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Valuing the distant future under stochastic resettings: the effect on discounting

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Abstract
We investigate the effects of resetting mechanisms when valuing the future in economic terms through the discount function. Discounting is specially significant in addressing environmental problems and in evaluating the sense of urgency to act today to prevent or mitigate future losses due to climate change effects and other disasters. Poissonian resetting events can be seen in this context as a way to intervene the market, it modifies the discount function and it can facilitate a specific climate policy. We here obtain the exact expression of the discount function in Laplace space and attain the expression of the long-run interest rate, a crucial value in environmental economics and climate policy. Both quantities are obtained without assuming any model for the evolution of the market. Model specific results are achieved for diffusion processes and in particular for the Ornstein–Uhlenbeck and Feller processes. The effect of Poissonian resetting events is non-trivial in these cases. The overall lesson we can learn from the obtained results is that effective policies to favor climate action should be resolute and frequent enough in time: the frequency of the interventions is critical for actually observing the desired consequences in the long-run interest rate.

Keywords: resettings, discount function, long-run rates

(Some figures may appear in colour only in the online journal)

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1. Introduction

During the last decade a great amount of work has been devoted to stochastic resetting, a kind of composite process consisting in combining a given random process with resetting events which randomly bring the process into some fixed value. The resetting mechanism exhibits at least two significant advantages. First, it usually stabilizes the underlying process in the sense that the composite process may become stationary even when the underlying is not. Second, and surely more important, the resetting mechanism may substantially reduce the mean first-passage time to some critical value. It is this last characteristic which opens the way to significant applications in many different fields because it optimizes any search strategy based on the composite process. This approach is not only relevant in one dimensional problems. It is of interest for wide variety of settings and contexts such as the case of protein identification in DNA [1–5], animal foraging [6, 7] or data mining [8–10], just to name a few. Aside from few prior works in physics (e.g. [1, 11]) and in the mathematics literature (see [12] for more information) the topic has been reviewed and extensively developed by Evans, Majumdar and collaborators [13–19] along with an ever increasing number of different investigators of which we cite a very small sample [20–25] out of a huge literature.

As far as we know, resetting processes have been mostly studied when the underlying process is the Brownian motion although some generalizations include the continuous-time random walks [20], Lévy flights [22], some bounded diffusion processes [26, 27] and, quite recently, telegraphic processes [28] and anomalous diffusion [29, 30]. Situations of relevance to provide more context to the present paper are those processes such as energy or current (or area) which involve an additive process (see e.g. [31]). In these situations, long-time asymptotics of the integral (area) of the Ornstein–Uhlenbeck (OU) process under resetting can be obtained [31] and the conditions for first-order phase transitions due to resetting can be identified [32]. In most cases addressed, and for a wide variety of underlying dynamics, the resetting mechanism is governed by a Poisson process and it is seen as a way to stabilize the process and reduce the first-passage time as mentioned above and developed from diverse points of view in references [23, 24, 29, 33–39].

Another remarkable feature of resetting is that it can optimize the probability of success in Bernoulli trials [23] and, because of the universal character of Bernoulli trials in modeling countless phenomena, such optimization enhances the importance of stochastic resettings from theoretical as well as practical points of view and it results in many applications to several branches of physical, socio-economic sciences and technology. In this paper we will develop one of such applications and investigate the effect of resetting on the process of valuing the future in economic terms.

In economics, estimating future prices is done through the process of discounting which weights future values relative to the present and the weighting procedure is usually carried out through a discount function. As a simple example, under a constant interest rate r, continuously compounded, a dollar invested today at time $t = 0$ yields e^{rt} dollars at time t, hence, one dollar in any future time t is worth e^{-rt} today. In this case the discount function, $D(t)$, connecting future and present values is given by $D(t) = e^{-rt}$.

The importance of discounting does not reside exclusively in purely financial issues, its greatest consequences have to be found in the intergenerational equity as, for example, long-term environmental planning, a crucial issue in the combat of climate change. Indeed, an environmental problem that costs X to fix at some distant time t would costs $e^{-rt}X$ today, then if the (long-run) interest rate r is substantial this implies a negligible investment today and we would not have to worry about taking immediate action. Letting interest rates be a proxy for economic growth, a different version of the same argument is that the technologies of the future will be so powerful that they will surpass anything we can achieve with present-day technologies. Following this line of reasoning it would be more effective to follow policies that foster economic growth than trying to combat global warming now.

It is thus little wonder that the estimation of long-run discount rate has vast repercussions and it has been the object of intense work and controversy over conflicting estimates between relatively low rates, as the ones advocated by Stern [40], and the higher rates of Nordhaus [41, 42]. The choice of a proper long-run discount rate has enormous repercussions on long-run environmental planning and in latter years a number of empirical results have appeared on this matter [43–51] and the issue is far from being settled. Most recent discussions, specially from those that call for immediate action, argue that climate, intergenerational and financial uncertainties are not properly handled and that more work is necessary when exploring in practical terms the effect of specific interventions to, for instance, carbon prices [52–57].

Let us remark that interest rates are uncertain, particularly in the long run. The assumption that rates are fixed and constant, or even a deterministic function of time, is totally unrealistic and (as it is done in short-time finance [58]) rates $r = r(t)$ are much better described by random functions of time. In this situation, the discount function $D(t)$ linking present and future values is defined as an average over all possible realizations of the rate $r(t)$. The simplest and most common assumption is to consider interest rates as stationary diffusion process and, hence, Markovian and continuous [58]. We have addressed the discount problem using three of the most well-established diffusion models for describing interest rates, the OU model, the Feller model and the log-normal model [59, 60]. We have also performed a rather exhaustive empirical survey on a number of countries which shows that real rates (i.e. nominal rates, usually positive, corrected by inflation) are negative around 25% of the time [46, 50].

However, stationary and continuous diffusion models do not fully describe the rate evolution in a completely satisfactory manner. We have thus generalized the continuous models to include cases where rates occasionally suffer discontinuities in the form of sudden jumps, a model that may be useful for studying the effect of 'catastrophic events' on discounting [61] (such as, for instance, the Covid-19 pandemics). A further generalization of the stationary diffusion models that we are currently working on consists in a non-stationary model in which the normal level (i.e. the stationary mean value) is itself a random process [62].

In the present work we will address another generalization which consists in assuming that, superposed to the normal diffusive behavior, interest rates also suffer random resettings. The motivation for this generalization lies in the fact that central banks occasionally (and randomly) set interest rates to some fixed value. We thus want to elucidate which are the consequences on long-run discount of such fixings. The introduction of resetting events to the discount function analysis further enriches the current debates in environmental economics and climate action that asks for a deeper treatment of the involved uncertainties [57]. The financial interventions modeled as random resettings in the interest rates add an additional layer to climate mitigation set of actions in a financial level [56], in a different level than modifying carbon prices [54, 55].

2. The discount function

We first briefly summarize the main traits of discounting. The reader familiar with the concept may skip this section and go directly to the next section. On the other hand, the reader who wants a more complete information on the subject is referred to [59] and the surveys contained in [60, 63].

We denote by $M = M(t)$ the quantity of wealth at time t. In economics its variation is given by the phenomenological law:

$$dM(t) \propto M(t)dt, \tag{2.1}$$

which is based on the empirical observation that the bigger $M(t)$ is, the greater its variation together with the simpler assumption that such a variation is linear in M. Let us note that the linearity of this law implies that the rate, defined as the relative time variation of wealth:

$$r(t) \equiv \frac{1}{M(t)}\frac{dM(t)}{dt} = \frac{d\ln M(t)}{dt}, \tag{2.2}$$

is independent of wealth. In the simplest case $r(t) = r$ is constant we have the familiar exponential law

$$M(t) = M(t_0)e^{r(t-t_0)},$$

which connects wealth at some initial time t_0, for instance today (we will usually take $t_0 = 0$) to wealth at some future time $t > t_0$.

As we have mentioned in the introduction, discounting relates wealth at different times which can be carried out by a discount function defined by

$$\delta(t) = \frac{M(t_0)}{M(t)}.$$

For constant rates this function is simply given by $\delta(t) = e^{-r(t-t_0)}$. Note that if rates vary with time then the phenomenological law (2.1) reads

$$dM(t) = r(t)M(t)dt$$

and hence

$$\delta(t) = \exp\left(-\int_{t_0}^{t} r(t')dt'\right).$$

However, as we have mentioned above, it is not realistic to represent interest rates by constants or deterministic functions of time, specially in the long run. We thus assume that rates are described by random processes and define the effective discount function as:

$$D(t) = \mathbf{E}\left[\exp\left(-\int_{t_0}^{t} r(t')dt'\right)\right], \tag{2.3}$$

where the average $\mathbf{E}[\cdot]$ is taken over all possible realizations of the random process $r(t)$.

In terms of $D(t)$, the long-run discount rate r_∞ is defined by the limit

$$r_\infty \equiv -\lim_{t\to\infty} \frac{\ln D(t)}{t}. \tag{2.4}$$

Note that if the limit exists the discount function can be asymptotically written in the familiar form

$$D(t) \simeq e^{-r_\infty t}, \qquad (t \to \infty). \tag{2.5}$$

In terms of the auxiliary random process

$$x(t) = \int_{t_0}^{t} r(t')dt' \tag{2.6}$$

the effective discount function (2.3) can be written as $D(t) = \mathbf{E}[e^{-x(t)}]$, so that

$$D(t) = \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} e^{-x} p(x, r, t|x_0, r_0, t_0) dx, \tag{2.7}$$

where $p(x, r, t|x_0, r_0, t_0)$ is the probability density function (PDF) of the bidimensional random process $\mathbf{u}(t) = \big(x(t), r(t)\big)$, $x_0 = x(t_0)$ and $r_0 = r(t_0)$ is the initial value of the rate (note that by the definition (2.6) $x_0 = 0$).

Obtaining $D(t)$ turns out to be rather straightforward using the characteristic function of $\mathbf{u}(t)$ instead of the PDF, the former defined as the Fourier transform of the latter:

$$\tilde{p}(\omega_1, \omega_2, t|x_0, r_0, t_0) = \int_{-\infty}^{\infty} e^{-i\omega_1 x} dx \int_{-\infty}^{\infty} e^{-i\omega_2 r} p(x, r, t|x_0, r_0, t_0) dr. \tag{2.8}$$

Indeed, comparison of (2.7) and (2.8) shows at once that the effective discount function is simply given by

$$D(t) = \tilde{p}\big(\omega_1 = -i, \omega_2 = 0, t\big). \tag{2.9}$$

The representation of discount in terms of the characteristic function is very useful in linear problems and particularly when the joint process $\mathbf{u} = (x, r)$ is Gaussian since in this case we can obtain a closed expression for $D(t)$ in terms of the first two moments of the accumulated rate $x(t)$. In effect, if (x, r) is a bidimensional Gaussian process its characteristic function has the general form (to lighten notation we omit the dependence on the initial rate r_0)

$$\begin{aligned} \tilde{p}(\omega_1, \omega_2, t) = \exp\big\{ & -\sigma_x^2(t)\omega_1^2/2 - \sigma_y^2(t)\omega_2^2/2 - \sigma_{xy}(t)\omega_1\omega_2 \\ & - i m_x(t)\omega_1 - i m_y(t)\omega_2 \big\}, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} \sigma_x^2(t) &= \mathbf{E}\left[x^2(t)\right] - \mathbf{E}\left[x(t)\right]^2, \\ \sigma_y^2(t) &= \mathbf{E}\left[r^2(t)\right] - \mathbf{E}\left[r(t)\right]^2, \\ \sigma_{xy}(t) &= \mathbf{E}\left[x(t)r(t)\right] - \mathbf{E}\left[x(t)\right]\mathbf{E}\left[r(t)\right], \end{aligned}$$

and

$$m_x(t) = \mathbf{E}\left[x(t)\right], \qquad m_y(t) = \mathbf{E}\left[r(t)\right].$$

Note that because of (2.6) these quantities are related to each other. In particular,

$$m_x(t) = \int_0^t m_y(t')dt', \quad \sigma_x^2(t) = 2\int_0^t dt' \int_0^{t'} \sigma_y^2(t'')dt'', \quad \sigma_{xy}(t) = \int_0^t \sigma_y^2(t')dt'.$$

From (2.9) and (2.10) we find

$$D(t) = \exp\left\{-\left[m_x(t) - \sigma_x^2(t)/2\right]\right\} \tag{2.11}$$

showing that in the Gaussian case discount depends solely on the first two moments of the accumulated rate $x(t)$. Note that the long-run rate defined in (2.4) is now given by

$$r_\infty = \lim_{t\to\infty} \frac{1}{t}\left[m_x(t) - \sigma_x^2(t)/2\right], \tag{2.12}$$

as long as the limit exists.

3. Discounting under resetting

Let us now assume that in the evolution of interest rates there are resetting events which instantaneously bring rates to some fixed value r^*. Resettings occur at random instants of time and denote by $\psi(\tau)$ the PDF of the time interval τ between two consecutive events which are supposed to be identically distributed. In what follows we will assume that resetting events are Poissonian and hence

$$\psi(\tau) = \lambda e^{-\lambda\tau},$$

where $\lambda > 0$ is the rate (or frequency) of resetting, so that λ^{-1} is the average time interval between two consecutive resettings. The probability that the time interval between resettings is greater than τ is

$$\Psi(\tau) = \int_\tau^\infty \psi(\tau')d\tau' = e^{-\lambda\tau}.$$

Consider the bidimensional discount process $\mathbf{u} = (x, r)$ and let us denote by $p_0(\mathbf{u}, t|\mathbf{u}_0, t_0)$ the propagator of $\mathbf{u}$ without resettings:

$$p_0(\mathbf{u}, t|\mathbf{u}_0)dxdr = \text{Prob}\{x < x(t) \leqslant x + dx, r < r(t) \leqslant r + dr|\mathbf{u}(t_0) = \mathbf{u}_0; \\ \text{no resettings}\}.$$

We also denote by $p(\mathbf{u}, t|\mathbf{u}_0)$ the propagator under resettings and the central objective is obtaining p knowing p_0.

Let us first remark that resettings are assumed only on the rate $r(t)$ but not on $x(t)$. Therefore, if at some instant of time t' the bidimensional process has reached the value $\mathbf{u}' = (x', r')$ where $x' = x(t' - 0)$ and $r' = r(t' - 0)$ and a resetting event occurs, then right after t' the process has the values (x', r^*) where $x' = x(t' + 0)$ (x is continuous) and $r^* = r(t' + 0)$.

For Poissonian resettings the propagator p of the entire process including resetting events obeys the integral equation

$$\begin{aligned} p(x, r, t|x_0, r_0, t_0) &= e^{-\lambda(t-t_0)}p_0(x, r, t|x_0, r_0, t_0) \\ &\quad + \lambda \int_{t_0}^t e^{-\lambda(t-t')}dt' \int_{-\infty}^\infty dx' \int_{-\infty}^\infty p_0(x, r, t|x', r^*, t') \\ &\quad \times p(x', r', t'|x_0, r_0, t_0)dr', \end{aligned} \tag{3.1}$$

where the first term on the right hand side accounts for the evolution with no resetting events between t_0 and t, while the second term accounts for the probability that the last resetting event occurred at t' and no reset after t'.

We next assume that the problem is homogenous in time and x, so that

$$\begin{aligned} p(x, r, t|x_0, r_0, t_0) &= p(x - x_0, r, t - t_0|r_0), \\ p_0(x, r, t|x', r^*, t') &= p_0(x - x', r, t - t'|r^*), \end{aligned}$$

which allows us to take $t_0 = 0$ without loss of generality ($x_0 = 0$ by the definition (2.6)). Thus (3.1) reads

$$\begin{aligned} p(x, r, t|r_0) &= e^{-\lambda t}p_0(x, r, t|r_0) + \lambda \int_0^t e^{-\lambda(t-t')}dt' \int_{-\infty}^\infty dx' \\ &\quad \times \int_{-\infty}^\infty p_0(x - x', r, t - t'|r^*)p(x', r', t'|r_0)dr'. \end{aligned} \tag{3.2}$$

In the appendix A we obtain the general solution of the integral equation (3.2) for the Laplace transform of the characteristic function defined as (cf equation (2.8))

$$\hat{\tilde{p}}(\omega_1,\omega_2,s|r_0) = \int_0^\infty e^{-st}dt \int_{-\infty}^\infty e^{-i\omega_1 x}dx \int_{-\infty}^\infty e^{-i\omega_2 r}p(x,r,t|r_0)dr. \tag{3.3}$$

This general solution is given by equation (A.3) of appendix A:

$$\begin{aligned}\hat{\tilde{p}}(\omega_1,\omega_2,s|r_0) = \Big\{&\hat{\tilde{p}}_0(\omega_1,\omega_2,\lambda+s|r_0) - \lambda\Big[\hat{\tilde{p}}_0(\omega_1,\omega_2,\lambda+s|r_0)\\ &\times \hat{\tilde{p}}_0(\omega_1,0,\lambda+s|r^*) - \hat{\tilde{p}}_0(\omega_1,\omega_2,\lambda+s|r^*)\hat{\tilde{p}}_0(\omega_1,0,\lambda+s|r_0)\Big]\Big\}\\ &\times \Big[1-\lambda\hat{\tilde{p}}_0(\omega_1,0,\lambda+s|r^*)\Big]^{-1},\end{aligned} \tag{3.4}$$

and it allows us to attain the discount function (in Laplace space) and the long-run interest rate as we will see next.

3.1. The discount function

Let us denote by $D(t)$ the discount function when resettings are present and by $D_0(t)$ the discount without resetting events in the rate. In terms of the corresponding characteristic functions $\tilde{p}(\omega_1,\omega_2,t)$ and $\tilde{p}_0(\omega_1,\omega_2,t)$ we have (cf equation (2.9))

$$D(t) = \tilde{p}\big(\omega_1 = -i, \omega_2 = 0, t\big), \qquad D_0(t) = \tilde{p}_0\big(\omega_1 = -i, \omega_2 = 0, t\big).$$

Setting $\omega_1 = -i$ and $\omega_2 = 0$ in equation (3.4) we have

$$\hat{D}(s|r_0) = \frac{\hat{D}_0(\lambda+s|r_0)}{1-\lambda\hat{D}_0(\lambda+s|r^*)}, \tag{3.5}$$

where $\hat{D}(s|r_0)$ is the Laplace transform

$$\hat{D}(s|r_0) = \int_0^\infty e^{-st}D(t|r_0)dt, \tag{3.6}$$

and similarly for $\hat{D}_0(s|r_0)$.

Equation (3.5) allows us to know, via Laplace transform, the discount function with resettings in the interest rates in terms of the discount function without resettings. Let us note that this expression is completely general since it does not imply the assumption of any particular model for the evolution of interest rates (other than Poissonian resettings and time homogeneity) and it constitutes one of the main results of the present work.

3.2. Long-run interest rate

We next perform an asymptotic analysis on discounting as $t \to \infty$. To this end we will use the asymptotic Tauberian theorems which relate the long time analysis of a function of time with the short s behavior of its Laplace transform. That is [64, 65]

$$\hat{f}(s) \sim \hat{g}(s) \quad (s \to 0) \quad \Longleftrightarrow \quad f(t) \sim g(t) \quad (t \to \infty).$$

Let us thus obtain the small s behavior of $\hat{D}(s|r_0)$. Expanding (3.5) up to first order in s and rearranging terms we have

$$\hat{D}(s|r_0) = \frac{\hat{D}_0(\lambda|r_0)}{1-\lambda\hat{D}_0(\lambda|r^*)}\left\{1 + \left[\frac{\hat{D}_0'(\lambda|r_0)}{\hat{D}_0(\lambda|r_0)} + \frac{\lambda\hat{D}_0'(\lambda|r^*)}{1-\lambda\hat{D}_0(\lambda|r^*)}\right]s\right\} + O(s^2),$$

which within the same degree of approximation can be written as

$$\hat{D}(s|r_0) = \frac{\hat{D}_0(\lambda|r_0)}{1-\lambda\hat{D}_0(\lambda|r^*)}\left\{\frac{1}{1-\left[\frac{\hat{D}_0'(\lambda|r_0)}{\hat{D}_0(\lambda|r_0)}+\frac{\lambda\hat{D}_0'(\lambda|r^*)}{1-\lambda\hat{D}_0(\lambda|r^*)}\right]s}\right\} + O(s^2), \tag{3.7}$$

where

$$\hat{D}_0'(\lambda|r_0) = -\int_0^\infty te^{-\lambda t}D_0(t|r_0)dt < 0.$$

The Laplace inversion of (3.7) is straightforward and due to Tauberian theorems it provides the asymptotic discount function:

$$D(t|r_0) \simeq \frac{\hat{D}_0(\lambda|r_0)r_\infty}{1-\lambda\hat{D}_0(\lambda|r^*)}e^{-r_\infty t}, \qquad (t\to\infty), \tag{3.8}$$

where

$$r_\infty \simeq \frac{-1}{\frac{\hat{D}_0'(\lambda|r_0)}{\hat{D}_0(\lambda|r_0)}+\frac{\lambda\hat{D}_0'(\lambda|r^*)}{1-\lambda\hat{D}_0(\lambda|r^*)}} \tag{3.9}$$

is the long-run rate (cf equations (2.4) and (2.5)) which we can also write in the form

$$r_\infty \simeq \frac{-\hat{D}_0(\lambda|r_0)[1-\lambda\hat{D}_0(\lambda|r^*)]}{\hat{D}_0'(\lambda|r_0)+\lambda[\hat{D}_0(\lambda|r_0)\hat{D}_0'(\lambda|r^*)-\hat{D}_0(\lambda|r^*)\hat{D}_0'(\lambda|r_0)]}. \tag{3.10}$$

This general expression—independent of any particular market model—constitutes the second key result of the present work.

4. Diffusion processes

Let us recall that up to now our analysis has been rather general and, aside Poissonian resettings and time homogeneity, no stochastic model for the evolution of interest rates between consecutive resettings has been assumed. The two key results obtained, (3.5) and (3.10), relate discount and long-run rates of the complete process with resets to those of the reset-free process and without any specification on the nature of the underlying interest rate process $r(t)$. As mentioned in section 1, one of the simplest and most common assumption consists in taking rates as Markovian processes with continuous sample paths [58]. That is to say, diffusion process which are solutions to (Itô) stochastic equations of the form

$$dr = f(r)dt + g(r)dW(t), \tag{4.1}$$

where $W(t)$ is the standard Wiener process. In this case the bidimensional process $\mathbf{u}(t) = (x(t), r(t))$ is described by the following pair of stochastic differential equations (cf equations (2.6) and (4.1))

$$\begin{aligned} dx &= rdt, \\ dr &= f(r)dt + g(r)dW(t), \end{aligned} \tag{4.2}$$

with initial conditions $x(0) = 0$ and $r(0) = r_0$. The joint PDF (in the absence of resetting events) obeys the Fokker–Planck equation (FPE) [63, 66]

$$\frac{\partial p_0}{\partial t} = -r\frac{\partial p_0}{\partial x} - \frac{\partial}{\partial r}[f(r)p_0] + \frac{1}{2}\frac{\partial^2}{\partial r^2}[g^2(r)p_0], \tag{4.3}$$

with the initial condition

$$p_0(x,r,0|r_0) = \delta(x)\delta(r-r_0). \tag{4.4}$$

In mathematical finance [58] the standard approach to obtain the discount function $D_0(t|r_0)$ is based on solving the Feynman–Kac equation, a backward partial differential equation which for diffusive rates reads [58, 63]

$$\frac{\partial D_0}{\partial t} = -r_0 D_0 + f(r_0)\frac{\partial D_0}{\partial r_0} + \frac{1}{2}g^2(r_0)\frac{\partial^2 D_0}{\partial r_0^2}, \tag{4.5}$$

with initial condition

$$D_0(0|r_0) = 1. \tag{4.6}$$

However, even in problems where $f(r_0)$ and $g(r_0)$ are linear functions, this equation is rather difficult to solve. In linear problems it turns out to be much easier to obtain discount through the characteristic function of the process $\tilde{p}_0(\omega_1,\omega_2,t|r_0)$ via equation (2.9). In these linear cases the equation for $\tilde{p}$ is analytically solvable and discount is simply obtained by

$$D_0(t|r_0) = \tilde{p}_0(\omega_1 = -i, \omega_2 = 0, t|r_0).$$

In recent works (see for instance [59]) we have obtained exact expressions for the discount function for two different linear models of rates: the OU model and the Feller model.

4.1. The Ornstein-Uhlenbeck model

In the theory of financial interest rates the OU model was proposed in [67] and it is sometimes referred to as the Vasicek model. The model is a diffusion process characterized by linear drift and constant noise intensity

$$f(r) = -\alpha(r-m), \qquad g(r) = k, \tag{4.7}$$

the parameter m, sometimes referred to as 'normal level' is the mean value to which the process reverts, $k>0$ is the amplitude of fluctuations, and $\alpha>0$ is the strength of the reversion to the mean. These parameters have to be estimated from empirical data [46, 50, 51, 60]. Substituting (4.7) into (4.3) yields a linear FPE whose double Fourier transform (cf equation (2.8)) results in a linear and first-order equation for the characteristic function [59]

$$\frac{\partial \tilde{p}_0}{\partial t} = (\omega_1 - \alpha\omega_2)\frac{\partial \tilde{p}_0}{\partial \omega_2} - \left(i\alpha m\omega_2 + \frac{k^2}{2}\omega_2^2\right)\tilde{p}_0, \tag{4.8}$$

with the initial condition

$$\tilde{p}_0(\omega_1,\omega_2,0|r_0) = e^{-i\omega_2 r_0}. \tag{4.9}$$

In [59] we have shown that the solution to this problem leads to a Gaussian characteristic function which, by means of equation (2.9), results in the following expression for the reset-free discount

$$D_0(t|r_0) = \exp\left\{-\mu t - \nu(r_0) + \rho(r_0)e^{-\alpha t} - \kappa e^{-2\alpha t}\right\}, \tag{4.10}$$

where

$$\mu = m - \frac{k^2}{2\alpha^2}, \quad \nu(r_0) = \frac{1}{\alpha}\left(r_0 + \frac{3k^2}{4\alpha^2} - m\right),$$
$$\rho(r_0) = \frac{1}{\alpha}\left(r_0 + \frac{k^2}{\alpha^2} - m\right), \quad \kappa = \frac{k^2}{4\alpha^3}. \tag{4.11}$$

Note that the parameter μ is the long-run rate of the reset-free process $r_\infty^{(0)}$. Indeed, from equations (2.4) and (4.10), we see that

$$r_\infty^{(0)} \equiv -\lim_{t\to\infty} \frac{\ln D_0(t)}{t} = \mu. \tag{4.12}$$

In this case the Laplace transform $\hat{D}_0(s|r_0)$ is given by

$$\hat{D}_0(s|r_0) = \int_0^\infty \exp\left\{-(s+\mu)t - \nu(r_0) + \rho(r_0)e^{-\alpha t} - \kappa e^{-2\alpha t}\right\} dt. \tag{4.13}$$

Solving this integral and substituting the result into (3.5) and (3.10) leads to both the discount function (via Laplace transform) and the long-run interest rate. Unfortunately obtaining an exact expression for the integral (4.13) seems to be out of reach except for two particular cases which correspond to (a) r_0 is such that $\rho(r_0) = 0$,

$$\hat{D}_0(s|r_0) = \frac{1}{2\alpha}\kappa^{-\frac{s+\mu}{2\alpha}} e^{-\kappa}\gamma\left(\frac{s+\mu}{2\alpha}, \kappa\right), \tag{4.14}$$

and (b) $\kappa = 0$ (the deterministic case)

$$\hat{D}_0(s|r_0) = \frac{1}{\alpha}\left(\frac{m-r_0}{\alpha}\right)^{-\frac{s+\mu}{\alpha}} e^{\frac{m-r_0}{\alpha}}\gamma\left(\frac{s+\mu}{\alpha}, \frac{m-r_0}{\alpha}\right), \tag{4.15}$$

where $\gamma(s,x)$ is the lower incomplete Gamma function,

$$\gamma(s,x) = \int_0^x u^{s-1}e^{-u}du,$$

valid when $s \geqslant 0$ and $x \geqslant 0$, with analytic continuation to $x < 0$. For the general case, however, we have to resort to approximations as we will see next.

4.2. Analysis of the Ornstein-Uhlenbeck model under resetting

Recall that as $t \to \infty$ the reset discount function tends to (cf equation (2.5))

$$D(t|r_0) \sim e^{-r_\infty t}, \tag{4.16}$$

where r_∞ is given by (3.10). For the OU model the expressions for $\hat{D}_0(\lambda|r_0)$ and $\hat{D}_0'(\lambda|r_0)$ appearing in (3.10) are obtained from (4.13) after setting $s = \lambda$,

$$\hat{D}_0(\lambda|r_0) = e^{-\nu(r_0)}\int_0^\infty \exp\left\{-(\lambda+\mu)z + \rho(r_0)e^{-\alpha z} - \kappa e^{-2\alpha z}\right\} dz, \tag{4.17}$$

and similarly for $\hat{D}_0'(\lambda|r_0)$.

In order to get approximate expressions for the long-run rate r_∞ we need to distinguish between the cases where λ is small or large. We show in appendix B that for small values of λ, the expression of the long-run rate reads

$$r_\infty \simeq \mu + \lambda\left(1 - e^{-\nu^*}\right) \qquad (\lambda \to 0), \tag{4.18}$$

with $\nu^* = \nu(r^*)$ (cf equation (4.11)) while for large values of λ one has

$$r_\infty \simeq \frac{[\lambda^2 + \lambda(r_0 + r^*) + r_0 r^*]r^*}{\lambda^2 + \lambda(r_0 + r^*) + r^{*2}} \qquad (\lambda \to \infty). \tag{4.19}$$

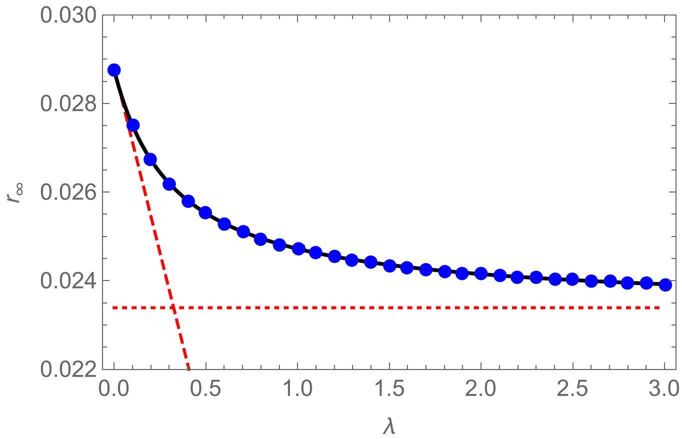

Figure 1. Long-run discount rate r_∞ as a function of the intensity λ of the resetting mechanism for the OU case. Blue dots were obtained by the numerical inversion of $\hat{D}(s|r_0)$, with $t = 300$ years. The continuous black line represents formula (3.10), which in this case can be directly evaluated from $\hat{D}_0(s|r_0)$ in (4.14). The red dashed line and the red dotted line correspond to the limiting approximate expressions (4.18) and (4.20), respectively. The value of the rest of parameters are $m = 0.0342$, $\alpha = 0.1635$, $k = 0.017$ and $r_0 = r^* \approx 0.0234$, what renders $\rho(r_0) = \rho(r^*) = 0$ and $r_\infty^{(0)} \approx 0.0288$. These values have annual units.

Observe that the long-run rate for small frequencies is a linear function of λ. Let us also note that the long-run rate tends to the reset-free rate when $\lambda \to 0$, i.e.

$$\lim_{\lambda\to 0} r_\infty = \mu = r_\infty^{(0)},$$

as otherwise expected. On the other hand, as $\lambda \to \infty$ one has $r_\infty \to r^*$, that is

$$\lim_{\lambda\to\infty} r_\infty = r^*, \tag{4.20}$$

which confirms intuition because if resetting events are infinitely frequent, the interest rate becomes fixed to the resetting value. We here also want to mention that [31] has performed an analysis to obtain a value that can be straightforwardly linked to r_∞. The authors obtain specific results for large or small fluctuations assuming the OU process. Their motivation is however not to study the discount and the interest rates but physical magnitudes such as energy or current.

In figure 1 we can check the goodness of the different approximations we have done with the aid of equation (4.14) for $\hat{D}_0(s|r_0)$. Note that, by setting

$$r_0 = r^* = m - \frac{k^2}{\alpha^2},$$

one gets $\rho(r_0) = \rho(r^*) = 0$, and the exact expression for the composite process $\hat{D}(s|r_0)$ can be recovered from $\hat{D}_0(s|r_0)$ and $\hat{D}_0(s|r^*)$ thanks to formula (3.5). Therefore, by numerical inversion of the Laplace transform of $\hat{D}(s|r_0)$, one can evaluate r_∞ without performing any analytical approximation. As it can be observed, the results found are in excellent agreement with those obtained with the Tauberian expression (3.10), as well as with the small- and large-λ approximations, equations (4.18) and (4.20). The values of the parameters, m, α and k, were chosen from the data reported for UK in table 3 of [50].

As mentioned in section 1 the magnitude of the long-run rate is very relevant for the long-run economic planning, specially in environmental problems. Therefore, a key question would be to ask whether resetting events increase or decrease the long-run rate. From (4.18) we see that

$$\frac{\partial r_\infty}{\partial \lambda} \simeq 1 - e^{-\nu^*} \qquad (\lambda \to 0).$$

That is, for small values of λ we have from (4.18) that r_∞ will be an increasing function of the resetting frequency λ if $1 - e^{-\nu^*} > 0$, that is, if $\nu^* > 0$ which implies a high value for r^* (cf equation (4.11)) $r^* > m - 3k^2/4\alpha^2$ or, in terms of $r_\infty^{(0)} = \mu$,

$$r^* > r_\infty^{(0)} - \frac{k^2}{4\alpha^2}.$$

On the other hand, as we can check immediately, the long-run rate is a decreasing function of the resetting frequency if

$$r^* < r_\infty^{(0)} - \frac{k^2}{4\alpha^2}.$$

Let us discuss now the behavior of the approximate expression for large λ. From (4.19) we have

$$r_\infty \simeq \left[1 + (r_0 - r^*) r^* \lambda^{-2} + O(\lambda^{-3})\right] r^* \qquad (\lambda \to \infty), \tag{4.21}$$

then

$$\frac{\partial r_\infty}{\partial \lambda^{-1}} \simeq 2(r_0 - r^*) r^{*2} \lambda^{-1} \to 0 \qquad (\lambda \to \infty), \tag{4.22}$$

and the first correction to the limiting value for $\lambda \to \infty$ is zero.

Therefore, in the case

$$r_\infty^{(0)} - \frac{k^2}{4\alpha^2} < r^* < r_\infty^{(0)}, \tag{4.23}$$

one has that r_∞ is equal to $r_\infty^{(0)}$ for $\lambda \to 0$, with an initial upward slope and an asymptotic limit r^* which is smaller than the starting point: as r_∞ is a continuous function of λ, it must *attain a maximum value at some critical frequency.* Note that when the situation is reversed, the condition for the long-run discount rate to show a minimum would be

$$r_\infty^{(0)} < r^* < r_\infty^{(0)} - \frac{k^2}{4\alpha^2},$$

which is not feasible.

This means that in this last case, and other possible scenarios, the previous analysis is not conclusive: we must not forget that the above reasoning is based on approximate expressions for r_∞. The main result is therefore that the monotonicity of r_∞ shown in figure 1 is not a general trait. In this example, we set $r_\infty^{(0)} \approx 0.0288$ and $r_0 = r^* \approx 0.0234$. If we increase the value of r^* up to $r^* = 0.0285$, and keep the rest of parameters fixed, condition (4.23) is satisfied because $r^* < r_\infty^{(0)}$ and

$$r_\infty^{(0)} - \frac{k^2}{4\alpha^2} \approx 0.0261 < r^*,$$

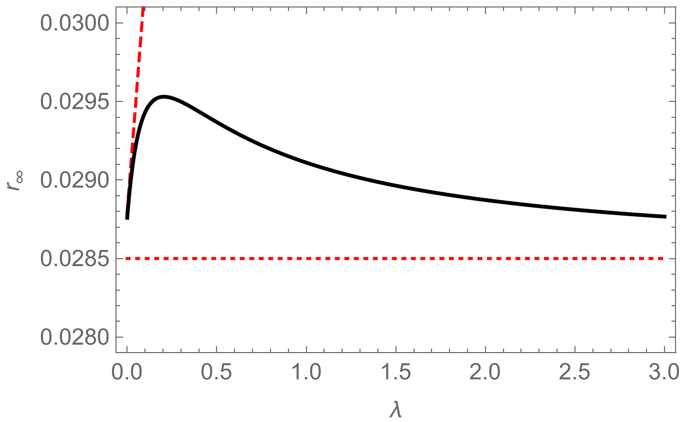

Figure 2. Non-monotonous behavior of r_∞ as a function of the intensity λ for the OU case. The continuous black line is obtained from the numerical computation of (3.10) for the OU model, equation (4.13). The red dashed line and the red dotted line correspond to the limiting approximate expressions (4.18) and (4.20), respectively. The value of the parameters are $m = 0.0342$, $\alpha = 0.1635$, $k = 0.017$, $r_0 = 0.0234$ and $r^* = 0.0285$. These values have annual units.

and consequently r_∞ exhibits a maximum, as seen in figure 2. In plotting this figure we have evaluated numerically the integral expression of equation (4.13) and its derivative (for r_0 and r^*) and used the Tauberian formula (3.10).

4.3. The Feller model

Let us next address the discount problem when the stochastic evolution between resettings is given by the Feller process which is an alternative linear diffusion process (known in finance as the Cox–Ingersoll–Ross model of interest rates [68]). The drift and noise intensity of such a process are [69]

$$f(r) = -\alpha(r - m), \qquad g(r) = k\sqrt{r}. \tag{4.24}$$

In this case the bidimensional process (x, r) obeys the stochastic system

$$dx = rdt,$$
$$dr = -\alpha(r - m)dt + k\sqrt{r}dW(t),$$

with initial conditions $x(0) = 0$ and $r(0) = r_0$ and the joint PDF in the absence of resettings, $p_0(x, y, t|r_0)$, obeys the linear Fokker–Planck equation

$$\frac{\partial p_0}{\partial t} = -r\frac{\partial p_0}{\partial x} + \alpha\frac{\partial}{\partial r}[(r - m)p_0] + \frac{k^2}{2}\frac{\partial^2}{\partial r^2}[rp_0], \tag{4.25}$$

with the initial condition

$$p_0(x, r, 0|r_0) = \delta(x)\delta(r - r_0). \tag{4.26}$$

Similarly to the OU model, the linear drift results in a restoring force which, in the absence of noise, makes the process decay towards the normal level m, while the state-dependent noise intensity $k\sqrt{r}$ magnifies the effect of noise for large values of r but as r goes to zero this effect vanishes. Thus, as the process approaches the origin, the drift drags r towards m. Hence, since $m > 0$, if the process starts at some positive value $r_0 > 0$ it cannot attain negative values, with the overall result that the Feller process remains always positive. It is this characteristic of the Feller process that has made the model a convenient tool for pricing bonds which are hardly negative [58]. In previous works [59, 70] we have reviewed rather thoroughly the properties of the Feller process and refer the reader to these works for more information. Contrary to the OU process, the Feller process is not Gaussian and the stationary PDF as $t \to \infty$ is the Gamma distribution [59]

$$p_0^{(st)}(r) = \frac{(2\alpha/k^2)^\theta}{\Gamma(\theta)} r^{\theta-1} e^{-(2\alpha/k^2)r}, \tag{4.27}$$

where

$$\theta = \frac{2\alpha m}{k^2} \tag{4.28}$$

is a positive and dimensionless constant which combines all the parameters of the model.

The double Fourier transform of (4.25) and (4.26) ends up in the simpler problem

$$\frac{\partial \tilde{p}_0}{\partial t} = \left(\omega_1 - \alpha\omega_2 - i\frac{k^2}{2}\omega_2^2\right)\frac{\partial \tilde{p}_0}{\partial \tilde{\omega}_2} - i\alpha m\omega_2\tilde{p}_0, \tag{4.29}$$

with

$$\tilde{p}_0(\omega_1, \omega_2, 0|r_0) = e^{-i\omega_2 r_0} \tag{4.30}$$

which is a partial differential equation of first order whose solution can be obtained by the method of characteristics and we refer the interested reader to our previous work [59] for detailed information. Once we know the solution $\tilde{p}_0(\omega_1, \omega_2, t|r_0)$, the discount function is then obtained through (2.9) with the result [59]

$$D_0(t) = \left[\frac{2\gamma e^{-(\gamma-\alpha)t/2}}{(\gamma+\alpha)+(\gamma-\alpha)e^{-\gamma t}}\right]^\theta \exp\left\{-\frac{2(1-e^{-\gamma t})r_0}{(\gamma+\alpha)+(\gamma-\alpha)e^{-\gamma t}}\right\}, \tag{4.31}$$

where θ is defined in (4.28) and

$$\gamma = \sqrt{\alpha^2 + 2k^2}. \tag{4.32}$$

The reset-free long-run rate is given by the limit (cf equations (2.4) and (4.31))

$$r_\infty^{(0)} = -\lim_{t\to\infty} \frac{\ln D_0(t)}{t} = \frac{1}{2}(\gamma - \alpha)\theta, \tag{4.33}$$

which, substituting for (4.28) and using (4.32), can be written as

$$r_\infty^{(0)} = \frac{2\alpha m}{\gamma + \alpha}. \tag{4.34}$$

4.4. Analysis of the Feller model under resetting

As we have seen, when resetting events are present the Laplace transform of the discount function, $\hat{D}(s|r_0)$, is given by (3.5) in terms of the Laplace transform of the reset-free discount $\hat{D}_0(s|r_0)$. As $t \to \infty$ we have asymptotic expression (cf equation (2.4))

$$D(t|r_0) \sim e^{-r_\infty t},$$

with r_∞ shown in (3.10). In the Feller case, $D_0(t|r_0)$ is given by (4.31) and then

$$\hat{D}_0(\lambda|r_0) = (2\gamma)^\theta \int_0^\infty \frac{dz}{[(\gamma+\alpha)+(\gamma-\alpha)e^{-\gamma z}]^\theta} \times \exp\left\{-\left(\lambda + r_\infty^{(0)}\right)z - \frac{2(1-e^{-\gamma z})r_0}{(\gamma+\alpha)+(\gamma-\alpha)e^{-\gamma z}}\right\}. \tag{4.35}$$

Getting an analytical expression for this quantity seems to be even more difficult than for the OU case discussed above. We will thus obtain approximate formulas which allow us to get a more convenient expression for the long-run rate defined in (3.10).

We proceed similarly as in the OU case and leave the details for the study of small and large resetting frequencies to appendix C. There, we derive the following approximate expression for r_∞:

$$r_\infty \simeq r_\infty^{(0)} + \lambda\left[1 - \left(\frac{2\gamma}{\gamma+\alpha}\right)^\theta e^{-2r^*/(\gamma+\alpha)}\right], \qquad (\lambda \to 0), \tag{4.36}$$

showing that, analogously to the OU model, the long-run rate is a linear function of the resetting frequency for λ small. Note that $r_\infty \to r_\infty^{(0)}$ when $\lambda \to 0$, as otherwise expected. For large values of λ one has, in turn, that the long-run rate will also be given by (4.19):

$$r_\infty \simeq \frac{\left[\lambda^2 + \lambda(r_0 + r^*) + r_0 r^*\right] r^*}{\lambda^2 + \lambda(r_0 + r^*) + r^{*2}} \to r^*, \qquad (\lambda \to \infty).$$

Likewise the OU case, the long-run rate can exhibit a maximum at some critical frequency. Taking the derivative with respect to λ in equation (4.36) we write

$$\frac{\partial r_\infty}{\partial \lambda} \simeq 1 - \left(\frac{2\gamma}{\gamma+\alpha}\right)^\theta e^{-2r^*/(\gamma+\alpha)}, \qquad (\lambda \to 0).$$

Thus, r_∞ is an increasing function of λ for small frequencies if

$$\left(\frac{2\gamma}{\gamma+\alpha}\right)^\theta e^{-2r^*/(\gamma+\alpha)} < 1 \qquad \Rightarrow \qquad \frac{2r^*}{\gamma+\alpha} > \theta \ln\left(\frac{2\gamma}{\gamma+\alpha}\right).$$

That is, for

$$r^* > \frac{1}{2}\theta(\gamma+\alpha)\ln\left(\frac{2\gamma}{\gamma+\alpha}\right) = r_\infty^{(0)} \cdot \frac{\gamma+\alpha}{\gamma-\alpha}\ln\left(\frac{2\gamma}{\gamma+\alpha}\right), \tag{4.37}$$

the long-run rate r_∞ is an increasing function of λ, otherwise r_∞ decreases with λ. Moreover, since $u \geqslant \ln(1+u)$, the right-hand-side of equation (4.37) is never larger than $r_\infty^{(0)}$, indeed

$$\frac{\gamma+\alpha}{\gamma-\alpha}\ln\left(\frac{2\gamma}{\gamma+\alpha}\right) = \frac{\gamma+\alpha}{\gamma-\alpha}\ln\left(1 + \frac{\gamma-\alpha}{\gamma+\alpha}\right) \leqslant 1,$$

which implies that when the sufficient condition

$$r_\infty^{(0)} \cdot \frac{\gamma+\alpha}{\gamma-\alpha}\ln\left(\frac{2\gamma}{\gamma+\alpha}\right) < r^* < r_\infty^{(0)} \tag{4.38}$$

is satisfied, the long-run rate will present a maximum, being undecidable the presence of an extreme in the most general situation.

Consider for instance the example in figure 3: in this case, the values of the parameters, $m = 0.0342$, $\alpha = 0.1635$, $k = 0.1$ and $r^* = 0.03$, have been chosen in such a way that only the

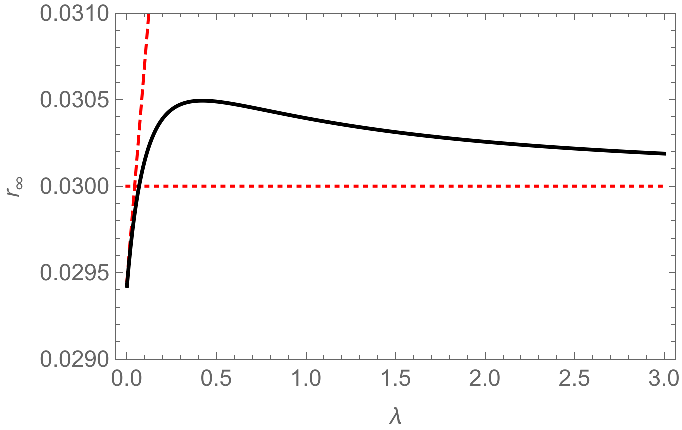

Figure 3. Non-monotonous behavior of r_∞ as a function of the intensity λ for the Feller case. The continuous black line is obtained from the numerical computation of (3.10) for the Feller model, equation (4.35). The red dashed line corresponds to the approximate expression (4.36) whereas the red dotted line marks the level $r^* = 0.03$. The value of the rest of parameters are $m = 0.0342$, $\alpha = 0.1635$, $k = 0.1$ and $r_0 = 0.025$, what implies that $r_\infty^{(0)} \approx 0.02946$. These values have annual units.

first inequality in (4.38) holds, i.e. r_∞ will grow with λ initially, but $r^* > r_\infty^{(0)} \approx 0.2946$. Even thus, the long-run rate shows a maximum.

5. Concluding remarks

In this paper we have studied the effect of resettings on the evolution of interest rates and specially on long-run discounting, a crucial issue in any environmental planning. We have assumed that interest rates evolve following some time-homogeneous stochastic process but that at random instants of times (which we have assumed to be Poissonian) resettings to a fixed value of the rate occur. As we have remarked in section 1 this combination of stochastic evolution plus resettings can be a useful model for interest rates since to the usual random evolution due to market forces there may be superposed, at random times, the recurrence to some fixed value by central banks.

For Poissonian resetting events we have obtained the expression of the discount function in terms of the discount function of the reset-free process (in the Laplace space, equation (3.5)). Moreover, and within this general framework, we have also obtained the real time expression for the discount function as $t \to \infty$, equation (3.8), together with the exact expression for the long-run rate, equation (3.10). All these expressions are valid regardless the kind of random evolution of interest rates between resettings as long as time homogeneity holds.

We have also obtained more explicit expressions, specially for the long-run discount rate r_∞, when interest rates between consecutive resetting are described by diffusion processes. We have studied in detail two widespread models, the OU process and the Feller process, being the former more convenient for real rates (which can be positive and negative) while the latter is more adapted to the modeling of nominal rates (mostly positive). For both models the long-run rate r_∞ may present a non-monotonous behavior in terms of the resetting frequency, or intensity, λ (which, is the inverse of the mean time between consecutive resettings), depending on the value of the reset rate r^*.

Discussions in environmental economics are very often grounded on a proper calibration of the long-run rate r_∞ and the lower the rate, the more intense is the call for immediate action to mitigate climate change. As shown in figure 1, one may initially guess that the higher the resetting frequency is the easier we can reach a pre-established reset rate r^* for the long-run rate r_∞. Therefore, one may hypothetically imagine that intervening the interest rates could become an instrument in policies to favor climate action and thus mitigate climate change. However, the frequency λ of interventions to favor lower r_∞ rates must be handled with care. If λ is too small (no more than one intervention every 3–4 years in figures 2 and 3), the intervention can lead to just the opposite effect, that is, an increase of r_∞. The lesson we can learn is that effective policies to favor climate action with interest rates should be resolute and frequent enough in time. The frequency of the interventions is critical to actually observe the desired consequences in r_∞.

One might think that we have raised here only an hypothetical scenario within environmental planning from an economic perspective. However, there are many situations where resetting is applied to interest rates. Indeed resetting can be seen as a debt cancellation or debt clemency and has been present in a wide variety of cultures, ancient cultures included. In old Mesopotamia and old Egypt, debt clemency was frequently applied by their rulers to bring justice to the oppressed. These sort of considerations are actually permeating into current climate action discourse being a fundamental part of the United Nations 2030 Agenda for Sustainable Development. Climate emergency is expected to quickly increase in the coming years and it will impact differently among countries and among social groups [71].

We finally remark that the combination of stochastic evolution plus resettings can be a useful model for interest rates because, apart to the usual random evolution of rates due to market forces, there may be superposed, at random times, the return to some value imposed by central banks. We have obtained discount for a fixed reset value r^* of the rate, obviously a more realistic approach would be to assume that r^* is also a random variable which can be, for instance, uniformly distributed over some finite interval. This generalization is under consideration.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Acknowledgments

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Appendix A. Solution to the integral equation (3.2)

Taking the Laplace transform in time and the double Fourier transform in x and r (cf equation (3.3)) of equation (3.2), using the property:

$$\mathcal{L}\{e^{-\lambda t}f(t), s\} = \hat{f}(\lambda + s),$$

and the convolution theorem for the Fourier transform in x, we eventually obtain

$$\hat{\tilde{p}}(\omega_1, \omega_2, s|r_0) = \hat{\tilde{p}}_0(\omega_1, \omega_2, \lambda + s|r_0) + \lambda\hat{\tilde{p}}_0(\omega_1, \omega_2, \lambda + s|r^*) \times \int_{-\infty}^{\infty} \hat{p}(\omega_1, r', s|r_0)dr',$$

but

$$\int_{-\infty}^{\infty} \hat{p}(\omega_1, r', s|r_0)dr' = \hat{\tilde{p}}(\omega_1, \omega_2 = 0, s|r_0),$$

hence

$$\hat{\tilde{p}}(\omega_1, \omega_2, s|r_0) = \hat{\tilde{p}}_0(\omega_1, \omega_2, \lambda + s|r_0) + \lambda\hat{\tilde{p}}_0(\omega_1, \omega_2, \lambda + s|r^*)\hat{\tilde{p}}(\omega_1, 0, s|r_0). \tag{A.1}$$

Setting $\omega_2 = 0$ in this expression we get

$$\hat{\tilde{p}}(\omega_1, 0, s|r_0) = \frac{\hat{\tilde{p}}_0(\omega_1, 0, \lambda + s|r_0)}{1 - \lambda\hat{\tilde{p}}_0(\omega_1, 0, \lambda + s|r^*)}, \tag{A.2}$$

and substituting back into equation (A.1) we finally achieve

$$\hat{\tilde{p}}(\omega_1, \omega_2, s|r_0) = \Big\{\hat{\tilde{p}}_0(\omega_1, \omega_2, \lambda + s|r_0) - \lambda\Big[\hat{\tilde{p}}_0(\omega_1, \omega_2, \lambda + s|r_0) \times \hat{\tilde{p}}_0(\omega_1, 0, \lambda + s|r^*) - \hat{\tilde{p}}_0(\omega_1, \omega_2, \lambda + s|r^*)\hat{\tilde{p}}_0(\omega_1, 0, \lambda + s|r_0)\Big]\Big\} \times \Big[1 - \lambda\hat{\tilde{p}}_0(\omega_1, 0, \lambda + s|r^*)\Big]^{-1}, \tag{A.3}$$

which is the general solution to the integral equation (3.2) and provides the characteristic function (in Laplace space) of the complete bidimensional process (x, r) in terms of the reset-free distribution. Note that when the resetting interest rate coincides with the initial value, $r^* = r_0$, the general solution (A.3) simplifies to

$$\hat{\tilde{p}}(\omega_1, \omega_2, s|r_0) = \frac{\hat{\tilde{p}}_0(\omega_1, \omega_2, \lambda + s|r_0)}{1 - \lambda\hat{\tilde{p}}_0(\omega_1, 0, \lambda + s|r_0)}. \tag{A.4}$$

Observe how the denominator in this last expression, and by extension in equation (A.2), is always non-zero for $s > 0$ and $\omega_1 \in \mathbb{R}$, since

$$\begin{aligned}\left|\hat{\tilde{p}}_0(\omega_1, 0, \lambda + s|r_0)\right| &= \left|\int_{-\infty}^{\infty} dx e^{-i\omega_1 x}\int_{-\infty}^{\infty} \hat{p}_0(x, r, \lambda + s|r_0)dr\right| \\ &\leqslant \int_{-\infty}^{\infty} dx \left|e^{-i\omega_1 x}\int_{-\infty}^{\infty} \hat{p}_0(x, r, \lambda + s|r_0)dr\right| \\ &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \hat{p}_0(x, r, \lambda + s|r_0)dr = \frac{1}{\lambda + s}.\end{aligned} \tag{A.5}$$

Appendix B. Approximate expressions in section 4.2

The long-run rate r_∞ can be written, cf equation (3.10), in the form:

$$r_\infty \simeq \frac{-\hat{D}_0(\lambda|r_0)[1 - \lambda\hat{D}_0(\lambda|r^*)]}{\hat{D}_0'(\lambda|r_0) + \lambda[\hat{D}_0(\lambda|r_0)\hat{D}_0'(\lambda|r^*) - \hat{D}_0(\lambda|r^*)\hat{D}_0'(\lambda|r_0)]}. \tag{B.1}$$

For the OU model the expressions for $\hat{D}_0(\lambda|r_0)$ and $\hat{D}_0'(\lambda|r_0)$ appearing in (B.1) are obtained from (4.13) after setting $s = \lambda$,

$$\hat{D}_0(\lambda|r_0) = e^{-\nu(r_0)} \int_0^\infty \exp\left\{-(\lambda+\mu)z + \rho(r_0)e^{-\alpha z} - \kappa e^{-2\alpha z}\right\} dz, \tag{B.2}$$

and similarly for $\hat{D}_0'(\lambda|r_0)$. In order to get approximate expressions for the long-run rate we need to distinguish the cases in which (a) λ is small and (b) λ is large.

(a) Suppose that λ is small (in fact we should assume that $\lambda + \mu$ is small or at least not too large). We incidentally note that since λ^{-1} is the average time between consecutive resettings and $\mu = r_\infty^{(0)}$, we see that this assumption implies that resettings are not very frequent and the reset-free long-run rate is not too large. In such a case the major contribution to the integral (B.2) comes from $z \to \infty$ and the exponential terms $e^{-\alpha z}$ and $e^{-2\alpha z}$ become negligible very quickly. Under these circumstances (B.2) is approximated by

$$\hat{D}_0(\lambda|r_0) \simeq e^{-\nu(r_0)} \int_0^\infty e^{-(\lambda+\mu)z} dz = \frac{e^{-\nu(r_0)}}{\lambda+\mu}$$

and

$$\hat{D}_0'(\lambda|r_0) \simeq -\frac{e^{-\nu(r_0)}}{(\lambda+\mu)^2}.$$

Substituting these expressions into (B.1) provides and approximate expression for the long-run rate which after simplifying reads

$$r_\infty \simeq \mu + \lambda\left(1 - e^{-\nu^*}\right), \qquad (\lambda \to 0), \tag{B.3}$$

which is equation (4.18).

(b) For large values of λ, the main contribution to the integral in (B.2) comes now from $z \to 0$. Thus, $e^{-\alpha z} \simeq 1 - \alpha z$ and $e^{-2\alpha z} \simeq 1 - 2\alpha z$ and (B.2) is approximated by

$$\hat{D}_0(\lambda|r_0) \simeq e^{-\nu(r_0)+\rho(r_0)-\kappa} \int_0^\infty \exp\{-[\lambda + \mu + \alpha\rho(r_0) - 2\alpha\kappa]z\} dz,$$

but, cf equation (4.11),

$$-\nu(r_0) + \rho(r_0) - \kappa = 0 \qquad \text{and} \qquad \mu + \alpha\rho(r_0) - 2\alpha\kappa = r_0,$$

hence

$$\hat{D}_0(\lambda|r_0) \simeq \int_0^\infty e^{-(\lambda+r_0)z} dz,$$

and whence

$$\hat{D}_0(\lambda|r_0) \simeq \frac{1}{\lambda + r_0} \quad \Rightarrow \quad \hat{D}_0'(\lambda|r_0) \simeq -\frac{1}{(\lambda+r_0)^2}, \qquad (\lambda \to \infty). \tag{B.4}$$

The approximate expression for the long-run rate is then obtained by substituting (B.4) into (B.1) which, after elementary manipulations and cancelations, yields

$$r_\infty \simeq \frac{[\lambda^2 + \lambda(r_0 + r^*) + r_0 r^*]r^*}{\lambda^2 + \lambda(r_0 + r^*) + r^{*2}}, \qquad (\lambda \to \infty), \tag{B.5}$$

which is equation (4.19).

Appendix C. Approximate expressions in section 4.4

Recall, cf equation (3.10), that the long-run rate r_∞ can be written as

$$r_\infty \simeq \frac{-\hat{D}_0(\lambda|r_0)[1 - \lambda\hat{D}_0(\lambda|r^*)]}{\hat{D}_0'(\lambda|r_0) + \lambda[\hat{D}_0(\lambda|r_0)\hat{D}_0'(\lambda|r^*) - \hat{D}_0(\lambda|r^*)\hat{D}_0'(\lambda|r_0)]}. \tag{C.1}$$

In the Feller case, $\hat{D}_0(\lambda|r_0)$ is given by (4.35),

$$\hat{D}_0(\lambda|r_0) = (2\gamma)^\theta \int_0^\infty \frac{dz}{[(\gamma+\alpha) + (\gamma-\alpha)e^{-\gamma z}]^\theta} \times \exp\left\{-\left(\lambda + r_\infty^{(0)}\right)z - \frac{2(1-e^{-\gamma z})r_0}{(\gamma+\alpha) + (\gamma-\alpha)e^{-\gamma z}}\right\}. \tag{C.2}$$

Again, we will consider separately the cases in which (a) λ is small and (b) λ is large.

(a) When λ is small (as well as $r_\infty^{(0)} = \theta(\gamma-\alpha)/2$ not too large) the main contribution to the integral (C.2) comes from large values of z for which $e^{-\gamma z} \simeq 0$ and we write

$$\hat{D}_0(\lambda|r_0) \simeq \left(\frac{2\gamma}{\gamma+\alpha}\right)^\theta e^{-2r_0/(\gamma+\alpha)} \int_0^\infty e^{-(\lambda + r_\infty^{(0)})z} dz, \qquad (\lambda \to 0),$$

that is

$$\hat{D}_0(\lambda|r_0) \simeq \left(\frac{2\gamma}{\gamma+\alpha}\right)^\theta e^{-2r_0/(\gamma+\alpha)} \frac{1}{\lambda + r_\infty^{(0)}} \tag{C.3}$$

and

$$\hat{D}_0'(\lambda|r_0) \simeq -\left(\frac{2\gamma}{\gamma+\alpha}\right)^\theta e^{-2r_0/(\gamma+\alpha)} \frac{1}{\left(\lambda + r_\infty^{(0)}\right)^2}. \tag{C.4}$$

Substituting (C.3) and (C.4) into (C.1) and simple manipulations yield

$$r_\infty \simeq r_\infty^{(0)} + \lambda\left[1 - \left(\frac{2\gamma}{\gamma+\alpha}\right)^\theta e^{-2r^*/(\gamma+\alpha)}\right], \qquad (\lambda \to 0), \tag{C.5}$$

which is equation (4.36).

(b) When λ is large, the main contribution to the integral (C.2) comes from small values of z and accordingly we will expand the integrand in (C.2) in powers of z up to first order. We thus start from (4.35) which can be rewritten as

$$\hat{D}_0(\lambda|r_0) = (2\gamma)^\theta \int_0^\infty \exp\left\{-\left(\lambda + r_\infty^{(0)}\right) z - \frac{2(1-e^{-\gamma z})r_0}{(\gamma+\alpha)+(\gamma-\alpha)e^{-\gamma z}} - \theta \ln\left[(\gamma+\alpha)+(\gamma-\alpha)e^{-\gamma z}\right]\right\},$$

and then

$$\hat{D}_0(\lambda|r_0) = (2\gamma)^\theta \int_0^\infty \exp\left\{-\left(\lambda + r_\infty^{(0)}\right) z - r_0 z - \theta \ln 2\gamma + \theta(\gamma-\alpha)z/2 + O(z^2)\right\}.$$

Taking into account that

$$\exp\{-\theta \ln(2\gamma)\} = (2\gamma)^{-\theta} \quad \text{and} \quad r_\infty^{(0)} = \theta(\gamma-\alpha)/2,$$

we finally get

$$\hat{D}_0(\lambda|r_0) \simeq \int_0^\infty \exp\{-(\lambda+r_0)z\}dz, \quad (\lambda \to \infty).$$

Hence

$$\hat{D}_0(\lambda|r_0) \simeq \frac{1}{\lambda+r_0} \quad \text{and} \quad \hat{D}_0(\lambda|r_0) \simeq -\frac{1}{(\lambda+r_0)^2}, \qquad (\lambda \to \infty), \quad \text{(C.6)}$$

which are same expressions as those of the OU model (cf equation (B.4)). Therefore, the long-run rate will also be given by (4.19):

$$r_\infty \simeq \frac{\left[\lambda^2 + \lambda(r_0+r^*) + r_0 r^*\right] r^*}{\lambda^2 + \lambda(r_0+r^*) + r^{*2}}, \qquad (\lambda \to \infty).$$

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