Dynamic Information Design under Constrained Communication Rules

By Caio Lorecchio and Daniel Monte*

An information designer wishes to persuade agents to invest in a project of unknown quality. To do so, she must induce investment and collect feedback from these investments. Motivated by data regulations and simplicity concerns, our designer faces communication constraints. These constraints hinder her without benefiting the agents: they impose an upper bound on the induced belief spread, limiting persuasion. Nevertheless, two-rating systems (direct recommendations) (i) are the optimal design when experimentation is needed to generate information, and (ii) approximate the designer's first-best payoff, for specific feedback structures. When the designer has altruistic motives, constrained rules significantly decrease welfare. (JEL D82, D83, L15, M3)

In dynamic interactions, information design is crucial for the success of an information intermediary in motivating actions. For example, online platforms employ a variety of forms to recommend products and disclose feedback data to customers. Yet importantly, in many instances, not all information generated is used. First, there may be data storage constraints: keeping track of the decisions, activities, and reviews of every user in every period could be too costly or impractical. Indeed, for large matrices of data with information on products and consumers, designing algorithms that minimize memory requirements may be essential in large-scale settings.[1] Constraints might also come from legal pressure for platforms to limit data archives.[2] Second, there is an important debate on whether platforms and artificial intelligence systems should be subjected to stricter transparency obligations, such as requirements that customers have access to the same information used in recommendation algorithms.[3] How do these restrictions affect communication? Do they benefit or hinder the players? This paper addresses such questions.

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[1] See, for example, Aggarwal (2016), section 7.3.8.

[2] The European Union's General Data Protection Regulation (GDPR) is an illustration of such pressure. Article 5.1 of GDPR sets the principles relating to the processing of personal data, with one being "storage limitation" (5.1.e). The 2021 legislative proposal in India to regulate non-personal data storage as well is another illustration (https://bit.ly/3nieCMG).

[3] For example, the European Commission's recent proposal for a regulation establishing rules on artificial intelligence includes increased transparency obligations (https://bit.ly/3t3i8Ov). Moreover, the GDPR has defined customers' right to access and receive a copy of their personal data, and other supplementary information. Information has to be provided in a concise and user-friendly format. California also has an illustrative transparency law, ruling that customers have access to data about them held by technology companies.

In our baseline model, a long-run designer faces an infinite sequence of short-lived agents, who must each decide whether or not to invest in a project. The project is of a fixed quality, which we assume to be either good or bad. This underlying quality is unknown to the designer and the agents. The designer's payoff in the stage game is state-independent: She gets a positive payoff whenever an agent invests and zero otherwise, regardless of the true underlying quality. If an agent invests, the project randomly generates a payoff to that agent. A good quality project yields positive expected payoffs, whereas a bad quality project yields negative expected payoffs. If the agent does not invest, she gets a zero payoff. Any given agent will invest if and only if her belief that the project is of bad quality is sufficiently low.

The designer commits to a set of messages, an initial distribution over messages and a Markov transition rule, mapping each message and each current agent's payoff to a distribution over messages from which the next period message will be drawn. Before making an investment decision, each agent's only source of information comes from the observed message. In fact, we assume that agents do not know their place in the queue, that is, they do not know the calendar time and, instead, form beliefs about the actual history.[4,5]

An important feature of our class of communication rules is that past information may be encoded in the messages, but cannot be used directly in the transition rule. Therefore, the probability of each subsequent message does not depend directly on any information from past agents, only on the observed message by a current agent and the feedback collected after her decision. We further restrict our class of communication to include only finite message sets. As each message stores past information, a finite message set constrains communication to have finite memory (and finite computational complexity).[6] Thus, we say that communication has *memory restrictions*.

One way of thinking about our class of communication rules is as rating systems, as in the commonly used star-rating systems. In this case, each message is a different rating. Indeed, for ease of visualization, from now on we will refer to our communication rules as rating systems, and each message as a rating. In the special case where the designer chooses to use only two messages, the output of our system is simply a recommendation.

[4]This assumption is imposed to simplify matters. When agents do not know the calendar time, there is an issue on how to compute beliefs. We postpone this discussion to Section II. Note also that if agents knew the calendar time, there would be a cold-start problem if the prior is above a given threshold: the first agent would not invest, so there would be no belief updating and no other agents would invest either. This is easily dealt with in our extension of changing types (section V).

[5]This class of communication rules includes (a) any bounded recall rule in which each agent fully observes the last t investment decisions and resulting payoffs; and, if the message set is countable, (b) the case in which every agent observes the entire history of payoffs and investments.

[6]Our communication rule is implemented by a finite state automaton, and ratings are states of this machine. Our measure of complexity can be motivated by the notion of computational complexity in Moore machines (which was used to capture the complexity of implementing a strategy in the earlier automata literature in game theory, such as by Rubinstein, 1986; Abreu and Rubinstein, 1988; and Kalai, 1990).

We characterize how these restrictions affect the optimal communication and what are the implications for the designer and the agents. The optimal rating system is similar to the optimal static information design, as we discuss below, but with some fundamental differences. We also show the implications of these communication restrictions on the designer's welfare, the agents' welfare and how the optimal system depends on the conditional payoff distributions.

To understand our setting more intuitively, let us consider the same stage game within a Bayesian persuasion framework.[7] Think of the static problem in which the designer wishes to maximize the agent's probability of investing in the project. If she could credibly commit to a map from the true project's type to a set of messages, she would choose to "split the posteriors" of the agent in one posterior that makes the agent indifferent about investing or not,[8] and another (extreme) posterior in which the agent is convinced about the type of the project being bad. If we return to our designer with finite ratings, ideally she would like to end up with two ratings that induce beliefs to reproduce the Bayesian persuasion benchmark. However, our designer cannot commit to a map from the project's type to ratings (which in turn induce the posteriors), and it is only through the agents' induced actions that the designer can (partially) inform about the type of the project.

We show that the belief spread induced by rating systems cannot be as large as in the static Bayesian persuasion literature. The intuition is as follows.[9] Once the rating system is designed, the set of ratings can be partitioned into two subsets: ratings in which the induced belief is such that agents invest and ratings in which the agents do not invest. At least one rating from one subset must interact with at least one rating from the other subset, a result that comes directly from irreducibility. The beliefs of these two ratings in consideration can be far apart only insofar as the actions played in each of them can give enough information about the underlying quality of the project. However, in one of these ratings agents are not investing, so no information is being generated. Thus, the beliefs in these non-investment ratings are formed solely based on (i) the beliefs of the ratings that are connected to them and (ii) the information generated by the actions played on those connecting ratings. This allows us to compute a maximum bound on the highest belief about the project being bad.[10]

[7]See, for example, Rayo and Segal (2010) or Kamenica and Gentzkow (2011).

[8]Indifference can be easily accommodated, so for now we ignore indifference.

[9]For now, we focus on irreducible systems, but in the main part of the paper we prove the result for the general case.

[10]Inducing the indifference belief through a rating system is not a problem. If every rating in the investment subset has an associated belief higher than the indifference belief, the designer could change the transitions out the non-investment subset to have the system spending more time in the investment subset. By doing so, she would increase the project's investment probability.

Most importantly, we show that only two ratings are needed for the optimal system, which we interpret as a direct recommendation to invest or not to invest. Although this looks like the revelation principle in the Bayesian persuasion literature, it is actually not a consequence in our environment, as we argue in the next paragraph. Ratings serve the dual purpose of generating information and inducing payoff relevant actions, but without investment, no information is generated. Moreover, to induce investment it is sufficient to have a rating with a belief at which the agent is indifferent between investing and not investing, and generating further information is not optimal.

To fully understand the role that ratings play, we also solve for the optimal system in a world in which information is generated even when there is no investment. In this case, we do not have a revelation principle of this sort; instead, more ratings imply a higher payoff. Such a result also holds when the designer's objective function is to maximize social welfare, as, in this case, more information is important even in the set of ratings that induce investment.[11] These extensions allow us to show that the results in our benchmark model depend on the interaction of our assumptions on the environment (that informative payoffs are only generated following investment) and on the restrictions on the designer's communication rule.

Our model also enables us to pin down an important relation between the signal structure and Bayesian persuasion. The relevant statistic for persuading is the strength of bad news, that is, the informativeness of the signal with the highest likelihood between the bad and the good type of project. As this informativeness increases, the designer's payoff becomes closer to the Bayesian persuasion payoff. In contrast, good news is not relevant for persuasion.[12]

Finally, we extend our model to the case in which the project's quality changes according to a Markov process. It is reasonable to consider that the quality of an item that a platform recommends changes over time. Thus, our constraints may seem less restrictive in a world in which old feedback has less informational value. Nevertheless, we prove that not only do our results regarding bounds on the belief spread hold qualitatively, but the set of parameter values for which a rating system contributes to persuasion is also smaller. Moreover, although only bad news matters for the optimal system, the designer's payoff from it does not coincide with the Bayesian persuasion payoff, even if bad news is perfectly informative.

[11] An interesting implication of optimality and Bayes-plausibility, is that even when the designer uses the many ratings to improve her payoff, the optimal rule implies staying most often at the extreme ratings. This result is consistent with the results in the bounded memory literature, such as Hellman and Cover (1970) and Wilson (2014).

[12] The intuition is that persuasion requires agents to be very convinced of the bad type of project, but only partially convinced by the good type. Thus, the designer would like to have an inflated belief for the bad type and an indifference belief for the good type. Because information is not generated in the non-investment rating, the only way to have a high belief in it is through an informative transition about the bad type from the investment rating to the non-investment one. In contrast, in the no experimentation case in which the designer is capable of generating information even when there is no investment, good news is also relevant; indeed, the more informative the agents' payoffs are (regardless of whether good or bad news), the closer to the Bayesian persuasion payoff the designer will be.

A. Literature review

Our paper combines Bayesian persuasion with constrained rules and memory considerations. Thus, perhaps the most related papers to ours are the ones that combine dynamic games with limited record keeping. Ekmekci (2011) studies a rating system in a moral hazard game, and the system structure is very close to ours - yet with some fundamental differences. For instance, in his model, there is a permanent flow of informative signals about the long-run player's type. In credence goods markets, however, the investment is needed to have informative signals, so there might not exist such a permanent flow of signals.

Our paper is related to the literature of learning under limited memory. Hellman and Cover (1970) and Wilson (2014) consider a bounded memory agent that has to design the optimal memory for a two-hypothesis testing problem. Monte (2013, 2014) studies a model in which a bounded memory player is playing a reputation game against an informed player. We study a similar problem, but under the perspective of a designer that must persuade agents to invest at the same time that she must learn. In our model there is a subset of the memory system in which there is no information revelation and the designer must choose how long to spend in those ratings. Monte and Said (2014) extend the problem of hypothesis testing under bounded memory to the case in which the states of the world change over time.

Kovbasyuk and Spagnolo (2021) study information transmission in dynamic markets with limited record keeping as well. Types are constantly changing over time in their environment, thus it is natural to look at stationary rules, like they do. In our benchmark case, types are fixed throughout time, but the stationarity comes from the constraint on the set of possible communication. As an extension to the benchmark case, we also consider information transmission with changing types. Nevertheless, our extended setting differs from their paper in the monitoring technology available to the designer and the restrictions we impose. More specifically, our designer neither perfectly learns the state through the agents' feedback nor conditions messages on information that is not publicly available to each current agent.[13]

We also connect with the literature on information design and Bayesian persuasion. Aumann, Maschler and Stearns (1995) study this question in repeated games with incomplete information. Recent papers include Brocas and Carrillo (2007), Rayo and Segal (2010), Kamenica and Gentzkow (2011, 2014), Bergemann and Morris (2016), Ely (2017), Mathevet, Perego and Taneva (2020), Matysková and Montes (2021), Lipnowski, Ravid and Shishkin (2018), Taneva (2019), Li and Norman (2021) and Doval and Ely (2020).[14]

[13]Other related papers on information transmission with limited record keeping are Liu (2011) and Liu and Skrzypacz (2014). In the former, past information is costly to observe and in latter the focus is on how reputation evolves in a dynamic market with limited record keeping.

[14]See also Bergemann and Morris (2019) and Kamenica (2019) for two recent surveys on information design.

We tackle a related problem, but our designer informs the agents through ratings, which are updated using the payoffs from actions taken in each period. In particular, bad ratings imply periods with no investment, which in turn do not generate payoffs informative of the state.

Le Treust and Tomala (2019) study Bayesian persuasion with constrained communication as well. However, in their setting, the designer has the ability to condition any informative disclosure on the state of the world, and the communication constraints come from distortions on the messages agents receive, as well as limitations on the size of the message space relative to the size of the state space. In our model, information transmission is noiseless and the designer is free to choose any (finite) size of the message space. Constraints come instead from information generation, as the designer faces memory restrictions and feedback from agents is imperfectly revealing.

By studying optimal rating structures from the principal's point of view, our work also relates to the literature of reputation and optimal information design (Hörner and Lambert, 2016; Smolin, 2017; and Bhaskar and Thomas, 2017). Also related is Halac, Kartik and Liu (2017), which studies the design of optimal disclosure policies in dynamic contests. Our underlying model is similar to Kremer, Mansour and Perry (2014) and Che and Horner (2017), in which a recommender system must learn from the product feedback. The key difference between our paper and theirs is our restrictions on the communication. Moreover, we consider a different signal structure and different objective function for the designer. [15]

Our environment is also similar to Glazer, Kremer and Perry (2021), but in their paper, agents must pay a cost to observe the signal. Vong (2021) studies optimal ratings in a repeated game in which there is adverse selection and moral hazard. Other recent related papers are Lillethun (2017) and Sperisen (2018). Salmi, Laiho and Murto (2020) study a problem in which a decision maker chooses in every period whether or not to increase the consumption of a variable of an unknown and fixed type. The more she experiments, the more she learns about the type.

Our rating system might resemble Best and Quigley (2017)'s review aggregator, but there are fundamental differences. In their paper, the long-run designer cannot commit to a rule and the state of the world is changing over time. Thus, their review aggregator is meant to provide the designer a way to approach the Bayesian persuasion payoff in a world with no commitment and changing types (states of the world). In contrast, our rating system is the information policy to which the designer credibly commits to and is intended to solve a joint learning problem between the designer and the sequence of agents.

[15] There is a literature in computer science and algorithmic game theory studying information design problems and complexity. Dughmi (2017) provides a recent survey on the topic.

I. Model

An infinite sequence of myopic agents enter the market, one at a time. Each agent chooses whether or not to invest in a given project. The project is one of two types: $\Omega = \{B, G\}$. The type is fixed throughout the game and unknown to the agents.[16] Each type randomly generates a payoff to an agent every period in which an agent invests. If the agent does not invest, her payoff is zero. Suppose that there are $M \geq 2$ possible payoff realizations: $X = \{x_1, x_2, ..., x_M\}$. If the project's type is B, payoffs are drawn according to $\Pr(x_m|B) = \gamma_m^B$, $m \in \{1, 2, ..., M\}$ whereas if it is G, payoffs are drawn according to $\Pr(x_m|G) = \gamma_m^G$.[17] Therefore, the conditional expectation of investing if the type is B is $\sum_{m=1}^{M} \gamma_m^B x_m$, and similarly if the type is G. We assume that the conditional expected payoff of investing is such that

$$\sum_{m=1}^{M} \gamma_m^G x_m > 0 > \sum_{m=1}^{M} \gamma_m^B x_m.$$

Every agent's initial prior probability that the type of the project is bad is denoted by ρ, with $\rho \in (0, 1)$. In a one-shot interaction, an agent decides to invest if and only if

$$\rho \sum_{m=1}^{M} \gamma_m^B x_m + (1 - \rho) \sum_{m=1}^{M} \gamma_m^G x_m \geq 0.$$

This inequality defines an indifference threshold, ρ^*, which is given by

$$\rho^* = \frac{\sum_{m=1}^{M} \gamma_m^G x_m}{\sum_{m=1}^{M} \gamma_m^G x_m - \sum_{m=1}^{M} \gamma_m^B x_m}.$$

Whenever $\rho > \rho^*$, the agent in a static problem does not invest. In our model, agents do not observe calendar time and their only information is the rating sent by the designer. We will come back to this in section II. For now, to gain intuition, consider a communication rule in which the designer communicates the entire sequence of previous actions and payoff realizations to each agent. Then, if $\rho > \rho^*$, the first agent does not invest and since she does not invest, no information is generated and subsequent agents will also choose not to invest. The market breaks down.

[16]We solve for the case of changing types in section V.

[17]Our results extend naturally to more general distributions F^B and F^G, such as the case in which F^B and F^G are absolutely continuous with respect to the Lebesgue measure, with corresponding continuous densities f^B and f^G uniformly bounded away from 0 and ∞. The discrete case that we use in the paper has the advantage of allowing us to construct the exact optimal rating system, instead of the ε-optimal system that may be required if the payoff distribution is continuous.

If $\rho \leq \rho^*$, it is optimal for the first agent to invest. If subsequent agents have access to the full history of the game, they will update their beliefs on the quality of the project as new information arrives. As long as the posterior is smaller than ρ^*, agents keep investing. As soon as the posterior becomes higher than ρ^*, agents cease to invest and the market collapses. A rating system might overcome this problem if agents have limited information about past outcomes. It is noteworthy that a straightforward implication of our model is that if $\rho \leq \rho^*$, then a rating system that conceals all information will induce investment every period. For this reason, we will focus on the more interesting case, that is, when $\rho > \rho^*$.

We will consider the information design problem faced by a designer restricted to a system with finitely many messages and a stationary rule. We will denote this restricted system as a rating system.

DEFINITION 1 (Rating System): *A rating system is a tuple $\phi = (\mathscr{R}, \varphi_0, \varphi)$, where $\mathscr{R} = \{1, 2, ..., R\}$ is a finite set of ratings; $\varphi_0 \in \Delta(\mathscr{R})$ is an initial probability distribution over ratings; and $\varphi : \mathscr{R} \times [X \cup \{\emptyset\}] \to \Delta(\mathscr{R})$ is a transition rule, where $\emptyset$ represents the event in which there was no investment and X is the set of investment returns.*

Because agents do not observe calendar time, but only the current rating,[18] each agent's strategy is a map from the current rating to a probability of investment. We refer to their strategies as $\alpha : \mathscr{R} \to [0, 1]$.

Agents compute the probability distribution over the histories as if the game had been going on for a long time, and their beliefs are computed using steady-state probabilities (or time-average convergence). To give a more detailed description of how beliefs are computed, first note that any given rating system ϕ together with a given strategy α defines Markov matrices $T^G = (\tau^G_{i,j})$ and $T^B = (\tau^B_{i,j})$ for the good and bad types of the project, respectively. The transition $\tau^\omega_{i,j}$ from rating i to rating j given ω is

$$\tau^\omega_{i,j} = \alpha_i \sum_{m=1}^{M} \gamma^\omega_m \varphi^m_{i,j} + (1 - \alpha_i)\varphi_{i,j},$$

where $\varphi^m_{i,j}$ represents the transition from i to j upon the observation of payoff x_m and $\varphi_{i,j}$ stands for the transition when there is no investment.

For now, we focus on irreducible systems, i.e., systems in which no subsets are absorbing. We will show in Proposition 3 that this focus on irreducible systems is without loss of generality. A well-known result in Markov processes is that in an irreducible system, there is a unique invariant distribution.[19] Thus, for

[18] Strictly speaking, they also observe the resulting payoff of their investment, but since they are short-lived, this is irrelevant for what follows.
[19] For the proof see, for example, Stokey, Lucas and Prescott (1989), Theorem 11.2.

each ω, there will be a unique conditional distribution $f^{\omega} = (f_i^{\omega})_{i\in\mathscr{R}}$ over the ratings. They will not depend on the initial distribution φ_0, so we will not be explicit about the initial distribution whenever constructing irreducible systems.

The invariant distributions will be used to calculate both the designer's expected payoff and the agents' beliefs at any given rating. The designer seeks to maximize the *ex-ante* probability of investment, so her expected payoff given a rating system ϕ and a strategy α, is given by

$$\Pi = \rho \sum_{i\in\mathscr{R}} f_i^B \alpha_i + (1-\rho) \sum_{i\in\mathscr{R}} f_i^G \alpha_i.$$

For every rating i that is reachable, that is, $f_i^{\omega} > 0$ for some ω, the updated belief is given by

$$\rho_i = \frac{\rho f_i^B}{\rho f_i^B + (1-\rho) f_i^G}.$$

For any given rating system we define the equilibrium as a strategy α and beliefs $(\rho_i)_{i\in\mathscr{R}}$ such that every investor is taking her optimal action given her beliefs. We state the concept below.

DEFINITION 2 (Equilibrium): *Given a rating system ϕ, an equilibrium is a strategy α as well as beliefs $(\rho_i)_{i\in\mathscr{R}}$ such that, for every $i \in \mathscr{R}$:*

1) $\alpha_i > 0 \Leftrightarrow \rho_i \sum_{m=1}^{M} \gamma_m^B x_m + (1-\rho_i) \sum_{m=1}^{M} \gamma_m^G x_m \geq 0$;

2) ρ_i *is consistent whenever reachable (that is, derived from Bayes' rule and the stationary distributions generated by* ρ, α, *and* ϕ*).*

We will focus on informative rating systems, that is, a rating system with at least two ratings and for which there is an equilibrium such that at least one rating is an investment rating and at least one rating is a non-investment rating.

DEFINITION 3 (Informative Systems): *An informative system is a rating system with $R \geq 2$ ratings that induces an equilibrium such that at least one rating is an investment rating and at least one rating is a non-investment rating.*

The name is adequate because, in such systems, there will be at least one rating in which agents will have more information to support their non-investment decision, and at least one rating in which they will have more information to support their investment decision.

To better understand the mechanics of our model and how to compute beliefs, consider the example.

EXAMPLE 1 (Binary Payoffs and Deterministic Transition Rules): *Consider a binary payoff environment in which there is a good realization, denoted by* h, *and a bad realization, denoted by* l. *Then, assume that* $\Pr(h|G) = \gamma_h > \frac{1}{2}$, *and* $\Pr(h|B) = \gamma_l < \frac{1}{2}$. *Construct a rating system with two ratings and a deterministic transition rule. That is,* $\varphi^h_{2,1} = 0, \varphi^l_{2,1} = 1$, *and* $\varphi_{1,2} = 1$. *For such a system, the induced Markov matrices are given by*

$$T^G = \begin{pmatrix} 0 & 1 \\ 1-\gamma_h & \gamma_h \end{pmatrix} \text{ and } T^B = \begin{pmatrix} 0 & 1 \\ 1-\gamma_l & \gamma_l \end{pmatrix}.$$

Below we depict this system for the purpose of illustration.

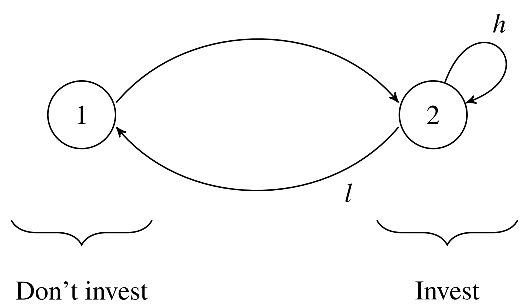

FIGURE 1. DETERMINISTIC RATING SYSTEM

Because the system is irreducible, we compute the stationary distributions by solving $f^\omega \cdot T^\omega = f^\omega$ *for each* ω, *leading to*

$$f_1^G = \frac{1-\gamma_h}{2-\gamma_h}, \qquad f_1^B = \frac{1-\gamma_l}{2-\gamma_l}.$$

Finally, the posterior beliefs are given by

$$\rho_1 = \frac{\rho(1-\gamma_l)(2-\gamma_h)}{\rho(1-\gamma_l)(2-\gamma_h)+(1-\rho)(1-\gamma_h)(2-\gamma_l)};$$
$$\rho_2 = \frac{\rho(2-\gamma_h)}{\rho(2-\gamma_h)+(1-\rho)(2-\gamma_l)}.$$

This rating system is irreducible and informative if $\rho_2 \leq \rho^*$ and $\rho_1 > \rho^*$. We note that this is just an example of a rating system, but it is generically not the optimal one.

II. Optimal Rating Systems

In this section, we construct the optimal rating system. To do this, we first derive a bound on the belief spread induced by the designer's constrained communication rule. This in turn leads to an upper bound on her optimal value of information, which is proved in Theorem 1. We then contrast these results with what a designer would be able to get if she could condition messages on the project's true type. We also discuss how the informativeness of the payoffs plays a role in the designer's value of information.

Without loss of generality, we can label the ratings such that $\rho_1 \geq \rho_2 \geq ... \geq \rho_R$. This implies that

$$\frac{f_1^B}{f_1^G} \geq \frac{f_2^B}{f_2^G} \geq ... \geq \frac{f_R^B}{f_R^G}.$$

For what follows, it will also be convenient to define

$$\lambda = \frac{\rho^*}{1-\rho^*}\frac{1-\rho}{\rho},$$

where note that $\lambda < 1$ if $\rho > \rho^*$. If i is a rating in which there is investment, then it must be that $\rho_i \leq \rho^*$. Note as well that $\rho_i \leq \rho^*$ if and only if $\frac{f_i^B}{f_i^G} \leq \lambda$. This represents the prior bias against investment. Let $\mathscr{N}$ represent the set of non-investment ratings and $\mathscr{I}$ refer to the set of investment ratings. We can write the incentive compatibility inequalities as:

$$\frac{f_i^B}{f_i^G} > \lambda \ \ \forall\, i \in \mathscr{N}; \tag{1}$$

$$\frac{f_i^B}{f_i^G} \leq \lambda \ \ \forall\, i \in \mathscr{I}. \tag{2}$$

We will now work out a third restriction for equilibria with at least one investment rating. We will show that there is an upper bound on the belief spread in the induced beliefs. The intuition for the fact that beliefs cannot be too far apart is that if the designer tries to construct a rating that induces a belief that is too high (meaning that it is likely that the project is bad), it will affect the beliefs in the other rating, violating the obedience constraint (incentives for investing) in the other rating. The next proposition formalizes this intuition, but before we state it, we need some additional notation. Let us define

$$\eta = \max_{m} \frac{\gamma_m^B}{\gamma_m^G}.$$

This represents the strength of bad news, or the informativeness of the payoff with the highest likelihood between the bad and the good type of project. For the remainder of this section, let us assume that $\eta < \infty$, so that all payoffs that occur with positive probability under project B also occur with positive probability under project G. Note that $\eta > 1$. The case with $\eta = \infty$ will be discussed separately (see Example 2).

It will also be useful to define

$$\nu = \min_{m} \frac{\gamma_m^B}{\gamma_m^G}.$$

As we will see, perhaps surprisingly, our results do not depend on ν. We will postpone this discussion. Before we prove our proposition, let us state the following well-known result for Markov processes:

LEMMA 1: *If $\mathscr{R}$ is partitioned into any two subsets S and S', then in the steady state the probability of transition from S to S' must equal the probability of transition from S' to S. Precisely stated, for each ω,*

$$\sum_{i\in S}\sum_{j\in S'} f_i^{\omega}\tau_{i,j}^{\omega} = \sum_{i\in S'}\sum_{j\in S} f_j^{\omega}\tau_{i,j}^{\omega}.$$

PROPOSITION 1 (Upper Bound on the Belief Spread): *In any informative rating system, all non-investment ratings i obey*

$$\frac{f_i^B}{f_i^G} \leq \lambda\eta.$$

PROOF: Let $\mathscr{N}_1$ be the set of non-investment ratings i such that $\frac{f_i^B}{f_i^G} > \lambda\eta$ and $\mathscr{N}_2$ the set of non-investment ratings such that $\frac{f_i^B}{f_i^G} \leq \lambda\eta$. We want to show that $\mathscr{N}_1 = \emptyset$. Assume by way of contradiction that $\mathscr{N}_1$ is non-empty. With a slight abuse of notation, let $\tau_{i,S}^{\omega}$ be the transition chance from rating i to a set of ratings S conditional on $\omega \in \{B, G\}$. For every non-investment rating, we eliminate the superscript ω because the transition chance will be uninformative. From Lemma 1, the steady-state probabilities obey the following:

$$\frac{\sum_{i\in\mathscr{N}_1} f_i^B\left(\tau_{i,\mathscr{N}_2}+\tau_{i,\mathscr{I}}\right)}{\sum_{i\in\mathscr{N}_1} f_i^G\left(\tau_{i,\mathscr{N}_2}+\tau_{i,\mathscr{I}}\right)} = \frac{\sum_{i\in\mathscr{N}_2} f_i^B\tau_{i,\mathscr{N}_1}+\sum_{i\in\mathscr{I}} f_i^B\tau_{i,\mathscr{N}_1}^B}{\sum_{i\in\mathscr{N}_2} f_i^G\tau_{i,\mathscr{N}_1}+\sum_{i\in\mathscr{I}} f_i^G\tau_{i,\mathscr{N}_1}^G}.$$

The left-hand side exceeds $\lambda\eta$ by assumption since it is a weighted average of the ratios $\frac{f_i^B}{f_i^G}$, $i \in \mathscr{N}_1$ - each weight is $\frac{f_i^G(\tau_{1,\mathscr{N}_2}+\tau_{i,\mathscr{I}})}{\sum_{i\in\mathscr{N}_1} f_i^G(\tau_{1,\mathscr{N}_2}+\tau_{i,\mathscr{I}})}$. If the right-hand side is at most $\lambda\eta$, a contradiction follows. But the right-hand side is a weighted average of ratios $\frac{f_i^B}{f_i^G}$, $i \in \mathscr{N}_2$ and ratios $\frac{f_i^B \tau_{i,\mathscr{N}_1}^B}{f_i^G \tau_{i,\mathscr{N}_1}^G}$, $i \in \mathscr{I}$. The former are at most $\lambda\eta$ by construction and the latter are at most $\lambda\eta$ by the incentive compatibility constraint (investment requires $\frac{f_i^B}{f_i^G} \leq \lambda$ for every $i \in \mathscr{I}$) and information constraint (investment payoff likelihoods satisfy $\frac{\tau_{i,\mathscr{N}_1}^B}{\tau_{i,\mathscr{N}_1}^G} \leq \eta$). ■

Proposition 1 implies that in all non-investment ratings, agents will hold induced beliefs about a bad project no higher than $\bar{\rho}$, defined by

$$\bar{\rho} = \frac{\eta\rho^*}{1-\rho^*+\eta\rho^*}.$$

We also derive an upper bound on the designer's payoff and prove that this upper bound can be achieved, provided that the rating system induces all investment beliefs to be the lowest possible (that is, equal to ρ^*) and all non-investment beliefs to be highest possible (equal to $\bar{\rho}$). This is done in Theorem 1 below.

THEOREM 1 (Payoff Upper Bound): *For any informative rating system, there exists an upper bound on the designer's payoff:*

$$\Pi \leq \frac{(1-\rho)}{(1-\rho^*)} \frac{\lambda\eta - 1}{\lambda(\eta-1)}.$$

This upper bound is achieved iff beliefs in all non-investment ratings equal $\bar{\rho}$ and beliefs in all investment ratings equal ρ^*.

PROOF: Incentive compatibility of investing in $\mathscr{I}$ implies

$$\frac{\sum_{i\in\mathscr{I}} f_i^B}{\sum_{i\in\mathscr{I}} f_i^G} = \frac{1-\sum_{i\in\mathscr{N}} f_i^B}{1-\sum_{i\in\mathscr{N}} f_i^G} \leq \lambda.$$

Therefore,

$$\sum_{i\in\mathscr{N}} f_i^B \geq 1-\lambda+\lambda \sum_{i\in\mathscr{N}} f_i^G. \tag{3}$$

By Proposition 1,

$$\frac{\sum_{i\in\mathscr{N}} f_i^B}{\sum_{i\in\mathscr{N}} f_i^G} \leq \lambda\eta.$$

Therefore,

$$\sum_{i \in \mathcal{N}} f_i^G \geq \frac{1}{\lambda \eta} \sum_{i \in \mathcal{N}} f_i^B. \tag{4}$$

Substituting equation (4) into equation (3) yields

$$\sum_{i \in \mathcal{N}} f_i^B \geq \eta \left[\frac{1-\lambda}{\eta - 1} \right]. \tag{5}$$

Substituting equation (5) back into equation (4) yields

$$\sum_{i \in \mathcal{N}} f_i^G \geq \frac{1}{\lambda} \left[\frac{1-\lambda}{\eta - 1} \right]. \tag{6}$$

By equations (5) and (6), the designer's payoff - the steady state-investment probability - has the bound:

$$\rho \left(1 - \sum_{i \in \mathcal{N}} f_i^B \right) + (1-\rho) \left(1 - \sum_{i \in \mathcal{N}} f_i^G \right) \leq \rho \left(1 - \eta \left[\frac{1-\lambda}{\eta - 1} \right] \right) + (1-\rho) \left(1 - \frac{1}{\lambda} \left[\frac{1-\lambda}{\eta - 1} \right] \right).$$

Using $\lambda \equiv \frac{\rho^*}{1-\rho^*} \frac{1-\rho}{\rho}$, this simplifies to the desired upper bound. Such bound is achieved if and only if equations (3) and (4) both hold with equality. Finally, (3) holds with equality if and only if every investment rating i has $\frac{f_i^B}{f_i^G} = \lambda$, which holds iff ρ_i is the exactly cutoff belief for investment, i.e. $\rho_i = \rho^*$. Likewise, (4) holds with equality if and only if every non-investment rating i has $\frac{f_i^B}{f_i^G} = \lambda \eta$, which holds if and only if ρ_i is exactly the upper bound belief for non-investment, i.e. $\rho_i = \bar{\rho}$. ■

We will show in Proposition 2 that we can construct an optimal rating system with only two ratings. This system induces beliefs $\rho_1 = \bar{\rho}$ and $\rho_2 = \rho^*$. But before constructing such system, let us discuss three main differences from the results from our model with results from the standard Bayesian persuasion framework.

First, if the designer could commit to a distribution of messages conditional on the project's type, it would also be sufficient for him to induce only two beliefs in equilibrium, to maximize the investment probability: one equal to the cutoff for investment and the other as strongly as possible against investment. The designer could induce these beliefs by (i) recommending investment for sure when the type is G and (ii) randomizing between an investment and a non-investment recommendation when the type is B with the exact probability that would induce the cutoff posterior following the investment recommendation.

Binary messages are optimal in our model for different reasons. More specifically, our result is *not* a revelation principle. It is instead a consequence of the fact that (i) in non-investment ratings no information is generated; and (ii) in investment ratings there is no need to provide additional incentives through more information. Indeed, we will show in the next section that, when information is obtained even without investment, the designer's payoff increases with more ratings.

Second, to mimic a standard Bayesian persuasion designer, a rating system would ideally induce steady-state probabilities that would lead to $\frac{f_2^B}{f_2^G} = \lambda$ and $\frac{f_1^B}{f_1^G} = \infty$. From Proposition 1, our constrained designer cannot obtain the ratio of rating 1. As a result, the designer's investment payoff Π^{BP} from Bayesian persuasion is higher than the designer's constrained payoff Π from Theorem 1:

$$\Pi^{BP} = \rho\lambda + (1-\rho) = \frac{1-\rho}{1-\rho^*} > \frac{1-\rho}{1-\rho^*}\left[\frac{\lambda\eta - 1}{\lambda(\eta-1)}\right] = \Pi.$$

Nevertheless, as $\eta \to \infty$, we can see from the above equation that Π converges to Π^{BP}. This means that the relevant statistic for the designer is the likelihood η. Intuitively, a perfectly informative investment payoff about the bad type does make it possible to fully convince the agents that the project is bad. Note that, because the upper bound on Proposition 1 does not depend on ν, even a very informative investment payoff about the good type of project adds nothing to the expected designer's payoff.

Third, in the Bayesian persuasion framework, for every initial prior $\rho \in (\rho^*, 1)$, the designer can induce posteriors leading to the optimal value of information at ρ. This is true because the optimal value of information is the concave closure of the designer's payoff without any communication.[20] This is not the case in the constrained framework. The corollary of the lemma below will help us better understand this result.

LEMMA 2: *In any rating system, in equilibrium, at least one rating has a belief (weakly) lower than the prior, and at least one rating must have a belief (weakly) higher than the prior. Formally $\rho_i \leq \rho$ and $\rho_j \geq \rho$ for some $i, j \in \mathscr{R}$.*

PROOF: Suppose not. That is, suppose that $\rho_i > \rho$, $\forall i$. From the consistency requirement, we can write: $\frac{\rho f_i^B}{\rho f_i^B + (1-\rho) f_i^G} > \rho$, which implies that $f_i^B > f_i^G$, $\forall i$. If we sum for all i, we have that $\sum_i f_i^B > \sum_i f_i^G$. However, $\sum_i f_i^B = \sum_i f_i^G = 1$, so we have a contradiction. Same logic holds for contradicting $\rho_i < \rho$. ■

A corollary of this result (together with the implication of Proposition 1) is that whenever $\rho \geq \bar{\rho}$ there is no equilibrium in an informative system. From now on, we focus on the more interesting case, that is,

[20] That is, the pointwise infimum of affine functions that are weakly higher than the no-communication payoff.

ASSUMPTION 1: *The prior belief about the bad type is such that $\rho < \bar{\rho}$.*

Figure 2 and Figure 3 summarize the comparison between our constrained framework and the persuasion framework. In the left figure, the solid line and the thick dashed line represent the expected payoff the designer gets from standard Bayesian persuasion. In the right figure, the thick dashed line represents the payoff the designer gets from an optimal rating system when the prior is above ρ^*. It can also be seen in the right figure that as $\eta \to \infty$, the prior belief region for which there is an equilibrium with a rating system expands and the payoff converges to the Bayesian persuasion payoff.

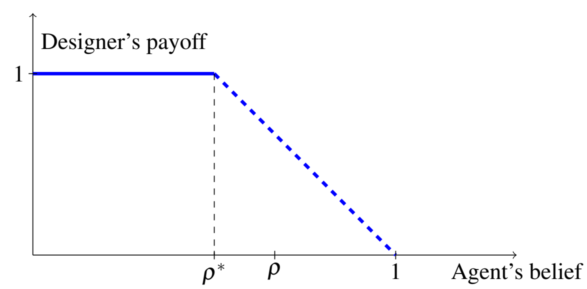

FIGURE 2. BAYESIAN PERSUASION PAYOFF

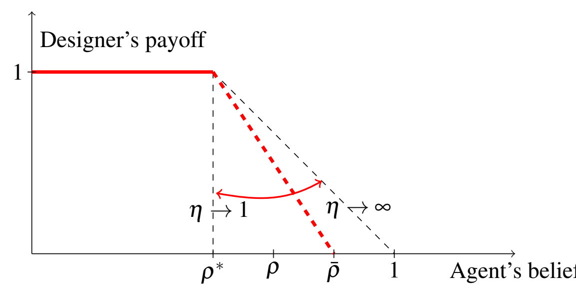

FIGURE 3. RATING SYSTEM PAYOFF

We now turn to the construction of the optimal rating system. We do this by defining transition rules so that f_1^B and f_1^G are as low as possible. Applying equation (3) for only one non-investment rating, we have

$$f_1^B \geq 1 - \lambda + \lambda f_1^G. \tag{7}$$

Because $\lambda < 1$, if the steady-state probabilities satisfy equation (7), then non-investment compatibility constraint (1) is satisfied and we can ignore it. From the investment compatibility constraint (2) and the constraint from Proposition 1, we derive

$$f_1^B \left(1 - f_1^G\right) \leq f_1^G \eta \left(1 - f_1^B\right) \Rightarrow f_1^B \leq \frac{\eta f_1^G}{1 + f_1^G (\eta - 1)}. \tag{8}$$

To maximize the steady-state investment probability, equations (7) and (8) must bind. So it must be that

$$1 - \lambda + \lambda f_1^G = \frac{\eta f_1^G}{1 + f_1^G (\eta - 1)}.$$

Indeed, we can see from Figure 4 that there are two points that satisfy the above equation, namely, when $f_1^G = f_1^B = 1$ (although this does not interest us), and another point at which $f_1^B < 1$ and $f_1^G < 1$.

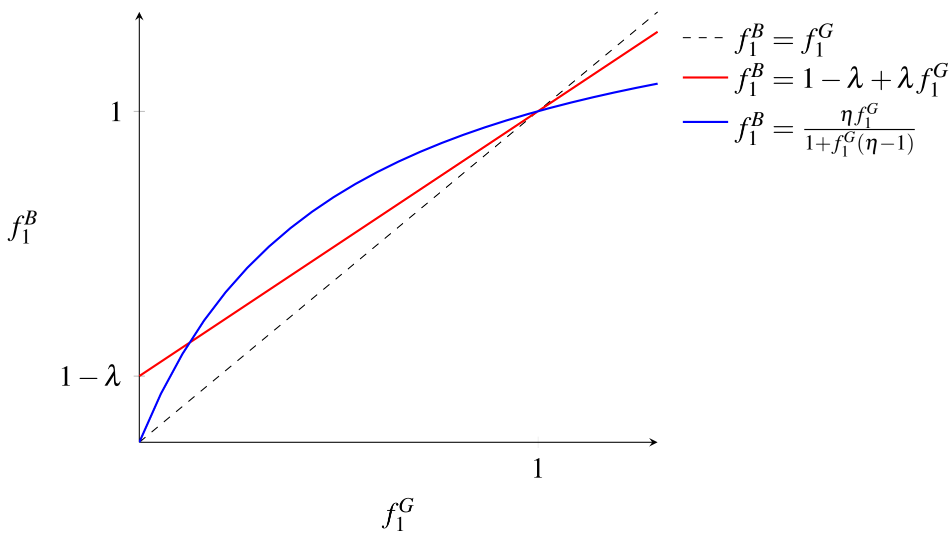

FIGURE 4. OPTIMAL STATIONARY PROBABILITIES

In Proposition 2 below, we construct rating systems that achieve this upper bound. The particular choice of transition rules will depend on the parameters, but it is possible to construct the optimal system for any given set of parameters satisfying Assumption 1.

PROPOSITION 2 (Constructing an Optimal System): *For any set of parameters satisfying Assumption 1, there exists a binary rating system that achieves the payoff bound from Theorem 1.*

PROOF: Let m be such that $\frac{\gamma_m^B}{\gamma_m^G} = \eta$. From rating 1 there is a random exit probability $\varphi_{12} = \tau$ and from rating 2 the transition rule is: (i) $\varphi_{21}^m = \kappa$ and (ii) $\varphi_{21}^n = 0\ \forall n \neq m$. Then, these transitions induce the following Markov transition matrices:

$$T^B = \begin{pmatrix} 1-\tau & \tau \\ \gamma_m^B \kappa & (1-\gamma_m^B)+\gamma_m^B(1-\kappa) \end{pmatrix} \text{ and } T^G = \begin{pmatrix} 1-\tau & \tau \\ \gamma_m^G \kappa & (1-\gamma_m^G)+\gamma_m^G(1-\kappa) \end{pmatrix}.$$

We can compute the stationary distributions by solving $f^\omega \cdot T^\omega = f^\omega$ for each $\omega \in \{B, G\}$. This leads to

$$f_2^B = \frac{\tau}{\tau+\gamma_m^B \kappa}; \qquad f_2^G = \frac{\tau}{\tau+\gamma_m^G \kappa}.$$

We will choose τ and κ appropriately so that $\frac{f_2^B}{f_2^G} = \lambda$. This gives us $\frac{\tau+\gamma_m^G\kappa}{\tau+\gamma_m^B\kappa} = \lambda$, which in turn implies that:

$$\frac{\tau}{\kappa} = \frac{\lambda\gamma_m^B - \gamma_m^G}{1-\lambda}.$$

Recall from Assumption 1 that we are only interested in the case in which $\rho \leq \bar{\rho}$, which implies $\lambda\eta > 1$, and in turn implies $\lambda\gamma_m^B - \gamma_m^G > 0$. We can conclude that both denominator and numerator are between 0 and 1. Thus, we can set $\tau = \lambda\gamma_m^B - \gamma_m^G$ and $\kappa = 1-\lambda$. This leads to the desired ratios $\frac{f_2^B}{f_2^G} = \lambda$ and $\frac{f_1^B}{f_1^G} = \eta\lambda$. ■

The optimal rating system derived in Proposition 2 is illustrated in Figure 5.

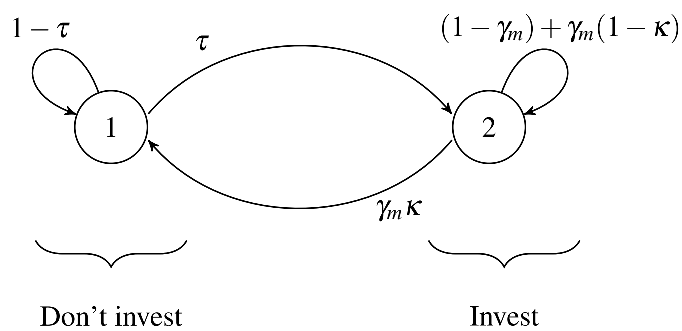

FIGURE 5. OPTIMAL RATING SYSTEM

We still have to deal with $\eta = \infty$. This is easily done, as it can be seen in Example 2. In this case, the rating system achieves the Bayesian persuasion payoff.

EXAMPLE 2 (Special Case: Perfectly Bad News): *Consider the case in which $\eta = \infty$, that is, there is a payoff that happens with positive probability under B, but not under G.*[21] *Since in this case the bad type can be fully learned, we might call it as the perfectly bad news case. A simple construction will suffice to achieve the Bayesian persuasion payoff. Let x_m be a payoff for which $\gamma_m^B > 0$ and $\gamma_m^G = 0$. Consider a binary rating system with transitions given by $\varphi_{2,1}^m = \kappa$, $\varphi_{2,1}^n = 0$ for all $n \neq m$, and $\tau_{1,2} = \tau$, where, recall, that this last transition must be independent of the payoff realization, since on-equilibrium path, there is no investment in rating 1. With this rating system, the stationary distributions are given by $f_1^B = \frac{\gamma_m^B\kappa}{\tau+\gamma_m^B\kappa}$ and $f_1^G = 0$. It follows immediately that $\rho_1 = 1$. To get the Bayesian persuasion payoff, we need to choose τ and κ such*

[21] The designer's optimal value from information does not depend on the strength of good news. Thus, even in the special case in which only the good type is fully revealing, the designer's payoff bound remains the same obtained in Theorem 1.

that $\rho_2 = \rho^*$. It must be that

$$\frac{\kappa}{\tau} = \frac{\rho - \rho^*}{\gamma_m^B \rho^* (1-\rho)}.$$

Since both the numerator and denominator are positive and strictly less than one, it suffices to set $\kappa = \rho - \rho^*$ and $\tau = \gamma_m^B \rho^* (1-\rho)$.

Before concluding this section, we prove that restricting attention to irreducible systems was without loss of generality. First, note that a rating system that has a unique recurrent class (i.e., one with a unique absorbing set of ratings) must be such that eventually the system reaches that class and remains there thereafter, regardless of the underlying type of project. This implies that this smaller set of ratings - the set of recurrent ratings - works in a fashion similar to an irreducible system, and we have shown that for such systems it is sufficient to consider two ratings.

Our next task is to show that the multiple recurrent classes cannot all be solely composed of investment ratings. Suppose, by contradiction, that all recurrent classes are composed exclusively of ratings in which the recommendation is to invest. The system will eventually reach some of these ratings, and the belief of at least one of these ratings must be at least as high as the original prior, which we are assuming to be a non-investment prior. Thus, at least one recurrent class has at least one non-investment rating, which is a contradiction. This result is similar to the one in Lemma 2.

In recurrent classes with both investment and non-investment ratings, we repeat the analysis of our general rating system with irreducible ratings. Thus, in each recurrent class, it must be that beliefs are bounded as shown in Proposition 1. Thus, the designer cannot improve upon a two-rating machine. Finally, it remains to show that any system with a recurrent class composed exclusively of non-investment ratings does not improve on our general irreducible system. This is shown in Proposition 3, and the proof is in the Appendix.

PROPOSITION 3 (Weak Sub-optimality of Reducible Systems): *A reducible system with a recurrent class of non-investment ratings cannot improve upon the optimal irreducible system.*

III. No Experimentation

In this section we look at the case where information about the type of the project arrives independently of actions. That is, if the agent does not invest, she still gets a payoff of zero, but there is an observable realization from the set $X = \{x_1, x_2, ..., x_M\}$ following the same distribution as specified in section I. We are interested in this environment since it helps us understand the forces driving our main results in the previous section. We show that there exists an upper bound on beliefs than can be achieved in equilibrium, but this bound is increasing in both the number of ratings and in the strength of good and bad news. Then we derive an upper bound on designer's payoff and construct a rating system that gets arbitrarily close to it.

As before, we label the ratings in a decreasing belief order, that is, $\rho_1 \geq \rho_2 \geq ... \geq \rho_R$. This implies that the likelihood ratios of the steady-state probabilities are decreasing as well. Hellman and Cover (1970) provided the following two lemmas[22], which will be useful in this section. The first derives a bound on the ratio of the transition rules. The second derives an upper bound on the likelihood ratios from rating i to rating $i+1$.

LEMMA 3: *The ratio of the transition from rating i to rating j under type B to the same transition under G satisfies*

$$\eta \geq \frac{\tau_{i,j}^B}{\tau_{i,j}^G} \geq \nu.$$

LEMMA 4: *For every rating $i \leq R-1$,*

$$\frac{f_i^B}{f_i^G} \leq \frac{\eta}{\nu}\frac{f_{i+1}^B}{f_{i+1}^G}.$$

PROPOSITION 4 (Upper Bound on the Belief Spread under No Experimentation): *In any informative rating system,*

$$\frac{f_1^B}{f_1^G} \leq \left(\frac{\eta}{\nu}\right)^{R-1}\lambda. \tag{9}$$

PROOF: Because there must be investment in at least one rating, it is true that $\frac{f_R^B}{f_R^G} \leq \lambda$. From Lemma 4, it is also true that $\frac{f_{R-1}^B}{f_{R-1}^G} \leq \left(\frac{\eta}{\nu}\right)\lambda$. Proceeding recursively, we find that $\frac{f_1^B}{f_1^G} \leq \left(\frac{\eta}{\nu}\right)^{R-1}\lambda$, as desired. ■

This proposition implies that the upper bound on the highest belief induced in equilibrium is increasing in the number of ratings R, the strength of bad news η and decreasing in the good news parameter ν. Indeed, the bound given by equation 9 can be written in terms of belief and is given by

$$\bar{\rho} = \frac{\eta^{R-1}\rho^*}{\eta^{R-1}\rho^* + \nu^{R-1}(1-\rho^*)}.$$

Note that $\bar{\rho} \to 1$ when $\eta \to \infty$, a result that is similar to the one we had in the previous section. However, we also have $\bar{\rho} \to 1$ when $\nu \to 0$ or when $R \to \infty$. The parameter ν affects the belief bound in a non-

[22]Lemmas 3 and 4 in our paper are lemmas 1 and 2 in Hellman and Cover (1970), respectively.

experimentation environment because induced beliefs in non-investment ratings need not be all the same anymore. Thus, a high belief in non-investment rating is possible even if transitions from good ratings to bad ratings are not very informative, provided that R is big enough. The number of ratings affects the belief bound because every non-investment rating $i \leq R-1$ can keep track of the history of payoffs up to $R-1$. Therefore, the higher the number of ratings, the easier it is to convince the agents that only the bad type of project can visit infinitely often the first ratings, by assigning downgrades to more informative payoffs about the bad type and upgrades to more informative payoffs about the good type.

Theorem 2 below is the analogous result of Theorem 1 for this section.

THEOREM 2 (Payoff Upper Bound under No Experimentation): *For any rating system with some rating i^* such that there is no investment in any rating $i \leq i^*-1$ and investment in any rating $i \geq i^*$, there exists an upper bound on the information designer's payoff:*

$$\Pi \leq \frac{1-\rho}{1-\rho^*}\left[\frac{\lambda-\left(\frac{\nu}{\eta}\right)^{i^*-1}}{\lambda\left(1-\left(\frac{\nu}{\eta}\right)^{i^*-1}\right)}\right].$$

PROOF: As before, $\mathscr{N}$ refers to the set of non-investment ratings and $\mathscr{I}$ to the set of investment ratings. Let $i^* = \min\{i : i \in \mathscr{I}\}$. From the same steps of the proof of Theorem 1, but now considering the bound on every $\frac{f_i^B}{f_i^G}$ from Proposition 4; in particular, $\sum_{i\in\mathscr{N}} f_i^G \geq \frac{1}{\lambda}\left(\frac{\nu}{\eta}\right)^{i^*-1}\sum_{i\in\mathscr{N}} f_i^B$, the designer's payoff has the following upper bound:

$$\Pi \leq \rho\left(1-\left[\frac{1-\lambda}{1-\left(\frac{\nu}{\eta}\right)^{i^*-1}}\right]\right)+(1-\rho)\left(1-\frac{\left(\frac{\nu}{\eta}\right)^{i^*-1}}{\lambda}\left[\frac{1-\lambda}{1-\left(\frac{\nu}{\eta}\right)^{i^*-1}}\right]\right).$$

Using $\lambda \equiv \frac{\rho^*}{1-\rho^*}\frac{1-\rho}{\rho}$, this simplifies to the desired upper bound. ■

To achieve this upper bound, it must be that every investment rating i has $\frac{f_i^B}{f_i^G} = \lambda$, which holds if and only ρ_i is the exact cutoff belief for investment, i.e. $\rho_i = \rho^*$. Therefore, it is sufficient to have only one investment rating with an indifference belief, that is, $\mathscr{I} = \{R\}$ and $\rho_R = \rho^*$. However, it is not true anymore that all non-investment ratings must have the same induced beliefs.

We can construct a rating system that gets arbitrarily close to the upper bound derived in Theorem 2 by making intermediary steady-state probabilities arbitrarily close to zero - that is, $f_1^B + f_R^B = f_1^G + f_R^G \approx 1$ -

and extreme beliefs close to the optimal bounds - that is, $\rho_R \approx \rho^*$ and $\rho_1 \approx \bar{\rho}$. The construction is done in Proposition 5 below (the proof is in the Appendix). Here is an intuition.

We first choose $\bar{R}$ high enough so that $\rho \leq \bar{\rho}$. This means that the range of priors for which an informative rating system exists under no experimentation expands to $(\rho^*, 1)$. Then, we partition the payoff space in two subsets such that the likelihood of aggregated payoffs in one set is higher than the likelihood of aggregated payoffs in the other set. They will work as a proxy for two relevant statistics for the transition rules. The constructed rating system for given $\bar{R}$ is such that: (i) at intermediate ratings, every payoff in one payoff subset leads to a downgrade and every payoff in another payoff subset leads to an upgrade; (ii) at extreme ratings 1 and $\bar{R}$, upgrades and downgrades are governed by parameters τ and κ, respectively. Those parameters are chosen appropriately so that even if $\tau \approx 0$ and $\kappa \approx 0$, the belief in $\bar{R}$ is kept at ρ^* and the belief in rating 1 is well defined. For a number $R \geq \bar{R}$ large enough, the designer's payoff from such system will be close enough to the bound derived in Theorem 2.

Because the upper bound on the designer's payoff in that theorem is increasing in the number of ratings and converges to the Bayesian persuasion payoff as R grows large, a corollary of Proposition 5 is that the constructed system also approximates the Bayesian persuasion payoff. In that case, the induced extreme beliefs will be such that $\rho_R \to \rho^*$ and $\rho_1 \to 1$ as $R \to \infty$.

PROPOSITION 5: *For any set of parameters and for every $\varepsilon > 0$, there exists a number $\bar{R}$ and an informative rating system with $R \geq \bar{R}$ ratings that is ε-close to the payoff bound from Theorem 2, that is,*

$$\Pi \geq \frac{1-\rho}{1-\rho^*}\left[\frac{\lambda - \left(\frac{\nu}{\eta}\right)^{R-1}}{\lambda\left(1-\left(\frac{\nu}{\eta}\right)^{R-1}\right)}\right] - \varepsilon.$$

In such system, intermediary ratings are almost never visited, that is, $f_1^\omega + f_R^\omega \approx 1$ for $\omega \in \{B, G\}$.

IV. Altruistic Designer

We argued before that the reason for the optimality of two ratings is a consequence of the fact that (i) information is not generated in non-investment ratings and (ii) there is no need to generate more information in investment ratings. In the previous section, we showed that the number of ratings matters when (i) is relaxed. In this section, we show this will also be the case when (ii) is relaxed. We do this by assuming that the designer is now a benevolent social planner, so she wants agents to have more information about the type of the project when investing.

Let us return to the case in which experimentation is needed to generate information. The agents' *ex-ante* payoff under the designer's rating system can be written as follows:

$$\Pi^A = \rho f_2^B \sum_{m=1}^{M} \gamma_m^B x_m + (1-\rho) f_2^G \sum_{m=1}^{M} \gamma_m^G x_m = f_2 \left[\rho_2 \sum_{m=1}^{M} \gamma_m^B x_m + (1-\rho_2) \sum_{m=1}^{M} \gamma_m^G x_m \right].$$

We obtain the second equality by dividing and multiplying Π^A by $f_2 \equiv \rho f_2^B + (1-\rho) f_2^G$. We know from Theorem 1 that at rating 2, the agent is indifferent between investing and not investing. Therefore, $\Pi^A = 0$.

Consider now the case in which the designer is altruistic. That is, her payoff is equivalent to the *ex-ante* payoff of the agents, which we write as

$$\Pi = \Pi^A = \rho \sum_{i \in \mathscr{I}} f_i^B \sum_{m=1}^{M} \gamma_m^B x_m + (1-\rho) \sum_{i \in \mathscr{I}} f_i^G \sum_{m=1}^{M} \gamma_m^G x_m.$$

Ideally, the altruistic designer would like to have extreme beliefs - zero for investment ratings and one for non-investment ratings. However, from Proposition 1 we know that the highest belief is bounded above by $\bar{\rho}$ and we compute an upper bound for the altruistic designer's expected payoff, which is given in the theorem below.

THEOREM 3: *For any $\rho \leq \bar{\rho}$, there exists an upper bound on the altruistic designer's payoff:*

$$\Pi \leq \sum_{m=1}^{M} \gamma_m^G x_m \left[\frac{\bar{\rho} - \rho}{\bar{\rho}} \right].$$

PROOF: This follows from

$$\begin{aligned}
\Pi &= \rho \sum_{i \in \mathscr{R}} \alpha_i f_i^B \sum_{m=1}^{M} \gamma_m^B x_m + (1-\rho) \sum_{i \in \mathscr{R}} \alpha_i f_i^G \sum_{m=1}^{M} \gamma_m^G x_m, \\
&= \sum_{i \in \mathscr{R}} \alpha_i f_i \left[\rho_i \sum_{m=1}^{M} \gamma_m^B x_m + (1-\rho_i) \sum_{m=1}^{M} \gamma_m^G x_m \right], \\
&\leq \sum_{m=1}^{M} \gamma_m^G x_m \left[\sum_{i \in \mathscr{R}} \alpha_i f_i \right].
\end{aligned}$$

The second line follows from the definitions of the belief ρ_i and f_i, which denotes the stationary probability of being at rating i, i.e., $f_i \equiv \rho f_i^B + (1-\rho) f_i^G$. The third line follows from $\sum_{m=1}^{M} \gamma_m^G x_m$ being the upper bound for the expression in brackets. Such value is maximal if and only if every investment rating i has $\rho_i = 0$. However, note that every irreducible rating system satisfies[23]

$$\sum_{i \in \mathscr{R}} f_i \rho_i = \rho.$$

Using the above equation but considering the split into non-investment ratings $i \in \mathscr{N}$ - for which $\rho_i \leq \bar{\rho}$ from Proposition 1 - and investment ratings $\mathscr{I}$ - for which we want to set $\rho_i = 0$, we get

$$\rho = \sum_{i \in \mathscr{R}} f_i \rho_i \leq \bar{\rho} \left[\sum_{i \in \mathscr{N}} f_i \right] \Rightarrow \sum_{i \in \mathscr{N}} f_i \geq \frac{\rho}{\bar{\rho}}.$$

Therefore,

$$\sum_{i \in \mathscr{I}} f_i \leq \frac{\bar{\rho} - \rho}{\bar{\rho}}.$$

Recall that $\alpha_i = 1$ for $i \in \mathscr{I}$ (and 0 otherwise). Thus, substituting the above inequality in the bound derived for Π at the beginning of the proof, we have the result. ■

This is best seen in the figure below. The solid line represents the expected payoff an altruistic designer would get without any informative system, as a function of beliefs. The outer dashed line represents what she would get if she could condition messages directly on the types of the project, that is, what she would get under a Bayesian persuasion framework. The inner dashed line represents the payoff upper bound derived in the previous theorem, under our constrained communication rule. Examining the figure, we can conclude that the altruistic designer and, consequently, the agents, are worse-off under constrained communication rules.

[23]This is analogous to the martingale property (Aumann, Maschler and Stearns, 1995) or Bayes plausibility (Kamenica and Gentzkow, 2011).

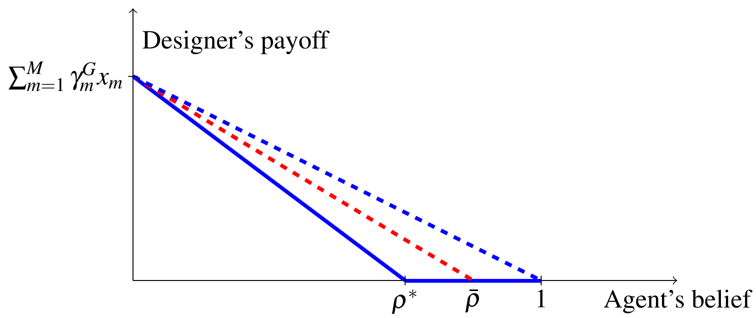

FIGURE 6. ALTRUISTIC DESIGNER

We can construct a rating system that approximates the payoff upper bound given by Theorem 3. The construction is very similar to the optimal rating system under no experimentation; thus, intermediary ratings will be required to bring extreme beliefs ρ_1 and ρ_R close to $\bar{\rho}$ and 0, respectively, but the system will stay most often on ratings 1 and R. The main difference now is that in rating 1, the exit rate must be independent of payoffs, since there is no investment in this rating. The higher the number of ratings, the closer this system will be to the upper bound.[24]

V. Changing Types

In this section, we consider an environment in which the project's type may change in every period. As such, old information is less relevant to induce current investment and the environment itself is stationary. Therefore, our communication rules with their stationary transitions and resulting invariant distributions may seem more reasonable.[25] We prove that many of our results hold qualitatively with evolving types. Specifically, we derive an upper bound on the maximum belief induced by a rating system, which also leads to an upper bound on the designer's value from a constrained communication. Additionally, we show that the irrelevance of good news holds in this new environment. Unlike other sections, however, we show that rating systems may be more restrictive when comparing with the Bayesian persuasion framework, because (1) the payoff upper bound does not converge to the Bayesian persuasion payoff as bad news gets arbitrarily revealing and (2) the set of priors for which it is possible to construct an informative system is smaller than this set under a fixed type environment.

[24]We omit the construction of such system here, but it is available upon request.

[25]In the case in which beliefs tend toward investment over time, it is optimal for the designer in the long-run not to engage in any strategic information disclosure. The more interesting aspect of the environment with changing types is when the stationary belief induces no investment. In this case, even when assuming away cold-start issues, there is still a non-trivial information design problem, because beliefs tend toward non-investment over time.

We now assume that types change according to a Markov process in which

$$\mathrm{P}(\omega_{t+1} = \omega_t) = 1 - \mu.$$

Thus, $1-\mu$ captures the persistence of the types of the project and we assume that $\mu < 1/2$. Without any rating system, relying only on the information that types are changing over time, the belief process converges to $\frac{1}{2}$. If $\rho^* \geq \frac{1}{2}$, in the long run, the designer has no need to disclose any additional information. Otherwise, that is, if $\rho^* < \frac{1}{2}$, then even in the long run there is room for improvement under an informative split of beliefs. We focus on this second case.

The timing of the problem given any rating system $\phi = (\mathscr{R}, \varphi_0, \varphi)$ is as follows. At $t = 1$ Nature selects the bad type with probability ρ and the initial rating is observed following the chosen φ_0. For each $t > 1$:

1) A new agent arrives, observes the current rating and chooses whether or not to invest.
2) If the agent invested, a payoff $x \in X$ is realized according to the distribution $\mathrm{P}(x_m|\omega_t)$. If the agent does not invest, no payoff is realized.
3) The rating system updates to a new rating using the current rating, the observed payoff (if any) and the specified transition rule from the rating system.
4) Nature draws a new type $\omega_{t+1} \in \Omega$ and we proceed to period $t+1$.

Notice that the combination of type transitions and rating transitions generates a Markov process on an extended state space $\mathscr{R} \times \Omega$. We concentrate on rating systems that generate a unique stationary distribution over the extended space. $\mathscr{N} \subset \mathscr{R}$ refer to the subset of non-investment ratings and $\mathscr{I} \subset \mathscr{R}$ refer to the subset of investment ratings. For any given rating system and its induced equilibrium, we can partition the set of ratings into $\mathscr{N}$ and $\mathscr{I}$. Then, the stationary distribution must satisfy

$$\sum_{i \in \mathscr{I}} f_i^B + \sum_{j \in \mathscr{N}} f_j^B = \sum_{i \in \mathscr{N}} f_i^G + \sum_{j \in \mathscr{I}} f_j^G = \frac{1}{2}, \tag{10}$$

where f_i^ω stands for the joint stationary probability of $(i, \omega) \in \mathscr{R} \times \Omega$. Without loss of generality, we can label the ratings such that $\rho_1 \geq \rho_2 \geq ... \geq \rho_R$. This implies that

$$\frac{f_1^B}{f_1^G} \geq \frac{f_2^B}{f_2^G} \geq ... \geq \frac{f_R^B}{f_R^G}.$$

From the incentive compatibility constraints, it must be the case that $\rho_i \leq \rho^*$ for $\forall i \in \mathscr{I}$ and $\rho_j > \rho^*$ for $\forall j \in \mathscr{N}$. Equivalently,

$$\frac{f_i^B}{f_i^G} > \frac{\rho^*}{1-\rho^*} \quad \forall i \in \mathscr{N}, \tag{11}$$

$$\frac{f_j^B}{f_j^G} \leq \frac{\rho^*}{1-\rho^*} \quad \forall j \in \mathscr{I}. \tag{12}$$

In the proposition below we prove that there is a lower bound for induced beliefs in any informative system. Intuitively, it is not possible for the designer to induce an informative rating system in which there is a rating for which agents are arbitrarily confident that the project is good. The reason is that the project's type has a positive probability of changing every period, which naturally bounds the beliefs.

PROPOSITION 6 (Lower Bound on the Belief Spread under Changing Types): *In any informative system, all ratings satisfy*

$$\frac{f_i^B}{f_i^G} > \frac{\mu}{1-\mu}.$$

PROOF: The stationary probabilities for (R,B) and (R,G) are given by:

$$f_R^B = (1-\mu)\left(\sum_{i\in\mathscr{N}} f_i^B \tau_{i,R} + \sum_{i\in\mathscr{I}} f_i^B \tau_{i,R}^B\right) + \mu\left(\sum_{i\in\mathscr{N}} f_i^G \tau_{i,R} + \sum_{i\in\mathscr{I}} f_i^G \tau_{i,R}^G\right),$$

$$f_R^G = (1-\mu)\left(\sum_{i\in\mathscr{N}} f_i^G \tau_{i,R} + \sum_{i\in\mathscr{I}} f_i^G \tau_{i,R}^G\right) + \mu\left(\sum_{i\in\mathscr{N}} f_i^B \tau_{i,R} + \sum_{i\in\mathscr{I}} f_i^B \tau_{i,R}^B\right).$$

Therefore, we can write:

$$\frac{f_R^B}{f_R^G} = \frac{\mu\left(\sum_{i\in\mathscr{N}} f_i^G \tau_{i,R} + \sum_{i\in\mathscr{I}} f_i^G \tau_{i,R}^G\right) + (1-\mu)\left(\sum_{i\in\mathscr{N}} f_i^B \tau_{i,R} + \sum_{i\in\mathscr{I}} f_i^B \tau_{i,R}^B\right)}{(1-\mu)\left(\sum_{i\in\mathscr{N}} f_i^G \tau_{i,R} + \sum_{i\in\mathscr{I}} f_i^G \tau_{i,R}^G\right) + \mu\left(\sum_{i\in\mathscr{N}} f_i^B \tau_{i,R} + \sum_{i\in\mathscr{I}} f_i^B \tau_{i,R}^B\right)}.$$

The right hand side of the above equation is increasing in $\sum_{i\in\mathscr{N}} f_i^B \tau_{i,R} + \sum_{i\in\mathscr{I}} f_i^B \tau_{i,R}^B$. Moreover, there must exists at least one rating communicating with R, so this term is strictly above zero. Therefore,

$$\frac{f_R^B}{f_R^G} > \frac{\mu\left(\sum_{i\in\mathscr{I}} f_i^G \tau_{i,R} + \sum_{i\in\mathscr{I}} f_i^G \tau_{i,R}^G\right)}{(1-\mu)\left(\sum_{i\in\mathscr{N}} f_i^G \tau_{i,R} + \sum_{i\in\mathscr{I}} f_i^G \tau_{i,R}^G\right)} = \frac{\mu}{1-\mu}.$$

Since $\frac{f_i^B}{f_i^G} \geq \frac{f_R^B}{f_R^G}$ by construction, we get the desired result. ■

Note that Proposition 6 implies that in all ratings, agents will hold induced beliefs about project being bad no lower than μ. This means that if $\rho^* < \mu$, there can be no rating systems for which investment is possible in equilibrium. From now on, we will focus on the interesting case in which $\rho^* > \mu$.

ASSUMPTION 2: *The critical belief about the bad type is such that $\rho^* > \mu$.*

This assumption is not sufficient to ensure the existence of an informative rating system, however. It is also necessary that the belief upper bound from previous sections is higher than $\frac{1}{2}$ (the stationary prior). We prove this in Lemma 6 in the Appendix. From now on, we focus on environments satisfying this condition.

ASSUMPTION 3: *The critical belief about the bad type is such that $\frac{\eta\rho^*}{1-\rho^*+\eta\rho^*} > \frac{1}{2}$.*

With changing types, there is also an upper bound the the maximum spread in the induced beliefs, as next proposition states (and whose proof is in the Appendix). As before, the strength of good news does not matter for the derivation of the upper bound. However, different from previous sections, the upper bound depends on how persistent the project's type is over time.

PROPOSITION 7 (Upper Bound on the Belief Spread under Changing Types): *In any informative system, all ratings satisfy*

$$\frac{f_i^B}{f_i^G} \leq \frac{\mu(1-\rho^*)+(1-\mu)\eta\rho^*}{(1-\mu)(1-\rho^*)+\mu\eta\rho^*}.$$

Proposition 7 implies that in all non-investment ratings, agents will hold induced beliefs about a bad project no higher than a new threshold $\tilde{\rho}$, defined by

$$\tilde{\rho} = \frac{\eta\rho^*}{1-\rho^*+\eta\rho^*} + \mu\frac{1-\rho^*-\eta\rho^*}{1-\rho^*+\eta\rho^*} \tag{13}$$

As usual, we derive an upper bound on information designer's payoff and prove that this upper bound can be achieved, provided that the rating system induces all investment beliefs to be the lowest possible (that is, equal to $\tilde{\rho}$) and all non-investment beliefs to be highest possible (equal to ρ^*).

THEOREM 4 (Payoff Upper Bound under Changing Types): *For any informative rating system, there exists an upper bound on the designer's payoff:*

$$\Pi \leq \frac{1}{2(1-\rho^*)}\left[\frac{2\tilde{\rho}-1}{\tilde{\rho}-\frac{\rho^*}{1-\rho^*}(1-\tilde{\rho})}\right],$$

This upper bound is achieved if and only if and only if beliefs in all non-investment ratings equal $\tilde{\rho}$ and beliefs in all investment ratings equal ρ^*.

PROOF: From the properties of a stationary distribution in the extended space, $\frac{\sum_{i\in\mathscr{I}} f_i^B}{\sum_{i\in\mathscr{I}} f_i^G} = \frac{\frac{1}{2}-\sum_{i\in\mathscr{N}} f_i^B}{\frac{1}{2}-\sum_{i\in\mathscr{N}} f_i^G} \leq \frac{\rho^*}{1-\rho^*}$. Therefore,

$$\sum_{i\in\mathscr{N}} f_i^B \geq \frac{1}{2}\left(1-\frac{\rho^*}{1-\rho^*}\right)+\frac{\rho^*}{1-\rho^*}\sum_{i\in\mathscr{N}} f_i^G. \tag{14}$$

From Proposition 7, $\frac{\sum_{i\in\mathscr{N}} f_i^B}{\sum_{i\in\mathscr{N}} f_i^G} \leq \frac{\mu+(1-\mu)\frac{\rho^*}{1-\rho^*}\eta}{(1-\mu)+\mu\frac{\rho^*}{1-\rho^*}\eta} = \frac{\tilde{\rho}}{1-\tilde{\rho}}$. Therefore,

$$\sum_{i\in\mathscr{N}} f_i^G \geq \left[\frac{1-\tilde{\rho}}{\tilde{\rho}}\right]\sum_{i\in\mathscr{N}} f_i^B. \tag{15}$$

Substituting equation (15) into equation (14) yields

$$\sum_{i\in\mathscr{N}} f_i^B \geq \frac{1}{2}\left[\frac{1-\frac{\rho^*}{1-\rho^*}}{1-\frac{\rho^*}{1-\rho^*}\left(\frac{1-\tilde{\rho}}{\tilde{\rho}}\right)}\right]. \tag{16}$$

Substituting equation (16) back into equation (15) yields

$$\sum_{i\in\mathscr{N}} f_i^G \geq \frac{1-\frac{\rho^*}{1-\rho^*}}{2}\left[\frac{\frac{1-\tilde{\rho}}{\tilde{\rho}}}{1-\frac{\rho^*}{1-\rho^*}\left(\frac{1-\tilde{\rho}}{\tilde{\rho}}\right)}\right]. \tag{17}$$

By equations (16) and (17), the designer's payoff - the steady-state investment probability - is such that

$$\begin{aligned}\left(\frac{1}{2}-\sum_{i\in\mathscr{N}} f_i^B\right)+\left(\frac{1}{2}-\sum_{i\in\mathscr{N}} f_i^G\right) &\leq \frac{1}{2}\left[\frac{\frac{\rho^*}{1-\rho^*}\left(1-\frac{1-\tilde{\rho}}{\tilde{\rho}}\right)}{1-\frac{\rho^*}{1-\rho^*}\left(\frac{1-\tilde{\rho}}{\tilde{\rho}}\right)}\right]+\frac{1}{2}\left[\frac{1-\frac{1-\tilde{\rho}}{\tilde{\rho}}}{1-\frac{\rho^*}{1-\rho^*}\left(\frac{1-\tilde{\rho}}{\tilde{\rho}}\right)}\right].\\ &= \frac{1}{2}\left[\frac{(1+\frac{\rho^*}{1-\rho^*})(1-\frac{1-\tilde{\rho}}{\tilde{\rho}})}{1-\frac{\rho^*}{1-\rho^*}\left(\frac{1-\tilde{\rho}}{\tilde{\rho}}\right)}\right],\\ &= \frac{1}{2(1-\rho^*)}\left[\frac{2\tilde{\rho}-1}{\tilde{\rho}-\frac{\rho^*}{1-\rho^*}(1-\tilde{\rho})}\right].\end{aligned}$$

Such upper bound is achieved if and only if equations (14) and (15) both hold with equality. Finally, (14) holds with equality if and only if every investment rating i has $\frac{f_i^B}{f_i^G} = \frac{\rho^*}{1-\rho^*}$, which holds iff ρ_i is the exactly cutoff belief for investment, i.e. $\rho_i = \rho^*$. Likewise, 15 holds with equality if and only if every non-investment rating i has $\frac{f_i^B}{f_i^G} = \frac{\mu(1-\rho^*)+(1-\mu)\eta\rho^*}{(1-\mu)(1-\rho^*)+\mu\eta\rho^*}$, which holds if and only if ρ_i is exactly the upper bound belief for non-investment, i.e. $\rho_i = \tilde{\rho}$. ■

As usual, we compare our results with the Bayesian persuasion framework, but now with evolving types of project. A hypothetical advisor that could perfectly forecast the next period's type conditional on the current type and could commit to a map from types to messages at each period would still split the prior beliefs into posteriors to achieve the concave closure of the expected investment probability at each period.[26] The optimal profit of our constrained designer does not converge to the long-run Bayesian persuasion profit as $\eta \to +\infty$, however. This is best seen through the bound in the ratio of the stationary probabilities. It increases as $\eta \to \infty$ and approaches:

$$\lim_{\eta \to +\infty} \frac{\mu(1-\rho^*)+(1-\mu)\eta\rho^*}{(1-\mu)(1-\rho^*)+\mu\eta\rho^*} = \frac{1-\mu}{\mu}.$$

This happens since even a sufficiently convinced agent must account for the fact that according to the underlying hidden Markov chain, with probability μ the world will have changed. Indeed, $\lim_{\eta \to +\infty} \frac{f_1^B}{f_1^G} = \frac{1-\mu}{\mu}$ implies that $\lim_{\eta \to +\infty} \rho_1 = 1-\mu$. Figure 7 illustrates the upper bound given by the proposition above.

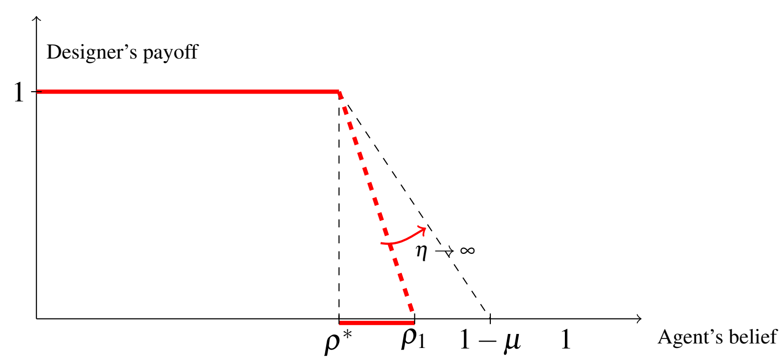

FIGURE 7. RATING SYSTEM PAYOFF WITH CHANGING TYPES OF PROJECT

[26] See results in Renault, Solan and Vieille (2017) for two types.

We now turn to the construction of the optimal rating system. We do this by defining transition rules so that f_1^B and f_1^G are as low as possible. Applying equation (14) for only one non-investment rating, we have

$$f_1^B \geq \frac{1}{2}\left[1 - \frac{\rho^*}{1-\rho^*}\right] + \frac{\rho^*}{1-\rho^*} f_1^G.$$

Because $\frac{\rho^*}{1-\rho^*} < 1$, if the stationary probabilities satisfy the above equation, then the non-investment compatibility constraint (11) is satisfied and we can ignore it. Applying equation (15) for only one non-investment rating, we have

$$f_1^B \leq \left[\frac{1-\tilde{\rho}}{\tilde{\rho}}\right] f_1^G$$

The pair of stationary distributions $\left(f_1^B, f_1^G\right)$ that maximizes the steady state probability of investing is obtained when both equations are binding. Thus, the distribution is

$$f_1^B = \frac{\tilde{\rho}}{2}\left[\frac{1-\frac{\rho^*}{1-\rho^*}}{\tilde{\rho} - \frac{\rho^*}{1-\rho^*}(1-\tilde{\rho})}\right], \qquad f_1^G = \frac{1-\tilde{\rho}}{2}\left[\frac{1-\frac{\rho^*}{1-\rho^*}}{\tilde{\rho} - \frac{\rho^*}{1-\rho^*}(1-\tilde{\rho})}\right],$$

$$f_2^B = \frac{\rho^*}{2(1-\rho^*)}\left[\frac{2\tilde{\rho}-1}{\tilde{\rho} - \frac{\rho^*}{1-\rho^*}(1-\tilde{\rho})}\right], \qquad f_2^G = \frac{1}{2}\left[\frac{2\tilde{\rho}-1}{\tilde{\rho} - \frac{\rho^*}{1-\rho^*}(1-\tilde{\rho})}\right].$$

We can construct rating systems that reach these probabilities, but only for some values of parameters. This is in contrast with the construction of optimal systems when the project's type is fixed. The construction of this optimal rating system is only possible if $\frac{1}{\gamma_m^G}\frac{1-2\rho^*}{1-2\mu}\frac{1}{\eta\rho^*-(1-\rho^*)} < 1$, where m refers to the most informative payoff of a bad type (that is, $\frac{\gamma_m^B}{\gamma_m^B} = \eta$). We state this in Proposition 8 below and prove it in the Appendix. However, we can discuss its intuition. Suppose that $\frac{1}{\gamma_m^G}\frac{1-2\rho^*}{1-2\mu}\frac{1}{\eta\rho^*-(1-\rho^*)} > 1$. Another way of writing this condition is

$$\rho^*(\gamma_m^B + \gamma_m^G) - \gamma_m^G < \frac{1-2\rho^*}{1-2\mu}.$$

The left hand side of this inequality says that the signal payoff m is relatively rare. If the rating system specifies that, from any given rating, the transition out of it happens only after such signal, it means that the system will be in that given rating for a long time, waiting for such signal to arrive. In an environment with changing types, waiting for too long in a rating implies that it is likely that the type will have changed while waiting, thus breaking down the incentive compatibility constraint. It might still be possible to construct an informative rating system, but its payoff will be bounded away from the one obtained in Theorem 4.

PROPOSITION 8: *There exists a binary rating system that achieves the bound from Theorem 4 only if*

$$\frac{1}{\gamma_m^G}\frac{1-2\rho^*}{1-2\mu}\frac{1}{\eta\rho^* - (1-\rho^*)} < 1,$$

where m is such that $\frac{\gamma_m^B}{\gamma_m^G} = \eta$.

PROOF: See the Appendix. ■

VI. Conclusion

In many of our everyday transactions, we rely on information intermediaries. However, communication is often impaired by data storage constraints (computational or legal) as well as transparency obligations. In this paper, we consider an information designer subjected to barriers to communication, seeking to motivate actions from uninformed agents she interacts with over time. Specifically, the designer in our baseline model uses a finite set of ratings and a transition rule over them, conditioned only on the rating each current agent sees and the feedback she provides after investing in a project of unknown, but fixed, quality.

We construct the optimal rating system and obtain one negative result and a set of positive results. Our negative result is that induced beliefs cannot be too far apart, limiting the scope of persuasion. This happens in our model because when agents do not invest, no feedback is collected. The need for informative signals about the project's quality generates a bound on how different a bad belief must be from a good belief, and this bound generates the maximum posterior belief for which there is an informative equilibrium. In our set of positive results, we show that (i) direct recommendation (a two-rating system) is the optimal design when experimentation is needed to generate information, and (ii) for specific signal structures, simple rules can approximate Bayesian persuasion. However, each rating matters when the designer collects data about the project's quality independently of investment decisions, or when the designer and the agents share same preferences (as a social planner would).

Our results imply that communication restrictions decrease the maximum payoff the designer can get from the interaction, without benefiting the agents. Thus, our model evidences an adverse consequence of strict requirements to data storage, which are increasingly advocated in recent legislation proposals. Our results shed light on what type of feedback a designer must pay attention to when communicating to induce data generation. Specifically, bad-news type of feedback is always useful, but good-news feedback is not.

As an extension, we analyze implications of communication barriers when the types of the project change over time, in a Markov fashion. Our main results hold qualitatively in this environment. Specifically, there will exist bounds on the belief spread that can be induced in equilibrium, leading to a bound on the designer's maximum payoff. Moreover, only bad news matters for the construction of the optimal system.

There are other extensions worth exploring in further research. For example, we assumed the realized payoffs agents obtain from investing are always the same. It would be interesting to study what would happen in a world in which quality is endogenous. That is, the designer uses the investments to improve the project, which would encourage agents to invest by improving the outcomes of the good and the bad type.

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Appendix

PROOF OF PROPOSITION 3: First, let $\mathscr{C}_1$ be a recurrent class of $j-$ratings, all of which induce no investment. Then, it must be the case that $\rho_1 = \rho_2 = ... = \rho_j$. Since no investments are made in these ratings, the transitions among them are independent of the payoffs (because there are no payoffs). And given that this subset $\mathscr{C}_1$ is irreducible, the stationary distribution is independent of the initial distribution.

Second, we argue that the prior in these ratings must be weakly smaller than the highest posterior of the ratings out of $\mathscr{C}_1$ that transition into $\mathscr{C}_1$. Ratings that transition into $\mathscr{C}_1$ are either investment ratings or non-investment ratings. If they are non-investment ratings, their priors are at least the posterior of an investment rating that led to this rating, since there are no payoffs in those non-investment ratings. If they are investment ratings, their posterior must be at most ρ^*. Thus, the posterior belief of an investment rating i entering the recurrent class $\mathscr{C}_1$ must be

$$\frac{\rho_i \sum_{m=1}^{M} \gamma_m^B \varphi_{i,j}^m}{\rho_i \sum_{m=1}^{M} \gamma_m^B \varphi_{i,j}^m + (1-\rho_i) \sum_{m=1}^{M} \gamma_m^G \varphi_{i,j}^m} \leq \frac{\eta \rho^*}{1+\rho^*(\eta-1)} = \bar{\rho}.$$

Given that the prior in $\mathscr{C}_1$ must be at most as high as the highest posterior of ratings transitioning into it, we have that the prior in $\mathscr{C}_1$ cannot be higher than $\bar{\rho}$. ■

PROOF OF PROPOSITION 5: First, note that we can always find some $\bar{R}$ such that

$$\rho \leq \frac{\eta^{\bar{R}-1}\rho^*}{\nu^{\bar{R}-1}(1-\rho^*)+\eta^{\bar{R}-1}\rho^*}.$$

Therefore, for every prior value ρ, we can construct a finite, irreducible rating system. Now partition the payoff space into X_ℓ and X_h and the index space into $\mathscr{M}_h = \{m : x_m \in X_h\}$ and $\mathscr{M}_\ell = \{m : x_\ell \in X_\ell\}$ such that

$$\frac{\sum_{m\in\mathscr{M}_\ell}\gamma_m^B}{\sum_{m\in\mathscr{M}_\ell}\gamma_m^G} > \frac{\sum_{m\in\mathscr{M}_h}\gamma_m^B}{\sum_{m\in\mathscr{M}_h}\gamma_m^G}.$$

In words, we divide the payoff space into two aggregation of payoffs, such that one aggregation leads to higher aggregated likelihood than the other. Such aggregations will work as an aggregate payoff (signal). Denote by φ_{ij}^n the chosen transition rule from rating i to rating j conditional on the observation of any $x \in X_n$, with $n \in \{\ell, h\}$. Define as well $\delta^\omega = \sum_{m\in\mathscr{M}_h}\gamma_m^\omega$, for each ω.

Consider the following (irreducible) system with $\bar{R}$ ratings: (1) $\varphi_{1,2}^h = \tau$, $\varphi_{1,1}^h = 1-\tau$, $\varphi_{1,1}^\ell = 1$; (2) $\varphi_{\bar{R},\bar{R}-1}^\ell = \kappa$, $\varphi_{\bar{R},\bar{R}}^\ell = 1-\kappa$, $\varphi_{\bar{R},\bar{R}}^h = 1$; (3) $\varphi_{i,i+1}^h = \varphi_{i,i-1}^\ell = 1$ for every $i \notin \{1,\bar{R}\}$. With these transition rules, we obtain the following steady-state probabilities, for each ω:

$$f_2^\omega = \tau\left(\frac{\delta^\omega}{1-\delta^\omega}\right)f_1^\omega, \qquad f_i^\omega = \left(\frac{\delta^\omega}{1-\delta^\omega}\right)f_{i-1}^\omega \;\; \forall\, i \notin \{1,\bar{R}\}, \qquad f_{\bar{R}}^\omega = \frac{1}{\kappa}\left(\frac{\delta^\omega}{1-\delta^\omega}\right)f_{\bar{R}-1}^\omega.$$

In particular, we have $f_{\bar{R}}^\omega = \frac{\tau}{\kappa}\left(\frac{\delta^\omega}{1-\delta^\omega}\right)^{\bar{R}-1}f_1^\omega$. This generates a system of equations in which

$$f_1^\omega + \tau\left(\frac{\delta^\omega}{1-\delta^\omega}\right)f_1^\omega + \tau\left(\frac{\delta^\omega}{1-\delta^\omega}\right)^2 f_1^\omega + \ldots + \tau\left(\frac{\delta^\omega}{1-\delta^\omega}\right)^{\bar{R}-2} f_1^\omega + \frac{\tau}{\kappa}\left(\frac{\delta^\omega}{1-\delta^\omega}\right)^{\bar{R}-1} f_1^\omega = 1.$$

Solving for f_1^ω leads to

$$f_1^\omega = \left[1 + \tau\left(\frac{\delta^\omega}{1-\delta^\omega}\right) + \tau\left(\frac{\delta^\omega}{1-\delta^\omega}\right)^2 + \ldots + \tau\left(\frac{\delta^\omega}{1-\delta^\omega}\right)^{\bar{R}-2} + \frac{\tau}{\kappa}\left(\frac{\delta^\omega}{1-\delta^\omega}\right)^{\bar{R}-1}\right]^{-1}. \tag{A1}$$

We want to set $\frac{f^B_{\bar{R}}}{f^G_{\bar{R}}} = \lambda$. To achieve this, we find that $\frac{f^B_1}{f^G_1}$ must be such that

$$\frac{f^B_{\bar{R}}}{f^G_{\bar{R}}} = \left(\frac{\delta^B}{\delta^G}\frac{1-\delta^G}{1-\delta^B}\right)^{\bar{R}-1}\frac{f^B_1}{f^G_1} = \lambda \quad \Rightarrow \frac{f^B_1}{f^G_1} = \lambda\left(\frac{\delta^G}{\delta^B}\frac{1-\delta^B}{1-\delta^G}\right)^{\bar{R}-1}.$$

Note that $\frac{f^B_1}{f^G_1} \leq \left(\frac{\eta}{\nu}\right)^{\bar{R}-1}\lambda$ from the definition of δ^ω. Moreover, $\frac{f^B_2}{f^G_2} = \lambda\left(\frac{\delta^G}{\delta^B}\frac{1-\delta^B}{1-\delta^G}\right)^{\bar{R}-2} \leq \left(\frac{\eta}{\nu}\right)^{\bar{R}-2}\lambda$ as well as $\frac{f^B_i}{f^G_i} \leq \left(\frac{\eta}{\nu}\right)^{\bar{R}-i}\lambda$ for every $i \notin \{1,\bar{R}\}$, because $f^\omega_i = \left(\frac{\delta^\omega}{1-\delta^\omega}\right)f^\omega_{i-1}$. This means that all non-investment steady state probabilities satisfy the constraint obtained from Proposition 4. Also note that substituting equation A1 into the expression for f^ω_2, we have

$$f^\omega_2 = \tau\left(\frac{\delta^\omega}{1-\delta^\omega}\right)f^\omega_1 = \frac{\tau}{\frac{1-\delta^\omega}{\delta^\omega} + \tau + \left(\frac{\delta^\omega}{1-\delta^\omega}\right)\tau + \ldots + \left(\frac{\delta^\omega}{1-\delta^\omega}\right)^{\bar{R}-3}\tau + \left(\frac{\delta^\omega}{1-\delta^\omega}\right)^{\bar{R}-2}\frac{\tau}{\kappa}}.$$

Therefore, $\lim_{\tau\to 0} f^\omega_2 = 0$ as well as $\lim_{\tau\to 0} f^\omega_i = 0$, for $i = 3,4,\ldots,\bar{R}-1$. Furthermore, if $\tau \to 0$ an $\kappa \to 0$, but $\frac{\tau}{\kappa} > 0$, then

$$\lim_{\tau\to 0,\kappa\to 0}\frac{f^B_1}{f^G_1} = \frac{1+\left(\frac{\delta^G}{1-\delta^G}\right)^{\bar{R}-1}\frac{\tau}{\kappa}}{1+\left(\frac{\delta^B}{1-\delta^B}\right)^{\bar{R}-1}\frac{\tau}{\kappa}}.$$

Given that we want $\frac{f^B_{\bar{R}}}{f^G_{\bar{R}}} = \lambda$, let us find the appropriate ratio $\frac{\tau}{\kappa}$ such that $\lim_{\tau\to 0,\kappa\to 0}\frac{f^B_1}{f^G_1} = \lambda\left(\frac{\delta^G}{\delta^B}\frac{1-\delta^B}{1-\delta^G}\right)^{\bar{R}-1}$. With some algebra, we get

$$\text{(A2)} \qquad \frac{\tau}{\kappa} = \frac{\lambda}{1-\lambda}\left(\frac{1-\delta^B}{\delta^B}\right)^{\bar{R}-1} - \frac{1}{1-\lambda}\left(\frac{1-\delta^G}{\delta^G}\right)^{\bar{R}-1}.$$

This value must be positive, so we need to guarantee that $\lambda > \left(\frac{\delta^B}{\delta^G}\frac{1-\delta^G}{1-\delta^B}\right)^{\bar{R}-1}$. But $\left(\frac{\delta^B}{\delta^G}\frac{1-\delta^G}{1-\delta^B}\right) < 1$ by construction, so as long as we set some $R \geq \bar{R}$ sufficiently large, this inequality must be satisfied. Therefore, we have the result. Take, for instance, the sequence $\kappa_t = \frac{1}{t}$ for each $t \in \mathbb{N}$ and set

$$\tau_t = \frac{1}{t}\left[\frac{\lambda}{1-\lambda}\left(\frac{1-\delta^B}{\delta^B}\right)^{R-1} - \frac{1}{1-\lambda}\left(\frac{1-\delta^G}{\delta^G}\right)^{R-1}\right].$$

As $t \to \infty$, we have that $\kappa_t \to 0, \tau_t \to 0$ and the ratio $\frac{\tau_t}{\kappa_t}$ equals the ratio A2 for every t. Thus,

$$\begin{aligned}
\lim_{t\to\infty} \Pi &= \rho f_R^B + (1-\rho) f_R^G, \\
&= f_R^G(\rho\lambda + 1 - \rho), \\
&= \frac{1-\rho}{1-\rho^*}\left[\frac{\tau}{\kappa}\left(\frac{\delta^G}{1-\delta^G}\right)^{R-1} f_1^G\right], \\
&= \frac{1-\rho}{1-\rho^*}\left[\frac{\frac{\tau}{\kappa}\left(\frac{\delta^G}{1-\delta^G}\right)^{R-1}}{1+\frac{\tau}{\kappa}\left(\frac{\delta^G}{1-\delta^G}\right)^{R-1}}\right].
\end{aligned}$$

The second equality follows from $f_R^B = \lambda f_R^G$; the third from the definition of λ and $f_R^G = \frac{\tau}{\kappa}\left(\frac{\delta^G}{1-\delta^G}\right)^{R-1} f_1^G$; the fourth from equation A1. Using the value of $\frac{\tau}{\kappa}$ as in A2,

$$\frac{\tau}{\kappa}\left(\frac{\delta^G}{1-\delta^G}\right) = \frac{\lambda}{1-\lambda}\left(\frac{\delta^G}{\delta^B}\frac{1-\delta^B}{1-\delta^G}\right)^{R-1} - \frac{1}{1-\lambda}.$$

Substituting this back to the limiting value of Π,

$$\lim_{t\to\infty}\Pi = \frac{1-\rho}{1-\rho^*}\left[\frac{\lambda\left(\frac{\delta^G}{\delta^B}\frac{1-\delta^B}{1-\delta^G}\right)^{R-1}-1}{\left(\frac{\delta^G}{\delta^B}\frac{1-\delta^B}{1-\delta^G}\right)^{R-1}-1}\right]\frac{1}{\lambda} = \frac{1-\rho}{1-\rho^*}\left[\frac{\lambda-\left(\frac{\delta^B}{\delta^G}\frac{1-\delta^G}{1-\delta^B}\right)^{R-1}}{\lambda\left(1-\left(\frac{\delta^B}{\delta^G}\frac{1-\delta^B}{1-\delta^B}\right)^{R-1}\right)}\right].$$

Now, $\left(\frac{\delta^B}{\delta^G}\frac{1-\delta^G}{1-\delta^B}\right) \geq \frac{\nu}{\eta}$, but $\lim_{R\to\infty}\left(\frac{\delta^B}{\delta^G}\frac{1-\delta^G}{1-\delta^B}\right) = \lim_{R\to\infty}\left(\frac{\nu}{\eta}\right)^{R-1} = 0$. Thus, for every $\varepsilon > 0$, we can find some $R \geq \bar{R}$ that still keeps equation A2 positive and gets $\left(\frac{\delta^B}{\delta^G}\frac{1-\delta^G}{1-\delta^B}\right)^{R-1}$ ε-close to $\left(\frac{\nu}{\eta}\right)^{R-1}$. ■

LEMMA 5: *In any informative rating system, it must be that* $\sum_{j\in\mathcal{N}} f_j^B > \sum_{j\in\mathcal{N}} f_j^G$.

PROOF: Take any $i \in \mathcal{N}$ and any $j \in \mathcal{I}$. From incentive compatibility, it must be that $f_i^B f_j^G > f_j^B f_i^G$. Summing over all investment ratings leads to

$$f_i^B\left(\sum_{j\in\mathcal{I}} f_j^G\right) > \left(\sum_{j\in\mathcal{I}} f_j^B\right) f_i^G.$$

Summing over all non-investment ratings leads to

$$\left(\sum_{i\in\mathcal{N}} f_i^B\right)\left(\sum_{j\in\mathcal{I}} f_j^G\right) > \left(\sum_{j\in\mathcal{I}} f_j^B\right)\left(\sum_{i\in\mathcal{N}} f_i^G\right).$$

We use the result on stationary probabilities given in equation 10 to obtain:

$$\left(\sum_{i\in\mathcal{N}} f_i^B\right)\left(\frac{1}{2}-\sum_{i\in\mathcal{N}} f_i^G\right) > \left(\frac{1}{2}-\sum_{i\in\mathcal{N}} f_i^B\right)\left(\sum_{i\in\mathcal{N}} f_i^G\right) \Rightarrow \sum_{i\in\mathcal{N}} f_i^B > \sum_{i\in\mathcal{N}} f_i^G.$$

■

LEMMA 6: *An informative rating system can only exist if* $\frac{\eta\rho^*}{1-\rho^*+\eta\rho^*} > \frac{1}{2}$.

PROOF: Consider the sets $\mathcal{N}\times\{B\}\cup\mathcal{I}\times\{G\}$ and $\mathcal{N}\times\{G\}\cup\mathcal{I}\times\{B\}$. They form a partition of the space $\mathcal{R}\times\Omega$. Then the result in stochastic processes that outflows must equal to inflows (Lemma 1) leads to

$$\begin{aligned}\sum_{j\in\mathcal{N}} f_j^B\left[\mu\tau_{j,\mathcal{N}}+(1-\mu)\tau_{j,\mathcal{I}}\right]+\sum_{i\in\mathcal{I}} f_i^G\left[\mu\tau_{i,\mathcal{N}}^G+(1-\mu)\tau_{i,\mathcal{I}}^G\right] &= \sum_{j\in\mathcal{N}} f_j^G\left[\mu\tau_{j,\mathcal{N}}+(1-\mu)\tau_{j,\mathcal{I}}\right]+\\ &+\sum_{i\in\mathcal{I}} f_i^B\left[\mu\tau_{i,\mathcal{N}}^B+(1-\mu)\tau_{i,\mathcal{I}}^B\right].\end{aligned}$$

Rearranging,

$$\sum_{j\in\mathcal{N}}(f_j^B-f_j^G)\left[\mu\tau_{j,\mathcal{N}}+(1-\mu)\tau_{j,\mathcal{I}}\right] = \sum_{i\in\mathcal{I}} f_i^B\left[(1-\mu)\sigma_{i,\mathcal{N}}^B+\mu\tau_{i,\mathcal{I}}^B\right]-\sum_{i\in\mathcal{I}} f_i^G\left[(1-\mu)\tau_{j,\mathcal{N}}^G+\mu\sigma_{j,\mathcal{I}}^G\right].$$

Note that the left hand side of the equation above is positive:

$$\begin{aligned}\sum_{j\in\mathcal{N}}(f_j^B-f_j^G)\left[\mu\tau_{j,\mathcal{N}}+(1-\mu)\tau_{j,\mathcal{I}}\right] &\geq \mu\sum_{j\in\mathcal{N}}(f_j^B-f_j^G)\left[\tau_{j,\mathcal{N}}+\tau_{j,\mathcal{I}}\right],\\ &= \mu\sum_{j\in\mathcal{N}}(f_j^B-f_j^G),\\ &> 0.\end{aligned}$$

The first inequality comes from $\mu < 1/2$; the second inequality comes from Lemma 5. Moreover, the right hand side of the equation is such that:

$$\sum_{i\in\mathscr{I}} f_i^B\left[(1-\mu)\sigma^B_{i,\mathscr{N}}+\mu\tau^B_{i,\mathscr{I}}\right]-\sum_{i\in\mathscr{I}} f_i^G\left[(1-\mu)\tau^G_{j,\mathscr{N}}+\mu\sigma^G_{j,\mathscr{I}}\right]\leq\frac{\eta\rho^*}{1-\rho^*}\sum_{j\in\mathscr{I}} f_j^G\left[(1-\mu)\tau^G_{j,\mathscr{N}}+\mu\tau^G_{j,\mathscr{I}}\right]-$$
$$-\sum_{j\in\mathscr{I}} f_j^G\left[(1-\mu)\tau^G_{j,\mathscr{N}}+\mu\tau^G_{j,\mathscr{I}}\right],$$
$$=\left(\frac{\eta\rho^*}{1-\rho^*}-1\right)\sum_{j\in\mathscr{I}} f_j^G\left[(1-\mu)\tau^G_{j,\mathscr{N}}+\mu\tau^G_{j,\mathscr{I}}\right].$$

The inequality comes from the compatibility constraint at non-investment ratings and $\tau^B_{j,\mathscr{N}}\leq\eta\tau^G_{j,\mathscr{N}}$ for every $i\in\mathscr{I}$. Now, since the left hand side is positive, the right hand side must also be positive, and since it is smaller than the last term in the inequality above, it follows that $(\frac{\eta\rho^*}{1-\rho^*}-1)$ must also be positive. ∎

PROOF OF PROPOSITION 7: Let $\mathscr{N}_1$ be the set of non-investment ratings i such that $\frac{f_i^B}{f_i^G}>\frac{\mu(1-\rho^*)+(1-\mu)\eta\rho^*}{(1-\mu)(1-\rho^*)+\mu\eta\rho^*}$ and $\mathscr{N}_2$ the set of non-investment ratings such that $\frac{f_i^B}{f_i^G}\leq\frac{\mu(1-\rho^*)+(1-\mu)\eta\rho^*}{(1-\mu)(1-\rho^*)+\mu\eta\rho^*}$. Assume by way of contradiction that $\mathscr{N}_1$ is non-empty. Consider the sets $\mathscr{N}_1\times\{B\}$ and $\mathscr{N}_1\times\{G\}\cup\mathscr{N}_1^{\complement}\times\{B,G\}$. Again from Lemma 1,

$$\sum_{i\in\mathscr{N}_1} f_i^B\left[\mu\tau_{i,\mathscr{N}_1}+\tau_{i,\mathscr{N}_2}+\tau_{i,\mathscr{I}}\right]=\mu\sum_{i\in\mathscr{N}_1} f_i^G\tau_{i,\mathscr{N}_1}+(1-\mu)\sum_{i\in\mathscr{N}_2} f_i^B\tau_{i,\mathscr{N}_1}+\mu\sum_{i\in\mathscr{N}_2} f_i^G\tau_{i,\mathscr{N}_1}+$$
$$+(1-\mu)\sum_{i\in\mathscr{I}} f_i^B\tau^B_{i,\mathscr{N}_1}+\mu\sum_{i\in\mathscr{I}} f_i^G\tau^G_{i,\mathscr{N}_1}. \quad \text{(A3)}$$

Consider now the sets $\mathscr{N}_1\times\{G\}$ and $\mathscr{N}_1\times\{B\}\cup\mathscr{N}_1^{\complement}\times\{B,G\}$. The steady state probabilities obey

$$\sum_{i\in\mathscr{N}_1} f_i^G\left[\mu\tau_{i,\mathscr{N}_1}+\tau_{i,\mathscr{N}_2}+\tau_{i,\mathscr{I}}\right]=\mu\sum_{i\in\mathscr{N}_1} f_i^B\tau_{i,\mathscr{N}_1}+(1-\mu)\sum_{i\in\mathscr{N}_2} f_i^G\tau_{i,\mathscr{N}_1}+\mu\sum_{i\in\mathscr{N}_2} f_i^B\tau_{i,\mathscr{N}_1}+$$
$$+(1-\mu)\sum_{i\in\mathscr{I}} f_i^G\tau^G_{i,\mathscr{N}_1}+\mu\sum_{i\in\mathscr{I}} f_i^B\tau^B_{i,\mathscr{N}_1}. \quad \text{(A4)}$$

Combining equations A3 and A4 and rearranging, we get that the left hand side (*LHS*) equals the right hand side (*RHS*), where

$$LHS=\frac{\sum_{i\in\mathscr{N}_1} f_i^B\left[\mu\tau_{i,\mathscr{N}_1}+\tau_{i,\mathscr{N}_2}+\tau_{i,\mathscr{I}}\right]}{\sum_{i\in\mathscr{N}_1} f_i^G\left[\mu\tau_{i,\mathscr{N}_1}+\tau_{i,\mathscr{N}_2}+\tau_{i,\mathscr{I}}\right]},$$

$$RHS = \frac{\mu \sum_{i\in\mathcal{N}_1} f_i^G \tau_{i,\mathcal{N}_1} + \mu \left[\sum_{i\in\mathcal{N}_2} f_i^G \tau_{i,\mathcal{N}_1} + \sum_{i\in\mathcal{I}} f_i^G \tau_{i,\mathcal{N}_1}^G\right] + (1-\mu)\left[\sum_{i\in\mathcal{N}_2} f_i^B \tau_{i,\mathcal{N}_1}^B + \sum_{i\in\mathcal{I}} f_i^B \tau_{i,\mathcal{N}_1}^B\right]}{\mu \sum_{i\in\mathcal{N}_1} f_i^B \tau_{i,\mathcal{N}_1} + \mu \left[\sum_{i\in\mathcal{N}_2} f_i^B \tau_{i,\mathcal{N}_1} + \sum_{i\in\mathcal{I}} f_i^B \tau_{i,\mathcal{N}_1}^B\right] + (1-\mu)\left[\sum_{i\in\mathcal{N}_2} f_i^G \tau_{i,\mathcal{N}_1}^G + \sum_{i\in\mathcal{I}} f_i^G \tau_{i,\mathcal{N}_1}^G\right]}.$$

The LHS exceeds $\frac{\mu(1-\rho^*)+(1-\mu)\eta\rho^*}{(1-\mu)(1-\rho^*)+\mu\eta\rho^*}$ by assumption since it is a weighted average of the ratios $\frac{f_i^B}{f_i^G}$, $i \in \mathcal{N}_1$ - each weight is $\frac{f_i^G(\mu\tau_{1,\mathcal{N}_1}+\tau_{1,\mathcal{N}_2}+\tau_{i,\mathcal{I}})}{\sum_{i\in\mathcal{N}_1} f_i^G(\mu\tau_{1,\mathcal{N}_1}+\tau_{1,\mathcal{N}_2}+\tau_{i,\mathcal{I}})}$. RHS is increasing in the term $\sum_{i\in\mathcal{N}_2} f_i^B \tau_{i,\mathcal{N}_1} + \sum_{i\in\mathcal{I}} f_i^B \tau_{i,\mathcal{N}_1}^B$. Moreover,

$$\begin{aligned}
\sum_{i\in\mathcal{N}_2} f_i^B \tau_{i,\mathcal{N}_1} + \sum_{i\in\mathcal{I}} f_i^B \tau_{i,1}^B &= \sum_{i\in\mathcal{N}_2} \left(\frac{f_i^B}{f_i^G}\right) f_i^G \tau_{i,\mathcal{N}_1} + \sum_{i\in\mathcal{I}} \left(\frac{f_i^B}{f_i^G}\right) f_i^G \tau_{i,\mathcal{N}_1}^B, \\
&\leq \frac{\mu(1-\rho^*)+(1-\mu)\eta\rho^*}{(1-\mu)(1-\rho^*)+\mu\eta\rho^*} \sum_{i\in\mathcal{N}_2} f_i^G \tau_{i,\mathcal{N}_1} + \frac{\eta\rho^*}{1-\rho^*} \sum_{i\in\mathcal{I}} f_i^G \tau_{i,\mathcal{N}_1}^G, \\
&\leq \frac{\eta\rho^*}{1-\rho^*} \left[\sum_{i\in\mathcal{N}_2} f_i^G \tau_{i,\mathcal{N}_1} + \sum_{i\in\mathcal{I}} f_i^G \tau_{i,\mathcal{N}_1}^G\right].
\end{aligned}$$

The second inequality follows from

$$\frac{\mu(1-\rho^*)+(1-\mu)\eta\rho^*}{(1-\mu)(1-\rho^*)+\mu\eta\rho^*} = \frac{\mu + (1-\mu)\frac{\eta\rho^*}{1-\rho^*}}{(1-\mu)+\mu\frac{\eta\rho^*}{1-\rho^*}} < \frac{\eta\rho^*}{1-\rho^*}.$$

Therefore,

$$\begin{aligned}
RHS &\leq \frac{\mu \sum_{i\in\mathcal{N}_1} f_i^G \tau_{i,\mathcal{N}_1} + \left(\mu + (1-\mu)\frac{\eta\rho^*}{1-\rho^*}\right)\left[\sum_{i\in\mathcal{N}_2} f_i^G \tau_{i,\mathcal{N}_1} + \sum_{i\in\mathcal{I}} f_i^G \tau_{i,\mathcal{N}_1}^G\right]}{\mu \sum_{i\in\mathcal{N}_1} f_i^B \tau_{i,\mathcal{N}_1} + \left((1-\mu) + \mu\frac{\eta\rho^*}{1-\rho^*}\right)\left[\sum_{i\in\mathcal{N}_2} f_i^G \tau_{i,\mathcal{N}_1} + \sum_{i\in\mathcal{I}} f_i^G \tau_{i,\mathcal{N}_1}^G\right]}, \\
&< \frac{\mu \sum_{i\in\mathcal{N}_1} f_i^B \tau_{i,\mathcal{N}_1} + \left(\mu + (1-\mu)\frac{\eta\rho^*}{1-\rho^*}\right)\left[\sum_{i\in\mathcal{N}_2} f_i^G \tau_{i,\mathcal{N}_1} + \sum_{i\in\mathcal{I}} f_i^G \tau_{i,\mathcal{N}_1}^G\right]}{\mu \sum_{i\in\mathcal{N}_1} f_i^B \tau_{i,\mathcal{N}_1} + \left((1-\mu) + \mu\frac{\eta\rho^*}{1-\rho^*}\right)\left[\sum_{i\in\mathcal{N}_2} f_i^G \tau_{i,\mathcal{N}_1} + \sum_{i\in\mathcal{N}^\complement} f_i^G \tau_{i,\mathcal{N}_1}^G\right]}, \\
&< \frac{\mu + (1-\mu)\eta\rho^*\eta}{(1-\mu)+\mu\eta\rho^*}.
\end{aligned}$$

The second inequality follows from $f_i^G < f_i^B$ for every $i \in \mathcal{N}_1$ and the third inequality follows from $\mu < \frac{1}{2}$, respectively. But then we reach a contradiction:

$$\frac{\mu+(1-\mu)\eta\rho^*}{(1-\mu)+\mu\eta\rho^*} < LHS = RHS < \frac{\mu+(1-\mu)\eta\rho^*}{(1-\mu)+\mu\eta\rho^*}.$$

■

PROOF OF PROPOSITION 8: Let m be such that $\frac{\gamma_m^B}{\gamma_m^G} = \eta$. Assume that, from rating 1 there is a random exit probability $\varphi_{1,2} = \tau$ and from rating 2 the transition rule is: (i) $\varphi_{2,1}^m = \kappa$ and (ii) $\varphi_{2,1}^n = 0\ \forall n \neq m$. Then, these transitions induce the following Markov transition matrix for states $\{(1,G),(1,B),(2,G),(2B)\}$:

$$T = \begin{pmatrix} (1-\mu)(1-\tau) & \mu(1-\tau) & (1-\mu)\tau & \mu\tau \\ \mu(1-\tau) & (1-\mu)(1-\tau) & \mu\tau & (1-\mu)\tau \\ (1-\mu)\gamma_m^G\kappa & \mu\gamma_m^G\kappa & (1-\mu)(1-\gamma_m^G\kappa) & \mu(1-\gamma_m^G\kappa) \\ \mu\eta\gamma_m^G\kappa & (1-\mu)\eta\gamma_m^G\kappa & \mu(1-\eta\gamma_m^G\kappa) & (1-\mu)(1-\eta\gamma_m^G\kappa) \end{pmatrix}.$$

As we are focusing on irreducible systems, we can compute the stationary distribution by solving $f \cdot T = f$. This leads to

(A5)
$$\frac{f_2^B}{f_2^G} = \frac{\tau[2\mu+\tau(1-2\mu)]+\gamma_m^G\kappa\tau(1-2\mu)}{\tau[2\mu+\tau(1-2\mu)]+\gamma_m^G\kappa\tau\eta(1-2\mu)}.$$

First note that with $\tau = 1$ we have that:

(A6)
$$\frac{f_2^B}{f_2^G} = \frac{1+\gamma_m^G\kappa(1-2\mu)}{1+\gamma_m^G\kappa\eta(1-2\mu)}.$$

Now, set κ to be such that the above ratio equals to the stationary distribution $\frac{\rho^*}{1-\rho^*}$.

$$\frac{1+\gamma_m^G\kappa(1-2\mu)}{1+\gamma_m^G\kappa\eta(1-2\mu)} = \frac{\rho^*}{1-\rho^*}.$$

Solving this, leads us to:

$$\kappa = \frac{1}{\gamma_m^G}\frac{1-2\rho^*}{1-2\mu}\frac{1}{\eta\rho^*-(1-\rho^*)}.$$

If this value is below 1, the value of κ is feasible, that is, $\kappa \in [0,1]$. Now, it remains to verify that with $\tau = 1$ and $\kappa = \frac{1}{\gamma_m^G}\frac{1-2\rho^*}{1-2\mu}\frac{1}{\eta\rho^*-(1-\rho^*)}$ we have that $\frac{f_2^B}{f_2^G} = \frac{\rho^*}{1-\rho^*}$, and:

$$\text{(A7)} \qquad \frac{f_1^B}{f_1^G} = \frac{f_2^G \mu \gamma_m^G \kappa + f_2^B (1-\mu)\eta\gamma_m^G \kappa}{f_2^G (1-\mu)\gamma_m^G \kappa + f_2^B \mu\eta\gamma_m^G \kappa}.$$

which gives us:

$$\text{(A8)} \qquad \frac{f_1^B}{f_1^G} = \frac{\mu + \frac{\rho^*}{1-\rho^*}(1-\mu)\eta}{(1-\mu) + \mu\eta\frac{\rho^*}{1-\rho^*}}.$$

We now claim that if $\tau < 1$ and $\frac{f_2^B}{f_2^G} = \frac{\rho^*}{1-\rho^*}$, then there is no rating system that achieves the upper bound payoff of Theorem 4. To see this, define $\lambda_1 = \frac{f_1^B}{f_1^G}$. Then,

$$\begin{aligned}
\lambda_1 &= \frac{f_1^G \mu(1-\tau) + \lambda_1 f_1^G (1-\mu)(1-\tau) + f_2^G \mu \sigma_{2,1}^G + \frac{\rho^*}{1-\rho^*} f_2^G (1-\mu)\sigma_{2,1}^B}{f_1^G (1-\mu)(1-\tau) + \lambda_1 f_1^G \mu(1-\tau) + f_2^G (1-\mu)\sigma_{2,1}^G + \frac{\rho^*}{1-\rho^*} f_2^G \mu \sigma_{2,1}^B}, \\
&= \frac{f_1^G (1-\tau)\left[\mu + \lambda_1(1-\mu)\right] + f_2^G \left[\mu\sigma_{2,1}^G + \frac{\rho^*}{1-\rho^*}(1-\mu)\sigma_{2,1}^B\right]}{f_1^G (1-\tau)\left[(1-\mu) + \lambda_1\mu\right] + f_2^G \left[(1-\mu)\sigma_{2,1}^G + \frac{\rho^*}{1-\rho^*}\mu\sigma_{2,1}^B\right]}.
\end{aligned}$$

Rearranging, we have:

$$\begin{aligned}
f_1^G (1-\tau)(\lambda_1^2 - 1)\mu &= f_2^G \left\{ \sigma_{2,1}^G \left[\mu - \lambda_1(1-\mu)\right] + \frac{\rho^*}{1-\rho^*}\sigma_{2,1}^B \left[(1-\mu) - \lambda_1\mu\right] \right\}, \\
&= f_2^G \sum_{k=1}^{m} \tau_{2,1}^k \left\{ \left[\gamma_k^G (\mu - \lambda_1(1-\mu)\right] + \frac{\rho^*}{1-\rho^*}\gamma_k^B \left[(1-\mu) - \lambda_1\mu\right] \right\}.
\end{aligned}$$

However, note that if $\lambda_1 = \frac{\mu(1-\rho^*)+(1-\mu)\eta\rho^*}{(1-\mu)(1-\rho^*)+\mu\eta\rho^*}$, then:

$$\begin{aligned}
\mu - \lambda_1(1-\mu) &= \mu - (1-\mu)\left[\frac{\mu(1-\rho^*) + (1-\mu)\eta\rho^*}{(1-\mu)(1-\rho^*) + \mu\eta\rho^*}\right], \\
&= \frac{\eta\rho^*\left[\mu^2 - (1-\mu)^2\right]}{(1-\mu)(1-\rho^*) + \mu\eta\rho^*}, \\
&= \frac{\eta\rho^*(2\mu - 1)}{(1-\mu)(1-\rho^*) + \mu\eta\rho^*}.
\end{aligned}$$

Likewise,

$$\begin{aligned}(1-\mu)-\lambda_1\mu &= (1-\mu)-\mu\left[\frac{\mu(1-\rho^*)+(1-\mu)\eta\rho^*}{(1-\mu)(1-\rho^*)+\mu\eta\rho^*}\right],\\ &= \frac{(1-\rho^*)\left[(1-\mu)^2-\mu^2\right]}{(1-\mu)(1-\rho^*)+\mu\eta\rho^*},\\ &= \frac{(1-\rho^*)(1-2\mu)}{(1-\mu)(1-\rho^*)+\mu\eta\rho^*}.\end{aligned}$$

Substituting back we have $\frac{f_1^G}{f_2^G}(1-\tau)(\lambda_1^2-1)\mu = \left[\frac{\rho^*(1-2\mu)}{(1-\mu)(1-\rho^*)+\mu\eta\rho^*}\right]\sum_{k=1}^m \tau_{2,1}^k\left(\gamma_k^B-\gamma_k^G\eta\right)$. Note that if $\tau < 1$ and $\lambda_1 > 1$, then the left hand side of this expression is positive. However, $\gamma_k^B - \gamma_k^G\eta \leq 0$, $\forall k$. ■