J. reine angew. Math. **424** (1992), 47—106

Journal für die reine und angewandte Mathematik

Prym varieties of bi-elliptic curves

By *Juan-Carlos Naranjo* at Barcelona

Introduction

Let $\pi: \tilde{C} \to C$ be an unramified double covering of a smooth curve of genus g. One defines the associated Prym variety as the abelian variety of dimension $g-1$

$$P(\tilde{C}, C) = \operatorname{Ker}(\mathrm{Nm}_\pi)^0 ,$$

where $\mathrm{Nm}_\pi: J(\tilde{C}) \to J(C)$ is the induced norm map of Jacobians. The principal polarization on $J(\tilde{C})$ restricts to twice a principal polarization Ξ on $P(\tilde{C}, C)$ ([Mul], p. 333). In the sequel we shall always consider $P(\tilde{C}, C)$ endowed with this canonical principal polarization. We denote by $\mathscr{R}_g$ and $\mathscr{A}_g$ the moduli spaces for pairs $(\tilde{C}, C)$ as above and for principally polarized abelian varieties of dimension g, respectively. The morphism:

$$P: \mathscr{R}_g \to \mathscr{A}_{g-1}$$

sending $(\tilde{C}, C)$ to $P(\tilde{C}, C)$ is called the Prym map. Beauville ([Be1]) introduces a partial compactification $\bar{\mathscr{R}}_g$ of $\mathscr{R}_g$ parametrizing allowable double coverings of stable curves of genus g and he extends P to a proper map

$$\bar{P}: \bar{\mathscr{R}}_g \to \mathscr{A}_{g-1} .$$

This map $\bar{P}$ is known to be surjective for $g \leqq 6$ and generically injective for $g \geqq 7$ ([F-S], [K], [We1], [De1]). On the other hand Donagi associates two allowable double coverings to an unramified double cover of a smooth tetragonal curve (i.e.: with a linear series g_4^1), the three coverings having the same Prym variety. This construction, called the tetragonal construction, shows that $\bar{P}$ is non-injective for all g. Donagi conjectured:

Tetragonal conjecture (Donagi, [Do]). If two elements $(\tilde{C}, C)$ and $(\tilde{C}', C')$ of $\mathscr{R}_g$ verify $P(\tilde{C}, C) \cong P(\tilde{C}', C')$ then $(\tilde{C}', C')$ is obtained from $(\tilde{C}, C)$ by successive applications of the tetragonal construction (we say that the pairs are "tetragonally related").

Debarre proved in [De2] that the conjecture is true for the fibre of P over the Prym variety of a sufficiently general tetragonal curve of genus $g \geqq 13$. However, it is known that in general the conjecture is not true: say that a smooth curve C is bi-elliptic if it can be represented as a ramified double covering of an elliptic curve, and denote by $\mathscr{R}_{B,g}$ the moduli space for the elements $(\tilde{C}, C) \in \mathscr{R}_g$ with C bi-elliptic. One has a decomposition into irreducible components

$$\mathscr{R}_{B,g} = \mathscr{R}'_{B,g} \cup \left(\bigcup_{t=0}^{[\frac{g-1}{2}]} \mathscr{R}_{B,g,t} \right).$$

Then, no elements of $\mathscr{R}'_{B,g}$ are tetragonally related to another element of $\mathscr{R}_g$ and the same holds for $\mathscr{R}_{B,g,0}$, but $P(\mathscr{R}'_{B,g}) \subset P(\mathscr{R}_{B,g,0})$ (see [De3] or § 2 for details).

Nevertheless, if $(\tilde{C}, C) \in \mathscr{R}_{B,g,0}$ and $(\tilde{C}', C') \in \mathscr{R}'_{B,g}$ verify $P(\tilde{C}, C) \cong P(\tilde{C}', C')$, there exists an allowable cover tetragonally related to both covers: there is a "tetragonal path" through an allowable cover connecting $(\tilde{C}, C)$ and $(\tilde{C}', C')$. In view of these remarks it seems convenient to extend the tetragonal construction to allowable covers. This is done in § 15 following ideas of Beauville ([Be 2]). Then it makes sense to consider the extended tetragonal conjecture by replacing $\mathscr{R}_g$ by $\bar{\mathscr{R}}_g$ in the above conjecture. Alas there are other counter-examples to this extended version: those given by Wirtinger coverings and those coming from the "bi-elliptic construction" explained in § 11. This seems to indicate that Donagi's picture is too optimistic.

The purpose of this paper is to check to what extent Donagi's conjecture holds for elements of $\mathscr{R}_{B,g}$ by studying the fibre of the extended Prym map over $P(\tilde{C}, C)$, where $(\tilde{C}, C)$ is a generic element of $\mathscr{R}_{B,g}$. We obtain a complete description of this fibre. The paper is divided into three parts. In the first part (The fibre of P over a generic element of $P(\mathscr{R}_{B,g})$) we prove the following:

Theorem ((5.11), (6.4), (7.9), (8.7) and (10.10)). *Let $(\tilde{C}, C)$ be a general element of $\mathscr{R}_{B,g}$ with $g \geqq 10$ and let $(\tilde{C}', C') \in \mathscr{R}_g$ be such that $P(\tilde{C}, C) \cong P(\tilde{C}', C')$. Then $(\tilde{C}', C') \in \mathscr{R}_{B,g}$, and $(\tilde{C}, C)$ and $(\tilde{C}', C')$ are tetragonally related. Moreover if $(\tilde{C}, C)$ belongs to $\mathscr{R}_{B,g,t}$ with $t \geqq 1$ then the pairs $(\tilde{C}, C)$ and $(\tilde{C}', C')$ are related by standard tetragonal constructions.*

We obtain also in this part an explicit injection of $\mathscr{R}'_{B,g}$ in $\mathscr{R}_{B,g,0}$ (cf. § 10).

In the second part (A bi-elliptic construction) we find allowable coverings $(\tilde{D}, D)$ with D non-tetragonal and such that $\bar{P}(\tilde{D}, D) \in P(\mathscr{R}_{B,g,4})$. This is a new counter-example to the injectivity of the Prym map, of non-tetragonal type.

Finally in the third part (The fibre of $\bar{P}$ on a generic element of $\mathscr{R}_{B,g}$) we obtain:

Theorem ((16.1)). *Let $(\tilde{C}, C)$ be a general element of $\mathscr{R}_{B,g}$ with $g \geqq 10$ and let $(\tilde{C}', C') \in \bar{\mathscr{R}}_g$ be such that $P(\tilde{C}, C) \cong \bar{P}(\tilde{C}', C')$. Then one of the following facts occurs:*

i) *$(\tilde{C}, C)$ and $(\tilde{C}', C')$ are tetragonally related, or*

ii) *$(\tilde{C}', C')$ is obtained from $(\tilde{C}, C)$ by the bi-elliptic construction. In particular $(\tilde{C}, C) \in \mathscr{R}_{B,g,4}$ in this case.*

That is to say, the tetragonal and the bi-elliptic constructions account for the whole fibre in the (generic) bi-elliptic case. As a summary, we give a complete description of the fibre of the extended Prym map at a generic element of $\mathscr{R}_{B,g}$ in § 20.

I am deeply indebted to Gerald E. Welters for his guidance during the preparation of this work. I wish also to stress the influence of the work of O. Debarre in the present paper. I am grateful to the referee for his suggestions, for his criticism, and for his careful reading of the manuscript.

1. Notation. Throughout this paper we work over the field of the complex numbers. We fix an integer g greater than or equal to 10. By a curve we shall mean a projective connected curve with at most double ordinary singularities. If C is a curve we shall denote by $g(C)$ the arithmetic genus of C. For a subspace F of $\mathscr{R}_g$, the symbol $\bar{F}$ denotes the closure of F in $\bar{\mathscr{R}}_g$.

For D, D' two divisors on a smooth curve C, the expression $D \equiv D'$ will indicate that they are linearly equivalent. We shall denote by $\mathrm{Pic}^d(C)$ the set of linear equivalence classes of degree d divisors on C. Usually we shall not make a distinction between a divisor and its linear equivalence class in $\mathrm{Pic}^d(C)$. For two non-negative integers r, d we shall consider the algebraic subsets of $\mathrm{Pic}^d(C)$:

$$W_d^r(C) = \{\zeta \in \mathrm{Pic}^d(C) \mid h^0(\zeta) \geqq r+1\} .$$

Let $\pi\colon \tilde{C} \to C$ be a double cover of a smooth curve, either unramified or ramified exactly at the points $\tilde{Q}_1, \ldots, \tilde{Q}_k \in \tilde{C}$. Let Δ be the discriminant divisor. Once C is given, the morphism π and the curve $\tilde{C}$ are determined by Δ and a unique element $\xi \in \mathrm{Pic}(C)$ satisfying $2\xi \equiv \Delta$ and $\pi^*(\xi) \equiv \sum_{i=1}^{k} \tilde{Q}_i$. We will refer to ξ and Δ as the class and the discriminant divisor, respectively attached to the covering.

A curve C is said to be hyperelliptic if it can be represented as a double covering of the projective line.

Let D, D_1 and D_2 be curves. The notation $D = D_1 \cup_k D_2$ means that $D = D_1 \cup D_2$ and $\# D_1 \cap D_2 = k$.

The symbols [ ] and $\sim$ will mean rational cohomology class and algebraic equivalence, respectively.

If A is an abelian variety and n is a positive integer, the group of the elements $x \in A$ such that $nx = 0$ will be written ${}_nA$. For a polarized abelian variety A the symbol L_A denotes an invertible sheaf defining the polarization, we call λ_A the isogeny $A \to \hat{A}$ induced by L_A (cf. [Mu 2]) and we denote by $H(L_A)$ its kernel. We shall denote by Ξ_A an effective divisor such that $\mathcal{O}_A(\Xi_A) \cong L_A$. When speaking of the Jacobian of a smooth curve N we shall use L_N and Θ_N instead of L_{JN} and Ξ_{JN}.

We shall set

$$\zeta_A = [\Xi_A]^{a-1}/(a-1)!,$$

where $a = \dim A$. If X is a subvariety of A we define $I(X) := \{x \in A \mid x + X \subset X\}$. This is a closed algebraic subgroup of A.

If (A, L_A) and (B, L_B) are two polarized abelian varieties, the divisor $\Xi_A \times B + A \times \Xi_B$ gives on $A \times B$ a polarization written $L_A \times L_B$. Let $(\tilde{C}, C) \in \mathscr{R}_g$ and P its associated Prym variety (cf. Introduction). There is a natural model (P^*, Ξ^*) of (P, Ξ) in $\mathrm{Pic}^{2g-2}(\tilde{C})$ described as follows ([Mu1])

$$P^* = \{\tilde{\zeta} \in \mathrm{Pic}^{2g-2}(\tilde{C}) \mid \mathrm{Nm}_\pi(\tilde{\zeta}) \equiv K_C,\ h^0(\tilde{\zeta}) \text{ even}\},$$
$$\Xi^* = \{\tilde{\zeta} \in P^* \mid h^0(\tilde{\zeta}) \geqq 2\}.$$

The singular locus of Ξ is described (loc. cit.) as:

$$\mathrm{Sing}\,\Xi^* = \mathrm{Sing}^\pi_{\mathrm{st}}\,\Xi^* \cup \mathrm{Sing}^\pi_{\mathrm{ex}}\,\Xi^*$$

where

$$\mathrm{Sing}^\pi_{\mathrm{st}}\,\Xi^* = \{\tilde{\zeta} \in P^* \mid h^0(\tilde{\zeta}) \geqq 4\}$$

and

$$\mathrm{Sing}^\pi_{\mathrm{ex}}\,\Xi^* = \{\tilde{\zeta} \in P^* \mid \tilde{\zeta} = \pi^*(\zeta) + \tilde{\zeta}_0,\ h^0(\tilde{\zeta}_0) \geqq 1,\ h^0(\zeta) \geqq 2\}.$$

The singularities of the first kind are called stable and the singularities of the second kind are called exceptional. These definitions depend on π.

We refer to [Be1] for the definition of allowable double covering. We shall assume (except in § 15) that we are in the stable case.

I. The fibre of P over a generic element of $P(\mathscr{R}_{B,g})$

2. Summary of known results. The following facts mostly are taken from [De3].

Let $\mathscr{B}_g$ be the moduli space for bi-elliptic curves of genus g and let $\mathscr{R}_{B,g}$ be the moduli space for unramified double coverings of bi-elliptic curves.

Let us fix an element $(\tilde{C}, C) \in \mathscr{R}_{B,g}$ and let $\varepsilon: C \to E$ be a morphism of degree two on a smooth elliptic curve E (ε is unique up to automorphisms of E if $g \geqq 6$). The Galois group of $\tilde{C}$ over E may be identified with either $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We shall denote by $\mathscr{R}'_{B,g}$ the subset of elements with Galois group $\mathbb{Z}/2\mathbb{Z}$.

(2.1) If $\mathrm{Gal}_E(\tilde{C}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, we write for the elements of the group: Id, ι, ι_1, ι_2, where ι is the involution which interchanges the sheets of the double cover π. Let $C_1 = \tilde{C}/(\iota_1)$, $C_2 = \tilde{C}/(\iota_2)$ be the quotient curves.

One has a commutative diagram:

(2.2)
$$\begin{array}{ccccc} & & \tilde{C} & & \\ & \overset{\pi}{\swarrow} & \downarrow{\scriptstyle \pi_1} & \overset{\pi_2}{\searrow} & \\ C & & C_1 & & C_2 \\ & \underset{\varepsilon}{\searrow} & \downarrow{\scriptstyle \varepsilon_1} & \underset{\varepsilon_2}{\swarrow} & \\ & & E & & \end{array}$$

where π_1, π_2, ε_1 and ε_2 are the obvious morphisms. We shall always assume that $g(C_1) \leqq g(C_2)$. Let $\mathscr{R}_{B,g,t}$ be the subset of $\mathscr{R}_{B,g}$ consisting of the elements $(\tilde{C}, C)$ with $\mathrm{Gal}_E(\tilde{C}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $g(C_1) = t+1$, $g(C_2) = g - t$.

One finds that $\mathscr{R}'_{B,g}, \mathscr{R}_{B,g,0}, \ldots, \mathscr{R}_{B,g,\left[\frac{g-1}{2}\right]}$ are the irreducible components of $\mathscr{R}_{B,g}$ and that each one has dimension $2g-2$.

(2.3) Let $(\tilde{C}, C) \in \mathscr{R}_{B,g,t}$. We fix the following notation:

i) τ, τ_1 and τ_2 are the involutions of C, C_1 and C_2 associated to ε, ε_1 and ε_2, respectively.

ii) Let $\bar{\Delta} = \sum_{i=1}^{2g-2} \bar{P}_i$ be the discriminant divisor of ε and let $P_1, \ldots, P_{2g-2}$ be the corresponding ramification points.

iii) $\bar{\xi} \in \mathrm{Pic}^{g-1}(E)$ is the class associated to ε. Hence $2\bar{\xi} \equiv \bar{\Delta}$.

iv) $\eta \in {}_2JC$ is the class associated to π.

We may assume that $\bar{P}_1, \ldots, \bar{P}_{2t}$ are the discriminant points of ε_1 and that $\bar{P}_{2t+1}, \bar{P}_{2g-2}$ are those of ε_2. We shall denote by $\bar{\Delta}_1$, $\bar{\Delta}_2$, $\bar{\xi}_1$ and $\bar{\xi}_2$ the discriminant divisors and the classes associated to ε_1 and ε_2, respectively.

(2.4) It is easy to check the following facts:

i) $\bar{\xi} = \bar{\xi}_1 + \bar{\xi}_2$, $\bar{\Delta} = \Delta_1 + \bar{\Delta}_2$.

ii) $\eta \equiv P_1 + \ldots + P_{2t} - \varepsilon^*(\bar{\xi}_1) \equiv P_{2t+1} + \ldots + P_{2g-2} - \varepsilon^*(\bar{\xi}_2)$.

iii) $\tilde{C} \cong C_1 \times_E C_2$.

(2.5) We keep the assumption $(\tilde{C}, C) \in \mathscr{R}_{B,g,t}$ and we write $P = P(\tilde{C}, C)$. We have the description:

$$\Xi^* = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in W_t^0(C_1),\ \zeta_2 \in W_{g-t-1}^0(C_2),\ \mathrm{Nm}_{\varepsilon_1}(\zeta_1) + \mathrm{Nm}_{\varepsilon_2}(\zeta_2) = \xi\}\,.$$

(2.6) For $g \geqq 7$ we shall define in (3.7) the subvarieties V, W_{-2}, W_0 and W_2 of Ξ^*. Then $\mathrm{Sing}\,\Xi^* \supseteqq V \cup W_{-2} \cup W_0 \cup W_2$ with equality if $\bar{\Delta}$ does not belong to the image of the addition map $|\bar{\xi}| \times |\bar{\xi}| \to |2\bar{\xi}|$ (this happens if $(\tilde{C}, C)$ is general). Otherwise a finite number of new isolated singularities appear.

(2.7) The following table contains relevant information to be used in the sequel:

t	0	1	2	3	$\geqq 4$
V	$\emptyset$	$\emptyset$	$\emptyset$	dim $g-7$	irred. dim $g-7$
W_{-2}	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	irred. dim $g-5$
W_0	$\emptyset$	$\emptyset$	irred. dim $g-5$	irred. dim $g-5$	irred. dim $g-5$
W_2	irred. dim $g-5$	irred. dim $g-5$	irred. dim $g-5$	irred. dim $g-5$	irred. dim $g-5$.

When $t = 3$ and $(\tilde{C}, C)$ is general V has two components (see (3.4)). The singularities corresponding to the elements of these varieties are stable for V, exceptional for W_0 and stable and exceptional for W_{-2} and W_2.

(2.8) Consider now the abelian varieties $P_1 := P(C_1, E) = \mathrm{Ker}(\mathrm{Nm}_{\varepsilon_1})$ (if $t \geqq 1$) and $P_2 := P(C_2, E) = \mathrm{Ker}(\mathrm{Nm}_{\varepsilon_2})$. We define the morphisms:

$$\varphi\colon P_1 \times P_2 \to P$$

by sending (ζ_1, ζ_2) to $\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2)$, if $t \geqq 1$, and

$$\psi\colon P_2 \to P$$

by sending ζ_2 to $\pi_2^*(\zeta_2)$ if $t = 0$. Then φ and ψ are isogenies and:

$$\mathrm{Ker}(\varphi) = \{(\varepsilon_1^*(\bar{\alpha}), \varepsilon_2^*(\bar{\alpha})) \mid \bar{\alpha} \in {}_2JE\}\,,$$

$$\mathrm{Ker}(\psi) = \{0, \varepsilon_2^*(\bar{\xi}_1)\}\,.$$

(2.9) Remark. The definitions of ι, τ, $\bar{P}_1, \ldots, \bar{P}_{2g-2}$, $\bar{\Delta}$, ξ and η given in (2.3) make still sense if $(\tilde{C}, C) \in \mathscr{R}'_{B,g}$ and we will use them throughout.

(2.10) Now we want to apply the tetragonal construction to an element $(\tilde{C}, C) \in \mathscr{R}_{B,g}$. Assuming first that $(\tilde{C}, C) \in \mathscr{R}_{B,g,t}$ and keeping the notation of (2.3), fix a linear series g_2^1 on E inducing an involution ν. Applying the tetragonal construction to $(\tilde{C}, C)$ with respect to $\varepsilon^*(g_2^1)$ one obtains two elements $(\tilde{C}', C')$ and $(\tilde{C}'', C'')$ of $\bar{\mathscr{R}}_g$ (cf. Introduction) verifying:

a) In terms of the data introduced in (2.1), one of the coverings, say $(\tilde{C}', C')$, can be described by the new set of data:

$$\begin{array}{ccccc} & & \tilde{C}' & & \\ & \swarrow & \downarrow & \searrow & \\ C' & & C_1 & & C_2 \\ & \searrow & \downarrow{\scriptstyle \varepsilon_1} & \swarrow{\scriptstyle \nu\circ\varepsilon_2} & \\ & & E & & \end{array}$$

Note that $(\tilde{C}', C') \cong (\tilde{C}, C)$ if $t = 0$.

If $\nu(\bar{P}_i) \neq \bar{P}_j$ for $1 \leqq i \leqq 2t < j \leqq 2g-2$, then $(\tilde{C}', C') \in \mathscr{R}_{B,g,t}$. In any case $(\tilde{C}', C') \in \bar{\mathscr{R}}_{B,g,t}$.

b) We now consider the second covering $(\tilde{C}'', C'')$. For $2 \leqq t \leqq \left[\frac{g-1}{2}\right]$ we define $\mathscr{H}'_{g,t} = \{(\tilde{\Gamma}, \Gamma) \in \bar{\mathscr{R}}_g \,|\, \Gamma = \Gamma_1 \cup_4 \Gamma_2$ with Γ_1, Γ_2 curves of genus $t-1, g-t-2$, respectively$\}$.

(Notice that $t - 1 \leqq g - t - 2$, since $t \leqq \left[\frac{g-1}{2}\right]$.) We call $\mathscr{H}_{g,t}$ the subspace defined by the additional condition of Γ_1, Γ_2 being irreducible and smooth. Then the second cover $(\tilde{C}'', C'')$ is an element of $\mathscr{H}'_{g,t}$ such that the components of C'' are hyperelliptic curves. If moreover $\nu(\bar{P}_i) \neq \bar{P}_j$ for $1 \leqq i \leqq 2t < j \leqq 2g-2$, then $(\tilde{C}'', C'') \in \mathscr{H}_{g,t}$.

For $t = 1$ we put

$$\mathscr{H}'_{g,1} = \{(\tilde{\Gamma}, \Gamma) \in \bar{\mathscr{R}}_g \,|\, \tilde{\Gamma} = \mathbb{P}^1 \cup_4 \Gamma_2 \text{ and } \Gamma_2 \text{ is a hyperelliptic curve}\}\,.$$

Again the additional condition of Γ_2 being irreducible and smooth defines a subspace $\mathscr{H}_{g,1}$. Then $(\tilde{C}'', C'') \in \mathscr{H}'_{g,1}$. When ν satisfies the same condition as above, then $(\tilde{C}'', C'') \in \mathscr{H}_{g,1}$.

Finally we define for $t = 0$

$$\mathscr{H}'_{g,0} = \{(\tilde{\Gamma}, \Gamma) \in \bar{\mathscr{R}}_g \,|\, \Gamma \text{ is obtained from a hyperelliptic curve by identifying two pairs of points}\}\,.$$

By imposing that the hyperelliptic curve being irreducible and smooth, and each pair being non-hyperelliptic we define a subspace $\mathscr{H}_{g,0}$. Then $(\tilde{C}'', C'') \in \mathscr{H}'_{g,0}$. If ν is general then $(\tilde{C}'', C'') \in \mathscr{H}_{g,0}$.

By applying the tetragonal construction to an element of $\mathscr{R}'_{B,g}$ we obtain two elements of $\mathscr{H}'_{g,0}$. Once again if the linear series g^1_2 is general, then they belong to $\mathscr{H}_{g,0}$.

The spaces $\mathscr{H}_{g,t}$ are irreducible and dense in $\mathscr{H}'_{g,t}$, $t = 0, \ldots, \left[\frac{g-1}{2}\right]$. We have also

$$\dim \mathcal{H}_{g,t} = 3g - 7 \qquad \text{for } t \geqq 2,$$
$$\dim \mathcal{H}_{g,1} = 2g - 2 \qquad \text{and}$$
$$\dim \mathcal{H}_{g,0} = 2g - 1.$$

Notice that our definition of $\mathcal{H}_{g,0}$ differs a bit of that of [De 3]. This change is necessary in order to have the next property.

(2.11) Any element of $\mathcal{H}_{g,0}$ can be obtained by means of the tetragonal construction from an element of $\mathcal{R}_{B,g,0}$. In fact, this is a consequence of the construction that will be given in § 15. On the other hand, notice that $P(\mathcal{R}'_{B,g}) \subset \bar{P}(\mathcal{H}_{g,0})$. Hence $P(\mathcal{R}'_{B,g}) \subset P(\mathcal{R}_{B,g,0})$.

(2.12) Finally we recall two lemmas borrowed from [Mu 1] and [De 2]. First we need a definition. Let $\pi\colon \tilde{C} \to C$ be a double cover of a smooth curve. We shall say that an effective divisor on $\tilde{C}$ is π-simple if it does not contain inverse images of effective divisors of C. Let $\zeta \in \mathrm{Pic}(C)$ be the class attached to π. With this notation one has:

(2.13) Lemma ([Mu 1], p. 338). *If $\mathcal{L}$ is an invertible sheaf on C and $\tilde{D}$ is an effective π-simple divisor on $\tilde{C}$ there exists an exact sequence*:

$$0 \to \mathcal{L} \to \pi_*(\pi^*(\mathcal{L}) \otimes_{\mathcal{O}_{\tilde{C}}} \mathcal{O}_{\tilde{C}}(\tilde{D})) \to \mathcal{L} \otimes_{\mathcal{O}_C} \mathcal{O}_C(\mathrm{Nm}_\pi(\tilde{D}) - \zeta) \to 0.$$

(2.14) Lemma ([De 2], p. 550). *Let $\pi\colon \tilde{C} \to C$ be an allowable double cover of a stable curve C, $\tilde{\mathcal{L}}$ an invertible sheaf on $\tilde{C}$ and D a reduced element of $|K_C \otimes (\mathrm{Nm}_\pi(\tilde{\mathcal{L}}))^{-1}|$ with non-singular support. Suppose that $h^0(\tilde{\mathcal{L}} \otimes_{\mathcal{O}_{\tilde{C}}} \mathcal{O}_{\tilde{C}}(\tilde{D})) \geqq 1$ for all effective divisors $\tilde{D}$ such that* $\mathrm{Nm}_\pi(\tilde{D}) = D$. *Then $h^0(\tilde{\mathcal{L}}) \geqq 1$.*

3. Some properties of bi-elliptic curves. This section deals with properties of bi-elliptic curves that will be used later on. In a first reading it may be skipped and kept for reference purposes.

Let $\varepsilon\colon C \to E$ be a 2-to-1 morphism of smooth curves where E is an elliptic curve. We denote by $\bar{\Delta}$ and $\bar{\xi}$ the discriminant divisor and the class determining ε. By Riemann-Hurwitz:

$$\deg \bar{\Delta} = 2g - 2, \quad \deg \bar{\xi} = g - 1.$$

Let $\tau\colon C \to C$ be the involution which interchanges the points of each fibre.

(3.1) Lemma. *Let $\bar{A}$, B be effective divisors on E and C, respectively. Assume that B is ε-simple* (*cf.* (2.12)). *Then*:

$$\deg(\bar{A}) + \deg(B) < g(C) - 1 \Rightarrow h^0(\varepsilon^*(\bar{A}) + B) = h^0(\bar{A}).$$

Proof. Use (2.13). □

(3.2) If $g(C) \geqq 5$, then C is not trigonal (cf. [Te]).

(3.3) If $g(C) \geqq 4$, then C is not hyperelliptic. Use (3.1).

(3.4) Assume that C is general, of genus 4. Then $W_3^1(C)$ has two different points.

(3.5) Assume that C is general, of genus 3. Then C is not hyperelliptic.

(3.6) We consider the following subvarieties of $\mathrm{Pic}^{g(C)-1}(C)$:

$$Z' = \{\zeta \in \mathrm{Sing}\,\Theta^* \mid \mathrm{Nm}_\varepsilon(\zeta) = \bar{\xi}\},$$
$$Z'' = \{\varepsilon^*(\bar{x}+\bar{y}) + \zeta' \mid \bar{x}, \bar{y} \in E,\ \zeta' \in W^0_{g(C)-5}\} \quad \text{if } g(C) \geqq 5,$$
$$A = \{\varepsilon^*(\bar{x}) + \zeta' \mid \bar{x} \in E,\ \zeta' \in W^0_{g(C)-3}\} \supset Z'' \quad \text{if } g(C) \geqq 3.$$

Remarks. i) If $g(C) \geqq 3$, then A is irreducible of dimension $g(C)-2$.

ii) If $g(C) \geqq 6$, then Z' and Z'' are irreducible of dimension $g(C)-4$ and $\mathrm{Sing}\,\Theta^* = Z' \cup Z''$ ([We2], Prop. 3.6). If $g(C) = 5$, then the equality holds but Z' is not always irreducible (in fact by [Te] there is a bijection between the set of its components and the set of bi-elliptic structures on C).

(3.7) Now we define the varieties V and W_a (where $a \in \{2, 0, -2\}$) mentioned in (2.5). We use the definitions of (3.6) applied to C_1 and C_2. In these terms:

$$V = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in Z_1',\ \zeta_2 \in Z_2'\},$$
$$W_{-2} = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in Z_1'',\ \zeta_2 \in \Theta_2^*,\ \mathrm{Nm}_{\varepsilon_1}(\zeta_1) + \mathrm{Nm}_{\varepsilon_2}(\zeta_2) = \bar{\xi}\},$$
$$W_0 = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in A_1,\ \zeta_2 \in A_2,\ \mathrm{Nm}_{\varepsilon_1}(\zeta_1) + \mathrm{Nm}_{\varepsilon_2}(\zeta_2) = \bar{\xi}\},$$
$$W_2 = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in \Theta_1^*,\ \zeta_2 \in Z_2'',\ \mathrm{Nm}_{\varepsilon_1}(\zeta_1) + \mathrm{Nm}_{\varepsilon_2}(\zeta_2) = \bar{\xi}\}.$$

(3.8) Lemma ([De3], Lemma 5.2.10). *Assume* $g(C) \geqq 6$ *and fix* $\bar{\lambda} \in \mathrm{Pic}^{g(C)-1}(E)$. *Then* $\{\zeta \in Z'' \mid \mathrm{Nm}_\varepsilon(\zeta) = \bar{\lambda}\}$ *is irreducible of dimension* $g(C)-5$.

In particular $Z' \cap Z''$ is irreducible.

The following facts will be used throughout.

(3.9) Proposition. *One has the following equalities*:

i) *If* $g(C) \geqq 3$ *then* $I(A) = \{\varepsilon^*(\bar{\alpha}) \mid \bar{\alpha} \in \mathrm{Pic}^0(E)\}$.

ii) *If* $g(C) \geqq 5$ *then*:

$$\begin{aligned}\{a \in JC \mid a + Z'' \subset A\} &= \{a \in JC \mid a + Z'' \subset \Theta^*\} \\ &= \{a \in JC \mid a + Z' \cap Z'' \subset A\} = \{a \in JC \mid a + Z' \cap Z'' \subset \Theta^*\} \\ &= \{\varepsilon^*(\bar{x}) - r - s \mid \bar{x} \in E,\ r, s \in C\}.\end{aligned}$$

iii) *If* $g(C) \geqq 5$ *then*:

$$I(Z'') = \{a \in JC \mid a + Z' \cap Z'' \subset Z''\} = \{\varepsilon^*(\bar{\alpha}) \mid \bar{\alpha} \in \mathrm{Pic}^0(E)\}.$$

Proof. i) The inclusion of the second member in the first one is clear. Let $a \in JC$ such that $a + A \subset A$. In particular, for all $\bar{x} \in E$ and $D \in C^{(g-3)}$ one has $h^0(a + \varepsilon^*(\bar{x}) + D) > 0$. Then $h^0(a + \varepsilon^*(\bar{x})) > 0$.

Now we may write $a \equiv D - \varepsilon^*(\bar{x})$ where D is an effective divisor of degree two verifying

$$h^0(D + \varepsilon^*(\bar{\alpha})) > 0 \text{ for all } \bar{\alpha} \in \operatorname{Pic}^0(E).$$

By applying (2.13) we conclude that $D \in \operatorname{Im}(\varepsilon^*)$, thereby proving i).

ii) All the equalities are an easy consequence of the following one:

$$\{a \in JC \mid a + Z' \cap Z'' \subset \Theta^*\} = \{\varepsilon^*(\bar{x}) - r - s \mid \bar{x} \in E,\ r, s \in C\}.$$

This fact was proved by Debarre in [De 5]. We give here a sketch of the proof. We only prove the inclusion of the left hand side member in the right hand side member. Write $a \equiv D - \varepsilon^*(\bar{A})$, where $\bar{A} \in \operatorname{Pic}^r(E)$ and D is effective. If we assume that D is ε-simple then $2r \leqq g + 1$. In fact it is not necessary to consider the case $2r = g + 1$. It suffices to obtain a contradiction if $r \geqq 2$.

Suppose that $2r \leqq g - 2$. For a generic element $\bar{B} \in \operatorname{Pic}^r(E)$ there exists $D' \geqq 0$ such that:

- $D + D'$ is ε-simple.
- $2\bar{B} + \operatorname{Nm}_\varepsilon(D') \equiv \bar{\xi}$.

Then $\varepsilon^*(\bar{B}) + D' \in Z' \cap Z''$. By applying (2.13)

$$\begin{aligned} 0 < h^0(a + \varepsilon^*(\bar{B}) + D') &= h^0(D + D' + \varepsilon^*(\bar{B} - \bar{A})) \\ &\leqq h^0(\bar{B} - \bar{A}) + h^0(\operatorname{Nm}_\varepsilon(D + D') + \bar{B} - \bar{A} - \bar{\xi}) \\ &= h^0(\bar{B} - \bar{A}) + h^0(\operatorname{Nm}_\varepsilon(D) - \bar{A} - \bar{B}) \end{aligned}$$

which is a contradiction because $\bar{B}$ is generic. The cases $2r = g - 1, g$ are similar.

Part iii) follows from ii). □

4. A key lemma. Let $f\colon \tilde{N} \to N$ be a (2:1) morphism of smooth curves with ramification divisor $\sum_{i=1}^{k} \tilde{Q}_i$. We denote by σ the involution of $\tilde{N}$ attached to f.

Let $\tilde{L}$ be a line bundle on $\tilde{N}$ with $\tilde{L} \cong \sigma^*(\tilde{L})$. Choose an isomorphism φ normalized in such a way that:

$$\sigma^*(\varphi) \circ \varphi = \operatorname{Id}_{\tilde{L}}.$$

Writing $\tilde{L}[\tilde{x}]$ for the pointwise fibre of $\tilde{L}$ over $\tilde{x} \in \tilde{N}$, one obtains isomorphisms:

$$\varphi[\tilde{Q}_i] : \tilde{L}[\tilde{Q}_i] \to \sigma^*(\tilde{L})[\tilde{Q}_i] = \tilde{L}[\sigma(\tilde{Q}_i)] = \tilde{L}[\tilde{Q}_i], \quad i \in \{1, \ldots, k\},$$

given by multiplication with constants λ_i with $\lambda_i^2 = 1$. We attach to $\tilde{L}$ a vector $v(\tilde{L}) = (\lambda_1, \ldots, \lambda_k) \in (\mu_2)^k$ which depends on the choice of φ. The ambiguity disappears when we pass to the quotient modulo μ_2 by the diagonal action. Then we have a homomorphism of groups:

$$v\colon \operatorname{Ker}(\sigma^* - 1) \to \frac{(\mu_2)^k}{\mu_2}.$$

We use the notation $v(\tilde{D})$ and $v(\tilde{\mathscr{L}})$ for $\tilde{D}$ a divisor and $\tilde{\mathscr{L}}$ an invertible sheaf on $\tilde{N}$.

(4.1) Proposition. *There exists a line bundle L on N such that $f^*(L) \cong \tilde{L}$ iff* $v(\tilde{L}) = \overline{(1, \ldots, 1)}$.

Proof. It suffices to use [G], Th. 1, p. 17. □

(4.2) Proposition. *Let $\tilde{\mathscr{L}}$ be an invertible sheaf on $\tilde{N}$ such that $\sigma^*(\tilde{\mathscr{L}}) \cong \tilde{\mathscr{L}}$. Then there exists a divisor $\tilde{D}$ on $\tilde{N}$ with $0 \leqq \tilde{D} \leqq \sum_{i=1}^{k} \tilde{Q}_i$ and an invertible sheaf $\mathscr{L}$ on N such that*

$$f^*(\mathscr{L}) \cong \tilde{\mathscr{L}} \otimes \mathscr{O}_N(-\tilde{D}).$$

Proof. By using the exact sequence:

$$0 \to \mathscr{O}_{\tilde{N}}(-\tilde{Q}_i) \to \mathscr{O}_{\tilde{N}} \to \mathscr{O}_{\tilde{Q}_i} \to 0$$

and by observing that $\mathscr{O}_{\tilde{N}}(-\tilde{Q}_i) \notin \operatorname{Im}(f^*)$ (hence by (4.1) $v(-\tilde{Q}_i) \neq \overline{(1, \ldots, 1)}$) one has $v(-\tilde{Q}_i) = \overline{(1, \ldots, -1, \ldots, 1)}$. Then, by tensoring $\tilde{\mathscr{L}}$ with suitable sheaves $\mathscr{O}_{\tilde{N}}(-\tilde{Q}_i)$ we can make all the coordinates of the corresponding vector equal. □

Let $(\tilde{C}, C) \in \mathscr{R}_{B,g}$. We keep the notations of §2. In particular $\eta \in {}_2JC$ is the class determining $\pi\colon \tilde{C} \to C$.

(4.3) Corollary. *One has $(\tilde{C}, C) \in \mathscr{R}'_{B,g}$ iff $\tau^*(\eta) \neq \eta$.*

Proof. By (2.4) ii), $\tau^*(\eta) = \eta$ when $(\tilde{C}, C) \notin \mathscr{R}'_{B,g}$. Conversely suppose $\tau^*(\eta) = \eta$. Applying (4.2) we may write:

$$\eta = D - \varepsilon^*(\bar{A}) \quad \text{with} \quad 0 \leqq D \leqq \sum_{i=1}^{2g-2} P_i.$$

Let C_1 (resp. C_2) be the double cover on E given by the class of $\bar{A}$ (resp. $\xi - \bar{A}$) and the discriminant divisor $\operatorname{Nm}_\varepsilon(D)$ (resp. $\bar{\Delta} - \operatorname{Nm}_\varepsilon(D)$). Observe that:

$$\varepsilon^*(\operatorname{Nm}_\varepsilon(\eta)) = 2\eta = 0.$$

So due to the injectivity of ε^*:

$$\mathrm{Nm}_\varepsilon(\eta) = 0 \quad \text{and} \quad 2\bar{A} \equiv \mathrm{Nm}_\varepsilon(D)\,.$$

Then $\tilde{C} \cong C_1 \times_E C_2$ and $C \cong \tilde{C}/(\iota_1 \circ \iota_2)$, ι_1 and ι_2 being the involutions of $\tilde{C}$ attached to the projections on C_1 and C_2, respectively. Hence $(\tilde{C}, C) \in \mathscr{R}_{B,g,t}$ for some t. □

(4.4) Lemma. *Assume $t > 0$. We consider the commutative diagram:*

$$\begin{array}{ccc} JC_1 & \xrightarrow{\pi_1^*} & J\tilde{C} \\ {\scriptstyle \varepsilon_1^*}\uparrow & & \uparrow{\scriptstyle \pi_2^*} \\ JE & \xrightarrow[\varepsilon_2^*]{} & JC_2\,. \end{array}$$

Then $\pi_1^*(JC_1) \cap \pi_2^*(JC_2) = \{\pi^*(\varepsilon^*(\bar{\alpha})) \mid \bar{\alpha} \in \mathrm{Pic}^0(E)\}$.

Proof. Fix $\tilde{\beta} \in \mathrm{Im}(\pi_1^*) \cap \mathrm{Im}(\pi_2^*)$ and $\beta_1 \in JC_1$, $\beta_2 \in JC_2$ such that

$$\tilde{\beta} = \pi_1^*(\beta_1) = \pi_2^*(\beta_2).$$

Then $\pi_1^*(\beta_1) = \pi_1^*(\tau_1^*(\beta_1))$ and $\pi_2^*(\beta_2) = \pi_2^*(\tau_2^*(\beta_2))$. Since $t > 0$, the morphisms π_1 and π_2 are ramified hence $\beta_1 = \tau_1^*(\beta_1)$ and $\beta_2 = \tau_2^*(\beta_2)$. Applying (4.2), there exist divisors D_1 on C_1, D_2 on C_2 and classes $\bar{\alpha}_1, \bar{\alpha}_2 \in \mathrm{Pic}^0(E)$ such that:

$$\beta_1 \equiv \varepsilon_1^*(\bar{\alpha}_1) - D_1, \quad \beta_2 \equiv \varepsilon_2^*(\bar{\alpha}_2) - D_2 \tag{4.5}$$

where $0 \leqq D_i \leqq$ ramification divisor of ε_i, $\quad i = 1, 2$.

Hence:

$$\pi^*(\varepsilon^*(\bar{\alpha}_1 - \bar{\alpha}_2)) \equiv \pi_1^*(D_1) - \pi_2^*(D_2)\,.$$

Let R_1 and R_2 be effective divisors on C such that

$$\pi^*(R_1) = \pi_1^*(D_1), \quad \pi^*(R_2) = \pi_2^*(D_2)$$

thus

$$0 \leqq R_1 \leqq \sum_{i=1}^{2t} P_i, \quad 0 \leqq R_2 \leqq \sum_{i=2t+1}^{2g-2} P_i\,.$$

From

$$\pi^*(\varepsilon^*(\bar{\alpha}_1 - \bar{\alpha}_2)) \equiv \pi^*(R_1 - R_2)\,,$$

two possibilities appear:

$$\text{either i)} \quad \varepsilon^*(\bar{\alpha}_1 - \bar{\alpha}_2) \equiv R_1 - R_2$$

$$\text{or ii)} \quad \varepsilon^*(\bar{\alpha}_1 - \bar{\alpha}_2) \equiv R_1 - R_2 + \eta\,.$$

We first suppose i). From (4.1) we have $v(R_1 - R_2) = \overline{(1, \dots, 1)}$, i.e.: $v(R_1) = v(R_2)$. By applying the proof of (4.2) we can compute these vectors:

$$v(R_1) = \overline{(\lambda_1, \dots, \lambda_{2t}, 1, \dots, 1)} \quad \text{with} \quad \lambda_i = -1 \quad \text{iff} \quad P_i \in \operatorname{Supp}(R_1),$$

$$v(R_2) = \overline{(1, \dots, 1, \lambda_{2t+1}, \dots, \lambda_{2g-2})} \quad \text{with} \quad \lambda_i = -1 \quad \text{iff} \quad P_i \in \operatorname{Supp}(R_2).$$

We conclude that $\lambda_1 = \dots = \lambda_{2t} = \lambda_{2t+1} = \dots = \lambda_{2g-2}$, that is to say, either $R_1 = R_2 = 0$ or $R_1 = \sum_{i=1}^{2t} P_i$, $R_2 = \sum_{i=2t+1}^{2g-2} P_i$. If $R_1 = R_2 = 0$, then $D_1 = D_2 = 0$ and we finish by taking $\bar{\beta} = \bar{\alpha}_1 = \bar{\alpha}_2$. Similarly, if $R_1 = \sum_{i=1}^{2t} P_i$, $R_2 = \sum_{i=2t+1}^{2g-2} P_i$ we get $D_1 \equiv \varepsilon_1^*(\bar{\xi}_1)$ and $D_2 \equiv \varepsilon_2^*(\bar{\xi}_2)$ (see (2.3)). By replacing in (4.5):

$$\beta_1 = \varepsilon_1^*(\bar{\alpha}_1 - \bar{\xi}_1), \quad \beta_2 = \varepsilon_2^*(\bar{\alpha}_2 - \bar{\xi}_2).$$

On the other hand, by (2.4) ii):

$$\varepsilon^*(\bar{\alpha}_1 - \bar{\alpha}_2) \equiv \sum_{i=1}^{2t} P_i - \sum_{i=2t+1}^{2g-2} P_i \equiv \varepsilon^*(\bar{\xi}_1 - \bar{\xi}_2)$$

and one finally obtains $\bar{\beta} = \bar{\alpha}_1 - \bar{\xi}_1 = \bar{\alpha}_2 - \bar{\xi}_2$.

In the case ii) we can imitate the above proof by replacing η by the expression of (2.4) ii). □

5. The components $\mathscr{R}_{B,g,t}$ for $t \geqq 4$. In this paragraph $(\tilde{C}, C)$ is an element of $\mathscr{R}_{B,g,t}$ with $t \geqq 4$ and $P = P(\tilde{C}, C)$. We keep the notations of §§ 1 and 2. In particular $g \geqq 10$.

In order to describe the fibre of the Prym map over P we shall use ideas from [We 1] and [De 2]. We perform intrinsic geometrical constructions to get information on the covering from the Prym variety. We will use the components of Sing Ξ^*.

Recalling the descriptions of (3.7) one has:

(5.1) Proposition. *The variety $W_{-2} \cap W_2$ is irreducible of dimension $g - 9$ and one has the equality*:

$$W_{-2} \cap W_2 = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in Z_1'', \zeta_2 \in Z_2'', \operatorname{Nm}_{\varepsilon_1}(\zeta_1) + \operatorname{Nm}_{\varepsilon_2}(\zeta_2) = \xi\}.$$

Proof. We check first the equality. Clearly the second member is contained in the first one. Conversely, any $\zeta \in W_2 \cap W_{-2}$ can be written as

$$\zeta = \pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) = \pi_1^*(\zeta_1') + \pi_2^*(\zeta_2') \tag{5.2}$$

where

$$\zeta_1 \in \Theta_1^*, \quad \zeta_2 \in Z_2'', \quad \zeta_1' \in Z_1'', \quad \zeta_2' \in \Theta_2^*$$

and

$$\mathrm{Nm}_{\varepsilon_1}(\zeta_1) + \mathrm{Nm}_{\varepsilon_2}(\zeta_2) = \mathrm{Nm}_{\varepsilon_2}(\zeta_1') + \mathrm{Nm}_{\varepsilon_2}(\zeta_2') = \xi .$$

Then:

$$\pi_1^*(\zeta_1 - \zeta_1') = \pi_2^*(\zeta_2 - \zeta_2') .$$

By (4.4) there exists $\bar{\alpha} \in \mathrm{Pic}^0(E)$ such that:

$$\zeta_1 - \zeta_1' = \varepsilon_1^*(\bar{\alpha}) ,$$
$$\zeta_2' - \zeta_2 = \varepsilon_2^*(\bar{\alpha}) .$$

In particular $\zeta_1 = \varepsilon_1^*(\bar{\alpha}) + \zeta_1'$ and replacing this in (5.2) we are done.

Consider now the morphism:

$$\begin{aligned} \Psi\colon Z_1'' \times Z_2'' &\to \mathrm{Pic}^{2g-2}(\tilde{C}) , \\ (\zeta_1, \zeta_2) &\to \pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) . \end{aligned}$$

Let us define $T = \{(\zeta_1, \zeta_2) \in Z_1'' \times Z_2'' \mid \mathrm{Nm}_{\varepsilon_1}(\zeta_1) + \mathrm{Nm}_{\varepsilon_2}(\zeta_2) = \xi\}$. Clearly $\Psi(T) = W_{-2} \cap W_2$. Since each fibre of the induced map $T \to W_{-2} \cap W_2$ is isomorphic to E (use (4.4)) it suffices to prove that T is irreducible of dimension $g-8$. To see this look at the first projection: $T \to Z_1''$. Clearly Z_1'' is irreducible and by (3.8) the fibres are irreducible of dimension $g-t-5$ (note that $g \geqq 10$, $t \geqq 4$ and $t+1 \leqq g-t$ imply $g-t \geqq 6$). Thus T is irreducible and $\dim T = \dim Z_1'' + g - t - 5 = t - 3 + g - t - 5 = g - 8$. □

(5.3) Proposition. *The varieties $W_0 \cap W_{-2}$ and $W_0 \cap W_2$ are both irreducible of dimension $g-7$ and they are described as follows*:

$$W_0 \cap W_{-2} = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in Z_1'',\ \zeta_2 \in A_2,\ \mathrm{Nm}_{\varepsilon_1}(\zeta_1) + \mathrm{Nm}_{\varepsilon_2}(\zeta_2) = \xi\} ,$$
$$W_0 \cap W_2 = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in A_1,\ \zeta_2 \in Z_2'',\ \mathrm{Nm}_{\varepsilon_1}(\zeta_1) + \mathrm{Nm}_{\varepsilon_2}(\zeta_2) = \xi\} .$$

Proof. By symmetry only one variety has to be considered, for instance $W_0 \cap W_2$. Imitating the proof of (5.1) one finds the equality. The irreducibility and dimension may be obtained as above replacing Ψ by the morphism:

$$\begin{aligned} \Psi'\colon A_1 \times Z_2'' &\to \mathrm{Pic}^{2g-2}(\tilde{C}) , \\ (\zeta_1, \zeta_2) &\to \pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \end{aligned}$$

and T by $T' = \{(\zeta_1, \zeta_2) \in A_1 \times Z_2'' \mid \mathrm{Nm}_{\varepsilon_1}(\zeta_1) + \mathrm{Nm}_{\varepsilon_2}(\zeta_2) = \xi\}$. □

(5.4) Remark. The second statement of Proposition (5.3) still holds true if $t \geqq 2$.

(5.5) We put

$$\Lambda_a = \{\tilde{x} \in P \mid \tilde{x} + W_0 \cap W_a \subset W_0\}$$

where $a = 2, -2$.

Because of (5.1) and (5.3) we can tell W_0 among the three components of Sing Ξ^* of dimension $g-5$. Hence $\Lambda_{-2} \cup \Lambda_2$ is intrinsically recovered from P. Our next aim is to determine Λ_{-2} and Λ_2.

(5.6) Proposition. *One has the equalities*:

$$\text{i)}\quad \Lambda_{-2} = \{\pi_1^*(\varepsilon_1^*(\bar{x}) - r - s) \mid \bar{x} \in E,\ r, s \in C_1,\ 2\bar{x} \equiv \varepsilon_1(r) + \varepsilon_1(s)\}\,,$$
$$\text{ii)}\quad \Lambda_2 = \{\pi_2^*(\varepsilon_2^*(\bar{x}) - r - s) \mid \bar{x} \in E,\ r, s \in C_2,\ 2\bar{x} \equiv \varepsilon_2(r) + \varepsilon_2(s)\}\,.$$

Proof. We only prove the second one, the first one being equivalent. Looking at (5.3) it is easy to check that the second member of this equality is contained in the first one (by (2.8) its elements belong to P). We show the opposite inclusion. Fix $\tilde{a} \in \Lambda_2$. By using (2.8) we may write $\tilde{a} = \pi_1^*(a_1) + \pi_2^*(a_2)$ with $\mathrm{Nm}_{\varepsilon_1}(a_1) = \mathrm{Nm}_{\varepsilon_2}(a_2) = 0$. Let $\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \in W_0 \cap W_2$ where $\zeta_1 \in A_1$, $\zeta_2 \in Z_2''$ and $\mathrm{Nm}_{\varepsilon_1}(\zeta_1) + \mathrm{Nm}_{\varepsilon_2}(\zeta_2) = \bar{\xi}$ (cf. (5.3)). Applying Lemma (4.4) there exist elements $\zeta_1' \in A_1$, $\zeta_2' \in A_2$ and $\bar{\alpha} \in \mathrm{Pic}^0(E)$ such that:

$$a_1 + \zeta_1 = \zeta_1' + \varepsilon_1^*(\bar{\alpha}) \in A_1\,,$$
$$a_2 + \zeta_2 = \zeta_2' - \varepsilon_2^*(\bar{\alpha}) \in A_2\,.$$

Therefore $a_1 + A_1 \subset A_1$ and $a_2 + Z_2'' \subset A_2$. Then by using (3.9) i) and (3.9) ii) we finish the proof. □

(5.7) Proposition. *Assume $t \geqq 4$. The sets $\Lambda_{-2} \cap 2\Lambda_{-2}$ and $\Lambda_2 \cap 2\Lambda_2$ are two symmetric irreducible curves. Their normalizations are C_1 and C_2, respectively, and τ_1 and τ_2 are the involutions induced by the (-1) map of P.*

Proof. We first observe that:

$$2\Lambda_{-2} = \{\pi_1^*(x + y - \tau_1(x) - \tau_1(y)) \mid x, y \in C_1\}\,,$$
$$2\Lambda_2 = \{\pi_2^*(x + y - \tau_2(x) - \tau_2(y)) \mid x, y \in C_2\}\,.$$

Now, it suffices to consider the set $\Lambda_{-2} \cap 2\Lambda_{-2}$. One has:

$$\Lambda_{-2} \cap 2\Lambda_{-2} = \{\pi_1^*(x - \tau_1(x)) \mid x \in C_1\}\,.$$

Indeed, since τ_1 has fixed points, $\pi_1^*(x - \tau_1(x)) \in 2\Lambda_{-2}$ for all $x \in C_1$. Moreover:

$$\pi_1^*(x - \tau_1(x)) = \pi_1^*(\varepsilon_1^*(\varepsilon_1(x)) - 2\tau_1(x)) \in \Lambda_{-2}\,.$$

So the right hand side member of the equality is contained in the left hand side member. To see the opposite inclusion, take $\bar{x} \in E$ and $r, s \in C_1$ such that $2\bar{x} \equiv \varepsilon_1(r) + \varepsilon_2(s)$ and suppose that $\pi_1^*(\varepsilon_1^*(\bar{x}) - r - s) \in 2\Lambda_{-2}$. We obtain a linear equivalence:

$$\pi_1^*(\varepsilon_1^*(\bar{x}) - r - s) \equiv \pi_1^*(y + z - \tau_1(y) - \tau_1(z))$$

where $y, z \in C_1$. Since π_1^* is injective:

$$\varepsilon_1^*(\bar{x}) + \tau_1(y) + \tau_1(z) \equiv y + z + r + s\,. \tag{5.8}$$

By assumption $t \geqq 4$ and then (3.1) implies that $h^0(\varepsilon_1^*(\bar{x}) + \tau_1(y) + \tau_1(z)) = 1$ iff $\tau_1(z) \neq y$. If $y = \tau_1(z)$ the initial element belongs to the right hand side member trivially. Thus we can assume that (5.8) is an equality of divisors and then either $y = \tau_1(z)$ or $y = \tau_1(y)$ or $z = \tau_1(z)$. In any case the inclusion follows.

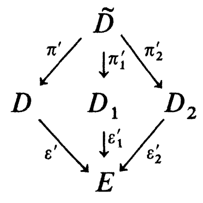
Now, taking the morphism

$$\begin{aligned} \varphi_1 \colon C_1 &\to \Lambda_{-2} \cap 2\Lambda_{-2}, \\ x &\to \pi_1^*(x - \tau_1(x)) \end{aligned}$$

the statement follows by observing that φ_1 is birational (C_1 is not hyperelliptic by (3.3)) and that $\varphi_1(\tau_1(x)) = -\varphi_1(x)$. □

(5.9) Let $\pi' \colon \tilde{D} \to D$ be an unramified double cover of smooth curves such that $P(\tilde{D}, D) \cong P$. Since the singular locus of the theta divisor of P has dimension $g - 5 = \dim P - 4$, D is either trigonal or bi-elliptic (cf. [Mu1], p. 344). If D is trigonal P is the Jacobian of a curve (cf. [Re]). Then, by [Sh1] C has to be either hyperelliptic or trigonal, which contradicts either (3.2) or (3.3). Thus D is bi-elliptic.

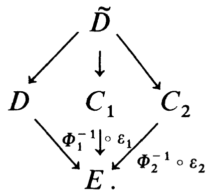
Moreover, table (2.7) and observation (2.11) show that $(\tilde{D}, D) \in \mathscr{R}_{B,g,s}$ with $s \geqq 4$. Let D_1 and D_2 be the bi-elliptic curves of genus $s+1$ and $g-s$ attached to $(\tilde{D}, D)$ in the usual way (cf. (2.1)). Since as we have seen in (5.7), (C_1, τ_1) and (C_2, τ_2) can be recovered from P, one has isomorphisms $\varphi_i \colon D_i \to C_i$, $i = 1, 2$, commuting with the corresponding involutions. In particular the base elliptic curve is the same and $s = t$. Summarizing, if the diagram attached to $(\tilde{D}, D)$ is:



there exist $\Phi_i \in \mathrm{Aut}(E)$, $i = 1, 2$, such that

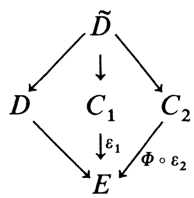
$$\begin{array}{ccc} D_i & \xrightarrow{\varphi_i} & C_i \\ {\scriptstyle \varepsilon'_i}\downarrow & & \downarrow{\scriptstyle \varepsilon_i} \\ E & \xrightarrow[\Phi_i]{} & E. \end{array}$$

Thus we obtain a diagram



Composing with a suitable automorphism of E we get

(5.10)



where $\Phi \in \mathrm{Aut}(E)$ and $\Phi(\bar{P}_i) \neq \bar{P}_j$, for all $1 \leqq i \leqq 2t < j \leqq 2g-2$.

(5.11) Theorem. *Let $(\tilde{C}, C)$ be a general element of $\mathcal{R}_{B,g,t}$ with $t \geqq 4$ and $g \geqq 10$. Let $(\tilde{D}, D) \in \mathcal{R}_g$ such that $P(\tilde{D}, D) \cong P(\tilde{C}, C)$. Then $(\tilde{D}, D) \in \mathcal{R}_{B,g,t}$ and $(\tilde{C}, C)$ and $(\tilde{D}, D)$ are tetragonally related.*

Proof. By (5.9) it only remains to see that each diagram (5.10) can be obtained by applying successively the tetragonal construction starting from the initial element $(\tilde{C}, C)$. By (2.10) it suffices to see the following fact:

Lemma. *Assume that E is general. Then the set*

$$\Gamma = \{\Phi \in \mathrm{Aut}(E) \mid \Phi(\bar{P}_i) \neq \bar{P}_j, \text{ for } 1 \leqq i \leqq 2t < j \leqq 2g-2\}$$

is generated multiplicatively by the elements of Γ that correspond to the linear series g_2^1 of E.

Proof. Left to the reader. □

6. The component $\mathcal{R}_{B,g,3}$. This section is devoted to proving the analogue of the Theorem (5.11) for the component $\mathcal{R}_{B,g,3}$. We begin with a general result valid for any t.

(6.1) Lemma. *One has the equalities (cf. § 1 and (2.8) for notations, part* iii) *will not be needed here, but later on):*

i) $I(W_2) = \pi_1^*(P_1)$ *for* $t \geqq 1$,

ii) $I(W_0) = \pi^*(\varepsilon^*({}_2JE))$ *for* $t \geqq 2$,

iii) $I(W_{-2}) = \pi_2^*(P_2)$ *for* $t \geqq 4$.

Proof. We show first the equality i). To prove the inclusion of $\pi_1^*(P_1)$ in the left hand side member we consider $\pi_1^*(\beta) \in \pi_1^*(P_1)$ and we take an element $\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \in W_2$ where $\zeta_1 \in \Theta_1^*$, $\zeta_2 \in Z_2''$ and $\mathrm{Nm}_{\varepsilon_1}(\zeta_1) + \mathrm{Nm}_{\varepsilon_2}(\zeta_2) = \bar{\xi}$. Since the map

$$\mathrm{Pic}^0(E) \times C_1^{(t)} \to \mathrm{Pic}^t(C_1)$$

is surjective, we may write

$$\beta + \zeta_1 \equiv \zeta_1' + \varepsilon_1^*(\bar{\varrho}), \quad \text{where} \quad \zeta_1' \in \Theta_1^*, \bar{\varrho} \in \mathrm{Pic}^0(E).$$

Then $\pi_1^*(\beta) + \pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) = \pi_1^*(\zeta_1') + \pi_2^*(\zeta_2 + \varepsilon_2^*(\bar{\varrho})) \in W_2$. To see the opposite inclusion take $\tilde{a} = \pi_1^*(a_1) + \pi_2^*(a_2) \in P$ with $a_1 \in P_1$, $a_2 \in P_2$ and such that $\tilde{a} + W_2 \subset W_2$. By applying Lemma (4.4) as before (see for instance the proof of (5.6)) we get $a_2 + Z_2'' \subset Z_2''$. By (3.9) iii) there exists $\bar{\alpha} \in_2 JE$ such that $a_2 = \varepsilon_2^*(\bar{\alpha})$. Therefore $\tilde{a} \in \pi_1^*(P_1)$.

In ii), the inclusion of the right hand side member in the left hand side member is obvious. Take now $\tilde{a} = \pi_1^*(a_1) + \pi_2^*(a_2)$ with $a_1 \in P_1$ and $a_2 \in P_2$. Assume that $\tilde{a} + W_0 \subset W_0$. Again as a consequence of Lemma (4.4) one has $a_1 + A_1 \subset A_1$ and $a_2 + A_2 \subset A_2$. By using (3.9) i) we obtain that $\tilde{a} \in \pi^*(\varepsilon^*(JE))$. This ends the proof of the inclusion $I(W_0) \subset \pi^*(\varepsilon^*({}_2JE))$.

Part iii) is analogous to part i). □

We now assume $t = 3$. Let $(\tilde{C}, C)$ be a general element of $\mathscr{R}_{B,g,3}$. There are two components of dimension $g - 5$ in $\mathrm{Sing}\,\Xi^*$: W_0 and W_2 (cf. (2.7)). Lemma (6.1) shows that we may distinguish between W_0 and W_2 because the dimension of $I(W_0)$ and $I(W_2)$ are different.

(6.2) Proposition. *One has*:

$$\bigcup_{\tilde{\zeta} \in W_0} ((W_0)_{-\tilde{\zeta}} \cap \pi_1^*(P_1)) = \{\pi_1^*(\varepsilon_1^*(\bar{x}) - r - s) \mid \bar{x} \in E;\, r, s \in C_1 \text{ and } 2\bar{x} \equiv \varepsilon_1(r) + \varepsilon_1(s)\}.$$

Proof. Let $\tilde{\zeta} = \pi_1^*(\varepsilon_1^*(\bar{z}) + r) + \pi_2^*(\zeta_2)$ be an element of W_0, where $\bar{z} \in E$, $r \in C_1$, $\zeta_2 \in A_2$ and such that $\mathrm{Nm}_{\varepsilon_1}(\varepsilon_1^*(\bar{z}) + r) + \mathrm{Nm}_{\varepsilon_2}(\zeta_2) = \bar{\xi}$. Suppose $a_1 \in P_1$ satisfies $\pi_1^*(a_1) + \tilde{\zeta} \in W_0$. By Lemma (4.4) this implies that $a_1 + \varepsilon_1^*(\bar{z}) + r \in A_1$. Hence $a_1 = r' - r + \varepsilon_1^*(\bar{\alpha})$ where $\bar{\alpha} \in JE$, $r, r' \in C_1$. By replacing $\bar{\alpha}$ by $\bar{x} - \varepsilon_1(r')$ for some $\bar{x} \in E$ we get

$$(6.3) \qquad (W_0)_{-\tilde{\zeta}} \cap \pi_1^*(P_1) \subset \{\pi_1^*(\varepsilon_1^*(\bar{x}) - r - s) \mid \bar{x} \in E, s \in C_1 \text{ and } 2\bar{x} \equiv \varepsilon_1(r) + \varepsilon_1(s)\}.$$

The inclusion of the right hand side member in the left hand side member in (6.3) is trivial. The equality in (6.3) clearly implies the equality we wanted to prove. □

(6.4) Theorem. *Let $(\tilde{C}, C)$ be a generic element of $\mathscr{R}_{B,g,3}$ with $g \geqq 10$ and let $(\tilde{D}, D) \in \mathscr{R}_g$ such that $P(\tilde{D}, D) \cong P(\tilde{C}, C)$. Then $(\tilde{D}, D) \in \mathscr{R}_{B,g,3}$ and $(\tilde{C}, C)$ and $(\tilde{D}, D)$ are tetragonally related.*

Proof. First we observe that the methods used in the section 5 (i.e.: for $(\tilde{C}, C) \in \mathscr{R}_{B,g,t}$, $t \geqq 4$) to recover the set of data (C_2, τ_2) are still valid (cf. (5.4), (5.6) ii) and (5.7)). On the other hand we have seen in (6.2) how to recognize intrinsically in P the set

$$\{\pi_1^*(\varepsilon_1^*(\bar{x}) - r - s) \mid \bar{x} \in E, r, s \in C_1 \text{ and } 2\bar{x} \equiv \varepsilon_1(r) + \varepsilon_1(s)\}.$$

Since it coincides with the set obtained in (5.6) i) we can also imitate the process given in (5.7) to obtain the set of data (C_1, τ_1). Then the proof continues as in (5.11). □

7. The component $\mathscr{R}_{B,g,2}$. In this paragraph we wish to prove the analogue of Theorem (5.11) for the component $\mathscr{R}_{B,g,2}$. In addition to the ideas of § 5 we shall use some intersections $\Xi^* \cap \Xi^*_{\tilde{a}}$ to recover (C_1, τ_1). We keep the assumptions and notation of § 1 and § 2.

Let us denote by $(\tilde{C}, C)$ a general element of $\mathscr{R}_{B,g,2}$. From (2.6) and (2.7) we may suppose that:

$$\operatorname{Sing} \Xi^* = W_0 \cup W_2 .$$

(7.1) Because of (6.1) we can make a difference between both components.

(7.2) Remark. Imitating § 5, one gets from P^* the pair (C_2, τ_2) and the subvariety $\pi_1^*(P_1)$.

We shall now describe a subvariety of $\pi_1^*(P_1)$ that determines the curve C_1.

(7.3) Proposition. *One has the following equalities*:

i) *If $\tilde{a} = \pi_2^*(x - \tau_2(x))$, where $x \in C_2$, then $\Xi^* \cap \Xi^*_{\tilde{a}} = F \cup X(\tilde{a})$, where*

$$X(\tilde{a}) = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in \Theta_1^*, \zeta_2 \in \Theta_2^*, h^0(\zeta_2 - x) > 0 \text{ and } \operatorname{Nm}_{\varepsilon_1}(\zeta_1) + \operatorname{Nm}_{\varepsilon_2}(\zeta_2) = \xi\}$$

is the moving part of this algebraic system and F is the fixed part (see below for a description of F).

ii) *Let $N = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in \Theta_1^*, \operatorname{Nm}_{\varepsilon_1}(\zeta_1) = \xi_1, \zeta_2 \in Z_2'\}$. Then*:

$$\bigcap_{\tilde{a} \in \Lambda_2 \cap 2\Lambda_2 - \{0\}} X(\tilde{a}) = W_0 \cup W_2 \cup N,$$

and N is the union of the irreducible components not contained in $W_0 \cup W_2$.

iii) *If $\tilde{a} = \pi_1^*(a_1)$, where $a_1 \in P_1 - \{0\}$, then*:

$$N \cap \Xi^*_{\tilde{a}} = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in \Theta_1^* \cap (\Theta_1^*)_{a_1}, \operatorname{Nm}_{\varepsilon_1}(\zeta_1) = \xi_1, \zeta_2 \in Z_2'\}$$
$$\cup \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in \Theta_1^*, \operatorname{Nm}_{\varepsilon_1}(\zeta_1) = \xi_1, \zeta_2 \in Z_2' \cap Z_2''\}.$$

Proof. i) Let $\tilde{\zeta} = \pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \in \Xi^* \cap \Xi_{\tilde{a}}^*$ with $\tilde{a} = \pi_2^*(x - \tau_2(x))$. By applying Lemma (4.4) we find elements $\zeta_1' \in \Theta_1^*, \zeta_2' \in \Theta_2^*$ and $\bar{\varrho} \in \mathrm{Pic}^0(E)$ such that:

$$\zeta_1 \equiv \zeta_1' + \varepsilon_1^*(\bar{\varrho}), \qquad (7.4)$$
$$\tau_2(x) - x + \zeta_2 \equiv \zeta_2' - \varepsilon_2^*(\bar{\varrho}).$$

Suppose first that $\bar{\varrho} = 0$. Then

$$\zeta_2 \in \Theta_2^* \cap (\Theta_2^*)_{x - \tau_2(x)} = \{\zeta_2 \in \Theta_2^* \mid h^0(\zeta_2 - x) > 0\} \cup \{\zeta_2 \in \Theta_2^* \mid h^0(\zeta_2 + \tau_2(x)) \geqq 2\}.$$

If ζ_2 belongs to the second set, by Riemann-Roch one has

$$h^0(K_{C_2} - \zeta_2 - \tau_2(x)) > 0.$$

Define $\bar{\lambda} = \bar{\xi}_2 - \mathrm{Nm}_{\varepsilon_2}(\zeta_2)$, $\beta_1 = \zeta_1 - \varepsilon_1^*(\bar{\lambda})$ and $\beta_2 = \zeta_2 + \varepsilon_2^*(\bar{\lambda})$. Then

$$h^0(\beta_1) = h^0(\zeta_1 - \varepsilon_1^*(\bar{\xi}_2 - \mathrm{Nm}_{\varepsilon_2}(\zeta_2))) = h^0(-\tau_1^*(\zeta_1) + \varepsilon_1^*(\bar{\xi}_1)) = h^0(\tau_1(\zeta_1)) > 0,$$
$$h^0(\beta_2 - x) = h^0(K_{C_2} - \tau_2^*(\zeta_2) - x) = h^0(K_{C_2} - \zeta_2 - \tau_2(x)) > 0.$$

Therefore $\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) = \pi_1^*(\beta_1) + \pi_2^*(\beta_2) \in X(\tilde{a})$.

On the other hand if $\bar{\varrho} \neq 0$ then (cf. [De4], p. 9)

$$\zeta_1 \in \Theta_1^* \cap (\Theta_1^*)_{\varepsilon_1^*(\bar{\varrho})} = A_1 \cup \{\zeta_1 \in \Theta_1^* \mid \mathrm{Nm}_{\varepsilon_1}(\zeta_1) = \bar{\xi}_1 + \bar{\varrho}\}.$$

If $\mathrm{Nm}_{\varepsilon_1}(\zeta_1) = \bar{\xi}_1 + \bar{\varrho}$ then $\bar{\varrho} = \mathrm{Nm}_{\varepsilon_1}(\zeta_1) - \bar{\xi}_1 = \bar{\xi}_2 - \mathrm{Nm}_{\varepsilon_2}(\zeta_2)$ and by replacing in (7.4) one has

$$\tau_2^*(\zeta_2) + x - \tau_2(x) \equiv K_{C_2} - \zeta_2'.$$

Thus $\zeta_2 \in \Theta_2^* \cap (\Theta_2^*)_{x - \tau_2(x)}$ and proceeding as above we conclude that $\tilde{\zeta} \in X(\tilde{a})$. We have proved the inclusion $\Xi^* \cap \Xi_{\tilde{a}}^* \subset F \cup X(\tilde{a})$, where

$$F = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in A_1, \zeta_2 \in \Theta_2^*, \mathrm{Nm}_{\varepsilon_1}(\zeta_1) + \mathrm{Nm}_{\varepsilon_2}(\zeta_2) = \bar{\xi}\}$$

(note that $F = \emptyset$ if $t \leqq 1$). The inclusion of $X(\tilde{a})$ in the left hand side member is trivial. Take now $\tilde{\zeta} = \pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \in F$. Since the map

$$\mathrm{Pic}^0(E) \times C_2^{(g-3)} \to \mathrm{Pic}^{g-3}(C_2),$$
$$(\bar{\alpha}, D) \to \varepsilon_2^*(\bar{\alpha}) + D$$

is surjective we can write

$$x - \tau_2(x) + \zeta_2 \equiv D + \varepsilon_2^*(\bar{\alpha})$$

and then $\pi_2^*(x - \tau_2(x)) + \tilde{\zeta} \equiv \pi_1^*(\zeta_1 + \varepsilon_1^*(\bar{\alpha})) + \pi_2^*(D) \in \Xi^*$.

The reader may observe that F and $X(\tilde{a})$ have pure dimension $g-3$ and that $\dim(F \cap X(\tilde{a})) = g-4$ for all $\tilde{a}$. This concludes the proof of i).

ii) The inclusion

$$W_0 \cup W_2 \cup N \subset \bigcap_{\tilde{a} \in \Lambda_2 \cap 2\Lambda_2 - \{0\}} X(\tilde{a})$$

is left to the reader.

To see the opposite inclusion let $\tilde{\zeta} = \pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \in X(\tilde{a})$ for all $\tilde{a} \in \Lambda_2 \cap 2\Lambda_2 - \{0\}$. Then for all $x \in C_2$ there exist $\zeta_1' \in \Theta_1^*$, $\zeta_2' \in \Theta_2^*$ and $\bar{\varrho} \in \mathrm{Pic}^0(E)$ such that

$$\begin{aligned} &h^0(\zeta_2' - x) > 0\,, \\ &\zeta_1 \equiv \zeta_1' + \varepsilon_1^*(\bar{\varrho})\,, \\ &\zeta_2 \equiv \zeta_2' - \varepsilon_2^*(\bar{\varrho})\,. \end{aligned}$$

There exists an irreducible component T of the fibre over ζ_2 of the map

$$\begin{aligned} \mathrm{Pic}^0(E) \times C_2 \times C_2^{(g-4)} &\to \mathrm{Pic}^{g-3}(C_2)\,, \\ (\bar{\varrho}, x, D) &\to x + D - \varepsilon_2^*(\bar{\varrho}) \end{aligned}$$

which dominates C_2. Suppose that the projection $T \to \mathrm{Pic}^0(E)$ is constant and let $\bar{\varrho}_0$ be the image. Then for all $x \in C_2$ we find an effective divisor D such that:

$$\zeta_2 \equiv x + D - \varepsilon_2^*(\bar{\varrho}_0)\,.$$

Therefore $h^0(\zeta_2 + \varepsilon_2^*(\bar{\varrho}_0) - x) > 0$ for all $x \in C_2$ and hence $\zeta_2 \in \mathrm{Sing}\,\Theta_2^* = Z_2' \cup Z_2''$. So $\tilde{\zeta}$ belongs to $W_2 \cup N$.

If $T \to \mathrm{Pic}^0(E)$ is surjective we find that

$$h^0(\zeta_2 + \varepsilon_2^*(\bar{\varrho})) > 0$$

for all $\bar{\varrho} \in \mathrm{Pic}^0(E)$. Hence $\zeta_2 \in A_2$. Now it is not hard to deduce that $\tilde{\zeta} \in W_0 \cup W_2$.

From the descriptions it is clear that no components of N are contained in $W_0 \cup W_2$. This finishes the proof of ii).

iii) The inclusion of the right hand side member in the left hand side member is left to the reader. To see the opposite inclusion let $\zeta_1 \in \Theta_1^*$ with $\mathrm{Nm}_{\varepsilon_1}(\zeta_1) = \xi_1$ and $\zeta_2 \in Z_2'$ and suppose that

$$\pi_1^*(-a_1 + \zeta_1) + \pi_2^*(\zeta_2) \in \Xi^*\,.$$

Again there exist $\zeta_1' \in \Theta_1^*$, $\zeta_2' \in \Theta_2^*$ and $\bar{\varrho} \in \mathrm{Pic}^0(E)$ with

$$-a_1 + \zeta_1 \equiv \zeta_1' + \varepsilon_1^*(\bar{\varrho}),$$
$$\zeta_2 \equiv \zeta_2' - \varepsilon_2^*(\bar{\varrho}).$$

If $\bar{\varrho} = 0$ then $\zeta_1 \in \Theta_1^* \cap (\Theta_1^*)_{a_1}$. On the other hand $\bar{\varrho} \neq 0$ implies that

$$\zeta_2 \in \Theta_2^* \cap (\Theta_2^*)_{-\varepsilon_2^*(\bar{\varrho})} = A_2 \cup \{\zeta_2 \in \Theta_2^* \mid \mathrm{Nm}_{\varepsilon_2}(\zeta_2) = \bar{\xi}_2 - \bar{\varrho}\}.$$

Since $\zeta_2 \in Z_2'$, only $\xi_2 \in A_2$ is possible and then $\xi_2 \in Z_2' \cap Z_2''$. □

(7.5) We shall define for $\tilde{a} = \pi_1^*(a_1)$, $a_1 \in P_1 - \{0\}$

$$N(\tilde{a}) = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in \Theta_1^* \cap (\Theta_1^*)_{a_1},\ \mathrm{Nm}_{\varepsilon_1}(\zeta_1) = \bar{\xi}_1, \zeta_2 \in Z_2'\}.$$

This set is recovered from $N \cap \Xi_a^*$ as the union of the components not contained in W_2. Our next goal is to distinguish points in $\pi_1^*(P_1)$ looking at the number of components of $N(\tilde{a})$. We will see below that the set $\Theta_1^* \cap (\Theta_1^*)_{a_1} \cap \mathrm{Nm}_{\varepsilon_1}^{-1}(\bar{\xi}_1)$ is finite. The cardinal of this set coincides with the number of irreducible components of $N(\tilde{a})$.

(7.6) Let D be the ample divisor induced by Θ_1 on the abelian surface P_1. By Riemann-Roch

$$h^0(D) = \frac{D^2}{2} \quad \text{and} \quad h^0(D)^2 = \deg(\lambda_D).$$

By using [Mu1], p. 330 we obtain $\deg(\lambda_D) = 4$ and therefore $D^2 = 4$.

(7.7) Let Σ be the curve given by the pull-back diagram:

$$\begin{array}{ccc} \Sigma & \longrightarrow & C_1^{(2)} \\ \downarrow & & \downarrow \varepsilon_1^{(2)} \\ |\bar{\xi}_1| & \longrightarrow & E^{(2)} \end{array}$$

the horizontal arrows being inclusions. Since C_1 is general it is easy to obtain (cf. § 11) that Σ is a smooth curve of genus 3 and the quotient $\Sigma / \tau_1^{(2)}$ is an elliptic curve not isomorphic to E.

We shall denote by Σ_0 the image of the map

$$\Sigma \to P_1,$$
$$x + y \to x + y - \tau_1(x) - \tau_1(y).$$

(7.8) Proposition. *One has:*

$$\{\tilde{a} \in \pi_1^*(P_1) \mid \textit{number comp. } N(\tilde{a}) < 4\} = \Pi \cup \pi_1^*(\Sigma_0)$$

where $\Pi = \{\pi_1^*(x - \tau_1(x)) \mid x \in C_1\}$.

Proof. By (7.5) we must study the cardinal of the set $\Theta_1^* \cap (\Theta_1^*)_{a_1} \cap \mathrm{Nm}_{\varepsilon_1}^{-1}(\bar{\xi}_1)$ when $a_1 \in P_1$. It is easy to see the inclusion of Π in the left hand side member. To prove the rest of the statement we shall need the following properties of the quartic plane curve C_1:

- The lines determined by the divisors $\varepsilon_i^*(\bar{x})$ with $\bar{x} \in E$ all pass through a common point $O \in \mathbb{P}^2$, where $O \notin C_1$. (In fact $O = \mathbb{P}(H^0(E, \mathcal{O}_E(\bar{\xi}_1))^\perp) \subset \mathbb{P}H^0(C_1, K_{C_1})^*$).

- The ramification points $P_1', \ldots, P_4'$ of ε_1 belong to a line l and $O \notin l$.

- If $x, y \in C_1$ verify $\varepsilon_1(x) + \varepsilon_1(y) \equiv \bar{\xi}_1$ then $O \in \overline{xy}$.

Take now a point $x + y \in \Theta_1^* \cap (\Theta_1^*)_{a_1} \cap \mathrm{Nm}_{\varepsilon_1}^{-1}(\bar{\xi}_1)$. The following equalities are well-known:

$$\overline{xy} = \mathbb{P}T_{\Theta_1^*}(x+y) \subset \mathbb{P}T_{JC_1}(x+y) \cong \mathbb{P}H^0(C_1, K_{C_1})^*,$$
$$\overline{rs} = \mathbb{P}T_{(\Theta_1^*)_{a_1}}(x+y) \quad \text{where} \quad r + s \in |x + y - a_1|.$$

Since $\varepsilon_1(x) + \varepsilon_1(y) \equiv \varepsilon_1(r) + \varepsilon_1(s) \equiv \bar{\xi}_1$ both lines pass through O. They are equal iff the following equality of divisors holds $x + y + \tau_1(x) + \tau_1(y) = r + s + \tau_1(r) + \tau_1(s)$, that is to say iff $\pi_1^*(a_1) \in \Pi \cup \pi_1^*(\Sigma_0)$.

Assume first that $\pi_1^*(a_1) \notin \Pi \cup \pi_1^*(\Sigma_0)$. In this case the curve $\Theta_1^* \cap (\Theta_1^*)_{a_1}$ is not singular at $x + y$ and it suffices to show that $O \notin \mathbb{P}T_{P_1}(0)$ in order to obtain transversality in the intersection. Indeed:

$$T_{P_1}(0) = (H^0(C_1, K_{C_1})^-)^* = H^0(E, \mathcal{O}_E(\bar{\xi}_1))^* = H^0(E, \mathcal{O}_E)^\perp \subset H^0(C_1, K_{C_1})^*.$$

On the other hand, if s_R is an equation for the ramification divisor $R = \sum_{i=1}^{4} P_i'$ then the inclusion

$$H^0(E, \mathcal{O}_E) \hookrightarrow H^0(C_1, K_{C_1}),$$
$$s \to \varepsilon_1^*(s) s_R$$

induces an equality $\mathbb{P}H^0(E, \mathcal{O}_E) = \{R\}$. By dualizing we get $O \notin l = \mathbb{P}T_{P_1}(0)$. Observe in particular that it follows that the set $\Theta_1^* \cap (\Theta_1^*)_{a_1} \cap \mathrm{Nm}_{\varepsilon_1}^{-1}(\bar{\xi}_1)$ is finite. Combining (7.6) with transversality we find

$$\pi_1^*(a_1) \notin \Pi \cup \pi_1^*(\Sigma_0) \Rightarrow \text{number comp. } N(\tilde{a}) = 4.$$

Finally if $a_1 \in \Sigma_0$ then $\mathbb{P}T_{\Theta_1^*}(x+y) = \mathbb{P}T_{(\Theta_1^*)_{a_1}}(x+y)$. Thus $\Theta_1^* \cap (\Theta_1^*)_{a_1}$ is singular at $x+y$.

Therefore $a_1 \in \Sigma_0 \Rightarrow$ number comp. $N(\tilde{a}) < 4$. □

(7.9) Theorem. *Let $(\tilde{C}, C)$ be a generic element of $\mathscr{R}_{B,g,2}$ with $g \geqq 10$ and let $(\tilde{D}, D) \in \mathscr{R}_g$ be such that $P(\tilde{D}, D) \cong P(\tilde{C}, C)$. Then $(\tilde{D}, D) \in \mathscr{R}_{B,g,2}$ and $(\tilde{D}, D)$ and $(\tilde{C}, C)$ are tetragonally related.*

Proof. In view of the proof of (5.11) it suffices to show how to recognize (C_1, τ_1) and (C_2, τ_2) from P. Observe that (7.2) says how to recover (C_2, τ_2). In particular we recover the curve E. By combining (7.1), (7.2), (7.3) and (7.8) we recover the set $\Pi \cup \pi_1^*(\Sigma_0)$ intrinsically. By (7.7) the normalization of $\pi_1^*(\Sigma_0)$ is an irreducible curve of genus $\leqq 3$. If it has genus < 3 we distinguish Π as the component of the set with normalization of genus 3. Otherwise since the quotient of Σ by the involution given by symmetry is not isomorphic to E we also recover Π. Now by normalizing the symmetric curve Π we obtain (C_1, τ_1). □

8. The component $\mathscr{R}_{B,g,1}$.

In this section $(\tilde{C}, C)$ is a general element of $\mathscr{R}_{B,g,1}$. By (2.6) and (2.7) we can assume that $\operatorname{Sing} \Xi^* = W_2$ is irreducible of dimension $g-5$.

(8.1) Proposition. *One has the following equality:*

$$\{\tilde{a} \in P \mid \tilde{a} + W_2 \subset \Xi^*\}$$
$$= \{\pi_1^*(a_1) + \pi_2^*(\varepsilon_2^*(\bar{x}) - r - s) \mid a_1 \in P_1, \bar{x} \in E, r, s \in C_2, 2\bar{x} \equiv \varepsilon_2(r) + \varepsilon_2(s)\}.$$

Proof. The inclusion of the second set in the first one is clear. To see the opposite inclusion take $\tilde{a} = \pi_1^*(a_1) + \pi_2^*(a_2) \in P$ where $a_1 \in P_1$, $a_2 \in P_2$ and such that $\tilde{a} + W_2 \subset \Xi^*$. Let $\zeta = \pi_1^*(x) + \pi_2^*(\zeta_2) \in W_2$, with $x \in C_1$, $\zeta_2 \in Z_2''$ and $\varepsilon_1(x) + \operatorname{Nm}_{\varepsilon_2}(\zeta_2) = \xi$. By applying Lemma (4.4) one finds elements $x' \in C_1$, $\zeta_2' \in W_{g-2}^0(C_2)$ and $\bar{\varrho} \in \operatorname{Pic}^0(E)$ such that

$$\begin{aligned} a_1 + x &\equiv x' + \varepsilon_1^*(\bar{\varrho}), \\ a_2 + \zeta_2 &\equiv \zeta_2' - \varepsilon_2^*(\bar{\varrho}). \end{aligned} \tag{8.2}$$

Let us define the following subvariety of $C_1 \times Z_2''$

$$Y = \{(x, \zeta_2) \in C_1 \times Z_2'' \mid \varepsilon_1(x) + \operatorname{Nm}_{\varepsilon_2}(\zeta_2) \equiv \xi\}.$$

Consider now the morphism:

$$\begin{aligned} \Psi\colon \operatorname{Pic}^0(E) \times C_1 \times C_2^{(g-2)} &\to \operatorname{Pic}^1(C_1) \times \operatorname{Pic}^{g-2}(C_2), \\ (\bar{\varrho}, x', D) &\to (x' + \varepsilon_1^*(\bar{\varrho}) - a_1, D - \varepsilon_2^*(\bar{\varrho}) - a_2). \end{aligned}$$

The equivalences of (8.2) read: $Y \subset \operatorname{Im}(\Psi)$. Since Y is irreducible (apply (3.8) to the fibres of the projection map from Y to C_1) there exists an irreducible component X of $\Psi^{-1}(Y)$ such that the induced map

$$\tilde{\Psi}\colon X \to Y$$

is dominant. If $q\colon X \to \mathrm{Pic}^0(E)$ is the first projection we call $Y_{\bar\varrho} := \tilde\Psi(q^{-1}(\bar\varrho))$ for all $\bar\varrho \in \mathrm{Pic}^0(E)$. Two cases are possible:

$$\begin{aligned} &\text{either a)} \quad Y_{\bar\varrho} = Y \quad \text{for some} \quad \bar\varrho \in \mathrm{Pic}^0(E)\\ &\text{or b)} \quad Y_{\bar\varrho} \neq Y \quad \text{for all} \quad \bar\varrho \in \mathrm{Pic}^0(E)\,. \end{aligned}$$

In case a) define

$$b_1 = a_1 - \varepsilon_1^*(\bar\varrho) \quad \text{and} \quad b_2 = a_2 + \varepsilon_2^*(\bar\varrho)\,.$$

Then (8.2) says:

$$h^0(b_1 + x) > 0\,, \quad h^0(b_2 + \zeta_2) > 0 \quad \text{for all} \quad (x, \zeta_2) \in Y\,.$$

Hence $b_1 = 0$ and $b_2 + Z_2'' \subset \Theta_2^*$. Therefore by using (3.9) ii) we finish the proof.

In case b) we write $\lambda\colon Y \to C_1 \subset \mathrm{Pic}^1(C_1)$ for the first projection. We claim that $\lambda_{|Y_{\bar\varrho}}$ is non-surjective for general $\bar\varrho \in \mathrm{Pic}^0(E)$. Otherwise for all $x \in C_1$ one finds an element $\zeta_2 \in Z_2''$ such that $(x, \zeta_2) \in Y_{\bar\varrho}$. In particular $h^0(a_1 + x - \varepsilon_1^*(\bar\varrho)) > 0$ and $a_1 = \varepsilon_1^*(\bar\varrho)$, which cannot hold for a general $\bar\varrho$.

Now since $Y_{\bar\varrho}$ has codimension 1 in Y, it follows from the claim that, for a general $\bar\varrho$, there exists $x_0 \in C_1$ such that $\lambda^{-1}(x_0) \subset Y_{\bar\varrho}$. Hence (8.2) reads:

$$h^0(a_1 + x_0 - \varepsilon_1^*(\bar\varrho)) > 0 \quad \text{and} \quad h^0(a_2 + \zeta_2 + \varepsilon_2^*(\bar\varrho)) > 0$$

for all $\zeta_2 \in Z_2''$ with $\mathrm{Nm}_{\varepsilon_2}(\zeta_2) = \bar\xi - \varepsilon_1(x_0)$. In particular

$$a_2 + \varepsilon_2^*(\bar\varrho) + \{\zeta_2 \in Z_2'' \,|\, \mathrm{Nm}_{\varepsilon_2}(\zeta_2) \equiv \bar\xi - \varepsilon_1(x_0)\} \subset \Theta_2^*\,.$$

The proof ends by observing that

$$\{\zeta_2 \in Z_2'' \,|\, \mathrm{Nm}_{\varepsilon_2}(\zeta_2) = \bar\xi - \varepsilon_1(x_0)\} = \varepsilon_2^*(\bar\alpha) + Z_2' \cap Z_2''$$

where $2\bar\alpha = \bar\xi_1 - \varepsilon_1(x_0)$, and applying (3.9) ii). □

We shall denote by B the set described in (8.1).

(8.3) Proposition. *The abelian variety $\pi_1^*(P_1)$ acts on $B \cap 2B$ by translation and the quotient*

$$\frac{B \cap 2B}{\pi_1^*(P_1)} \subset \frac{P}{\pi_1^*(P_1)}$$

is a symmetric curve with normalization C_2. The reflection on $P/\pi_1^*(P_1)$ induces on C_2 the involution τ_2.

Proof. By using the arguments of § 5 one has:

$$B \cap 2B = \{\pi_1^*(a_1) + \pi_2^*(x - \tau_2(x)) \mid a_1 \in P_1, x \in C_2\}.$$

Now the morphism

$$\lambda\colon C_2 \to \frac{B \cap 2B}{\pi_1^*(P_1)},$$

$$x \to \overline{\pi_2^*(x - \tau_2(x))}$$

is birational and verifies $\lambda(\tau_2(x)) = -\lambda(x)$. □

(8.4) The reader can prove without much work the following properties:

- $P_1 \subset JC_1$ is an elliptic curve.
- The morphism

$$\mu\colon C_1 \to P_1,$$

$$x \to x - \tau_1(x)$$

is a double cover with two ramification points inducing on C_1 a new bi-elliptic structure. The attached involution τ_1' is the composition of τ_1 with the hyperelliptic involution.

- We shall write Q_1 and Q_2 for the fixed points of τ_1' and P_1', P_2' for the ramification points of ε_1. With the notations of (2.1):

$$Q_1 + Q_2 \equiv P_1' + P_2' \equiv K_{C_1}$$

and

$$\tau_1(Q_1) = Q_2, \quad \varepsilon_1(Q_1) = \varepsilon_1(Q_2) \in |\xi_1|,$$
$$\tau_1'(P_1') = P_2', \quad \mu(P_1') = \mu(P_2') = 0.$$

We write $\bar{Q}_1 = \mu(Q_1)$ and $\bar{Q}_2 = \mu(Q_2)$. Let $\bar{P}_0$ the element of $|\xi_1|$.

- Note that $\bar{Q}_1 = \mu(Q_1) = Q_1 - \tau_1(Q_1) = -(Q_2 - \tau_1(Q_2)) = -\mu(Q_2) = -\bar{Q}_2$. Moreover $\mu^*(0) = P_1' + P_2' \equiv Q_1 + Q_2$.

Summarizing we obtain (composing with $\pi_1^*\colon P_1 \to \pi_1^*(P_1)$) that C_1 can be represented as the double cover of $\pi_1^*(P_1)$ associated to the class of the origin (as a point of the abelian subvariety of P) and the discriminant divisor $\pi_1^*(\bar{Q}_1) + \pi_1^*(\bar{Q}_2)$. Since the class is trivially recovered from $\pi_1^*(P_1)$, we only need to find the divisor inside P. Moreover the involution τ_1 will appear when composing the canonical involution of C_1 with the involution attached to this cover.

(8.5) Proposition. *Let $\tilde{a} = \pi_1^*(x - \tau_1(x)) \neq 0$ where $x \in C_1$. Then:*

i) $\Xi^* \cap \Xi_a^* = F' \cup R(\tilde{a})$ *where*

$$F' = \{\pi_1^*(y) + \pi_2^*(\zeta_2) \mid y \in C_1, \zeta_2 \in A_2, \varepsilon_1(y) + \mathrm{Nm}_{\varepsilon_2}(\zeta_2) \equiv \xi\},$$
$$R(\tilde{a}) = \{\pi_1^*(x) + \pi_2^*(\zeta_2) \mid \zeta_2 \in \Theta_2^*, \varepsilon_1(x) + \mathrm{Nm}_{\varepsilon_2}(\zeta_2) \equiv \xi\}.$$

ii) $\dim(\mathrm{Sing}(\Xi^* \cdot \Xi_a^*) - F') > 0$ *iff* $\varepsilon_1(x) \equiv \xi_1$.

Proof. Debarre proved in [De5] that

$$\Xi^* \cdot \Xi_a^* = \{\zeta \in \Xi^* \mid h^0(\zeta - \pi_1^*(x)) \geqq 1\} \text{ for } \tilde{a} = \pi_1^*(x - \tau_1(x))$$

and

$$\mathrm{Sing}(\Xi^* \cdot \Xi_a^*) \supset \{\zeta \in \Xi^* \mid h^0(\zeta - \pi_1^*(x)) \geqq 2\}.$$

Part i) comes from the equality of sets

$$F' \cup R(\tilde{a}) = \{\zeta \in \Xi^* \mid h^0(\zeta - \pi_1^*(x)) \geqq 1\}.$$

This is straightforward.

Next note that $\{\zeta \in \Xi^* \mid h^0(\zeta - \pi_1^*(x)) \geqq 1\}$ is the special subvariety associated to the linear system $|K_C - \mathrm{Nm}_\pi(\pi_1^*(x))|$ (cf. [Be2], [We3]). A characterization of Welters (loc. cit.) of the singularities of the special subvarieties gives the inclusion

$$\text{(8.6)}\quad \mathrm{Sing}(\Xi^* \cdot \Xi_{\tilde{a}}^*) \subset \{\zeta \in \Xi^* \mid h^0(\zeta - \pi_1^*(x)) \geqq 2\} \cup \{\pi_1^*(x) + \pi^*(A) + \tilde{D} \text{ such that } A, \tilde{D} \geqq 0,\ h^0(A + \varepsilon^*(\varepsilon_1(x))) > 1\}.$$

To prove ii) it suffices to show the following facts:

a) If $\varepsilon_1(x) \in |\xi_1|$, then $\dim(R(\tilde{a}) - F') \cap \{\zeta \in \Xi^* \mid h^0(\zeta - \pi_1^*(x)) \geqq 2\} > 0$.

b) If $\varepsilon_1(x) \notin |\xi_1|$, then $R(\tilde{a}) - F'$ intersects the second member of (8.6) in a finite number of points.

To see a) observe that the set

$$\{\pi_1^*(x) + \pi_2^*(\zeta_2) \mid \zeta_2 \in Z_2' - Z_2' \cap Z_2'', \varepsilon_1(x) \equiv \xi_1\}$$

of dimension $g - 6$ is contained in the above intersection.

Assume now that $\varepsilon_1(x) \notin |\xi_1|$ and take $\zeta = \pi_1^*(x) + \pi_2^*(\zeta_2)$ such that $\zeta_2 \notin A_2$. Then

$$h^0(\zeta - \pi_1^*(x)) = h^0(\pi_2^*(\zeta_2)) = h^0(\zeta_2) + h^0(\zeta_2 - \varepsilon_2^*(\xi_1)) = h^0(\zeta_2).$$

So if $h^0(\tilde{\zeta} - \pi_1^*(x)) \geqq 2$, it implies $\zeta_2 \in \operatorname{Sing} \Theta_2^* = Z_2' \cup Z_2''$. Since $\zeta_2 \notin A_2 \supset Z_2''$ and $\operatorname{Nm}_{\varepsilon_2}(\zeta_2) \neq \xi_2$. This is a contradiction.

Suppose now there exists a divisor $A \geqq 0$ on C such that

$$h^0\left(\pi_2^*(\zeta_2) - \pi^*(A)\right) > 0 \text{ and } h^0\left(A + \varepsilon^*(\varepsilon_1(x))\right) \geqq 2\,.$$

In particular $A \neq 0$. By using (3.1) the second inequality says that either A is not ε-simple or $\deg(A) = g-2$. In the first case we conclude that we may write

$$\pi_2^*(\zeta_2) \equiv \pi^*\left(\varepsilon^*(\bar{A})\right) + \tilde{B}$$

where $\bar{A}$ and $\tilde{B}$ are effective divisors on E and $\tilde{C}$, respectively, and $\bar{A}$ is not trivial. Then

$$0 < h^0\left(\pi_2^*\left(\zeta_2 - \varepsilon_2^*(\bar{A})\right)\right) = h^0\left(\zeta_2 - \varepsilon_2^*(\bar{A})\right) + h^0\left(\zeta_2 - \varepsilon_2^*(\bar{A}) - \varepsilon_2^*(\xi_1)\right),$$

which contradicts that $\zeta_2 \notin A_2$. On the other hand, if $\deg(A) = g-2$ then $\pi^*(A) = \pi_2^*(\zeta_2)$. Taking norms one obtains that $2A \equiv \varepsilon^*\left(\varepsilon_2(\zeta_2)\right)$. By (3.1) there exists an effective divisor $\bar{A}_0$ of degree $g-2$ on E such that $2A = \varepsilon^*(\bar{A}_0)$. As above A not ε-simple leads to a contradiction. If A is ε-simple, then it has support in the ramification locus of ε_1, which leaves a finite number of possibilities. □

(8.7) Theorem. *Let $(\tilde{C}, C)$ be a generic element of $\mathscr{R}_{B,g,1}$ and let $(\tilde{D}, D) \in \mathscr{R}_g$ such that $P(\tilde{C}, C) \cong P(\tilde{D}, D)$. Then $(\tilde{D}, D) \in \mathscr{R}_{B,g,1}$, and $(\tilde{C}, C)$ and $(\tilde{D}, D)$ are tetragonally related.*

Proof. By using the arguments of (5.9) we conclude that $(\tilde{D}, D) \in \mathscr{R}_{B,g}$. Then, from the number of irreducible components of $\operatorname{Sing} \Xi^*$ (cf. (2.7)) we conclude that

$$(\tilde{D}, D) \in \mathscr{R}_{B,g}' \cup \mathscr{R}_{B,g,0} \cup \mathscr{R}_{B,g,1}\,.$$

As we shall see (independently) in (9.1) i) (combined with (2.11)) the property

$$\dim\{\tilde{a} \in P \mid \tilde{a} + \operatorname{Sing} \Xi^* \subset \operatorname{Sing} \Xi^*\} = 1$$

(cf. (5.12) i)) does not hold for the elements of the components $\mathscr{R}_{B,g}'$ and $\mathscr{R}_{B,g,0}$. So $(\tilde{D}, D) \in \mathscr{R}_{B,g,1}$. Arguing as in (5.11) it suffices to explain how to recover (C_1, τ_1) and (C_2, τ_2) from P. The latter is recovered using Propositions (8.1) and (8.3), the former by combining (8.4) and (8.5). □

9. The component $\mathscr{R}_{B,g,0}$. Let $(\tilde{C}, C) \in \mathscr{R}_{B,g,0}$. We keep the notations of § 1 and § 2. In this section we do not need the assumption of generality. Although W_2 is not equal to $\operatorname{Sing} \Xi^*$, it is its unique component of positive dimension. Recall that $t = 0$ implies that ε_1 and π_2 are unramified. We shall denote by λ the non trivial element of $\pi^*\left(\varepsilon^*({}_2JE)\right)$.

(9.1) Proposition. *One has the equalities:*

i) $I(W_2) = \{0, \lambda\}$.

ii) $\{\tilde{a} \in P \mid \tilde{a} + W_2 \subset \Xi^*\} = \{\pi_2^*(\varepsilon_2^*(\bar{x}) - r - s) \mid \bar{x} \in E, r, s \in C_2, 2\bar{x} \equiv \varepsilon_2(r) + \varepsilon_2(s)\}$.

Proof. Part i) is proved in [De 3], (5.6.5). Part ii) is left to the reader. □

Let us denote by S the set described in (9.1) ii). Then

(9.2) Proposition. *The set $S \cap 2S$ is a symmetric curve with normalization C_2. Moreover τ_2 is the involution induced on C_2 by the (-1) map of P.*

Proof. It is easy to prove the following:

$$S \cap 2S = \{\pi_2^*(x - \tau_2(x)) \mid x \in C_2\}.$$

All the statements are a consequence of this equality. In fact, only the birationality of the map

$$\begin{aligned} \varphi\colon C_2 &\to S \cap 2S, \\ x &\to \pi_2^*(x - \tau_2(x)) \end{aligned}$$

needs to be proved. Assume that $\varphi(x) = \varphi(y)$. Then

$$x + \tau_2(y) - \tau_2(x) - y \in \operatorname{Ker}(\pi_2^*) = \{0, \varepsilon_2^*(\xi_1)\}.$$

Hence:

$$2x + 2\tau_2(y) \equiv 2y + 2\tau_2(x).$$

Equality of divisors would lead to either $x = y$ or $x = \tau_2(x)$. So we can suppose that $h^0(2x + 2\tau_2(y)) \geqq 2$. Since all g_4^1's on C_2 come from g_2^1's on E one finds a divisor $\bar{A} \in E^{(2)}$ such that $2x + 2\tau_2(y) = \varepsilon^*(\bar{A})$ and then we have again either $x = y$ or $x = \tau_2(x)$. □

(9.3) Remark. The data (C_2, τ_2) do not determine the initial element $(\tilde{C}, C)$. However, by recovering the class $\varepsilon_2^*(\xi_1)$, the curve C_1 (hence $(\tilde{C}, C)$) may be reconstructed from our information.

(9.4) Theorem. *Let $(\tilde{C}, C)$ and $(\tilde{D}, D)$ be two elements of $\mathscr{R}_{B,g,0}$ verifying the condition $P(\tilde{C}, C) \cong P(\tilde{D}, D)$. Then: $(\tilde{C}, C) \cong (\tilde{D}, D)$.*

Proof. By (9.1), (9.2) and (9.3) it suffices to recover $\varepsilon_2^*(\xi_1)$ from P. Going back to the proof of (9.2) one finds a morphism:

$$C_2 \to P$$

inducing a morphism:

$$j\colon JC_2 \to P.$$

By construction one can factorize j into $j' \circ h$, where

$$h\colon JC_2 \to \operatorname{Im}(\operatorname{Id} - \tau_2^*) \cong P_2$$

is the obvious map and $j' = \pi_{2|P_2}^*$. Then $\operatorname{Ker}(j') = \{0, \varepsilon_2^*(\bar{\xi}_1)\}$. Hence we obtain $\varepsilon_2^*(\bar{\xi}_1) \in P_2 \subset JC_2$. □

10. The component $\mathcal{R}'_{B,g}$. Let $(\tilde{C}, C) \in \mathcal{R}'_{B,g}$. We keep the notations and assumptions of § 1 and § 2 (see specially (2.9) and (2.11)). In particular $g \geqq 10$. Recall that by (4.3) one has $\tau^*(\eta) \neq \eta$.

(10.1) Proposition. *With the above notations,* $\operatorname{Sing} \Xi^*$ *has a unique irreducible component of dimension* $g-5$. *This component is*:

$$W = \{\pi^*(\varepsilon^*(\bar{x} + \bar{y})) + \tilde{\zeta} \in P^* \mid \bar{x}, \bar{y} \in E, \tilde{\zeta} \in W_{2g-10}^0(\tilde{C})\}\,.$$

Proof. It suffices to check that $\dim W = g - 5$. □

(10.2) Proposition. *One has the equality*:

$$\{\tilde{a} \in P \mid \tilde{a} + W \subset \Xi^*\} = \{\pi^*(\varepsilon^*(\bar{x})) - \tilde{\zeta} \in P \mid \bar{x} \in E, \tilde{\zeta} \in W_4^0(\tilde{C})\}\,.$$

Proof. The inclusion of the right hand side member in the left hand side member is trivial. By (9.1), $\{\tilde{a} \in P \mid \tilde{a} + W \subset \Xi^*\}$ has dimension 2. Hence it is enough to show that it is irreducible. This follows from the description of (9.1) and from the fact that for generic x in E, the Galois group of the composition of ε with the g_2^1 given by $|2x|$ is $\mathbb{Z}/2\mathbb{Z}$. □

Let us denote by S' the set $\{\tilde{a} \in P \mid \tilde{a} + W \subset \Xi^*\}$.

(10.3) Proposition. *The following inclusions hold*:

$$S' \cap 2S' \subset T' = \{\tilde{D} - \iota^*(\tilde{D}) \in J\tilde{C} \mid \tilde{D} \in W_2^0(\tilde{C}), \operatorname{Nm}_\pi(\tilde{D}) \in \operatorname{Im}(\varepsilon^*)\} \subset S'\,.$$

Proof. Let us define

$$U = \{\tilde{D} - \iota^*(\tilde{D}) \mid \tilde{D} \in W_4^0(\tilde{C}), \operatorname{Nm}_\pi(\tilde{D}) \in \operatorname{Im}(\varepsilon^*)\}\,.$$

By (10.2) one has $2S' \subset U$. So, our statements follow from the claim:

$$U \cap S' = T'\,.$$

The inclusion $T' \subset U \cap S'$ is clear. We prove the opposite inclusion. Let $\tilde{D} - \iota^*(\tilde{D}) \in U$ and $\bar{r}, \bar{s} \in E$ such that $\operatorname{Nm}_\pi(\tilde{D}) = \varepsilon^*(\bar{r} + \bar{s})$. If we suppose that $\tilde{D} - \iota^*(\tilde{D}) \in S'$ then one finds elements $\tilde{D}' \in \tilde{C}^{(4)}$ and $\bar{x} \in E$ such that

$$\iota^*(\tilde{D}) + \tilde{D}' \equiv \tilde{D} + \pi^*(\varepsilon^*(\bar{x}))\,. \tag{10.4}$$

We may write $\tilde{D} = \pi^*(A) + \tilde{B}$ where $\tilde{B} \geqq 0$ is π-simple and A is effective. Looking at the degree of A we have three possibilities:

a) $\deg(A) = 2$. In this case $\tilde{D} - \iota^*(\tilde{D}) = 0 \in T'$.

b) $\deg(A) = 1$. Therefore $\deg(\tilde{B}) = 2$. By replacing in (10.4)

$$\tilde{D}' + \iota^*(\tilde{B}) \equiv \tilde{B} + \pi^*(\varepsilon^*(\bar{x})).$$

The equality of divisors would imply $\tilde{B} \leqq \pi^*(\varepsilon^*(\bar{x}))$. Since $\tilde{B}$ is π-simple, $\mathrm{Nm}_\pi(\tilde{B}) = \varepsilon^*(\bar{x})$ and then

$$\tilde{D} - \iota^*(\tilde{D}) \equiv \tilde{B} - \iota^*(\tilde{B}) \in T'.$$

We suppose now that $2 \leqq h^0(\tilde{B} + \pi^*(\varepsilon^*(\bar{x})))$. By applying (2.13)

$$2 \leqq h^0(\varepsilon^*(\bar{x})) + h^0(\mathrm{Nm}_\pi(\tilde{B}) + \varepsilon^*(\bar{x}) - \eta) = 1 + h^0(\mathrm{Nm}_\pi(\tilde{B}) + \varepsilon^*(\bar{x}) - \eta).$$

On the other hand $\mathrm{Nm}_\pi(\tilde{B}) = \mathrm{Nm}_\pi(\tilde{D}) - 2A = \varepsilon^*(\bar{r} + \bar{s}) - 2A$. So

$$0 < h^0(\mathrm{Nm}_\pi(\tilde{B}) + \varepsilon^*(\bar{x}) - \eta) = h^0(\varepsilon^*(\bar{r} + \bar{s} + \bar{x}) - 2A - \eta).$$

Then we get $\tau^*(\eta) = \eta$, which is a contradiction.

c) $\deg(A) = 0$. Then $\tilde{D}$ is π-simple. We go back to (10.4). If there is an equality, then $\tilde{D} = \pi^*(\varepsilon^*(\bar{x}))$ and one has a contradiction. Otherwise, by applying (2.13)

$$2 \leqq h^0(\tilde{D} + \pi^*(\varepsilon^*(\bar{x}))) \leqq 1 + h^0(\varepsilon^*(\bar{x}) + \mathrm{Nm}_\pi(\tilde{D}) - \eta).$$

Since $\mathrm{Nm}_\pi(\iota^*(\tilde{D})) = \varepsilon^*(\bar{r} + \bar{s})$ one has $h^0(\varepsilon^*(\bar{x} + \bar{r} + \bar{s}) - \eta) > 0$. Again this implies $\tau^*(\eta) = \eta$, which is a contradiction. □

(10.5) By (9.2), $S' \cap 2S'$ is a symmetric irreducible curve and its normalization has genus g. Since T' is also a curve we conclude that $S' \cap 2S'$ is an irreducible component of T'.

In order to study the curve T' we define T as the variety given by following pull-back diagram:

$$\begin{array}{ccc} T & \longrightarrow & \tilde{C}^{(2)} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi^{(2)}} \\ E & \xrightarrow{\varepsilon^*} & C^{(2)}. \end{array}$$

It is not hard to see that the morphism

$$\tilde{C}^{(2)} \to P,$$
$$\tilde{D} \to \tilde{D} - \iota^*(\tilde{D})$$

sends T birationally to T'. We shall denote by j the involution of T induced by $\iota^{(2)}$.

(10.6) Proposition. *T is an irreducible smooth curve of genus g and the equality $T' = S' \cap 2S'$ holds. Moreover T' is symmetric and the multiplication by -1 induces on T the involution j.*

Proof. Because the Galois group of $\varepsilon \circ \pi$ is $\mathbb{Z}/2\mathbb{Z}$, T is irreducible. A local computation shows that $\varepsilon^*(E)$ is transverse to the diagonal, therefore T is smooth, hence T' is irreducible and equal to $S' \cap 2S'$. □

(10.7) Comparing with the construction made in § 9, we note that (T, j) play the role of (C_2, τ_2). There we obtained a point of ${}_2(JC_2)$ which allowed us to reconstruct C_1. By translating this to the present context we can conclude that there exists an intrinsic way to recognize a certain element of ${}_2JT$. Moreover this class appears in $\mathrm{Im}(f_1^*)$, where f_1 is the map $T \to T/j$.

Our next aim is to compute this point in terms of the initial data. To do this we imitate the proof of (9.4).

Let $\gamma\colon T \to P$ be the composition of the normalization map with the inclusion $T' \hookrightarrow P$. The induced map between JT and P factorizes through a morphism

$$\tilde{\gamma}\colon (\mathrm{Id} - j^*)(JT) = \mathrm{Ker}(\mathrm{Nm}_{f_1}) \to P .$$

We want to find the kernel of $\tilde{\gamma}$.

(10.8) Proposition. $\mathrm{Ker}(\tilde{\gamma}) = f^*({}_2JE)$.

Proof. Let $\zeta \in \mathrm{Pic}^2(\tilde{C})$. Consider the morphism $T \hookrightarrow \tilde{C}^{(2)} \xrightarrow{-\zeta} J\tilde{C}$ and the induced morphism $v\colon JT \to J\tilde{C}$. Then: $\mathrm{Im}(v_{|\mathrm{Ker}(\mathrm{Nm}_{f1})}) \subset P$. A straightforward computation shows that the restriction $\tilde{v}\colon \mathrm{Ker}(\mathrm{Nm}_{f1}) \to P$ is $\tilde{\gamma}$.

On the other hand it is easy to see that $v \circ f^*\colon JE \to J\tilde{C}$ coincides with $2(\varepsilon \circ \pi)^*$. Therefore

$$\mathrm{Ker}(v_{|\mathrm{Im} f_1^*}) = \mathrm{Ker}(v_{|\mathrm{Im} f^*}) = f^*({}_2JE) .$$

Since the unique non zero element of the kernel of $\tilde{v}$ appears in $\mathrm{Im}(f_1^*) = \mathrm{Im}(f^*)$ one has $\mathrm{Ker}(\tilde{v}) = f^*({}_2JE)$ and we are done. □

(10.9) Theorem. *Let $(\tilde{C}, C), (\tilde{D}, D) \in \mathscr{R}'_{B,g}$ such that $P(\tilde{C}, C) \cong P(\tilde{D}, D)$. Then $(\tilde{C}, C) \cong (\tilde{D}, D)$.*

Proof. It suffices to show that the initial data are determined by T, j and $f^*({}_2JE)$. Indeed the non-zero element of $f^*({}_2JE)$ gives a point of ${}_2J(T/j)$ that allows us to recover the morphism $f_2\colon T/j \to E$ (where $f = f_2 \circ f_1$).

Now consider the pull-back diagram

$$\begin{array}{ccc} X & \longrightarrow & T^{(2)} \\ \downarrow & & \downarrow \\ E & \xrightarrow{f_2^*} & (T/j)^{(2)}. \end{array}$$

Then, the morphism

$$\tilde{C} \to X,$$
$$\tilde{x} \to (\tilde{x} + \tilde{x}') + (\tilde{x} + \iota(\tilde{x}'))$$

where $\pi(\tilde{x}) + \pi(\tilde{x}') \in \mathrm{Im}(\varepsilon^*)$, is an isomorphism and the involution $j^{(2)}$ of $T^{(2)}$ induces on $\tilde{C}$ the involution ι. □

(10.10) Theorem. *Let* $(\tilde{C}, C) \in \mathcal{R}_{B,g,0} \cup \mathcal{R}'_{B,g}$ *and let* $(\tilde{D}, D) \in \mathcal{R}_g$ *such that* $P(\tilde{D}, D) \cong P(\tilde{C}, C)$. *Then* $(\tilde{D}, D) \in \mathcal{R}_{B,g,0} \cup \mathcal{R}'_g$ *and* $(\tilde{C}, C)$ *and* $(\tilde{D}, D)$ *are tetragonally related (in the general sense explained in the Introduction).*

Proof. By arguing as in (5.9) one obtains that D is bi-elliptic. The table (2.7) implies that

$$(\tilde{D}, D) \in \mathcal{R}_{B,g,1} \cup \mathcal{R}_{B,g,0} \cup \mathcal{R}'_{B,g}.$$

By comparing (6.1) i) with (9.1) i) we exclude the first possibility. If $(\tilde{C}, C)$ and $(\tilde{D}, D)$ belong to the same component, then the statement is a consequence of (9.4) and (10.9). If they belong to different components, say $(\tilde{D}, D) \in \mathcal{R}'_{B,g}$ and $(\tilde{C}, C) \in \mathcal{R}_{B,g,0}$, then after two tetragonal constructions starting in $(\tilde{D}, D)$ (via $\mathcal{H}_{g,0}$, cf. (2.11) and § 15) one finds an element $(\tilde{D}_0, D_0) \in \mathcal{R}_{B,g,0}$ with $P(\tilde{D}, D) \cong P(\tilde{D}_0, D_0)$. By (9.4), $(\tilde{C}, C) = (\tilde{D}_0, D_0)$ and we are done. □

We now compare the constructions used to prove theorems (9.4) and (10.9) in order to obtain an injection from $\mathcal{R}'_{B,g}$ in $\mathcal{R}_{B,g,0}$ commuting with the Prym map. A posteriori (see proof of (10.10)) the injection is obtained by two tetragonal constructions (via $\mathcal{H}_{g,0}$).

Let $(\tilde{C}', C') \in \mathcal{R}'_{B,g}$. Suppose that $\varepsilon'\colon C' \to E'$ is a bi-elliptic structure of C'.

Construct the pull-back diagram

$$\begin{array}{ccc} T & \longrightarrow & \tilde{C}'^{(2)} \\ \downarrow & & \downarrow \\ E' & \xrightarrow{(\varepsilon')^*} & C'^{(2)}. \end{array}$$

The involution $\iota^{(2)}$ restricts to an involution j of T. Then T/j is an elliptic curve. We call $\varepsilon_1 \colon E' \to T/j$ to the transposed map. By taking again a pull-back diagram we get

$$\begin{array}{ccc} \tilde{C} & \longrightarrow & T \\ \big\downarrow & & \big\downarrow{\scriptstyle \varepsilon_2} \\ E' & \xrightarrow{\varepsilon_1} & T/j\,. \end{array}$$

The curve $\tilde{C}$ has two involutions attached to the projections; call ι the composition of this involutions. Then $(\tilde{C}, \tilde{C}/\iota) \in \mathscr{R}_{B,g,0}$ is the image of $(\tilde{C}', C')$.

There is a natural way of inverting the injection above: Start with an element $(\tilde{C}, C) \in \mathscr{R}_{B,g,0}$. With the notations of § 2, observe that $t = 0$ implies that C_1 is also elliptic. We call $f_1 \colon E \to C_1$ to the transposed morphism. Then the pull-back diagram

$$\begin{array}{ccc} \tilde{C}' & \longrightarrow & C_2^{(2)} \\ \big\downarrow & & \big\downarrow{\scriptstyle \varepsilon_2^{(2)}} \\ C_1 & \xrightarrow{f_1^*} & E^{(2)} \end{array}$$

gives an element $(\tilde{C}', C') \in \bar{\mathscr{R}}_g$, where $C' = \tilde{C}'/\iota$, ι being the restriction to $\tilde{C}'$ of the involution $\iota^{(2)}$. In general this element belongs to $\mathscr{R}'_{B,g}$ and in this case its image by the injection given above is $(\tilde{C}, C)$. In any case $(\tilde{C}', C') \in \bar{\mathscr{R}}'_{B,g}$ and C' is a double covering of a smooth curve of genus 1.

II. A bi-elliptic construction

For all this part we fix a generic element $(\tilde{C}, C)$ of $\mathscr{R}_{B,g,4}$ and a linear series g_2^1 on the elliptic curve E (we keep the notations of §§ 1 and 2). The first section (§ 11) is devoted to the description of four allowable covers constructed from this set of data. These covers belong to the fibre of $\bar{P}$ over $P(\tilde{C}, C)$. The proof of this fact is given in § 13.

11. The construction. We shall give the description of the attached coverings in three steps.

Step 1. The curve C_1 is bi-elliptic of genus 5. Since it is general it has a unique bi-elliptic structure. It is well known that (cf. [A-C-G-H], p. 270, or remark ii in (3.6))

$$W_4^1(C_1) = \tilde{D}_1 \cup \varepsilon_1^*(\mathrm{Pic}^2(E))$$

where $\tilde{D}_1 = \{\zeta \in W_4^1(C_1) \mid \mathrm{Nm}_{\varepsilon_1}(\zeta) = \xi_1\}$ is a smooth curve of genus 7. The intersection

$$\tilde{D}_1 \cap \varepsilon_1^*(\mathrm{Pic}^2(E)) = \{\varepsilon_1^*(\bar{x} + \bar{y}) \mid \bar{x}, \bar{y} \in E \text{ and } 2\bar{x} + 2\bar{y} \equiv \xi_1\}$$

consists of four different points.

The variety $W_4^1(C_1)$ is invariant by the action $\zeta \to K_{C_1} - \zeta$ and, by passing to the quotient, we get an allowable double cover $W_4^1(C_1) \to D_1 \cup l$, where D_1 is a smooth irreducible plane quartic and l is a line intersecting D_1 in four different points.

There is an isomorphism of principally polarized abelian varieties ([Ma], [Be 3] and [K-K]) $JC_1 \cong P(W_4^1(C_1), D_1 \cup l)$.

Step 2. Let us consider the commutative pull-back diagram

$$\begin{array}{ccc} \tilde{D}_2 & \longrightarrow & C_2^{(2)} \\ \downarrow & & \downarrow \varepsilon_2^{(2)} \\ \mathbb{P}^1 & \xrightarrow{g_2^1} & E^{(2)} . \end{array} \tag{11.1}$$

The involution $\tau_2^{(2)}$ leaves invariant the curve $\tilde{D}_2$. Call D_2 the quotient curve. For simplicity we will suppose that the linear series g_2^1 is general. Then $\tilde{D}_2$ and D_2 are smooth, connected by (16.1), and D_2 is hyperelliptic of genus $g-6$.

Step 3. To construct an allowable cover $(\tilde{D}, D)$ from the pairs $(\tilde{D}_1, D_1)$ and $(\tilde{D}_2, D_2)$ we identify the ramification points of both covers (and the discriminant points correspondingly) in the following way:

Let $\bar{\eta}_i \in \operatorname{Pic}^2(E)$, such that $2\bar{\eta}_i \equiv \bar{\zeta}_1$, $i = 1, \ldots, 4$. The classes $\varepsilon_1^*(\bar{\eta}_i)$ correspond to the ramification points of $\tilde{D}_1 \to D_1$. Note that

$$\{0, \bar{\eta}_1 - \bar{\eta}_2, \bar{\eta}_1 - \bar{\eta}_3, \bar{\eta}_1 - \bar{\eta}_4\} = {}_2JE .$$

On the other hand the ramification points of $\tilde{D}_2 \to D_2$ are $\varepsilon_2^*(\bar{x}_i) \in C_2^{(2)}$ where $2\bar{x}_i \in g_2^1$, $i = 1, \ldots, 4$. One has also $\{0, \bar{x}_1 - \bar{x}_2, \bar{x}_1 - \bar{x}_3, \bar{x}_1 - \bar{x}_4\} = {}_2JE$.

(11.2) Let σ be a bijection

$$\{\bar{\eta}_i\}_{i=1,\ldots,4} \to \{\bar{x}_i\}_{i=1,\ldots,4},$$
$$\bar{\eta}_i \to \sigma(\bar{\eta}_i)$$

such that $\bar{\eta}_i - \bar{\eta}_j$ and $\sigma(\bar{\eta}_i) - \sigma(\bar{\eta}_j)$ coincide in ${}_2JE$. It is easy to see that four such bijections exist. We then identify $\varepsilon_1^*(\bar{\eta}_i)$ with $\varepsilon_2^*(\sigma(\bar{\eta}_i))$, $i = 1, \ldots, 4$, thus obtaining an allowable covering $(\tilde{D}, D)$. The corresponding covering map will be denoted by $p\colon \tilde{D} \to D$. Moreover, after changing the indices of the $\bar{x}_i$ we may assume that $\bar{x}_i = \sigma(\bar{\eta}_i)$, $i = 1, \ldots, 4$.

(11.3) Theorem. There exists an isomorphism of principally polarized abelian varieties

$$P(\tilde{C}, C) \cong P(\tilde{D}, D) .$$

The proof will be given in §13.

(11.4) Remark. Observe that the curve D is neither tetragonal nor stable reduction of a tetragonal curve. Therefore $(\tilde{D}, D)$ and $(\tilde{C}, C)$ are not tetragonally related (cf. § 15 for the definition of tetragonal relation).

12. The isogenies g_i and h_i. In this section we keep the notations $p: \tilde{D} \to D$, $(\tilde{D}_i, D_i)$, $i = 1, 2$, to refer to the coverings constructed in § 11. We put $p_i := p_{|D_i}$, $i = 1, 2$.

For a line bundle $\tilde{L}$ on $\tilde{D}_i$ invariant by the covering involution we defined in § 4 an element

$$v_i(\tilde{L}) \in \frac{(\mu_2)^4}{\mu_2}, \quad i = 1, 2 .$$

We shall take the ordering of the factors of $(\mu_2)^4$ for v_1 and v_2 compatible with the identifications made in Step 3 of § 11.

The aim of this section is to prove the following technical result:

(12.1) Proposition. *There exist isogenies*

$$g_i\colon P(\tilde{D}_i, D_i) \to P(C_i, E)$$

and

$$h_i\colon P(C_i, E) \to P(\tilde{D}_i, D_i) \quad \textit{for } i = 1, 2$$

satisfying $h_i \circ g_i = 2$ and such that

i) $\operatorname{Ker}(g_i) = p_i^*({}_2JD_i)$,

ii) $g_i({}_2P(\tilde{D}_i, D_i)) = \varepsilon_i^*({}_2JE)$,

iii) $g_i^*(L_{P(C_i,E)}) \sim L_{P(\tilde{D}_i,D_i)}^{\otimes 2}$,

iv) *if $\tilde{\alpha}_i \in {}_2P(\tilde{D}_i, D_i)$, then*

$$v_1(\tilde{\alpha}_1) = v_2(\tilde{\alpha}_2) \textit{ iff } \exists \varrho \in {}_2JE \textit{ such that } g_i(\tilde{\alpha}_i) = \varepsilon_i^*(\varrho) ,$$

i′) $\operatorname{Ker}(h_i) = \varepsilon_i^*({}_2JE)$,

ii′) $h_i({}_2P(C_i, E)) = p_i^*({}_2JD_i)$,

iii′) $h_i^*(L_{P(\tilde{D}_i,D_i)}) \sim L_{P(C_i,E)}^{\otimes 2}$,

for $i = 1, 2$, where $L_{P(\tilde{D}_i,D_i)}$ and $L_{P(C_i,E)}$ are the polarizations induced by the inclusions in the respective Jacobians.

Proof. We first consider the case $i = 1$. The inclusion $\tilde{D}_1 \hookrightarrow \operatorname{Nm}_{\varepsilon_1}^{-1}(\xi_1) \cong P(C_1, E)$, yields a morphism $g_1'\colon J\tilde{D}_1 \to P(C_1, E)$. We define $g_1 := (g_1')_{|P(\tilde{D}_1, D_1)}$.

It is convenient to describe the map g_1' explicitely. Let $\tilde{z} \in \tilde{D}_1$. We denote by $\langle \tilde{z} \rangle$ the corresponding element of $\operatorname{Pic}^4(C_1)$. Then

$$g_1'\left(\sum_i n_i \tilde{z}_i\right) = \sum_i n_i \langle \tilde{z}_i \rangle, \quad \text{where} \quad \sum_i n_i = 0\,.$$

(12.2) Lemma. *One has* $g_1'\left(p_1^*(JD_1)\right) = 0$. *In particular* $g_1\left(p_1^*({}_2JD_1)\right) = 0$.

Proof. Let $\sum_i n_i z_i \in JD_1$ with $\sum_i n_i = 0$. Then

$$\begin{aligned} g_1'\left(p_1^*\left(\sum_i n_i z_i\right)\right) &= g_1'\left(\sum_i n_i p^*(z_i)\right) = \sum_i n_i \langle p^*(z_i)\rangle \\ &= \sum_i n_i K_{C_1} = \left(\sum_i n_i\right) \cdot K_{C_1} = 0\,. \quad \square \end{aligned}$$

On the other hand in Proposition (4.7) of [C-G-T] the following result is proved: for a general bi-elliptic curve Γ the Jacobian $J\Gamma$ is isogenous to a product of an elliptic curve by a simple abelian variety. Thus $g_1 \neq 0$ implies that g_1 is an isogeny. To study the behaviour of g_1 with respect to the points of order two we use the following result:

(12.3) Lemma. *One has an equality*

$$\begin{aligned} {}_2P(\tilde{D}_1, D_1) = p_1^*({}_2JD_1) \cup \{p_1^*(\gamma) - \tilde{z}_1 - \tilde{z}_2 \in P(\tilde{D}_1, D_1) \mid \gamma \in \operatorname{Pic}^1(D_1), \\ \tilde{z}_1, \tilde{z}_2 \in \tilde{D}_1 \textit{ ramification points of } p_1\}\,. \end{aligned}$$

Proof. Let $\tilde{\alpha} \in {}_2P(\tilde{D}_1, D_1)$. Since it is invariant by the involution on $\tilde{D}_1$ we can apply Proposition (4.2). We get that there exists an effective divisor $\tilde{A}$ contained in the ramification divisor of p_1 such that $\tilde{\alpha} + \tilde{A} \in p_1^*\left(\operatorname{Pic}(D_1)\right)$. In particular $0 \leqq \deg \tilde{A} \leqq 4$ and $\deg \tilde{A}$ is even. Since the ramification divisor belongs to $p_1^*\left(\operatorname{Pic}(D_1)\right)$, the cases $\deg \tilde{A} = 0, 4$ imply $\tilde{\alpha} \in p_1^*({}_2JD_1)$. When $\deg \tilde{A} = 2$ there exist two ramification points $\tilde{z}_1$, $\tilde{z}_2$ such that $\tilde{\alpha} + \tilde{z}_1 + \tilde{z}_2 \in p_1^*\left(\operatorname{Pic}^1(D_1)\right)$ and we are done. $\square$

(12.4) Corollary. *Let* $\tilde{z}_1, \tilde{z}_2$ *be two ramification points of* p_1 *such that* $\langle \tilde{z}_1 \rangle = \varepsilon_1^*(\bar{\eta}_1)$, $\langle \tilde{z}_2 \rangle = \varepsilon_1^*(\bar{\eta}_2)$ *and* $p_1^*(\gamma) - \tilde{z}_1 - \tilde{z}_2 \in P(\tilde{D}_1, D_1)$ *for some* $\gamma \in \operatorname{Pic}^1(D_1)$. *Then*

$$g_1\left(p_1^*(\gamma) - \tilde{z}_1 - \tilde{z}_2\right) = \varepsilon_1^*(\bar{\eta}_1 - \bar{\eta}_2)\,.$$

Proof. By using the explicit description of g_1' one has

$$\begin{aligned} g_1\left(p_1^*(\gamma) - \tilde{z}_1 - \tilde{z}_2\right) &= \langle p_1^*(\gamma)\rangle - \varepsilon_1^*(\bar{\eta}_1) - \varepsilon_1^*(\bar{\eta}_2) \\ &= \varepsilon_1^*(\bar{\xi}_1) - \varepsilon_1^*(\bar{\eta}_1) - \varepsilon_1^*(\bar{\eta}_2) = \varepsilon_1^*(\bar{\eta}_1 - \bar{\eta}_2)\,. \quad \square \end{aligned}$$

Clearly this implies ii) of Proposition (12.1).

To prove i) we shall see that $\deg(g_1) = 2^6 (= \#\, {}_2JD_1)$. This will be enough because of (12.2). We begin with:

(12.5) Lemma. *In $P(C_1, E)$ one has the equality of cohomology classes (cf. §1 for notation)*

$$[\tilde{D}_1] = \zeta_{P(C_1,E)}\,.$$

Proof. One has an exact sequence

$$\begin{array}{ccccccccc} 0 \to {}_2JE & \xrightarrow{(\varepsilon_1^*,1)} & P(C_1,E)\times JE & \xrightarrow{\sigma} & JC_1 \to 0\,, \\ & & (x,y) & \longrightarrow & x+\varepsilon_1^*(y) \end{array}$$

and $\sigma^*\Theta_{C_1} \sim \Xi_{P(C_1,E)} \times JE + 2P(C_1,E)\times\{0\}$ (cf. [Mu1], p. 330). On the other hand the following equality holds in JC_1

$$[\tilde{D}_1 + E] = [W_4^1(C_1)] = 2\zeta_{C_1}$$

(cf. [A-C-G-H], p. 320, Th. 4.4). By applying σ^*:

$$\begin{aligned} 4[\tilde{D}_1\times\{0\}] + 4[\{0\}\times JE] &= 2\sigma^*(\zeta_{C_1}) = \frac{2}{4!}\sigma^*([\Theta_{C_1}^4]) \\ &= \frac{2}{4!}[\Xi_{P(C_1,E)}\times JE + 2P(C_1,E)\times\{0\}]^4 \\ &= \frac{2}{4!}[\Xi^4_{P(C_1,E)}\times JE] + \frac{2}{4!}\cdot 4\cdot 2[\Xi^3_{P(C_1,E)}\times\{0\}]\,. \end{aligned}$$

Therefore $[\tilde{D}_1\times\{0\}] = [\Xi_{P(C_1,E)}]^3/3!\times\{0\}$ and we are done. □

(12.6) Lemma. *The isogeny g_1 has degree 2^6.*

Proof. Taking quotient by a maximal isotropic subgroup of $H(L_{P(C_1,E)}) = \varepsilon_1^*({}_2JE)$ we get an isogeny of degree 2

$$c\colon P(C_1,E)\to A$$

where A has a principal polarization L_A such that $c^*(L_A)\sim L_{P(C_1,E)}$. By the projection formula $c_*(\zeta_{P(C_1,E)}) = 2\zeta_A$. Thus (12.5) implies that $c_*(\tilde{D}_1)$ is twice the minimal class in A. Hence the principal polarization of $(J\tilde{D}_1)\hat{}$ induces on $\hat{A}$ twice the principal polarization, that is to say, there is a commutative diagram

(12.7)
$$\begin{array}{ccccc} \hat{A} & \xrightarrow{\hat{c}} & P(C_1,E)\hat{} & \xrightarrow{(g_1')\hat{}} & (J\tilde{D}_1)\hat{} \\ {\scriptstyle 2\lambda_A^{-1}}\downarrow & & {\scriptstyle \mu}\downarrow & & {\scriptstyle \lambda_{J\tilde{D}_1}^{-1}}\downarrow \\ A & \xleftarrow{c} & P(C_1,E) & \longleftarrow & J\tilde{D}_1\,. \end{array}$$

In particular

$$\deg(\mu) = \frac{\deg(2\lambda_A^{-1})}{\deg(c)^2} = \frac{2^{2\dim A}}{4} = 2^6 .$$

On the other hand, since g_1 is an isogeny and $g_1'(p_1^*(JD_1)) = 0$, we get $(\operatorname{Ker} g_1')^0 = p_1^*(JD_1)$. Now let us consider the diagram

$$
(12.8)\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & (\operatorname{Ker} g_1')^0 & \xrightarrow{\cong} & JD_1 & & & & \\
 & & \downarrow & & {\scriptstyle p_1^*}\downarrow & & & & \\
0 & \longrightarrow & \operatorname{Ker} g_1' & \longrightarrow & J\tilde{D}_1 & \xrightarrow{g_1'} & P(C_1, E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \operatorname{Ker} g_1'/(\operatorname{Ker} g_1')^0 & \longrightarrow & J\tilde{D}_1/p_1^*(JD_1) & \xrightarrow{g_0} & P(C_1, E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0\,. & & & &
\end{array}
$$

Combining (12.7) and the dual diagram of (12.8) one gets a commutative diagram

$$
\begin{array}{ccccccc}
 & & P(C_1, E) & \xleftarrow{g_1'} & & & J\tilde{D}_1 \\
 & {\scriptstyle \mu}\nearrow & & & & \overset{\cong}{\nearrow} & \uparrow \\
P(C_1, E)\hat{} & & \xrightarrow{(g_1')\hat{}} & & J\tilde{D}_1\hat{} & & \\
 & & & & \uparrow & \overset{\cong}{\nearrow} & P(\tilde{D}_1, D_1) \\
P(C_1, E) & & \xrightarrow{\hat{g}_0} & & (J\tilde{D}_1/p_1^*(JD_1))\hat{} & &
\end{array}
$$

where v is the inclusion map ($g_1 = g_1' \circ v$) and the commutative diagram

$$
\begin{array}{ccc}
J\tilde{D}_1\hat{} & \xrightarrow{\cong} & J\tilde{D}_1 \\
\uparrow & & {\scriptstyle v}\uparrow \\
(J\tilde{D}_1/p_1^*(JD_1))\hat{} & \xrightarrow{\cong} & P(\tilde{D}_1, D_1)
\end{array}
$$

is a consequence of the relation $(p_1^*)\hat{} = \lambda_{D_1} \circ \operatorname{Nm}_{p_1} \circ \lambda_{\tilde{D}_1}^{-1}$ (cf. [Mu1], p. 328). Then $2^6 = \deg(\mu) = \deg(g_0) \cdot \deg(g_1)$. By (12.2) we have $\deg(g_1) \geqq 2^6$. Thus $\deg(g_0) = 1$ and $\deg(g_1) = 2^6$. This finishes the proof of Lemma (12.6) and hence of part i) of Proposition (12.1). □

To prove iii) we use part i). One has $\operatorname{Ker} g_1 = H(L_{P(\tilde{D}_1, D_1)})$. Hence there exists an isomorphism of abelian varieties $\alpha\colon P(\tilde{D}_1, D_1)\hat{} \to P(\tilde{C}_1, C_1)$ such that $\alpha \circ \lambda_{P(\tilde{D}_1, D_1)} = g_1$.

From part ii) it then follows that

$$\alpha\left(\lambda_{P(\tilde{D}_1, D_1)}\left({}_2P(\tilde{D}_1, D_1)\right)\right) = \varepsilon_1^*({}_2JE) = H(L_{P(C_1, E)})\,.$$

We then have the following diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & p_1^*({}_2JD_1) & \longrightarrow & {}_2P(\tilde D_1, D_1) & \longrightarrow & H(L_{P(C_1,E)}) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & p_1^*({}_2JD_1) & \longrightarrow & P(\tilde D_1, D_1) & \xrightarrow{\alpha\circ\lambda_{P(D_1,D_1)}} & P(C_1, E) & \longrightarrow & 0 \\
 & & & & \cdot 2\downarrow & & \downarrow \lambda_{P(C_1,E)} & & \\
 & & & & P(\tilde D_1, D_1) & \underset{\cong}{\xrightarrow{\beta}} & P(C_1,E)\hat{} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with β an isomorphism of abelian varieties, and

$$\hat g_1 \circ \lambda_{P(C_1,E)} \circ g_1 = \hat\lambda_{P(\tilde D_1,D_1)} \circ \hat\alpha \circ \lambda_{P(C_1,E)} \circ \alpha \circ \lambda_{P(\tilde D_1,D_1)} = 2\hat\lambda_{P(\tilde D_1,D_1)} \circ \hat\alpha \circ \beta\,.$$

Since $\mathrm{End}\big(P(C_1,E)\big) \cong \mathbb{Z}$, one has $\hat\alpha \circ \beta = \mathrm{Id}$ and

$$g_1^*(L_{P(C_1,E)}) \sim L_{P(\tilde D_1,D_1)}^{\otimes 2},$$

so part iii) follows.

The isogeny $h_1\colon P(C_1,E) \to P(\tilde D_1, D_1)$ is defined by the condition $h_1 \circ g_1 = 2$. It is then easy to deduce i′), ii′) and iii′) from i), ii) and iii). All this for $i = 1$, of course.

We consider now the case $i = 2$. The inclusion $\tilde D_2 \hookrightarrow C_2^{(2)}$ gives a map $g_2'\colon J\tilde D_2 \to P(C_2, E)$. That is

(12.9) $$g_2'\Big(\sum_i n_i \tilde z_i\Big) = \sum_i n_i(z_{i,1} + z_{i,2})$$

where $\sum_i n_i = 0$ and $z_{i,1} + z_{i,2} \in C_2^{(2)}$ is the divisor corresponding to the point $\tilde z_i \in \tilde D_2$.

It is straightforward to check that $g_2'\big(p_1^*(JD_2)\big) = 0$. Let $g_2 = g'_{2|P(\tilde D_2, D_2)}$. As in the case $i = 1$, g_2 is an isogeny and $g_2\big(p_2^*({}_2JD_2)\big) = 0$.

We can reverse the construction of diagram (11.1): by using the linear series g_2^1 on D_2 given by the hyperelliptic structure and normalizing the curve obtained from the natural pull-back diagram we get

$$\begin{array}{ccc}
C_2 & \longrightarrow & \tilde D_2^{(2)} \\
\downarrow & & \downarrow \\
\mathbb{P}^1 & \xrightarrow{g_2^1} & D_2^{(2)}\,.
\end{array}$$

Moreover the involution of $\tilde D_2^{(2)}$ induces on C_2 an involution that coincides with τ_2. Imitating the construction of g_2 we get an isogeny $h_2\colon P(C_2,E) \to P(\tilde D_2, D_2)$ verifying

$h_2(\varepsilon_2^*({}_2JE)) = 0$. By using the descriptions of g_2 and h_2 we obtain $h_2 \circ g_2 = 2$. Now parts i), ii), i') and ii') are obvious. Part iii) is (as in case $i = 1$) a formal consequence of i) and ii) and the fact that $\mathrm{End}(P(C_2, E)) = \mathbb{Z}$. Now $h_2 \circ g_2 = 2$ and iii) give iii').

It remains only to prove iv). First of all we note that the Lemma (12.3) is still valid for ${}_2P(\tilde{D}_2, D_2)$. Let $\tilde{\alpha}_2 = p_2^*(\gamma) - \tilde{x}_i - \tilde{x}_j \in {}_2P(\tilde{D}_2, D_2)$ with $\tilde{x}_i, \tilde{x}_j \in \tilde{D}_2$ ramification points given by the divisors $\varepsilon_2^*(\bar{x}_i)$ and $\varepsilon_2^*(\bar{x}_j)$, respectively. Then by using (12.9) one has

$$(12.10) \qquad g_2(\tilde{\alpha}_2) = \varepsilon_2^*(\bar{x}_i - \bar{x}_j)\,.$$

Let $\tilde{\alpha}_1 = p_1^*(\gamma') - \varepsilon_1^*(\bar{\eta}_{i'}) - \varepsilon_1^*(\bar{\eta}_{j'})$, $\gamma' \in \mathrm{Pic}^1(D_1)$. From (12.4) and (12.10) it follows

$$(12.11) \qquad \exists \bar{\varrho} \text{ such that } g_1(\tilde{\alpha}_1) = \varepsilon_1^*(\bar{\varrho}) \text{ and } g_2(\tilde{\alpha}_2) = \varepsilon_2^*(\bar{\varrho}) \Leftrightarrow \bar{\eta}_{i'} - \bar{\eta}_{j'} = \bar{x}_i - x_j\,.$$

Hence, by (11.2)

$$\exists \bar{\varrho} \text{ such that } g_1(\tilde{\alpha}_1) = \varepsilon_1^*(\bar{\varrho}) \text{ and } g_2(\tilde{\alpha}_2) = \varepsilon_2^*(\bar{\varrho})$$
$$\Leftrightarrow \text{ either } \{i, j, i', j'\} = \{1, 2, 3, 4\} \text{ or } \{i, j\} = \{i', j'\} \text{ or } i = j \text{ and } i' = j'\,.$$

On the other hand $v_1(\tilde{\alpha}_1)$ (resp. $v_2(\tilde{\alpha}_2)$) gives -1 in the entries i' and j' (resp. i and j) when $i' \neq j'$ (resp. $i \neq j$). If $i = j$ (resp. $i' = j'$), then $v_1(\tilde{\alpha}_1) = \overline{(1,1,1,1)}$ (resp. $v_2(\tilde{\alpha}_2) = \overline{(1, 1, 1, 1)}$). We finally get

$$(12.12) \qquad \exists \bar{\varrho} \text{ such that } g_1(\tilde{\alpha}_1) = \varepsilon_1^*(\bar{\varrho}) \text{ and } g_2(\tilde{\alpha}_2) = \varepsilon_2^*(\bar{\varrho}) \Leftrightarrow v_1(\tilde{\alpha}_1) = v_2(\tilde{\alpha}_2)\,.$$

This ends the proof of Proposition (12.1). □

By combining (12.11) and (12.12) one finds:

(12.13) Remark. Once a bijection σ

$$\{\bar{\eta}_i\}_{i=1,\ldots,4} \to \{\bar{x}_i\}_{i=1,\ldots,4}\,,$$
$$\bar{\eta}_i \to \sigma(\bar{\eta}_i)$$

(cf. § 11 for definitions) is given, the following two facts are equivalent:

i) $\bar{\eta}_i - \bar{\eta}_j$ and $\sigma(\bar{\eta}_i) - \sigma(\bar{\eta}_j)$ coincide in ${}_2JE$ for all $i, j = 1, \ldots, 4$,

ii) for all $\tilde{\alpha}_1 \in {}_2P(\tilde{D}_1\, D_1)$ and $\tilde{\alpha}_2 \in {}_2P(\tilde{D}_2, D_2)$:

$$v_1(\tilde{\alpha}_1) = v_2(\tilde{\alpha}_2) \text{ iff } \exists \bar{\varrho} \in {}_2JE \text{ such that } g_i(\tilde{\alpha}_i) = \varepsilon_i^*(\bar{\varrho}),\ i = 1, 2\,.$$

In other words, the property we require in (11.2) and property (12.1) iv) are equivalent.

13. Proof of Theorem (11.3).

We define the morphism

$$\Phi\colon P(\tilde{D}, D) \xrightarrow{\tilde{f}^*} P(\tilde{D}_1, D_1) \times P(\tilde{D}_2, D_2) \xrightarrow{g_1 \times g_2} P(C_1, E) \times P(C_2, E) \xrightarrow{\varphi} P(\tilde{C}, C)$$

where $\tilde{f}\colon \tilde{D}_1 \sqcup \tilde{D}_2 \to \tilde{D}$ is the desingularization of $\tilde{D}$, g_1, g_2 are the isogenies defined in § 12 and φ is the map given in (2.8). In [De3], Debarre proves that

$$\varphi^*(L_{P(C,C)}) \sim L_{P(C_1,E)} \times L_{P(C_2,E)}.$$

By (12.1) iii)

$$(g_1 \times g_2)^* \varphi^*(L_{P(\tilde{C},C)}) \sim L^{\otimes 2}_{P(\tilde{D}_1,D_1)} \times L^{\otimes 2}_{P(\tilde{D}_2,D_2)}.$$

On the other hand the pull-back of the polarization of $P(\tilde{D}_1, D_1) \times P(\tilde{D}_2, D_2)$ induces on $P(\tilde{D}, D)$ twice the principal polarization (cf. [Be1]). Thus:

$$\Phi^*(L_{P(\tilde{C},C)}) \sim L^{\otimes 4}_{P(\tilde{D},D)}. \tag{13.2}$$

Theorem (11.3) follows in an obvious way from (13.2) and the next

(13.3) Lemma. *The following equality holds*: $\operatorname{Ker}(\Phi) = {}_2P(\tilde{D}, D)$.

Proof. Since $\deg \tilde{f}^* = \deg \varphi = 4$ and $\deg(g_1 \times g_2) = 2^{2(g(D_1)+g(D_2))}$ (cf. (12.1) i)) we get $\deg \Phi = \#\, {}_2P(\tilde{D}, D)$. Therefore the statement can be written alternatively

$$\tilde{f}^*({}_2P(\tilde{D}, D)) \subset \operatorname{Ker}(\varphi \circ (g_1 \times g_2)) = (g_1 \times g_2)^{-1}(\operatorname{Ker}\varphi). \tag{13.4}$$

Since $\operatorname{Ker}\varphi = \{(\varepsilon_1^*(\bar{\alpha}), \varepsilon_2^*(\bar{\alpha})) \mid \bar{\alpha}_2 \in JE\}$ (see (2.8)), one has

$$\begin{aligned}(13.5)\quad &(g_1 \times g_2)^{-1}(\operatorname{Ker}\varphi)\\ &= \{(\tilde{\alpha}, \tilde{\beta}) \in P(\tilde{D}_1, D_1) \times P(\tilde{D}_2, D_2) \mid g_1(\tilde{\alpha}) = \varepsilon_1^*(\bar{\alpha}),\ g_2(\tilde{\beta}) = \varepsilon_2^*(\bar{\alpha}) \text{ and } \bar{\alpha} \in {}_2JE\}\\ &= \{(\tilde{\alpha}, \tilde{\beta}) \in {}_2P(\tilde{D}_1, D_1) \times {}_2P(\tilde{D}_2, D_2) \mid g_1(\tilde{\alpha}) = \varepsilon_1^*(\bar{\alpha}),\ g_2(\tilde{\beta}) = \varepsilon_2^*(\bar{\alpha}) \text{ and } \bar{\alpha} \in {}_2JE\}\end{aligned}$$

(in the second equality use (12.1) ii)). If we prove that

$$\tilde{f}^*({}_2P(\tilde{D}, D)) = \{(\tilde{\alpha}, \tilde{\beta}) \in {}_2P(\tilde{D}_1, D_1) \times {}_2P(\tilde{D}_2, D_2) \mid v_1(\tilde{\alpha}) = v_2(\tilde{\beta})\} \tag{13.6}$$

then (13.3) will follow from (13.6) and (12.1) iv).

We check equality (13.6). We first prove the inclusion of the left hand side member in the right hand side member. Let $(\tilde{\alpha}, \tilde{\beta}) \in \tilde{f}^*({}_2P(\tilde{D}, D))$. Denote by $L(\tilde{\alpha})$ and $L(\tilde{\beta})$ the corresponding line bundles on $\tilde{D}_1$ and $\tilde{D}_2$, respectively. Then there exists a line bundle $\tilde{L} \in P(\tilde{D}, D)$ such that $\tilde{L}^{\otimes 2}$ is trivial and $\tilde{f}^*(\tilde{L}) = (L(\tilde{\alpha}), L(\tilde{\beta}))$. Let $\tilde{x} \in \tilde{D}_1 \cap \tilde{D}_2$. We call $\tilde{x}_1$ (resp. $\tilde{x}_2$) the point $\tilde{x}$ when viewed as a point of $\tilde{D}_1$ (resp. $\tilde{D}_2$). Taking pointwise fibres we obtain an isomorphism $\lambda\colon L(\tilde{\alpha})[\tilde{x}_1] \xrightarrow{\cong} L(\tilde{\beta})[\tilde{x}_2]$ as the composition of the natural identification $L(\tilde{\alpha})[\tilde{x}_1] \xrightarrow{\cong} L[\tilde{x}] \xleftarrow{\cong} L(\tilde{\beta})[\tilde{x}_2]$.

Since $\operatorname{Nm}_p(\tilde{L}) = 0$, $\tilde{L} \otimes \iota^*(\tilde{L})$ is trivial So $\iota^*(\tilde{L}) \cong \tilde{L}^{-1} \cong \tilde{L}$. We choose an isomorphism $\varphi\colon \tilde{L} \to \iota^*(\tilde{L})$ normalized in order to have $\iota^*(\varphi) \circ \varphi = \mathrm{Id}$. The morphism φ induces by restriction

$$\varphi_1\colon L(\tilde{\alpha}) \xrightarrow{\cong} \iota^*(L(\tilde{\alpha})),$$

$$\varphi_2\colon L(\tilde{\beta}) \xrightarrow{\cong} \iota^*(L(\tilde{\beta})).$$

By construction one has a commutative diagram

$$\begin{array}{ccc} L(\tilde{\alpha})[\tilde{x}_1] & \xrightarrow[\cong]{\lambda} & L(\tilde{\beta})[\tilde{x}_2] \\ \cong \downarrow \varphi_1[\tilde{x}_1] & & \cong \downarrow \varphi_2[\tilde{x}_2] \\ \imath^*(L(\tilde{\alpha}))[\tilde{x}_1] & & \imath^*(L(\tilde{\beta}))[\tilde{x}_2] \\ \| & & \| \\ L(\tilde{\alpha})[\tilde{x}_1] & \xrightarrow[\cong]{\lambda} & L(\tilde{\beta})[\tilde{x}_2]\,. \end{array}$$

Thus $v_1(L(\tilde{\alpha})) = v_2(L(\tilde{\beta}))$ (see § 4 for the definition of v_1) and therefore

$$\tilde{f}^*(\tilde{L}) \in \{(\tilde{\alpha}, \tilde{\beta}) \in {}_2P(\tilde{D}_1, D_1) \times {}_2P(\tilde{D}_2, D_2) \mid v_1(\tilde{\alpha}) = v_2(\tilde{\beta})\}\,.$$

Now, to obtain (13.6) we prove that both sets have the same cardinality. Form (12.3) (applied to both $P(\tilde{D}_1, D_1)$ and $P(\tilde{D}_2, D_2)$) one gets

$$v_1({}_2P(\tilde{D}_1, D_1)) = v_2({}_2P(\tilde{D}_2, D_2))\left(= \{\overline{(\lambda_1, \ldots, \lambda_4)} \in (\mu_2)^4/\mu_2 \mid \prod_{i=1}^{4} \lambda_i = 1\}\right).$$

Since $\operatorname{Ker}(v_i) = p_i^*({}_2JD_i)$, $i = 1, 2$ (cf. (4.1)) we conclude

$$\# \{(\tilde{\alpha}, \tilde{\beta}) \in {}_2P(\tilde{D}_1, D_1) \times {}_2P(\tilde{D}_2, D_2) \mid v_1(\tilde{\alpha}) = v_2(\tilde{\beta})\}$$

$$= \# {}_2P(\tilde{D}_1, D_1) \cdot \# \operatorname{Ker}(v_2) = \frac{1}{4} \# {}_2P(\tilde{D}_1, D_1) \cdot \# {}_2P(\tilde{D}_2, D_2) = \# \tilde{f}^*({}_2P(\tilde{D}, D))\,.$$

This finishes the proof of Theorem (11.3). □

III. The fibre of P over a generic element of $P(\mathscr{R}_{B,g})$

This part is devoted to studying the fibre of the extended Prym map for generic elements of $\mathscr{R}_{B,g}$. The results we obtain are summarized in Theorem (16.1). Essentially we prove that the elements described in Part II yield the unique counterexamples to the extended tetragonal conjecture that exist generically in the bi-elliptic case.

Some results on special subvarieties of divisors for ramified double coverings appear in § 14. In § 15 we extend the tetragonal construction to allowable covers and we apply this construction to the coverings considered in our situation. In § 16 we start the proof of Theorem (16.1). In § 20 we give a complete description of the fibre of $\bar{P}$ over $P(\tilde{C}, C)$ with $(\tilde{C}, C)$ a generic element of $\mathscr{R}_{B,g}$.

14. Special subvarieties of divisors for ramified double coverings. In this section we shall collect various results. They are generalizations of known results (cf. [We 3], [Be 2]). The proofs are not given because they are similar to those of [We 3].

Let N be a projective irreducible smooth curve of genus g and let $\pi\colon \tilde{N} \to N$ be a double cover ramified at the points $\tilde{R}_1, \ldots, \tilde{R}_{2n}$. Let Λ be a linear system on N of degree d (not necessarily complete) of dimension $\geqq 1$. The special subvariety determined by Λ is, by definition, the variety X_Λ given by the following pull-back diagram:

$$\begin{array}{ccc} X_\Lambda & \xrightarrow{i} & \tilde{N}^{(d)} \\ {\scriptstyle \pi^{(d)}_{|X_\Lambda} =: \pi_\Lambda} \downarrow & & \downarrow {\scriptstyle \pi^{(d)}} \\ \Lambda & \xrightarrow{j} & N^{(d)}. \end{array}$$

(14.1) Proposition (Connectedness criterion). *If Λ is base-point-free, then X_Λ is connected.*

(14.2) Proposition (Irreducibility criterion). *If Λ is base-point-free and the codimension of* Sing X_Λ *in X_Λ is greater than or equal to* 2, *then X_Λ is irreducible.*

(14.3) Proposition (Smoothness criterion). *Assume that Λ is complete and base-point-free. Let $D \in \Lambda$ and let $\tilde{D} \in X_\Lambda$ such that $\pi_\Lambda(\tilde{D}) = D$. Put*

$$\tilde{D} = \pi^*(A) + \tilde{B} + \tilde{R}_{i_1} + \ldots + \tilde{R}_{i_k}, \quad i_j \neq i'_j, \text{ if } j \neq j'$$

with A, $\tilde{B}$ effective and $\tilde{B}$ simple with respect to π and not containing ramification points. Then X_Λ is smooth at $\tilde{D}$ if and only if

$$h^0(D - A - \pi(\tilde{R}_{i_1}) - \ldots - \pi(\tilde{R}_{i_k})) = h^0(D) - \deg(A) - k.$$

15. The generalized tetragonal construction. In this section we give a natural way to extend the tetragonal construction (cf. [Do], [Be2]) to allowable double covers. We follow the idea suggested by Beauville in [Be2], Remarque 4, p. 364. We do not need here the hypothesis of stability on the curves. We do not give the proofs.

Let $\pi\colon \tilde{D} \to D$ an allowable double covering with $c_e(\tilde{D}, D) = 0$ (cf. [Be1]) and ι the associated involution on $\tilde{D}$. We say that D is tetragonal if it can be represented as a four-to-one cover of the projective line. We denote by $\mathrm{Div}^d(\tilde{D})$ and $\mathrm{Div}^d(D)$ the varieties which parametrize the effective Cartier divisors of degree d on $\tilde{D}$ and D, respectively. Recall that the group of Cartier divisors on $\tilde{D}$ is:

$$\mathrm{Div}(\tilde{D}) = \bigoplus_{x \in \tilde{C}_{\mathrm{reg}}} \mathbb{Z}x + \bigoplus_{s \, \mathrm{singular}} \tilde{K}_s^* / \mathcal{O}_s^*$$

where $\tilde{K}$ is the ring of rational functions on $\tilde{D}$. Choosing uniformizing parameters t_1 and t_2 at the preimages $\tilde{s}_1$ and $\tilde{s}_2$ in the normalization of $\tilde{D}$ of a singular point $\tilde{s}$ one finds an isomorphism $\tilde{K}_{\tilde{s}}^* / \mathcal{O}_{\tilde{s}}^* \xrightarrow{\cong} \mathbb{C}^* \times \mathbb{Z} \times \mathbb{Z}$.

The four-to-one covering $\gamma\colon D \to \mathbb{P}^1$ induces an inclusion $\mathbb{P}^1 \xrightarrow{\gamma^*} \mathrm{Div}^4(D)$. On the other hand there exists a norm map ([Be1], p. 158):

$$\mathrm{Nm}_\pi\colon \mathrm{Div}^4(\tilde{D}) \to \mathrm{Div}^4(D).$$

Imitating the tetragonal construction for the smooth case (cf. [Do]), we obtain two allowable double covers $(\tilde{X}_1, X_1)$ and $(\tilde{X}_2, X_2)$, where X_1 and X_2 are tetragonal.

(15.1) Proposition. *The following properties hold*:

i) *The tetragonal construction applied to* $(\tilde{X}_1, X_1)$ (*resp.* $(\tilde{X}_2, X_2)$) *with its inherited tetragonal structure yields* $(\tilde{X}_2, X_2)$ (*resp.* $(\tilde{X}_1, X_1)$) *and* $(\tilde{D}, D)$.

ii) $P(\tilde{X}_1, X_1) \cong P(\tilde{X}_2, X_2) \cong P(\tilde{D}, D)$.

(15.2) Next we indicate how to apply the tetragonal construction to a covering $(\tilde{D}, D) \in \mathscr{H}'_{g,0}$ such that D is obtained from an irreducible hyperelliptic curve H by identifying two non-hyperelliptic pairs of points x_1, x_2 and y_1, y_2. The curve D is tetragonal in two different ways:

a) The curve D is the stable reduction of the curve $D' = \mathbb{P}^1 \cup H \cup \mathbb{P}^1$ where H intersects the first copy of $\mathbb{P}^1$ in two points: x_1 and x_2, the second copy in the points y_1 and y_2 and the two $\mathbb{P}^1$ are disjoint. The curve D' is clearly tetragonal. Applying the tetragonal construction we obtain a single cover. One shows that it belongs to $\mathscr{R}_{B,g,0}$.

b) Let $\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2 \in \mathbb{P}^1$ be the images of x_1, x_2, y_1, y_2 by the hyperelliptic morphism. There is a unique double covering $\mathbb{P}^1 \xrightarrow{(2:1)} \mathbb{P}^1$ sending each pair $\bar{x}_1, \bar{x}_2$ and $\bar{y}_1, \bar{y}_2$ to a single point. The four-to-one covering $H \to \mathbb{P}^1$ obtained by composing the hyperelliptic map with the (2:1) morphism above factorizes through D. In this case the tetragonal construction gives two covers: one in $\mathscr{H}'_{g,0}$ and the other in $\bar{\mathscr{R}}'_{B,g}$ (compare with (2.10)) (in fact, with the notations of (16.3), this second element belongs to $\mathscr{R}''_{B,g}$).

16. The Main Theorem.

In this section we state the central Theorem of Part III.

(16.1) Theorem. *Let* $(\tilde{C}, C)$ *be a generic element of* $\mathscr{R}_{B,g}$ *and let* $(\tilde{D}, D) \in \bar{\mathscr{R}}_g$ *such that* $P(\tilde{C}, C) \cong P(\tilde{D}, D)$. *Then one (and only one) of the following two facts occurs*:

i) $(\tilde{C}, C)$ *and* $(\tilde{D}, D)$ *are tetragonally related.*

ii) $(\tilde{C}, C) \in \mathscr{R}_{B,g,4}$ *and* $(\tilde{D}, D)$ *is obtained from* $(\tilde{C}, C)$ *as in the bi-elliptic construction* (*see* § 11).

Let $(\tilde{C}, C)$ be a generic element of $\mathscr{R}_{B,g}$. Let $(\tilde{D}, D) \in \bar{\mathscr{R}}_g$ be such that $P(\tilde{D}, D) \cong P(\tilde{C}, C)$. The theta divisor of $P(\tilde{D}, D)$ is singular in codimension 3 and $P(\tilde{D}, D)$ is not a Jacobian (cf. [Sh1] and (3.2), (3.3)). Then, [Be1], Th. 5.4 implies that $c_e(\tilde{D}, D) = 0$. On the other hand in Th. (4.10) of loc. cit. there is a list of coverings with $c_e = 0$ and dimension of the singular locus of the theta divisor equal to $g-5$. Since $P(\tilde{C}, C)$ is not a Jacobian and $g \geqq 10$, we are in, at least, one of the following cases:

(16.2) a) D is a double cover of a stable curve of genus 1,

b) $(\tilde{D}, D) \in \mathscr{H}'_{g,0}$,

c) $(\tilde{D}, D) \in \mathscr{H}'_{g,1}$,

d) $(\tilde{D}, D) \in \mathscr{H}'_{g,t}$ where $2 \leqq t \leqq \left[\frac{g-1}{2}\right]$

(cf. (2.10) for definitions).

(16.3) Remark. We shall use the notations

$$\mathscr{R}'_{B,g,t} = \{(\tilde{\Gamma}, \Gamma) \in \bar{\mathscr{R}}_{B,g,t} \mid \Gamma \text{ verifies (16.2) a)}\}, \quad t = 0, \ldots, \left[\frac{g-1}{2}\right],$$

$$\mathscr{R}''_{B,g} = (\mathscr{R}_{B,g}')' = \{(\tilde{\Gamma}, \Gamma) \in \bar{\mathscr{R}}'_{B,g} \mid \Gamma \text{ verifies (16.2) a)}\}.$$

The spaces $\mathscr{H}'_{g,t}$, $\mathscr{R}'_{B,g,t}$ for $t = 0, \ldots, \left[\frac{g-1}{2}\right]$ and $\mathscr{R}''_{B,g}$ are not closed in $\bar{\mathscr{R}}_g$.

The aim of this section is to prove the theorem in the cases (16.2) a), (16.2) b) and (16.2)c). The possibility (16.2) d) will be considered in sections 17, 18 and 19.

We first treat the possibility (16.2) b).

(16.4) Proposition. *Let $(\tilde{C}, C)$ be a generic element of $\mathscr{R}_{B,g}$. Let $(\tilde{D}, D) \in \mathscr{H}'_{g,0}$ be such that $P(\tilde{D}, D) \cong P(\tilde{C}, C)$. Then $(\tilde{C}, C)$ and $(\tilde{D}, D)$ are tetragonally related.*

Proof. Let H be a hyperelliptic curve such that D is constructed from H by identifying two pairs of points. If any of the pairs is hyperelliptic, then D is obtained from a hyperelliptic curve by identifying a pair of points. By (4.10) in [Be1], $P(\tilde{D}, D)$ is a Jacobian and we get a contradiction. Now, an easy dimension count shows that the genericity of $(\tilde{C}, C)$ implies that H is irreducible. By (15.2), the tetragonal construction gives a cover $(\tilde{C}', C') \in \mathscr{R}_{B,g,0}$ tetragonally related with $(\tilde{D}, D)$. Then by (10.10) and (9.4) either $(\tilde{C}', C') = (\tilde{C}, C)$ or $(\tilde{C}, C)$ is tetragonally related with $(\tilde{C}', C')$ (and hence with $(\tilde{D}, D)$). □

Now we treat the possibility (16.2) a).

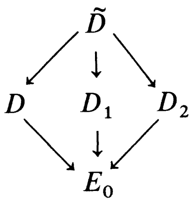
(16.5) Proposition. *Let $(\tilde{C}, C)$ be a general element of $\mathscr{R}_{B,g}$ and let $(\tilde{D}, D) \in \bar{\mathscr{R}}_g$ be such that D is a double cover of a stable curve E_0 of genus 1 and $P(\tilde{D}, D) \cong P(\tilde{C}, C)$. Then $(\tilde{C}, C)$ and $(\tilde{D}, D)$ are tetragonally related.*

Proof. If D is smooth, then the statement is a consequence of the results of Part I. Assume that D is singular. Observe that a stable curve of genus 1 is irreducible with, at most, one double point.

If D is reducible, it consists in the union of two curves of genus $\leqq 1$ intersecting in, at most, $g+1$ points, hence belongs to a subspace of codimension at least 2 in $\bar{\mathscr{R}}_{B,g}$. But this is impossible since $\dim P(\mathscr{R}_{B,g,t}) \geqq 2g-3$ and $(\tilde{C}, C)$ is generic. Therefore D is irreducible. For the same reasons D either has one singularity or two singularities with image a singularity of E_0. In the second case the element $(\tilde{D}, D)$ belongs to $\mathscr{H}'_{g,0}$ and by (16.4) the statement follows. In the rest of the proof we assume that D has one singularity.

If E_0 is singular then D is obtained by identifying a pair of points on a hyperelliptic curve. By [Be 1], (4.10) this implies that $P(\tilde{D}, D)$ is the Jacobian of a curve and we get a contradiction with [Sh 1]. Hence E_0 is smooth.

We treat first the case $\mathrm{Gal}_{E_0}(\tilde{D}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. There exist two involutions ι'_1 and ι'_2 on $\tilde{D}$ lifting the involution on D. By construction, ι'_1 and ι'_2 exchange the branches of the singularity of $\tilde{D}$. Then one obtains the following commutative diagram:



where $D_i := \tilde{D}/\iota'_i$, $i = 1, 2$, are smooth curves and the discriminant divisors of $D_1 \to E$ and $D_2 \to E_0$ intersect in a point (in particular $t \geqq 1$). By (2.10) this element is obtained by applying the tetragonal construction to an element of $\mathscr{R}_{B,g,t}$ for some t. By the results of Part I $(\tilde{D}, D)$ and $(\tilde{C}, C)$ are tetragonally related.

Finally assume that $\mathrm{Gal}_E(\tilde{D}) \cong \mathbb{Z}/2\mathbb{Z}$. Then $(\tilde{D}, D) \in \mathscr{R}''_{B,g}$ (cf. (16.3)). Proposition (16.5) is now a consequence of the following Lemma and the results of Part I.

(16.6) Lemma. *With these assumptions, there exists an element $(\tilde{C}', C') \in \mathscr{R}_{B,g,0}$ tetragonally related to $(\tilde{D}, D)$.*

Proof. It is easy to check that the injection $j\colon \mathscr{R}'_{B,g} \hookrightarrow \mathscr{R}_{B,g,0}$ (commuting with the Prym map) given in § 10 extends to $\mathscr{R}''_{B,g}$ (replace the symmetric products $\tilde{D}^{(2)}$, $D^{(2)}$ by the varieties of effective Cartier divisors of degree 2 $\mathrm{Div}^2(\tilde{D})$, $\mathrm{Div}^2(D)$). Since for all $(\tilde{D}, D)$, the elements $(\tilde{D}, D)$ and $j(\tilde{D}, D)$ are tetragonally related (cf. § 10) we are done. □

Before proceeding to cases (16.2) c) and (16.2) d), we prove the following two facts, which will be very useful in the rest of the paper.

(16.7) Lemma. *Let $(\tilde{C}, C)$ be a general element of $\mathscr{R}_{B,g,t}$ with $t \geqq 1$. Then $P(\tilde{C}, C)$ is isogenous to a product of two simple abelian varieties of dimensions t and $g - t - 1$. If $(\tilde{C}, C)$ is a generic element of $\mathscr{R}_{B,g,0} \cup \mathscr{R}'_{B,g}$, then $P(\tilde{C}, C)$ is simple.*

Proof. By (2.8) and (2.11) all we have to prove is simplicity. This is a consequence of Proposition (4.7) in [C-G-T]. □

(16.8) Corollary. *Let $(\tilde{C}, C)$ be a generic element of $\mathscr{R}_{B,g}$ and let $(\tilde{D}, D) \in \mathscr{H}'_{g,t}$ with $t \geqq 1$ such that $P(\tilde{C}, C) \cong P(\tilde{D}, D)$. We write $D = D_1 \cup_4 D_2$ where $g(D_1) = t - 1$ and $g(D_2) = g - t - 2$. Then:*

a) *the curves* D_1 *and* D_2 *are irreducible,*

b) $(\tilde{C}, C) \in \mathscr{R}_{B,g,t}$.

Proof. It is left to the reader. □

Next we consider the case (16.2) c).

(16.9) Proposition. *Let* $(\tilde{C}, C)$ *be a generic element of* $\mathscr{R}_{B,g}$ *and let* $(\tilde{D}, D) \in \mathscr{H}'_{g,1}$ *be such that* $P(\tilde{D}, D) \cong P(\tilde{C}, C)$. *Then* $(\tilde{C}, C)$ *and* $(\tilde{D}, D)$ *are tetragonally related.*

Proof. We write $D = \mathbb{P}^1 \cup_4 D_2$ where D_2 is a hyperelliptic curve (cf. (2.10)). By (16.8) a) D_2 is irreducible. By applying the tetragonal construction (see § 15) one finds an element $(\tilde{C}', C') \in \mathscr{R}'_{B,g,1}$ tetragonally related to $(\tilde{D}, D)$. By (16.5) we are done. □

17. The case (16.2) d). The aim of this section is to prove the following (compare with (16.2)):

(17.1) Proposition. *Let* $(\tilde{C}, C)$ *be a general element of* $\mathscr{R}_{B,g}$ *and let*

$$(\tilde{D}, D) \in \mathscr{H}'_{g,t} - \Big(\mathscr{H}'_{g,0} \cup \mathscr{H}'_{g,1} \cup \Big(\bigcup_{s=0}^{\left[\frac{g-1}{2}\right]} \mathscr{R}'_{B,g,s}\Big) \cup \mathscr{R}''_{B,g}\Big) \text{ with } t \geqq 2$$

such that $P(\tilde{C}, C) \cong P(\tilde{D}, D)$ *(see* (16.3)*). Then* $(\tilde{C}, C)$ *and* $(\tilde{D}, D)$ *are tetragonally related or at least one of the following facts occurs*:

a) $\tilde{D} = \tilde{D}_1 \cup_4 \tilde{D}_2$, $D = D_1 \cup_4 D_2$ *and* D_1 *is an irreducible plane quartic. Writing* $D_1 \cap D_2 = \{x_1 + \ldots + x_4\}$, *one has* $\mathcal{O}_{D_1}(x_1 + \ldots + x_4) = \omega_{D_1}$. *The curve* D_2 *is irreducible and hyperelliptic of genus* $g-5$. *In this case* $(\tilde{C}, C) \in \mathscr{R}_{B,g,4}$.

b) $\tilde{D} = \tilde{D}_1 \cup_4 \tilde{D}_2$ *and* $D = D_1 \cup_4 D_2$ *with* D_1, D_2 *irreducible hyperelliptic curves of genus* $t-1$ *and* $g-t-2$ *respectively, with* $t \geqq 2$. *In this case* $(\tilde{C}, C) \in \mathscr{R}_{B,g,t}$.

(17.2) Remark. In § 18 we shall prove that possibility (17.1) a) implies that $(\tilde{D}, D)$ is constructed from $(\tilde{C}, C)$ as in § 11. In § 19 we shall see that possibility (17.1) b) implies that $(\tilde{C}, C)$ and $(\tilde{D}, D)$ are tetragonally related. These facts complete the proof of (16.1).

Proof. Recall that $P(\tilde{C}, C)$ is not a Jacobian and that $g \geqq 10$. By (16.8) b) $(\tilde{C}, C) \in \mathscr{R}_{B,g,t}$. On the other hand $D = D_1 \cup_4 D_2$ where D_1 and D_2 are irreducible (cf. (16.8) a)).

The following fact is a particular case of (5.12) in [Sh 2]:

(17.3) Proposition. *Let* $\pi: \tilde{D} \to D$ *as above and let* X *an irreducible component of* $\operatorname{Sing} \Xi$ *of dimension* $g-5$. *Then we are in one of the cases* a), b), c), d), e) *below and* X, *thought in the natural model* Ξ^*, *is contained in the respective varieties* Z_a, Z_b, Z_c, Z_d, *or* Z_e *(cf.* [Sh 2], (3.21) *and* § 1 *for definitions)*:

a) *D is obtained by identifying two pairs of points on a curve H. There exists a morphism* $\gamma\colon H \to \mathbb{P}^1$ *of degree* 2 *over the generic point of* $\mathbb{P}^1$. *Let*

$$\begin{array}{ccc} \tilde{H} & \xrightarrow{\tilde{h}} & \tilde{D} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \pi} \\ H & \longrightarrow & D \end{array}$$

be the partial desingularizations. Then

$$Z_a = \text{closure of } \{\tilde{L} \in P(\tilde{D}, D)^* \mid \tilde{h}^0(\tilde{L}) = q^*(\gamma^*(\mathcal{O}_{\mathbb{P}^1}(1)))(\tilde{A}) \text{ where } \tilde{A} \text{ is an effective divisor with non singular support}\}.$$

b) *Let* $\tilde{D} = \tilde{D}_1 \cup_4 \tilde{D}_2$. *If* $\tilde{f}$ *is the partial desingularization of* $\tilde{D}$ *at* $\tilde{D}_1 \cap \tilde{D}_2$, *then*

$$Z_b = (\tilde{f}^0)^{-1}(\Xi_1^* \times \Xi_2^*).$$

In this case the codimension of Ξ_i^* *in* $P(\tilde{D}_i, D_i)^*$, $i = 1, 2$ *is exactly* 2 *and* $\dim Z_b = g - 5$.

c) *Let* $\tilde{D} = \tilde{D}_1 \cup_4 \tilde{D}_2$. *A component of D, say,* D_1 *is hyperelliptic with* γ *the attached* (2:1) *map. If* $\tilde{f}$ *is the partial desingularization of* $\tilde{D}$ *at* $\tilde{D} \cap \tilde{D}_2$, *then*

$$Z_c = (\tilde{f}^0)^{-1}(ex_1^* \times P(\tilde{D}_2, D_2)^*),$$

where

$$ex_1^* = \text{closure of } \{\pi^*(\gamma^*(\mathcal{O}_{\mathbb{P}^1}(1)))^*(\tilde{A}) \in P(\tilde{D}_1, D_1) \mid \text{ where } \tilde{A} \text{ is an effective divisor with non singular support}\}.$$

d) D_1 *a plane quartic. Writing* $D_1 \cap D_2 = \{x_1 + \ldots + x_4\}$, *it is*

$$\mathcal{O}_{D_1}(x_1 + \ldots + x_4) = \omega_{D_1}.$$

One has

$$Z_d = \text{closure of } \{\tilde{L} = \pi^*(M)(\tilde{A}) \in P(\tilde{D}, D)^* \mid \tilde{A} \text{ is an effective divisor with non singular support and } M \in \operatorname{Pic}^4(D) \text{ with } h^0(M) \geqq 2 \text{ and } M_{|D_1} = \omega_{D_1}\}.$$

e) *There exists a morphism* $\varepsilon\colon D \to E_0$ *onto a curve* E_0 *consisting of at most two irreducible components; the genus of* E_0 *is equal to* 1 *and the morphism* ε *has degree* 2 *over the generic points of* E_0. *We will not need the description of* Z_e.

We shall call in each case Z_a^m, Z_b^m, Z_c^m, Z_d^m, $(\Xi^*)^m$ and $(ex^*)^m$ the union of the components of maximal dimension.

We use (17.3) to identify the components of Sing Ξ of dimension $g - 5$ in P$(\tilde{D}, D)$. Note that $t \geqq 2$ implies that $W_0 \neq \emptyset$ for all t.

(17.4) Lemma. *Let $\tilde{C}$, C) and $(\tilde{D}, D)$ be as above. Then Z_b^m is irreducible and via the isomorphism $P(\tilde{D}, D) \cong P(\tilde{C}, C)$ it corresponds to the component W_0 of* $\operatorname{Sing} \Xi^*$ (*cf.* (2.7) *and* (17.3) *for definitions and notations*).

Proof. Indeed, let X_1 and X_2 be components of $(\Xi_1^*)^m$ and $(\Xi_2^*)^m$, respectively. Then $(\tilde{f}^0)^{-1}(X_1 \times X_2)$ is irreducible: if not, different components of $\operatorname{Sing} \Xi^*$ of dimension $g-5$ would be exchanged by translations. From the definitions of W_i, $i = -2, 0, 2$ (cf. (2.6), (3.7)) it is easy to check this is not possible in $P(\tilde{C}, C)$ and we get a contradiction.

On the other hand

$$\tilde{f}^*(I((\tilde{f}^0)^{-1}(X_1 \times X_2))) = I(X_1) \times I(X_2).$$

By (16.7), $P(\tilde{D}_1, D_1)$ and $P(\tilde{D}_2, D_2)$ are simple. Thus, for $i = 1, 2$ either $I(X_i)$ is finite or $I(X_i) = P(\tilde{D}_i, D_i)$. Let $\tilde{L}_i$ be a generic element of X_i, $i = 1, 2$. Then $h^0(\tilde{L}_i) = 1$ (recall that $\operatorname{codim}_{P(\tilde{D}_i, D_i)} X_i = 2$). Now (cf. e.g. (3.14) of [Sh 2]) $h^0(\tilde{L}_i(\tilde{x}_i - \iota'(\tilde{x}_i))) = 0$, where $\tilde{x}_i$ is a generic point in $\tilde{D}_i$ and ι' is the natural involution. Therefore $\tilde{x}_i - \iota'(\tilde{x}_i) \notin I(X_i)$. We conclude that $I(X_1)$, $I(X_2)$ and $I((\tilde{f}^0)^{-1}(X_1 \times X_2))$ are finite. Hence $(\tilde{f}^0)^{-1}(X_1 \times X_2)$ is an irreducible component of $\operatorname{Sing} \Xi^*$ invariant only by a finite group. Only the component W_0 verifies this property (cf. (6.1)), therefore $X_i = (\Xi_i^*)^m$, for $i = 1, 2$ and Z_b^m is an irreducible component of $\operatorname{Sing} \Xi^*$ corresponding to W_0. □

In the situation of (17.4), $\deg(\tilde{f}^*) = 4$ (cf. [Be 1], (3.6)), thus from the proof of (17.4) one also obtains that $I((\Xi_i^*)^m) = 0$, $i = 1, 2$, and $I(Z_b^m) = \ker \tilde{f}^*$.

(17.5) Lemma. *Assume that one of the components of D, say D_1, is hyperelliptic and that* $\dim Z_c = g - 5$ (*cf.* (17.3)). *Then the corresponding variety Z_c^m is irreducible.*

Proof. Arguing as in Lemma (17.4), if X is a component of $(ex_1^*)^m$, then $(\tilde{f}^0)^{-1}(X \times P(\tilde{D}_2, D_2)^*)$ is irreducible. Suppose that Y is another component of $(ex_1^*)^m$. Since Z_b^m is non empty and corresponds to W_0, then the isomorphism $P(\tilde{D}, D) \cong P(\tilde{C}, C)$ sends $(\tilde{f}^0)^{-1}(X \times P(\tilde{D}_2, D_2)^*) \cup (\tilde{f}^0)^{-1}(Y \times P(\tilde{D}_2, D_2)^*)$ to $W_{-2} \cup W_2$. On the other hand

$$\tilde{f}^*(I((\tilde{f}^0)^{-1}(X \times P(\tilde{D}_2, D_2)^*))) \cap I((\tilde{f}^0)^{-1}(Y \times P(\tilde{D}_2, D_2)^*)) \supset \{0\} \times P(\tilde{D}_2, D_2).$$

Hence we get a contradiction because

$$I(W_2) \cap I(W_{-2}) \text{ is finite}.$$

Therefore $(ex_1^*)^m$ and Z_c^m are irreducible. □

(17.6) Lemma. *With our hypothesis, if $(\tilde{D}, D)$ verifies also* (17.3) a), *then* $\dim Z_a^m < g - 5$.

Proof. The unique configuration of the type of (17.3) a) compatible with $D = D_1 \cup_4 D_2$, D_1 and D_2 irreducible, and $(\tilde{D}, D) \notin \mathscr{H}'_{g,0}$ is the following one:

The normalization of D at two points of $D_1 \cap D_2$ is a curve H admiting a (2:1) map $\gamma: H \to \mathbb{P}^1$ which is constant on one of the curves, say D_2.

Assume that $\dim Z_a^m = g-5$. We call $\tilde{H}$ the curve obtained by normalizing $\tilde{D}$ at the two points corresponding to the above ones, and we write q for the double cover $\tilde{H} \to H$. Let $\tilde{d}_1, \tilde{d}_2 \in \tilde{H}$ be the preimages of the remaining points in $\tilde{D}_1 \cap \tilde{D}_2$. Let $\tilde{g}$ the partial desingularization of $\tilde{H}$ in $\tilde{d}_1, \tilde{d}_2$. One has the isogenies (cf. [Sh2], (3.21))

$$P(\tilde{D}, D)^* \xrightarrow{\tilde{h}^0} P(\tilde{H}, H)^* \xrightarrow{\tilde{g}^0} P(\tilde{D}_1, D_1)^* \times P(\tilde{D}_2, D_2)^*$$

where $\tilde{h}$ is the desingularization of $\tilde{D}$ at $\tilde{D}_1 \cap \tilde{D}_2$. Let $\tilde{L}$ be a general element of Z_a, then $\tilde{h}^0(\tilde{L}) = q^*(\gamma^*(\mathcal{O}_{\mathbb{P}_1}(1)))(\tilde{A})$, with $\tilde{A}$ an effective divisor with non singular support. Thus

$$\begin{aligned}
\tilde{g}^0(\tilde{h}^0(\tilde{L})) &= \tilde{g}^0(q^*(\gamma^*(\mathcal{O}_{\mathbb{P}^1}(1)))(\tilde{A}))\\
&= (q^*(\gamma^*(\mathcal{O}_{\mathbb{P}^1}(1)))(\tilde{A})_{|\tilde{D}_1}(-\tilde{d}_1-\tilde{d}_2), q^*(\gamma^*(\mathcal{O}_{\mathbb{P}_1}(1)))(\tilde{A})_{|\tilde{D}_2}(-\tilde{d}_1-\tilde{d}_2))\\
&= (\mathcal{O}_{\tilde{D}_1}(2\tilde{d}_1+2\tilde{d}_2)(\tilde{A}_1)(-\tilde{d}_1-\tilde{d}_2), \mathcal{O}_{\tilde{D}_2}(-\tilde{d}_1-\tilde{d}_2)(\tilde{A}_2))\\
&= (\mathcal{O}_{\tilde{D}_1}(\tilde{d}_1+\tilde{d}_2)(\tilde{A}_1), \mathcal{O}_{\tilde{D}_2}(-\tilde{d}_1-\tilde{d}_2)(\tilde{A}_2)),
\end{aligned}$$

where $\mathcal{O}_{\tilde{D}}(\tilde{A})_{|\tilde{D}_i} = \mathcal{O}_{\tilde{D}_i}(\tilde{A}_i)$, $i = 1, 2$. Hence:

$$\tilde{g}^0\tilde{h}^0(Z_a) \subset \{\tilde{L}_1 \in \Xi_1^* \,|\, h^0(\tilde{L}_1(-\tilde{d}_1-\tilde{d}_2)) > 0\} \times \{\tilde{L}_2 \in P(\tilde{D}_2, D_2)^* \,|\, h^0(\tilde{L}_2(\tilde{d}_1+\tilde{d}_2)) > 0\}.$$

It is easy to check that the dimensions of the sets on the right hand side are less than or equal to (a posteriori equal to) $\dim P(\tilde{D}_1, D_1) - 3$ and $\dim P(\tilde{D}_2, D_2) - 1$, respectively. Therefore, if X is a component of Z_a^m, there exist irreducible components X_1 and X_2 of the sets on the right hand side such that $\tilde{g}^0(\tilde{h}^0(X))\ X_1 \times X_2$. Arguing as in Lemma (17.4), one finds that $\tilde{x} - \iota'(\tilde{x})$ does not belong to $I(X_i)$ if $\tilde{x}$ is general in $\tilde{D}$ and ι' is the involution. Therefore the simplicity of $P(\tilde{D}_i, D_i)$ (cf. (16.7)) implies that $I(X_i)$ is finite for $i = 1, 2$. In particular $I(X)$ is finite. Hence X corresponds to W_0 by the isomorphism $P(\tilde{D}, D) \cong P(\tilde{C}, C)$. Since the components Z_a^m and Z_b^m are different (take $f = g \circ h$ and compare $\tilde{f}^0(Z_a)$ computed above with $\tilde{f}^0(Z_b) = \Xi_1^* \times \Xi_2^*$) one gets a contradiction with (17.4).

(17.7) Lemma. *Keeping our assumptions, suppose that* $(\tilde{D}, D)$ *verifies* (17.3) d) *and that* $\dim Z_d = g - 5$. *Then* Z_d *is irreducible* (*in particular* $Z_d = Z_d^m$).

Proof. Writing $\tilde{f}$ for the partial normalization of $\tilde{D}$ at $\tilde{D}_1 \cap \tilde{D}_2$ one easily checks that

$$\tilde{f}^0(Z_d) \subset \{\tilde{l}\} \times P(\tilde{D}_2, D_2)^*$$

where $\tilde{l}$ is the ramification divisor of $\tilde{D}_1 \to D_1$. Since $(\tilde{f}^0)^{-1}(\{\tilde{l}\} \times P(\tilde{D}_2, D_2)^*)$ is irreducible and has dimension $g-5$ the result follows. □

Now we end the proof of Proposition (17.1). We can apply (17.3) in order to recognize the components of maximal dimension in $\operatorname{Sing} \Xi^*$. By (17.4) the component W_0 corresponds to Z_b^m. Since $t \geqq 2$ other components of maximal dimension exist (cf. (2.7)). According to (17.6), case (17.3) a) does not provide any component. Let us consider case e). One obtains that the only configuration of type (17.3) e) compatible with our hypothesis is:

D_1, D_2 are two hyperelliptic curves and $D_1 \cap D_2$ consists of two pairs of hyperelliptic points for both curves.

These elements parametrize a subspace of $\bar{\mathcal{R}}_g$ of dimension $2g - 4$ and this contradicts the genericity of $(\tilde{C}, C)$.

We conclude that $(\tilde{D}, D)$ verifies the hypothesis of (17.3) c) or (17.3) d). By (17.5) and (17.7) the components W_2 and W_{-2} correspond to types Z_c^m when $t \neq 4$, that is to say: the curves D_1 and D_2 are hyperelliptic. If $t = 4$, then one has a new possibility: one of the components corresponds to a variety of type Z_d^m, therefore the pair $(\tilde{D}, D)$ verifies (17.1) a). This finishes the proof of (17.1). □

18. The plane quartic case. This section is devoted to prove the following.

(18.1) Proposition. *Let $(\tilde{C}, C)$ be a generic element of $\mathcal{R}_{B,g}$ and let $(\tilde{D}, D) \in \mathcal{H}'_{g,4}$ be such that $P(\tilde{D}, D) \cong P(\tilde{C}, C)$, $D = D_1 \cup_4 D_2$ and D_1 is an irreducible plane quartic. Suppose also that if $D_1 \cap D_2 = \{x_1, \ldots, x_4\}$, then $\mathcal{O}_{D_1}(x_1 + \ldots + x_4) = \omega_{D_1}$, and that the curve D_2 is irreducible and hyperelliptic of genus $g - 5$.*

Then $(\tilde{D}, D)$ is constructed from $(\tilde{C}, C)$ as in the bi-elliptic construction of § 11.

Proof. It follows from (16.8) b) that $(\tilde{C}, C) \in \mathcal{R}_{B,g,4}$. From the proof of (17.1) we get that the isomorphism $P(\tilde{D}, D) \cong P(\tilde{C}, C)$ identifies Z_b^m with W_0, Z_c^m with W_2 and $Z_d^m = Z_d$ with W_{-2} (see (17.3)).

We shall use again the variety

$$\Lambda_2 = \{\tilde{a} \in P(\tilde{C}, C) \mid \tilde{a} + W_0 \cap W_2 \subset W_0\}$$

defined in (5.5).

One has

(18.2) Lemma. *With the hypothesis of* (18.1) *the following facts hold*:

a) *The curve $\Lambda_2 \cap 2\Lambda_2$ is birational to the curve $\tilde{B}_2$ obtained by the pull-back diagram*

$$\text{(18.3)} \qquad \begin{array}{ccc} \tilde{B}_2 & \longrightarrow & \tilde{N}_2^{(2)} \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{g_2^1} & E^{(2)}. \end{array}$$

where $\tilde{N}_2$ and N_2 are the normalizations of $\tilde{D}_2$ and D_2 respectively, and g_2^1 is the linear series induced by the hyperelliptic structure of D_2.

b) *The curve C_2 (see (2.1)) is the normalization of $\tilde{B}_2$.*

c) *The involution τ_2 in C_2 corresponds to the involution of $\tilde{B}_2$ given by the restriction of the natural involution of $\tilde{N}_2^{(2)}$.*

d) *There exists a linear series g_2^1 on E such that one gets a pull-back diagram*

$$\begin{array}{ccc} \tilde{D}_2 & \longrightarrow & C_2^{(2)} \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{g_2^1} & E^{(2)} . \end{array}$$

Moreover the involution $(\tau_2^{(2)})_{|\tilde{D}_2}$ exchanges the sheets of $\tilde{D}_2$.

Proof. We first see a). By using the identifications $W_0 = Z_b^m$ and $W_2 = Z_c^m$, and the definitions of Z_b^m, Z_c^m (cf. (17.3)) it is easy to see that

$$W_0 \cap W_2 = (\tilde{f}^*)^{-1}\left(({\Xi_1^*})^m \times \left((ex_2^*)^m \cap ({\Xi_2^*})^m\right)\right),$$

where $\tilde{f}$ is the normalization of $\tilde{D}$ at $\tilde{D}_1 \cap \tilde{D}_2$. On the other hand, by (5.3) the dimension of this set is $g-7$. This forces to have $(ex_2^*)^m \subset (\Xi_2^*)^m$. Hence

$$\Lambda_2 = (\tilde{f}^*)^{-1}\left(\{(\tilde{a}_1, \tilde{a}_2) \in P(\tilde{D}_1, D_1) \times P(\tilde{D}_2, D_2) | \tilde{a}_1 + (\Xi_1^*)^m \subset (\Xi_1^*)^m, \tilde{a}_2 + (ex_2^*)^m \subset (\Xi_2^*)^m\}\right).$$

In the proof of (17.4) we saw that $I\left((\Xi_1^*)^m\right) = (0)$. Therefore

$$\Lambda_2 = (\tilde{f}^*)^{-1}\left(\{0\} \times \{\tilde{a}_2 \in P(\tilde{D}_2, D_2) | \tilde{a}_2 + (ex_2^*)^m \subset (\Xi_2^*)^m\}\right).$$

Since $(\Xi_2^*)^m$ is irreducible (cf. (17.4)) and Sing Ξ^* has no components of dimension $g-6$, it is not hard to see that $(\Xi_2^*)^m$ is the closure of the set of effective divisors with non-singular support $\tilde{A}$ such that $\mathrm{Nm}(\tilde{A}) = \omega_{D_2}$. By using this one checks the inclusion

$$\{\tilde{x} + \tilde{y} - \tilde{r} - \tilde{s} \in P(\tilde{D}_2, D_2) | \tilde{x}, \tilde{y}, \tilde{r}, \tilde{s} \in (\tilde{D}_2)_{\mathrm{reg}}, \mathrm{Nm}(\tilde{x} + \tilde{y}) \in g_2^1\} + ex_2^* \subset (\Xi_2^*)^m .$$

Thus one has

$$(\tilde{f}^*)^{-1}\left(\{0\} \times \text{closure}\,\{\tilde{x} + \tilde{y} - \tilde{r} - \tilde{s} \in P(\tilde{D}_2, D_2) | \tilde{x}, \tilde{y}, \tilde{r}, \tilde{s} \in (\tilde{D}_2)_{\mathrm{reg}}, \mathrm{Nm}(x + y) \in g_2^1\}\right) \subset \Lambda_2 .$$

From this inclusion a straightforward computation gives

$$\begin{aligned} \{0\} \times \text{closure}\,\{\tilde{x} + \tilde{y} - \iota'(\tilde{x}) - \iota'(\tilde{y}) \in P(\tilde{D}_2, D_2) | \tilde{x}, \tilde{y} \in (\tilde{D}_2)_{\mathrm{reg}}, \mathrm{Nm}(\tilde{x} + \tilde{y}) \in g_2^1\} \\ \subset \tilde{f}^*(\Lambda_2 \cap 2\Lambda_2), \end{aligned}$$

where ι' is the natural involution on $\tilde{D}_2$. Since the curve on the right hand side is irreducible (cf. (5.7)) one has an equality. By using the description of $\Lambda_2 \cap 2\Lambda_2$ in $P(\tilde{C}, C)$ one obtains that $\Lambda_2 \cap 2\Lambda_2$ is birationally isomorphic to $\Lambda_2 \cap 2\Lambda_2 / \pi^*\left(\varepsilon^*({}_2JE)\right) = \tilde{f}^*(\Lambda_2 \cap 2\Lambda_2)$ (recall that $\mathrm{Ker}(\tilde{f}^*) = \pi^*\left(\varepsilon^*({}_2JE)\right)$). On the other hand there exists a natural map from the normalization of $\tilde{B}_2$ to the set of the left hand side in the inclusion above. Since C_2 is the normalization of $\Lambda_2 \cap 2\Lambda_2$ we get a morphism from the normalization of $\tilde{B}_2$ to C_2. Using (14.3), one checks that the genus of the normalization of $\tilde{B}_2$ is $g(C_2)$. Therefore C_2 and $\tilde{B}_2$ are isomorphic and a) is proved.

Part b) is a corollary of a). To see c) it suffices to recall that the multiplication by (-1) induces on C_2 the involution τ_2. Note that in this context this multiplication coincides on $\tilde{B}_2$ with the restriction of the involution on $\tilde{N}_2^{(2)}$.

Finally, we prove d). We first observe that c) implies that E is the normalization of $\tilde{B}_2/(\text{involution})$. Since this last curve has an obvious hyperelliptic structure given by diagram (18.2) we obtain on E a linear series g_2^1. The rest is left to the reader. □

As a consequence $(\tilde{D}_2, D_2)$ is obtained from $((C_2, E), g_2^1)$ as in Step 2 of §11.

Next we concentrate on the relation between (C_1, E) and $(\tilde{D}_1, D_1)$. We shall consider as above the surface

$$\Lambda_{-2} = \{\tilde{a} \in P(\tilde{C}, C) \mid \tilde{a} + W_0 \cap W_{-2} \subset W_0\}$$

defined in (5.5). From the descriptions of Z_b^m and Z_d (cf. (17.3)) one gets

$$\Lambda_{-2} = (\tilde{f}^*)^{-1}(((\Xi_1^*)^m - \{\tilde{l}\}) \times \{0\})$$

where $\tilde{l}$ is the ramification divisor of $\tilde{D}_1 \to D_1$. We call S the surface $((\Xi_1^*)^m - \{\tilde{l}\} \times \{0\})$. That is to say the group

$$\operatorname{Ker} \tilde{f}^* = I(W_0) = \pi^*(\varepsilon^*({}_2JE))$$

acts on Λ_{-2} and the quotient is S. We study first this surface in the more transparent context of $P(\tilde{C}, C)$.

(18.4) Proposition. *The surfaces S and $C_1^{(2)}$ are birationally equivalent.*

Proof. We borrow from (5.6) the equality

$$\Lambda_{-2} = \{\pi_1^*(\varepsilon_1^*(\bar{x}) - r - s) \mid \bar{x} \in E, r, s \in C_1, 2\bar{x} \equiv \varepsilon_1(r) + \varepsilon_1(s)\}.$$

Let $X \subset C_1^{(2)} \times E$ be the preimage of Λ_{-2} by the morphism

$$\begin{aligned} &C_1^{(2)} \times E \to J\tilde{C}, \\ &(r+s, \bar{x}) \to \pi_1^*(r + s - \varepsilon_1^*(\bar{x})). \end{aligned}$$

Then X is an unramified covering of degree 4 of $C_1^{(2)}$. One obtains the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \Lambda_{-2} \\ \downarrow & & \downarrow \\ C_1^{(2)} & \longrightarrow & \Lambda_{-2}/\pi^*(\varepsilon^*({}_2JE)) = S. \end{array}$$

The morphism $C_1^{(2)} \to S$ is an isomorphism away from the origin 0 and the preimage of 0 is the irreducible curve $\varepsilon_1^*(E)$, of positive genus. Thus S is exactly singular at the origin and $C_1^{(2)}$ is the minimal resolution of the singularity. □

We shall consider the plane quintic given by the union of D_1 and the line r containing the discriminant points of $\tilde{D}_1 \to D_1$. We call E' the elliptic curve obtained as the double cover of r with discriminant divisor $r \cap D_1$. By identifying in the natural way the ramification points of $\tilde{D}_1 \to D_1$ and $E' \to r$ one constructs an allowable double cover of the plane quintic mentioned above. By [Be 3], Proposition (6.23), there exists a smooth non hyperelliptic curve Γ of genus 5 such that

$$\begin{array}{ccc} W_4^1(\Gamma) & \xrightarrow{\cong} & \tilde{D}_1 \cup E' \\ \downarrow & & \downarrow \\ W_4^1(\Gamma)/\text{involution} & \xrightarrow{\cong} & D_1 \cup r. \end{array}$$

Now to prove that $(\tilde{D}_1, D_1)$ is constructed from C_1 as in Step 1 of § 11 it suffices to show that $\Gamma \cong C_1$.

(18.5) Proposition. *The surfaces S and $\Gamma^{(2)}$ are birationally equivalent.*

Proof. The description of S as a subset of $P(\tilde{D}_1, D_1) \times P(\tilde{D}_2, D_2)$ (cf. (17.3)) gives the isomorphism $S \cong (\Xi_1^*)^m$. The general element of $(\Xi_1^*)^m$ is an effective divisor of degree 4 with non-singular support. Its norm is a divisor on D_1 consisting of 4 points on a line. By construction the general point of $\tilde{D}_1$ corresponds to a linear series g_4^1 on Γ that does not come from linear series on E'.

Let x, y be general points of Γ. To contain the line $\overline{xy}$ is a linear condition for a quadric containing the canonical image of Γ in $\mathbb{P}^4$. The intersection of the pencil of quadrics so obtained with D_1 provides four singular quadrics containing $\overline{xy}$. Consequently there exist exactly four linear series g_4^1 on Γ passing through the divisor $x + y$. These four linear series define an effective divisor of degree 4 on $\tilde{D}_1$ and the image in D_1 are four collinear points. We obtain a generically injective rational map from $\Gamma^{(2)}$ to $(\Xi_1^*)^m$ and we are done. □

(18.6) Corollary. *The curves C_1 and Γ are isomorphic.*

Proof. By (18.4) and (18.5) it follows that $C_1^{(2)}$ and $\Gamma^{(2)}$ are birationally equivalent. Now the result is a consequence of a Theorem of Martens ([M]). □

Having established that $(\tilde{D}_i, D_i)$ is obtained from (C_i, E), $i = 1, 2$, as in Part II we end the proof of (18.1) showing that $(\tilde{D}, D)$ comes from $(\tilde{D}_1, D_1)$ and $(\tilde{D}_2, D_2)$ as in the Step 3 of § 11. Note first that the results just obtained make possible to use all the parts of (12.1) except the part iv). All we have to do to end the proof of (18.1) is to show that (12.1) iv) holds. Keeping this strategy in mind one constructs a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}_2JE & \longrightarrow & \varepsilon_1^*({}_2JE)\times\varepsilon_2^*({}_2JE) & \longrightarrow & \pi^*(\varepsilon^*({}_2JE)) = \operatorname{Ker}\tilde{f}^* & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}_2JE & \longrightarrow & P(C_1,E)\times P(C_2,E) & \xrightarrow{\varphi} & P(\tilde{C},C)\cong P(\tilde{D},D) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle h_1\times h_2} & & \downarrow{\scriptstyle \tilde{f}^*} & & \\
 & & & & P(\tilde{D}_1,D_1)\times P(\tilde{D}_2,D_2) & \underset{\cong}{\xrightarrow{\delta}} & P(\tilde{D}_1,D_1)\times P(\tilde{D}_2,D_2) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $\tilde{f}$ is the normalization of $\tilde{D}$ at $\tilde{D}_1 \cap \tilde{D}_2$ (cf. (2.8) for the definition of φ and cf. (17.4) and (6.1) for the top right corner). Since $\operatorname{End} P(\tilde{D}_i, D_i) \cong \mathbb{Z}$ (cf. [C-G-T], (4.7)), $\delta = (\pm \mathrm{Id}) + (\pm \mathrm{Id})$. Hence

$$(18.7) \qquad \tilde{f}^*({}_2P(\tilde{D},D)) = (h_1 \times h_2)(\varphi^{-1}({}_2P(\tilde{C},C))).$$

In (13.6) we saw that

$$\tilde{f}^*({}_2P(\tilde{D},D)) = \{(\tilde{\alpha}_1,\tilde{\alpha}_2) \in {}_2P(\tilde{D}_1,D_1) \times {}_2P(\tilde{D}_2,D_2) \mid v_1(\tilde{\alpha}_1) = v_2(\tilde{\alpha}_2)\}$$

(cf. §§4 and 12 for definitions). On the other hand it is easy to check that

$$\varphi^{-1}({}_2P(\tilde{C},C))$$
$$= \{(\tilde{\alpha}_1,\tilde{\alpha}_2) \in {}_2P(C_1,E) \times {}_2P(C_2,E) \mid \exists \bar{\varrho} \in {}_2JE \text{ such that } 2\tilde{\alpha}_1 = \varepsilon_1^*(\bar{\varrho}),\, 2\tilde{\alpha}_2 = \varepsilon_2^*(\bar{\varrho})\}.$$

Thus by applying $g_1 \times g_2$ to (18.7) one has

$$(18.8) \qquad g_1 \times g_2(\{(\tilde{\alpha}_1,\tilde{\alpha}_2) \in {}_2P(\tilde{D}_1,D_1) \times P(\tilde{D}_2,D_2) \mid v_1(\tilde{\alpha}_1) = v_2(\tilde{\alpha}_2)\})$$
$$= \{(\varepsilon_1^*(\bar{\varrho}), \varepsilon_2^*(\bar{\varrho})) \mid \bar{\varrho} \in {}_2JE\}.$$

Finally we show that (18.8) implies

$$v_1(\tilde{\alpha}_1) = v_2(\tilde{\alpha}_2) \quad \text{iff} \quad \exists \bar{\varrho} \in {}_2JE \quad \text{such that} \quad g_i(\tilde{\alpha}_i) = \varepsilon_i^*(\bar{\varrho})$$

for all $\tilde{\alpha}_1 \in P(\tilde{D}_1, D_1)$ and $\tilde{\alpha}_2 \in P(\tilde{D}_2, D_2)$. The part $\Rightarrow$ is clear. Suppose that $g_1(\tilde{\alpha}_1) = \varepsilon_1^*(\bar{\varrho})$ and $g_2(\tilde{\alpha}_2) = \varepsilon_2^*(\bar{\varrho})$ for $\bar{\varrho} \in {}_2JE$. Then by (18.8) there exist $(\tilde{\alpha}_1', \tilde{\alpha}_2')$ such that $v_1(\tilde{\alpha}_1') = v_2(\tilde{\alpha}_2')$ and $g_1(\tilde{\alpha}_1) = g_1(\tilde{\alpha}_1')$, $g_2(\tilde{\alpha}_2) = g_2(\tilde{\alpha}_2')$. Since $\operatorname{Ker} g_i = p_i^*({}_2JD_i)$, $i = 1, 2$ (cf. (12.1) i)) and these elements do not change the value of v_i the part $\Leftarrow$ follows. This finishes the proof of (18.1). □

19. The hyperelliptic case. In this section we end the proof of Theorem (16.1). Recall that (16.4), (16.5), (16.9) and (17.1) reduced the proof to two cases. In (18.1) we have treated the first. So, to finish the proof of Theorem it suffices to prove the following

(19.1) Proposition. *Let $(\tilde{C}, C)$ be a general element of $\mathscr{R}_{B,g}$ and let $(\tilde{D}, D) \in \mathscr{H}'_{g,t}$, $t \geqq 2$ such that $P(\tilde{C}, C) \cong P(\tilde{D}, D)$. We write $D = D_1 \cup_4 D_2$. Assume that D_1, D_2 are irreducible hyperelliptic curves of genus $t-1$ and $g-t-2$, respectively. Then $(\tilde{C}, C)$ and $(\tilde{D}, D)$ are tetragonally related.*

(19.2) Remark. Recall that in this case $(\tilde{C}, C) \in \mathscr{R}_{B,g,t}$ and with the notations of (17.3), the isomorphism $P(\tilde{D}, D) \cong P(\tilde{C}, C)$ identifies Z_b^m with W_0 and the two varieties of type Z_c^m corresponding to the two hyperelliptic components with W_2 and W_{-2} (one of them is empty exactly when $W_{-2} = \emptyset$.

Proof. If we prove that D is tetragonal we can apply the tetragonal construction to $(\tilde{D}, D)$ and we find elements of $\mathscr{R}'_{B,g,t}$ tetragonally related with $(\tilde{D}, D)$. Then, by (16.5), these elements will be tetragonally related to elements of $\mathscr{R}_{B,g,t}$ and $(\tilde{C}, C)$ and $(\tilde{D}, D)$ will be tetragonally related. Therefore the proposition is a consequence of the following fact.

(19.3) Proposition. *There exists a finite morphism of degree four, $\gamma\colon D \to \mathbb{P}^1$, whose restrictions to D_1 and D_2 coincide with the respective hyperelliptic morphism and such that $\gamma(D_1 \cap D_2)$ consists of four different points.*

Proof. What we have to do is to glue the hyperelliptic morphisms $\gamma_i\colon D_i \to \mathbb{P}^1$. Let $D_1 \cap D_2 = \{d_1, \ldots, d_4\}$. It suffices to prove the equality of cross ratios

$$(19.4) \qquad |\gamma_1(d_1) : \gamma_1(d_2) : \gamma_1(d_3) : \gamma_1(d_4)| = |\gamma_2(d_1) : \gamma_2(d_2) : \gamma_2(d_3) : \gamma_2(d_4)|\,.$$

Recall that we obtained in (18.2) that the irreducible curve $\Lambda_2 \cap 2\Lambda_2$ (cf. (5.5) and (5.7)) is birationally equivalent to the curve $\tilde{B}_2$ given by the pull-back diagram

$$\begin{array}{ccc} \tilde{B}_2 & \longrightarrow & \tilde{N}_2^{(2)} \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \longrightarrow & N_2^{(2)} \end{array}$$

where $\tilde{N}_2$ and N_2 are the normalizations of $\tilde{D}_2$ and D_2, respectively. Moreover the involution on $\Lambda_2 \cap 2\Lambda_2$ attached to the multiplication by -1 equals the involution on $\tilde{B}_2$ inherited from the involution of $\tilde{N}_2^{(2)}$. According to (5.7) we have that C_2 is the normalization of $\tilde{B}_2$ and therefore E is the normalization of $\tilde{B}_2/(\text{involution})$. Then from the analysis of the diagram (19.5) we get that the cross ratio $|\gamma_1(d_1) : \gamma_1(d_2) : \gamma_1(d_3) : \gamma_1(d_4)|$ coincides with the cross ratio of the four discriminant points of the obvious two-to-one covering $E \to \mathbb{P}^1$. In particular the points $\gamma(d_i^i)$, $i = 1, \ldots, 4$, are all different.

When $t \geqq 4$ the same argument works when replacing $\Lambda_2 \cap 2\Lambda_2$ by $\Lambda_{-2} \cap 2\Lambda_{-2}$ and $\tilde{B}_2$ by the curve $\tilde{B}_1$ given by the pull-back diagram analogous to (19.5). So the cross ratio at the right hand side in (19.4) also equals the cross ratio of the four discriminant points of certain two-to-one morphism from E to a projective line. This clearly implies the equality (19.4).

To conclude the proof we only need to consider cases $t = 2$ and $t = 3$.

Assume first $t = 3$. We denote by $\tilde{f}$ the desingularization of $\tilde{D}$ at $\tilde{D}_1 \cap \tilde{D}_2$. We call π_1 and π_2 to the ramified double covers $\tilde{D}_i \to D_i$, $i = 1, 2$, induced by the partial desingularization. One has (compare with (6.1) i) and (6.2)):

(19.6) Lemma. *The following equalities hold* (*cf.* (17.3) *for definitions*):

a) $I(Z_c^m) = (\tilde{f}^*)^{-1}(P(\tilde{D}_1, D_1) \times \{0\})$ (*this is true for* $t \geqq 1$).

b) $\bigcup_{\tilde{L} \in Z_b^m} ((Z_b^m)_{-\tilde{L}} \cap I(Z_c^m)) = (\tilde{f}^*)^{-1}(\{\tilde{L} - \tilde{M} \in P(\tilde{D}_1, D_1) \mid \tilde{L}, \tilde{M} \in (\Xi_1^*)^m\} \times \{0\})$.

Proof. We first see a). According to (6.1) and (19.2), the set $I(Z_c^m)$ is an abelian variety of dimension t containing $I(W_0) = I(Z_b^m) = \operatorname{Ker}(\tilde{f}^*)$ (see (17.4)). On the other hand the very definitions imply that $\tilde{f}^*(I(Z_c^m)) \supset P(\tilde{D}_1, D_1) \times \{0\}$. Hence

$$I(Z_c^m) \supset (\tilde{f}^*)^{-1}(P(\tilde{D}_1, D_1) \times \{0\}).$$

Equality of dimensions concludes the proof of a).

In part b) we only show the inclusion of the left hand side member in the right hand side member. The opposite inclusion is left to the reader. Fix $\tilde{L} \in Z_b^m$. By definition $\tilde{f}^0(\tilde{L}) = (\tilde{L}_1, \tilde{L}_2) \in (\Xi_1^*)^m \times (\Xi_2^*)^m$. Then

$$\begin{aligned}(Z_b^m)_{-\tilde{L}} \cap I(Z_c^m) &= \{\tilde{\alpha} \in P(\tilde{D}, D) \mid \tilde{f}^*(\tilde{\alpha}) = (\tilde{\alpha}_1, 0) \text{ and } \tilde{\alpha} + \tilde{L} \in Z_b^m\} \\ &= \{\tilde{\alpha} \in P(\tilde{D}, D) \mid \tilde{f}^*(\tilde{\alpha}) = (\tilde{\alpha}_1, 0) \text{ and } \tilde{\alpha}_1 + \tilde{L}_1 \in (\Xi_1^*)^m\}\end{aligned}$$

and we are done. □

Let us denote by Λ_{-2} the 2-dimensional variety obtained in (19.6) b) (observe that $\dim(\Xi_1^*)^m = \dim P(\tilde{D}_1, D_1) - 2 = t - 2 = 1$).

(19.7) Lemma. *One has the equality*:

$$\tilde{f}^*(\Lambda_{-2} \cap 2\Lambda_{-2}) = \{\tilde{L} - \iota_1^*(\tilde{L}) \in P(\tilde{D}_1, D_1) \mid \tilde{L} \in (\Xi_1^*)^m, \operatorname{Nm}_{\pi_1}(\tilde{L}) = \gamma_1^*(\mathcal{O}_{\mathbb{P}^1}(1))\} \times \{0\}.$$

Proof. One has $\tilde{f}^*(\Lambda_{-2} \cap 2\Lambda_{-2}) = \tilde{f}^*(\Lambda_{-2}) \cap 2\tilde{f}^*(\Lambda_{-2})$. This set is an irreducible curve. Since both sets in the equality of the statement have dimension 1, we only have to prove the inclusion of the right hand side member in the left hand side member and this is straightforward. □

Observe that the normalization of the curve $\tilde{B}_1$ given by the pull-back diagram

$$\begin{array}{ccc} \tilde{B}_1 & \longrightarrow & \tilde{N}_1^{(2)} \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \longrightarrow & N_1^{(2)} \end{array}$$

has a natural morphism onto $\{\tilde{L} - \iota_1^*(\tilde{L}) \mid \tilde{L} \in (\Xi_1^*)^m, \operatorname{Nm}_{\pi_1}(\tilde{L}) = \gamma_1^*(\mathcal{O}_{\mathbb{P}^1}(1))\}$. Since C_1 is the normalization of $\Lambda_{-2} \cap 2\Lambda_{-2}$ and $\Lambda_{-2} \cap 2\Lambda_{-2}$ is birationally equivalent to

$\tilde{f}^*(\Lambda_{-2} \cap 2\Lambda_{-2})$ (use the explicit description of $\Lambda_{-2} \cap 2\Lambda_{-2}$ in $P(\tilde{C}, C)$ and that $\operatorname{Ker} \tilde{f}^* = \pi^*(\varepsilon^*({}_2JE))$) we obtain a morphism from the normalization of $\tilde{B}_1$ to C_1. By comparing genera one gets that C_1 is also the desingularization of $\tilde{B}_1$. The proof of (19.3) follows as in the case $t \geqq 4$.

Finally we observe that in case $t = 2$ the curve D is always tetragonal. Indeed, in this case the genus of D_1 is 1. To simplify assume it is smooth. Then the cross ratio of the images of the four points $D_1 \cap D_2$ by the two-to-one morphisms $D_1 \to \mathbb{P}^1$ induced by the linear series g_2^1 on D_1 is not constant. Hence with a suitable such morphism we construct a four-to-one morphism $D \to \mathbb{P}^1$. This concludes the proof of (19.3) and therefore of Theorem (16.1). □

20. Description of the fibre. As a consequence of the description (2.10), the construction of §11 and Theorems (5.11), (6.4), (7.9), (8.7), (10.10) and (16.1) we get a description of the fibre of $\bar{P}$ over a generic element $(\tilde{C}, C)$ of $\mathscr{R}_{B,g}$ (we keep the notation of § 2, in particular E is the elliptic curve associated with the unique bi-elliptic structure of C):

a) If $t \neq 0, 1, 4$, it is the disjoint union of
- two copies of E contained in $\mathscr{R}'_{B,g,t}$,
- a copy of $E \times E$ contained in $\mathscr{H}'_{g,t}$.

b) If $t = 4$, it is the disjoint union of
- two copies of E contained in $\mathscr{R}'_{B,g,4}$,
- a copy of $E \times E$ contained in $\mathscr{H}'_{g,4}$,
- a curve contained in $\mathscr{H}'_{g,4}$.

c) If $t = 1$, it is the disjoint union of
- two copies of E contained in $\mathscr{R}'_{B,g,1}$,
- an irreducible curve contained in $\mathscr{H}'_{g,1}$.

d) If $t = 0$ or $(\tilde{C}, C) \in \mathscr{R}'_{B,g}$
- a single point in each component $\mathscr{R}_{B,g,0}$ and $\mathscr{R}'_{B,g}$,
- a copy of E contained in $\mathscr{H}'_{g,0}$.

References

[A-C-G-H] *E. Arbarello, M. Cornalba, P.A. Griffiths, J. Harris,* Geometry of Algebraic Curves, vol. 1, Grundl. math. Wiss. **267**, Berlin–Heidelberg–New York 1985.

[Be 1] *A. Beauville,* Prym varieties and the Schottky problem, Invent. Math. **41** (1977), 149–196.

[Be 2] *A. Beauville,* Sous-variétés spéciales des variétés de Prym, Comp. Math. **45** (1982), 357–383.

[Be 3] *A. Beauville,* Variétés de Prym et Jacobiennes Intermédiaires, Ann. Sci. E.N.S. **10** (1977), 304–392.

[C-G-T] *C. Ciliberto, G.v. Geer, M. Teixidor,* On the number of parameters of curves whose Jacobians possess non-trivial endomorphisms, Preprint, University of Amsterdam, 1989.

[De1] *O. Debarre,* Sur le problème de Torelli pour les variétés de Prym, Amer. J. Math. **111** (1989), 111–134.
[De2] *O. Debarre,* Sur les variétés de Prym des courbes tétragonales, Ann. Sci. E. N. S. **21** (1988), 545–559.
[De3] *O. Debarre,* Sur les variétés abéliennes dont le diviseur thêta est singulier en codimension 3, Duke Math. J. **57** (1988), 221–273.
[De4] *O. Debarre,* Sur la démonstration de A. Weil du Théorème de Torelli pour les courbes, Comp. Math. **38** (1986), 3–11.
[De5] *O. Debarre,* Variétés de Prym, conjecture de la trisécante et ensembles d'Andreotti et Mayer, Thèse, Université Paris Sud, Centre d'Orsay, 1986.
[Do] *R. Donagi,* The tetragonal construction, Bull. Amer. Math. Soc. **4** (1981), 181–185.
[D-S] *R. Donagi, R. Smith,* The structure of the Prym map, Acta Math. **146** (1981), 25–102.
[F-S] *R. Friedman, R. Smith,* The generic Torelli Theorem for the Prym map, Invent. Math. **67** (1982), 473–490.
[G] *A. Grothendieck,* Techniques de descente et Théorèmes d'existence en Géométrie Algébrique I, Sem. Bourbaki **190** (1959–1960).
[K] *V. Kanev,* The global Torelli theorem for Prym varieties at a generic point, Math. USSR Izvestija **20** (1983), 235–258.
[K-K] *V. Kanev, L. Katsarkov,* Universal properties of Prym varieties of singular curves, C. R. Acad. Bulgare Sci. **41** (1988), 25–27.
[M] *H.H. Martens,* An extended Torelli Theorem, Amer. J. Math. **87** (1965), 257–260.
[Ma] *L. Masiewicki,* Universal properties of Prym varieties with an application to algebraic curves of genus five, Trans. Amer. Math. Soc. **222** (1976), 221–240.
[Mu1] *D. Mumford,* Prym varieties I, in Contributions to Analysis, New York 1974.
[Mu2] *D. Mumford,* Abelian varieties, London 1970.
[R] *S. Recillas,* Jacobians of curves with g^1_4's are the Pryms of trigonal curves, Bol. Soc. Mat. Mexicana **19** (1974), 9–13.
[Sh1] *V. V. Shokurov,* Distinguishing Prymians from Jacobians, Invent. Math. **65** (1981), 209–219.
[Sh2] *V. V. Shokurov,* Prym varieties: Theory and applications, Math. USSR Izvestija **23** (1984), 83–147.
[Te] *M. Teixidor,* For which Jacobi varieties is Sing^{Θ} reducible?, J. reine angew. Math. **354** (1984), 141–149.
[We1] *G. Welters,* Recovering the curve data from a general Prym variety, Amer. J. Math. **109** (1987), 165–182.
[We2] *G. Welters,* The surface *C-C* on Jacobi varieties and 2nd order theta functions, Acta Math. **157** (1986), 1–22.
[We3] *G. Welters,* Abel-Jacobi isogenies for certain types of Fano threefolds, MC Tract 141, CWI, Amsterdam 1981.

Departament d'Àlgebra i Geometria, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain

Eingegangen 12. September 1990, in revidierter Fassung 16. Januar 1991