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Connectedness Bertini Theorem via numerical equivalence

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Abstract: Let X be an irreducible projective variety and let $f : X \to \mathbb{P}^n$ be a morphism. We give a new proof of the fact that the preimage of any linear variety of dimension $k \geq n+1-\dim f(X)$ is connected. We show that the statement is a consequence of the Generalized Hodge Index Theorem using easy numerical arguments that hold in any characteristic. We also prove the connectedness Theorem of Fulton and Hansen as an application of our main theorem.

Keywords: Bertini Theorem, connectedness, numerical equivalence, Hodge Index Theorem.

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1 Introduction

In this note we consider the Bertini Theorem. Eugenio Bertini (1846–1933) was a student of Luigi Cremona, one of the founders of the Italian school of Algebraic Geometry, who lived and worked in Pavia from 1880 to 1892. In his paper [1] dated 1880 and published in 1882, he proved that for a nonsingular projective variety $X \subseteq \mathbb{P}^n$ the general hyperplane section is nonsingular, that is, there is a nonempty Zariski open set $U \subseteq (\mathbb{P}^n)^\vee$ such that for every $H \in U$, the subvariety $H \cap X$ is regular at every point. Moreover, if the dimension of the variety is at least two, the general hyperplane section is connected. For more details on the life and work of Bertini and on the history of this theorem we refer the reader to the paper of Kleiman [9]. This famous theorem has been generalized in many directions, especially in the context of the theory of linear series. For further details, see the book of Jouanolou [8].

We consider here a slightly more general statement: let X be an irreducible projective variety defined over an algebraically closed field of any characteristic and let $f : X \longrightarrow \mathbb{P}^n$ be a morphism. Then the preimage of a linear variety $L \subseteq \mathbb{P}^n$ is connected if $\dim L + \dim f(X) > n$.

Over the complex numbers, the statement is usually proved using the Generic Smoothness Theorem, which fails in positive characteristic (see for instance [7, Corollary 10.7] for the statement of the Generic Smoothness Theorem and [11, Theorem 3.3.1, Theorem 3.3.3] for a proof of the Bertini Theorem in this case). A characteristic free proof of the irreducibility is [14, Theorem 17, IX.6] and [12, Theorem 9]. Our aim is to give a new direct proof of the connectedness statement without using the Generic Smoothness Theorem. The main interest of our approach is the use of numerical connectedness to deduce topological connectedness on the preimages of linear varieties. In particular, we show that the connectedness is a consequence of the Generalized Hodge Index Theorem. Hence our proof works in any characteristic. More precisely, we prove the following theorem; see also [7, Theorem 7.1] and [11, Theorem 3.3.1].

Theorem 1.1. *Let X be an irreducible projective variety and let $f : X \to \mathbb{P}^n$ be a morphism. Then for any linear subvariety $L \in \mathbb{G}(k, n)$ of dimension $k \geq n + 1 - \dim f(X)$, the preimage $f^{-1}(L)$ is connected.*

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The steps of our proof are the following. First, in Section 3.1, we prove that we can assume that f is dominant and that the linear variety L is general in the corresponding Grassmannian. Then in Section 3.2 we prove the statement when $k = 1$: first we assume that $n = 2$ and use the fact that big and nef divisors are 1-connected, see Proposition 3.3; then we generalize to any n. Finally, we conclude with a proof by induction on the dimension of L (Section 3.4).

In Section 5, following an idea of Deligne, we deduce the connectedness Theorem of Fulton and Hansen [4]. This theorem is a striking generalization of the Bertini Theorem with many interesting geometric applications, see [5]. For instance, Zak [15] has used the connectedness theorem to establish a famous result on tangencies to a smooth subvariety $X \subseteq \mathbb{P}^m$ of dimension n, from which he deduced that if $3n > 2(m-1)$ then X is linearly normal, as predicted by Hartshorne's conjecture. Moreover, over the complex numbers, it is possible to obtain information about the fundamental groups by applying these connectivity results to a covering space of the variety. In this context, Deligne [3] generalized the connectedness theorem to a statement about the fundamental group π_1 and π_0, and then its work has been extended to higher homotopy groups, see [5, Section 9] and [13]. It would be interesting to see whether using our methods some of these problems can be generalized to the algebraic setting; many nice questions remain open.

2 Preliminaries

In this paper all the varieties are considered with the Zariski topology. We recall some definitions.

Definition 2.1. Let X be a complete variety and D a divisor on X, we say that D is nef (numerically effective) if $D \cdot C \geq 0$ for all irreducible curves $C \subseteq X$.

Definition 2.2. A line bundle L on an irreducible projective variety X is big if it has maximal Kodaira dimension $k(X, L) = \dim X$.

Definition 2.3. A line bundle L on an irreducible projective variety X is semiample if there exists an integer r such that $L^{\otimes r}$ is globally generated.

We will use the following characterization of bigness for nef divisors, see [11, I, Theorem 2.2.16].

Theorem 2.4. *Let D be a nef divisor on an irreducible projective variety X of dimension n. Then D is big if and only if its top self-intersection is strictly positive: $D^n > 0$.*

A central tool in the proof of numerical connectedness on surfaces is the following version of the Hodge Index Theorem, which is more general than the usual one for surfaces and can be easily deduced from the standard version (see [7, V, Theorem 1.9]):

Theorem 2.5 (Hodge Index Theorem). *Let S be a smooth projective surface and let H be a divisor with $H^2 > 0$. Let D be a divisor such that $D \cdot H = 0$. Then either $D^2 < 0$ or D is numerically trivial.*

Generalizations of this classical result have arisen in many directions. We will use the following inequality, see [11, Theorem 1.6.1, Formula (1.24)]:

Theorem 2.6 (Generalized inequality of Hodge type). *Let X be an irreducible complete variety of dimension n, and let $\beta_1, \dots, \beta_{n-1}, h$ be numerical classes of nef divisors. Then*

$$(\beta_1 \cdot \dots \cdot \beta_{n-1} \cdot h)^{n-1} \geq ((\beta_1)^{n-1} \cdot h) \dots ((\beta_{n-1})^{n-1} \cdot h).$$

One of the main problems in positive characteristic is the lack of resolution of singularities for varieties of dimension greater than three. Instead of the resolution of singularities, we will use several times the existence of alterations.

Definition 2.7 (See [2]). Let X be a variety over an algebraically closed field k. An alteration of X is a proper dominant morphism $X' \to X$ of varieties over k with $\dim X' = \dim X$. An alteration is regular if X' is smooth.

Thanks to [2, Theorem 3.1], there always exists a regular alteration.

Remark 2.8. In Theorem 1.1 we can assume X to be nonsingular. Indeed, assume by contradiction that there exists a variety X, a morphism $f : X \to \mathbb{P}^n$ and a linear subvariety L such that $f^{-1}(L)$ is disconnected. Consider a regular alteration $a : \tilde{X} \to X$. Then the components of $a^{-1}f^{-1}(L)$ are disconnected.

3 The main theorem

This section is devoted to the proof of Theorem 1.1. Observe that for $n = 1$ there is nothing to prove because the only possible choice is $L = \mathbb{P}^1$; so we can assume $n \geq 2$. By Remark 2.8 we can assume that X is nonsingular.

3.1 Reduction to f dominant and L general

It is enough to prove the theorem for a general linear variety. This is a consequence of an argument of Jouanolou; see the proof of [11, Theorem 3.3.3]. So from now on we assume the generality of L.

Let $f : X \to \mathbb{P}^n$ be a morphism and $L \subseteq \mathbb{P}^n$ a linear variety; we assume that $k = \dim L \geq n + 1 - \dim f(X)$. If f is not dominant and L is general, L is not contained in $f(X)$. Then there exists a point $p \in L \setminus f(X)$. We project from the point p to a hyperplane $H = \mathbb{P}^{n-1}$ and let $f' : X \to \mathbb{P}^{n-1}$ be the composition of f with this projection. The intersection $L \cap H$ is a linear subvariety $L' \subset H$ such that

$$\dim L' = k - 1 \geq \dim H + 1 - \dim f'(X)$$

(observe that $\dim f'(X) = \dim f(X)$). If the theorem is true for the map f', then $f'^{-1}(L') = f^{-1}(L)$ is connected. Therefore it is enough to prove the theorem for f'. If f' is not dominant, we perform successive projections until we reach a dominant map. So we can assume for the rest of the proof that f is dominant.

3.2 Connectedness of the preimage of a line by a generically finite map

The aim of this subsection is to prove the following particular case of Theorem 1.1.

Proposition 3.1. *Let X be a projective variety of dimension n defined over an algebraically closed field and let $f : X \to \mathbb{P}^n$ be a generically finite map. Let $L \subseteq \mathbb{P}^n$ be a general line. Then $f^{-1}(L)$ is connected.*

We start with the simplest case.

3.2.1 Case $n = 2$

Since the map f is generically finite, this implies that X is a surface. We need some numerical connectedness considerations:

Definition 3.2. Let X be a projective surface. A divisor H is 1-connected if for any nontrivial effective divisors A and B such that A and B do not have common components and $H = A + B$, we have $A \cdot B \geq 1$.

The proof of the following lemma is contained in Lemma 3.11 in [10].

Proposition 3.3. *Let H be a big and nef divisor on a smooth projective surface X. Then H is 1-connected.*

Proof. Let H be a big and nef divisor and let A and B be nontrivial effective divisors such that $H = A + B$. Since X is projective, there exist very ample divisors on X, hence A and B are not numerically trivial. H nef gives

$$A^2 + A \cdot B = H \cdot A \geq 0$$
$$A \cdot B + B^2 = H \cdot B \geq 0.$$

Now if $A \cdot B \leq 0$ then $A^2B^2 \geq (A \cdot B)^2 \geq 0$ contradicts the Hodge Index Theorem, see Theorem 2.6. Thus $A \cdot B \geq 1$ and H is 1-connected. □

Let X be a smooth projective surface, let $f : X \to \mathbb{P}^2$ be a dominant map and let $L \subset \mathbb{P}^2$ be a line. Let $H = f^*(L)$ be the pullback of L through f. We claim that H is big and nef. Indeed, let C be any irreducible curve on X; then by the projection formula

$$H \cdot C = f^*(L) \cdot C = L \cdot f_*(C) \geq 0.$$

Therefore H is nef. The selfintersection of H is

$$H^2 = f^*(L) \cdot f^*(L) = f_*(f^*(L)) \cdot L = \deg(f)L^2 > 0.$$

Thus by Theorem 2.4 H is big. Let us suppose by contradiction that the preimage of a line is not connected. Then there exist two effective nontrivial divisors A and B such that $A + B = H$ and $A \cdot B = 0$. This contradicts Proposition 3.3.

3.2.2 Proof of Proposition 3.1

It is enough to show how to reduce to the case $n = 2$. Let $H \subseteq \mathbb{P}^n$ be a general hyperplane. Let $D := f^*(H)$ be the preimage of H. The divisor $D \subset X$ might be non-irreducible, so we write $D = \sum_i n_i D_i$ where the D_i are all the irreducible components of D. Since H is a general ample divisor, D is nef, big and semiample. Moreover, the restriction of $f^*(H)$ to D_i gives a linear series without base points, because if there was a base point it would be also a base point of $|f^*(H)|$. Thus the D_i have to move algebraically and given any point p we can find D_j numerically equivalent to D_i that does not have p in its support. This implies that f does not contract D_i: otherwise, since D_i covers X, the fiber of the general point of $\mathbb{P}^n$ would have dimension greater than one. We set $f_*(D_i) = k_i H$ with $k_i > 0$.

We will use the following notation: $A = \sum_{i \in S} n_i D_i$ and $B = \sum_{i \in T} n_i D_i$ such that $S \cap T = \emptyset$ and $A + B = D$.

Lemma 3.4. *With the notation above, $A \cdot D^{n-1} > 0$ and $B \cdot D^{n-1} > 0$.*

Proof. It is enough to prove that $D_i \cdot D^{n-1} > 0$ for each i. We have

$$D_i \cdot D^{n-1} = D_i \cdot f^*(H^{n-1}) = f_*(D_i) \cdot H^{n-1} = k_i H^n = k_i > 0.$$ □

Lemma 3.5. *With the above notation, the following holds:*

$$H^{n-2} \cdot f_*(A \cdot B) = f^*(H^{n-2})(A \cdot B) = D^{n-2} \cdot A \cdot B > 0$$

Proof. We assume by contradiction that $D^{n-2} \cdot A \cdot B = 0$. So we have

$$(A + B)^{n-2} \cdot A \cdot B = \sum_{k=0}^{n-2} \binom{n-2}{k} A^{k+1} B^{n-1-k} = 0. \tag{1}$$

Since A and B are nef, all the coefficients are positive and all the addends of the sum are zero. In particular

(i) $A^{n-1} \cdot B = 0$ corresponding to $k = n - 2$;
(ii) $A \cdot B^{n-1} = 0$ corresponding to $k = 0$.

Using now the generalized inequality of Hodge type, see 2.6, we have

$$\begin{aligned} 0 &= (A \cdot B \cdot \underbrace{D \dots D}_{n-2})^{n-1} \\ &\geq (A^{n-1} \cdot D)(B^{n-1} \cdot D) \underbrace{(D^{n-1} \cdot D) \dots (D^{n-1} \cdot D)}_{n-2} \geq 0. \end{aligned}$$

Since $D^n > 0$ by Theorem 2.4, either $A^{n-1} \cdot D = 0$ or $B^{n-1} \cdot D = 0$. We assume to be in the first case; the proof is the same for the second case. Using (i), we have that $A^{n-1} \cdot D = A^n + A^{n-1} \cdot B = A^n = 0$. Therefore $A \cdot D^{n-1} = \sum_{k=0}^{n-1} A^{k+1} B^{n-1-k} = 0$, since the term corresponding to $k = n - 1$ is A^n and all the other addends are zero because they appear in (1). This is in contradiction with Lemma 3.4. □

Now we finish the proof of Proposition 3.1. We proceed by induction on $n \geq 2$. The initial step has been proved in Subsection 3.2.1. Let $L \subseteq \mathbb{P}^n$ be a general line and let H be a general hyperplane containing L. As before, we write $f^*(H) =: D = \sum_i n_i D_i$ and the restrictions of f to D_i

$$f_i : D_i \longrightarrow \mathbb{P}^{n-1} = H$$

are dominant. Since D_i might be singular we have to consider a regular alteration $a_i : \tilde{D}_i \to D_i$ where the $\tilde{D}_i$ are smooth. By the induction hypothesis we can assume that the $a_i^{-1} f_i^{-1}(L)$ are connected curves. In particular the $f_i^{-1}(L) = C_i$ are also connected (otherwise the preimages via a_i of the components of C_i would disconnect $a_i^{-1} f_i^{-1}(L)$). We have to prove that $C := \cup_i C_i$ is connected. Let $Z := \cup_{i \in S} C_i$ and $Y := \cup_{i \in T} C_i$ with $S \cap T = \emptyset$ and let A and B be two divisors on X whose supports are

$$[A] := \bigcup_{i \in S} D_i \quad \text{and} \quad [B] := \bigcup_{i \in T} D_i.$$

We will prove that $Z \cap Y \neq \emptyset$. We set $\Sigma := [A] \cap [B] \subseteq X$. Then $\Gamma := f(\Sigma)$ is a subvariety in $\mathbb{P}^n$ such that $\Gamma \cap H^{n-2}$ is a curve, otherwise $H^{n-2} \cdot f_*(A \cdot B) = 0$ in contradiction with Lemma 3.5. Since H^{n-2} is a plane in $\mathbb{P}^n$ that contains L, Γ has to intersect L. Let $p \in \Gamma \cap L$. We can find a point $q \in X$ such that $f(q) = p$ and $q \in [A] \cap [B] \cap f^{-1}(L) = Z \cap Y$. Thus $Z \cap Y \neq \emptyset$ and $f^{-1}(L)$ is connected.

This concludes the proof of the proposition.

3.3 Connectedness of the preimage of a line for a dominant morphism

We want now to prove the following proposition:

Proposition 3.6. *Let X be a projective variety of dimension n defined over an algebraically closed field and let $f : X \to \mathbb{P}^n$ be a dominant morphism. Let $L \subseteq \mathbb{P}^n$ be a general line. Then $f^{-1}(L)$ is connected.*

Proof. Using the Stein Factorization of f, see [7, III, Corollary 11.5], we obtain the following diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f'"'] & & \mathbb{P}^n \\ & Y \arrow[ur, "g"'] & \end{tikzcd}$$

where f' has connected fibers and g is a finite map into $\mathbb{P}^n$. Let $\lambda : \tilde{Y} \to Y$ be a regular alteration of Y.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f'"'] & & \mathbb{P}^n \\ & Y \arrow[ur, "g"'] & \\ & \tilde{Y} \arrow[u, "\lambda"] \arrow[uur, "\tilde{g}"'] & \end{tikzcd}$$

Since the map $\tilde{g}$ is generically finite and $\tilde{Y}$ is nonsingular, using Proposition 3.1 we get that $\tilde{g}^{-1}(L)$ is connected, for any general line $L \subseteq \mathbb{P}^n$. Thus $g^{-1}(L)$ is connected, otherwise $\tilde{g}^{-1}(L) = \lambda^{-1} g^{-1}(L)$ would not be connected. Since the fibers of f' are connected, $f^{-1}(L)$ is connected. □

3.4 Connectedness of a linear variety of higher dimension

Finally, we have to consider the case where L has arbitrary dimension. Let $H \in (\mathbb{P}^n)^\vee$ be a hyperplane and let $p \notin H$ a general point. We consider the projection $\Pi : \mathbb{P}^n \setminus \{p\} \to H$. We can identify H with the projective space of the lines through p. Then the blow-up of $\mathbb{P}^n$ in the point p can be seen as

$$B_p = \{(x, l) \in \mathbb{P}^n \times H \mid x \in l\}.$$

The exceptional divisor is $E_p = \{p\} \times H$ and the first projection gives the birational map $\varepsilon_p : B_p \to \mathbb{P}^n$. Since we resolved the indeterminacy locus, the map $\varepsilon_p \circ \Pi$ is now a morphism.

We can now consider the fiber product $\tilde{X} := X \times_{\mathbb{P}^n} B_p$. We obtain a well-defined dominant morphism $f' : \tilde{X} \to \mathbb{P}^{n-1}$. The fiber $f^{-1}(p)$ might be singular because of the failure of the Generic Smoothness Theorem in positive characteristic. Therefore singularities might arise in $\tilde{X}$. As before, we consider a regular alteration $f'' : X'' \to \tilde{X}$ of $\tilde{X}$, where X'' is a smooth irreducible variety.

$$\begin{tikzcd}
X'' \arrow[d, "f''"] & \\
\tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{\varepsilon}"] \arrow[ddr, "f'"] & B_p \arrow[d, "\varepsilon_p"] \\
X \arrow[r, "f"] & \mathbb{P}^n \arrow[d, dashed, "\Pi"] \\
 & \mathbb{P}^{n-1}
\end{tikzcd}$$

We put $h := f'' \circ f'$. Assuming the connectedness statement for h, we get that $h^{-1}(L)$ is connected for any linear variety $L \subseteq \mathbb{P}^{n-1}$ of dimension $k \geq 1$. Thus $f''(h^{-1}(L))$ is connected too, otherwise the preimages via f'' of the components would disconnect $h^{-1}(L)$. Since $f^{-1}(L \vee p) = \tilde{\varepsilon}(f''(h^{-1}(L)))$, the connectedness statement is true for any linear variety of $\mathbb{P}^n$ containing p of dimension $k' \geq 2$. To obtain the assertion for every linear variety L contained in $\mathbb{P}^n$ of dimension $k' \geq 2$, it is sufficient to consider a point $p \in L$ and repeat the previous construction. Observe that with this procedure we do not reach the one-dimensional case that we have proved in Proposition 3.6. This concludes the proof of Theorem 1.1.

4 The classical Bertini Theorem

If in Theorem 1.1 we consider the inclusion map, we get the classical version of the Bertini Theorem. The theorem states that a general hyperplane section of a nonsingular variety in a projective space is again nonsingular and connected if the dimension of the variety is greater than two. It holds over an arbitrary algebraically closed field of any characteristic and can be proved by computing the dimension of nontransverse hyperplanes to the variety; see [7, Theorem II.8.18].

Theorem 4.1 (Bertini Theorem). *Let X be a nonsingular closed subvariety of $\mathbb{P}^n_k$, where k is an algebraically closed field. Then there exists an hyperplane $H \subseteq \mathbb{P}^n_k$, not containing X, such that the scheme $H \cap X$ is regular at every point. If $\dim X \geq 2$, then $H \cap X$ is also connected. Furthermore, the set of hyperplanes with this property is an open dense subset of the complete linear system $|H|$, considered as projective space.*

5 The connectedness theorem of Fulton and Hansen

We give a very short proof of the connectedness theorem of Fulton and Hansen where we re-elaborate an idea of Deligne; see also [11, Theorem 3.3.6].

Theorem 5.1. *Let X be an irreducible projective variety and let $F : X \longrightarrow \mathbb{P}^n \times \mathbb{P}^n$ be a morphism such that* $\dim F(X) > n$. *Let $\Delta \subseteq \mathbb{P}^n \times \mathbb{P}^n$ be the diagonal. Then the inverse image $F^{-1}(\Delta) \subseteq X$ of the diagonal is connected.*

Proof. We consider $\mathbb{P}^n \times \mathbb{P}^n$ with homogenous coordinates $[z_i]$ and $[w_j]$. Let $f : X \to \mathbb{P}^n$ and $g : X \to \mathbb{P}^n$ be morphisms such that $F = (f, g)$. Then we can consider the following line bundles on X:

$$L := f^*\mathcal{O}_{\mathbb{P}^n}(1), \quad M := g^*\mathcal{O}_{\mathbb{P}^n}(1)$$

with sections $s_i = f^*(z_i)$ and $t_j = g^*(w_j)$. Let $E := L \oplus M$ be the rank two bundle and let $\pi : \mathbb{P}(E) \to X$ be the associated projective space bundle. We have the tautological exact sequence

$$0 \longrightarrow \mathcal{N} \longrightarrow \pi^*(E) \xrightarrow{\varphi} \mathcal{O}_E(1) \longrightarrow 0$$

where $\mathcal{N}$ is defined as the kernel of the map φ. The bundle $\pi^*(E)$ is generated by the $2n + 2$ sections $\pi^*(s_i)$ and $\pi^*(w_j)$, and their images via the map φ generate $\mathcal{O}_E(1)$. Therefore we have the map $Q : \mathbb{P}(E) \to \mathbb{P}^{2n+1}$ associated to $\pi^*(E)$, in coordinates:

$$P \longmapsto (\pi^* s_0(P), \dots, \pi^* s_n(P), \pi^* t_0(P), \dots, \pi^* t_n(P)).$$

We remark that $\dim Q(\mathbb{P}(E)) = \dim F(X) + 1$. If we consider the embedding of $\mathbb{P}^n \times \mathbb{P}^n$ into $\mathbb{P}^{2n+1}$, the image of the diagonal Δ is given by the n-dimensional linear subspace $L \subseteq \mathbb{P}^{2n+1}$ defined be the equations $\{z_i = w_i\}$. Then

$$F^{-1}(\Delta) = \pi(Q^{-1}(L))$$

and the assertion follows from Theorem 1.1. □

From Theorem 1.1 we can deduce a connectedness theorem for flag manifolds and Grassmannians.

Theorem 5.2 ([6, Section 1]). *Let $\mathbb{F}$ be any flag manifold in $\mathbb{P}^n$, and let Δ_F be the image of the diagonal embedding of $\mathbb{F}$ in $\mathbb{F} \times \mathbb{F}$. Further, let X be an irreducible variety and let $f : X \to \mathbb{F} \times \mathbb{F}$ be a morphism. Then $f^{-1}(\Delta_F)$ is connected if* $\operatorname{codim}(f(X), \mathbb{F} \times \mathbb{F}) < n$.

For the proof it is enough to follow the construction in [6] and then apply Theorem 5.1.

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References

[1] E. Bertini, Sui sistemi lineari. *Ist. Lombardo, Rend., II. Ser.* **15** (1882), 24–29. Zbl 14.0433.02
[2] A. J. de Jong, Smoothness, semi-stability and alterations. *Inst. Hautes Études Sci. Publ. Math.* no. **83** (1996), 51–93. MR1423020 Zbl 0916.14005

[3] P. Deligne, Le groupe fondamental du complément d'une courbe plane n'ayant que des points doubles ordinaires est abélien (d'après W. Fulton). In: *Bourbaki Seminar, Vol. 1979/80*, volume 842 of *Lecture Notes in Math.*, 1–10, Springer 1981. MR636513 Zbl 0478.14008

[4] W. Fulton, J. Hansen, A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings. *Ann. of Math.* (2) **110** (1979), 159–166. MR541334 Zbl 0389.14002

[5] W. Fulton, R. Lazarsfeld, Connectivity and its applications in algebraic geometry. In: *Algebraic geometry* (*Chicago, Ill.,* 1980), volume 862 of *Lecture Notes in Math.*, 26–92, Springer 1981. MR644817 Zbl 0484.14005

[6] J. Hansen, A connectedness theorem for flagmanifolds and Grassmannians. *Amer. J. Math.* **105** (1983), 633–639. MR704218 Zbl 0544.14034

[7] R. Hartshorne, *Algebraic geometry*. Springer 1977. MR0463157 Zbl 0367.14001

[8] J.-P. Jouanolou, *Théorèmes de Bertini et applications*. Birkhäuser 1983. MR725671 Zbl 0519.14002

[9] S. L. Kleiman, Bertini and his two fundamental theorems. *Rend. Circ. Mat. Palermo* (2) *Suppl.* no. **55** (1998), 9–37. MR1661859 Zbl 0926.14001

[10] J. Kollár, editor, *Complex algebraic geometry*, volume 3 of *IAS/Park City Mathematics Series*. Amer. Math. Soc. 1997. MR1442521 Zbl 0866.00043

[11] R. Lazarsfeld, *Positivity in algebraic geometry.* Springer 2004. MR2095471 Zbl 1093.14501 Zbl 1066.14021

[12] A. Seidenberg, The hyperplane sections of normal varieties. *Trans. Amer. Math. Soc.* **69** (1950), 357–386. MR0037548 Zbl 0040.23501

[13] A. J. Sommese, A. Van de Ven, Homotopy groups of pullbacks of varieties. *Nagoya Math. J.* **102** (1986), 79–90. MR846130 Zbl 0564.14010

[14] A. Weil, *Foundations of algebraic geometry*. Amer. Math. Soc. 1962. MR0144898 Zbl 0168.18701

[15] F. L. Zak, *Tangents and secants of algebraic varieties*, volume 127 of *Translations of Mathematical Monographs*. Amer. Math. Soc. 1993. MR1234494 Zbl 0795.14018