On the topological index of irregular surfaces

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Abstract

We study the topological index of some irregular surfaces that we call generalized Lagrangian. We show that under certain hypotheses on the base locus of the Lagrangian system the topological index is non-negative. For the minimal surfaces of general type with $q = 4$ and $p_g = 5$ we prove the same statement without any hypothesis.

1 Introduction.

We consider complex smooth algebraic varieties X of dimension n equipped with an element ω in the kernel of the map

$$\psi_{2,\Omega^1_X} = \psi_2 : \Lambda^2 H^0(X, \Omega^1_X) \longrightarrow H^0(X, \Omega^2_X).$$

Let $V \subset H^0(X, \Omega^1_X)$ be such that $w \in \Lambda^2 V$ and V has minimal dimension with this property. Then V has even dimension $2k$ and we say that this is the *rank of* w. Notice that rank 2 means decomposable.

The non-triviality of the kernel of

$$\Lambda^2 H^1(X, \mathbb{C}) \longrightarrow H^2(X, \mathbb{C})$$

determines the existence of nilpotent towers in the fundamental group of the variety (see [7] and also [1]). Hence, we obtain topological consequences of the non-injectivity of ψ_2. This is especially clear when an element of rank $< 2n$ appears in $Ker(\psi_2)$; then, using the results of Catanese in [4], one sees that X is fibred on a variety of lower dimension. If $n = 2$ this is the classical Castelnuovo-de Franchis Theorem. Our principal interest is to find

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numerical restrictions on the topological invariants of the varieties (mainly surfaces) where $Ker(\psi_2) \neq 0$.

Consider the image of the evaluation map on V:

$$V \otimes \mathcal{O}_X \longrightarrow \Omega^1_X.$$

The image defines a torsion free sheaf $\overline{\Omega^1_X}$.

Definition 1.1. We say that an n-dimensional variety X is *generalized Lagrangian* if there exists $w \in Ker(\psi_2)$ of rank $2n$ and moreover $rank\,\overline{\Omega^1_X} = n$. In other words, there exist $\omega_1, \dots, \omega_{2n} \in H^0(X, \Omega^1_X)$ generating generically Ω^1_X and such that $\omega_1 \wedge \omega_2 + \cdots + \omega_{2n-1} \wedge \omega_{2n} = 0$ in $H^0(X, \Omega^2_X)$.

Following [15], we say that X is *Lagrangian* if there exists a degree 1 map

$$a : X \longrightarrow a(X) \subset A,$$

where A is an Abelian variety of dimension $2n$, and a $(2,0)$ form w of maximal rank $2n$ on A such that $a^*(w) = 0$. This is a particular case of our definition 1.1.

We notice that in this case the vector space V appears as the pull-back of the cotangent space of A.

We want to give restrictions on the invariants of these varieties, mainly regarding their topological significance. We obtain results of two types: first we work with generalized Lagrangian varieties of any dimension and we prove that under some hypotheses the second degree part of the Chern character is non-negative. In fact, this is a consequence of a more general result on reflexive sheaves (see 3.4). Second we focus on the 2-dimensional case and we assume that $X = S$ is minimal of general type. It is natural to ask what kind of geometry appears if a relation of type

$$\omega_1 \wedge \omega_2 + \omega_3 \wedge \omega_4 = 0,$$

holds. In other words, the study of generalized Lagrangian surfaces corresponds naturally to a higher rank analogon of the classical situation elucidated by Castelnuovo and de Franchis. In fact, we are mainly interested in the non fibred generalized Lagrangian surfaces.

Attached to the vector space V we have a natural subsystem of the canonical system: the one given by the image of $\Lambda^2 V$ in $H^0(S, \omega_S)$. Let F_V be the corresponding base divisor. Assume that F_V is reduced. We say that F_V is contracted by V if the map

$$V \otimes \mathcal{O}_S \longrightarrow \Omega^1_S \longrightarrow \Omega^1_{S_{|F_V}} \longrightarrow \omega_{F_V}$$

vanishes. Then we prove (see 4.1 and 5.4):

Theorem 1.2. *Assume that $F_V = 0$ or F_V is a reduced connected divisor with normal crossings contracted by V. Then the topological index $\tau(S)$ satisfies $\tau(S) \geq 0$.*

We will see that the condition "F_V contracted by V" is necessary in general (see section 6). It is natural to state:

Conjecture 1: Let S be a minimal generalized Lagrangian surface of general type. Assume that F_V is contracted by V. Then $\tau(S) \geq 0$.

Conjecture 2: Let S be a Lagrangian surface. Then $\tau(S) \geq 0$.

Observe that, by 1.2, Conjecture 2 is true when $a : X \longrightarrow A$ is an immersion, and more generally when a does not have a branch divisor.

Notice that the existence of non decomposable w implies $q(S) \geq 4$. As a test case to our conjecture, the first values where non-fibred surfaces could appear are $q(S) = 4$ and $p_g(S) = 5$. In this case the map ψ_2 has non-trivial kernel for dimensional reasons. We see that with these invariants no hypothesis is needed to obtain the inequality $\tau(S) \geq 0$. In fact, since $K_S^2 + c_2(S) = 24$ and $\tau(S) = \frac{1}{3}(K_S^2 - 2c_2(S))$, we can state our result as follows:

Theorem 1.3. *Let S be a minimal smooth projective surface of general type with $p_g(S) = 5$ and $q(S) = 4$. Then the following inequality holds:*

$$K_S^2 \geq 8\,\chi(\mathcal{O}_S) = 16.$$

If S has an irreducible pencil, then $K_S^2 = 16$ and we have a complete classification of these surfaces (see 7.3).

Recall the Bogomolov-Miyaoka-Yau inequality $K_S^2 \leq 9\chi(\mathcal{O}_S) = 18$, therefore the only possible values are $16, 17$ and 18.

We believe that this is in itself an interesting classification result on irregular surfaces. The result 1.2 is crucial in order to show it. The proof is rather involved and uses counting quadrics containing the canonical image, an intensive use of Reider's Theorem ([12]) and of the Hodge-Index theorem. The method exploits heavily the particular geometry of the surfaces with these invariants and does not seem suited to approaching the general situation posed in our conjecture.

The paper is organized as follows: in section 3 we study the kernel of the map $\psi_{2,\mathcal{E}}$ for any reflexive sheaf $\mathcal{E}$. At the end of the section we find the results on generalized Lagrangian varieties of any dimension.

In section 4 we focus on the rank 2 case and we introduce basic concepts for the rest of the paper. In section 5 we assume that the base divisor F_V is a divisor with normal crossings and we find a lower bound for the invariant $\delta(S)$. Section 6 is devoted to the construction of examples.

In sections 7 and 8 we prove Theorem 1.3. First we consider the surfaces with a fibration of higher base genus, which can be studied directly by analyzing the relative invariants. Here $K_S^2 = 16$ and a structure theorem is obtained. For the general case we use the geometry of the canonical map, which, due to the previous results, factors through the Albanese morphism composed with the Gauss map.

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2 Notation

All the varieties considered in this paper are reduced, irreducible, projective and defined over the complex numbers.

For any coherent sheaf $\mathcal{E}$ on a variety X we denote by $\psi_{2,\mathcal{E}}$ the wedge map

$$\psi_{2,\mathcal{E}} : \Lambda^2 H^0(X,\mathcal{E}) \longrightarrow H^0(X,\Lambda^2\mathcal{E}).$$

If no confusion arises we will simply write ψ_2.

Given a coherent sheaf $\mathcal{E}$ on a variety we denote by

$$\delta(\mathcal{E}) = \frac{1}{2}(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})) \in H^4(X,\mathbb{Q})$$

the codimension 2 piece of the Chern character. Recall that δ is additive for exact sequences and for a locally free sheaf $\mathcal{F}$, $\delta(\mathcal{F}) = \delta(\mathcal{F}^*)$.

We shall say that a cohomological class $\eta \in H^{2r}(X,\mathbb{Q})$ is pseudo effective, and we shall write $\eta \geq 0$, if

$$\eta \cdot H_1 \cdots H_{n-r} \geq 0$$

for any ample divisors $H_1, \ldots H_{n-r}$. The notation $\eta_1 \geq \eta_2$ means $\eta_1 - \eta_2 \geq 0$.

We say that a property holds on X in codimension k if it holds outside of a closed set of codimension at least $k+1$.

Throughout the paper S is a smooth minimal surface.

We shall say that S has a fibration of higher base genus if there exists a fibration $f : S \longrightarrow B$, where B is a smooth irreducible curve of genus at least 2.

The Albanese map of S is written as $a : S \longrightarrow Alb(S)$.

We denote by $\tau(S)$ the topological index of S. Recall that $\tau(S) = \frac{1}{3}(K_S^2 - 2c_2(S))$. Hence $\delta(\Omega^1_S) = \frac{3}{2}\tau(S)$.

We denote by F the fixed divisor of the canonical system, and by $M := K_S - F$ the moving part.

The canonical map $S - - \rightarrow \mathbb{P}(H^0(S, \omega_S)^*)$ induced by M is denoted by φ_M or simply φ. We put $\Sigma := \varphi_M(S)$.

3 On the kernel of $\psi_{2,\mathcal{E}}$ for a reflexive $\mathcal{E}$

The goal of this section is to provide geometrical consequences of the existence of non trivial elements in the kernel of $\psi_{2,\mathcal{E}}$. We are especially interested in the non decomposable elements. Although the main applications will be for surfaces and when $\mathcal{E}$ is the sheaf of differentials, our aim is to work in a general framework. For this reason we will start in a quite general setting and we add hypotheses when the theory needs them to progress.

Fix a n-dimensional smooth variety X and a reflexive sheaf $\mathcal{E}$. In particular, $\mathcal{E}$ and its subsheaves are torsion-free. Let $w \in Ker(\psi_{2,\mathcal{E}})$ and let $V \subset H^0(X, \mathcal{E})$ be such that $w \in \Lambda^2 V$ and V has minimal dimension with this property. Then V has even dimension $2k$ and we say that this is the rank of w.

Let us now consider the evaluation map restricted to V:

$$V \otimes \mathcal{O}_X \longrightarrow \mathcal{E}.$$

The image of this map is a torsion-free sheaf $\mathcal{E}_V \subset \mathcal{E}$.

Dualizing the surjective map $V \otimes \mathcal{O}_X \to \mathcal{E}_V$, we obtain the following short exact sequence which serves as definition of the sheaf N:

$$0 \longrightarrow \mathcal{E}_V^* \longrightarrow V^* \otimes \mathcal{O}_X \longrightarrow N \longrightarrow 0.$$

Dualizing again we get an exact sequence

$$0 \longrightarrow N^* \longrightarrow V \otimes \mathcal{O}_X \xrightarrow{e} \mathcal{E}_V^{**} \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_X}(N, \mathcal{O}_X) \longrightarrow 0.$$

Notice that

$$V \otimes \mathcal{O}_X \longrightarrow \mathcal{E}_V^{**} \longrightarrow \mathcal{E}^{**} \cong \mathcal{E}$$

is again the evaluation map restricted to V, therefore the image of e is $\mathcal{E}_V \subset \mathcal{E}_V^{**}$. Hence the exact sequence splits into:

$$0 \longrightarrow N^* \longrightarrow V \otimes \mathcal{O}_X \longrightarrow \mathcal{E}_V \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{E}_V \longrightarrow \mathcal{E}_V^{**} \longrightarrow \mathcal{C}_V := \mathcal{E}xt^1_{\mathcal{O}_X}(N, \mathcal{O}_X) \longrightarrow 0.$$

Observe that contraction with w defines an isomorphism $V^* \otimes \mathcal{O}_X \longrightarrow V \otimes \mathcal{O}_X$. Then the condition $\psi_2(w) = 0$ translates into the vanishing of the composition

$$\mathcal{E}_V^* \longrightarrow V^* \otimes \mathcal{O}_X \longrightarrow V \otimes \mathcal{O}_X \longrightarrow \mathcal{E}_V.$$

Therefore the following diagram is induced

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}_V^* & \longrightarrow & V^* \otimes \mathcal{O}_X & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & N^* & \longrightarrow & V \otimes \mathcal{O}_X & \longrightarrow & \mathcal{E}_V & \longrightarrow & 0. \end{array}$$

Observe (by diagram chasing) that α is injective and β is surjective. Moreover $Coker(\alpha) \cong Ker(\beta)$. Hence $rank\, \mathcal{E}_V \leq rank\, N$ and

$$rank\, \mathcal{E}_V + rank\, N = \dim V = 2k.$$

In particular

Lemma 3.1. *With the above notations we have:*

$$rank\, \mathcal{E}_V \leq k = \frac{1}{2} \dim V.$$

The exact sequences above imply several easy consequences for the invariant δ of the sheaves involved. We present them in the following lemma:

Lemma 3.2. *With the above notations:*

a) $\delta(\mathcal{E}_V^*) = \delta(\mathcal{E}_V^{**})$.

b) $\delta(\mathcal{E}_V^{**}) \geq \delta(\mathcal{E}_V)$.

c) $\delta(\mathcal{E}_V) + \delta(N^*) = 0$.

d) $\delta(\mathcal{E}_V^*) + \delta(N) = 0$.

Proof. Since $\mathcal{E}_V^*$ is reflexive, then it is locally free in codimension 2. Therefore, in the computation of δ it can be considered as locally free. This justifies a). Moreover the sheaf $\mathcal{C}_V$ is supported in codimension 2, hence its restriction to a general surface $S = H_1 \cap \dots H_{n-2}$, H_i ample, satisfies $\delta(\mathcal{C}_{V|S}) \geq 0$. Therefore $\delta(\mathcal{E}_V^{**}) - \delta(\mathcal{E}_V) = \delta(\mathcal{C}_V) \geq 0$, then b) follows. The rest is obvious. □

Remark 3.3. We are especially interested in the case where $X = S$ is a surface. In this situation the inequality of b) becomes an inequality of integers. Moreover all reflexive sheaves, such as $\mathcal{E}$, $\mathcal{E}_V^*$ and N^*, are locally free.

To go further we need the following
Hypothesis: We assume $rank\, \mathcal{E}_V = k$. Notice that this is equivalent to saying that $rank\, N = k$.

With this assumption, the map

$$\alpha : \mathcal{E}_V^* \hookrightarrow N^*$$

is an injection of reflexive sheaves of the same rank. Moreover $c_1(N^*) = -c_1(\mathcal{E}_V) = c_1(\mathcal{E}_V^*)$ ($\mathcal{E}_V$ is torsion-free, therefore locally free in codimension 1). Hence α is an isomorphism in codimension 2. Therefore:

$$\delta(N^*) = \delta(\mathcal{E}_V^*) = \delta(\mathcal{E}_V^{**}).$$

By implementing this in 3.2 we obtain the following result:

Theorem 3.4. *Let X be a smooth variety and let $\mathcal{E}$ be a reflexive sheaf on X. We assume that there exists $w \in Ker(\psi_{2,\mathcal{E}})$ of rank $2k$ and let $V \subset H^0(X, \mathcal{E})$ a subspace of dimension $2k$ such that $w \in \Lambda^2 V$. Let $\mathcal{E}_V$ be the image of the evaluation map restricted to $V \otimes \mathcal{O}_X$.*
If $rank\, \mathcal{E}_V = k$, then:

a) $\delta(\mathcal{E}_V) + \delta(\mathcal{E}_V^{**}) = 0$

b) $\delta(\mathcal{E}_V^{**}) \geq 0$ *and* $\delta(\mathcal{E}_V) \leq 0$.

c) Assume that $\mathcal{E}_V \hookrightarrow \mathcal{E}$ is an isomorphism in codimension 1. Then $\delta(\mathcal{E}) \geq 0$

Proof. a) and b) follow from 3.2. To see part c) we observe that now also $\mathcal{E}_V^{**} \hookrightarrow \mathcal{E}$ is an isomorphism in codimension 2, then $\delta(\mathcal{E}) = \delta(\mathcal{E}_V^{**}) \geq 0$. □

In general the class $\delta(\mathcal{E})$ is not positive, even for globally generated sheaves. Consider, for instance, the symmetric product $S^2(C)$ of a non-hyperelliptic curve C. Then the Hodge numbers are easily computed and the equality $\tau(S^2(C)) = \delta(\Omega^1_{S^2(C)}) = -g(C) + 1$ is obtained.

Special cases where the inequality $\delta(\mathcal{E}) \geq 0$ is true are discussed by Lazarsfeld in [16], ex. 8.3.18.

We apply 3.4 to generalized Lagrangian varieties. So we assume $\mathcal{E} = \Omega^1_X$. We write $\overline{\Omega^1_X}$ instead of $\Omega^1_{X,V}$. Then we have:

Corollary 3.5. *Let X be a generalized Lagrangian variety such that $\overline{\Omega^1_X} \hookrightarrow \Omega^1_X$ is an isomorphism in codimension 1. Then $\delta(\Omega^1_X) \geq 0$.*

Remark 3.6. If X is Lagrangian with $a : X \longrightarrow A$ an immersion, then we obtain the well-known isomorphism between the dual of the tangent and the normal bundle of X in A, providing the self-dual exact sequence

$$0 \longrightarrow \Omega^{1*}_X \longrightarrow V^* \otimes \mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow 0.$$

Therefore the pieces of even codimension in the Chern character of Ω^1_X vanish. If a does not have a branch divisor then $\delta(\Omega^1_X) \geq 0$ by the last Corollary. In particular, if $X = S$ is a Lagrangian surface with these conditions, then $\tau(S) \geq 0$.

4 Rank 2 vector bundles

We apply the results of the preceding section in the following situation:

We consider $\mathcal{E}$ a locally free sheaf of rank 2 with determinant L, and

$$w = \omega_1 \wedge \omega_2 + \omega_3 \wedge \omega_4$$

a rank 4 element in $Ker(\psi_{2,\mathcal{E}})$. We also assume that $\omega_1, \ldots, \omega_4$ are linearly independent sections generating a vector space $V \subset H^0(X, \mathcal{E})$. The evaluation map $V \otimes \mathcal{O}_X \longrightarrow \mathcal{E}$ is assumed to be generically surjective, in other words $\mathcal{E}_V$ has rank 2.

We now introduce some notation. Let us consider the natural map

$$\Lambda^2 V \otimes \mathcal{O}_X \longrightarrow L.$$

The image can be written as $L(-F_V) \otimes \mathcal{I}_{Z_V}$, where F_V is a divisor and Z_V is a subscheme of codimension ≥ 2. In other words, the image of $\Lambda^2 V \longrightarrow H^0(X, L)$ defines a subsystem of $|L|$. Then F_V is its base divisor and Z_V is the base locus of its moving part.

With this notation, part c) of 3.4 translates into

Theorem 4.1. *With the above notations, assume $F_V = 0$. Then:*

$$\delta(\mathcal{E}) \geq 0.$$

In particular, if $X = S$ is a generalized Lagrangian surface with $F_V = 0$, then $\delta(\Omega^1_S) \geq 0$.

We want to extend this result to Lagrangian surfaces with some conditions on F_V. We will need the following:

Proposition 4.2. *The following equality hold:*

$$det\,\overline{\Omega^1_S}^{**} = \omega_S(-F_V).$$

Proof. Since

$$V \otimes \mathcal{O}_S \twoheadrightarrow \overline{\Omega^1_S} \subset \overline{\Omega^1_S}^{**} \subset \Omega^1_S,$$

and

$$\overline{\Omega^1_S} \subset \overline{\Omega^1_S}^{**}$$

is an isomorphism in codimension 1, then

$$\Lambda^2 V \otimes \mathcal{O}_S \twoheadrightarrow \omega_S(-F_V) \otimes \mathcal{I}_{Z_V} \subset \Lambda^2 \overline{\Omega^1_S}^{**}$$

being the last inclusion an isomorphism in codimension 1. Hence:

$$\omega_S(-F_V) = c_1(\omega_S(-F_V) \otimes \mathcal{I}_{Z_V}) = c_1(\Lambda^2 \overline{\Omega^1_S}^{**}) = det\,\overline{\Omega^1_S}^{**}.$$

□

5 Bounding $\delta(\Omega^1_S)$ in the case where F_V is a reduced connected divisor with normal crossings and rational components

We want to bound from below $\delta(\Omega^1_S)$ under some conditions on F_V. In particular we shall assume that F_V is a reduced divisor with normal crossings. The principal ingredient will be a relation of $\overline{\Omega^1_S}$ with a sheaf of logarithmic differentials, as we shall see in the proposition below.

We next recall a definition given in the Introduction.

Definition 5.1. An effective reduced divisor D on the smooth minimal surface S is contracted by the vector space $V \subset H^0(S, \Omega^1_S)$ if the composition:

$$V \otimes \mathcal{O}_S \longrightarrow \Omega^1_S \longrightarrow \Omega^1_{S_{|D}} \longrightarrow \omega_D$$

vanishes.

When $V = H^0(S, \Omega^1_S)$ this composition is the dual of the differential of the Albanese map a restricted to D. Therefore D is contracted by $H^0(S, \Omega^1_S)$ if and only if $a(D)$ has dimension 0.

On the other hand, if the components of D are rational, then, by the very definition, D is contracted by any V.

Proposition 5.2. *Assume F_V is a reduced divisor with normal crossings, with smooth irreducible components which is contracted by V. Then*

$$\overline{\Omega^1_S}^{**} \cong \Omega^1_S(log(F_V))(-F_V).$$

Proof. Let p be a smooth point of F_V and let x, y local coordinates at p such that $y = 0$ is the local equation of F. Observe that $\Omega^1_S(log(F))(-F)$ is generated at p by ydx and dy.

A global 1-form $\omega \in V$ is locally at p: $\omega = a(x,y)dx + b(x,y)dy$. Since F_V is contracted by V, ω vanishes in the tangent direction $\partial/\partial x$ to F_V, hence $a(x,y) = ya_0(x,y)$.

Then in the complementary of a finite set of points the elements of V belong to $H^0(S, \Omega^1_S(log(F_V))(-F_V))$. Since they are sections of locally free sheaves we conclude that $V \subset H^0(S, \Omega^1_S(log(F_V))(-F_V))$.

The commutativity of the diagram

$$\begin{array}{ccc}
V \otimes \mathcal{O}_S & \hookrightarrow & H^0(S, \Omega^1_S(log(F_V))(-F_V)) \otimes \mathcal{O}_S \\
\downarrow ev & & \downarrow ev \\
\overline{\Omega^1_S} & & \Omega^1_S(log(F_V))(-F_V) \\
\cap & & \cap \\
\overline{\Omega^1_S}^{**} & \hookrightarrow & \Omega^1_S,
\end{array}$$

and the surjectivity of the left evaluation map gives the inclusion of subsheaves of Ω^1_S:

$$\overline{\Omega^1_S}^{**} \subset \Omega^1_S(log(F_v))(-F_V).$$

Since both have the same determinant $\omega_S(-F_V)$ (see 4.2), they are equal. □

Corollary 5.3. *With the same hypotheses on F_V, one has*

$$\delta(\Omega^1_S) \geq K_S F_V + \frac{1}{2} F_V^2.$$

Proof. We consider the well-known short exact sequence:

$$0 \longrightarrow \Omega^1_S \longrightarrow \Omega^1_S(log(F_V)) \longrightarrow \mathcal{O}_{F_V} \longrightarrow 0.$$

Tensoring with $\mathcal{O}_F(-F)$ and applying 5.2 we obtain:

$$0 \longrightarrow \Omega^1_S(-F_V) \longrightarrow \overline{\Omega^1_S}^{**} \longrightarrow \mathcal{O}_{F_V}(-F_V) \longrightarrow 0.$$

Then, by using 3.4, b) (recall that here $\mathcal{E}_V = \overline{\Omega^1_S}$) and the previous exact sequence, we have:

$$\begin{aligned}
0 \leq \delta(\overline{\Omega^1_S}^{**}) &= \delta(\Omega^1_S(-F_V)) + \delta(\mathcal{O}_{F_V}(-F_V)) \\
&= \delta(\Omega^1_S) - K_S F_V + F_V^2 + \delta(\mathcal{O}_{F_V}(-F_V)) = \delta(\Omega^1_S) - K_S F_V + F_V^2 - \frac{3}{2}F_V^2 \\
&= \delta(\Omega^1_S) - K_S F_V - \frac{1}{2}F_V^2.
\end{aligned}$$

□

One of the more interesting consequences of this corollary is the following result:

Theorem 5.4. *Assume F_V is a reduced connected divisor with normal crossings contracted by V. Then $\tau(S) \geq 0$.*

Proof. Since F_V is contracted by V we can apply 5.3 to compute the index:

$$\tau(S) \geq \frac{2}{3}K_S F_V + \frac{1}{3}F_V^2.$$

Moreover $K_S F_V \geq -2 - {F_V}^2$. So

$$\tau(S) \geq -\frac{4}{3} - \frac{1}{3}F_V^2.$$

Since $F_V^2 \leq -2$, then $\tau(S) \geq -\frac{2}{3}$. Therefore the integer $\tau(S)$ is non-negative.

□

Remark 5.5. In fact, it is a straightforward computation to check that the theorem is true in a more general setting: it is enough to assume that F_V is reduced contracted by V and it contains at most one connected component with arithmetic genus 0. For instance, if F is a reduced rational connected divisor with normal crossings, then F_V is contracted and $\tau(S) \geq 0$.

6 Examples

The main purpose of this section is to investigate how far the topological statement $\tau(S) \geq 0$ remains true when the condition $F_V = 0$ in 4.1 is weakened. In the previous section we have seen that F_V rational, connected and with normal crossings again implies $\tau(S) \geq 0$ (see 5.4).

Nevertheless, the next example shows a way to construct generalized Lagrangian surfaces with negative index. In these examples the divisor F_V is not contracted by V.

Example 6.1. Let $f : X \longrightarrow Y$ a double covering of a surface Y. Let C be the discriminant divisor. We assume that X, Y and C are smooth. Then

$$2\chi_{top}(Y) - \chi_{top}(C) = \chi_{top}(X).$$

So $2c_2(Y) - (2 - 2g(Y)) = 2c_2(Y) + C^2 + CK_Y = c_2(X)$. On the other hand $K_X^2 = 2K_Y^2 + 2CK_Y + \frac{1}{2}C^2$. Together, this gives

$$\delta(X) = 2\delta(Y) - \frac{3}{4}C^2.$$

Now, we choose as Y the product of two curves of genus ≥ 2, and C a smooth curve on Y with $C^2 > 0$. Then X inherits the two fibrations on Y, hence there are four differential forms $\omega_1, \ldots, \omega_4$ such that $\omega_1 \wedge \omega_2 = 0$ and $\omega_3 \wedge \omega_4 = 0$. The sum of these two relations provides a rank 4 element in $Ker(\psi_{2,\Omega_S^1})$. But it is easy to check $\delta(Y) = 0$ and hence $\delta(X) < 0$.

In the rest of this section we give some examples of generalized Lagrangian surfaces.

Example 6.2. The most obvious examples are surfaces with two fibrations of higher base genus, for instance those which are isogenous to a product of curves (see [3]). Indeed, there are two different decomposable elements in the kernel of ψ_{2,Ω_S^1} provided by the two fibrations. The addition gives a rank 4 element in this kernel.

Example 6.3. Arguing as in Example 6.1 we see that coverings $f : X \longrightarrow Y$ of generalized Lagrangian surfaces are also generalized Lagrangian. Moreover if the covering f is étale then $\tau(X)$ is a multiple of $\tau(Y)$, hence $\tau(X) \geq 0$ if and only if $\tau(Y) \geq 0$.

Example 6.4. In the paper [15], Lagrangian surfaces embedded in Abelian four-folds are constructed. The authors proved that some of these examples have not fibrations of higher base genus.

We also recall the important examples by Campana [5] and Sommese-Van de Ven [9] of non-trivial kernel of the cohomology map. In particular, the paper [5] revived interest in the nilpotent completion of Kähler groups, showing examples of exotic nilpotent Kähler (conjectured not to exist).

Example 6.5. If S is a minimal surface with $p_g \leq 4q - 11$, then the kernel of ψ_{2,Ω_S^1} meets the set of elements in $\Lambda^2 H^0(S, \Omega_S^1)$ with rank less or equal than 4. So either they have a fibration of higher base genus or they are generalized Lagrangian. In particular this happens when $p_g = 5$ and $q = 4$ and we will see that $\tau(S) \geq 0$ in this case.

Example 6.6. We consider the surface constructed in [14] (pages 85-86) by means of the Prym map of simple cyclic triple coverings of elliptic curves. Roughly speaking, the moduli space

$$\mathcal{M} := \{\pi : C \longrightarrow E \,|\, g(C) = 4, deg\,(\pi) = 3, g(E) = 1, \pi \text{ cyclic}\}/\cong$$

and the map

$$\begin{aligned} P : \mathcal{M} &\longrightarrow \mathcal{A}_3(\gamma) \\ \pi &\longmapsto P(C, E) = P(\pi) = J(C)/\pi^*(J(E)), \end{aligned}$$

are considered. The variety $A_3(\gamma)$ is the moduli space of Abelian threefolds with a polarization of a convenient type γ.

Then the general fibre has dimension 1. In particular there are 1-dimensional families of coverings $\{\pi_t\}_{t\in B}$ with fixed $P(\pi_t)$. Hence there exist a smooth surface S, a fibration $f : S \longrightarrow B$ and an action of $\mathbb{Z}/(3)$ on S which restricts to the generic fibre C_t with quotient an elliptic curve E_t. Moreover $P(C_t, E_t)$ is a constant polarized Abelian variety P, which is exactly the kernel of the surjective map $Alb(S) \longrightarrow Alb(B) = J(B)$, in particular $q(S) = g(B) + 3$.

By choosing a generator ρ of the group of characters we have a decomposition as follows

$$H^0(S, \Omega^1_S) = \pi^*(H^0(B, \omega_B)) \oplus H^0(S, \Omega^1_S)^\rho \oplus H^0(S, \Omega^1_S)^{\rho^2}.$$

We can assume, possibly changing the generator, that $dim\, H^0(S, \Omega^1_S)^\rho = 1$ generated by α and $dim\, H^0(S, \Omega^1_S)^{\rho^2} = 2$ generated by β_1 and β_2.

Now the equivariant map

$$\Lambda^2 H^0(S, \Omega^1_S) \longrightarrow H^0(S, \omega_S) \longrightarrow H^0(C_t, \omega_{C_t})$$

sends

$$\Lambda^2 H^0(S, \Omega^1_S)^\rho = \pi^*(H^0(B, \omega_B)) \wedge \alpha \oplus \langle \beta_1 \wedge \beta_2 \rangle$$

to the 1-dimensional space $H^0(C_t, \omega_{C_t})^\rho$. Therefore there exists a 1-form η on B such that the 2-form on S: $\pi^*(\eta) \wedge \alpha + \beta_1 \wedge \beta_2$ vanishes when it restricts to the generic fibre C_t. It is not difficult to see that this produces an element $w \in Ker(\psi_{2,\Omega^1_S})$ of rank 4.

Finally we observe that this element "does not come from fibrations", in other words $\beta_1 \wedge \beta_2 \neq 0$. Indeed, otherwise there would exist a fibration of higher base genus $S \longrightarrow \Gamma$ such that $C_t \longrightarrow \Gamma$ is a covering. Notice that $g(\Gamma) \neq 4$ (otherwise $C_t \cong \Gamma$ and f would be isotrivial) and $g(\Gamma) \neq 3$ (by Riemann-Hurwitz). Moreover the existence of β_1, β_2 guarantees that $g(\Gamma) \geq 2$. Hence $g(\Gamma) = 2$. This is impossible since for a generic element $(C_t \longrightarrow E_t) \in \mathcal{M}$ the curve C_t does not admit non-trivial maps on curves of genus 2.

The previous construction can be used to obtain an actual Lagrangian surface in $P \times E$, where E is the elliptic curve, which is the algebraic part of the Intermedian Jacobian associated to $H^3(P, \mathbb{Z})$. We do not know any Lagrangian subvariety of a simple Abelian variety.

7 Surfaces of general type with $q = 4$, $p_g = 5$ and a fibration of higher base genus

For dimensional reasons, this is the case with the lowest invariants in which we can ensure the existence of a non-decomposable 2-form in the kernel of the map

$$\psi_2 : \Lambda^2 H^0(S, \Omega^1_S) \longrightarrow H^0(S, \omega_S).$$

(at least when there is not a fibration of higher base genus).

The following is the main result of the rest of the paper

Theorem 7.1. *Let S be a minimal complex projective surface over $\mathbb{C}$, of general type, with $q = 4$, $p_g = 5$. Then $\tau(S) \geq 0$.*

Remark 7.2. Notice that $K_S^2 + c_2(S) = 24$, therefore $\tau(S) \geq 0$ is equivalent to $K_S^2 \geq 16$. This means that $16 \leq K_S^2 \leq 18$, the upper bound given by the Miyaoka-Yau inequality.

Observe that, with these invariants, the kernel of ψ_2 is non-trivial. In order to prove the theorem we consider separately the two natural cases depending on the existence or not of decomposable elements in the kernel. In this section we give a complete classification of the first case assuming the existence of a fibration of higher base genus on S, and we postpone the rest of the proof until the next section.

Theorem 7.3. *The minimal surfaces of general type with a fibration of higher base genus and irregularity 4 and geometric genus 5 are isotrivial, of the form $(C \times H)/\mathbb{Z}_2$, where:*

a) C and H are two curves of genus 3 equipped with involutions without fixed points, or

b) C is a curve of genus 5 equipped with an involution without fixed points and H is a bielliptic curve of genus 2.

In both cases $\mathbb{Z}_2$ acts diagonally.

Both examples give irreducible families of surfaces of dimensions 6 and 8 respectively.

All of them satisfy $K_S^2 = 16$. In particular $\tau(S) = 0$.

Proof. Let $f : S \longrightarrow B$ be a fibration with $g(B) = b \geq 2$ and fibre H of genus $g \geq 2$. We consider the relative invariant

$$0 \leq \chi_f = \chi(\mathcal{O}_S) - (b-1)(g-1) = 2 - (b-1)(g-1).$$

There are two cases:

1. $b = g = 2$, then $q(S) = 4 = b + g$ and so $S \cong B \times H$ which contradicts $p_g(S) = 5$.

2. $b = 2, g = 3$ or $b = 3, g = 2$. Therefore $\chi_f = 0$ and then also $0 = K^2_{S|B} = K^2_S - 8(g-1)(b-1)$. So $K^2_S = 16$.

We can go further in the analysis of the fibrations in case 2. First we recall that $\chi_f = 0$ implies f isotrivial and smooth. Then we can apply the general results on these surfaces and, in particular, the very explicit description provided in [8]. For instance we know that S is the quotient of a product of two smooth curves $C \times H$ by a finite group G of order $n > 1$ acting diagonally and the map $C \longrightarrow C/G$ is unramified (since f is smooth). Then there are two fibrations $g_i : S \longrightarrow D_i/G$ which are the composition of

$$S \cong (C \times H)/G \longrightarrow C/G \times H/G$$

with the projections. The initial fibration f equals g_1, so $B = C/G$.

Observe that

$$\begin{aligned} q(S) &= q((C \times H)/G) = h^0((C \times H)/G, \Omega^1_{(C\times H)/G}) = \\ &\dim H^0(C \times H, \Omega^1_{C\times H})^G = g(C/G) + g(H/G) = b + g(H/G), \end{aligned}$$

hence $4 = b + g(H/G)$.

Assume first $b = g(H/G) = 2$. Then, Riemann-Hurwitz for $H \longrightarrow H/G$ implies $n = 2$. On the other hand

$$\begin{aligned} &p_g(S) = p_g((C \times H)/G) = h^0((C \times H)/G, \omega_{(C\times H)/G}) = \\ &\dim H^0(C, \omega_C)^+ \cdot \dim H^0(H, \omega_H)^+ + \dim H^0(C, \omega_C)^- \cdot \dim H^0(H, \omega_H)^- \\ &= b \cdot g(H/G) + (g(C) - b)(g(H) - g(H/G)) = 4 + (g(C) - 2). \end{aligned}$$

Therefore $g(C) = 3$ and $C \longrightarrow B$ is unramified.

Assume now $b = 3$ (hence $g = 2$), $g(H/G) = 1$. In this situation $S \longrightarrow H/G$ has two singular fibres since $H \longrightarrow H/G$ has two ramification points. These fibres are of the form $(\#H_i)C/H_i$, $i = 1, 2$, where H_i are subgroups of G. Then, by using the formula of theorem 4.1 in [8] for K_S,

$$\begin{aligned} &2g(H) - 2 = 2 = H^2 + K_S H = K_S H = \\ &(\#H_1 - 1)H \cdot (C/H_1) + (\#H_2 - 1)H \cdot (C/H_2). \end{aligned}$$

Since $H \cdot (C/H_i) \neq 0$, we deduce $H \cdot (C/H_i) = 1$ and $\#H_i = 2$. On the other hand

$$n = HC = (\#H_1)H \cdot C/H_1 = 2.$$

Hence $G \cong \mathbb{Z}_2$. Computing $p_g(S)$ as above we find $g(C) = 5$ and that $C \longrightarrow B$ is unramified. $\square$

Notice that the previous considerations allow us to prove that the Albanese dimension is always 2. Indeed, S would have otherwise a fibration on a curve of genus 4, which is not compatible with the range of genus found above.

8 Surfaces of general type with $q = 4$, $p_g = 5$ and without a fibration of higher base genus

In this section we finish the proof of 7.1. We assume now that S has not a fibration of higher base genus and the dimension of $a(S)$ is 2. In particular, they are generalized Lagrangian (see 6.5). There exists a non-decomposable

$$w \in \Lambda^2 H^0(S, \Omega^1_S),$$

unique, up to constant, which vanishes in $H^0(S, \omega_S)$. Then we can use the notations and theorems of §4. Here $V = H^0(S, \Omega^1_S)$, $F_V = F$ the base divisor of the canonical system and ψ_2 is surjective. Also by 4.1 we can restrict ourselves to the hypothesis $F \neq 0$. In particular, by 2-connectivity, $MF \geq 2$, where M is the moving part of the canonical system.

Now we analyze the canonical map of S. Note that a general type surface with $q \geq 3$ has a canonical image of dimension 2 ([11]).

Observe we have a short exact sequence of vector spaces

$$0 \longrightarrow \langle w \rangle \longrightarrow \Lambda^2 V \longrightarrow H^0(S, \omega_S) \longrightarrow 0.$$

By dualizing, we see $\mathbb{P}(H^0(S, \omega_S)^*)$ as the hyperplane $H_w \subset \mathbb{P}(\Lambda^2 V^*)$ naturally attached to w. Notice that H_w is not tangent to the Grassmannian $Gr(2, V^*)$, since w is not decomposable.

We attach to a point in S the geometric fibre of $\overline{\Omega^1_S}^* \subset V^* \otimes \mathcal{O}_S$ at this point. We see this 2-dimensional vector space as an element in the Grassmannian naturally embedded in $\mathbb{P}(\Lambda^2 V^*)$. By construction we have a commutative diagram of rational maps:

$$\begin{array}{ccc} S & \longrightarrow & Gr(2,V^*) \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ H_w & \longrightarrow & \mathbb{P}(\Lambda^2 V^*). \end{array}$$

As a consequence we obtain that the image Σ of the canonical map is contained in the smooth quadric $Q := Gr(2,V^*) \cap H_W$ of $\mathbb{P}^4 = H_W = \mathbb{P}(H^0(S,\omega_S)^*)$.

Any divisor on a smooth quadric of $\mathbb{P}^4$ is a complete intersection, so there exists a hypersurface T of degree t such that $\Sigma = Q \cap T$. Denote $e = \deg(\varphi)$. We have

$$M^2 = \deg(\Sigma)\deg(\varphi) + b = 2\,t\,e\, +\, b.$$

where b is the contribution of the base points of the linear system $|M|$. The surface Σ is not degenerated by construction, hence t is at least 2.

Proposition 8.1. *Under our hypothesis*

a) $M^2 \geq 10$.

b) If $M^2 = 10, 11$, *then* $b = 0, 1$ *respectively.*

Proof. If $te \geq 5$ we are done. We assume $t\,e\, \leq 4$. There are the following possibilities:

Case A: $t = 2$, $e = 2$.

Here S is a double covering of the intersection of two quadrics Q, T in $\mathbb{P}^4$. Since one of them is smooth it is a (possibly singular) del Pezzo surface. Hence S is covered by a linear system of hyperelliptic curves. This contradicts the results of [10] (see also [13]).

Case B: $t = 3$, $e = 1$.

In this case the canonical map is a birational morphism with a surface Σ which is not of general type; this contradicts our hypothesis.

Case C: $t = 4$, $e = 1$.

We consider a general hyperplane section of Σ. It is the complete intersection of a smooth quadric and a quartic surface in $\mathbb{P}^3$. So its arithmetical genus is 9. Its preimage is a general curve C in $|M|$. We have $M^2 + MK_S \geq 16 + MF \geq 18$. If C is smooth, then genus C is at least 10, and it is a desingularization of its image, φ being birational, a contradiction. If the general element C is singular, then by Bertini's theorem it must be singular at the base points of the linear system. In this case, necessarily $b \geq 4$ and so $M^2 \geq 12$. □

Now we study some properties of special curves on S.

Proposition 8.2. *Under our hypotheses*

a) There is not an effective divisor D on S with $D^2 = 0$ and $K_S D \leq 2$.

b) If C is a rational curve on S, then $C \subseteq F$.

c) Let k be the number of (-2)-curves on S. Then

$$K_S^2 \leq 18 - \frac{3}{4}k.$$

If equality holds then the (-2)-curves are disjoint.

Proof. a) The arithmetical genus of D is at most 2. Let $\Gamma = a(D)$ be the image of D by the Albanese map. Since the square of D is not negative it cannot be contracted, hence Γ is a curve with arithmetical genus 1 or 2. By definition the image of D by the map $a' : S \longrightarrow Alb(S)/\langle\Gamma\rangle$ is a point, hence, since $D^2 = 0$, $a'(S)$ is a curve, and generates the Abelian variety in the image, so its genus is at least 2. Then a' defines a fibration of higher base genus and this contradicts our hypothesis on S.

b) The global differential 1-forms vanish along the tangent directions to a rational curve. Then the kernel of the form at each point of the curve is 1-dimensional and the wedge product of two of them is zero. In other words, a rational curve is contained in the base locus of $\Lambda^2 H^0(S, \Omega_S^1) = H^0(S, \omega_S)$, which is F.

c) The result follows from [6], Theorem 1.1, taking $D = 0$ and E the divisor of (-2) curves on S. Let $E = E_1 + ... + E_l$ be its decomposition in connected components. Let k_i be the number of irreducible components of E_i (we have then that $k = k_1 + ...k_l$). Then the Zariski decomposition of $K_S + E$ is exactly $P = K_S$, $N = E$. For any i define

$$\nu_i = e(E_i) - \frac{1}{|G_i|}$$

where G_i is the finite group of the quotient singularity produced contracting E_i. Clearly $e(E_i) = 1 + k_i$ and $|G_i| \geq 2$ (and when equality holds E_i is a single (-2)-curve). Then, applying Theorem 1.1 in [6], we obtain

$$\frac{1}{2}l + k \leq \Sigma_i \nu_i \leq c_2(S) - \frac{1}{3}K_S^2 - \frac{1}{4}E^2 = 24 - \frac{4}{3}K_S^2 + \frac{1}{2}l$$

and so

$$K_S^2 \leq 18 - \frac{3}{4}k.$$

If equality holds, then $|G_i| = 2$ for all i and so all the connected components of E are single (-2)-curves.

□

In the next result we present useful information on the base locus of the linear system $|K_S + M|$.

Proposition 8.3. *Under our hypotheses, assume $K_S^2 \leq 15$. Then*

a) If $K_S F = 1$, $F^2 = -1$, then $|K_S + M|$ has no base point.

b) Let p be a base point of $|K_S + M|$. Then, there exists an effective divisor C through p, with $C^2 = -1\,, MC = 0\,, FC = 1$. If $KF = 0$ we can assume C is an elliptic smooth curve. If, moreover, $F^2 = -2$, then such a curve C is unique with the above numerical properties.

c) The base divisor B of $|K_S + M|$ is formed by elliptic curves in F contracted by $|M|$. In particular, since $MF \geq 2$, $F \neq B$.

Proof. From the previous results, we know that $M^2 \geq 10$, $MF \geq 2$.

Let p be a base point of $|K_S + M|$. Since M is big and nef we can apply Reider's theorem and obtain an effective divisor C passing through p with one of the following numerical properties:

- $C^2 = 0$, $MC = 1$
- $C^2 = -1$, $MC = 0$

Since M is big, the Hodge-Index theorem says

$$det \begin{pmatrix} M^2 & MF & MC \\ MF & F^2 & x \\ MC & x & C^2 \end{pmatrix} \geq 0$$

where $x = FC$. Since $1 + x = KC + C^2$ is even, x must be odd and, since $KC = MC + x \geq 0$ we obtain that $x \geq -MC$. Observe that, in fact, $x \geq 1$. Indeed, if $x = -1$, then $KC = 0$ and so C is a chain of (-2)-curves, which is impossible since $C^2 = 0$.

Moreover we have

$$12 \leq M^2 + MF \leq K_S^2 \leq 15$$

and so $MF \leq 5$. Computing the determinant, we obtain in the first case

$$-x^2 M^2 + 2xMF - F^2 \geq 0$$

which gives

$$10x^2 + 2 \leq (x^2 - 1)M^2 + 12 \leq 2(x+1)MF \leq 10(x+1)$$

As x is odd and positive, this can only happen if $x = 1$. But then observe that $KC = 2$ and $C^2 = 0$, which contradicts proposition 8.2 a).

So, only the second numerical possibility for C may occur. In this case, the determinant above gives

$$-F^2M^2 - M^2x^2 + (MF)^2 \geq 0$$

and so,

$$F^2 + x^2 \leq \frac{(MF)^2}{M^2}$$

Hence

$$-MF + x^2 \leq F^2 + x^2 \leq 2$$

which produces $x^2 \leq 2 + MF \leq 7$ and so $x = 1$.

If $KF = 1$, $F^2 = -1$ we have $(F + C)^2 = 0$ and $K(F + C) = 2$ which again contradicts proposition 8.2. This proves a).

If $KF = 0$, let C_0 be an irreducible component of C meeting F ($x = FC = 1$). Since M is nef, we still have $MC_0 = 0$. Since C_0 is irreducible and M big we obtain $-2 \leq C_0^2 \leq -1$ by the Hodge-Index theorem. If $C_0^2 = -2$ then C_0 would be rational and so, by 8.2, we would obtain $C_0 \leq F$. But this is impossible, since $1 = KC_0 \leq KF = 0$. So C_0 is a smooth elliptic curve with $MC_0 = 0$ and $C_0^2 = -1$. Finally, note that C_0 also contains necessarily p.

If, moreover, $F^2 = -2$ and we had two curves C_i with $C_i^2 = -1$, $MC_i = 0$ and $FC_i = 1$ note that, being orthogonal to M, the Hodge-Index theorem would give

$$det \begin{pmatrix} C_1^2 & C_1C_2 \\ C_1C_2 & C_2^2 \end{pmatrix} > 0$$

and hence $C_1C_2 = 0$. So

$$(F + C_1 + C_2)^2 = 0$$

$$K(F + C_1 + C_2) = 2$$

which contradicts proposition 8.2. This finishes the proof of b).

In order to prove c) observe that, since $|M|$ has no base divisor, the base divisor B of $|K_S + M|$ must be contained in F. Let D be an irreducible component of B. For any point $p \in D$ there must exist an effective divisor C_p containing it, such that $MC_p = 0$ by b). We can assume C_p is irreducible with this property. Such a C_p cannot move with p otherwise S would be covered by contracted curves. So $D = C_p$ for any p. But then

$$0 \leq MD \leq MC_p = 0$$

and so $MB = 0$. Since $MF \geq 2$, we obtain that necessarily $B \neq F$ which proves c). □

We can finally prove the main theorem of this part:

Proof. (of Theorem 7.1)

If S has a fibration of higher base genus, then the result is given in theorem 7.3. Otherwise, by the previous considerations, there exists an element $\omega_1 \wedge \omega_2 + \omega_3 \wedge \omega_4 = 0$.

Since K_S is nef and by 2-connectivity we obtain

$$K_S^2 = K_S(M+F) \geq K_S M = M^2 + MF \geq M^2 + 2,$$

therefore $M^2 \geq 14$ implies the theorem.

Moreover proposition 8.1 gives $M^2 \geq 10$. We have the following numerical facts:

1. MF is even.
2. $\chi(S, \mathcal{O}_S(M + K_S)) = h^0(S, \mathcal{O}_S(M + K_S)) = 2 + M^2 + \frac{1}{2}MF$.
3. $\chi(S, \mathcal{O}_S(2K_S)) = h^0(S, \mathcal{O}_S(2K_S)) = 2 + K_S^2$
4. $h^0(S, \mathcal{O}_S(2M)) \geq 13$. If $t \geq 3$, then $h^0(S, \mathcal{O}_S(2M)) \geq 14$
5. $h^0(S, 2M) < h^0(S, K_S + M)$

Indeed, (1) follows from adjunction formula for F, and (2) and (3) from Riemann-Roch formula and Ramanujam vanishing theorem. (5) is exactly 8.3 (iii). (4) is a consequence of counting quadrics containing Σ: there are 2 linearly independent quadrics through Σ if $t = 2$, and only 1 otherwise.

All these numerical restrictions configure the following possibilities if $K_S^2 \leq 15$:

M^2	MF	K_S^2	F^2	K_SF	$h^0(2M)$	$h^0(M+K_S)$	$h^0(2K_S)$
11	4	15	-4	0	14	15	17
12	2	14	-2	0	≥ 13	15	16
12	2	15	-1	1	$\geq$13	15	17
13	2	15	-2	0	$\geq$13	16	17

Notice that, following proposition 8.3, in all these cases either there are no base points of $|K_S + M|$ (when $KF = 1$) or there are no base divisor (when $KF = 0$ all the components of F are (-2)curves).

We will prove, by a case by case consideration, that none of these four possibilities may occur.

Case 1. $M^2 = 11$, $K_S^2 \le 15$ is not possible.

In this case $b = 1$ and $te = 5$. Let p be the unique base point of $|M|$. It is necessarily a simple base point. On the other hand, $t = 5$ implies that Σ is contained in a unique quadric of $\mathbb{P}^4$ and so the map

$$S^2 H^0(S, M) \longrightarrow H^0(S, 2M)$$

is surjective, by dimension counting. Hence p is a singular base point of the linear system $|2M|$.

Consider the inclusions

$$H^0(S, 2M) \subseteq H^0(S, \mathcal{I}_p(K_S + M)) \subseteq H^0(S, K_S + M).$$

Note that the first subspace has dimension 14 and the last 15. So the subspace in the middle coincides with one of them. If $H^0(S, \mathcal{I}_p(K_S + M)) = H^0(S, K_S + M)$, then p is a base point of $|K_S + M|$. By proposition 8.3 there exists an elliptic smooth curve C through p with $MC = 0$. Since p is a base point of $|M|$ and it has no base component, this is not possible.

If $H^0(S, 2M) = H^0(S, \mathcal{I}_p(K_S + M))$, then p is not a base point of $|K_S + M|$ and all the divisors of this linear system passing through p are singular at p. We can apply the second part of Reider's theorem: given any vector $v \in T_pS$, there exists a divisor D through p, such that $v \in T_pD$ and $MD \le 2$.

Assume first that D is smooth at p and, in particular, there is only one irreducible component D_v of D passing through p. Since it must have v as a tangent vector, and it is non-singular at p, this component moves with v. But this is impossible, since $MD_v \le 2$ and so, p being a simple base point of $|M|$, either the curves D_v are contracted via the canonical map or they are rational curves.

Therefore we can assume that D is singular at p. In particular $MD \ge 2$ and hence $MD = 2$. Looking at the possibilities of Reider's theorem we have $D^2 = 0$.

Now we compute $x = FD$ by using the Hodge-Index theorem. We consider the determinant of the matrix given by the intersection products of M, F and D:

$$det \begin{pmatrix} 11 & 4 & 1 \\ 4 & -4 & x \\ 1 & x & 0 \end{pmatrix} \ge 0$$

and we obtain $x = 0, 1, 2$. Notice that $KD + D^2 = MD + FD = 2 + x$ is even, so x is either 0 or 2. Therefore $p_a(D)$ is either 2 or 3 and considering the quotient $Alb(S)/ < a(D) >$, we construct a fibration $f : S \longrightarrow B$, $g(B) = 1, 2$ with fibre a multiple of D. Since F consists of rational curves, then $f(D) = 0$. Hence $x = FD = 0$. In particular $KD = 2$ (and $D^2 = 0$). By 8.2, part a) we obtain a contradiction.

Case 2. $M^2 = 12$, $K_S^2 \leq 15$ is not possible.

Assume first $|K_S + M|$ has base points. Then, by proposition 8.3 (i) we necessarily have $MF = 2\,, K_S^2 = 14\,, KF = 0$.

Observe that $|K_S + M|$ has at least one base point on F. Indeed, if $|M|$ is base point free so it is $|2M|$ and then this is immediate. If $b \neq 0$, then $t \neq 2$ and hence $h^0(S, 2M) = 14$. Since $h^0(S, K_S + M) = 15$ we have that $H^0(S, 2M)) + F \subseteq H^0(S, K_S + M)$ is a hyperplane. From $(K_S + M)F = 2$ we obtain some base point p on F.

By proposition 8.3 there exists an elliptic curve through p with $FC = 1$. Moreover, this C is unique so there are no more base points on F. Let F_0 be the component of F containing p.

Notice that $MF_0 = 0$ implies that $F_0 + C$ satisfies the same numerical conditions as C and this contradicts proposition 8.3 (ii). If $MF_0 = 2$, then

$$0 = K_S F_0 = MF_0 + FF_0 = 2 + FF_0.$$

The classification of [2] gives $F = F_0$ is rational and irreducible, in particular we can use remark 5.5 and we obtain $\tau(S) \geq 0$.

Finally assume $MF_0 = 1$. Observe that F_0 has to be a simple component, since $CF_0 = CF = 1$. The equality $MF = 2$ implies the existence of another component F_1 with MF_1. Then $h^0(S, M+K_S-F_1) = 14$ ($|M+K_S|$ does not have base divisor) and therefore there is a new base point on F_1. Applying again proposition 8.3 we find an elliptic curve C' through this point. Since $C \neq C'$ this contradicts that C is unique.

Then $FF_0 = -1$. We use again the classification in [2] to obtain that F is reduced and with normal crossings. By using 5.5 we obtain $\tau(S) \geq 0$.

Assume now that $|M+K_S|$ is base point free. Since $(M+K_S)F \neq 0$, then $\varphi_{M+K_S}(F)$ is not a point. The dimension of the linear subspace generated by $\varphi_{M+K_S}(F)$ is precisely $h^0(S, K_S+M) - h^0(S, 2M) - 1$. Hence, necessarily $h^0(S, \mathcal{O}_S(2M)) = 13$ and therefore $\varphi_{M+K_S}(F)$ is a line l.

We decompose $F = F_0 + F'$, F_0 consisting in the components of F not contracted by M; observe that F_0 has at most two components. Then, the restriction to F_0 of the map φ_{M+K_S} gives $h : F_0 \longrightarrow l$ of degree $(M + K_S)F_0 \geq 2$. We now apply the second part of Reider's theorem to the general fibre of h: there exists an effective divisor E such that either $ME = 0$, $E^2 = -1, -2$ or $ME = 1$, $-1, 0$ or $ME = 2$, $E^2 = 0$.

Assume that ME is either 0 or 1. Then if E moves with the fibres of h the surface is covered by rational or contracted curves: a contradiction. Therefore $E \geq F_0$ (if F_0 has two components we apply Reider's theorem to couples of points in different components), hence $ME \geq MF_0 = 2$. We deduce that only the case $ME = 2$, $E^2 = 0$ is possible.

Observe that $EF = 0$. Indeed, denote $x = EF$. Since $E^2 + K_S E = ME + x = 2 + x$ is even, we obtain that x is even. Moreover the determinant of the matrix of products of M, F, E:

$$\begin{pmatrix} 12 & 2 & 2 \\ 2 & F^2 & x \\ 2 & x & 0 \end{pmatrix}$$

is positive, hence $8x - 4F^2 - 12x^2 > 0$. Therefore $x = 0$.

Now $K_S E = ME = 2$ and $E^2 = 0$ implies the existence of a fibration of higher base genus (see proposition 8.2), a contradiction.

Case 3. $M^2 = 13$, $K_S^2 \leq 15$ is not possible.

Since $KF = 0$, F_{red} is exactly the divisor of (-2)-curves on S. If k is the number of such curves, and we have $K_S^2 = 15$, proposition 8.2 c) says that $k \leq 3$ (if $k = 4$, then we have equality and then the 4 (-2)-curves are disjoint: this is impossible if $F^2 = -2$). But then $F^2 = -2$ implies that $F = A_k$ and corollary 5.4 finishes the proof. □

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