On the other hand Y is, in certain cases as in Proposition 3.3, Theorem 3.2, Proposition 4.2 and Theorem 4.2, a local martingale when in equilibrium. Consequently (5.1) becomes the *canonical decomposition* of Y when we enlarge the filtration $\mathbb{F}^Y$ with the process V. We are then into a problem of *enlargement of filtrations.* However, in our problem Z is fixed in advance and we want to obtain Y as a function of Z, given V, so we look in fact for *strong* solutions of (5.1), whereas the results on enlargement of filtrations provide weak solutions. Then we can call upon the Yamada-Watanabe's theorem, when Z is Gaussian, to obtain strong solutions from weak solutions. See, for instance, Theorem 1.5.4.4. in Jeanblanc *et al.* (2009).

The following various examples correspond to different models, which are all extensions of the Kyle-Back model and where the results about enlargement of filtrations can be applied. We will not enter, however, into the details on the derivation of a strong solution in the corresponding stochastic differential equations appearing in equilibrium.

Example 5.1. (Aase et al. 2012a) Assume that $\tau = 1$ and suppose that Z is given by

$$Z_t = \int_0^t \sigma_s dW_s$$

where σ is deterministic. In equilibrium, if the strategy of the insider is optimal $V_1 = H(1, Y_1)$. Since $H(1, \cdot)$ can be chosen freely because it is the boundary condition of equation (3.15) and if V_1 has a continuous cumulative distribution function, we can assume without loss of generality that $Y_1 \equiv \mathrm{N}(0, \int_0^1 \sigma_s^2 \mathrm{d}s)$. It is assumed that V_1 (and consequently Y_1) is independent of Z. Then by Jeulin (1980), page 51,

$$Y_t = Z_t + \int_0^t \frac{Y_1 - Y_s}{\int_s^1 \sigma_u^2 \mathrm{d}u} \sigma_s^2 \mathrm{d}s,$$

has the same law as Z. Then

$$X_t = \int_0^t \frac{Y_1 - Y_s}{\int_s^1 \sigma_u^2 \mathrm{d}u} \sigma_s^2 \mathrm{d}s$$

is the optimal strategy. As a particular case we find the study of Back (1992) where $\sigma_s^2 \equiv \sigma^2$.

Example 5.2. (Campi & Çetin 2007) If we want both the total aggregate demand process Y to be a Brownian motion that reaches the value -1 for the first time at time $\bar{\tau}$, and the aggregate demand of the liquidity traders Z to be also a Brownian motion, then by Example 3 in Jeulin and Yor (1985), page 306, we can take the process Y to be

$$Y_t = Z_t + \int_0^t \left(\frac{1}{1 + Y_s} - \frac{1 + Y_s}{\bar{\tau} - s} \right) \mathbf{1}_{[0,\bar{\tau}]}(s) \mathrm{d}s.$$

So, in this case, we can refer to our framework by taking $\eta_t \equiv \bar{\tau}$, $V_t \equiv \mathbf{1}_{\{\bar{\tau} > 1\}}$ and the release time $\tau = \bar{\tau} \wedge 1$, which is known to the insider at $t = 0$.

Example 5.3. Another interesting example is that of Campi et al. (2013). There, the authors consider a defaultable stock. The default time is modeled as the first time that a Brownian motion, say B, hits the barrier -1, as in the above Example 5.2. In this case the default time, $\bar{\tau} = \inf\{t \geq 0, B_t = -1\}$, is not known to the insider at $t = 0$, but it is a stopping time for every trader. Instead, the insider observes the process $\left(B_{r(t)}\right)$ where $r(t)$ is a deterministic, increasing function with $r(t) > t$ for $t \in (0, 1)$, $r(0) = 0$, and $r(1) = 1$. This circumstance allows the insider to know in advance the default time. The horizon of the market is $t = 1$. The authors also consider a payoff of the kind $f(B_1)$ in case of no default. Note that $\bar{\tau} = r(\delta)$, where $\delta = \inf\{0 \leq t \leq 1, B_{r(t)} = -1\}$ and $\delta = \infty$ if the previous set is empty and then $\bar{\tau} > 1$. Then, setting this model in our framework, we have that the release time is $\tau = r(\delta \wedge 1)$, the signal is $\eta_t = B_{r(t)}$ and the fundamental value is

$$V_t = \mathbb{E}(f(B_1) \mathbf{1}_{\{\delta > \tau\}} | B_{r(t)}).$$

Moreover the aggregate demand of noise traders Z follows a Brownian motion, say W, so $Z = W$. In this case τ is a *predictable* stopping time, so the price pressure λ is constant and the optimal strategy moves prices to the fundamental one. To find the explicit form of an equilibrium strategy is not straightforward. However, if $\delta \leq s \leq \bar{\tau}$ (notice that then $\bar{\tau} = \tau \leq 1$) an equilibrium strategy is obtained from a strong solution of

$$Y_s = W_s + \int_0^s \left(\frac{1}{1 + Y_u} - \frac{1 + Y_u}{\bar{\tau} - u} \right) \mathrm{d}u,$$

as we deduce from Example 5.2 above. The difficult part is to see what happens for $s < \delta$. This requires a quite involved use of enlargement of filtrations and filtering techniques. See Campi et al. (2013b) for the details.

Another way of finding the equilibrium strategy is to consider first the rationality of prices and then to enter in a filtering problem. This approach follows the following point of view: market makers observe Y with dynamics

$$\mathrm{d}Y_t = \mathrm{d}Z_t + \theta(V_s, Y_s, 0 \leq s \leq t)\mathrm{d}t,$$

while V is not observed. Then, the dynamics of $m_t := \mathbb{E}(V_t|\mathcal{F}_t)$ can be obtained in certain cases from the filtering theory, see for instance Theorem 12.1 in Liptser and Shiryaev (1978). Now we can try to deduce $\theta(V_s, Y_s, 0 \leq s \leq t)$ from the equilibrium condition: $P_t = m_t$.

In the following example we use the filtering approach to find the equilibrium strategy.

Example 5.4. (Caldentey & Stacchetti 2010) The context is as follows. The release time $\tau \sim \exp(\mu)$ independent of (V, P, Z) and

$$\begin{aligned} \mathrm{d}V_t &= \sigma_v(t)\mathrm{d}B_t^v, \qquad V_0 \sim N(P_0, \Sigma_0) \\ \mathrm{d}Z_t &= \sigma_z(t)\mathrm{d}B_t^z, \qquad Z_0 = 0, \end{aligned}$$

where B^v and B^z are independent Brownian motions, independent of V_0 as well, and $\sigma_v(t)$ and $\sigma_z(t)$ are deterministic functions. If we look for strategies of the form

$$\mathrm{d}X_t = \beta(t)(V_t - P_t)\mathrm{d}t,$$

with $\beta(t)$ deterministic, we have that

$$\mathrm{d}Y_t = \beta(t)(V_t - P_t)\mathrm{d}t + \sigma_z(t)\mathrm{d}B_t^z.$$

Let $m_t := E(V_t|\mathcal{F}_t^Y)$. By standard filtering results (see for instance Theorem 12.1 in Lipster and Shiryayev (2001)) we have

$$\mathrm{d}m_t = \frac{\Sigma_t\beta(t)}{\sigma_z^2(t)}(\mathrm{d}Y_t - \beta(t)(m_t - P_t)\mathrm{d}t), \quad \frac{\mathrm{d}}{\mathrm{d}t}\Sigma_t = \sigma_v^2(t) - \frac{(\Sigma_t\beta(t))^2}{\sigma_z^2(t)},$$

where Σ_t is the filtering error. Now, since $P_t = m_t$ we have

$$P_t = P_0 + \int_0^t \lambda(t)\mathrm{d}Y_t$$

with $\lambda(t) := \frac{\Sigma_t\beta(t)}{\sigma_z^2(t)}$. Then

$$\Sigma_t = \Sigma_0 + \int_0^t \sigma_v^2(s)\mathrm{d}s - \int_0^t \sigma_z^2(s)\lambda^2(s)\mathrm{d}s, \;\; \beta(t) = \frac{\lambda(t)\sigma_z^2(t)}{\Sigma_t}.$$

Note that in particular we obtain that

$$Y_t = Z_t + \int_0^t \frac{\lambda(s)\sigma_z^2(s)\left(V_s - \int_0^s \lambda(u)\mathrm{d}Y_u\right)}{\Sigma_s}\mathrm{d}s,$$

is the canonical decomposition of the martingale Y in the filtration generated by (Z, V). Now if we assume $\sigma_z^2(t) = \sigma_z^2$, independent of t, and we take into account that in the equilibrium $\lambda(t) = \lambda_0 e^{-\mu t}$, we have that

$$\Sigma_t = \Sigma_0 + \int_0^t \sigma_v^2(s)\mathrm{d}s - \sigma_z^2\frac{\lambda_0^2}{2\mu}(1 - e^{-2\mu t}), \beta(t) = \frac{\sigma_z^2\lambda_0 e^{-\mu t}}{\Sigma_t}.$$

However λ_0 is not determined. We need an additional condition to fix λ_0. According to Theorem 4.2 we have

$$\lim_{t\to\infty} \Sigma_t = 0.$$

In such a case

$$0 = \Sigma_0 + \int_0^\infty \sigma_v^2(s)\mathrm{d}s - \sigma_z^2\frac{\lambda_0^2}{2\mu},$$

and

$$\lambda_0 = \sqrt{\frac{2\mu(\Sigma_0 + \int_0^\infty \sigma_v^2(s)\mathrm{d}s)}{\sigma_z^2}}.$$

Note that if $\sigma_v^2(t) = \sigma_v^2$ there is no solution!

Example 5.5. (Caldentey & Stacchetti 2010, continued) Hereafter we can discuss other types of strategies X in the same context of Example 5.4. For instance we can consider strategies involving a time T representing the time when the insider releases all the information to the market. With this kind of strategies, according with Proposition 4.2, the time T is such that the filtering error is

$$\Sigma_t = 0, \qquad \text{for all } t \geq T.$$

Then $P_t = V_t$ for $t \geq T$. But this implies, for $\sigma_v^2(t) = \sigma_v^2$,

$$\begin{aligned} 0 &= \Sigma_0 + \sigma_v^2 T - \sigma_z^2 \frac{\lambda_0^2}{2\mu}(1 - e^{-2\mu T}) \\ &= \Sigma_0 + \sigma_v^2 T - \sigma_z^2 \frac{\lambda_T^2}{2\mu}(e^{2\mu T} - 1). \end{aligned}$$

Now if we assume a smooth transition from the absolutely continuous strategy to the unbounded variation one, that is $\dot{\Sigma}_t = 0$, for all $t \geq T$, then $\sigma_v^2 - \sigma_z^2 \lambda^2(t) = 0$ and $\lambda(t) = \lambda_T = \frac{\sigma_v}{\sigma_z}$ for all $t \geq T$. Finally

$$\mathrm{d}P_t = \lambda(t)\mathrm{d}Y_t = \lambda(t)\mathrm{d}X_t + \lambda(t)\mathrm{d}Z_t = \mathrm{d}V_t, \quad t \geq T$$

so

$$\mathrm{d}X_t = \frac{\sigma_z}{\sigma_v}\mathrm{d}V_t - \mathrm{d}Z_t,$$

and T is the solution of

$$\Sigma_0 + \sigma_v^2 T = \frac{\sigma_v^2}{2\mu}(e^{2\mu T} - 1).$$

This is exactly what Caldentey & Stacchetti (2010) obtain. It is important to remark that the authors obtain a limit of optimal strategies when passing from the discrete time version of the model to the continuous one. This limit strategy is such that there is an endogenously determined time T such that, if $t \leq T$, then the limit strategy is absolutely continuous with respect to the Lebesgue measure and, if $t > T$, the strategy is not of bounded variation. In this case an insider's optimal strategy, between times T and τ, would yield to giving out the full information to the market by making the market prices match the fundamental value. The authors claim that this limit strategy is not optimal for the continuous time model and that we need to consider the discrete time model to realize about its existence. With respect to this point we remark that this limit strategy can be obtained as a limit of strategies for the continuous time model when we restrict the class of strategies to the set of those absolutely continuous and then we maximize the wealth. In fact, if we have a sequence of strategies $\left(X^{(n)}\right)_{n\geq 1}$, their corresponding wealth is given by

$$W_\tau^{(n)} = X_\tau^{(n)} V_\tau^{(n)} - \int_0^\tau P_{t-}^{(n)} \mathrm{d}X_t^{(n)} - [P^{(n)}, X^{(n)}]_\tau.$$

Then, if we assume that $(X^{(n)}, P^{(n)}, V^{(n)}) \underset{n\to\infty}{\overset{u.c.p}{\to}} (X, P, V)$ we obtain that

$$X_\tau^{(n)} V_\tau^{(n)} - \int_0^\tau P_{t-}^{(n)} \mathrm{d}X_t^{(n)} \underset{n\to\infty}{\overset{u.c.p}{\to}} X_\tau V_\tau - \int_0^\tau P_{t-} \mathrm{d}X_t$$

but, in general,

$$[P^{(n)}, X^{(n)}]_\tau \nrightarrow [P, X]_\tau.$$

For instance, if $X^{(n)}$ is a bounded variation process, then X is not necessarily a bounded variation one. Then the gain for this limit of strategies after T, on the set $\{\tau > T\}$, is given by

$$\begin{aligned}V_\tau X_\tau - V_T X_T - \int_T^\tau P_{t-}\mathrm{d}X_t &= \int_T^\tau X_{t-}\mathrm{d}V_t + \int_T^\tau V_{t-}\mathrm{d}X_t + \int_T^\tau \mathrm{d}[V,X]_t - \int_T^\tau P_{t-}\mathrm{d}X_t \\ &= \int_T^\tau (V_{t-} - P_{t-})\,\mathrm{d}X_t + \int_T^\tau \mathrm{d}[V,X]_t + \int_T^\tau X_{t-}\mathrm{d}V_t.\end{aligned}$$

Now if we take the conditional expectation, last term of the right-hand side cancels and we obtain that

$$\mathbb{E}\left(V_\tau X_\tau - V_T X_T - \int_T^\tau P_{t-}\mathrm{d}X_t \middle| \mathcal{H}_T \right) = \mathbb{E}\left(\int_T^\tau (V_{t-} - P_{t-})\,\mathrm{d}X_t + \int_T^\tau \mathrm{d}[V,X]_t \middle| \mathcal{H}_T \right).$$

Finally, since $V_{t-} = P_{t-}$, $t > T$ for the limit strategy, in the conditions of Example 4.1, we obtain that there is a profit after T given by

$$\mathbb{E}\left(\int_T^\infty e^{-\mu(t-T)}\mathrm{d}[V,X]_t \middle| \mathcal{H}_T \right) = \sigma_z\sigma_v \int_T^\infty e^{-\mu(t-T)}\mathrm{d}t = \frac{\sigma_z\sigma_v}{\mu} > 0.$$

Now we can justify the condition $\dot{\Sigma}_T = 0$. The expected wealth for the insider with this kind of strategies is given by

$$\begin{aligned}J(X) &= \mathbb{E}\left(\int_0^{T\wedge\tau} (V_t - P_t)\theta_t\mathrm{d}t\right) + \mathbb{E}\left(\int_{T\wedge\tau}^\tau \mathrm{d}[V,X]_t\right) = \mathbb{E}\left(\int_0^{T\wedge\tau} \beta_t(V_t - P_t)^2\mathrm{d}t\right) + \mathbb{E}\left(\int_{T\wedge\tau}^\tau \mathrm{d}[V,X]_t\right) \\ &= \mathbb{E}\left(\int_0^T \mathbf{1}_{[0,\tau]}(t)\beta_t(V_t - P_t)^2\mathrm{d}t\right) + \mathbb{E}\left(\int_T^\infty \mathbf{1}_{[0,\tau]}(t)\mathrm{d}[V,X]_t\right) = \int_0^T \mathbb{P}(\tau > t)\beta_t\Sigma_t\mathrm{d}t + \int_T^\infty \mathbb{P}(\tau > t)\frac{\sigma_v^2}{\lambda_t}\mathrm{d}t \\ &= \int_0^T e^{-\mu t}\beta_t\Sigma_t\mathrm{d}t + \sigma_v^2\int_T^\infty \frac{e^{-\mu t}}{\lambda_t}\mathrm{d}t = \sigma_z^2\int_0^T e^{-\mu t}\lambda_t\mathrm{d}t + \sigma_v^2\int_T^\infty \frac{e^{-\mu t}}{\lambda_t}\mathrm{d}t.\end{aligned}$$

Then if we impose that T is optimal, we have the condition

$$\sigma_z^2 e^{-\mu T}\lambda_T - \sigma_v^2\frac{e^{-\mu T}}{\lambda_T} = 0,$$

that is

$$\lambda_T = \frac{\sigma_v}{\sigma_z},$$

and this is equivalent to $\dot{\Sigma}_T = 0$. Note that other equilibria are possible by taking $\lambda_t \neq \lambda_T$ when $t > T$.

Example 5.6. The previous example can be modified in a more realistic way by assuming that

$$V_t = V_0 \exp\left\{\int_0^t \sigma_v(s)\mathrm{d}B_s^v - \frac{1}{2}\int_0^t \sigma_v^2(s)\mathrm{d}s\right\},\ \ \log V_0 = N(\log P_0 - \frac{1}{2}\Sigma_0, \Sigma_0).$$

Then we look for strategies of the form

$$\mathrm{d}X_t = \beta(t)\left(\log V_t - \left(\log P_t - \frac{1}{2}\Sigma_t\right)\right)\mathrm{d}t,$$

with $\beta(t)$ deterministic, and where Σ_t is the filtering error when we try to predict $\log V_t$ from Y_t,

$$\mathrm{d}Y_t = \beta(t)\left(\log V_t - \left(\log P_t - \frac{1}{2}\Sigma_t\right)\right)\mathrm{d}t + \sigma_z(t)\mathrm{d}B_t^z,$$

Then, since

$$\mathrm{d}\log V_t = \sigma_v(t)\mathrm{d}B_t^v - \frac{1}{2}\sigma_v^2(t)\mathrm{d}t,$$

if we $m_t := E(\log V_t|\mathcal{F}_t^Y)$, by the filtering results (see for instance Theorem 12.1 Lipster & Shiryayev 2001) we have

$$\mathrm{d}m_t = -\frac{1}{2}\sigma_v^2(t)\mathrm{d}t + \frac{\Sigma_t\beta(t)}{\sigma_z^2(t)}\left(\mathrm{d}Y_t - \beta(t)\left(m_t - \left(\log P_t - \frac{1}{2}\Sigma_t\right)\right)\mathrm{d}t\right), \quad \frac{\mathrm{d}}{\mathrm{d}t}\Sigma_t = \sigma_v^2(t) - \frac{(\Sigma_t\beta(t))^2}{\sigma_z^2(t)}.$$

Now, since

$$\log P_t = \log E(V_t|\mathcal{F}_t^Y) = m_t + \frac{1}{2}\Sigma_t,$$

we have that

$$\begin{aligned}\mathrm{d}\log P_t &= \frac{\Sigma_t\beta(t)}{\sigma_z^2(t)}\mathrm{d}Y_t - \frac{1}{2}\sigma_v^2(t)\mathrm{d}t + \frac{1}{2}\left(\sigma_v^2(t) - \frac{(\Sigma_t\beta(t))^2}{\sigma_z^2(t)}\right)\mathrm{d}t \\ &= \frac{\Sigma_t\beta(t)}{\sigma_z^2(t)}\mathrm{d}Y_t - \frac{(\Sigma_t\beta(t))^2}{\sigma_z^2(t)}\mathrm{d}t\end{aligned}$$

$$P_t = P_0 \exp\left(\int_0^t \lambda(s)\mathrm{d}Y_s - \frac{1}{2}\int_0^t \lambda^2(s)\sigma_z^2(s)\mathrm{d}s\right)$$

with $\lambda(t) := \frac{\Sigma_t\beta(t)}{\sigma_z^2(t)}$. Then

$$P_t = H\left(t, \int_0^t \lambda(s)\mathrm{d}Y_s\right),$$

with

$$H(t,x) = P_0 \exp\left\{x - \frac{1}{2}\int_0^t \lambda^2(s)\sigma_z^2(s)\mathrm{d}s\right\}$$

and satisfies (4.9) with $\sigma = \sigma_z$. Now, if we take $\lambda(t) = \lambda_0 e^{-\mu t}$ we have that

$$\Sigma_t = \Sigma_0 + \int_0^t \sigma_v^2(s)\mathrm{d}s - \lambda_0^2\int_0^t \sigma_z^2(s)e^{-2\mu s}\mathrm{d}s, \beta(t) = \frac{\sigma_z^2\lambda_0 e^{-\mu t}}{\Sigma_t}.$$

and according to Theorem 4.1 if we are in a equilibrium

$$\lim_{t\to\infty}\Sigma_t = 0.$$

In such a case

$$\lambda_0 = \sqrt{\frac{\Sigma_0 + \int_0^\infty \sigma_v^2(s)\mathrm{d}s}{\int_0^\infty \sigma_z^2(s)e^{-2\mu s}\mathrm{d}s}}.$$

A Appendix

Proof. (Theorem 4.1) Assume $(i) - (iii)$ hold true. We show that (H, λ, X) is an equilibrium. Define

$$\varsigma_t := \int_0^t \lambda(s)\sigma(s)\mathrm{d}B_s.$$

where B is a Brownian motion independent of τ. First if $H(t,y)$ is a solution of (4.9)

$$H(t,\varsigma_t) = H(0,0) + \int_0^t \partial_2 H(s,\varsigma_s)\lambda(s)\sigma(s)\mathrm{d}B_s$$

then, since $\int_0^T \mathbb{E}\left((\partial_2 H(s,\varsigma_s))^2\right)\lambda^2(s)\sigma^2(s)\mathrm{d}s < \infty$, $(H(t,\varsigma_t))_{t\geq 0}$ is a martingale (w.r.t. its own filtration). Then for $T \in [0,\infty]$ and $t < T$

$$H(t,y) = \mathbb{E}\left(H(T,\varsigma_T)|\,\varsigma_t = y\right) = \mathbb{E}\left(H(T, y + \varsigma_T - \varsigma_t)\right)$$

(This is well defined by condition $\int_0^T \mathbb{E}\left((\partial_2 H(s,\varsigma_s))^2\right)\lambda^2(s)\sigma^2(s)\mathrm{d}s < \infty$, $\lim_{\bar{T}\uparrow T} H\left(\bar{T},\varsigma_{\bar{T}}\right) = H\left(T,\varsigma_T\right)$ in L^2). Set now,

$$i(T,y,v) := \int_y^{H^{-1}(T,\cdot)(v)} \frac{v - H(T,x)}{c}\mathrm{d}x$$

with $c = \frac{\lambda(t)}{\mathbb{P}(\tau>t)}$and $H^{-1}(T,\cdot)(v) := \lim_{\bar{T}\uparrow T} H^{-1}(\bar{T},\cdot)(v)$. Define

$$I(t,y,v) := \mathbb{E}\left(i\left(T, y+\varsigma_T-\varsigma_t, v\right)\right), \quad t \geq 0,$$

we have that

$$\begin{aligned} \partial_2 I(t,y,v) &= \mathbb{E}\left(\partial_2 i(T,y+\varsigma_T-\varsigma_t,v)\right) \\ &= \mathbb{E}\left(-\frac{v-H(T,y+\varsigma_T-\varsigma_t)}{c}\right) = -\frac{v-H(t,y)}{c}. \end{aligned} \tag{A.1}$$

We can take the derivative under the integral sign because $H(T,\cdot)$ is monotone and $\mathbb{E}\left(H(T,\varsigma_T)\right) < \infty$. Then $I(t,y,v)$ is well defined and

$$\begin{aligned} I(t,y,v) &= \mathbb{E}\left(i\left(T,y+\varsigma_T-\varsigma_t,v\right)\right) \\ &= \mathbb{E}\left(\left. i\left(T,\varsigma_T,v\right)\right| \varsigma_t = y\right), \end{aligned}$$

and $\left(I\left(t,\varsigma_t,v\right)\right)_{t\geq 0}$ is a martingale (w.r.t. its own filtration), so

$$\partial_1 I\left(t,\varsigma_t,v\right) + \frac{1}{2}\partial_{22} I\left(t,\varsigma_t,v\right)\lambda(t)^2\sigma^2(t) = 0. \tag{A.2}$$

Now, consider any admissible strategy X, by using Itô's formula, we have

$$\begin{aligned} I(T,\xi_T,V_T) = I(0,0,V_0) &+ \int_0^T \partial_3 I(t,\xi_t,V_t)\mathrm{d}V_t + \int_0^T \partial_1 I(t,\xi_t,V_t)\mathrm{d}t \\ &+ \int_0^T \partial_2 I(t,\xi_{t-},V_t)\mathrm{d}\xi_t + \frac{1}{2}\int_0^T \partial_{22} I(t,\xi_t,V_t)\mathrm{d}[\xi^c,\xi^c]_t \\ &+ \int_0^T \partial_{23} I(t,\xi_t,V_t)\mathrm{d}[\xi^c,V]_t + \frac{1}{2}\int_0^T \partial_{33} I(t,\xi_t,V_t)\sigma_V^2(t)\mathrm{d}t \\ &+ \sum_{0\leq t<T}\left(\Delta I(t,\xi_t,V_t) - \partial_2 I(t,\xi_{t-},V_t)\Delta\xi_t\right). \end{aligned}$$

By construction, $\xi_0 = 0$, $\mathrm{d}\xi_t = \lambda(t)\mathrm{d}Y_t$. Now we have that

$$\mathrm{d}[\xi^c,\xi^c]_t = \lambda(t)^2\mathrm{d}[X^c,X^c]_t + 2\lambda(t)^2\mathrm{d}[X^c,Z]_t + \lambda(t)^2\sigma_Z^2(t)\mathrm{d}t.$$

Also by (A.1) and the fact that V and Z are independent,

$$\partial_{23} I(t,\xi_t,V_t)\mathrm{d}[\xi^c,V]_t = -\frac{1}{c}\mathrm{d}[\xi^c,V]_t = -\mathbb{P}(\tau>t)\mathrm{d}[X,V]_t,$$

then using (A.1) and (A.2), and the fact that Z has not jumps, we get

$$\begin{aligned} I(T,\xi_T,V_T) = I(0,0,V_0) &+ \int_0^T \partial_3 I(t,\xi_t,V_t)\mathrm{d}V_t + \int_0^T \mathbb{P}(\tau>t)(P_{t-}-V_t)(\mathrm{d}X_t+\mathrm{d}Z_t) \\ &+ \frac{1}{2}\int_0^T \partial_{22} I(t,\xi_t,V_t)\lambda(t)^2\mathrm{d}[X^c,X^c]_t - \int_0^T \mathbb{P}(\tau>t)\mathrm{d}[X,V]_t + \frac{1}{2}\int_0^\tau \partial_{33} I(t,\xi_t,V_t)\sigma_V^2(t)\mathrm{d}t \\ &+ \int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda(t)^2\mathrm{d}[X^c,Z]_t + \frac{1}{2}\int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda(t)^2\left(\sigma_Z^2(t)-\sigma^2(t)\right)\mathrm{d}t \\ &+ \sum_{0\leq t\leq\tau}\left(\Delta I(t,\xi_t,V_t) - \partial_2 I(t,\xi_{t-},V_t)\lambda(t)\Delta X_t\right). \end{aligned}$$

Subtracting $\int_0^T \mathbb{P}(\tau > t)\mathrm{d}[P,X]_t$ from both sides and rearranging the terms, we obtain

$$\int_0^T \mathbb{P}(\tau > t)(V_t - P_{t-})\mathrm{d}X_t - \int_0^T \mathbb{P}(\tau > t)\mathrm{d}[P,X]_t + \int_0^T \mathbb{P}(\tau > t)\mathrm{d}[X,V]_t$$
$$- \left(I(0,0,V_0) + \frac{1}{2}\int_0^T \partial_{33} I(t,\xi_t,V_t)\sigma_V^2(t)\mathrm{d}t \right)$$
$$= -I(T,\xi_T,V_T) + \int_0^T \partial_3 I(t,\xi_{t-},V_t)\mathrm{d}V_t + \int_0^T \mathbb{P}(\tau > t)(P_t - V_t)\mathrm{d}Z_t$$
$$+ \frac{1}{2}\int_0^T \partial_{22} I(t,\xi_t,V_t)\lambda(t)^2\mathrm{d}[X^c,X^c]_t + \int_0^T \partial_{22} I(t,\xi_t,V_t)\lambda(t)^2\mathrm{d}[X^c,Z]_t$$
$$+ \frac{1}{2}\int_0^T \partial_{22} I(t,\xi_t,V_t)\lambda(t)^2\left(\sigma_Z^2(t) - \sigma^2(t)\right)\mathrm{d}t$$
$$+ \sum_{0\le t<T} \left(\Delta I(t,\xi_t,V_t) - \partial_2 I(t,\xi_{t-},V_t)\lambda(t)\Delta X_t\right) - \int_0^T \mathbb{P}(\tau > t)\mathrm{d}[P,X]_t. \tag{A.3}$$

We have that

$$\mathbb{P}(\tau > t)\mathrm{d}[P,X]_t = \mathbb{P}(\tau > t)\mathrm{d}[P^c,X^c]_t + \mathbb{P}(\tau > t)\Delta P_t \Delta X_t.$$

Then Itô's formula for H shows that the continuous local martingale part of P is $\int \partial_2 H(t,\xi_t)\mathrm{d}\xi_t^c$, so by using (A.1), we obtain

$$\mathbb{P}(\tau > t)\mathrm{d}[P^c,X^c]_t = \frac{\lambda(t)}{c}\partial_2 H(t,\xi_t)\mathrm{d}\left[\xi^c,X^c\right]_t$$
$$= \partial_{22} I(t,\xi_t,V_t)\lambda(t)^2\mathrm{d}\left[X^c,X^c\right]_t + \int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda(t)^2\mathrm{d}\left[X^c,Z\right]_t,$$

and

$$\lambda(t)\partial_2 I(t,\xi_{t-},V_t)\Delta X_t + \mathbb{P}(\tau > t)\Delta P_t\Delta X_t = \frac{\lambda(t)}{c}(P_{t-} - V_t)\Delta X_t + \frac{\lambda(t)}{c}\Delta P_t\Delta X_t$$
$$= \frac{\lambda(t)}{c}(P_t - V_t)\Delta X_t = \lambda(t)\partial_2 I(t,\xi_t,V_t)\Delta X_t.$$

Substituting the above relationships in the right-hand side of the equation (A.3), and since we are in the class C, it becomes

$$- I(T,\xi_T,V_T) + \int_0^T \partial_3 I(t,\xi_t,V_t)\mathrm{d}V_t + \int_0^T \mathbb{P}(\tau > t)(P_t - V_t)\mathrm{d}Z_t - \frac{1}{2}\int_0^T \partial_{22} I(t,\xi_t,V_t)\lambda(t)^2\mathrm{d}[X^c,X^c]_t$$
$$+ \frac{1}{2}\int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda(t)^2\left(\sigma_Z^2(t) - \sigma^2(t)\right)\mathrm{d}t$$
$$+ \sum_{0\le t\le\tau} \left(I(t,\xi_t,V_t) - I(t,\xi_{t-},V_t) - \lambda(t)\partial_2 I(t,\xi_t,V_t)\Delta X_t\right)$$
$$= -I(T,\xi_T,V_T) + \int_0^T \partial_3 I(t,\xi_t,V_t)\mathrm{d}V_t + \int_0^T \mathbb{P}(\tau > t)(P_t - V_t)\mathrm{d}Z_t$$
$$+ \sum_{0\le t<T} \left(I(t,\xi_t,V_t) - I(t,\xi_{t-},V_t) - \lambda(t)\partial_2 I(t,\xi_t,V_t)\Delta X_t\right).$$

Now, observe that $\partial_{33} I(t,y,v)$ does not depend on y and so $\partial_{33} I(t,\xi_t,V_t)$ does not depend of ξ. Then $I(0,0,V_0) + \frac{1}{2}\int_0^T \partial_{33} I(t,\xi_t,V_t)\sigma_V^2(t)\mathrm{d}t$ has values that does not depend on the strategy. Then on the left-hand side of (A.3) only the term

$$\int_0^T \mathbb{P}(\tau > t)(V_t - P_{t-})\mathrm{d}X_t - \int_0^T \mathbb{P}(\tau > t)\mathrm{d}[P,X]_t + \int_0^T \mathbb{P}(\tau > t)\mathrm{d}[X,V]_t$$

depends on the strategy and its expectation is just the value of (2.11). Then we show that, taken the expectation, the right-hand side of (A.3) achieves its maximum value at X. The result follows from the following points.

1. (ii) guarantees that $\Delta X_t = 0$
2. The processes $\int_0^{\cdot} \partial_3 I(t,\xi_t,V_t)\mathrm{d}V_t$ and $\int_0^{\cdot}(P_t - V_t)\mathrm{d}Z_t$ are $\mathbb{H}$-martingales by (A5) and (A2) in Definition 2.2, hence they have null expectation
3. We know that $c\partial_{22} I(\bar{T},\xi_{\bar{T}},V_{\bar{T}}) = \partial_2 H(\bar{T},\xi_{\bar{T}}) > 0$ and that $c\partial_2 I(\bar{T},\xi_{\bar{T}},V_{\bar{T}}) = -V_{\bar{T}} + H(\bar{T},\xi_{\bar{T}})$ so by (i) we have a maximum value of $-I(T,\xi_T,V_T)$ for our strategy.

Assumption (iii) and (i) together with condition (A2) in Definition 2.2 guarantee the rationality of prices, given X. In fact from (4.9)

$$\mathrm{d}P_t = \lambda(t)\partial_2 H(t,\xi_t)\mathrm{d}Y_t + \frac{1}{2}\lambda(t)^2\left(\sigma_Y^2(t) - \sigma^2(t)\right)\partial_{22}H(t,\xi_t)\mathrm{d}t$$

and by (ii)

$$\begin{aligned}\mathrm{d}P_t &= \lambda(t)\partial_2 H(t,\xi_t)\mathrm{d}Y_t + \lambda(t)^2\left(\sigma_M^2(t) + \sigma_{M,Z}(t)\right)\partial_{22}H(t,\xi_t)\mathrm{d}t \\ &= \lambda(t)\partial_2 H(t,\xi_t)\left(\mathrm{d}Y_t + \lambda(t)\left(\sigma_M^2(t) + \sigma_{M,Z}(t)\right)\frac{\partial_{22}H(t,\xi_t)}{\partial_2 H(t,\xi_t)}\mathrm{d}t\right)\end{aligned}$$

so, P is an $\mathbb{F}$-local martingale and, by condition (A2) in Definition 2.2, it is an $\mathbb{F}$-martingale. Then from (i), and on the set $\{t \le \tau\}$ we have

$$\mathbb{E}\left(H(T,\xi_T)\middle|\mathcal{F}_t\right) = \mathbb{E}\left(V_T\middle|\mathcal{F}_t\right) = \mathbb{E}\left(\mathbb{E}\left(V_T\middle|\mathcal{H}_t\right)\middle|\mathcal{F}_t\right) = \mathbb{E}\left(V_t\middle|\mathcal{F}_t\right).$$

□

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