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Fourier transform and Prym varieties

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Abstract. Let P be the Prym variety attached to an unramified double covering $\tilde{C} \to C$. Let $X = X(\tilde{C}, C)$ be the variety of special divisors which birationally parametrizes the theta divisor in P. We prove that X is the projectivization of the Fourier-Mukai transform of a coherent sheaf $p_*(M)$, where M is an invertible sheaf on $\tilde{C}$ and $p : \tilde{C} \to P$ is the natural embedding. We apply this fact to give an algebraic proof of the following Torelli type statement proved by Smith and Varley in the complex case: under some hypothesis the variety X determines the covering $\tilde{C} \to C$.

1. Introduction

The Jacobian variety $J(C)$ of a smooth irreducible projective curve C of genus g is a principally polarized abelian variety (ppav in the sequel). This means that $J(C)$ comes equipped with a class, modulo algebraic equivalence, of an ample line bundle L, with $h^0(L) = 1$. Let Θ be the effective divisor in the linear series $|L|$. The image of the Abel map

$$C^{(g-1)} \to \mathrm{Pic}^{g-1}(C),$$
$$D \mapsto \mathcal{O}_C(D)$$

is a divisor Θ^{can} and the Riemann's Parametrization Theorem says that Θ^{can} corresponds to Θ under a convenient translation isomorphism

$$\mathrm{Pic}^{g-1}(C) \to \mathrm{Pic}^0(C) = J(C).$$

It was proved by Schwarzenberger in [8] that $C^{(g-1)}$ is the projectivization of (the image by an automorphism on $J(C)$ of) a coherent sheaf $\mathcal{F}$ on $J(C)$ supported on Θ called Picard sheaf (see §2 for a precise statement). In terms of the usual Fourier-Mukai transform $\mathcal{F}_{J(C)}$ attached to the normalized Poincaré bundle on $J(C) \times \widehat{J(C)}$, the sheaf $\mathcal{F}$ can be seen, up to a shift, as $\mathcal{F}_{J(C)}\big(j_*(\mathcal{M})\big)$, where $\mathcal{M} \in \mathrm{Pic}^{g-1}(C)$ and $j : C \hookrightarrow J(C)$.

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In a recent paper, Beilinson and Polishchuk ([2]) find an intrinsic characterization of the sheaves $\mathcal{F}_{J(C)}\big(j_*(\mathcal{M})\big)$ in terms of the ppav $\big(J(C),\Theta\big)$. By applying the involutive property of the Fourier-Mukai transform they recover the curve C, hence the Torelli Theorem is proved.

In this paper we study the "Prym Picard sheaves" $\mathcal{F}_P\big(p_*(M)\big)$, where P is the Prym variety attached to an unramified double covering

$$\tilde{C} \to C,$$

p is a natural embedding of the curve $\tilde{C}$ in P, and M a convenient invertible sheaf on $\tilde{C}$. In analogy with the Jacobian situation one has a variety $X(\tilde{C},C) \subset \tilde{C}^{2g-2}$ (see 2.10 for a precise definition) and a map

$$X(\tilde{C},C) \to P^{\mathrm{can}} \subset \mathrm{Pic}^{2g-2}(\tilde{C})$$

such that the fibre at $\mathcal{D}$ is $\mathbb{P}H^0(\tilde{C},\mathcal{D})$. Then the translation to P of the image of this map is a divisor representing the polarization. We will prove in section 3 that $X(\tilde{C},C)$ is isomorphic to the projectivization (of the image by an automorphism on P) of a Prym Picard sheaf. By using again the involutive property of the Fourier transform we get in section 4 an algebraic proof of the following theorem proved by Smith and Varley in the complex case ([9]): under some weak hypothesis $(\tilde{C},C)$ can be recovered from the variety $X(\tilde{C},C)$.

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2. Notation, preliminaries and statements

2.1. All the varieties are defined over an algebraically closed field of characteristic $\neq 2$. The projections of a product of varieties $X \times Y$ into each of its factors will be denoted by π_X and π_Y respectively.

2.2. Fourier-Mukai transform. Let (A,Θ_A) be a ppav of dimension g. We will denote by $\hat{A}$ the corresponding dual ppav. The polarization induces an isomorphism

$$\lambda_{\Theta_A} : A \to \hat{A}, \quad a \mapsto P_a := \mathcal{O}_A(\Theta_a - \Theta).$$

Translation by $a \in A$ will be denoted by $\tau_a : A \to A$, $\tau_a(x) = x + a$.

Let $\mathcal{P}_A$ be the normalized Poincaré bundle on $A \times \hat{A}$, that is:

$$\mathcal{P}_{A\,|\,\{0\}\times\hat{A}} \cong \mathcal{O}_{\hat{A}}, \quad \mathcal{P}_{A\,|\,A\times\{\xi\}} \cong \xi, \quad \forall \xi \in \hat{A}.$$

Following Mukai ([5]), one can define a functor $\hat{\mathcal{S}}$ of $\mathcal{O}_A$-modules into the category of $\mathcal{O}_{\hat{A}}$-modules by

$$\hat{\mathcal{S}}(-) = \pi_{\hat{A}*}\big(\pi_A^*(-) \otimes \mathcal{P}_A\big).$$

The derived functor $\mathbf{R}\hat{\mathscr{S}}$ of $\hat{\mathscr{S}}$ induces an equivalence of categories between the two derived categories $D(A)$ and $D(\hat{A})$ ([5], Theorem 2.2).

The Weak Index Theorem (W.I.T.) is said to hold for a coherent sheaf $\mathscr{M}$ on A if there exists an index $i(\mathscr{M})$ such that $R^i\hat{S}(\mathscr{M}) = 0$ for all $i \neq i(\mathscr{M})$. We will denote the coherent sheaf $\lambda_{\Theta_A}^*\big(R^{i(\mathscr{M})}\hat{S}(\mathscr{M})\big)$ on A by $\mathscr{F}_A(\mathscr{M})$.

2.3. Involutive property of the Fourier-Mukai transform. The Corollary 2.4 in [5] says, with our notation: If W.I.T. holds for $\mathscr{M}$, then so does $\mathscr{F}_A(\mathscr{M})$ and $i\big(\mathscr{F}_A(\mathscr{M})\big) = g - i(\mathscr{M})$. Moreover $\mathscr{F}_A\big(\mathscr{F}_A(\mathscr{M})\big)$ is isomorphic to $(-1_A)^*(\mathscr{M})$.

2.4. Translation property of the Fourier-Mukai transform. Let $\lambda_{\Theta_A}(a) = P_a \in \hat{A}$ be an algebraically trivial line bundle on A, then

$$\mathscr{F}_A(\mathscr{M} \otimes P_a) \cong \tau_a^*\big(\mathscr{F}_A(\mathscr{M})\big).$$

2.5. Here we recall the main Theorem of [8].

Let C be a smooth irreducible projective curve and fix a point $c \in C$. Consider the Poincaré bundle $\mathscr{L}$ over $C \times J(C)$ normalized by the condition $\mathscr{L}_{|\{c\} \times J(C)} \cong \mathscr{O}_{J(C)}$. We see $C^{(g-1)}$ as a $J(C)$-variety with the morphism

$$\begin{aligned} C^{(g-1)} &\to J(C), \\ D &\mapsto \mathscr{O}_C\big(D - (g-1)c\big). \end{aligned}$$

Then there is an isomorphism of $J(C)$-varieties

$$\mathbb{P}\big((-1_{J(C)})^*\tau^*_{\omega_C((2-2g)c)}R^1\pi_{J(C)*}\big(\pi_C^*\big(\mathscr{O}_C\big((g-1)c\big)\big) \otimes \mathscr{L}\big)\big) \cong C^{(g-1)}.$$

2.6. We want to express the last theorem in terms of the Fourier-Mukai transform.

By seesaw lemma it is easy to see that $\mathscr{L} \cong (j \times \lambda_{\Theta_{J(C)}})^*(\mathscr{P}_{J(C)})$, where $j : C \to J(C)$ is the natural embedding $x \mapsto \mathscr{O}_C(x - c)$. Since $j \times \lambda_{\Theta_{J(C)}}$ is a closed embedding, then $\mathbf{R}(j \times \lambda_{\Theta_{J(C)}})_* = (j \times \lambda_{\Theta_{J(C)}})_*$. Hence

$$\begin{aligned} &R^1\pi_{J(C)*}\big(\pi_C^*\big(\mathscr{O}_C\big((g-1)c\big)\big) \otimes \mathscr{L}\big) \\ &\overset{(1)}{\cong} R^1\pi_{J(C)*}\big(\pi_C^*\big(\mathscr{O}_C\big((g-1)c\big)\big) \otimes (j \times \lambda_{\Theta_{J(C)}})^*\mathscr{P}_{J(C)}\big) \\ &\overset{(2)}{\cong} \lambda^*_{\Theta_{J(C)}}R^1\pi_{\widehat{J(C)}*}(j \times \lambda_{\Theta_{J(C)}})_*\big(\pi_C^*\big(\mathscr{O}_C\big((g-1)c\big)\big) \otimes (j \times \lambda_{\Theta_{J(C)}})^*\mathscr{P}_{J(C)}\big) \\ &\overset{(3)}{\cong} \lambda^*_{\Theta_{J(C)}}R^1\pi_{\widehat{J(C)}*}\big((j \times \lambda_{\Theta_{J(C)}})_*\pi_C^*\big(\mathscr{O}_C\big((g-1)c\big)\big) \otimes \mathscr{P}_{J(C)}\big) \\ &\overset{(4)}{\cong} \lambda^*_{\Theta_{J(C)}}R^1\pi_{\widehat{J(C)}*}\big(\pi_{J(C)}^*j_*\big(\mathscr{O}_C\big((g-1)c\big)\big) \otimes \mathscr{P}_{J(C)}\big) \\ &\overset{(5)}{\cong} \mathscr{F}_{J(C)}\big(j_*\big(\mathscr{O}_C\big((g-1)c\big)\big)\big). \end{aligned}$$

In (1) we have replaced $\mathscr{L}$ by $(j \times \lambda_{\Theta_{J(C)}})^*(\mathscr{P}_{J(C)})$, in (2) we use that the projection $\pi_{J(C)} : C \times J(C) \to J(C)$ agrees with $\lambda^*_{\Theta_{J(C)}} \circ \pi_{\widehat{J(C)}} \circ (j \times \lambda_{\Theta_{J(C)}})$, in (3) we have applied projection formula, (4) follows from base change with respect to the diagram

$$\begin{array}{ccc} C \times J(C) & \xrightarrow{j \times \lambda_{\Theta_{J(C)}}} & J(C) \times \widehat{J(C)} \\ {\scriptstyle \pi_C}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{J(C)}} \\ C & \xrightarrow{\quad j \quad} & J(C) \end{array}$$

and (5) is the definition of $\mathscr{F}_{J(C)}$.

Observe that W.I.T. holds with index 1 for $j_*\big(\mathscr{O}_C\big((g-1)c\big)\big)$ by Corollary 2 in [8].

2.7. We can also rewrite the isomorphism by using 2.4 to delete the translation:

$$\tau^*_{\omega_C((2-2g)c)} \mathscr{F}_{J(C)}\big(j_*\big(\mathscr{O}_C\big((g-1)c\big)\big)\big) \cong \mathscr{F}_{J(C)}\big(j_*\big(\mathscr{O}_C\big((g-1)c\big)\big) \otimes P_{\omega_C((2-2g)c)}\big).$$

Hence, by projection formula and the equality $j^* = \lambda^{-1}_{\Theta_{J(C)}}$, this sheaf is isomorphic to

$$\mathscr{F}_{J(C)}\big(j_*\big(\omega_C\big((1-g)c\big)\big)\big).$$

2.8. All together gives the following statement, equivalent to 2.4 above:

$$\mathbb{P}\big((-1)^*_{J(C)} \mathscr{F}_{J(C)}\big(j_*\big(\omega_C\big((1-g)c\big)\big)\big)\big) \cong C^{(g-1)}.$$

2.9. Now we look to other structural maps

$$\phi_M : C^{(g-1)} \to J(C),$$
$$D \mapsto \mathscr{O}_C(D) \otimes M^{-1},$$

for any $M \in \mathrm{Pic}^{g-1}(C)$.

By applying to 2.8 a base change with respect to a convenient translation on $J(C)$, we obtain the following isomorphism of $J(C)$-varieties (via ϕ_M):

$$\mathbb{P}\big((-1_{J(C)})^* \mathscr{F}_{J(C)}\big(j_*(\omega_C \otimes M^{-1})\big)\big) \cong C^{(g-1)}.$$

2.10. Prym varieties. We recall some basic facts of the theory of Prym varieties. We quote [7] for the details.

Let $\tilde{C} \to C$ be an irreducible unramified double covering of an irreducible projective smooth curve C of genus g. The kernel of the attached norm map

$$Nm : J(\tilde{C}) \to J(C)$$

has two irreducible components. The component containing the origin is an abelian variety, $P = P(\tilde{C}, C)$, called the *Prym variety* attached to the covering. The principal polarization on $J(\tilde{C})$ restricts to twice a principal polarization on P, an effective divisor representing the polarization (determined up to translation) is denoted by Ξ. Therefore (P, Ξ) is a ppav.

We will fix a point $\tilde{c} \in \tilde{C}$ and we will denote by σ the involution in $\tilde{C}$ associated to the covering. We define $p : \tilde{C} \to P$ the embedding given by $p(\tilde{x}) = \tilde{x} - \sigma(\tilde{x}) + \tilde{c} - \sigma(\tilde{c})$.

The theta divisor can be defined canonically in

$$P^{\mathrm{can}} = P^{\mathrm{can}}(\tilde{C}, C) := \{L \in \mathrm{Pic}^{2g-2}(\tilde{C}) \mid Nm(L) = \omega_C, h^0(\tilde{C}, L) \text{ even}\},$$

as $\Xi^{\mathrm{can}} = \{L \in P^{\mathrm{can}} \mid h^0(\tilde{C}, L) > 0\}$.

The analogue of the symmetric product for the Prym variety is the subvariety $X = X(\tilde{C}, C)$ of $\tilde{C}^{(2g-2)}$ given by the effective divisors with norm in the complete linear system $|\omega_C|$ and h^0 even. If C is non hyperelliptic, then X is irreducible and normal (cf. [1]).

2.11. Statements. In the next section we will prove the following result:

Theorem 2.1. *For any Prym variety, and for an $M \in P^{\mathrm{can}}$ there is an isomorphism*

$$X \cong \mathbb{P}(-1_P)^*\big(\mathscr{F}_P\big(p_*\sigma^*(M)\big)\big).$$

By applying this theorem and the properties of the Fourier-Mukai transform we will prove in section 4 the following theorem:

Theorem 2.2. *Let $\tilde{C} \to C$ be an irreducible unramified double covering of a smooth complete irreducible curve of genus $g \geqq 3$, and let (P, Ξ) be its attached Prym variety. We assume C is non hyperelliptic and* $\dim \mathrm{Sing}(\Xi) \leqq g - 6$. *Then the variety $X(\tilde{C}, C)$ determines the covering $\tilde{C} \to C$.*

3. Prym Picard sheaves

In this section we will prove Theorem 2.1. We fix an unramified double covering $\tilde{C} \to C$ as in 2.8, and a point $\tilde{c}$. From now on the map $j : \tilde{C} \to J(\tilde{C})$ will be the embedding given by $j(\tilde{x}) = \mathcal{O}_{\tilde{C}}\big(\tilde{x} - \sigma(\tilde{c})\big)$. We keep the notations introduced in section 2.

3.1. Let $M \in P^{\mathrm{can}} \subset \mathrm{Pic}^{2g-2}(\tilde{C})$. Observe that the genus of $\tilde{C}$ is $2g - 1$, so $2g - 2 = g(\tilde{C}) - 1$. One has a pull-back diagram

$$\begin{array}{ccc} X = X(\tilde{C}, C) & \hookrightarrow & \tilde{C}^{(2g-2)} \\ \big\downarrow & & \big\downarrow \phi_M \\ P = P(\tilde{C}, C) & \hookrightarrow & J(\tilde{C}). \end{array}$$

Therefore, by applying the isomorphism in 2.9, we get

$$X \cong \mathbb{P}\big((-1_{J(\tilde{C})})^*\mathscr{F}_{J(\tilde{C})}\big(j_*(\omega_{\tilde{C}} \otimes M^{-1})\big)_{|P}\big).$$

3.2. Remark. The behaviour of the Fourier-Mukai transform with respect to maps of abelian varieties is only easy to handle in the case of isogenies. In general, it seems difficult to relate the tranforms $\mathscr{F}_{J(\tilde{C})}$ and $\mathscr{F}_P$. However we only need to compare the values on some coherent sheaves supported on the curve $\tilde{C}$, and in this case we have the following result.

Proposition 3.1. *For any $M \in P^{\mathrm{can}}$, the restriction to P of the sheaf*

$$\mathscr{F}_{J(\tilde{C})}\big(j_*(M)\big)$$

is isomorphic to:

$$\mathscr{F}_P\big(p_*(M)\big).$$

Notice that this proposition implies (replace M by $\omega_{\tilde{C}} \otimes M^{-1}$) the Theorem 2.1.

Proof. We begin by noticing that arguing as in 2.6, we can write:

3.3.

$$\mathscr{F}_{J(\tilde{C})}\big(j_*(M)\big) \cong R^1\pi_{J(\tilde{C})*}\big(\pi_{\tilde{C}}^*(M) \otimes \tilde{\mathscr{L}}\big),$$

where $\tilde{\mathscr{L}}$ is the Poincaré bundle on $\tilde{C} \times J(\tilde{C})$ normalized by the condition

$$\tilde{\mathscr{L}}_{|\{\sigma(\tilde{c})\}\times J(\tilde{C})} \cong \mathscr{O}_{J(\tilde{C})}.$$

We will prove a similar fact for the transform $\mathscr{F}_P$. We will use the next result.

Lemma 3.2. *For all $a \in P$ the following equality holds*:

$$\mathscr{O}_P(\Xi_a - \Xi)_{|\tilde{C}} = a.$$

Proof. This is a standard fact on Prym varieties (see for instance [9], section 8). For the convenience of the reader we give the sketch of a proof. By the Theorem of the square we can assume that Ξ is of the form $\Xi_{-\alpha}^{\mathrm{can}}$ where α is a convenient element in P^{can}. We assume that $h^0(\alpha) = 2$ and $h^0\big(\alpha + \tilde{c} - \sigma(\tilde{c})\big) = 1$. Put $p_1 + \cdots + p_{2g-2}$ for the effective divisor in the linear series $|\alpha + \tilde{c} - \sigma(\tilde{c})|$. Then $p\big(\sigma(p_i)\big) = \sigma(p_i) - p_i + \tilde{c} - \sigma(\tilde{c}) \in \Xi_{-\alpha}^{\mathrm{can}}$. Since $\tilde{C} \cdot \Xi = 2g - 2$, we get the isomorphism

$$\mathscr{O}_P(\Xi_{-\alpha}^{\mathrm{can}})_{|\tilde{C}} \cong \mathscr{O}_{\tilde{C}}\big(\sigma(\alpha) + \sigma(\tilde{c}) - \tilde{c}\big).$$

The statement follows easily. □

Corollary 3.3. *The invertible sheaf on $\tilde{C} \times P$*:

$$\mathscr{L}_P := (p \times \lambda_\Xi)^*(\mathscr{P}_P)$$

satisfies

$$\mathscr{L}_{P\,|\,\{\sigma(\tilde{c})\}\times P} = \mathscr{O}_P, \quad \mathscr{L}_{P\,|\,\tilde{C}\times\{a\}} = a.$$

Proof. Straightforward application of the lemma and seesaw. □

3.4. Now we can use the argument in 2.6 by replacing j by p, $\lambda_{\Theta_{J(C)}}$ by λ_{Ξ}, and using the corollary. Then:

$$\mathscr{F}_P\big(p_*(M)\big) \cong R^1\pi_{P*}\big(\pi_{\tilde{C}}^*(M) \otimes \mathscr{L}_P\big).$$

Now, by applying seesaw lemma once more,

3.5.

$$\tilde{\mathscr{L}}_{|\tilde{C}\times P} \cong \mathscr{L}_P.$$

Finally,

$$\begin{aligned}
\mathscr{F}_{J(\tilde{C})}\big(j_*(M)\big)_{|P} &\overset{(1)}{\cong} R^1\pi_{J(\tilde{C})*}\big(\big(\pi_{\tilde{C}}^*(M) \otimes \tilde{\mathscr{L}}\big)\big)_{|P} \\
&\overset{(2)}{\cong} R^1\pi_{P*}\big(\big(\pi_{\tilde{C}}^*(M) \otimes \tilde{\mathscr{L}}\big)_{|\tilde{C}\times P}\big) \\
&\overset{(3)}{\cong} R^1\pi_{J(\tilde{C})*}\big(\pi_{\tilde{C}}^*(M) \otimes \mathscr{L}_P\big) \\
&\overset{(4)}{\cong} \mathscr{F}_P\big(p_*(M)\big).
\end{aligned}$$

We have applied 3.3 in (1), 3.5 in (3), 3.4 in (4) and base change with respect to the following diagram in (2)

$$\begin{array}{ccc}
\tilde{C}\times P & \hookrightarrow & \tilde{C}\times J(\tilde{C}) \\
{\scriptstyle \pi_P}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{J(\tilde{C})}} \\
P & \hookrightarrow & J(\tilde{C}).
\end{array} \quad \square$$

3.6. Chern classes. In [8] the Chern classes of the Picard sheaves are computed. Since we work up to translation we are interested in the groups A^i of classes of cycles modulo algebraic equivalence. The statement in [8] can be translated to

$$c_i\big(\mathscr{F}_{J(\tilde{C})}\big(j_*(M)\big)\big) = [W_i] \in A^i\big(J(\tilde{C})\big)_{\mathbb{Q}}, \quad i = 1, \dots, \tilde{g}$$

where W_i stand for the Brill-Noether loci $W_i^0(\tilde{C})$.

Since $P \cdot \tilde{\Theta} = 2\Xi$ and $[W_i] = [\tilde{\Theta}]^i/i!$, then Proposition 3.1 implies that

$$c_i\big(\mathscr{F}_P\big(p_*(M)\big)\big) = 2^i \cdot [\Xi]^i/i!,$$

in particular

$$c_1\big(\mathscr{F}_P\big(p_*(M)\big)\big) = 2 \cdot [\Xi], \quad c_2\big(\mathscr{F}_P\big(p_*(M)\big)\big) = 2 \cdot [\Xi]^2.$$

4. A Torelli type theorem

The aim of this section is to prove Theorem 2.2. We assume in all the section the hypothesis of the theorem. In particular, C is non hyperelliptic of genus $g \geqq 3$. Therefore X is normal and irreducible.

In comparison with the argument given in [9], the main difference is that we represent the variety X by means of a coherent sheaf which comes from the theory of the Fourier-Mukai transform. This allows to finish quickly the proof by using the involutive property of the transform, instead of using the Narasimhan-Ramanan invariant.

4.1. The first step is to recover the ppav (P, Ξ) and the map $X \to \Xi$ from the variety X (see [9], section 4 for an analytic proof of the next proposition).

Proposition 4.1. *The morphism $X \to \Xi \subset P$ is an Albanese map for X.*

Proof. Restricting the map $X \to \Xi$ to a suitable open set $U \subset \Xi$ we get a product $U \times \mathbb{P}^1 \to U$. Since the Albanese variety is a birational invariant and $\mathrm{Alb}(U \times \mathbb{P}^1) = \mathrm{Alb}(U)$ we obtain that $\mathrm{Alb}(X) = \mathrm{Alb}(\Xi)$. Hence it suffices to show that the inclusion $\Xi \subset P$ is an Albanese map for Ξ. This is done in [6], Th. 1.2. □

4.2. We fix an Albanese map for X. This means that we look at X as a P-variety

$$X \to P, \quad D \mapsto \mathcal{O}_{\tilde{C}}(D) \otimes M^{-1},$$

$M \in P^{\mathrm{can}}$. Then we can apply Theorem 2.1 and express X as the projectivization of a coherent sheaf on P supported on $\Xi^{\mathrm{can}}_{-M} = \Xi$. To simplify notations we put

$$\mathscr{E}_0 := \mathscr{F}_P\big(p_* \sigma^*(M)\big),$$

hence $X \cong \mathbb{P}\big((-1_P)^* \mathscr{E}_0\big)$. In order to recover the sheaf $\mathscr{E}_0$ from X we need the following property (observe that in this lemma we only need the hypothesis $\dim \operatorname{Sing} \Xi \leqq g - 5 = \dim \Xi - 3$):

Lemma 4.2. *Let $u : \Xi_{\mathrm{sm}} \hookrightarrow \Xi$ be the inclusion of the open set of smooth points of Ξ. Then*

$$u_* u^* (\mathscr{E}_0) \cong \mathscr{E}_0.$$

Proof. Consider an effective divisor E on $\tilde{C}$ of degree $d \gg 0$. By applying the functors p_*, π_P^* and $\otimes \mathscr{P}_P$ to the exact sequence

$$0 \to M \to M \otimes \mathcal{O}_{\tilde{C}}(E) \to M \otimes \mathcal{O}_E \to 0$$

we get

$$0 \to \pi_P^*\big(p_*(M)\big) \otimes \mathscr{P}_P \to \pi_P^*\big(p_*\big(M \otimes \mathcal{O}_{\tilde{C}}(E)\big)\big) \otimes \mathscr{P}_P$$
$$\to \pi_P^*\big(p_*(M \otimes \mathcal{O}_E)\big) \otimes \mathscr{P}_P \to 0.$$

Now we apply the functor $\pi_{\hat{P}_*}$. Since

$$\pi_{\hat{P}_*}\big(\pi_P^*\big(p_*(M)\big) \otimes \mathscr{P}_P\big) = 0 = R^1\pi_{\hat{P}_*}\big(\pi_P^*\big(p_*\big(M \otimes \mathcal{O}_{\tilde{C}}(E)\big)\big) \otimes \mathscr{P}_P\big),$$

and

$$\pi_{\hat{P}_*}\big(\pi_P^*\big(p_*\big(M \otimes \mathcal{O}_{\tilde{C}}(E)\big)\big) \otimes \mathscr{P}_P\big), \quad \pi_{\hat{P}_*}\big(\pi_P^*\big(p_*(M \otimes \mathcal{O}_E)\big) \otimes \mathscr{P}_P\big)$$

are locally free, we obtain (by applying λ_Ξ^*) a short exact sequence:

$$0 \to F \to G \to \mathscr{E}_0 \to 0,$$

where F, G are locally free of the same rank. Notice that this implies that $\Xi = \operatorname{Supp} \mathscr{E}_0$ is a determinantal variety, hence locally Cohen-Macaulay.

In order to prove the lemma it suffices to check that

$$\mathscr{H}_Z^0(\mathscr{E}_0) = \mathscr{H}_Z^1(\mathscr{E}_0) = 0$$

where $Z := \Xi - \Xi_{\mathrm{sm}} = \operatorname{Sing} \Xi$ (cf. [3], section 3). By using the short exact sequence above one reduces to prove that

$$\mathscr{H}_Z^0(F) = \mathscr{H}_Z^1(F) = \mathscr{H}_Z^0(G) = \mathscr{H}_Z^1(G) = 0.$$

Since Ξ is locally Cohen-Macaulay

$$3 \leqq \operatorname{codim}_\Xi Z = \operatorname{depth}_Z(\mathcal{O}_\Xi) = \operatorname{depth}_Z(F) = \operatorname{depth}_Z(G)$$

and this implies the desired vanishing. □

4.3. We want to prove that the coherent sheaf $\mathscr{E}_0$ is determined, up to tensoring with an element of $\operatorname{Pic}^0(P)$, by the following intrinsic properties:

a) $\mathscr{E}_0$ is supported on a translated of the theta divisor and is locally free of rank 2 on the open set of the smooth points.

b) $u_*\big(u^*(\mathscr{E}_0)\big) \cong \mathscr{E}_0$, where u is the inclusion of the smooth open set in the theta divisor.

c) $\mathbb{P}\big((-1_P)^*\mathscr{E}_0\big) \cong X$, as P-varieties.

d) $c_1(\mathscr{E}_0) = 2 \cdot [\Xi]$, $c_2(\mathscr{E}_0) = 2 \cdot [\Xi]^2$.

Proposition 4.3. *Let $\mathscr{E}$ be a coherent sheaf on P satisfying the properties* a), b), c) *and* d). *Then, there exists an invertible sheaf $L \in \operatorname{Pic}^0(P)$ such that $\mathscr{E}_0 \cong \mathscr{E} \otimes L$.*

Proof. Let $\mathscr{E}$ as in the statement. Combining a) and c) we get that there exists an invertible sheaf L' over Ξ_{sm} such that

$$u^*(\mathscr{E}_0) \cong u^*(\mathscr{E}) \otimes L'.$$

Since the dimension of the singular locus of Ξ is $\leqq g-6 = \dim \Xi - 4$, then Ξ is locally factorial and L' can be extended to an invertible sheaf L on Ξ, so we replace L' by $u^*(L)$. By applying b) and projection formula one has $\mathscr{E}_0 \cong \mathscr{E} \otimes L$. By the Lefschetz theorem for Picard groups (cf. [4]), $\mathrm{Pic}(P) \cong \mathrm{Pic}(\Xi)$, hence we can extend L to an invertible sheaf on P and think of the last isomorphism in P.

To show that $L \in \mathrm{Pic}^0(P)$ we compare the chern character of $\mathscr{E}_0$ and that of $\mathscr{E} \otimes L$. By means of the property d) we get the relation $2\Xi \cdot c_1(L) = 0$. The intersection product with Ξ is injective, hence $c_1(L)$ is algebraically trivial, up to torsion. Since the Néron-Severi group of an abelian variety is torsion-free we arrive to $c_1(L) = 0$. □

4.4. Now, the involutive and the translation property of the Fourier-Mukai transform imply that we recover from X a coherent sheaf of the type $p_*(M)$. By taking the support we get, up to translation, the curve $\tilde{C}$ naturally embedded in the Prym variety. To complete the proof of the Theorem 2.2 we need to show how to recover the involution σ on $\tilde{C}$. This follows by an argument of Welters (cf. [10], (2.2), p. 96): the map $p : \tilde{C} \to P$ (composed with any translation) induces, by the universal property of the Albanese variety, a unique morphism $u : J(\tilde{C}) \to P$. Then ${}^t uu$, is the projection $1 - \sigma$, hence u determines the involution $\sigma = 1 - {}^t uu$ in $J(\tilde{C})$. Since $\tilde{C}$ is non hyperelliptic, by the strong Torelli Theorem, the involution σ in $\tilde{C}$ is recovered.

This finishes the proof of Theorem 2.2.

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