BOUNDS OF THE NUMBER OF RATIONAL MAPS BETWEEN VARIETIES OF GENERAL TYPE

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Abstract. We give a bound for the number of rational maps between algebraic varieties of general type under mild hypothesis on the canonical map. We use an idea inspired by Tanabe's work. Instead of attaching a morphism of Hodge structures to a rational map we simply associate to it a piece of the integral Hodge lattice. This procedure does not give an injective map, but by means of a geometric argument, we can estimate the number of maps with the same image.

1. Introduction. De Franchis proved in 1913 (see [3]) that the set of morphisms between two Riemann surfaces of genus at least 2 is finite. In other words, he showed the finiteness of the set

$$M(X,Z) = \{f\colon\ X \longrightarrow Z \mid f \text{ nonconstant}\},$$

where X, Z are curves of genus at least 2. Martens (cf. [9]) gave an effective bound of the number of elements $m(X,Z)$ of this set. Other estimates can be deduced from the effective bounds for the number elements of

$$M(X) = \{f\colon\ X \longrightarrow Y \mid f \text{ nonconstant, } Y \text{ smooth curve of genus } \geq 2\},$$

obtained in [6], [7] and [1].

Probably the most interesting open problem in the topic (see [5]) is whether $m(X,Z)$ can be bounded by a polynomial on the genus of the curves. Kani ([7]) showed that this is not true for $M(X)$.

In [10] Tanabe has improved the known bounds for $m(X,Z)$. In all the previous proofs morphisms of homological lattices were used to represent maps, or even correspondences, on curves. The main idea of Tanabe's work is to represent each map by a single element of the singular homology group of X. This enables him to control, with a geometric argument, the number of maps represented by the same element. In fact, his proof can be separated into two parts. In the first he shows that, fixing a holomorphic form α on the target, two maps in which

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the pull-backs of α are the same "differ" in a finite number of choices depending polynomially on the genus.

In the second part he assumes that α is the $(1,0)$-part of an element $\tilde{\alpha}$ of the integral lattice with minimal norm. Then he attaches to the map the pull-back of $\tilde{\alpha}$. This is an element in the integral lattice of the source. He then shows that we can reduce the lattice modulo d with d greater than twice the degree of the map, without losing information.

We refer to the first argument as the "geometric part" and to the second as the "torsion part" of Tanabe's work.

This paper concerns the same problem in a higher dimension, that is, we consider

$$M(X,Z) = \{f\colon X \dashrightarrow Z \mid f \text{ rational dominant map }\},$$

for X, Z varieties of general type of the same dimension. As above, $m(X,Z)$ denotes the number of elements of $M(X,Z)$.

It was proved by Kobayashi-Ochiai that $m(X,Z)$ is finite (see [8] and also [2]). Moreover there is an effective bound in [5] for complex manifolds with ample canonical bundle obtained by means of Chow varieties. This method provides necessarily a bound with a very high exponential.

In this paper we use an idea inspired by Tanabe's work. Instead of attaching a morphism of Hodge structures to a rational map we simply associate to it a piece of the integral Hodge lattice. This procedure does not give an injective map, but, by means of a geometric argument we can estimate the number of maps with the same image.

We do not need the restrictive hypothesis which guarantees the injectivity of the representation of the elements of $M(X,Z)$ as maps of Hodge structures. We can thus find good bounds under weak hypotheses. In fact, we find much better bounds for n-dimensional varieties than the ones currently known.

We use two approaches. The first works in dimensions 1, 2 and 3 and gives better results. The second applies in any dimension, under a more restrictive hypothesis.

Now we explain the ideas of the proofs: First we generalize the geometric part of Tanabe's work to surfaces with p_g at least 2 by using appropriate pencils of 2-forms on Z. Since $m(X,Z)$ is a birational invariant we may assume that X and Z are minimal. Next we represent the map using couples of elements in the transcendental lattice of the source variety. Roughly speaking, the transcendental lattice is the complement of the Neron-Severi group in the second cohomology group of the surface. The geometric part allows us to estimate the number of maps which are represented by the same couple of elements of the lattice. To do this we use the following fact, which is elementary but very useful: there exists an open set where all the maps are well-defined and such that for each point of this open set two different maps take different values. Then, by using the fact that the curves are moving in a pencil and thus cut this open set, one can reduce the proof to the one-dimensional case.

Next, instead of the torsion lemma we use a packing lemma due to Kani. To do so we give a lower estimate for the distance of two different elements. We obtain a bound of $m(X,Z)$ in terms of K_X^2, K_Z^2 and the second Betti number $b_2(X)$ (see 1.2). By combining Bogomolov-Miyaoka-Yau and Noether inequalities one can obtain an estimate in terms of the Euler characteristic (see 1.3).

Observe that, since we are not assuming that X, Z are canonical, the representation of the maps in $M(X,Z)$ as maps of transcendental Hodge structures is not injective in general.

Note also that using the packing lemma (instead of the torsion lemma) in the 1-dimensional case, we obtain a result which is slightly better than Tanabe's (see 1.1).

Then, with some additional hypotheses we can give a bound for threefolds following a similar argument. Note the difference in the arguments for surfaces and threefolds. In the first case, to prove the geometric lemma we reduce the proof to the one for curves. Instead, in the case of threefolds we need to use the full result on surfaces.

Apparently this "inductive procedure" does not extend to higher dimensions due to the method used and to the lack of a smooth minimal model in higher dimension.

In the last section we extend the torsion part in Tanabe's work. We use this to give a bound in general (see 1.5). This bound is clearly worse than the one obtained for surfaces and threefolds.

The paper is organized as follows: in §2 we give some preliminaries, mainly on Hodge structures. We also recall Kani's packing lemma.

To give the statements of the following theorems, we introduce the following function

$$P(a,e) = (a+1)^e - (a-1)^e, \qquad a \in \mathbb{R},\ e \in \mathbb{N}.$$

This is a polynomial on a. Its leading term is

$$2ea^{e-1}.$$

We also denote

$$\rho = \rho(X,Z) = \frac{K_X^n}{K_Z^n}$$

where X, Z are n-dimensional varieties (if $n = 1$, then $\rho = \frac{g(X)-1}{g(Z)-1}$).

Let $b_i(X)$ be the Betti number $\dim H^i(X,\mathbb{C})$.

In §3 we prove the theorem on curves:

THEOREM 1.1. *Let X, Z be smooth irreducible projective curves of genus ≥ 2. Then*

$$\begin{aligned} m(X,Z) &\leq 4(g(X)-1)\rho P(2\rho, 2g(X)) \\ &= 8(g(X)-1)\,\rho\left[\binom{2g(X)}{1}(2\rho)^{2g(X)-1}+\binom{2g(X)}{3}(2\rho)^{2g(X)-3}\cdots\right]. \end{aligned}$$

Section 4 is devoted to the proof of:

THEOREM 1.2. *Let X, Z be smooth irreducible projective minimal surfaces of general type. Assume $p_g(Z) \geq 2$. Then*

$$m(X,Z) \leq 4(K_X^2)^2\, P(4\sqrt{2}\,\rho, 2b_2(X)-2).$$

Since $\rho \leq K_X^2 \leq 9\chi(\mathcal{O}_X)$ (Bogomolov-Miyaoka-Yau) and

$$\begin{aligned} b_2(X) &= \chi_{\mathrm{top}}(X) + 4q(X) - 2 = 12\chi(\mathcal{O}_X) - K_X^2 + 4q(X) - 2 \\ &= 8\chi(\mathcal{O}_X) + 4p_g(X) - K_X^2 + 2 \leq 17\chi(\mathcal{O}_X) + 10 \end{aligned}$$

(we use Noether's formula and Noether's inequality) we immediately obtain a bound for surfaces in terms of the Euler characteristic:

COROLLARY 1.3. *Let X, Z be smooth irreducible projective surfaces of general type. Assume $p_g(Z) \geq 2$. Put $\chi = \chi(\mathcal{O}_X)$. Then*

$$m(X,Z) \leq 4\cdot 9^2\,\chi^2\, P(36\sqrt{2}\,\chi, 34\chi + 18).$$

In 1.3 we do not assume the surfaces are minimal. This will be useful in the proof of the next theorem for threefolds, which will be given in §5.

THEOREM 1.4. *Let X, Z be smooth irreducible projective complex threefolds of general type. Assume that K_X, K_Z are nef, $p_g(Z) \geq 2$ and the image of Z by the bicanonical map has dimension at least 2. Then*

$$m(X,Z) \leq 4\cdot 9^2\, h^2\, K_X^3\, P(36\sqrt{2}\,h, 34h+18)\cdot P(4\sqrt{2}\rho, 2b_3(X)),$$

where
$h = h^0(X, \mathcal{O}(2K_X)) + h^0(X, \Omega_X^2) - p_g(X) + 1.$

Finally in §6 we find a bound for n-dimensional varieties by extending Tanabe's torsion part to higher dimension. The result we obtain is:

THEOREM 1.5. *Let X, Z be two n-dimensional varieties of general type such that K_X, K_Z are nef and* $\dim(\varphi_{|K_Z|}(Z)) \geq n-1$. *Then:*

$$m(X,Z) \leq 2n(K_X^n)^2(2\rho+1)^{b_n(Z)\cdot b_n(X)}.$$

In the case of birational automorphisms we obtain a bound with a lower exponent than the one given in [5]:

COROLLARY 1.6. *Let X be a variety of general type with K_X nef and such that* $\dim(\varphi_{|K_X|}(X)) \geq n-1$, *then*

$$\#aut(X) \leq 2n\,(K_X^n)^2\,3^{b_n(X)^2}.$$

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2. Notations and preliminaries.

2.1. Notations. Throughout the paper we use the symbols $M(X,Z)$ and $m(X,Z)$ as in the introduction: the former is the set of rational dominant maps from to X to Z and the latter is its number of elements.

Analogously, $M_r(X,Z)$ is the subset of maps with fixed degree r and the number of its elements is $m_r(X,Z)$.

We also keep the notations:

$$\rho = \rho(X,Z) = \frac{K_X^n}{K_Z^n}, \qquad \text{where } n = \dim X = \dim Z,$$

and

$$b_i(X) = \dim H^i(X,\mathbb{C}).$$

We work over the complex numbers. In this paper variety means irreducible, smooth, projective, complex variety.

2.2. Hermitian spaces. Let (V, h_V) be a hermitian of finite dimension space we shall denote the norms for $v \in V$ by

$$\|v\| = \sqrt{h_V(v,v)}.$$

We recall that, for any map $f\colon\ V \to W$ between two hermitian spaces, we may define the adjoint map $g\colon\ W \to V$ as the unique linear map such for all $v \in V$ and $w \in W$

$$h_W(f(v), w) = h_V(v, g(w)).$$

2.3. Hodge structures and morphisms. Let X be a complete smooth algebraic variety of dimension n. Let $H^n(X)$ be the Hodge structure on X on the n-cohomology of X. The lattice $H_{\mathbb{Z}}$, is

$$H^n(X, \mathbb{Z})/torsion$$

and the standard Hodge decomposition: $H_{\mathbb{Z}} \otimes \mathbb{C} = H^n(X, \mathbb{C}) = \oplus H^{p,q}$, $H^{p,q} = \overline{H^{q,p}}$. Integration gives a natural polarization:

$$Q\colon\ H^n(X, \mathbb{C}) \times H^n(X, \mathbb{C}) \to \mathbb{C}$$

which is unimodular, by Poincaré duality, on $H_{\mathbb{Z}}$. We recall that a Hodge substructure R of H is given by a sublattice $R_{\mathbb{Z}} \subset H_{\mathbb{Z}}$ such that $R_{\mathbb{C}} = R_{\mathbb{Z}} \otimes \mathbb{C} = \oplus R^{p,q}$, where $R^{p,q} = H^{p,q} \cap R_{\mathbb{C}}$. The restriction of Q gives a polarization of R nonnecessarily unimodular over the integers. The polarization makes possible to define the orthogonal Hodge structure R'. Set $R'_{\mathbb{Z}} = \{\gamma \in H_{\mathbb{Z}} \mid Q(\gamma, \beta) = 0\ \forall \beta \in R_{\mathbb{Z}}\}$. One has $R_{\mathbb{C}} \oplus R'_{\mathbb{C}} = H_{\mathbb{C}}$.

Definition 2.1. The *transcendental* Hodge structure of X is the smallest Hodge substructure T_X of $H^n(X)$ containing $H^{n,0}(X)$. Its lattice $T_{\mathbb{Z},X}$ will be called the transcendental lattice of X. For any integer $d \geq 2$ let $T_{d,X} = T_{\mathbb{Z},X}/d \cdot T_{\mathbb{Z},X}$. Observe that if n is even then there exists a $(\frac{n}{2}, \frac{n}{2})$-integral class induced by a projective immersion; therefore $T_X \neq H^n(X, \mathbb{C})$ and $\dim T_X \leq b_n(X) - 1$.

Note that T_X is a birational invariant of X. Since T_X is contained in the primitive cohomology, then (due to the Hodge-Riemann relations, see [4], page 123) the cup-product modified with the Weil operator induces on T_X a hermitian product that we denote simply by $(,)$.

Let Z be another smooth complete variety of dimension n and

$$f\colon\ X \dashrightarrow Z$$

be a dominant rational map of degree $r = \deg f$. We then have two Hodge structure morphisms:

$$f^*\colon\ T_Z \to T_X, \quad f_*\colon\ T_X \to T_Z.$$

We have that they are adjoint maps; in other words:

$$(\alpha, f_*(\beta)) = (f^*(\alpha), \beta). \tag{2.2}$$

We may also define the group homomorphism f_d: $T_{d,Z} \to T_{d,X}$ induced by f^*. We have the following:

LEMMA 2.3.
(a) *If* $\gamma \in T_Z$ *then* $f_* f^*(\gamma) = r\gamma$.
(b) *If* $\gamma \in T_Z$ *then* $\|f^*(\gamma)\| = \sqrt{r}\,\|\gamma\|$.
(c) *For any* $\beta \in T_X$ $\|f_*(\beta)\| \leq \sqrt{r}\,\|\beta\|$ *and* $\|f_*(\beta)\| = \sqrt{r}\,\|\beta\|$ *if and only if there is* $\gamma \in T_Z$: $\beta = f^*(\gamma)$.

Proof. Parts (a) and (b) are well known. To see (c), we write $\beta = f^*(\beta_0) + \eta$, where η is orthogonal to the image of f^*. Therefore:

$$\begin{aligned}(f_*(f^*(\beta_0)+\eta), f_*(f^*(\beta_0)+\eta)) &= (f_*(f^*(\beta_0)), f_*(f^*(\beta_0)))\\ &= \deg(f)^2(\beta_0,\beta_0) = \deg(f)\,(f^*(\beta_0), f^*(\beta_0))\\ &\leq \deg(f)\,(\beta,\beta). \qquad \square\end{aligned}$$

2.4. Packing lemma. We will need the following lemma, which appears in [7]. To state it more clearly we define:

$$P(a,e) = (a+1)^e - (a-1)^e,$$

where $a \in \mathbb{R}$ and $e \in \mathbb{N}$. Observe that $a \leq a'$ implies $P(a,e) \leq P(a',e)$. Also $e \leq e'$ implies $P(a,e) \leq P(a,e')$.

LEMMA 2.4. *Let* $v_1, \ldots, v_N \in \mathbb{R}^v$, $\|v_i\| = R > 0$, $\forall i$. *Assume* $\|v_i - v_j\| \geq 2d$, $\forall i, j$, $i \neq j$, *then*

$$N \leq P\left(\frac{R}{d}, v\right) = 2\left[\binom{v}{1}\left(\frac{R}{d}\right)^{v-1} + \binom{v}{3}\left(\frac{R}{d}\right)^{v-3} + \cdots\right].$$

2.5. Degree of rational maps. Let X, Z be two n-dimensional varieties of general type such that K_X and K_Z are nef. One has:

LEMMA 2.5. *Let* f: $X \dashrightarrow Z$ *be a rational dominant map. Then*

$$\deg(f) \leq \rho(X,Z).$$

Proof. For $n = 1$ it is a consequence of Riemann-Hurwitz formula. Assume $n \geq 2$. Since K_X, K_Z are nef, by taking $l >> 0$, the linear systems $|lK_Z|$, $|lK_X|$ are base-point-free. Then, we can think of f as a linear projection in a projective space. Then the degree of f is bounded by the quotient of the degrees of $\varphi_{|lK_X|}(X)$ and $\varphi_{|lK_Z|}(Z)$; hence $deg(f) \leq K_X^n/K_Z^n$. $\square$

2.6. Rational domain. With two n-dimensional varieties of general type X, Z fixed, recall that $M(X,Z)$ is finite (see [8]). Assuming that it is not empty, we label its elements $M(X,Z) = \{f_i\}$, $i = 1, \dots, m(X,Z)$.

Definition 2.6. A Zariski open set $W \subset X$ will be called a rational domain for X and Z if any $f_i \in M(X,Z)$ defines a regular morphism $f_{i|W}\colon W \to Z$ and for any point $x \in W$ $f_i(x) = f_j(x)$ implies $i = j$.

A rational domain always exists since the closure of the sets
$f_i = f_j$, $i \neq j$ are proper algebraic subsets of X. Note for $x \in W$,

$$\#\{z_i = f_i(x)\} = m(X,Z).$$

3. Curves. We consider the case of curves, so $1 = \dim X = \dim Z$.

3.1. Tanabe's geometric lemma. Our first goal is to rewrite the geometric part of Tanabe (see [10]). We fix a holomorphic form on Z, $0 \neq \alpha \in H^0(Z, K_Z)$ and we say that two maps f, g are equivalent if and only if $f^*(\alpha) = g^*(\alpha)$. We want to give a bound of the number elements of the equivalence class $[f]$ of a map f.

Let x be a zero of $f^*(\alpha)$ and put $z = f(x) \in Z$. Let us denote by $\mathbb{D}$ the Poincaré disk and $p\colon \mathbb{D} \to X$ and $q\colon \mathbb{D} \to Z$ be the universal coverings such that $p(0) = x$ and $q(0) = z$. To any holomorphic map $f\colon X \to Z$ such that $f(x) = z$, there is a unique lifting holomorphic map $F\colon \mathbb{D} \to \mathbb{D}$ such $F(0) = 0$ and $q(F(t)) = f(p(t))$ for all $t \in \mathbb{D}$. Assume $g \in [f]$ is another nonconstant holomorphic function with $g(x) = z$ and lifting $G\colon \mathbb{D} \to \mathbb{D}$, $G(0) = 0$.

We give the following global version of Tanabe's lemma.

Lemma 3.1. *Under the previous hypothesis,*
(a) *There is a constant c such that $F(t) = c\,G(t)$.*
(b) *If n is the order of α at z then $c^{n+1} = 1$.*

Proof. Let us consider the pull-back of the form α on $\mathbb{D}$:

$$q^*(\alpha) = k(t)dt.$$

The condition $f^*(\alpha) = g^*(\alpha)$ translates into

$$k(F(t)) \cdot dF(t) = k(G(t)) \cdot dG(t).$$

If $K(t)\colon \mathbb{D} \to \mathbb{R}$ is the primitive of $k(t)$ such that $K(0) = 0$ we obtain:

$$K(F(t)) = K(G(t)).$$

Now if $k(t)$ has order n at zero, $K(t)$ has a zero of order $n + 1$ and we can find a function $w(t)$ defined near zero such that $w(K(t)) = t^{n+1}$. From $w(K(F(t))) =$

$w(K(G(t)))$ we obtain

$$F(t)^{n+1} = G(t)^{n+1}.$$

That is, $F(t) = c\,G(t)$, $c^{n+1} = 1$. □

Corollary 3.2. *The number of elements of* $[f]$ *is less than or equal to* $4(g(X) - 1)$.

Proof. Due to the lemma, for each zero x of $f^*(\alpha)$ we have at most

$$ord_{f(x)}(\alpha) + 1$$

maps of $[f]$ with the same image at x. Consider for any $x \in (f^*(\alpha))_0$ the set

$$A_x = \{g \in [f] \mid g(x) = z\},$$

where z is a fixed zero of α. Observe that

$$[f] = \bigcup_{x \in (f^*(\alpha))_0} A_x.$$

Therefore

$$\begin{aligned}\#[f] &\leq \sum_{x \in (f^*(\alpha))_0} (ord_{f(x)}(\alpha) + 1) \leq \sum_{x \in (f^*(\alpha))_0} (ord_x(f^*(\alpha)) + 1)\\ &\leq 2g(X) - 2 + \sum_{x \in (f^*(\alpha))_0} 1 \leq 4(g(X) - 1).\end{aligned}$$

□

3.2. Proof of Theorem 1.1. Let $\tilde{\alpha}$ be a nontrivial element in $T_{Z,\mathbb{Z}}$ with minimal norm. We denote by α its $(1,0)$-part. So

$$\tilde{\alpha} = \alpha + \overline{\alpha}.$$

We define the equivalence relation $\sim$ in $M(X,Z)$ as follows:

$$f \sim g \text{ if and only if } f^*(\tilde{\alpha}) = g^*(\tilde{\alpha}).$$

It is obvious that

$$f \sim g \text{ if and only if } f^*(\alpha) = g^*(\alpha).$$

In particular, the class of f under the relation $\sim$ is the set $[f]$ considered in the §3.1.

Let us fix a positive integer r. Observe that $\sim$ is in fact an equivalence relation in $M_r(X,Z)$, since $\|f^*(\tilde{\alpha})\| = \sqrt{deg(f)}\,\|\tilde{\alpha}\|$. By 2.5 the constant r is bounded above by ρ. So, due to 3.2, we get

$$m(X,Z) = \sum_{r=1}^{\rho} m_r(X,Z) \leq 4(g(X)-1)\sum_{r=1}^{\rho} \#(M_r(X,Z)/\sim).$$

Now we use the injection

$$\begin{aligned} M_r(X,Z)/\sim \;&\hookrightarrow\; H^1(X,\mathbb{Z})\otimes\mathbb{R} \\ f \;&\longmapsto\; v_f := (1/\|\tilde{\alpha}\|)f^*(\tilde{\alpha}) \end{aligned}$$

to bound the number of elements of the quotient $M_r(X,Z)/\sim$. Observe that the image belongs to the sphere of radius $\sqrt{r}$ centered at the origin in a real vector space of dimension $2g(X)$.

Proposition 3.3. *Let $f,g\colon X \longrightarrow Z$ be two maps of degree r such that $f^*(\alpha) \neq g^*(\alpha)$. Then*

$$\|v_f - v_g\| \geq \frac{1}{\sqrt{r}}.$$

Proof. Observe that

$$((f_* - g_*)(f^*(\tilde{\alpha}) - g^*(\tilde{\alpha})), \tilde{\alpha}) = (f^*(\tilde{\alpha}) - g^*(\tilde{\alpha}), f^*(\tilde{\alpha}) - g^*(\tilde{\alpha})) \neq 0,$$

hence we can assume $f_*(f^*(\tilde{\alpha}) - g^*(\tilde{\alpha})) \neq 0$. Therefore by using the minimality of the norm of $\tilde{\alpha}$:

$$\|\tilde{\alpha}\| \leq \|f_*(f^*(\tilde{\alpha}) - g^*(\tilde{\alpha}))\| \leq \sqrt{r}\,\|f^*(\tilde{\alpha}) - g^*(\tilde{\alpha})\|,$$

which implies the statement. □

By Lemma 2.4 with $d = \frac{1}{2\sqrt{r}}$, $v = 2g(X)$ and $R = \sqrt{r}$, we get

$$\#(M_r(X,Z)/\sim) \leq P(2r, 2g(X)) \leq P(2\rho, 2g(X)).$$

Together, this gives

$$m(X,Z) \leq 4(g(X)-1)\,\rho\,P(2\rho, 2g(X)),$$

proving 1.1.

Remark 3.4. Notice that

$$P(2\rho, 2g(X)) = \left[\binom{2g(X)}{1} (2\rho)^{2g(X)-1} + \binom{g(X)}{3} (2\rho)^{2g(X)-3} + \cdots \right],$$

so the dominant term of the bound has exponent $2g(X)-1$ instead of the exponent $2g(X)$ that appears in Tanabe's bound.

Remark 3.5. One can improve the bound given above by finding a better lower bound for $\|v_f - v_g\|$. In fact we can prove: $\|v_f - v_g\| \geq \sqrt{\frac{r^2+1}{r^3}}$.

4. Surfaces. In this section we assume that X and Z are surfaces of general type and that $p_g(Z) \geq 2$. The general strategy to find a bound for $m(X, Z)$ is similar to that used for curves: we find a bound for the number of maps which fix a pencil of 2-holomorphic forms minimal in some sense. Then we use the transcendental lattice to represent the maps and we prove a result similar to 3.3.

4.1. Generalization of the geometric lemma. We fix two independent $(2, 0)$ forms α and β on Z. We define the following equivalence relation on $M(X, Z)$:

$$f \sim g \iff f^*(\alpha) = g^*(\alpha) \text{ and } f^*(\beta) = g^*(\beta).$$

Remark 4.1. If $f \sim g$ then $|f^*(\beta)|^2 = |g^*(\beta)|^2$ then $\deg f = \deg g$, that is, the above relation gives a equivalence relation on $M_r(X, Z)$.

As in §3, we would like to evaluate the number of elements in an equivalence class $[f]$. To do so we take the pencil L generated by α and β. We also let B be the base curve (it could be $B = \emptyset$) of L. We may assume that β is the general element of L. Then the zero divisor $(\beta)_0$ of β can be written as

$$(\beta)_0 = B + \sum_{i=1}^{s} C_i,$$

where C_i are reduced and irreducible curve of geometric genus g with $g \geq 2$. We may also assume that $C_i \cdot C_j \geq 0$.

Now we denote by L' the pencil $f^*(L)$, which is independent of the choice of a map in $[f]$.

Then we obtain

$$(f^*(\beta))_0 = B' + \sum_{i=1}^{s'} C'_i,$$

where B' is the base divisor of the pencil and C'_i are irreducible reduced curves of genus $g' \geq 2$. We denote by N_i (respectively N_i') the normalization of the curve C_i (respectively C_i'). We have the following lemma:

LEMMA 4.2. *Let s be the number of irreducible components of $(\beta)_0 - B$. Then:*
(a) $s \leq K_Z^2$.
(b) $g(N_i) \leq K_Z^2 + 1$, $g(N_i') \leq K_X^2 + 1$.

Proof. (a) Since C_i moves, then $K_Z \cdot C_i \geq 1$. Therefore:

$$s \leq K_Z \cdot \sum_{i=1}^{s} C_i = K_Z \cdot (K_Z - B) \leq K_Z^2.$$

(b) The proof is given on Z. Observe that, since K_Z is nef:

$$(K_Z + C_i)(K_Z - C_i) \geq C_i(K_Z - C_i) \geq 0.$$

So, $K_Z^2 \geq C_i^2$. In fact, if there is more than one component, by 2-connectivity $C_i(K_Z - C_i) \geq 2$ and then $K_Z^2 \geq C_i^2 + 2$. Then we have

$$g(N_i) \ \leq \ p_a(C_i) = \frac{1}{2}(C_i^2 + C_i \cdot K_Z) + 1 \leq \frac{1}{2}(K_Z^2 + K_Z^2) + 1 = K_Z^2 + 1. \qquad \square$$

Let us consider $Z' \longrightarrow \mathbb{P}^1$ to be the minimal resolution of the pencil

$$Z \dashrightarrow \mathbb{P}^1.$$

Let $X' - - \rightarrow \mathbb{P}^1$ be the minimal resolution of the pencil on $X \times_Z Z'$. Then the map f and the forms α, β pull-back to f', α', β' and we obtain

$$[f] = [f'] = \{g'\colon\ X' \dashrightarrow Z' \mid f'^*(\alpha') = g'^*(\alpha'), f'^*(\beta') = g'^*(\beta')\}.$$

Observe that an irreducible component of a general fibre of the pencil on X' (resp.Z') is N_i' (resp. N_i).

Now we fix the component N_1'. Then $[f']$ is the union of the subsets of maps which send N_1' to N_i, $i = 1, \ldots, s$:

$$[f'] = \bigcup_i \{g \in [f'] \mid g(N_1') = N_i\}.$$

Observe that N_1' intersects the rational domain of X' and Z' (see 2.6) because it is a component of a generic element of a pencil. Moreover by taking the residue of $\alpha' \otimes \alpha'/\beta'$ along N_i a 1-form $\hat{\alpha}_i$ is induced on N_i (see [4], pp. 500–505). By definition, the pull-back of $\hat{\alpha}_i$ is the same for all the maps in $[f]$. Therefore

$$\{g \in [f'] \mid g(N_1') = N_i\} \subset \{g\colon\ N_1' \longrightarrow N_i \mid g^*(\hat{\alpha}_i) \text{ fixed}\}.$$

We are ready to prove:

PROPOSITION 4.3. *One has the inequality:*

$$\#[f] \leq 4K_Z^2 K_X^2.$$

Proof. We use 3.2 in the last inclusion of sets and we obtain, by means of 4.2:

$$\#[f] = \#[f'] \leq \sum_1^s 4(g(N_1') - 1) = 4s K_X^2 \leq 4K_Z^2 K_X^2. \qquad \square$$

4.2. Proof of 1.2. Let $\tilde{\alpha} \in T_{Z,\mathbb{Z}}$ be an element of the transcendental lattice in Z with minimal norm (see 2.1). Put α the $(2,0)$-component of $\tilde{\alpha}$. The smallest Hodge substructure containing $\tilde{\alpha}$ is denoted by $\langle\tilde{\alpha}\rangle$. If $\langle\tilde{\alpha}\rangle = T_Z$, then $\tilde{\beta}$ is any $(2,0)$-form linearly independent with α. If, on the contrary, $\langle\tilde{\alpha}\rangle \neq T_Z$ we can find a decomposition of Hodge structures $T_Z = \langle\tilde{\alpha}\rangle \oplus^{\perp} R$. Then we choose an element $\tilde{\beta} \in R_{\mathbb{Z}}$ with minimal norm. By construction its $(2,0)$-component β is linearly independent with α.

Definition 4.4. Two rational maps $f, g \in M(X,Z)$ are equivalent if and only if $f^*(\tilde{\alpha}) = g^*(\tilde{\alpha})$ and $f^*(\tilde{\beta}) = g^*(\tilde{\beta})$.

We denote this relation also by $\sim$, since by the next lemma it coincides with the relation given in 4.1.

LEMMA 4.5. *Let $f, g \in M(X,Z)$. With the notations above:*

$$f^*(\tilde{\alpha}) = g^*(\tilde{\alpha}) \text{ if and only if } f^*(\alpha) = g^*(\alpha)$$

and similarly for $\tilde{\beta}$.

Proof. One implication is obvious. In the opposite direction, we have

$$f^*(\alpha) = g^*(\alpha) \text{ and } f^*(\overline{\alpha}) = g^*(\overline{\alpha}).$$

Therefore $f^*(\tilde{\alpha}) - g^*(\tilde{\alpha})$ is a $(1,1)$ integral element, so it does not belong to the transcendental lattice. $\square$

Let us consider the injection

$$\begin{aligned} M_r(X,Z)/\sim \;\; &\hookrightarrow \;\; (T_{X,\mathbb{Z}} \times T_{X,\mathbb{Z}}) \otimes \mathbb{R} \\ [f] \;\; &\longmapsto \;\; v_f := \Big(\frac{1}{\|\tilde{\alpha}\|} f^*(\tilde{\alpha}), \frac{1}{\|\tilde{\beta}\|} f^*(\tilde{\beta})\Big). \end{aligned}$$

Then:

PROPOSITION 4.6. *Let $f, g \in M_r(X,Z)$ such that $g \notin [f]$. Then:*

$$\|v_f - v_g\| \geq \frac{1}{2\sqrt{r}}$$

Proof. Assume first that $f^*(\tilde{\alpha}) \neq g^*(\tilde{\alpha})$. Then, by arguing as in 3.3 we obtain

$$\|v_f - v_g\| \geq \|\frac{1}{\|\tilde{\alpha}\|}f^*(\tilde{\alpha}) - \frac{1}{\|\tilde{\alpha}\|}g^*(\tilde{\alpha})\| \geq \frac{1}{\sqrt{r}} > \frac{1}{2\sqrt{r}}.$$

If $f^*(\tilde{\alpha}) = g^*(\tilde{\alpha})$, then f and g coincide on $\langle\tilde{\alpha}\rangle$, which implies $\langle\tilde{\alpha}\rangle \neq T_X$. Observe that

$$((f_* - g_*)(f^*(\tilde{\beta}) - g^*(\tilde{\beta})), \tilde{\alpha}) = (f^*(\tilde{\beta}) - g^*(\tilde{\beta}), f^*(\tilde{\alpha}) - g^*(\tilde{\alpha})) = 0$$

and

$$((f_* - g_*)(f^*(\tilde{\beta}) - g^*(\tilde{\beta})), \tilde{\beta}) = \|f^*(\tilde{\beta}) - g^*(\tilde{\beta})\|^2 \neq 0.$$

Hence, $(f_* - g_*)(f^*(\tilde{\beta}) - g^*(\tilde{\beta}))$ is a nontrivial element in the lattice $R_{\mathbb{Z}}$, being $R = \langle\tilde{\alpha}\rangle^{\perp}$ the orthogonal Hodge structure to $\langle\tilde{\alpha}\rangle$ in T_X. Hence its norm is greater or equal to $\|\tilde{\beta}\|$. We get

$$\begin{aligned}
\|\tilde{\beta}\| &\leq \|(f_* - g_*)(f^*(\tilde{\beta}) - g^*(\tilde{\beta}))\| \\
&\leq \|f_*(f^*(\tilde{\beta}) - g^*(\tilde{\beta}))\| + \|g_*(f^*(\tilde{\beta}) - g^*(\tilde{\beta}))\| \leq 2\sqrt{r}\|f^*(\tilde{\beta}) - g^*(\tilde{\beta})\|
\end{aligned}$$

and the proposition follows. □

Finally, by using the packing lemma with $R = \sqrt{2r}$, $d = \frac{1}{4\sqrt{r}}$ and the fact that $r \leq \rho$ (see 2.5) and 4.3 we have:

$$\begin{aligned}
m(X,Z) &\leq 4K_Z^2 K_X^2 \sum_{r=1}^{\rho} \#(M_r(X,Z)/\sim) \\
&\leq 4K_Z^2 K_X^2 \, \rho \, P(4\sqrt{2}\,\rho, 2\dim T_X) \\
&\leq 4K_X^2 K_X^2 \, P(4\sqrt{2}\,\rho, 2b_2(X) - 2),
\end{aligned}$$

the last inequality comes from $\dim T_X \leq b_2(X) - 1$. Therefore the proof of 1.2 is finished.

5. Threefolds. We now consider the 3-dimensional case. As we will see below, we can concentrate on the geometric part of the proof, since the representation of $M_r(X,Z)/\sim$ in the transcendental lattice and the estimation of the distance work, word by word, in the same way.

We assume X, Z of general type, K_X, K_Z are nef, $p_g(Z) \geq 2$ and

$$\dim(\varphi_{|2K_Z|}(Z)) \geq 2.$$

Fix two linearly independent $(3,0)$ forms α and β. As in the precedent sections, given $f\colon X \dashrightarrow Z$ dominant, we focus in the estimation of the number of elements of

$$[f] = \{g\colon X \dashrightarrow Z \mid f^*(\alpha) = g^*(\alpha), f^*(\beta) = g^*(\beta)\}.$$

Remark 5.1. We use a pencil on Z to reduce the proof to the case of surfaces. We could instead fix 3 forms and try to reduce directly to curves. This method fails, since the corresponding map to $\mathbb{P}^2$ could not be dominant. Observe that we cannot choose generic forms since in order to apply packing arguments we need to fix them with some minimal properties.

We follow closely the case of surfaces: we have a pencil on Z, β is a general element of the pencil and its divisor of zeros is:

$$B + S_1 + \cdots + S_s,$$

where B is the base divisor.

In the same way, the divisor of zeros of $f^*(\beta)$ can be written:

$$B' + S'_1 + \cdots + S'_{s'},$$

where B' is the base divisor.

Denote $r = \deg(f)$. Then:

LEMMA 5.2. *One has the following inequality:*

$$s \leq K_Z^3.$$

Proof. Since S_i moves, a convenient pluricanonical map sends S_i to a surface. Therefore, for $l >> 0$, $(lK_Z)^2 S_i > 0$, so $K_Z^2 S_i > 0$. Hence, by the nefness of K_Z:

$$s \leq \sum_{i=1}^{s} K_Z^2 S_i = K_Z^2(K_Z - B) \leq K_Z^3. \qquad \square$$

Consider $Z' \longrightarrow Z$ to be the minimal resolution of the pencil $Z \dashrightarrow \mathbb{P}^1$ induced by α and β and let X' be the minimal resolution of the induced pencil on $X \times_Z Z'$. The general fibre of the pencil on Z' is a disjoint union of smooth surfaces $T_1, \ldots, T_s$, being T_i a desingularization of S_i. We have in the same way the smooth surfaces $T'_1, \dots, T'_{s'}$ on X'. Then the map f and the forms α, β pullback to f', α', β' and we obtain

$$[f] = [f'] = \{g'\colon X' \dashrightarrow Z' \mid f'^*(\alpha') = g'^*(\alpha'), f'^*(\beta') = g'^*(\beta')\}.$$

Now we divide the set $[f']$ into subsets depending on the image of the fixed surface T'_1:

$$[f'] = \bigcup_i \{g \in [f'] \mid g(T'_1) = T_i\} \subset \bigcup_i M(T'_1, T_i).$$

The second inclusion follows since the surface T'_1 intersects the rational domain for $X - - \rightarrow Z$.

PROPOSITION 5.3. *One has:*

$$\#[f] \leq 4 \cdot 9^2\, K_Z^3\, h^2\, P(36\sqrt{2}\, h, 34h + 18),$$

where $h = h^0(X, \mathcal{O}O(2K_X)) + h^0(X, \Omega_X^2) - p_g(X) + 1.$

Proof. By the inclusion above

$$\#[f] = \#[f'] \leq \sum_{i=1}^{s} m(T'_1, T_i).$$

The surfaces T'_1, T_i are of general type, since they move in a rational pencil and the threefolds are of general type.

Observe that, since the image of $\varphi_{|2K_Z|}$ has dimension ≥ 2 , there exist at least two elements $\alpha_1, \alpha_2 \in H^0(Z', \omega_{Z'}^{\otimes 2})$ such that the residues

$$\operatorname{Res}\,_{T_i}\left(\frac{\alpha_1}{\beta'}\right), \qquad \operatorname{Res}\,_{T_i}\left(\frac{\alpha_2}{\beta'}\right)$$

define on each component T_i two linearly independent holomorphic 2-forms. Therefore $p_g(T_i) \geq 2$. With these hypothesis we can apply corollary 1.3 to obtain

$$m(T'_1, T_i) \leq 4 \cdot 9^2 \chi^2\, P(36\sqrt{2}\, \chi, 34\chi + 18),$$

where χ is $\chi(\mathcal{O}_{T'_1})$.

To finish the proof we have to bound χ by h and use $s \leq K_Z^3$.

Let us consider the exact sequence of sheaves on X':

$$0 \to \omega_{X'} \longrightarrow \omega_{X'}(T_1') \longrightarrow \omega_{T_1'} \to 0.$$

By taking the attached long exact sequence in cohomology we obtain

$$p_g(T_1') \leq h^1(X, \omega_X) + h^0(X, \omega^{\otimes 2}) - p_g(X) = h^0(X, \Omega_X^2) + h^0(X, \omega^{\otimes 2}) - p_g(X).$$

Since $\chi \leq p_g(T_1') + 1$ we are done. □

To prove 1.5 we can imitate the proof of 1.2 given in the last section. The only difference is that the analogous statement to Proposition 4.5 is no longer true. However the obvious implication

$$f^*(\tilde{\alpha}) = g^*(\tilde{\alpha}) \Rightarrow f^*(\alpha) = g^*(\alpha)$$

is enough to ensure that the equivalence class of f is contained in $[f]$.

Then, by using 5.3:

$$m(X, Z) \leq 4 \cdot 9^2 K_Z^3 h^2 P(36\sqrt{2}\, h, 34h + 18)\, \rho\, P(4\sqrt{2}\rho, 2\dim T_X).$$

The statement of 1.4 follows replacing ρ with $\frac{K_X^3}{K_Z^3}$.

6. Torsion lemma. In this section we generalize the torsion part of Tanabe's work to higher dimensions. With some hypotheses, this allows us to produce bounds for $m(X, Z)$ in any dimension.

Let $f\colon X \to Z$ and $g\colon X \to Z$ be dominant maps of degree r. We let f_*, f^*, f_d, g_*, g^* and g_d induced maps (see §2).

We have the following:

LEMMA 6.1. *If $f_d = g_d$ for some $d > 2r$ then $f^* = g^*$.*

Proof. Let $h = f^* - g^*$ we have to prove that $T_Z = \ker(h)$. If not, let V be Hodge polarized structure orthogonal to $\ker h$. Let $h'\colon V \to T_Z$ be the restriction of h. Now h' is injective. Set $\mu \in V_{\mathbb{Z}}$ such that its norm is minimal in the lattice. We consider

$$\lambda = h'(\mu) = f^*(\mu) - g^*(\mu).$$

We would have that $\lambda \neq 0$. Moreover from the hypothesis $f_d = g_d$ we have that $\lambda = d \cdot \sigma$ where $\sigma \in T_{\mathbb{Z},Z}$ is an integral class.

We also consider $\beta_f = f_*(\lambda)$ and $\beta_g = g_*(\lambda)$. We have that β_f (and $\beta_{g)}$) are in V. To see this, first we remark that $f_* f^* = g_* g^*$, since

$$f_* f^*(\alpha) = r \cdot \alpha = g_* g^*(\alpha).$$

Now fix $\alpha \in \ker(h)$, $f^*(\alpha) = g^*(\alpha)$, we have

$$\begin{aligned}(\alpha, \beta_f) &= (\alpha, f_*(\lambda)) = (f^*(\alpha), \lambda) = (f^*(\alpha), f^*(\mu) - g^*(\mu)) \\ &= (f^*(\alpha), f^*(\mu)) - (f^*(\alpha), g^*(\mu)) = (f_*f^*(\alpha), \mu)) - (f^*(\alpha), g^*(\mu)) \\ &= (g_*g^*(\alpha), \mu)) - (f^*(\alpha), g^*(\mu)) = (g^*(\alpha), g^*(\mu)) - (f^*(\alpha), g^*(\mu)) \\ &= (h(\alpha), g^*(\mu)) = 0.\end{aligned}$$

Then we have

$$\beta_f - \beta_g = (f_* - g_*)(\lambda) = (f_* - g_*)(f^* - g^*)(\mu)$$

is not zero. Indeed

$$(\beta_f - \beta_g, \mu) = ((f^* - g^*)(\mu), f^* - g^*)(\mu)) = \|\lambda\|^2 \neq 0.$$

It follows that either β_f or β_g are not zero.

We may assume now that $\beta_f = f_*(\lambda) \neq 0$. Recall that we have that $\lambda = d \cdot \sigma$ where $\sigma \in T_{\mathbb{Z},Z}$. We have then

$$\|f_*((f^* - g^*)(\mu))\| = \|f_*(\lambda)\| = d \cdot \|f_*(\sigma)\| \geq d \cdot \|\mu\|,$$

by the minimality of $\|\mu\|$.

In addition:

$$\|f_*((f^* - g^*)(\mu))\| \leq \sqrt{r}\,\|(f^* - g^*)(\mu))\| \leq \sqrt{r}\,(\|f^*(\mu)\| + \|g^*(\mu)\|) = 2r\|\mu\|.$$

Hence

$$d \leq 2r. \qquad \square$$

The rest of the section is devoted to the proof of Theorem 1.5. We fix X, Z two n-dimensional varieties of general type, $n \geq 2$, such that K_Z is nef and $\dim(\varphi_{|K_Z|}(Z)) \geq n-1$ (in particular $p_g(Z) \geq n$).

Definition 6.2. We say that two maps $f, g \in M(X, Z)$ are equivalent if $f^* = g^*$ on T_Z.

As usual we would like to compute the number of elements of the class $[f]$ of a map f. We consider a general projection of the image of the canonical map of Z. Then we obtain a rational dominant map ϕ: $Z \dashrightarrow \mathbb{P}^{n-1}$. By definition $\phi \circ f = \phi \circ g$.

Observe ϕ can be written as $Z \dashrightarrow \mathbb{P}(V^*)$, where V is a n-dimensional vector space contained in $H^0(Z, \omega_Z(-F))$, F being the fixed divisor of the linear system attached to V. The general fibre of ϕ is

$$C_1 + \cdots + C_s$$

and can be thought of as the common zeros of $\{s_1, \ldots, s_{n-1}\}$, where $s_i \in V$. Let s_n be another element in V such that $s_1, \ldots, s_n$ is a basis of V.

The fibre of $\phi \circ f$ is of the form

$$C'_1 + \cdots + C'_{s'}.$$

We consider a resolution $\pi\colon\ Z' \longrightarrow Z$ of the singularities of ϕ. We put $\pi^*(s_i) = s'_i \cdot s_{E_0}$, where E_0 is the fixed divisor of the pull-back of the linear system and s_{E_0} is an equation for this divisor. Then

$$\langle s'_1, \ldots, s'_n \rangle \subset H^0(Z', \pi^*(\omega_Z(-F))(-E_0))$$

defines the map $\phi' = \phi \circ \pi\colon\ Z' \longrightarrow \mathbb{P}^n$. The normalizations N_i of C_i are the components of the general fibres of ϕ'. By construction we see $N_1 + \cdots + N_s$ as the common zeros of $\{s'_1, \ldots, s'_{n-1}\}$. In particular we have the rational equivalence of 1-cycles,

$$N_1 + \cdots + N_s \sim_{rat} (\pi^*(K_Z) - E)^{n-1},$$

where $E = E_0 + \pi^*(F)$ is an effective divisor (and $h^0(\pi^*(K_Z) - E) > 0$ by construction).

Notice also that $s_n'^{\otimes n}$ restricts to N_i giving a holomorphic form α.

Locally this form is computed as the following residue:

$$\mathrm{Res}_{\,N_i} \left(\frac{s'_n \cdot \cdots \cdot s'_n \cdot s_{E_0}}{s'_1 \cdot \cdots \cdot s'_{n-1}} \right).$$

Analogously we can resolve the singularities of the map $X \dashrightarrow \mathbb{P}^n$ and the general fibre is $N'_1 + \cdots + N'_{s'}$, being N'_i the desingularization of C'_i.

Then, since N'_1 intersects the rational domain of X and Z (defined in 2.6):

$$[f] = \bigcup_{i=1}^{s} \{h\colon\ N'_1 \longrightarrow N_i \mid h^*(\alpha) \text{ fixed}\}.$$

By 3.2 we obtain

$$\#[f] \leq s \cdot 4(g(N'_1) - 1).$$

To go further, we need the following

LEMMA 6.3. *With our hypothesis*
(a) $s \leq K_Z^n$.
(b) $g(N_i') \leq \frac{nK_X^n}{2} + 1$, $g(N_i) \leq \frac{nK_Z^n}{2} + 1$.

Proof. (a) Since C_i moves, $K_Z C_i \geq 1$. Then

$$s \leq K_Z \left(\sum_{i=1}^{s} C_i \right) = \pi^*(K_Z) \left(\sum_{i=1}^{s} N_i \right) = \pi^*(K_Z)(\pi^*(K_Z) - E)^{n-1}.$$

To see $s \leq K_Z^n$, it is enough to prove that

$$\pi^*(K_Z)(\pi^*(K_Z) - E)^{n-1} \leq K_Z^n,$$

which follows from the positivity of

$$\begin{aligned}&\pi^*(K_Z)(\pi^*(K_Z)^{n-1} - (\pi^*(K_Z) - E)^{n-1})\\ &\quad = \pi^*(K_Z)E(\pi^*(K_Z)^{n-2} + \pi^*(K_Z)^{n-3}(\pi^*(K_Z) - E) + \cdots)\end{aligned}$$

(using the fact that K_Z is nef).

(b) We give the proof on Z. Observe that $p_a(C_i) \leq p_a(\sum_i C_i)$ since $p_a(C_1) = \cdots = p_a(C_s) \geq 2$ and all the curves are reduced. Therefore

$$g(N_i) \leq p_a(C_i) \leq p_a(\sum_i C_i)$$

and

$$\begin{aligned}2p_a\left(\sum_i C_i\right) - 2 &= (K_Z + (n-1)(K_Z - F))(K_Z - F)^{n-1}\\ &= (nK_Z - (n-1)F)(K_Z - F)^{n-1} \leq nK_Z^n.\end{aligned}$$

The last inequality is proved as in the first part. We are done. □

A direct consequence of the lemma and the discussion above is the inequality:

PROPOSITION 6.4. *We have the bound:*

$$\#[f] \leq 2nK_X^n K_Z^n.$$

Now we fix a degree r and we consider the map

$$M_r(X, Z)/\sim \longrightarrow Hom(T_{Z,\mathbb{Z}}/(2r+1)T_{Z,\mathbb{Z}}, T_{X,\mathbb{Z}}/(2r+1)T_{X,\mathbb{Z}})$$

which sends f to f_d. The injectivity is an application of 6.1.

Then, by 6.4:

$$
\begin{aligned}
m(X,Z) &\leq 2n\, K_X^n K_Z^n \sum_{i=1}^{\rho} (2r+1)^{\dim T_Z \cdot \dim T_X} \\
&\leq 2n\, K_X^n K_Z^n\, \rho\, (2\rho+1)^{b_n(X)\, b_n(Z)}.
\end{aligned}
$$

This finishes the proof of 1.5.

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