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Zeros of Functions in Bergman Class

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Dedicado a mi familia
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Abstract

These notes are a survey of the results known of zero sets of functions in Bergman space.

Characterization of zero sets for functions in Bergman space remains being an open problem, which is closed in the case of $H^p(\mathbb{D})$ spaces, for $0 \leq p \leq \infty$ where H^0 corresponds to the Nevanlinna class $\mathcal{N}$, where we have an indistinguishable geometric characterization of zero sets of functions in terms of Blaschke products, as we will see it in Chapter 1.

In Chapter 2 we will introduce the weighted Bergman space $\mathcal{B}^p_\alpha$, which is a parametrization of the usual weighted Bergman space A^p_α. Such parametrization let us estimate a function $f \in \mathcal{B}^p_\alpha$ as

$$|f(z)| \leq \frac{C}{(1-|z|^2)^\alpha}, \quad z \in \mathbb{D}, \tag{1}$$

for some $C > 0$ constant. Notice that the estimation (1) does not depends on p.

Chapter 3 is devoted to study the basic properties of zero sets of functions in $\mathcal{B}^p_\alpha$ and a probabilistic model of random zero sets, by Gregory Bomash and apparently initiated by Emile Leblanc.

Furthermore, Chapter 3 will show us that characterization of zero sets of function in Bergman space is a hard problem. It is because by using Blaschke-type products, which involves only the modulus of the zeros, we can obtain necessary conditions that are far from being a sufficient condition or sufficient conditions that are far from being a necessary condition, which is the case of the sharp sufficient condition obtained by Bomash. Moreover, zero sets of the Bergman space $\mathcal{B}^p_\alpha$ are not necessary to be a zero set of a different Bergman space $\mathcal{B}^q_\gamma$, and union of zeros sets of $\mathcal{B}^p_\alpha$ are not necessary a zero set of $\mathcal{B}^p_\alpha$, which contrasts with the case of the spaces H^p.

Since working only with the modulus of the zeros is insufficient in order to obtain a characterization of zeros sets of functions in $\mathcal{B}^p_\alpha$ (i.e., a necessary and sufficient condition), in Chapter 4 we will introduce some notions of density, which join with the growth spaces $\mathcal{A}^{-\alpha}$ are the framework considered by Korenblum, who obtained the latest results about characterization of zero sets of funtions in Bergman space, whose necessary condition and whose sufficient condition are very close to be a characterization, as we will see it in Chapter 5.

Chapter 1

Zeros of functions in H^p class

1.1 Introduction

We are going to start this chapter introducing some notation; if f is any continuous function with domain $\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$, the open unit disk in the complex plane, we will define f_r on $\mathbb{T} := \{z : \ |z| = 1\}$, the unit circle in the complex plane, by

$$f_r(e^{i\theta}) = f(re^{i\theta}), \quad 0 \le r < 1,$$

and we let σ denote the Lebesgue measure on $\mathbb{T}$, so normalized that $\sigma(\mathbb{T}) = 1$. Accordingly, L^p-norms will refer to $L^p(\sigma)$. In particular,

$$\|f_r\|_p = \left(\int_{\mathbb{T}} |f_r|^p d\sigma \right)^{1/p}, \quad (0 < p < \infty),$$
$$\|f_r\|_\infty = \sup_\theta |f(re^{i\theta})|,$$

and we also introduce

$$\|f_r\|_0 = \exp \int_{\mathbb{T}} \log^+ |f_r| d\sigma,$$

where $\log^+ t = \log t$ if $t \ge 1$ and $\log^+ t = 0$ if $t < 1$.

Definition 1.1 *If $f \in H(\mathbb{D})$ and $0 \le p \le \infty$, we put*

$$\|f\|_p = \sup \{ \|f_r\|_p : \ 0 \le r < 1 \}.$$

If $0 < p \le \infty$, H^p is defined to be the class of all $f \in H(\mathbb{D})$ for which $\|f\|_p < \infty$.
The class $\mathcal{N} = \mathcal{N}(\mathbb{D})$ (for Nevanlinna) consists of all $f \in H(\mathbb{D})$ for which $\|f\|_0 < \infty$.

If it is required, we will also denote $\|f\|_{H^p} = \|f\|_p$, for $0 < p \le \infty$, and by $\|f\|_{\mathcal{N}} = \|f\|_0$.

Proposition 1.2 *Let $0 < s < p < \infty$, then $H^\infty \subset H^p \subset H^s \subset \mathcal{N}$.*

PROOF. On the one hand, if $f \in H^p$ we have that

$$\begin{aligned}
\|f\|_s &= \sup_{0\le r<1} \left(\int_{\mathbb{T}} |f_r|^s d\sigma \right)^{1/s} \\
&\le^{(a)} \sup_{0\le r<1} \left\{ \left(\int_{\mathbb{T}} (|f_r|^s)^{p/s}\, d\sigma \right)^{s/p} \left(\int_{\mathbb{T}} d\sigma \right)^{1/q} \right\}^{1/s} \\
&=^{(b)} \sup_{0\le r<1} \left(\int_{\mathbb{T}} |f_r|^p d\sigma \right)^{1/p} < \infty.
\end{aligned}$$

Where in (a) we have used the Hölder inequality with $q^{-1} = 1 - s/p$, and in (b) we have used that $\sigma(\mathbb{T}) = 1$.

On the other hand, see that the inclusion $H^s \subset \mathcal{N}$ is a consequence of Jensen's inequality,

$$\begin{aligned}
\sup_{0\le r<1} \exp \int_{\mathbb{T}} \log^+ |f_r| d\sigma &\le \sup_{0\le r<1} \int_{\mathbb{T}} \exp\left(\frac{s}{s} \log^+ |f_r| \right) d\sigma \\
&\le \sup_{0\le r<1} e^{1/s} \int_{\mathbb{T}} |f_r|^s d\sigma < \infty.
\end{aligned}$$

$\boxtimes$

1.2 Characterization of the zero sets of function in H^p class

In this section we will see that the zero set of a function in either class H^∞, H^p, for $0 < p < \infty$, and $\mathcal{N}$ have exactly the same geometric characterization. To be precise, we will see that $\{\alpha_n\} \subset \mathbb{D}$ is the zero set of a function f, which belongs to anyone of the above classes, if and only if $\{\alpha_n\}$ satisfies (1).

Definition 1.3 *Let $\{\alpha_n\}$ be a sequence of points in the unit disk $\mathbb{D}$. We will say that $\{\alpha_n\}$ satisfies the Blaschke condition if*

$$\sum_n 1 - |\alpha_n| < \infty. \tag{1}$$

We observe that each term in the sum on (1) is the distance between α_n and $\mathbb{T}$, so that,

$$\sum_n 1 - |\alpha_n| = \sum_n d(\alpha_n, \mathbb{T}),$$

it justify why we say that the zeros of functions in any of these classes have a geometric characterization.

Notice that by the Proposition 1.2, it is enough to see that the zero set $\{\alpha_n\}$ of functions $f \in \mathcal{N}$ satisfies (1), in order to conclude that the zero set of a functions in either class H^∞, H^p for $0 < p < \infty$, satisfies the Blaschke condition.

So, we will start seeing that the zeros of a function $f \in \mathcal{N}$ satisfies the Blaschke condition. To do that, we will need the following lemma.

Lemma 1.4 *Suppose* $0 \leq u_n < 1$, *then*

$$\prod_{n=1}^{\infty} (1-u_n) \quad \textit{converge and} \quad \prod_{n=1}^{\infty} (1-u_n) > 0$$

if and only if

$$\sum_{n=1}^{\infty} u_n < \infty.$$

PROOF. ($\Leftarrow$) Let

$$p_N = \prod_{k=1}^{N} (1-u_k).$$

It is clear that $\{p_N\}$ is an decreasing sequence of positive numbers; then the following limit exists,

$$p = \lim_{N\to\infty} p_N.$$

Moreover, since $\sum_{n=1}^{\infty} u_n < \infty$, we have that $u_n \to 0$ as $n \to \infty$. Because $u_n < 1$ for all n, we have that $\sup_n u_n = m < 1$. Notice that there is $c = c\,(m) > 0$ such that $\log\,(1-x) \geq -cx$ for all $x \in [0, m]$, it holds that

$$\sum_{n=1}^{\infty} \log\,(1-u_n) \geq -c \sum_{n=1}^{\infty} u_n > -\infty.$$

We deduce that $p = \prod\,(1-u_n) > 0$.

($\Rightarrow$) On the other hand

$$0 < p \leq p_N = \prod_{k=1}^{N} (1-u_k) \leq \exp\left\{\sum_{n=1}^{N} (-u_n)\right\},$$

and

$$\lim_{N\to\infty} \exp\left\{\sum_{n=1}^{N} (-u_n)\right\} = 0 \quad \text{if and only if} \quad \sum_{n=1}^{\infty} u_n = \infty,$$

so that

$$\sum_{n=1}^{\infty} u_n < \infty.$$

$\boxtimes$

Now that we have proved the Lemma 1.4, we are ready to see the Theorem 1.5.

Theorem 1.5 *Suppose $f \in \mathcal{N}$, f is not identically 0 in $\mathbb{D}$, and $\alpha_1, \alpha_2, \alpha_3, \cdots$ are the zeros of f, listed according to their multiplicities. Then, the zeros of f satisfies the Blaschke condition,*

$$\sum_{n\geq 1} (1 - |\alpha_n|) < \infty. \tag{2}$$

PROOF. To start, notice that we tacitly assume that f has infinitely many zeros in $\mathbb{D}$. If there are only finitely many, the above sum would be finite and there is nothing to prove. Moreover, we assume that $|\alpha_n| \leq |\alpha_{n+1}|$. Furthermore, if f has a zero of degree m at the origin and $g(z) = z^{-m} f(z)$, then $g \in \mathcal{N}$ and g has the same zeros as f, except at the origin. Hence we may assume, without loss of generality, that $f(0) \neq 0$. Let $n(r)$ be the number of zeros of f in $\mathrm{cl}(D(0;r))$, fix k, and fix $r < 1$ so that $n(r) > k$. Then Jensen's formula

$$|f(0)| \prod_{n=1}^{n(r)} \frac{r}{|\alpha_n|} = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta})| d\theta \right\} \tag{3}$$

implies that

$$|f(0)| \prod_{n=1}^{k} \frac{r}{|\alpha_n|} \leq \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta \right\}. \tag{4}$$

Our assumption that $f \in \mathcal{N}$ is equivalent to the existence of a constant $C < \infty$ which exceeds the right side of (4) for all r, $0 < r < 1$, that is,

$$|f(0)| \prod_{n=1}^{k} \frac{r}{|\alpha_n|} \leq C.$$

It follows that,

$$r^k |f(0)| C^{-1} \leq \prod_{n=1}^{k} |\alpha_n|. \tag{5}$$

We can see that in (5) the inequality persists, for every k, as $r \to 1$. Hence

$$\prod_{n=1}^{\infty} |\alpha_n| \ \geq \ C^{-1} |f(0)| > 0. \tag{6}$$

Finally, by the Lemma 1.4, (6) implies (2). ⊠

Since $H^\infty \subset H^p \subset H^s \subset \mathcal{N}$ $(0 < s < p < \infty)$ and by the Theorem 1.5, we have seen that the Blaschke condition is a necessary condition for $\{\alpha_n\}$ to be the zero set of a function in either of the classes H^∞, H^p, $\mathcal{N}$. Now, we will see in the Theorem 1.7 that, in fact, the Blaschke condition is also a sufficient condition; in the sense that if $\{\alpha_n\} \subset \mathbb{D}$ is any sequence satisfying (1), then there is a function $f \in H^\infty$ whose zero set is $\{\alpha_n\}$. To prove it, we will need the Theorem 1.6, which we will state it without proof.

Theorem 1.6 *Suppose $f_n \in H(\mathbb{D})$ for $n = 1, 2, 3, \cdots$, no f_n is identically 0, and*

$$\sum_{n=1}^{\infty} |1 - f_n(z)| \tag{7}$$

converge uniformly on compact subsets of $\mathbb{D}$. Then

(i) The infinite product

$$f(z) = \prod_{n=1}^{\infty} f_n(z) \tag{8}$$

converge uniformly in compact subsets of $\mathbb{D}$. Hence $f \in H(\mathbb{D})$.

(ii) We have that

$$m(f; z) = \sum_{n=1}^{\infty} m(f_n; z), \quad z \in \mathbb{D}, \tag{9}$$

where $m(f; z)$ is defined to be the multiplicity of the zeros of f at z (if $f(z) \neq 0$, then $m(f; z) = 0$).

Theorem 1.7 *If $\{\alpha_n\}$ is a sequence of points in $\mathbb{D}$ such that $\alpha_n \neq 0$ and*

$$\sum_{n \geq 1} (1 - |\alpha_n|) < \infty, \tag{10}$$

if k is a nonnegative integer, and if

$$B(z) = z^k \prod_{n=1}^{\infty} \frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \frac{|\alpha_n|}{\alpha_n} \quad (z \in U), \tag{11}$$

then $B \in H^\infty$, and B has no zeros except at the points α_n (and at the origin, if $k > 0$).

PROOF. If $|z| \leq r$, the nth term in the series

$$\sum_{n=1}^{\infty} \left| 1 - \frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \frac{|\alpha_n|}{\alpha_n} \right|$$

is

$$\left| \frac{\alpha_n + |\alpha_n| z}{(1 - \overline{\alpha_n} z)\, \alpha_n} \right| (1 - |\alpha_n|) \leq \frac{1 + r}{1 - r} (1 - |\alpha_n|).$$

Hence Theorem 1.6 shows that $B \in H(\mathbb{D})$ and that B has only the prescribed zeros. Since each factor in (10) has absolute value less than 1 in $\mathbb{D}$, it follows that $|B(z)| < 1$ for all $z \in \mathbb{D}$, then $B \in H^\infty$ and the proof is done. ⊠

To finish this chapter, let's state the following corollary.

Corollary 1.8 *Let $0 < p < \infty$.*

(i) The zero set of functions in either class H^∞, H^p, $\mathcal{N}$ have an indistinguishable geometric characterization.

(ii) Let f and g be functions of the same class (either H^∞, H^p or $\mathcal{N}$), $\{\alpha_n\}$ and $\{\beta_n\}$ be the zero set of f and g, respectively, then there is a function h of the same class as f and g such that $\{\alpha_n\} \cup \{\beta_n\}$ is the zero set of h.

(iii) If $\{\beta_n\} \subset \{\alpha_n\}$, where $\{\alpha_n\}$ is the zero set of a function f in either class H^∞, H^p, $\mathcal{N}$, then there is a function g of the same class as f such that $\{\beta_n\}$ is the zero set of g.

Chapter 2

The Bergman space

2.1 Introduction

In this chapter we introduce the Bergman space and concentrate on the general aspects of these spaces.

Throughout these notes we will denote the normalized area measure on $\mathbb{D}$ by dA. In terms of real (rectangular and polar) coordinates, we have

$$dA(z) = \frac{1}{\pi} dxdy = \frac{1}{\pi} rdrd\theta, \quad z = x + iy = re^{i\theta}.$$

The word positive will appear frequently throughout these notes. That a function f is positive means that $f(x) \geq 0$ for all values of x, and that a measure μ is positive means that $\mu(E) \geq 0$ for all measurable sets E. When we need to express the property that $f(x) > 0$ for all x, we say that f is strictly positive. These conventions apply to the word negative as well. Analogously, we prefer to speak of increasing and decreasing functions in the less strict sense, so that constant functions are both increasing and decreasing.

We use the symbol $\sim$ to indicate that two quantities have the same behavior asymptotically. Thus, $A \sim B$ means that A/B is bounded from above and below by two positive constants in the limit process in question.

For $0 < p < +\infty$ and $-1 < \alpha < +\infty$, the (weighted) Bergman space $A^p_\alpha = A^p_\alpha(\mathbb{D})$ of the disk is the space of analytic functions in $L^p(\mathbb{D}, dA_\alpha)$, where

$$dA_\alpha(z) = (\alpha + 1) \left(1 - |z|^2\right)^\alpha dA(z).$$

If f is in $L^p(\mathbb{D}, dA_\alpha)$, we write

$$\|f\|_{A^p_\alpha} = \left[\int_{\mathbb{D}} |f(z)|^p dA_\alpha(z)\right]^{1/p}.$$

Furthermore, for simplicity, from now when $\alpha = 0$, we will denote by $A^p = A^p_0$, $dA = dA_0$ and $\|\cdot\|_{A^p}$ the norm related to the space $L^p(\mathbb{D}, dA)$, so that, if $f \in L^p(\mathbb{D}, dA)$ we write

$$\|f\|^p_{A^p} = \int_{\mathbb{D}} |f(z)|^p dA(z).$$

Notice that when $1 \leq p < +\infty$, the space $L^P(\mathbb{D}, dA_\alpha)$ is a Banach space with the above norm; when $0 < p < 1$, the space $L^p(\mathbb{D}, dA_\alpha)$ is a complete metric space with the metric defined by

$$d(f,g) = \|f-g\|^p_{A^p_\alpha}.$$

Since $d(f,g) = d(f-g,0)$, the metric is invariant. The metric is also p-homogeneous, that is, $d(\lambda f, 0) = |\lambda|^p d(f,0)$ for scalars $\lambda \in \mathbb{C}$. Spaces of this type are called quasi-Banach spaces, because they share many properties of the Banach spaces.

We let $L^\infty(\mathbb{D})$ denote the space of (essentially) bounded functions on $\mathbb{D}$. For $f \in L^\infty$ we define

$$\|f\|_\infty = \operatorname{ess\,sup}\left\{|f(z)| : \ z \in \mathbb{D}\right\}.$$

The space $L^\infty(\mathbb{D})$ is a Banach space with the above norm. As usual, we let H^∞ denote the space of bounded analytic functions in $\mathbb{D}$. It is clear that H^∞ is closed in $L^\infty(\mathbb{D})$ and hence is a Banach space itself. For convenience we will define $A^\infty = H^\infty$.

Proposition 2.1 *Let $0 < p < \infty$. If $f \in H^p$, then $f \in A^p_\alpha$.*

PROOF. It is clear, since

$$\begin{aligned}
\|f\|_{A^p_\alpha} &= \int_{\mathbb{D}} |f(z)|^p (\alpha+1)(1-|z|^2)^\alpha dA(z) \\
&\leq (\alpha+1) \int_{\mathbb{D}} |f(z)|^p dA(z) \\
&\leq \sup_{0 \leq r < 1} (\alpha+1) \int_{\mathbb{T}} |f|^p d\sigma.
\end{aligned}$$

⊠

Proposition 2.2 *Suppose $0 < p < +\infty$, $-1 < \alpha < +\infty$, and that K is a compact subset of $\mathbb{D}$. Then there exists a positive constant $C = C(K, \alpha, n)$ such that*

$$\sup\left\{|f^{(n)}(z)| : \ z \in K\right\} \leq C\|f\|_{A^p_\alpha}$$

for all $f \in A^p_\alpha$ and all $n = 0, 1, 2, \cdots$. In particular, every point-evaluation in $\mathbb{D}$ is a bounded linear functional on A^p_α.

PROOF. Let $z \in K$. Without loss of generality assume $z \neq 0$ and consider $\sigma = (1-|z|)/2$ (for the case $z = 0$ we consider $\sigma = (1-r)/2$ with $0 < r < 1$). By the subarmonicity of $|f|^p$, we have that

$$|f(z)|^p \leq \frac{4}{(1-|z|)^2} \int_{B(z,\sigma)} |f(w)|^p dA(w) \leq \frac{16}{(1-|z|^2)^2} \int_{B(z,\sigma)} |f(w)|^p dA(w)$$

for all $z \in K$. Now, notice that

$$\frac{1}{2} \leq \frac{1-|w|^2}{1-|z|^2} \leq 2, \quad \text{for all } \ w \in B(z,\sigma).$$

It follows that

$$\begin{aligned}
|f(z)|^p &\leq \frac{16}{(1-|z|^2)^2} \int_{B(z,\sigma)} |f(w)|^p \left(2\frac{1-|w|^2}{1-|z|^2}\right)^{\alpha} dA(w) \\
&= \frac{2^{\alpha}16}{(\alpha+1)(1-|z|^2)^{\alpha+2}} \int_{B(z,\sigma)} |f(w)|^p (1+\alpha)\left(1-|w|^2\right)^{\alpha} dA(w) \\
&= \frac{2^{\alpha}16}{(\alpha+1)(1-|z|^2)^{\alpha+2}} \int_{B(z,\sigma)} |f(w)|^p dA_{\alpha}(w) \\
&\leq \frac{2^{\alpha}16}{(\alpha+1)(1-|z|^2)^{\alpha+2}} \|f\|^p_{A^p_{\alpha}}.
\end{aligned}$$

Finally, since K is a compact subset of the unit disk $\mathbb{D}$, we can write

$$C = \sup_{z\in K} \frac{2^{\alpha}16}{(\alpha+1)(1-|z|^2)^{(\alpha+2)/p}}$$

and we obtain that

$$\sup_{z\in K} |f(z)| \leq C\|f\|_{A^p_{\alpha}}.$$

By the special case we just proved, there exists a constant $M>0$ such that $|f(\xi)| \leq M\|f\|_{A^p_{\alpha}}$ for all $|\xi| = R$, where $R = (1+r)/2$ and $0<r<1$. Now if $z \in K$, then by Cauchy's integral formula,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|\xi|=R} \frac{f(\xi)d\xi}{(\xi - z)^{n+1}}.$$

It follows that

$$|f^{(n)}(z)| \leq \frac{n!MR}{\sigma^{n+1}} \|f\|_{A^p_{\alpha}}$$

for all $z \in K$ and $f \in A^p_{\alpha}$. ⊠

The above proposition show us that the growth of a function $f \in A^p_{\alpha}$ is controlled by

$$|f(z)| \leq \frac{C}{(1-|z|^2)^{(\alpha+2)/p}} \|f\|_{A^p_{\alpha}}, \tag{1}$$

where $C = C(\alpha)$ is some constant that depends on α.

For $\alpha > 1/p$, we will define $dB^p_{\alpha} := dA_{\alpha p - 2}$, it is,

$$dB^p_{\alpha}(z) = (\alpha p - 1)\left(1-|z|^2\right)^{\alpha p - 2} dA(z),$$

and the space $\mathcal{B}^p_{\alpha} = A^p_{\alpha p-2}$, hence if $f \in \mathcal{B}^p_{\alpha}$, we have

$$|f(z)| \leq \frac{C}{(1-|z|^2)^{\alpha}} \|f\|_{\mathcal{B}^p_{\alpha}}. \tag{2}$$

Notice that, with this notation, $\mathcal{B}^p_{2/p} = A^p_0$ corresponds to the classic Bergman space. Furthermore, for convenience, we will define $\mathcal{B}^{\infty} = H^{\infty}$.

Even more, the Proposition 2.2 give us another important consequence; if $f \in \mathcal{B}^p_\alpha$ and if $\{\beta_n\}$ is a subset of the zero set of f contained in some disk D internally tangent to the unit circle, then $\{\beta_n\}$ satisfies the Blaschke condition (1). If fact, we will see in the following corollary that f restricted on D belongs to the Nevanlinna class $\mathcal{N}(D)$.

Corollary 2.3 *Let $f \in \mathcal{B}^p_\alpha$ and let $D \subset \mathbb{D}$ be a disk centered at $z_0 \in \mathbb{D}$ and internally tangent to the unit circle. Then, f restricted on D lies in the Nevanlinna class $\mathcal{N}(D)$, it is,*

$$\sup_{0\le r<d} \exp \int_0^{2\pi} \log^+ |f_r| d\sigma < \infty, \tag{3}$$

where, here, $f_r(e^{i\theta}) = f(z_0 + re^{i\theta})$ and d is the radius of D.

PROOF. We can assume, without loss of generality, that D is centered at $0 < z_0 < 1$, it is,

$$D := \{z \in \mathbb{D} : \ |z_0 - z| < R\},$$

where

$$R = d(z_0, \mathbb{T}) = 1 - z_0.$$

Assume $f \in \mathcal{B}^p_\alpha$, then by the Proposition 2.2 we know that there is a constant $C = C(\alpha) > 0$ such that

$$|f(z)| \le \frac{C}{(1-|z|^2)^\alpha} \|f\|_{\mathcal{B}^p_\alpha}, \quad \text{for all } z \in \mathbb{D}.$$

To make easier the notation, let us put $z_r(\theta) = z_0 + re^{i\theta}$, where $0 \le r < R$. It follows that there is $C_1 > 0$ such that

$$\begin{aligned} \int_{\partial D} \log^+ |f_r| d\sigma &\le C_1 \int_0^{2\pi} \log^+ \left|\frac{1}{(1-|z_r(\theta)|^2)^\alpha}\right| d\theta \\ &= \alpha C_1 \int_0^{2\pi} \log^+ \left|\frac{1}{1-|z_r(\theta)|^2}\right| d\theta. \end{aligned} \tag{4}$$

Since $z \in D$ and

$$\begin{aligned} 1 - |z|^2 &= R^2 - r^2 + 2Rz_0 - 2z_0 r\cos(\theta) \\ &\ge 2z_0(R - r\cos(\theta)) \\ &\ge 2z_0 R(1-\cos(\theta)) \\ &\ge C_2\theta^2, \end{aligned} \tag{5}$$

for some $C_2 > 0$. It follows from (4) and (5) that

$$\begin{aligned} \int_{\partial D} \log^+ |f_r| d\sigma &\le C_3 \int_0^{2\pi} \log^+ \left|\frac{1}{\theta}\right| d\theta \\ &= C_3 \int_0^1 \log \left|\frac{1}{\theta}\right| d\theta < \infty, \end{aligned} \tag{6}$$

which give us (3) and we are done. ⊠

2.2 Relations between $\mathcal{B}^p_\alpha$ spaces

In this section we will see conditions over the parameters p, α, q and γ in order to determine when we can have the set inclusion $\mathcal{B}^p_\alpha \subset \mathcal{B}^q_\gamma$.

To start, we will study the case $p \leq q$, which will required the following lemma that we will state without a proof, but it ca be found in [7].

Lemma 2.4 *Suppose $a \in \mathbb{D}$, c real, $t > -1$ and define*

$$J_{c,t}(a) = \int_{\mathbb{D}} \frac{(1-|z|^2)^t}{|1-z\overline{a}|^{2+t+c}} dA(z).$$

When $c < 0$, then $J_{c,t}$ is bounded in $\mathbb{D}$.
When $c > 0$, then

$$J_{c,t}(a) \approx \frac{1}{(1-|z|^2)^c}.$$

Finally,

$$J_{0,t}(a) \approx \log \frac{1}{1-|z|^2}.$$

Theorem 2.5 *Let $p \leq q$. Then, $\mathcal{B}^p_\alpha \subseteq \mathcal{B}^q_\gamma$ if and only if $\alpha \leq \gamma$.*

PROOF. [⇒] Assume $\alpha \leq \gamma$ and $f \in \mathcal{B}^p_\alpha$. we have that

$$\begin{aligned} \|f\|_{\mathcal{B}^q_\gamma} &= \int_{\mathbb{D}} |f(z)|^p |f(z)|^{q-p} (1-|z|^2)^{\gamma q - 2} dA(z) = \\ &\leq C \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^{-\alpha(q-p)+\gamma q-2} dA(z). \end{aligned} \tag{7}$$

where $C > 0$ is some constant. Notice that in order to converge the last integral in (7), it is required the condition

$$-\alpha(q-p) + \gamma q - 2 \geq \alpha p - 2,$$

which is satisfied by hypothesis.

[⇐] By the counterreciprocal. Assume $\gamma < \alpha$. Chose $\beta > \alpha$ and define $f_{\beta,a}(z) = (1-z\overline{a})^{-\beta}$ and

$$h_{\beta,a}(z) = \frac{f_{\beta,a}(z)}{\|f_{\beta,a}\|_{\mathcal{B}^p_\alpha}}.$$

Notice that $\|h_{\beta,a}\|_{\mathcal{B}^p_\alpha} = 1$ for all $a \in \mathbb{D}$, moreover, if we take, for example, the sequence $\{a_n := n/(n+1)\}_{n=1}^{\infty}$, we see that h_{β,a_n} is a Cauchy sequence on $\mathcal{B}^p_\alpha$, so that, there is $h_\beta \in \mathcal{B}^p_\alpha$ such that $h_{\beta,a_n} \to h_\beta$ as $n \to \infty$.

On the other hand, by Lemma 2.4, we have that

$$\|h_{\beta,a_n}\|_{\mathcal{B}^q_\gamma} \approx (1-|a_n|^2)^{-q(\beta-\gamma)+p(\beta-\alpha)},$$

Now, since $\gamma < \alpha < \beta$ and $p < q$, we conclude that $h_\beta \notin \mathcal{B}^q_\gamma$. ⊠

Theorem 2.6 *Let $p \geq q$. If $\gamma - \dfrac{1}{q} > \alpha - \dfrac{1}{p}$, then $\mathcal{B}^p_\alpha \subset \mathcal{B}^q_\gamma$.*

PROOF. Assume $f \in \mathcal{B}^p_\alpha$. By Hölder inequality with exponent $p/q \geq 1$ we have that

$$\begin{aligned}
\|f\|_{\mathcal{B}^q_\gamma} &= \int_{\mathbb{D}} \left(|f(z)|^q(1-|z|^2)^{\alpha q - 2q/p}\right)\left((1-|z|^2)^{q(\gamma-\alpha)-2(1-q/p)}\right) dA(z) \\
&\leq \left(\int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^{\alpha p-2} dA(z)\right)^{q/p} \\
&\times \left(\int_{\mathbb{D}} (1-|z|^2)^{(\gamma-\alpha)qp/(p-q)-2} dA(z)\right)^{(p-q)/p}.
\end{aligned} \tag{8}$$

Because $f \in \mathcal{B}^p_\alpha$, the first integral in the right hand side of the inequality in (8) converges. To see the convergence of the second one, notice that it is required the condition

$$(\gamma-\alpha)qp/(p-q) - 2 > -1,$$

but it is satisfied by hypothesis.

Notice than when $\alpha - 1/p = 0$ the hypothesis becomes $f \in H^p$. Using $\|f\|^q_{H^q} \leq \|f\|^p_{H^p}$ and integrating in polar coordinates, we see that

$$\|f\|_{\mathcal{B}^q_\gamma} \leq \|f\|_{H^p} \int_0^1 (1-r^2)^{\gamma q-2} dr < \infty,$$

since $\gamma p - 2 \geq -1$. ⊠

In chapter 3 we will prove a theorem (Theorem 3.8) which implies that

$$\text{if} \quad q \leq p \quad \text{and} \quad \alpha - \frac{1}{p} > \beta - \frac{1}{q} \quad \text{then} \quad \mathcal{B}^p_\alpha \not\subseteq \mathcal{B}^q_\gamma.$$

Chapter 3

Basic properties of zero sets of functions in the Bergman space

The guide line of this chapter will be the paper [3], by Charles Horowitz. It will help us to understand why the characterization of zero sets of functions in Bergman spaces is a hard problem. In particular, we will see that the statements (i) and (ii) of the Corollary 1.8 for Hardy spaces in general do not hold in Bergman spaces, but the statement (iii) of the same corollary still holds in Bergman spaces.

Furthermore, in the last section of this chapter we will see a probabilistic model of random zero sets, which will give us a sharp sufficient condition for a sequence of points in $\mathbb{D}$ in order to be a zero set of a function in the usual Bergman space $A^2 = \mathcal{B}^p_{2/p}$.

3.1 Conditions on Taylor coefficients

Throughout this section, let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

be analytic in the unit disk. We will present some conditions on the Taylor coefficients a_k that are sufficient for f to belong to certain $\mathcal{B}^p_\alpha$ spaces. There is no difficulty for $\mathcal{B}^2_{2/p}$, since a direct application of Parseval's formula give us

$$\|f\|_{\mathcal{B}^2_{2/p}} = \sum_{k=0}^{\infty} \frac{|a_k|^2}{k+1}. \tag{1}$$

On the other hand, when $\alpha > 1/p$ and $\alpha \neq 2/p$, for $f \in \mathcal{B}^2_\alpha$ we have that

$$\begin{aligned} \|f\|_{\mathcal{B}^2_\alpha} &= \int_{\mathbb{D}} \Big|\sum_{k=0}^{\infty} a_k z^k\Big|^2 dB_\alpha(z) = \int_{\mathbb{D}} \left(\sum_{k=0}^{\infty} a_k z^k\right)\overline{\left(\sum_{k=0}^{\infty} a_k z^k\right)} dB_\alpha(z) \\ &= \sum_{k=0}^{\infty} |a_k|^2 \int_{\mathbb{D}} |z|^{2k} dB_\alpha(z) + \sum_{\substack{k,m=0\\ k\neq m}}^{\infty} a_k\overline{a_m} \int_{\mathbb{D}} z^k \overline{z^m} dB_\alpha(z) \\ &= (2\alpha-1)\sum_{k=0}^{\infty} |a_k|^2 \int_0^1 r^{2k}(1-r^2)^{2\alpha-2} 2r dr. \end{aligned} \tag{2}$$

Now, if in the last integral in (2) we consider $s = r^2$ we obtain that

$$\int_0^1 r^{2k}(1-r^2)^{2\alpha-2}2rdr = \int_0^1 s^k(1-s)^{2\alpha-2}ds =: B(k+1, 2\alpha-1), \tag{3}$$

which is the well known beta function. It follows from (2) and (3) that

$$\|f\|_{\mathcal{B}^2_\alpha} = (2\alpha-1)\sum_{k=0}^{\infty} |a_k|^2 B(k+1, 2\alpha-1). \tag{4}$$

It will be useful to keep in mind the following Stirling's approximation of the beta function,

$$B(n+1, \alpha p - 1) \sim (n+1)^{-(\alpha p-1)}, \quad \text{as } n \to \infty, \tag{5}$$

which, in fact, give us that

$$f \in \mathcal{B}^2_\alpha \quad \text{if and only if} \quad \sum_{k=0}^{\infty} |a_k|^2 (n+1)^{-(2\alpha-1)} < \infty.$$

Our theorems will be state in terms of the sums

$$S_N^{(q)} = \sum_{k=0}^{N} |a_k|^q, \quad 0 < q < \infty.$$

Theorem 3.1 *Let $f(z)$ be analytic in the unit disk and $0 < p \leq 2$. Suppose that $S_N^{(2)} = O(N^\eta)$ for some η. Then $f \in \mathcal{B}^p_\alpha$, for all $\alpha > \eta/2 + 1/p$.*

PROOF. Since all p's considered are less than or equal to 2, for $0 < r < 1$,

$$\begin{aligned} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{\pi}\right)^{1/p} &\leq \left(\int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{\pi}\right)^{1/2} = \left(\sum_{k=0}^{\infty} |a_k|^2 r^{2k}\right)^{1/2} \\ &= \left(\sum_{n=0}^{\infty} S_n^{(2)}(r^{2n} - r^{2n+2})\right)^{1/2}, \end{aligned} \tag{6}$$

by summation by parts. But our hypothesis is that $S_n^{(2)} \leq c(n+1)^\eta$ for all $n \geq 0$, and so the last expression is majorized by

$$\left(c(1-r^2) \sum_{n=0}^{\infty} (n+1)^\eta r^{2n} \right)^{1/2} \leq c_1 (1-r^2)^{-\eta/2},$$

where c_1 is a constant independent of r. It follows that

$$\begin{aligned} \|f\|_{\mathcal{B}_\alpha^p} &= (\alpha p - 1) \int_0^1 \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{\pi} \right) (1-r^2)^{\alpha p - 2} r dr \\ &\leq (\alpha p - 1) c_1 \int_0^1 (1-r^2)^{-p\eta/2 + \alpha p - 2} r dr < \infty, \end{aligned} \tag{7}$$

it is, $f \in \mathcal{B}_\alpha^p$ for all $\alpha > \eta/2 + 1/p$. ⊠

Theorem 3.2 *Let $p \geq 2$, and let p' be the conjugate exponent, with $1/p + 1/p' = 1$. If $S_N^{(p')} = O((N+1)^{(\alpha p - 1)(p'-1) - \epsilon})$ for some $\epsilon > 0$, then $f \in \mathcal{B}_\alpha^p$.*

PROOF. Let du be the measure which assigns to the nonnegative integer n the mass $B(n+1, \alpha p - 1)$ defined in (3). Using (4), we see that the mapping $T : L^{p'}(du) \to L^p(dA)$, defined by

$$T : \{a_n B(n+1, \alpha p - 1)\} \mapsto \sum_{n=0}^{\infty} a_n z^n,$$

is a bounded linear mapping when $p' = 1$ or $p' = 2$. In fact, $\|T\| \leq 1$ in both cases. By Riesz interpolation theorem (see [2]) the corresponding mapping of $L^{p'}(du)$ into $L^p(dA)$, for $1 \leq p' \leq 2$, also has a norm less than or equal to 1. Equivalently, for $1 \leq p' \leq 2$,

$$\begin{aligned} \|f\|_{\mathcal{B}_\alpha^p}^{p'} &= \|T\{a_n \mathcal{B}(n+1, \alpha p - 1)\}\|_{\mathcal{B}_\alpha^p}^{p'} \\ &\leq \|\{a_n \mathcal{B}(n+1, \alpha p - 1)\}\|_{L^{p'}}^{p'} = \sum_{n=0}^{\infty} |a_n|^{p'} \mathcal{B}(n+1, \alpha p - 1)^{p'} \\ &\leq C \sum_{n=0}^{\infty} |a_n|^{p'} \mathcal{B}(n+1, \alpha p - 1)^{p'-1}, \end{aligned}$$

where $C > 0$. If we apply summation by parts to the above formula and if we use Stirling's formula, we obtain that if

$$\sum_{n=0}^{\infty} S_n^{(p')} (n+1)^{(1-\alpha p)p'} < \infty \quad \text{while} \quad S_n^{(p')} (n+1)^{(1-\alpha p)(p'-1)} = O(1) \quad \text{as} \quad n \to \infty,$$

then $f \in \mathcal{B}_\alpha^p$. The result follows immediately. ⊠

3.2 Zero sets of functions in $\mathcal{B}^p_\alpha$

Let $f(z)$ be analytic in the unit disk, and let $\{z\}_{k=1}^{\infty}$ be its zeros, repeated according to multiplicity. The sequence $\{z_k\}$ is called the zero set of f. If $|z_1| \leq |z_2| \leq \cdots < 1$, then we call $\{z_k\}$ the ordered zeros of f. If $f \in \mathcal{B}^p_\alpha$, then $\{z_k\}$ is said to be an $\mathcal{B}^p_\alpha$ zero set. In this section we obtain a necessary condition for $\mathcal{B}^p_\alpha$ zero sets.

Lemma 3.3 *Let f be analytic for $|z| < 1$, and let $\{z_k\}$ be its ordered zeros. Assume moreover that $f(0) \neq 0$. Then for $0 < p < \infty$, for $0 \leq r < 1$, and for all positive integer N,*

$$|f(0)|^p \prod_{k=1}^{N} \frac{r^p}{|z_k|^p} \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta. \tag{8}$$

PROOF. Fix a value of p and r. Let $|z_{N_0}| < r \leq |z_{N_0+1}|$. By Jensen's formula,

$$\log|f(0)| + \sum_{k=1}^{N_0} \log\left(\frac{r}{|z_k|}\right) = \frac{1}{2\pi} \int_0^{2r} \log|f(re^{i\theta})| d\theta.$$

We multiply this equation by p and exponentiate to obtain

$$|f(0)|^p \prod_{k=1}^{N_0} \frac{r^p}{|z_k|^p} = \exp\left(\frac{1}{2\pi} \int_0^{2r} \log|f(re^{i\theta})|^p d\theta\right) \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta. \tag{9}$$

Since r and p are arbitrary in (9), the Lemma is proved if we can show that the integer N_0 in the inequality (9) may be replaced by an arbitrary integer N. So we can choose N ans seek to establish (8). There are two cases to be considered.

Case 1. $N < N_0$. Then $|z_k| \leq r$ for $N < k \leq N_0$. Hence

$$\prod_{k=1}^{N_0} \frac{r^p}{|z_k|^p} = \prod_{k=1}^{N} \frac{r^p}{|z_k|^p} \prod_{k=N+1}^{N_0} \frac{r^p}{|z_k|^p} \geq \prod_{k=1}^{N} \frac{r^p}{|z_k|^p},$$

and (8) follows from (9).

Case 2. $N > N_0$. Then $|z_k| \geq r$ for $N_0 < k \leq N$. Hence

$$\prod_{k=1}^{N} \frac{r^p}{|z_k|^p} = \prod_{k=1}^{N_0} \frac{r^p}{|z_k|^p} \prod_{k=N_0+1}^{N} \frac{r^p}{|z_k|^p} \leq \prod_{k=1}^{N_0} \frac{r^p}{|z_k|^p},$$

and once again (8) follows from (9). The proof is complete. ⊠

Theorem 3.4 *If $f \in \mathcal{B}^p_\alpha$, if $\{z_k\}$ are the ordered zeros of f, and if $f(0) \neq 0$, then*

$$\prod_{k=1}^{N} \frac{1}{|z_k|} = O(N^{\alpha - 1/p}). \tag{10}$$

PROOF. For $0 < p < \infty$, we integrate the inequality (8) with respect to $2(\alpha p-1)(1-r^2)^{\alpha p-2}rdr$, on the interval $[0,1]$. Thus, for any $N \geq 1$,

$$\left(|f(0)|^p \prod_{k=1}^{N} \frac{1}{|z_k|^p}\right) \int_0^1 r^{Np}(\alpha p-1)(1-r^2)^{\alpha p-2} 2rdr \leq \|f\|^p_{\mathcal{B}^p_\alpha},$$

or

$$\left(|f(0)|^p \prod_{k=1}^{N} \frac{1}{|z_k|^p}\right) B(Np/2+1, \alpha p-1) \leq \|f\|^p_{\mathcal{B}^p_\alpha}.$$

Thus we obtain that

$$\prod_{k=1}^{N} \frac{1}{|z_k|} \leq B(Np/2+1, \alpha p-1)^{-1/p} \frac{\|f\|_p}{|f(0)|}.$$

Finally, Stirling's formula give us (10). ⊠

Notice that the Hypothesis $f(0) \neq 0$ in Theorem 3.4 and Lemma 3.3 is clearly inessential, and was added only to simplify the statements of the results.

It should be note that (10) cannot possibly provide a sufficient condition for $\mathcal{B}^p_\alpha$ zeros sets $(0 < p < \infty)$. In fact, we know from Corollary 2.3 that if the zero set of a function in $\mathcal{B}^p_\alpha$ are contained in an internally tangent disk of $\mathbb{D}$, then such zero set must satisfy the Blaschke condition, although (10) does not require this.

Corollary 3.5 *Let $f \in \mathcal{B}^p_\alpha$, and let $\{z_k\}$ be the zero set of f. Let $b_k = 1 - |z_k|$. Then for all $\epsilon > 0$,*

$$\sum_{k\geq 1} b_k \left(\log \frac{1}{b_k}\right)^{-1-\epsilon} < \infty. \tag{11}$$

PROOF. without loss of generality, assume $f(0) \neq 0$ and let $\{z_k\}$ be the ordered zeros of f. By (10) we have

$$\prod_{k=1}^{N} \frac{1}{|z_k|} \leq cN^{\alpha-1/p}, \quad \text{for some} \quad c \geq 1.$$

So

$$\sum_{k=1}^{N} -\log|z_k| \leq \log c + (\alpha - \frac{1}{p})\log N \leq c_1 \log(N+1),$$

for some constant c_1 independent of N. From the inequality

$$1 - x \leq -\log x \quad (0 < x \leq 1)$$

we see that

$$\sum_{k=1}^{N} 1 - |z_k| \leq c_1 \log(N+1).$$

Letting $b_k = 1 - |z_k|$, we have

$$b_k \downarrow 0 \tag{12}$$

$$\sum_{k=1}^{N} b_k \leq c_1 \log (N+1) . \tag{13}$$

We now show that these two properties actually imply (11). Using (12) and (13) we have

$$N b_N \leq \sum_{k=1}^{N} b_k \leq c_1 \log (N+1) .$$

Thus

$$\log^{1+\epsilon}\left(\frac{1}{b_N}\right) \geq \log^{1+\epsilon}\left(\frac{N}{c_1 \log (N+1)}\right) \geq c_2 \log^{1+\epsilon}(N+1)$$

for N large, and it suffices to show that

$$\sum_{k=1}^{\infty} b_k \log^{-1-\epsilon}(k+1) < \infty.$$

We sum by parts, using (13) and the fact that

$$\log^{-1-\epsilon}(k+1) - \log^{-1-\epsilon}(k+2) \leq c_3 (k+1)^{-1} \log^{-2-\epsilon}(k+1),$$

which follows from the mean value theorem. Hence

$$\begin{aligned}
\sum_{k=1}^{\infty} b_k \log^{-1-\epsilon}(k+1) = \lim_{N\to\infty} &\left\{ \left[\sum_{k=1}^{N} b_k\right] \log^{-1-\epsilon}(N+1) \right. \\
&\left. + \sum_{k=1}^{N-1} \left[\sum_{m=1}^{k} b_m\right] \left(\log^{-1-\epsilon}(N+1) - \log^{-1-\epsilon}(N+2)\right) \right\} \\
&\leq \lim_{N\to\infty} c_1 \log(N+1) \log^{-1-\epsilon}(N+1) \\
&+ c_1 c_3 \sum_{k=1}^{\infty} \left[\log(k+1)\right] (k+1)^{-1} \log^{-2-\epsilon}(k+1) < \infty.
\end{aligned}$$

⊠

The interest of Corollary 3.5 is that it shows how "close" $\mathcal{B}^p_\alpha$ zeros sets are to the Blaschke sequences, which satisfy $\sum_{k\geq 1} b_k < \infty$.

3.3 Distinguishing zero sets of functions in $\mathcal{B}^p_\alpha$

In this section we show that for $p \neq q$ or $\alpha \neq \gamma$, $\mathcal{B}^p_\alpha$ zero sets are distinct from $\mathcal{B}^q_\gamma$ zero sets. the result contrasts with the case of H^p zero sets, which are the same for all p. Our proof is constructive. We consider functions of the form

$$f(z) = \prod_{k=0}^{\infty} \left(1 + \mu z^{\beta^k}\right), \tag{14}$$

where μ is an arbitrary positive number, and $\beta \geq 2$ is an integer. We begin by developing properties of the product in (14) which allow us to apply Theorems 3.1, 3.2 and 3.4 to our function $f(z)$.

Since $\sum_{k\geq 0}\mu z^{\beta k}$ converge absolutely and uniformly on compact subsets of the disk, f is well defined, analytic for $|z|<1$, and has zeros only at the zeros of the factors in the defining product.

Let

$$N_s=\sum_{k=0}^{s-1}\beta^k;\quad p_s(z)=\prod_{k=0}^{s-1}\left(1+\mu z^{\beta^k}\right). \tag{15}$$

Thus p_s is a polynomial of degree N_s, which is less than β^s, since $\beta\geq 2$. We have

$$p_{s+1}(z)=p_s\left(1+\mu z^{\beta^s}\right)=p_s(z)+\text{terms of higher degree}.$$

Thus every partial product for f is a partial sum of its Taylor series.

Proposition 3.6 *Let $\mu>1$, so that $f(z)$ has zeros in the disk. Let $\{z_k\}$ be the ordered zeros of such an f. Then there exists positive constants c_1 and c_2 independent of $N\geq 1$ such that*

$$c_1N^\eta\leq\prod_{k=1}^{N}\frac{1}{|z_k|}\leq c_2N^\eta,\quad \textit{where}\quad \eta=\frac{\log\mu}{\log\beta} \tag{16}$$

PROOF. With N_s as in (15), it is easily to see that $z_1,\cdots,z_{N_s}$ are precisely the zeros of $p_s(z)$. But p_s is a polynomial with constant term 1 and leading coefficient μ^s. Thus

$$\prod_{k=1}^{N_s}\frac{1}{|z_k|}=\mu^s.$$

Since $N_s<\beta^s<N_{s+1}$, we have

$$\mu^s\leq\prod_{k=1}^{\beta^s}\frac{1}{|z_k|}\leq\mu^{s+1},$$

for all integers $s\geq 0$. Now, if $\beta^s\leq N<\beta^{s+1}$, and if $\eta=\log\mu/\log\beta$, then

$$\frac{1}{\mu}N^\eta<\mu^s\leq\prod_{k=1}^{\beta^s}\frac{1}{|z_k|}\leq\prod_{k=1}^{N}\frac{1}{|z_k|}\leq\prod_{k=1}^{\beta^s+1}\frac{1}{|z_k|}\leq\mu^{s+2}\leq\eta^2N^\eta.$$

which proves (16) with $c_1=1/\mu$ and $c_2=\mu^2$. ⊠

Proposition 3.7 *Let f be as in* (14)*, we let $f(z)=\sum_{k\geq 0}a_kz^k$, and we define*

$$S_N^{(q)}=\sum_{k=0}^{N}|a_k|^q\quad(0<q<\infty),$$

as in section 3.1. Then we have that

$$S_{N_r}^{(q)}=(1+\mu^q)^r\quad(0<q<\infty,\ r>0\ \textit{an integer}), \tag{17}$$

where N_r is as in (15)*.*

PROOF. This is trivial for $r = 1$. Assume it has been proved for some $r \geq 1$; i.e., assume that

$$\sum_{k=0}^{N_r} |a_k|^q = (1+\mu^q)^r .$$

Now

$$\begin{aligned} p_{r+1}(z) &= \left(\sum_{k=0}^{N_r} a_k z^k\right)(1+\mu z^{\beta^r}) \\ &= \sum_{k=0}^{N_r} a_k z^k + \sum_{k=0}^{N_r} a_k \mu z^{(k+\beta^r)}. \end{aligned}$$

Thus

$$\begin{aligned} S^{(q)}_{N_{r+1}} &= \sum_{k=0}^{N_r} |a_k|^q + \sum_{k=0}^{N_r} |a_k \mu|^q \\ &= \sum_{k=0}^{N_r} |a_k|^q (1+\mu^q) = (1+\mu^q)^{r+1}, \end{aligned}$$

which proves (17) by induction. ⊠

Since N_r is approximately β^r, (17) says that $S^{(q)}_N$ is approximately N^s, where $\beta^s = (1+\mu^q)$. This can be made precise with arguments just like those used in proving (16). Thus, we see that there are positive constants c_3 and c_4, independent of N, such that

$$c_3 N^s \leq S^{(q)}_N \leq c_4 N^s, \tag{18}$$

where $\beta^s = 1+\mu^q$, or $s = (\log(1+\mu^q))/\log\beta$.

We have now laid the groundwork to obtain conditions over α, p, γ and q to distinguish the zero sets of these spaces $\mathcal{B}^p_\alpha$ and $\mathcal{B}^q_\gamma$.

Theorem 3.8 *Let $q \leq p$ and $\alpha - 1/p > \gamma - 1/q$. Then there exists an $\mathcal{B}^p_\alpha$ zero set which is not an $\mathcal{B}^q_\gamma$ zero set.*

PROOF. Consider functions of the form

$$f(z) = \prod_{k=1}^{\infty} (1+\mu z^{\beta k}),$$

as defined in (14).

[*Case 1:* $p < 2$] Choose μ and β such that

$$\gamma - \frac{1}{q} < \frac{\log\mu}{\log\beta} < \frac{\log(1+\mu^2)}{2\log\beta} < \alpha - \frac{1}{p},$$

Then (16) and Theorem 3.4 shows that the zero set of f is not an $\mathcal{B}^q_\gamma$ zero set. On the other hand, (18) and Theorem 3.1 shows that $f \in \mathcal{B}^p_\alpha$.

[*Case 1:* $p \geq 2$] Choose μ and β such that

$$\gamma - \frac{1}{q} < \frac{\log \mu}{\log \beta} < \frac{\log(1+\mu^{p'})}{p' \log \beta} < \alpha - \frac{1}{p},$$

where $1/p + 1/p' = 1$. As in the above case, the zero set of f is not an $\mathcal{B}^q_\gamma$ zero set. On the other hand,

$$\frac{\log(1+\mu^{p'})}{\log \beta} < (\alpha - 1/p)p' = (\alpha p - 1)(p' - 1),$$

and so, (16) together with Theorem 3.2 shows that $f \in \mathcal{B}^p_\alpha$. ⊠

Theorem 3.9 *Let $q \leq p$, $p \geq 2$ and $\alpha - 1/p = \gamma - 1/q$. Then there exists an $\mathcal{B}^p_\alpha$ zero set which is not an $\mathcal{B}^q_\gamma$ zero set.*

PROOF. Consider functions $f(z)$ as defined in (14).

Choose μ and β such that

$$\gamma - \frac{1}{q} < \frac{\log \mu}{\log \beta} < \frac{\log(1+\mu^{p'})}{p' \log \beta} < \alpha.$$

Then (16) and Theorem 3.4 shows that the zero set of f is not an $\mathcal{B}^q_\gamma$ zero set. On the other hand, since

$$\frac{\log(1+\mu^{p'})}{\log \beta} < \alpha p' = (\alpha p + 1)(p' - 1) - (p' - 1),$$

where $1/p + 1/p' = 1$. (16) together with Theorem 3.2 shows that $f \in \mathcal{B}^p_\alpha$. ⊠

In case $p < 2$ and $\alpha - 1/p = \beta - 1/q$, Theorem 3.4 does not distinguish between $\mathcal{B}^p_\alpha$ and $\mathcal{B}^q_\gamma$ zero sets.

3.4 Union of zero sets of functions in $\mathcal{B}^p_\alpha$

Theorem 3.10 *For all $0 < p < \infty$ and for all $1/p < \alpha < \infty$, there exists two $\mathcal{B}^p_\alpha$ zero sets whose union is not a $\mathcal{B}^p_\alpha$ zero set.*

PROOF. Choose $f \in \mathcal{B}^p_\alpha$, $f \neq 0$, whose ordered zeros satisfied

$$\prod_{k=1}^{N} \frac{1}{|z_k|} \geq CN^s, \quad \text{where} \quad \alpha - \frac{1}{p} > s > (\alpha - \frac{1}{p})\frac{1}{2}.$$

To see that it is possible to choose such a function f, we can proceed with the same arguments as in the proof of Theorem 3.8.

Now, Choose an angle θ such that the $\mathcal{B}^p_\alpha$ zero set $\left\{e^{i\theta} z_k\right\}$ is disjoint from $\{z_k\}$. Let

$$\left\{e^{i\theta} z_k\right\} \cup \{z_k\} = \{w_k\}, \quad 0 < |w_k| \leq |w_{k+1}| \leq \cdots$$

It follows that

$$\prod_{k=1}^{2N} \frac{1}{|w_k|} = \left(\prod_{k=1}^{N} \frac{1}{|z_k|}\right)^2 \geq (CN^s)^2 = C^2 N^{2s}.$$

But $2s > \alpha - 1/p$ and so, by Theorem 3.4 $\{w_k\}$ is not a $\mathcal{B}^p_\alpha$ zero set. ⊠

3.5 Subsets of zero sets of functions in $\mathcal{B}^p_\alpha$

Our main object of interest in this section is the question: is every subset of a $\mathcal{B}^p_\alpha$ zero set a $\mathcal{B}^p_\alpha$ zero set? We will give an affirmative answer in a particularly strong form.

For $|a| < 1$ let

$$C_a(z) = \frac{a - z}{1 - \overline{a}z}, \tag{19}$$

$$\nu_0(z) = z, \quad \text{and for } a \neq 0, \quad \nu_a(z) = \frac{|a|}{a} C_a(z) = \frac{|a|}{a} \frac{a - z}{1 - \overline{a}z}. \tag{20}$$

Let $f \in \mathcal{B}^p_\alpha$ and let z_k be an arbitrary subset of its zeros. Let

$$h(z) = \prod_{k=1}^{\infty} \nu_{z_k}(z) \left(2 - \nu_{z_k}(z)\right). \tag{21}$$

We will see that h is analytic in the unit disk with zeros $\{z_k\}$ and that the function

$$g(z) = \frac{f(z)}{h(z)} \tag{22}$$

lies in $\mathcal{B}^p_\alpha$.

Lemma 3.11 *Let $\{z_k\}$ be an arbitrary subset of an $\mathcal{B}^p_\alpha$ zero set ($0 < p < \infty$, $1/p < \alpha < \infty$). Then (21) defines a function $h(z)$ analytic in the unit disk with zero set $\{z_k\}$. Moreover, the value of the product defining $h(z)$ in independent of the order of the factors.*

PROOF. To start, notice that all the assertions of the lemma follows from the absolute convergence, uniform in compact subsets of the disk, of the product

$$\prod_{k=1}^{\infty} \nu_{z_k}(z)(2 - \nu_{z_k}(z)).$$

To proceed, we will apply the usual test for absolute convergence of a product, it is Theorem 1.6. We will assume without loss of generality that $0 \notin \{z_k\}$. Then, we have that

$$\begin{aligned} \sum_{k=1}^{\infty} |1 - \nu_{z_k}(z)(2 - \nu_{z_k}(z))| &= \sum_{k=1}^{\infty} |1 - \nu_{z_k}(z)|^2 \\ &= \sum_{k=1}^{\infty} \left| \frac{(1 - |z_k|)(z_k + z|z_k|)}{z_k(1 - \overline{z}_k z)} \right|^2. \end{aligned} \tag{23}$$

Since it follows immediately from Corollary 3.5 that

$$\sum_{k=1}^{\infty} (1 - |z_k|)^2 < \infty,$$

we have that in the disk $|z| \leq r$, the last expression in (23) is majorized by

$$\sum_{k=1}^{\infty} (1 - |z_k|)^2 \left(\frac{2}{1 - r}\right)^2 < \infty,$$

and we are done. ⊠

The following useful will let us to state Jensen's formula in a different way.

Lemma 3.12 *Let f be an analytic function on $\mathbb{D}$ with $|f(0)| = 1$ and a_k, $k = 1, 2, \cdots$, be its zeros. Then,*

$$N(r) = \int_0^r \frac{n(s)}{s} ds = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta,$$

where $0 < r < 1$ and $n(r)$ is the cardinal of the set $\{a_k : |a_k| \le r,\ k = 1, 2, \cdots\}$.

PROOF. To start, by Jensen's formula we have

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta = \sum_{k=1}^{n(r)} \log \frac{r}{|a_k|},$$

so, it is equivalent to see that

$$N(r) = \sum_{k=1}^{n(r)} \log \frac{r}{|a_k|}.$$

Define the measure

$$\mu = \sum_{k=1}^{\infty} \delta_{a_k}$$

where

$$\delta_{a_k}(z) = \begin{cases} 1 & \text{if } z = a_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $n(s) = \mu(\overline{D}(0, s))$ and we conclude

$$\begin{aligned} N(r) &= \int_0^r \int_{|z| \le s} \frac{1}{s} d\mu(z) ds = \int_{|z| \le r} \int_{|z|}^r \frac{1}{s} ds d\mu(z) \\ &= \int_{|z| \le r} \log \frac{r}{|z|} d\mu(z) = \sum_{k=1}^{n(r)} \log \frac{r}{|a_k|}. \end{aligned}$$

⊠

From Lemma 3.11 we know that the function g defined in (22) is at least analytic in the unit disk. The following lemma will enable us to estimate $g(0)$.

Lemma 3.13 *Let $f \in \mathcal{B}^p_\alpha$ and $\{z_k\}$ be its zeros. Assume $f(0) \ne 0$. Then there exists a constant $M(p, \alpha)$ such that*

$$|f(0)| \left(\prod_{k=1}^{\infty} |z_k| (2 - |z_k|) \right)^{-1} \le M(p, \alpha) \|f\|_{\mathcal{B}^p_{\alpha+2/p}}. \tag{24}$$

Notice that since $f \in \mathcal{B}^p_\alpha$, we will have that $\|f\|_{\mathcal{B}^p_{\alpha+2/p}} < \infty$.

PROOF. Without loss of generality assume $f(0) = 1$. Let $n(r)$ be the cardinality of the set $\{z_k : |z_k| < r\}$. It follows directly from Theorem 3.4 that

$$n(r) = O\left(\frac{1}{1-r}\log\left(\frac{1}{1-r}\right)\right) \quad \text{as} \quad r \to 1^-.$$

Then, taking the logarithm of right-hand side in (24), we can integrate by parts as follows

$$\begin{aligned}\sum_{k=1}^{\infty} -\log\left[|z_k|(2-|z_k|)\right] &= \int_0^1 -\log\left[|z_k|(2-|z_k|)\right] dn(r)\\ &= \int_0^1 \frac{2-2r}{2r-r^2} n(r)dr\\ &= \int_0^1 \frac{2}{(2-r)^2} N(r)dr,\end{aligned}$$

where, by Jensen's formula,

$$N(r) = \int_0^r \frac{n(s)}{s} ds = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta. \tag{25}$$

Thus, we have

$$\sum_{k=1}^{\infty} -\log\left[|z_k|(2-|z_k|)\right] = \frac{1}{2\pi}\int_0^1 \frac{2}{(2-r)^2}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta dr. \tag{26}$$

Let

$$M_1(p,\alpha) = \exp\left\{-\int_0^1 \frac{2}{(2-r)^2}\left(\log(1-r^2)^\alpha\right) dr\right\},$$

by (26) we have that

$$\begin{aligned}&\sum_{k=1}^{\infty} -\log\left[|z_k|(2-|z_k|)\right]\\ &\quad = \log M_1(p,\alpha) + \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \frac{2}{(2-r^2)}\log\left(|f(re^{i\theta})|(1-r^2)^\alpha\right) drd\theta.\end{aligned}$$

We multiply this equation by p, exponentiate, and apply the arithmetic-geometric mean inequality with respect to unit measure

$$\frac{1}{2\pi}\frac{2}{(2-r)^2}drd\theta.$$

Thus,

$$\begin{aligned}&\prod_{k=1}^{\infty}\left[|z_k|\,(2-|z_k|)\right]^{-p}\\ &\quad \le M_1^p(p,\alpha)\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \frac{2}{(2-r)^2}|f(re^{i\theta})|^p(1-r^2)^{\alpha p}drd\theta\\ &\quad \le 4M_1^p(p,\alpha)\|f\|^p_{\alpha+2/p},\end{aligned}$$

where the last inequality follows from the fact that $1 \le (2-r)^2 \le 4$, for all $0 \le r \le 1$. ⊠

Lemma 3.14 *Let $|v| < 1$, then there exists a constant $k(\alpha)$ independent of v such that*

$$\int_{\mathbb{D}} \left(1 - \left|\frac{v-w}{1-\overline{v}w}\right|^2\right)^{\alpha p} \left|\frac{1-|v|^2}{(1-\overline{v}w)^2}\right|^2 \left(\frac{1-|w|^2}{1-|v|^2}\right)^{\alpha p} dA(w) \leq k(\alpha).$$

PROOF. Let $z(w) = (v-w)/(1-\overline{v}w)$. Then

$$\left|\frac{1-|v|^2}{(1-\overline{v}w)^2}\right|^2 dA(w) = dA(z),$$

and our integral becomes

$$\int_{\mathbb{D}} (1-|z|^2)^{\alpha p} \left(1 - \left|\frac{v-z}{1-\overline{v}z}\right|^2\right) \frac{1}{(1-|v|^2)^{\alpha p}} dA(z). \tag{27}$$

We want to see that (27) is bounded for $|v| < 1$. But it follows from the fact that the integrand is uniformly bounded for all $|z| < 1$, $|v| < 1$. Indeed,

$$\begin{aligned} 1 - \left|\frac{v-z}{1-\overline{v}z}\right|^2 &= 1 - \frac{|v|^2 + |z|^2 - 2\mathrm{Re}\overline{v}z}{1 + |v|^2|z|^2 - 2\mathrm{Re}\overline{v}z} \\ &= \frac{1 + |v|^2|z|^2 - |v|^2 - |z|^2}{|1-\overline{v}z|^2} \\ &= \frac{(1-|z|^2)(1-|v|^2)}{|1-\overline{v}z|^2}. \end{aligned}$$

Thus

$$\frac{1-|z|^2}{1-|v|^2}\left(1 - \left|\frac{v-z}{1-\overline{v}z}\right|^2\right) = \frac{(1-|z|^2)^2}{|1-\overline{v}z|^2} \leq \frac{(1-|z|^2)^2}{(1-|z|)^2} < 4,$$

for all $|z| < 1$ and $|v| < 1$. It follows that the integral in (27) is smaller than $k(\alpha) = 4^{\alpha p+1}$ for all $|v| < 1$. ⊠

We are now prepared to prove the main result of this section.

Theorem 3.15 *Let $f \in B^p_\alpha$ $(0 < p < \infty,\ 1/p < \alpha < \infty)$. Let $\{z_k\}$ be an arbitrary subset of the zero set of f. Define $g(z)$ by (19)-(22). Then there is a constant $C(p,\alpha)$ such that*

$$\|g\|_{\mathcal{B}^p_\alpha} \leq C(p,\alpha)\|f\|_{\mathcal{B}^p_\alpha}.$$

In particular, every subset of an $\mathcal{B}^p_\alpha$ zero set is a $\mathcal{B}^p_\alpha$ zero set.

PROOF. Let

$$C_w(z) = \frac{w-z}{1-\overline{w}z}, \quad |w| < 1.$$

Let $\{a_k\}$ be the complete zero set of f. Then $f_w(z) = 0$ if and only if $c_w(z) = a_k$ for some k, i.e., if and only if $z = c_w(a_k)$, since $c_w = c_w^{-1}$. Also, $f_w(0) = f(w)$. Assuming $f(w) \neq 0$, Lemma 3.13 yields that

$$|f(w)| \left\{\prod_{k=1}^{\infty} |c_w(a_k)|(2 - |c_w(a_k)|)\right\}^{-1} \leq M(p,\alpha)\|f_w\|_{\mathcal{B}^p_{\alpha+2/p}}. \tag{28}$$

But

$$|c_w(a_k)| = \left|\frac{w-a_k}{1-\overline{w}a_k}\right| = \left|\frac{a_k-w}{1-\overline{a}_k w}\right| = |c_{a_k}(w)|,$$

and since for all $k \geq 1$, $|w| < 1$, we have

$$|c_{a_k}(w)|(2-|c_{a_k}(w)|) < 1,$$

it follows that

$$\begin{aligned}
|g(w)| &= |f(w)| \left\{\prod_{k=1}^{\infty} |\nu_{z_k}(w)||2-\nu_{z_k}(w)|\right\}^{-1} \\
&\leq |f(w)| \left\{\prod_{k=1}^{\infty} |\nu_{z_k}(w)|(2-|\nu_{z_k}(w)|)\right\}^{-1} \\
&= |f(w)| \left\{\prod_{k=1}^{\infty} |C_{z_k}(w)|(2-|C_{z_k}(w)|)\right\}^{-1} \\
&\leq |f(w)| \left\{\prod_{k=1}^{\infty} |c_{a_k}(w)|(2-|c_{a_k}(w)|)\right\}^{-1}.
\end{aligned}$$

Applying (28) we conclude that

$$|g(w)| \leq M(p,\alpha)\|f_w\|_{\mathcal{B}^p_{\alpha+2/p}}.$$

by continuity, this inequality holds even at points where $f(w) = 0$. Then, to estimate $\|g\|_{\mathcal{B}^p_\alpha}$, we need only compute

$$\int_{\mathbb{D}} \|f_w\|_{\mathcal{B}^p_{\alpha+2/p}}(1-|w|^2)^{\alpha p-2}dA(w). \tag{29}$$

Before doing so, we apply change of variables to the integral

$$\|f_w\|_{\mathcal{B}^p_{\alpha+2/p}} = \int_{\mathbb{D}} |f(C_w(z))|^p(1-|z|^2)^{\alpha p}dA(z).$$

Let

$$v(z) = C_w(z) = \frac{w-z}{1-\overline{w}z},$$

then

$$z = \frac{w-v}{1-\overline{w}v}$$

and

$$dA(z) = \left|\frac{1-|w|^2}{(1-\overline{w}v)^2}\right|^2 dA(v),$$

and we obtain that

$$\|f_w\|_{\mathcal{B}^p_{\alpha+2/p}} = \int_{\mathbb{D}} |f(v)|^p \left(1-\left|\frac{w-v}{1-\overline{w}v}\right|^2\right)^{\alpha p} \left|\frac{1-|w|^2}{(1-\overline{w}v)^2}\right|^2 dA(v).$$

Now, since $f \in B^p_\alpha$ and using Lemma 3.14, we obtain that

$$\begin{aligned}&\int_{\mathbb{D}} \|f_w\|^p_{\mathcal{B}^p_{\alpha+2/p}} (1-|w|^2)^{\alpha p-2} dA(w)\\ &= \int_{\mathbb{D}} |f(v)|^p (1-|v|^2)^{\alpha p-2} \int_{\mathbb{D}} \left(1-\left|\frac{w-v}{1-\overline{w}v}\right|^2\right)^{\alpha p} \left|\frac{1-|w|^2}{(1-\overline{w}v)^2}\right|^2 \left(\frac{1-|w|^2}{1-|v|^2}\right)^{\alpha p-2} dA(w)dA(v)\\ &< \infty,\end{aligned}$$

and we are done. ⊠

3.6 Random zero sets for Bergman Spaces

In this section we will see the result 3.17 which concerns a probabilistic characterization of zero sets of functions in the usual Bergman space $A^p = B^p_{2/p}$, due to Gregory Bomash.

For the definition of a random set, we will use the probability space $\Omega = \prod_{n=1}^{\infty} \Omega_n$, where Ω_n is the interval $[0, 2\pi)$ for each n. A_n is the σ-field of Lebesgue measurable sets and P_n is the (normalized) Lebesgue measure. An element of Ω is denoted by $w = (\theta_1,\ \theta_2, \cdots)$ where $0 \le \theta_n < 2\pi$ for all n. $\{\theta_1, \theta_2, \cdots\}$ is a sequence of random independent variables defined on Ω.

For every countable set $\Lambda = \{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{D}$ define a random set Λ_w as a map $\Omega \to 2^{\mathbb{D}}$, where for every $w \in \Omega$ the set Λ_w is obtained by a random rotation of each point $\lambda_n \in \Lambda$ through the angle θ_n:

$$\Lambda_w = \left\{\lambda_n e^{i\theta_n}\right\}_{n=1}^{\infty}. \tag{30}$$

This probabilistic approach was apparently initiated by Emilie Leblanc, who obtained the following result.

Theorem 3.16 *Let $\{r_n\}_{n=1}^{\infty}$ be a sequence in $(0,1)$ that satisfies the condition,*

$$\limsup_{\varepsilon\to 0} \frac{\sum_n (1-r_n)^{1+\varepsilon}}{\log 1/\varepsilon} < \frac{1}{2p}.$$

Then for almost all independent choices of $\{\theta_n\}_{n=1}^{\infty}$ the set $\left\{r_n e^{i\theta_n}\right\}_{n=1}^{\infty}$ is an A^p-zero set.

Following this idea of random zero sets of functions in A^p, Gregory Bomash obtained the following result, which is sharper than Leblanc's one.

Theorem 3.17 *Let $1 \le p \le 2$ and $\{r_n\}$ be a sequence in $(0,1)$ satisfying the condition*

$$\limsup_{\varepsilon\to 0} \frac{\sum_{r_n<1-\varepsilon}(1-r_n)}{\log 1/\varepsilon} < \frac{1}{p}.$$

Then for almost all independent choices of $\{\theta_n\}_{n=1}^{\infty}$ the set $\left\{r_n e^{i\theta}\right\}_{n=1}^{\infty}$ is an A^p-zero set. Moreover, the constant p is sharp.

In order to obtain the result 3.17, we denote by $\Lambda_r := \{\lambda \in \Lambda : |\lambda| \leq r\}$ the intersection of Λ with the disk of radius $0 < r < 1$ centered at the origin and define the following functions,

$$\begin{aligned}\varphi(r) &= \sum_{\lambda\in\Lambda_r}(1-|\lambda|),\\ \varphi_1(r) &= \sum_{\lambda\in\Lambda_r}\log\frac{r}{|\lambda|},\\ \varphi_2(r) &= \sum_{\lambda\in\Lambda_r}(1-|\lambda|^2),\\ n(r) &= \mathrm{card}\Lambda_r.\end{aligned} \tag{31}$$

From now on, we will restrict our considerations to the sets Λ which satisfy the condition 3.5, that is,

$$\sum_{\lambda_k\in\Lambda}(1-|\lambda_k|)\left(\log\frac{1}{1-|\lambda_k|}\right)^{-1-\varepsilon} < \infty, \quad \text{for all } \varepsilon > 0. \tag{32}$$

The following technical result will be useful.

Lemma 3.18 *Let $\Lambda \subset \mathbb{D}$ be a discrete set satisfying the condition* (32). *Then*

$$2\varphi(r) - \varphi_2(r) = O(1) \quad \textit{as} \quad r \to 1, \tag{33}$$

$$\varphi_1(r) + (1-r)n(r) - \varphi(r) = O(1) \quad \textit{as} \quad r \to 1 \tag{34}$$

$$n(r) = \int_0^r \frac{d\varphi(t)}{1-t}, \;\; r \in (0,1) \quad \textit{and} \quad n(r) \leq \frac{\varphi(r)}{1-r}. \tag{35}$$

PROOF. (33) and (35) are direct consequences of the definition (31). The expression in (34) is equal to

$$\sum_{\lambda\in\Lambda_r}(\log(1/|\lambda|) - 1 + |\lambda|) + n(r)(\log r - 1 + r). \tag{36}$$

The first term is bounded by the finite sum $\sum(1-|\lambda|)^2$. The second term is $O((1-r)^2 n(r))$ and hence $O((1-r)\varphi(r))$. The function $\varphi(r)$ admits the following estimate

$$\varphi(r) \leq \log^{1+\varepsilon}\frac{1}{1-r}\sum_{\lambda\in\Lambda_r}(1-|\lambda|)\log^{-1-\varepsilon}\left(\frac{1}{1-|\lambda|}\right).$$

We conclude that $(1-r)\varphi(r) = O(1)$ as $r \to 1$, and (34) is thus proved. ⊠

We need to construct a Blaschke-type product. For every $\lambda \in \mathbb{D}$ and $s \geq 1$ define

$$b_\lambda^{(s)}(z) = 1 - \frac{(1-|\lambda|^2)^s}{(1-\overline{\lambda}z)^s}. \tag{37}$$

when $s = 1$, this is equal to

$$b_\lambda^{(1)}(z) = B_\lambda(0)B_\lambda(z),$$

where B_λ is the classical Blaschke factor.

For every set $\Lambda \subset \mathbb{D}$ and every function $s = s(\lambda)$ we can define an infinite product

$$b_\Lambda^{(s)} = \prod_{\lambda\in\Lambda} b_\lambda^{(s(\lambda))}. \tag{38}$$

Suppose that the function s satisfies

$$\sum_{\lambda\in\Lambda}(1-|\lambda|)^{s(\lambda)} < \infty. \tag{39}$$

Then the product (38) represents a function holomorphic in $\mathbb{D}$ whose zeros are precisely on Λ. We will use these Blaschke-type products to prove the Teorem 3.22, which is somewhat more general than Theorem 3.17.

Lemma 3.19 *Let $0 < p \le 2$, $\lambda \in \mathbb{D}$ and $s \ge 1$. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} |b_\lambda^{(s)}(re^{i\theta})|^p d\theta \le \left(1 + \frac{\Gamma(2s-1)}{\Gamma^2(s)}\frac{(1-|\lambda|)^{2s}}{(1-|\lambda|^2r^2)^{2s-1}}\right)^{p/2}, \tag{40}$$

where Γ is the well-known gamma function.

PROOF. The function $b_\lambda^{(s)}$ defined by (37) has the following Taylor expansion:

$$b_\lambda^{(s)}(z) = 1 - (1-|\lambda|)^s \sum_{n=0}^{\infty} \frac{\Gamma(n+s)}{n!\Gamma(s)}(\bar{\lambda}z)^n. \tag{41}$$

Using this expansion we can easily compute the integral (40) when $p = 2$:

$$\frac{1}{2\pi}\int_0^{2\pi} |b_\lambda^{(s)}(re^{i\theta})|^2 d\theta = (1-(1-|\lambda|)^s)^2 + \sum_{n=1}^{\infty}(1-|\lambda|)^{2s}\left(\frac{\Gamma(n+s)}{n!\Gamma(s)}\right)^2 |\lambda r|^{2n}.$$

Since $\Gamma^2(n+s) \le \Gamma(n+1)\Gamma(n+2s-1)$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} |b_\lambda^{(s)}(re^{i\theta})|^2 d\theta \le 1 + \frac{\Gamma(2s-1)}{(\Gamma(s))^2}\frac{(1-|\lambda|)^{2s}}{(1-|\lambda|^2r^2)^{2s-1}},$$

which completes the proof in the case $p = 2$. For any $0 < p < 2$ one has to use the inequality $\|f\|_{H^p} \le \|f\|_{H^2}$. ⊠

Proposition 3.20 *The function*

$$h(s) = \Gamma(2s-1)\Gamma^2(s)$$

has the following properties :

(a) $h(1) = 1$; *(b)* $h'(0) = 0$; *(c)* $h''(1) = \dfrac{\pi^2}{3}$; *(d)* $h(s) \le 1 + 2(s-1)^2$ *for* $1 \le s \le 2$.

Proposition 3.21 *Let $0 < r < 1$, $\varepsilon > 0$ and r_1 such that*

$$1 - r_1 = (1 - r)\log^{-1-\varepsilon}\frac{1}{1-r}. \tag{42}$$

Then for every $|\lambda| \geq r_1$ and $s \geq 1$

$$(1 - |\lambda|)^{2s}(1 - |\lambda|^2 r^2)^{1-2s} \leq c d_\lambda \log^{-1-\varepsilon}(1/d_\lambda) \tag{43}$$

with some constant c independent of λ and r.

PROOF. Condition (42) and $|\lambda| \geq r_1$ imply that

$$(1 - |\lambda|) \leq 2(1 - r_1) = 2(1 - r)\log^{-1-\varepsilon}\frac{1}{1-r}.$$

The last inequality is equivalent to

$$\frac{1 - |\lambda|}{1 - r} \leq c_1 \log^{-1-\varepsilon}\frac{1}{1 - |\lambda|} \tag{44}$$

with some $c_1 > 0$.

We can also deduce from (42) that for $|\lambda| > r_1$

$$(1 - |\lambda|^2 r^2)^{-1} \leq c_2(1 - r)^{-1}. \tag{45}$$

Combining (44) with (45) we obtain

$$\frac{1 - |\lambda|}{1 - |\lambda|^2 r^2} \leq c \log^{-1-\varepsilon}\frac{1}{1 - |\lambda|}.$$

Now (43) follows from this and the condition $2s - 1 \geq 1$. ⊠

Now we are ready to state and prove the main result of this section.

Theorem 3.22 *Let $1 \leq p \leq 2$ and $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ be a discrete subset of the unit disk $\mathbb{D}$ that satisfies the condition*

$$\int_0^1 e^{p\varphi(r)} \log^\sigma \frac{1}{1-r} dr < \infty \tag{46}$$

for some $\sigma > 1$. Then for almost all independent choices of $\{\theta_n\}_{n=1}^{\infty}$ the set $\Lambda_w = \left\{\lambda_n e^{i\theta_n}\right\}$ is a zero set of a function in A^p.

PROOF. Consider the Banach space $L^p(\Omega, A^p)$ of all A^p-valued measurable functions on Ω with the norm

$$\|f\|_{\Omega,p} = \left(\int_\Omega \|f(w)\|_{A^p}^p dw\right)^{1/p}. \tag{47}$$

Let Λ be a subset of $\mathbb{D}$ that satisfies the condition (46), and Λ_w be the random set defined by (30). Our aim is to construct a sequence $s = \{s_n\}$ so that the product

$$B_{\Lambda_w}^{(s)}(z) = \prod_{n\geq 1}\left(1 - \left(\frac{1 - |\lambda_n|}{1 - \overline{\lambda}_n e^{i\theta_n} z}\right)^{s_n}\right) \tag{48}$$

converges to a holomorphic function in $\mathbb{D}$, which belongs to the space $L^p(\Omega, A^p)$. When this is done, the conclusion of the theorem will follow because the finitiness of the norm (47) for the product (48) implies that for almost all $w \in \Omega$ the function $B_{\Lambda_w}^{(s)}$ belongs to A^p. Hence for these $w's$ the set Λ_w is a zero set of a function in A^p.

Define a function g on $\mathbb{D}$ by

$$g(z) = \int_{\Omega} |B^{(s)}_{\Lambda_w}(z)|^p dw. \tag{49}$$

We can apply Fubini's theorem to obtain

$$\|B^{(s)}_{\Lambda_w}\|^p_{\Omega,p} = \int_{\mathbb{D}} g(z) dA(z). \tag{50}$$

Our goal is to established that $g \in L^1(\mathbb{D})$. To do that, fix a positive $\varepsilon < \sigma - 1$. For every $\lambda_n \in \Lambda$ we choose s_n as the root of the equation

$$(1 - |\lambda|)^{s_n} = (1 - |\lambda_n|) \log^{-1-\varepsilon} \frac{1}{1 - |\lambda|}, \tag{51}$$

that is,

$$s_n = 1 + (1 + \varepsilon) \frac{\log \log \dfrac{1}{1 - |\lambda_n|}}{\log \dfrac{1}{1 - |\lambda_n|}}. \tag{52}$$

First, note that condition (46) implies the convergence of the series

$$\sum_{\lambda \in \Lambda} (1 - |\lambda|) \log^{-1-\varepsilon} \frac{1}{1 - |\lambda|} < \infty, \tag{53}$$

for every positive ε. Indeed, this sum is equal to

$$\int_0^1 \log^{-1-\varepsilon} \frac{1}{1 - r} d\varphi(r).$$

Integration by parts and an application of Hölder's inequality then lead to (53).

Combining (53) and (51) we see that the product (48) converges for every $w \in \Omega$.

The independence of $\{\theta_n\}_{n=1}^{\infty}$ implies

$$\int_{\Omega} |B^{(s)}_{\Lambda_w}(z)|^p dw = \prod_{n=1}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} |b^{(s_n)}_{\lambda_n e^{i\theta_n}}(z)|^p d\theta_n.$$

Lemma 3.19 and the inequality $1 + x \leq \exp(x)$ result in the following estimate for the function g in (49):

$$g(z) \leq \exp \left\{ \frac{p}{2} \sum_{n=1}^{\infty} h(s_n)(1 - |\lambda_n|)^{2s_n} (1 - |\lambda_n|^2 r^2)^{1-2s_n} \right\}. \tag{54}$$

For a fixed $r = |z|$ we split the sum in (54) in two parts: the sum over $|\lambda| \geq r_1$ and the sum over $|\lambda| < r_1$. Let r_1 be defined as in Proposition 3.21. The sum over $|\lambda| \geq r_1$ is bounded by the finite sum

$$c \sum_{|\lambda| \geq r_1} (1 - |\lambda|) \log^{-1-\varepsilon} \frac{1}{1 - |\lambda|} \leq \text{ const.} \tag{55}$$

(Here we used (43) and (53)). For the sum over $|\lambda| < r_1$ (i.e. $\lambda \in \Lambda_{r_1}$) we have

$$\begin{aligned} \sum_{\lambda\in\Lambda_{r_1}} h(s_n)(1-|\lambda_n|) &= \sum_{\lambda\in\Lambda_{r_1}} (1-|\lambda_n|) + \sum_{\lambda\in\Lambda_{r_1}} (h(s_n)-1)(1-|\lambda_n|) \\ &\le \varphi_2(r_1) + 2\sum_{\lambda\in\Lambda_{r_1}} (s_n-1)^2(1-|\lambda_n|) \\ &\le 2\varphi(r_1) + c, \end{aligned} \tag{56}$$

where the first inequality follows from Proposition 3.20 and the last one follows from (33), (52) and (53). Combining (55) and (56) we obtain

$$g(z) \le c\exp(p\varphi(r_1)),$$

where r_1 depends on $r = |z|$ as in (42). Using (46) and the change of variable $r_1 = r_1(r)$ we obtain

$$\int_{\mathbb{D}} g(z)dA(z) \le c\int_0^1 \exp(p\varphi(r_1))dr \le c\int_0^1 \exp(p\varphi(r))\log^\sigma \frac{1}{1-r}dr < \infty.$$

Hence $B^{(s)}_{\Lambda_w}$ belongs to the space $L^p(\Omega, A^p)$ and the proof of Theorem (3.22) is complete. ⊠

Corollary 3.23 *(see Theorem 3.17.) If $1 \le p \le 2$ and*

$$\limsup_{\varepsilon\to 0} \frac{\sum_{r_k<1-\varepsilon}(1-r_k)}{\log 1/\varepsilon} < 1/p,$$

then for almost all $w \in \Omega$ the set $\left\{r_n e^{i\theta}\right\}_{n=1}^{\infty}$ is a zero set of a function if A^p.

The condition in Theorem 3.22 is far from being *necessary.* To see it, let Λ be an A^p-zero set, i.e. there exists a nonzero function $f \in A^p$ with $f(a) = 0$ for all $a \in A$. without loss of generality we can assume $f(0) \ne 0$. Recall from Lemma 9 that

$$|f(0)|^p \exp(p\varphi_1(r)) = \exp\left(\frac{1}{2\pi}\int_0^{2r} \log|f(re^{i\theta})|^p d\theta\right) \le \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta =: M_p^p(f,r). \tag{57}$$

(for the definition of φ_1 see (31)). For all functions $f \in A^p$ we have

$$\begin{aligned} 2\int_0^1 M_p^p(f,r)rdr &= \|f\|^p_{A^p} < \infty, \\ M_p^p(f,r) &\le \frac{\|f\|^p_{A^p}}{1-r}, \\ M_p^p(f,r) &= o((1-r)^{-1}). \end{aligned}$$

Thus, inequality (57) implies the two (generally speaking, not equivalent) necessary conditions for A^p-zero sets:

(1) The L^p-type condition

$$\int_0^1 \exp(p\varphi_1(r))dr < \infty. \tag{58}$$

(2) The L^∞-type condition

$$\exp(p\varphi_1(r)) = o((1-r)^{-1}). \tag{59}$$

Lemma (3.18) shows that if the set Λ satisfies the following growth condition for the function $n(r) = \operatorname{card}(\Lambda_r)$,

$$n(r) = O\left(\frac{1}{1-r}\right),$$

then the function φ_1 in (58) and (59) can be replaced by the function φ. In this case the necessary condition (58) differs from the probabilistic condition (46) in Theorem 3.22 by a logarithmic factor. In particular we see that the constant $1/p$ in Theorem 3.17 is sharp, i.e., it cannot be replaced by any larger one.

Chapter 4

Notions of density

In this chapter we will introduce several notions of density in order to understand the structure of the zero set of functions in $\mathcal{B}^p_\alpha$. The reason is that, as we have seen in the above chapter, the zero sets of function in the space $\mathcal{B}^p_\alpha$ can not be captured by a simple Blaschke-type condition in terms of the moduli: indeed, a spread-out zero set need not fulfill the Blaschke condition, whereas a concentrated one must do so − if, say, all the zeros contained in a finite union of Stolz angles, which is a consequence of Lemma 2.3.

4.1 Notions of density for sequences in the unit disk

For a point $z \in \mathbb{T}$, we let $\mathfrak{s}_z$ be the convex hull of the set

$$\{z\} \cup \left\{ w \in \mathbb{C} : \ |w| \leq 1/\sqrt{2} \right\},$$

with the vertex point z removed. $\mathfrak{s}_z$ is known as the standard relative closed Stolz angle in $\mathbb{D}$ with vertex at z and aperture $\pi/2$.

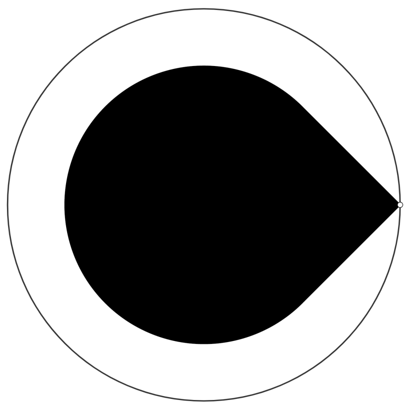

Figure 4.1: Representation of the Sotlz angle with vertex at $z = 1$ and aperture $\pi/2$.

For an arc $I \subset \mathbb{T}$, let $|I|$ be its arc length, and $|I|_s = |I|/(2\pi)$ its normalized arc length. The subscript s refers to the measure $ds(z) = |dz|/(2\pi)$. For a closed and proper subset F of $\mathbb{T}$ with complementary arcs $\{I_n\}_n$, we define

$$\hat{\kappa}(F) = \sum_n |I_n|_s \log \frac{e}{|I_s|_s},$$

where $e = 2.71828\cdots$ is the base for the natural logarithm. The quantity $\hat{\kappa}(F)$ will be called the entropy of F. We define $\hat{\kappa}(\emptyset) = 0$ for the empty set.

A closed subset F of $\mathbb{T}$ is called a Beurling-Carleson set if F is nonempty, has Lebesgue length measure zero, and has $\hat{\kappa}(F) < +\infty$ finite entropy. It is clear that $1 \leq \hat{\kappa}(F)$ for such sets, with equality only for one-point sets F.

Let $d_{\mathbb{T}}$ be the standard metric on the unit circle $\mathbb{T}$:

$$d_{\mathbb{T}}(z, w) = \left| \arg\left(\frac{z}{w}\right) \right|,$$

where the argument function is assume to take values in the interval $(-\pi, \pi]$. The distance to a closed subset F on $\mathbb{T}$ is then

$$d_{\mathbb{T}}(z, F) = \inf \{d_{\mathbb{T}}(z, w) : \ w \in F\},$$

and F is a Beurling-Carleson set if and only if

$$\hat{\kappa}(F) = \int_{\mathbb{T}} \log \frac{\pi}{d_{\mathbb{T}}(z, F)} ds(z) < +\infty.$$

Notice that if $d_{\mathbb{C}}$ stand for the Euclidean metric in $\mathbb{D}$ (i.e., $d_{\mathbb{C}}(z, w) = |z - w|$), then, for any closed subset F of $\mathbb{T}$,

$$\frac{2}{\pi} d_{\mathbb{T}}(z, F) \leq d_{\mathbb{C}}(z, F) \leq d_{\mathbb{T}}(z, F), \quad z \in \mathbb{T},$$

where the distance to sets is defined in terms of an infimum as for $d_{\mathbb{T}}$, so that by the above,

$$\hat{\kappa}(F) - \log \pi \leq \int_{\mathbb{T}} \log \frac{1}{d_{\mathbb{C}}(z, F)} ds(z) \leq \hat{\kappa}(F) - \log 2, \tag{1}$$

provided F has zero length.

For most of our discussion, we assume that F is a finite set. In association with F, we define the Stolz star domain $\mathfrak{s}_F$ as

$$\mathfrak{s}_F = \bigcup_{z \in F} \mathfrak{s}_z. \tag{2}$$

Let $A = \{a_n\}_{n=1}^{\infty}$ be a sequence of (not necessary distinct) points from $\mathbb{D}$. For an arbitrary subset E of $\mathbb{D}$, we form the partial Blaschke sum

$$\Sigma(A, E) = \frac{1}{2} \sum_{\substack{n=1 \\ a_n \in E}}^{\infty} 1 - |a_n|^2. \tag{3}$$

We note that for points $a \in \mathbb{D}$ close to $\mathbb{T}$, the quantities $(1 - |a|^2)/2$ and $1 - |a|$ are very close. Later on, we also need the related "logarithmic" sum

$$\Lambda(A, E) = \sum_{\substack{n=1 \\ a_n \in E}}^{\infty} \log \frac{1}{|a_n|}. \tag{4}$$

Again, for $a \in \mathbb{D}$ close to $\mathbb{T}$, the quantity $\log(1/|a|)$ is very close to $1 - |a|$.

We will sum over the Stolz stars $\mathfrak{s}_F$, where $F \subset \mathbb{T}$ is finite.

Definition 4.1 *Let A be a sequence of points in $\mathbb{D}$ and F be a finite subset of $\mathbb{T}$. Then the quantity*

$$D^+(A) = \limsup_{\hat{\kappa}(F)\to+\infty} \frac{\Sigma(A, \mathfrak{s}_F)}{\hat{\kappa}(F)}$$

is called the upper asymptotic κ-density of A.

4.2 Equivalent definitions of the upper asymptotic κ-density

We will start this section seeing another notion of density based on Carleson squares. For an open arc $I \subset \mathbb{T}$, with $|I| = 2\pi|I|_s < 1$, the associate Carleson square is the set

$$Q(I) = \{w \in \mathbb{D} \setminus \{0\} : \ 1 - |I| < |w|, \ w/|w| \in I\}\,;$$

for open arcs of bigger length, we let $Q(I)$ be the entire sector

$$Q(I) = \{w \in \mathbb{D} \setminus \{0\} : w/|w| \in I\}\,.$$

If $\{I_n\}_n$ are the complementary arcs of a finite set F in $\mathbb{T}$, we define

$$\mathfrak{q}_F = \mathbb{D} \setminus \cup_n Q(I_n).$$

We arrive to the following way of obtaining $D^+(A)$.

Proposition 4.2 *Let $A = \{a_n\}_n$ be any sequence of points in $\mathbb{D}$ and F be finite subsets of $\mathbb{T}$. Then*

$$D^+(A) = \limsup_{\hat{\kappa}(F)\to+\infty} \frac{\Sigma(A, \mathfrak{q}_F)}{\hat{\kappa}(F)}.$$

PROOF. Enlarge every finite set F by inserting on each complementary arc I of F additional points accumulating at the endpoints of I so that their distances from the nearest endpoint of I form a geometric progression with some fixed ratio q, $0 < q < 1$. So that, if we call F_1 the resulting enlarged set and if we let $0 < q < 1/2$, only to make easier the computations, we obtain that

$$\begin{aligned}
\hat{\kappa}(F_1) &= \sum_{n=1}^{\#F} \left\{ (1-2q)|I_n|_s \log \frac{e}{|I_n|_s(1-2q)} + 2|I_n|_s \sum_{k=1}^{\infty} (q^k - q^{k+1}) \log \frac{e}{|I_n|_s(q^k - q^{k+1})} \right\} \\
&= \sum_{n=1}^{\#F} \left\{ (1-2q)|I_n|_s \log \frac{e}{|I_n|_s} + (1-2q)|I_n|_s \log \frac{e}{1-2q} \right. \\
&\left. +2q(1-q)|I_n|_s \log \frac{e}{|I_n|_s} \sum_{k=0}^{\infty} q^k + 2(1-q)|I_n|_s \sum_{k=1}^{\infty} q^k \log \frac{e}{(q^k - q^{k+1})} \right\} \qquad (5) \\
&= \hat{\kappa}(F) + \left\{ (1-2q) \log \frac{e}{1-2q} + 2(1-q) \sum_{k=1}^{\infty} q^k \log \frac{e}{q^k(1-q)} \right\} \sum_{n=1}^{\#F} |I_n|_s \\
&= \hat{\kappa}(F) + \left\{ (1-2q) \log \frac{e}{1-2q} + 2(1-q) \sum_{k=1}^{\infty} q^k \log \frac{e}{q^k(1-q)} \right\},
\end{aligned}$$

where $\#F$ is the cardinal of F and I_n, for $n = 1, \cdots, \#F$, are the complementary arcs of F. It follows from (5) that there is a constant C depending only on the ratio q such that

$$\hat{\kappa}(F) < \hat{\kappa}(F_1) < \hat{\kappa}(F) + C.$$

We can also choose q such that

$$\mathfrak{q}_F \subset \mathfrak{s}_F \subset \mathfrak{q}_{F_1},$$

so that

$$\Sigma(A, \mathfrak{q}_F) \leq \Sigma(A, \mathfrak{s}_F) \leq \Sigma(A, \mathfrak{q}_{F_1}).$$

We are done. ⊠

Notice that replacing the Stolz angle $\mathfrak{s}_z$ by a general Stolz angle $\mathfrak{s}_{z,\alpha}$ with fixed aperture $0 < \alpha < \pi$ and making the corresponding changes in the definitions of $\mathfrak{s}_F$ and $\Sigma(A, \mathfrak{s}_F)$ will not alter the quantities $D^+(A)$. What is somewhat surprising is that the angle α can be reduced to 0 with no effect on $D^+(A)$. More specifically, for a finite set F and a sequence A of points in $\mathbb{D}$, we set

$$\mathfrak{r}_F = \{rz \in \mathbb{D} : \; 0 \leq r < 1, \; z \in F\}.$$

The set $\mathfrak{r}_F$ is the union of radii from 0 to the points of F. Then we have the following result.

Proposition 4.3 *Let $A = \{a_n\}_n$ be any sequence of points in $\mathbb{D}$ and F be finite subsets of $\mathbb{T}$. Then*

$$D^+(A) = \limsup_{\hat{\kappa}(F) \to +\infty} \frac{\Sigma(A, \mathfrak{r}_F)}{\hat{\kappa}(F)}.$$

At first glance this looks highly improbable, since the sum defining $\Sigma(A, \mathfrak{s}_F)$ involves all points from $\mathfrak{s}_F$, while the sum defining $\Sigma(A, \mathfrak{r}_F)$ involves only those points lying on one of the radii from 0 to points of F. However, a careful argument will prove the above proposition.

PROOF. Observe that $\mathfrak{r}_F \subset \mathfrak{s}_F$, and thus $\Sigma(A, \mathfrak{r}_F) \leq \Sigma(A, \mathfrak{s}_F)$, which implies

$$\limsup_{\hat{\kappa}(F) \to +\infty} \frac{\Sigma(A, \mathfrak{r}_F)}{\hat{\kappa}(F)} \leq \limsup_{\hat{\kappa}(F) \to +\infty} \frac{\Sigma(A, \mathfrak{s}_F)}{\hat{\kappa}(F)}.$$

By Proposition 4.2, the reverse inequality is equivalent to

$$\limsup_{\hat{\kappa}(F) \to +\infty} \frac{\Sigma(A, \mathfrak{q}_F)}{\hat{\kappa}(F)} \leq \limsup_{\hat{\kappa}(F) \to +\infty} \frac{\Sigma(A, \mathfrak{r}_F)}{\hat{\kappa}(F)}. \tag{6}$$

Let's see that (6) hold. Without loss of generality, we may assume that the lim sup on the left-hand side of the inequality (6) is positive. Let L be a positive number less than this lim sup. This implies that there are finite subsets F of $\mathbb{T}$ of arbitrarily large $\hat{\kappa}(F)$ such that

$$\Sigma(A, \mathfrak{q}_F) = \frac{1}{2} \sum_{a_k \in \mathfrak{q}_F} (1 - |a_k|^2) > L\hat{\kappa}(F).$$

Until the end of the proof, we will assume that F satisfies this inequality.

Let F_1 equal the set F plus the radial projections $z/|z|$ of points from the set $A \cap (\mathfrak{q}_F \setminus \mathfrak{r}_F)$, so that

$$\Sigma(A, \mathfrak{q}_F) \le \Sigma(A, \mathfrak{r}_{F_1}).$$

Let k_n be the number of such radial projections (counting multiplicities) that lie on I_n, where I_n is a complementary arc to the finite set $F \subset \mathbb{T}$. Observe now that the contribution to $\hat{\kappa}(F_1)$ from the complementary arcs of F_1 contained in I_n does not exceed the quantity

$$|I_n|_s \left[\log \frac{e}{|I_n|_s} + \log(k_n + 1)\right],$$

which corresponds to the case of k_n equidistant points of $(F_1 \setminus F) \cap I_n$. Therefore,

$$\hat{\kappa}(F) \le \hat{\kappa}(F_1) \le \hat{\kappa}(F) + r(F),$$

where $r(F)$ is the"remainder" term

$$r(F) = \sum_n |I_s|_s \log(k_n + 1).$$

Suppose the point $a_j \in A \cap (\mathfrak{q}_F \setminus \mathfrak{r}_F)$ is such that its radial projection lies on I_n. Then $|I_n| = 2\pi|I_n|_s < 1$ by the construction of the Carleson squares forming the complement of $\mathfrak{q}_F$ in $\mathbb{D}$, and moreover, we have $|a_j| \le 1 - |I_n|$. It follows that

$$|I_n|_s < \pi|I_n|_s = \frac{1}{2}|I_n| \le \frac{1}{2}(1 + |a_j|)(1 - |a_j|) = \frac{1}{2}(1 - |a_j|^2).$$

This leads to the conclusion

$$\sum_n k_n |I_n|_s \le \Sigma(A, \mathfrak{q}_F \setminus \mathfrak{r}_F) \le \Sigma(A, \mathfrak{q}_F).$$

We now show that the remainder term is small:

$$r(F) = o(\Sigma(A, \mathfrak{q}_F)) \quad \text{as} \quad \hat{\kappa}(F) \to +\infty.$$

To this end, we pick a positive integer N and split the sum defining $r(F)$ into two parts, keeping the above estimate in mind:

$$\begin{aligned} r(F) &= \left[\sum_{k_n \le N} + \sum_{k_n > N}\right] |I_n|_s \log(k_n + 1) \\ &\le \log(N+1) + \frac{\log(N+1)}{N} \sum_{k_n > N} k_n |I_n|_s \\ &\le \log(N+1) + \frac{\log(N+1)}{N} \Sigma(A, \mathfrak{q}_F). \end{aligned}$$

Letting $\hat{\kappa}(F) \to +\infty$, with $\Sigma(A, \mathfrak{q}_F) \to +\infty$, first holding N constant and then making $N \to +\infty$, we obtain $r(F) = o(\Sigma(A, \mathfrak{q}_F))$, as desired. Consequently,

$$\hat{\kappa}(F_1) = \hat{\kappa}(F) + o(\Sigma(A, \mathfrak{q}_F)) \quad \text{as} \quad \hat{\kappa}(F) \to \infty.$$

Since by the above, $\Sigma(A, \mathfrak{q}_F) \le \Sigma(A, \mathfrak{r}_{F_1})$, we get

$$\frac{\Sigma(A, \mathfrak{r}_{F_1})}{\hat{\kappa}(F_1)} \ge \frac{\Sigma(A, \mathfrak{q}_F)}{\hat{\kappa}(F) + o(\Sigma(A, \mathfrak{q}_F))}$$

as $\hat{\kappa}(F) \to +\infty$. This implies that

$$\limsup_{\hat{\kappa}(F)\to+\infty} \frac{\Sigma(A, \mathfrak{r}_{F_1})}{\hat{\kappa}(F_1)} \ge L.$$

Since F_1 above can be substituted for F, and L can be chosen arbitrarily close to

$$\limsup_{\hat{\kappa}(F)\to+\infty} \frac{\Sigma(A, \mathfrak{q}_F)}{\hat{\kappa}(F)},$$

the proposition has been proved. ⊠

Let $A = \{a_n\}_n$ be a sequence in $\mathbb{D}$, and fix a real parameter $\varrho \in (0, +\infty)$. If, for every finite subset F of $\mathbb{T}$,

$$\Sigma(A, \mathfrak{s}_F) \le \varrho\hat{\kappa}(F) + C,$$

for some constant C independent of F, then by the inclusion $\mathfrak{r}_F \subset \mathfrak{s}_F$, we also have

$$\Sigma(A, \mathfrak{r}_F) \le \varrho\hat{\kappa}(F) + C.$$

Conversely, if for every finite subset F of $\mathbb{T}$,

$$\Sigma(A, \mathfrak{r}_F) \le \varrho\hat{\kappa}(F) + C,$$

then by Proposition 4.3, $D^+(A) \le \varrho$, so that

$$\Sigma(A, \mathfrak{s}_F) \le (\varrho + \epsilon)\hat{\kappa}(F) + C'(\varepsilon)$$

for $\varepsilon > 0$, where $C'(\varepsilon)$ is a constant that is independent of the finite set $F \subset \mathbb{T}$, but may vary with ε.

We shall need a similar but more precise comparison between $\Sigma(A, \mathfrak{s}_F)$ and $\Sigma(A, \mathfrak{r}_F)$ for some slightly different asymptotic restrictions on the latter.

Proposition 4.4 *Fix $0 < \varrho, \eta < \infty$. Suppose that the sequence A in $\mathbb{D}$ is such that*

$$\Sigma(A, \mathfrak{r}_F) \le \varrho\hat{\kappa}(F) + \eta \log \hat{\kappa}(F) + C,$$

for every finite nonempty subset F of $\mathbb{T}$, where C is a constant. Then

$$\Sigma(A, \mathfrak{s}_F) \le \varrho\hat{\kappa}(F) + (\eta + \varrho) \log \hat{\kappa}(F) + C',$$

for every finite nonempty subset F of $\mathbb{T}$, for some other constant C'.

PROOF. As in the proofs of Propositions 4.2 and 4.3, we can show that the second inequality here is equivalent to a similar estimate with summation over Stolz stars $\mathfrak{s}_F$ replaced by summation over the regions $\mathfrak{q}_F$ with omitted Carleson squares:

$$\Sigma(A, \mathfrak{q}_F) \le \varrho\hat{\kappa}(F) + (\eta + \varrho) \log \hat{\kappa}(F) + O(1),$$

where $O(1)$ stands for a quantity that is bounded independently of the finite set F.

Let $F \subset \mathbb{T}$ be finite, and $\{I_n\}_n$ the collection of complementary arcs; $\{Q(I_n)\}_n$ are the associate Carleson squares. Project all points from $A \cap \mathfrak{q}_F$ (other than 0) radially to $\mathbb{T}$, and let

$$F' = \left\{ \frac{z}{|z|} \in \mathbb{T} : \ z \in A \cap \mathfrak{q}_F, \ z \neq 0 \right\}$$

be the resulting set, so that $\Sigma(A, \mathfrak{q}_F) \leq \Sigma(A, \mathfrak{r}_{F'})$. We put $k_n = \operatorname{card}(I_n \cap F')$, and note that

$$\hat{\kappa}(F) \leq \hat{\kappa}(F') \leq \hat{\kappa}(F) + \sum_n |I_n|_s \log(k_n + 1), \tag{7}$$

with equality only occurring in the right hand side inequality if the k_n points from $F' \cap I_n$ divide I_n into $k_n + 1$ equals subarcs. On the other hand, as we saw in the proof of Proposition 4.3,

$$\sum_n k_n |I_n|_s \leq \Sigma(A, \mathfrak{q}_F) \leq \Sigma(A, \mathfrak{r}_{F'}).$$

Since $\sum_n |I_n|_s = 1$, the concavity of the function $\log t$ (that is, the geometric-arithmetic mean value inequality) gives

$$\sum_n |I_n|_s \log(k_n + 1) \leq \log \left(1 + \sum_n k_n |I_n|_s \right) \leq \log(1 + \Sigma(A, \mathfrak{q}_F)). \tag{8}$$

Now, replace F with F' in the assumptions of the proposition and use the inequalities (7) and (8) to get

$$\begin{aligned} \Sigma(A, \mathfrak{q}_F) &\leq \Sigma(A, \mathfrak{r}_{F'}) \leq \varrho \hat{\kappa}(F') + \eta \log \hat{\kappa}(F') + O(1) \\ &\leq \varrho \hat{\kappa}(F) + \varrho \log \Sigma(A, \mathfrak{q}_F) + \eta \log(\hat{\kappa}(F) + \log \Sigma(A, \mathfrak{q}_F)) + O(1). \end{aligned}$$

From the proof of Propositions 4.2 and 4.3 we know that

$$\log \Sigma(A, \mathfrak{q}_F) \leq \log \hat{\kappa}(F) + O(1),$$

and thus

$$\Sigma(A, \mathfrak{q}_F) \leq \varrho \hat{\kappa}(F) + (\eta + \varrho) \log \hat{\kappa}(F) + O(1),$$

which is equivalent to the inequality stated at the beginning of the proof. ⊠

Chapter 5

The Growth Spaces $\mathcal{A}^{-\alpha}$ and $\mathcal{A}^{-\infty}$

5.1 Introduction

In this chapter we will introduce a class of Bergman-type spaces, denoted by $\mathcal{A}^{-\alpha}$ and $\mathcal{A}^{-\infty}$, which are closely to the spaces $\mathcal{B}_\alpha^p$ and are sometimes called growth spaces, and being the study of their zero sets.

Definition 5.1 *For any $\alpha > 0$, the space $\mathcal{A}^{-\alpha}$ consists of analytic functions f in $\mathbb{D}$ such that*

$$\|f\|_{\mathcal{A}^{-\alpha}} = \sup\left\{(1-|z|^2)^\alpha |f(z)| : \ z \in \mathbb{D}\right\} < +\infty.$$

It is easy to verify that $\mathcal{A}^{-\alpha}$ is a (nonseparable) Banach space with the norm defined above. Each space $\mathcal{A}^{-\alpha}$ clearly contains all the bounded analytic functions. The closure in $\mathcal{A}^{-\alpha}$ of the set of polynomials will be defined by $\mathcal{A}_0^{-\alpha}$, which is a separable Banach space and consists of exactly those functions f in $\mathcal{A}^{-\alpha}$ with

$$\lim_{|z| \to 1^-} (1-|z|^2)^\alpha |f(z)| = 0.$$

We will also consider the space

$$\mathcal{A}^{-\infty} = \bigcup_{0<\alpha<+\infty} \mathcal{A}^{-\alpha}.$$

It is clear that an analytic function f in $\mathbb{D}$ belongs to $\mathcal{A}^{-\infty}$ if and only if there exist positive constant C and N such that

$$|f(z)| \leq \frac{C}{(1-|z|^2)^N}, \quad \text{for all } z \in \mathbb{D}.$$

It is also clear that

$$\mathcal{A}^{-\infty} = \bigcup_{1/p<\alpha<\infty} \mathcal{B}_\alpha^p$$

for any $p \in (0, \infty)$. The space $\mathcal{A}^{-\infty}$ is a topological algebra when endowed with the inductive-limit topology.

For an analytic function f in $\mathbb{D}$ that is not identically zero, we define its hyperbolic exponential type

$$\mathfrak{t}(f) = \limsup_{|z|\to 1^-} \frac{\log |f(z)|}{\log \dfrac{1}{1-|z|}}.$$

The function f is said to be of finite hyperbolic exponential type if $\mathfrak{t}(F) < +\infty$. It is clear that

$$\mathfrak{t}(f) = \inf \left\{\alpha : \; f \in \mathcal{A}^{-\alpha}\right\}.$$

When $f \in \mathcal{A}^{-\alpha}$ for $\alpha = \mathfrak{t}(f)$, we say that f is of exact type. If $\mathfrak{t}(f) = 0$, we say that f is of minimal type. Clearly, $\mathfrak{t}(f) = 0$ if and only if $f \in \mathcal{A}^{-\alpha}$ for all $\alpha > 0$.

The space $\mathcal{A}^{-\infty}$ then consists of 0 and functions of finite hyperbolic exponential type.

To better formulate the main results about zero sets for Bergman-type space, we introduce two additional type spaces. Thus, we set

$$\mathcal{A}_+^{-\alpha} = \bigcap_{\beta:\beta>\alpha} \mathcal{A}^{-\beta} = \{0\} \cup \{f \in H(\mathbb{D}) : \; \mathfrak{t}(f) \leq \alpha\}$$

and

$$\mathcal{A}_-^{-\alpha} = \bigcup_{\beta:\beta<\alpha} \mathcal{A}^{-\beta} = \{0\} \cup \{f \in H(\mathbb{D}) : \; \mathfrak{t}(f) < \alpha\}.$$

It is clear that

$$\mathcal{A}_-^{-\alpha} \subset \mathcal{A}_0^{-\alpha} \subset \mathcal{A}^{-\alpha} \subset \mathcal{A}_+^{-\alpha}.$$

We can now state the main results of this chapter; the next two sections are devoted to their proofs.

Theorem 5.2 *Let $A = \{a_n\}_n$ be a sequence in $\mathbb{D}$. Then A is a zero set for $\mathcal{A}_+^{-\alpha}$ if and only if $\mathbb{D}^+(A) \leq \alpha$.*

In concrete terms, we prove that the condition $D^+(A) \leq \alpha$ is necessary and the condition $D^+(A) < \infty$ is sufficient for A to be an $\mathcal{A}^{-\alpha}$ zero set. This clearly implies the following.

Corollary 5.3 *A sequence $A = \{a_n\}_n$ in $\mathbb{D}$ is an $\mathcal{A}_-^{-\alpha}$ zero set if and only if $D^+(A) < \alpha$.*

Corollary 5.4 *A sequence $A = \{a_n\}_n$ in $\mathbb{D}$ is an $\mathcal{A}^{-\infty}$ zero set if and only if $D^+(A) < \infty$.*

5.2 Zero sets of functions in $\mathcal{A}^{-\alpha}$, necessary conditions

We begin the proof of the necessity of the condition $D^+(A) \leq \alpha$ for zero sets of functions in $\mathcal{A}^{-\alpha}$ with the following balayage-type estimate, which enables us to“sweep” zeros of an analytic function f radially to the circumference $\mathbb{T}$ and convert them into singular masses without increasing $|f|$ in a certain critical region.

Lemma 5.5 *Let $\mathfrak{s}_1$ be the standard Stolz angle at $z = 1$. Then*

$$\left|\frac{a-z}{1-az}\right| \geq \exp\left[(\log a)\frac{1-|z|^2}{|1-z|^2}\right]$$

for all $0 < a < 1$ and $z \in \mathbb{D} \setminus \mathfrak{s}_1$.

PROOF. Using the transform

$$w = \phi(z) = \frac{1+z}{1-z}$$

from $\mathbb{D}$ onto the right half-plane $\mathbb{C}_+ = \{w \in \mathbb{C} : \ \mathrm{Re} w > 0\}$, we can rewrite the desired inequality as

$$\left|\frac{b-w}{b+w}\right| \geq \exp\left[\left(\log\frac{b-1}{b+1}\right)u\right],$$

where $w = u + iv \in \mathbb{C}_+ \setminus \phi(\mathfrak{s}_1)$ and

$$b = \frac{1+a}{1-a} > 1.$$

We are going to take the logarithm on both sides of this second inequality and show that it actually holds for w in the larger set $\mathbb{C}_+ \setminus \Omega$, where

$$\Omega = \{w = u + iv : \ u > 1, \ |v| < u\}.$$

To see that Ω is smaller than $\phi(\mathfrak{s}_1)$, observe that $\partial(\phi^{-1}(\Omega))$ consists of parts of two orthogonal circles through 1 and -1 and an arc of the circle through 0 and 1 tangent to $\mathbb{T}$ at 1. Then it is geometrically obvious that $\phi^{-1}(\Omega) \subset \mathfrak{s}_1$.

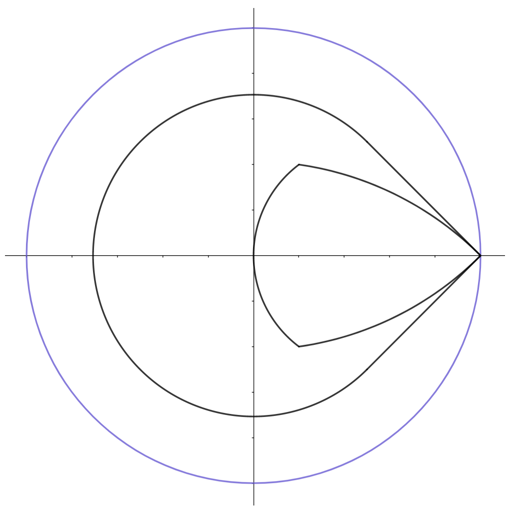

Figure 5.1: Representation of $\partial(\phi^{-1}(\Omega))$ and $\partial\mathfrak{s}_1$.

We show that

$$\frac{1}{u}\log\frac{b^2+u^2+v^2+2bu}{b^2+u^2+v^2-2bu} \leq 2\log\frac{b+1}{b-1},$$

where $b > 1$ and $w = u + iv \in \mathbb{C}_+ \setminus \Omega$. It is easy to check that the left-hand side above decreases, for any fixed u, as $|v|$ increases; and for $v = 0$, it is an increasing function of u. Thus, the inequality above holds in the strip $0 \le u \le 1$ with equality attained at $u = 1$ and $v = 0$. It remains to verify the case $|v| = u$:

$$\frac{1}{u}\log\frac{b^2+2u^2+2bu}{b^2+2u^2-2bu} \le 2\log\frac{b+1}{b-1}$$

for $u \ge 1$.

Let $u = bt$. It then suffices to show that

$$\frac{1}{t}\log\frac{1+2t^2+2t}{1+2t^2-2t} \le 2b\log\frac{b+1}{b-1}$$

for $b > 1$ and $t > 0$. The right-hand side here is decreasing in b and tends to 4 as $b \to +\infty$. So it is enough to show that

$$\frac{1}{t}\log\frac{1+2t^2+2t}{1+2t^2-2t} \le 4$$

for all $t > 0$. Since it is an elementary exercise, we leave the details to the reader. ⊠

Given a finite subset E of the punctured disk $\mathbb{D} \setminus \{0\}$, we define the push-out measure $d\Lambda_E$:

$$d\Lambda_E = \sum_{z\in E}\log\frac{1}{|z|}d\delta_{z^*},$$

where $z^* = z/|z| \in \mathbb{T}$ is the pushed-out point and $d\delta_\zeta$ stands for the unit point mass at $\zeta \in \mathbb{T}$. This measure is related with the logarithmic sum defined in (4) of Chapter 4. For a finite Borel measure μ on $\mathbb{T}$, the Poisson extension is defined as

$$P[\mu](z) = \int_{\mathbb{T}} P(z,w)d\mu(w), \quad z \in \mathbb{D},$$

where

$$P(z,w) = \frac{1-|z|^2}{|1-z\overline{w}|^2}$$

is the Poisson kernel. Lemma 5.5 states that the following assertion holds for one-point set A: the general case follows by iteration.

Corollary 5.6 *Suppose $f \in \mathcal{A}^{-\alpha}$ and $A = \{a_1, \cdots, a_n\} \subset \mathbb{D} \setminus \{0\}$ are some of the zeros of f. Let B_A be the Blaschke product associated with A, and let $A^* = \{a_1/|a_1|, \cdots, a_n/|a_n|\}$ be the pushed-out sequence on $\mathbb{T}$. Then*

$$\Big|\frac{f(z)}{B_A(z)}\Big| \le \frac{\|f\|_{\mathcal{A}^{-\alpha}}}{(1-|z|^2)^\alpha}\exp\left(P[\Lambda_A](z)\right), \quad z \in \mathbb{D} \setminus \mathfrak{s}_{A^*}.$$

We will need some estimates for several auxiliary harmonic functions. Recall that for a closed set F in $\mathbb{T}$,

$$d_{\mathbb{C}}(z,F)=\inf\left\{|z-\zeta|:\ \zeta\in F\right\},\quad z\in\overline{\mathbb{D}},$$

is the Euclidean distance from z to F. Also, recall that ds is the normalized arc-length measure on $\mathbb{T}$, it is, $ds(z)=|dz|/(2\pi)$.

Lemma 5.7 *Suppose F is a finite set in $\mathbb{T}$ and its complementary arcs $I_1,\cdots,I_n$ satisfy $|I_k|=2\pi|I_k|_s<1$, for all $k=1,\cdots,n$. Then the harmonic function*

$$U_F(z)=\int_{\mathbb{T}}\frac{1-|z|^2}{|\zeta-z|^2}\log\frac{1}{d_{\mathbb{C}}(\zeta,F)}ds(\zeta),\quad z\in\mathbb{D},$$

is positive and satisfies

$$\log\frac{1}{d_{\mathbb{C}}(z,F)}\leq U_F(z),\quad z\in\mathbb{D}.$$

PROOF. We have

$$\log\frac{1}{d_{\mathbb{C}}(z,F)}=\max_{\zeta\in F}\log\frac{1}{|z-\zeta|},\quad z\in\mathbb{D},$$

so that the left hand side expresses a positive subharmonic function on $\mathbb{D}$ whose boundary values equal those of $U_F(z)$. Hence the desired inequality follows from the maximum principle. ⊠

For $0<p<1$, consider the harmonic function

$$V_p(z,\zeta)=\left(\sec\frac{p\pi}{2}\right)\operatorname{Re}(1-\overline{\zeta}z)^{-p},\quad z\in\mathbb{D},$$

where ζ is a point on $\mathbb{T}$. The choice of the constant factor involving the secant function ensures that

$$|1-z\overline{\zeta}|^{-p}\leq V_p(z,\zeta),\quad (z,\zeta)\in\mathbb{D}\times\mathbb{T}. \tag{1}$$

Also, for $\zeta\in\mathbb{T}$, and $0<c<\dfrac{1}{4}$, let $\gamma(\zeta,p,c)$ be the curve

$$\gamma(\zeta,p,c)=\left\{z\in\overline{\mathbb{D}}:1-|z|^2=c|\zeta-z|^{2-p}\right\},$$

which makes one loop around the origin and touches the unit circle exactly at ζ. More generally, for a finite subset F of $\mathbb{T}$, we define the curve

$$\gamma(F,p,c)=\left\{z\in\overline{\mathbb{D}}:\ 1-|z|^2=cd_{\mathbb{C}}(z,F)^{2-p}\right\},$$

which encloses a star-shape domain touching the unit circle exactly at the points of F (see Figure 5.2).

Lemma 5.8 *Fix $0<p<1$ and $0<c<1/4$. Then, for fixed $\zeta\in\mathbb{T}$,*

$$\frac{1-|z|^2}{|\zeta-z|^2}=P(z,\zeta)<cV_p(z,\zeta)$$

for all z in the region between $\mathbb{T}$ and $\gamma(\zeta,p,c)$.

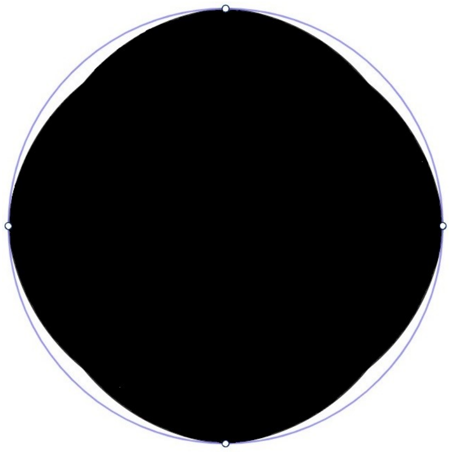

Figure 5.2: Representation of the region enclosed by the curve $\gamma(F, p, c)$ with $F = \{\pm 1,\ \pm i\}$.

PROOF. In the region between $\gamma(\zeta, p, c)$ and $\mathbb{T}$, we have

$$1 - |z|^2 < c|\zeta - z|^{2-p},$$

and there

$$\frac{1-|z|^2}{|\zeta - z|^2} < c|1 - \overline{\zeta} z|^{-p} \leq cV_p(z, \zeta),$$

by the inequality (1). ⊠

for a finite Borel measure μ on $\mathbb{T}$, let

$$V_p[\mu](z) = \int_{\mathbb{T}} V_p(z, \zeta) d\mu(\zeta), \quad z \in \mathbb{D},$$

be corresponding potential, which represents a harmonic function on $\mathbb{D}$.

Remark: We restrict the parameters p and c to $0 < p < 1$ and $0 < c < 1/4$, and assume that the finite set F has complementary arcs $\{I_k\}_k$ satisfying $|I_k| = 2\pi |I_k|_s < 1$ for all k.

Lemma 5.9 *Let μ be a finite positive Borel measure on $\mathbb{T}$, supported on a finite set F. Then the inequality*

$$P[\mu](z) \leq cV_p[\mu](z)$$

holds for all z between $\mathbb{T}$ and the curve $\gamma(F, p, c)$.

PROOF. The function $P[\mu]$ is a finite sum of Poisson kernels; apply Lemma 5.8 to each term. As the set of points between $\mathbb{T}$ and $\gamma(F, p, c)$ is the intersection of the domains described in Lemma 5.8 over $\zeta \in F$, the assertion is immediate. ⊠

The key to our necessary condition for $\mathcal{A}^{-\alpha}$ zero sets if the following Jessen-type inequality. Recall the definition (4) in Chapter 4 of the logarithmic sum

$$\Lambda(A, E) = \sum_{\substack{j=1 \\ a_j \in E}}^{\infty} \log \frac{1}{|a_j|},$$

where $A = \{a_j\}_j$, counting multiplicities.

Theorem 5.10 *Let f be a nonzero function in $\mathcal{A}^{-\alpha}$ having zeros (counting multiplicities) at $A = \{a_n\}_n$ with $0 \notin A$. Then, for any finite set F in $\mathbb{T}$,*

$$\begin{aligned}\Lambda(A, \mathfrak{r}_F) - \alpha \log \Lambda(A, \mathfrak{r}_F) \\ \le \alpha\left[\hat{\kappa}(F) + \log \hat{\kappa}(F)\right] - \alpha(\log \alpha - 2) + \log \|f\|_{\mathcal{A}^{-\alpha}} - \log |f(0)|\end{aligned}$$

whenever $4\alpha < \Lambda(A, \mathfrak{r}_F)\hat{\kappa}(F)$.

PROOF. We can assume A to be a finite sequence. By Corollary 5.6,

$$\log\left|\frac{f(z)}{B_{A\cap\mathfrak{r}_F}(z)}\right| \le \log \|f\|_{\mathcal{A}^{-\alpha}} + \alpha \log \frac{1}{1-|z|^2} + P[\Lambda_{A\cap\mathfrak{r}_F}](z), \quad z \in \gamma(F, p, c),$$

where $B_{A\cap\mathfrak{r}_F}(z)$ is the Blaschke product for the zeros $A \cap \mathfrak{r}_F$ and the push-out measure $d\Lambda_{A\cap\mathfrak{r}_F}$ is as before. We now use the geometric properties of $\gamma(F, p, c)$ and apply Lemmas 5.7, 5.9 to obtain

$$\log\left|\frac{f(z)}{B_{A\cap\mathfrak{r}_F}(z)}\right| \le \alpha(2-p)U_F(z) + \alpha \log \frac{1}{c} + cV_p[\Lambda_{A\cap\mathfrak{r}_F}](z) + \log \|f\|_{\mathcal{A}^{-\alpha}}$$

for $z \in \gamma(F, p, c)$; the function U_F is as in Lemma 5.7. The left-hand side here is a subharmonic function in the region enclosed by the curve $\gamma(F, p, c)$. Note that

$$\log |B_{A\cap\mathfrak{r}_F}(0)| = -\Lambda(A, \mathfrak{r}_F) \quad \text{and} \quad V_p[\Lambda_{A\cap\mathfrak{r}_F}](0) = \left(\sec \frac{p\pi}{2}\right) \Lambda(A, \mathfrak{r}_F).$$

Hence, by the maximum principle, we then have

$$\begin{aligned}\log\left|\frac{f(0)}{B_{A\cap\mathfrak{r}_F}(0)}\right| &= \log |f(0)| + \Lambda(A, \mathfrak{r}_F) \\ &\le \alpha(2-p)U_F(0) + \alpha \log \frac{1}{c} + cV_p[\Lambda_{A\cap\mathfrak{r}_F}](0) + \log \|f\|_{\mathcal{A}^{-\alpha}} \\ &= \alpha(2-p)\int_{\mathbb{T}} \log \frac{1}{d_{\mathbb{C}}(\zeta, F)} ds(\zeta) + \alpha \log \frac{1}{c} \\ &\quad + \left(c \sec \frac{p\pi}{2}\right) \Lambda(A, \mathfrak{r}_F) + \log \|f\|_{\mathcal{A}^{-\alpha}}.\end{aligned}$$

By the inequality (1) of Chapter 4, the integral expression above is less than or equal to $\hat{\kappa}(F)$, and it is elementary that

$$\sec \frac{p\pi}{2} < \frac{1}{1-p}.$$

Thus,

$$\log |f(0)| \le \alpha(2-p)\hat{\kappa}(F) + \left(\frac{c}{1-p} - 1\right) \Lambda(A, \mathfrak{r}_F) + \alpha \log \frac{1}{c} + \log \|f\|_{\mathcal{A}^{-\alpha}}.$$

To minimize the right-hand side, we put

$$1 - p = \frac{1}{\hat{\kappa}(F)}, \quad c = \frac{\alpha}{\Lambda(A, \mathfrak{r}_F)\hat{\kappa}(F)}.$$

The desired result then follows. ⊠

Note that the result above implies that $\Lambda(A, \mathfrak{r}_F) < +\infty$ for every finite subset F of $\mathbb{T}$.

We now prove two necessary conditions for zero sets of function in $\mathcal{A}^{-\alpha}$.

Theorem 5.11 *If $A = \{a_n\}_n$ is the zero sequence of a function in $\mathcal{A}^{-\alpha}$, then*

$$\Sigma(A, \mathfrak{r}_F) \leq \alpha \left[\hat{\kappa}(F) + 2\log \hat{\kappa}(F)\right] + O(1),$$

where $O(1)$ stands for a quantity which is uniformly bounded independently of the finite nonempty subset F of $\mathbb{T}$.

PROOF. Since

$$\frac{1}{2}(1 - t^2) < \log \frac{1}{t}, \quad 0 < t < 1,$$

a comparison of the summation functions Σ and Λ shows that by Theorem 5.10,

$$\Sigma(A, \mathfrak{r}_F) - \alpha \log^+ \Sigma(A, \mathfrak{r}_F) \leq \alpha \left[\hat{\kappa}(F) + 2\hat{\kappa}(F)\right] + O(1),$$

which give us the result of the theorem. ⊠

Theorem 5.12 *If $A = \{a_n\}_n$ is the zero sequence of a function in $\mathcal{A}^{-\alpha}$, then*

$$\Sigma(A, \mathfrak{s}_F) \leq \alpha \left[\hat{\kappa}(F) + 2\log \hat{\kappa}(F)\right] + O(1),$$

where $O(1)$ stands for a quantity which is uniformly bounded independently of the finite nonempty subset F of $\mathbb{T}$.

PROOF. This is a direct consequence of the preceding theorem and Proposition 4.4. ⊠

We derive two useful corollaries from the above necessary conditions. The first one is a formulation of Corollary 3.5 in terms of zero sets of functions in $\mathcal{A}^{-\infty}$.

Corollary 5.13 *Let $A = \{a_n\}_n$ be an $\mathcal{A}^{-\infty}$ zero sequence. Then*

$$S(r) = \sum_{|a_n|<r} (1 - |a_n|) = O\left(\log \frac{1}{1-r}\right) \quad as \quad r \to 1^-,$$

and for each $\epsilon > 0$, we have

$$\sum_n \frac{1 - |a_n|}{\left[\log \dfrac{e}{1 - |a_n|}\right]^{1+\epsilon}} < +\infty.$$

PROOF. Taking

$$F = \{\exp(2k\pi i/N) : \ 1 \leq k \leq N\}$$

in Theorem 5.12 and letting $N \to +\infty$ yields the first estimate, because the Stolz star $\mathfrak{s}_F$ will then cover a disk of radius $1 - \pi/N$, and a simple computation reveals that $\hat{\kappa}(F) = 1 + \log N$. Since

$$\sum_n \frac{1 - |a_n|}{\left[\log \dfrac{e}{1 - |a_n|}\right]^{1+\epsilon}} = \int_0^1 \frac{dS(r)}{\left[\log \dfrac{e}{1-r}\right]^{1+\epsilon}}$$

$$= S(0) + (1+\epsilon)\int_0^1 \frac{S(r)dr}{(1-r)\left[\log \dfrac{e}{1-r}\right]^{2+\epsilon}},$$

The second estimate then follows from the first one. ⊠

Corollary 5.14 *If A is the zero set of a function in $\mathcal{A}^{-\alpha}$, then $D^+(A) \leq \alpha$.*

5.3 Zero sets of functions in $\mathcal{A}^{-\alpha}$, a sufficient condition

In this section we will present a sufficient condition for a sequence A in $\mathbb{D}$ to be a zero set of a function in $\mathcal{A}^{-\alpha}$. The proof of the main theorem consists of two key ideas: an "oblique" projection technique, and a technique from Linear Programming.

Throughout this section, we let $\mathfrak{s}_\zeta$ denote the Stolz angle with the vertex at $\zeta \in \mathbb{T}$ and an arbitrary but fixed aperture φ with $\pi/2 \leq \varphi < \pi$. Thus, $\mathfrak{s}_\zeta$ is the convex hull of

$$\{\zeta\} \cup \{z \in \mathbb{C} : \ |z| \leq \sin(\varphi/2)\},$$

with the vertex ζ removed. As before, for a finite subset F of $\mathbb{T}$,

$$\mathfrak{s}_F = \bigcup_{\zeta \in F} \mathfrak{s}_\zeta$$

is the corresponding Stolz star domain.

Given $\lambda \in \mathbb{D}$, contained in the annulus $\sin(\varphi/2) < |\lambda| < 1$, there are exactly two Stolz angles $\mathfrak{s}_\xi$ (with $\xi \in \mathbb{T}$) such that $\lambda \in \partial\mathfrak{s}_\xi$. Let ξ_1 and ξ_2 be the corresponding points of $\mathbb{T}$, which of course depend on λ. Given another point $\zeta \in \mathbb{T}$, we pick the one (out of ξ_1, ξ_2) which is the fartest away from ζ, and call it the oblique projection $\varpi_\zeta(\lambda)$ of λ. This can be done unless λ is on the straight line connecting ζ and $-\zeta$; however, we shall mainly be interested in $\lambda \in \mathbb{D} \setminus \mathfrak{s}_{\zeta,-\zeta}$. We also need the concept of a tend: for an open arc $I \subset \mathbb{T}$ with endpoint w_1 and w_2, we define the tent $\mathfrak{h}_I$ as the component of $\mathbb{D} \setminus \mathfrak{s}_{w_1,w_2}$ abutting on I. The geometric situation is illustrated in the following Figure.

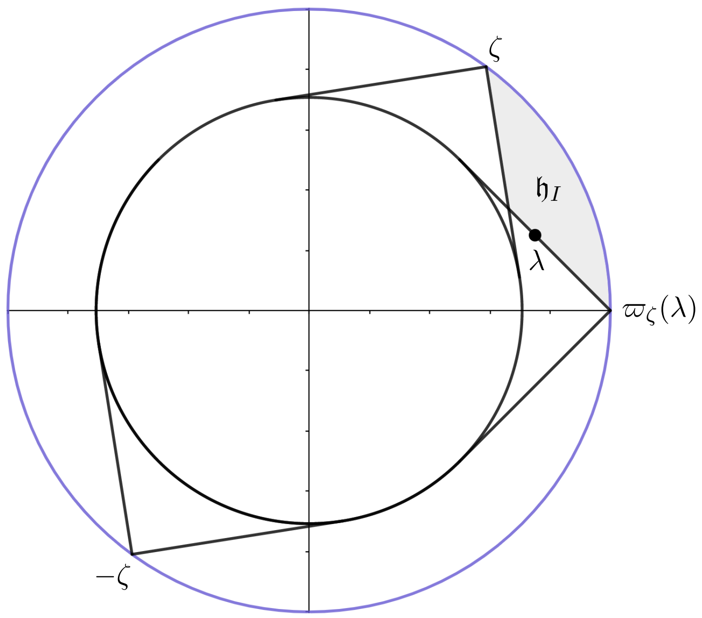

Figure 5.3: Representation of the oblique projection $\varpi_\zeta(\lambda)$ and the tend $\mathfrak{h}_I$.

The following lemma will show us that on the radius $\{z = t\zeta :\ 0 < t < 1\}$, the Blaschke factor $(\lambda - z)/(1 - \overline{\lambda} z)$ is dominated in modulus by the singular inner function $\exp\left[-\sigma(\varpi + z)/(\varpi - z)\right]$, where $\sigma = (1 - |\lambda|^2)/2$.

Lemma 5.15 *Fix the aperture of the Stolz angles $\varphi \in [3\pi/5, \pi)$. Then for all $z = t\zeta$, $0 < t < 1$, and $\lambda \in \mathbb{D} \setminus \mathfrak{s}_{\{\zeta, -\zeta\}}$, we have*

$$\log\left|\frac{\lambda - z}{1 - \overline{\lambda} z}\right| + \frac{(1 - |z|^2)(1 - |\lambda|^2)}{2|1 - \overline{\varpi} z|^2} \leq 0,$$

where $\varpi = \varpi_\zeta(\lambda)$.

PROOF. Using the identity

$$1 - \left|\frac{\lambda - z}{1 - \overline{\lambda} z}\right|^2 = \frac{(1 - |\lambda|^2)(1 - |z|^2)}{|1 - \overline{\lambda} z|^2}$$

we can rewrite the desired inequality as

$$\log(1 - 2\sigma a_2) + 2\sigma a_1 \leq 0,$$

where

$$2\sigma = 1 - |\lambda|^2, \quad a_1 = \frac{1 - |z|^2}{|1 - \overline{z}\varpi|^2}, \quad a_2 = \frac{1 - |z|^2}{|1 - \overline{z}\lambda|^2}.$$

Since

$$\begin{aligned}
\log(1 - 2\sigma a_2) + 2\sigma a_1 &= -\sum_{n=1}^{+\infty} \frac{(2\sigma a_2)^n}{n} + 2\sigma a_1 \\
&\leq 2\sigma(a_1 - a_2) - \frac{(2\sigma a_2)^2}{2(1 - \sigma a_2)} \\
&= \frac{2\sigma(a_1 - a_2 - \sigma a_1 a_2)}{1 - \sigma a_2},
\end{aligned}$$

it suffices for us to prove

$$a_1 - a_2 - \sigma a_1 a_2 \leq 0,$$

which is equivalent to

$$\frac{1}{a_2} - \frac{1}{a_1} \leq \sigma,$$

that is,

$$\left|\frac{1}{\overline{z}} - \lambda\right|^2 - \left|\frac{1}{\overline{z}} - \varpi\right|^2 \leq \frac{1}{2}(1 - |\lambda|^2)\left(\frac{1}{|z|^2} - 1\right). \tag{2}$$

Let

$$\beta = |\arg(\zeta/\varpi)|, \quad \gamma = |\arg(\zeta/\lambda)|,$$

where as usual the argument takes values in the interval $(-\pi, \pi]$. The definition of oblique projection implies that $0 < \beta/2 \leq \gamma \leq \beta < \pi$, and a geometric consideration reveals that

$$1 - |\lambda| \leq (\beta - \gamma)\cos\frac{\varphi}{2} < \frac{1}{2}(\pi - \varphi)(\beta - \gamma).$$

Using the expansions

$$\left|\frac{1}{\bar{z}} - \lambda\right|^2 = \frac{1}{|z|^2} + |\lambda|^2 - 2\frac{|\lambda|}{|z|}\cos\gamma$$

and

$$\left|\frac{1}{\bar{z}} - \varpi\right|^2 = \frac{1}{|z|^2} + 1 - \frac{2}{|z|}\cos\beta$$

we can reformulate (2) as

$$\cos\beta - |\lambda|\cos\gamma \leq \frac{1}{4}\left(|z| + \frac{1}{|z|}\right)(1 - |\lambda|^2).$$

Since

$$|z| + \frac{1}{|z|} > 2, \quad z \in \mathbb{D} \setminus \{0\},$$

it is enough to prove

$$\cos\beta - |\lambda|\cos\gamma \leq \frac{1}{2}(1 - |\lambda|^2), \quad \lambda \in \mathbb{D} \setminus \mathfrak{s}_\zeta.$$

We can further assume $\beta < \pi/3$; otherwise, the above inequality holds for all $\lambda \in \mathbb{D}$. Solving the quadratic inequality, we are lead to check that

$$0 \leq 1 - |\lambda| \leq (1 - \cos\gamma) + \left[(1 - \cos\gamma)^2 + 4\sin\frac{\beta - \gamma}{2}\sin\frac{\beta + \gamma}{2}\right]^{1/2}.$$

The right-hand side is actually greater than $(2/\pi)(\beta - \gamma)$. For $3\pi/5 \leq \varphi \leq \pi$, we have

$$1 - |\lambda| < \frac{1}{2}(\pi - \varphi)(\beta - \gamma) \leq \frac{\pi}{5}(\beta - \gamma) < \frac{2}{\pi}(\beta - \gamma),$$

which completes the proof of the lemma. ☒

In the remainder of this section, we assume that the aperture φ of the Stolz angles in chosen in the interval $[3\pi/5, \pi)$, so that the conclusion of Lemma 5.15 holds true.

Given an arc I of the circle $\mathbb{T}$, let $\kappa(I)$ be the quantity

$$\kappa(I) = |I|_s \log\frac{e}{|I|_s}.$$

Definition 5.16 *Suppose $A = \{a_n\}_n$ is a finite sequence in $\mathbb{D}$, w_0 is a point in $\mathbb{T}$, and α is a positive number. A positive Borel measure μ on $\mathbb{T}$ is (A, α, w_0)-admissible if*

(i) $\mu(\{w_0\}) = 0$;

(ii) for each open arc $I \subset \mathbb{T}$, with $w_0 \notin I$, the following inequality holds:

$$0 \leq \mu(I) \leq \alpha\kappa(I) + \Sigma(A, \mathfrak{h}_I), \tag{3}$$

where $\mathfrak{h}_I$ is the tent associated with I.

The set of all (A, α, w_0)-admissible measures will be denoted by $\mathcal{M}(A, \alpha, w_0)$, or just $\mathcal{M}$. The second condition above, (ii), clearly implies that $\mu(\{\zeta\}) = 0$ for any $\zeta \in \mathbb{T}$, not just for $\zeta = w_0$.

Lemma 5.17 *Let $A \subset \mathbb{D}$ be a finite set, w_0 a point in $\mathbb{T}$, α a positive number and $\mu \in \mathcal{M}(A, \alpha, w_0)$. Define F as the set consisting of w_0 and the "oblique projections" of the points $a \in A \cap \{\sin(\varphi/2) < |z| < 1\}$, where $\varphi \in [3\pi/5, 1)$ is fixed.*

If $\tilde{\mu}$ is the measure with constant density on each complementary interval I_k of F and $\tilde{\mu}$ is such that $\tilde{\mu}(I_k) = \mu(I_k)$, then $\tilde{\mu} \in \mathcal{M}(A, \alpha, w_0)$.

PROOF. By definition of F, for all $a \in A \cap \{\sin(\varphi/2) < |z| < 1\}$ there exists $\zeta \in F$ such that $a \in \partial\mathfrak{s}_\zeta$. It follows that if I_k is a complementary arc of F, then

$$\mu(I_k) \leq \alpha\kappa(I_k) + \Sigma(A, \mathfrak{h}_{I_k}) = \alpha\kappa(I_k). \tag{4}$$

Now consider I an open arc of $\mathbb{T}$ with $w_0 \notin I$.

[*case 1.*] Assume I is contained in some I_k. Since $\tilde{\mu}$ is a measure with constant density and $\tilde{\mu}(I_k) = \mu(I_k)$, if we define $0 < \delta = |I|_s/|I_k|_s \leq 1$, we have that

$$\begin{aligned}\tilde{\mu}(I) = \delta\mu(I_k) &\leq \delta\alpha\kappa(I_k) \\ &\leq \alpha\left(\delta|I_k|_s \log\frac{e}{\delta|I_k|_s}\right) \\ &\leq \alpha\kappa(I) + \Sigma(A, \mathfrak{h}_I).\end{aligned}$$

[*case 2.*] Assume $I \nsubseteq I_k$ for all k, then, there are $I_{k_1} \neq I_{k_n}$ two complementary arcs of F such that one endpoint of I is contained in I_{k_1} and the other one is contained in I_{k_2}. Hence, there exists $n-1$ different points $\zeta_i \in \mathbb{T}$, $\zeta_i \neq w_0$ for $i = 1, \cdots, n-1$, such that

$$I = (I_{k_1} \cup \{\zeta_1\} \cup I_{k_2} \cup \cdots \cup \{\zeta_{n-1}\} \cup I_{k_n}) \cap I.$$

It follows from the above case that

$$\tilde{\mu}(I \cap I_{k_i}) \leq \alpha\kappa(I \cap I_{k_i}) + \Sigma(A, \mathfrak{h}_{I\cap I_i}), \quad i = 1, \cdots, n,$$

hence,

$$\tilde{\mu}(I) \leq \sum_{i=1}^{n} \left(\alpha\kappa(I \cap I_{k_i}) + \Sigma(A, \mathfrak{h}_{I\cap I_i})\right). \tag{5}$$

On the one hand, by the definition of the operator Σ and the tend $\mathfrak{h}_I$, it is clear that

$$\Sigma(A, \mathfrak{h}_{I\cap I_1}) + \cdots + \Sigma(A, \mathfrak{h}_{I\cap I_n}) \leq \Sigma(A, \mathfrak{h}_I). \tag{6}$$

On the other hand, since

$$|I \cap I_{k_i}| = \delta_i|I|, \quad i = 1, \cdots, n,$$

where $0 < \delta_i$ and $\delta_1 + \cdots + \delta_n = 1$, by the concavity of κ we have that

$$\sum_{i=1}^{n} \kappa(I \cap I_{k_i}) \leq \kappa(I). \tag{7}$$

It follows from (5), (6) and (7) that

$$\tilde{\mu}(I) \leq \alpha\kappa(I) + \Sigma(A, \mathfrak{h}_I).$$

⊠

Proposition 5.18 *Suppose* $0 < \alpha < +\infty$*,* $w_0 \in \mathbb{T}$*, and* $A = \{a_n\}_n$ *is a finite sequence in* $\mathbb{D}$*. Then*

$$\sup\{\mu(\mathbb{T}) : \ \mu \in \mathcal{M}\} = \inf\{\alpha\hat{\kappa}(F) + \Sigma(A, \mathbb{D} \setminus \mathfrak{s}_F) : \ F \in \mathbb{T} \text{ is finite and } w_0 \in F\}, \qquad (8)$$

where $\mathcal{M} = \mathcal{M}(A, \alpha, w_0)$ *is the set of all* (A, α, w_0)*-admissible measures. Furthermore, there is at least one maximal admissible measure* μ_0 *for which*

$$\mu_0(\mathbb{T}) = \inf\{\alpha\hat{\kappa}(F) + \Sigma(A, \mathbb{D} \setminus \mathfrak{s}_F) : \ F \in \mathbb{T} \text{ is finite and } w_0 \in F\} \qquad (9)$$

PROOF. The set $\mathbb{D} \setminus \mathfrak{s}_F$ is a disjoint union of tends of the kind $\mathfrak{h}_I$, with $w_0 \notin I$, so by the definition of (A, α, w_0)-admissible measure, the "sup" on the left hand side is less than or equal to the "inf" on the right hand side.

To see the other direction of the inequality, define a finite set F_0 consisting of w_0 and all those points on $\mathbb{T}$ which are "oblique projections" of points of A in the annulus $\sin(\varphi/2) < |z| < 1$. Moreover, let $\tilde{\mu}$ denote the measure with constant density on each I_k and such that $\tilde{\mu}(I_k) = \mu(I_k)$ for all k, where $\{I_k\}_{k=1}^N$ are the complementary arcs of F_0 and N is the cardinality of F_0. Then, Lemma 5.17 give us that $\tilde{\mu}$ is a (A, α, w_0)-admissible measure.

Each measure $\tilde{\mu}$ is described by a vector $x = (x_1, \cdots, x_N)$, where $x_k = \mu(I_k)$. We are thus led to a standard optimization problem from Linear Programming: maximize the functional

$$L(x) = x_1 + \cdots + x_N,$$

where the positive vector $x = (x_1, \cdots, x_N)$ satisfies $N(N+1)/2$ restrictions of the type

$$x_k + x_{k+1} + \cdots + x_l \leq b_{k,l}, \quad 1 \leq k \leq l \leq N, \qquad (10)$$

where each quantity $b_{k,l}$ is given by

$$b_{k,l} = \alpha\kappa(I_{k,l}) + \Sigma(A, \mathfrak{h}_{I_{k,l}}),$$

where $I_{k,l}$ is the arc obtained by filling in finitely many points in the union $I_k \cup I_{k+1} \cup \cdots \cup I_l$. Notice that the inequality (10) corresponds to the inequality of Definition 5.16 with arcs I whose endpoints are in F_0. We will refer to this as the optimization problem.

Let $\mathcal{C}$ denote the closed convex polyhedron in $\mathbb{R}^N$ defined by the above restrictions

$$x_k + x_{k+1} + \cdots + x_l \leq b_{k,l}, \quad 1 \leq k \leq l \leq N,$$

and denote by $\mathcal{C}_+$ its intersection with $\mathbb{R}_+^N = \prod^N \mathbb{R}_+$, where $\mathbb{R}_+$ is the half-axis $\mathbb{R}_+ = [0, +\infty)$.

See that the "inf" over all finite subsets $F \subset \mathbb{T}$ appearing in the formulation of the proposition can only get bigger if we restrict F to be subsets of the "oblique projected" set F_0, so it is clearly enough to prove the equality under the additional restriction $F \subset F_0$.

Thus, in terms of the optimization problem stated above, the assertion of the lemma can now be reformulated as follows:

$$\max\{L(x) : \ x \in \mathcal{C}_+\} = \min \sum_\nu b_{k_\nu, I_\nu}, \qquad (11)$$

where the minimum is taken over all simple covering $\{[k_\nu, I_\nu]\}_\nu$ of $\mathbb{N}_N = \{1, 2, \cdots, N\}$. We will refer to (11) as the min-max equation. Note that we here deviate from standard notation and let $[k, l]$ stand for an interval consisting of integers and not of reals.

It is at least clear that on $\mathcal{C}_+$, $L(x)$ assumes its maximum somewhere. We claim that the maximum is in fact assumed at some point $x = (x_1, \cdots, x_N) \in \mathcal{C}_+$ with $x_j > 0$ for all $j = 1, \cdots, N$. To this end, take a point $x \in \mathcal{C}_+$, with $x_j = 0$ for some j. There may be a few zero slots clustering together, so say that $x_j = 0$ on the "interval" $k < j < l$, but that at the endpoints we have $x_k > 0$ and $x_l > 0$. For a small parameter $\varepsilon > 0$, consider the point

$$x' = (x_1, \cdots, x_{k-1}, x_k - \varepsilon(l-k-1), \varepsilon, \cdots, \varepsilon, x_l, \cdots, x_N).$$

We now use the property of the given quantities $b_{k,l}$, namely that they are positive and strictly monotonically increasing in the interval $[k,l]$: $b_{k,l} < b_{k',l'}$, whenever $[k,l]$ is strictly contained in $[k',l']$. It follows that the competing point x' is in $\mathcal{C}_+$ for sufficiently small ε, and moreover, $L(x') = L(x)$. If x is the point whenever $L(x)$ assume its maximum, we treat all clusters of zeros the same way, and find a (perhaps different) point $x' \in C_+$ with $L(x')$ maximal, and $x'_j > 0$ for all $j = 1, \cdots, N$.

We next claim that $L(x) \le L(x')$ for all $x \in \mathcal{C}$, in fact, for all $x \in \mathcal{C}_+$. Suppose for the moment that at some point $x^0 \in \mathcal{C}$, the inequality $L(x^0) > L(x')$ holds. Then we consider points x close to x' along the line segment connecting x' with x^0. Such x will be in $\mathcal{C}$ by convexity, and they are in $\mathbb{R}^N_+$, and hence in $\mathcal{C}_+$. the value of $L(x)$ must be slightly bigger than $L(x')$, a contradiction.

We can now apply the standard duality theorem of linear programming due to Gale, Kuhn, and Tucker [8]. To formulate the result, we write the $N(N+1)/2$ inequalities defining $\mathcal{C}$ as

$$\langle x, e^j \rangle \le b_j, \quad j = 1, 2, \cdots, N(N+1)/2,$$

where b_j equals $b_{k,l}$ for the pair (k,l) numbered by j, and similarly, e^j stands for the vector $(0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0)$ in $\mathbb{R}^N$, with 1's precisely on the interval $[k,l]$ associated with the index j. Here $\langle \cdot, \cdot \rangle$ is the usual inner product of $\mathbb{R}^N$.

We also write $L(x) = \langle x, L \rangle$, where $L = (1, 1, \cdots, 1)$. With this notations, the duality theorem assure us that

$$\begin{aligned}\max\{\langle x, L\rangle : \ x \in \mathcal{C}_+\} &= \max\{\langle x, L\rangle : \ x \in \mathcal{C}\} \\ &= \min\left\{\sum_j \theta_j b_j : \ \theta_j \in \mathbb{R}_+ \text{ for all } j, \ \sum_j \theta_j e^j = L\right\}.\end{aligned} \tag{12}$$

The min-max equation (12) claims that the above minimum is achieved with coefficients $\theta_j \in \{0,1\}$. The points $\theta = (\theta_1, \cdots, \theta_{N(N+1)/2}) \in \mathbb{R}^{N(N+1)/2}$ with

$$\sum_j \theta_j e^j = L$$

constitute –by inspection of the vectors involved (the e^j's and L)– a closed convex lower-dimensional polyhedron $\mathcal{S}$ contained in the cube $[0,1]^{N(N+1)/2}$. We show that the polyhedron $\mathcal{S}$ is the closed convex hull of "edge points" $\theta \in \mathcal{S}$ of the type that $\theta_j \in \{0,1\}$ for every j. The min-max equation then follows easily. Points θ with positive rational coordinates are dense in

$\mathcal{S}$, and it suffices to obtain that there are in the convex hull of the "edge points". Multiplying by the least common denominator n of the positive rationals $\theta_1, \cdots, \theta_{N(N+1)/2}$, we have

$$\sum_j \vartheta_j e^j = nL, \tag{13}$$

where $\vartheta_j = n\theta_j \in \mathbb{Z}_+$. Here, $\mathbb{Z}_+ = \{1, 2, 3, \cdots\}$ stands for the set of all positive integer. We interpret the above situation in terms of covering. Let $\mathcal{J}$ stand for the set of all closed intervals $J = [k, l]$ in the integers $\mathbb{Z}$ whose end points are integers satisfying $1 \leq k \leq l \leq N$. A system $\mathcal{P} = \{\mathcal{J}_\nu\}_\nu = \{[k_\nu, l_\nu]\}_\nu$ of such intervals (repetitions are allowed) is called an n-fold covering (or n-covering, to shorter the notation) of $\mathbb{N}_N = \{1, 2, \cdots, N\}$ if every $n \in \mathbb{N}_N$ belongs to exactly n intervals from $\mathcal{P}$ (if $n = 1$, we speak of a simple covering). In (13), we have an n-fold covering of $\mathbb{N}_N$ supplied by the various support intervals of the coordinates of the vectors e^j, with multiplicities as expressed by θ_j. we now claim:

Every n-covering $\mathcal{P}$ of $\mathbb{N}_N$ is the union of n simple coverings. In fact, every interval $J = [k, l] \in \mathcal{P}$ with $l < N$ has the property that $l + 1$ is covered n times by $\mathcal{P} \setminus \{J\}$ while l is covered only $n - 1$ times. This is possible only if there is an interval in $\mathcal{P} \setminus \{J\}$ whose left endpoint is $l + 1$. The rest is done by induction.

This means that the integer-valued vector $\vartheta = (\vartheta_1, \cdots, \vartheta_{N(N+1)2})$ can be written as a sum of n vectors of the type $\epsilon = (\epsilon_1, \cdots, \epsilon_{N(N+1)/2})$, where $\epsilon_j \in \{0, 1\}$ for all j and

$$\sum_j \epsilon_j e^j = L;$$

each ϵ is then an "edge point" of $\mathcal{S}$. that is, θ is a convex combination of "edge points", as claimed. The proof is complete. ⊠

Lemma 5.19 *Let λ be a measure on $\mathbb{T}$ such that*

$$\lambda(I) \leq \alpha\kappa(I) + \alpha(\log 2)|I|_s, \tag{14}$$

where $\alpha > 0$ and $I \subset \mathbb{T}$ is an open arc. Then

$$\int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|} d\lambda(\zeta) \leq \alpha \log \frac{1}{1 - |z|} + C,$$

where C is a constant that only depends on α and $z \in \mathbb{D}$.

PROOF. Let $P_z(\zeta) = P(z, \zeta)$ be the Poisson kernel at $z \in \mathbb{D}$. By Fubini's theorem we have that

$$\begin{aligned} \int_{\mathbb{T}} P_z(\zeta) d\lambda(\zeta) &= \int_{\zeta \in \mathbb{T}} \left(\int_0^{P_z(\zeta)} dt \right) d\lambda(\zeta) = \int_0^\infty \left(\int_{\{P_z(\zeta) > t\}} d\lambda(\zeta) \right) dt \\ &= \int_0^\infty \lambda(\{P_z > t\}) dt. \end{aligned} \tag{15}$$

Now, by simplicity, let's assume that $z = r$ with $0 < r < 1$. It follows from (14) and (15) that

$$\begin{aligned}
\int_{\mathbb{T}} P_z(\zeta) d\lambda(\zeta) &= (1-r^2) \int_0^{1/(1-r)^2} \lambda\left(\left\{\zeta \in \mathbb{T} :\ |\zeta - r| < \sqrt{1/t}\right\}\right) \\
&\leq \alpha(1-r^2) \int_0^{1/(1-r)^2} \left(\frac{\sqrt{1/t}}{2\pi} \log \frac{2\pi e}{\sqrt{1/t}} + (\log 2) \frac{\sqrt{1/t}}{2\pi}\right) dt \\
&\leq \alpha(1-r^2) \frac{2 \log \dfrac{1}{1-r} + 2\log(2\pi)}{2\pi(1-r)} + C_1 \\
&\leq \alpha \log \frac{1}{1-r} + C_2.
\end{aligned}$$

where C_1, C_2 are some constant values that only depends on α and r. ⊠

We can now prove the main theorem of this section

Theorem 5.20 *Suppose $A = \{a_n\}_n$ is a sequence in $\mathbb{D}$. Suppose*

$$\Sigma(A, \mathfrak{s}_F) \leq \alpha \hat{\kappa}(F) + O(1)$$

holds for all finite subsets F of $\mathbb{T}$, where $O(1)$ is bounded independently of F. Then there exist a function $f \in \mathcal{A}^{-\alpha}$ such that A is its zero set.

PROOF. Without loss of generality, we can assume that $0 \notin A$. Let A_0 be a finite subsequence of A. Now we choose an arbitrary $w_0 \in \mathbb{T}$, construct as in Proposition 5.18 a maximal (A_0, α, w_0)-admissible measure μ_0, and for the function

$$f_0(z) = B_{A_0}(z)\Phi(z),$$

where B_{A_0} is the Blaschke product for A_0 and Φ is the outer function

$$\Phi(z) = \exp\left(\int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\mu_0(\xi)\right).$$

We are going to obtain an upper estimate for $\|f_0\|_{\mathcal{A}^{-\alpha}}$ and a lower estimate for $|f_0(0)|$, both independent of $A_0 \subset A$. To this end, we fix a point $\zeta \in \mathbb{T}$ and consider two subsequences of A_0: $A_0' = A_0 \cap \mathfrak{s}_{\{\zeta, -\zeta\}}$ and $A'' = A_0 \setminus A_0'$. Let $B_{A_0'}$ and $B_{A_0''}$ be the Blaschke products for A_0' and A_0'', respectively. For each $a_n \in A_0''$, let $\varpi_n = \varpi_\zeta(a_n)$ be its oblique projection. Form an atomic measure σ on $\mathbb{T}$ by placing at each ϖ_n a point mass of magnitude $\sigma_n = (1 - |a_n|^2)/2$, and let $\Psi = \Phi S_\sigma$, where S_σ is the singular inner function

$$S_\sigma(z) = \exp\left(-\int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\sigma(\xi)\right).$$

From its definition, we see that the measure σ has

$$\sigma(I) = \Sigma(A_0'', \mathfrak{h}_I) = \Sigma(A_0, \mathfrak{h}_I),$$

for each open arc I in the punctured circle $\mathbb{T} \setminus \{\zeta, -\zeta\}$. The (A_0, α, w_0)-admissibility of μ_0 means that

$$\mu_0(I) \leq \alpha\kappa(I) + \Sigma(A_0, \mathfrak{h}_I),$$

for any open arc I in $\mathbb{T} \setminus \{w_0\}$. We need this inequality for arcs that contain the point w_0, too. This is achieved by the following argument, if we pay a small price. If we partition an arc $I \subset \mathbb{T}$ into two arcs I_1 and I_2, then

$$|I|_s \log \frac{e}{|I|_s} \leq |I_1|_s \log \frac{e}{|I_1|_s} + |I_2| \log \frac{e}{|I_2|_s} \leq |I|_s \log \frac{e}{|I|_s} + (\log 2)|I|_s.$$

This implies that

$$\mu_0(I) \leq \alpha\kappa(I) + \alpha(\log 2)|I|_s + \Sigma(A_0, \mathfrak{h}_I)$$

holds for all arcs I, also those containing the point w_0.

The boundary measure for the zero-free function Ψ is $\mu_0 - \sigma$, and putting the above observations together, we have

$$(\mu_0 - \sigma)(I) \leq \alpha\kappa(I) + \alpha(\log 2)|I|_s$$

for every arc I in $\mathbb{T} \setminus \{\zeta, -\zeta\}$. We apply this to arcs having ζ as one endpoint. It follows from Lemma 5.19 that

$$\begin{aligned}
|\Psi(z)| &= \exp\left(\operatorname{Re} \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \, (d\mu_0 - d\sigma)\,(\xi)\right) \\
&= \exp\left(\int_{\mathbb{T}} P(z, \zeta)\,(d\mu_0 - d\sigma)\,(\xi)\right) \\
&\leq \exp\left(\alpha \log \frac{1}{1 - |z|} + C_1\right) \\
&= \frac{C_2}{(1 - |z|^2)^\alpha}
\end{aligned}$$

where C_1 and C_2 are some constants that only depends on α and $z = t\zeta$, for $0 \leq t < 1$. At this point, we apply Lemma 5.15, to get

$$|B_{A_0''}(z)| \leq |S_\sigma(z)|, \quad z = t\zeta,$$

for $0 \leq t < 1$. The point $\zeta \in \mathbb{T}$ is arbitrary, and hence $\|f_0\|_{\mathcal{A}^{-\alpha}} \leq C_2$.

We note that

$$\log |B_{A_0}| = -\Lambda(A_0, \mathbb{D}) \quad \text{and} \quad \log |\Phi(0)| = \mu_0(\mathbb{T}),$$

where the logarithmic sum function Λ is as in (4), so that for the function $f_0 = B_{A_0}\Phi$, we have

$$\log |f_0(0)| = -\Lambda(A_0, \mathbb{D}) + \mu_0(\mathbb{T}).$$

By Proposition 5.18 and the maximality of μ_0,

$$\mu_0(\mathbb{T}) = \inf \left\{\alpha\hat{\kappa}(F) + \Sigma(A, \mathbb{D} \setminus \mathfrak{s}_F) : \ F \subset \mathbb{T} \text{ is finite and } w_0 \in F\right\}.$$

We obtain

$$\begin{aligned}
\log \frac{1}{|f_0(0)|} &= \Lambda(A_0, \mathbb{D}) - \mu_0(\mathbb{T}) \\
&= -\inf \left\{\alpha\hat{\kappa}(F) + \Sigma(A, \mathbb{D} \setminus \mathfrak{s}_F) : \ F \subset \mathbb{T} \text{ is finite and } w_0 \in F\right\} \\
&\quad + \Sigma(A_0, \mathbb{D}) + [\Lambda(A_0, \mathbb{D}) - \Sigma(A_0, \mathbb{D})] \\
&= \sup \left\{\Sigma(A, \mathfrak{s}_F) - \alpha\hat{\kappa}(F) : \ F \subset \mathbb{T} \text{ is finite and } w_0 \in F\right\} \\
&\quad + [\Lambda(A_0, \mathbb{D}) - \Sigma(A_0, \mathbb{D})].
\end{aligned}$$

Since

$$0 \leq \log \frac{1}{t} - \frac{1}{2}(1-t^2) = O\left[(1-t)^2\right] \quad \text{as} \quad t \to 1,$$

and the assumption on the sequence $A = \{a_n\}_n$ easily implies (see the proof of Corollary 5.13)

$$\sum_n (1-|a_n|)^2 < +\infty,$$

we have that

$$\Lambda(A_0, \mathbb{D}) - \Sigma(A_0, \mathbb{D}) \sim \sum_{a_n \in A_0} (1-|a_n|) - \frac{1}{2} \sum_{a_n \in A_0} (1-|a_n|^2) = O(1),$$

with a bound that is independent of which particular finite subsequence A_0 we have picked. From the assumption of the theorem, we thus have

$$\log \frac{1}{|f_0(0)|} = O(1),$$

with a bound independent of $A_0 \subset A$.

Now, take a nested sequence of a finite subsequence of A, $A_1 \subset A_2 \subset A_3 \subset \cdots$, with $A = \cup_n A_n$, and construct as above functions f_n for each A_n. The functions $\{f_n\}$ form a normal family. Hence there is a subsequence $\{f_{n_k}\}_k$ converging to an analytic function f uniformly on compact subsets of $\mathbb{D}$; the function f is in $\mathcal{A}^{-\alpha}$ and its zero sequence is A. ⊠

Corollary 5.21 *Suppose A is a sequence in $\mathbb{D}$ with $\mathbb{D}^+(A) < \alpha$. Then A is the zero set of a function in $\mathcal{A}^{-\alpha}$.*

5.4 Zero sets of functions in $\mathcal{B}^p_\alpha$

In this section we consider zero sets for the spaces $\mathcal{B}^p_\alpha$. The main work was done in the previous sections and in Chapter 3; we only have to define what is an inner function for $\mathcal{B}^p_\alpha$ spaces and a proposition which involves such functions.

Definition 5.22 *A function φ in $\mathcal{B}^p_\alpha$ is called a $\mathcal{B}^p_\alpha$-inner function if*

$$\int_{\mathbb{D}} (|\varphi(z)|^p - 1) z^n dB^p_\alpha(z) = 0$$

for all nonngative integers n.

It follows easily from the above definition that a function φ in $\mathcal{B}^p_\alpha$ is an $\mathcal{B}^p_\alpha$-inner function if and only if

$$\int_{\mathbb{D}} |\varphi(z)|^p q(z) dB^p_\alpha(z) = q(0)$$

for every polynomial q, and this condition is clearly equivalent to

$$\int_{\mathbb{D}} |\varphi(z)|^p h(z) dB^p_\alpha(z) = h(0),$$

where h is any bounded harmonic function in $\mathbb{D}$. In particular, every $\mathcal{B}^p_\alpha$-inner function is a unit vector in $\mathcal{B}^p_\alpha$.

An obvious example of an $\mathcal{B}^p_\alpha$-inner function is a constant times a monomial.

The following lemma will show us that every $\mathcal{B}^p_\alpha$-inner function is a contractive multiplier from the classical Hardy space H^p into $\mathcal{B}^p_\alpha$ and we will obtain an estimate for such $\mathcal{B}^p_\alpha$-inner functions.

Lemma 5.23 *If φ is a $\mathcal{B}^p_\alpha$-inner function, then φ is a contractive multiplier from H^p into $\mathcal{B}^p_\alpha$, and consequently,*

$$|\varphi(z)| \leq \frac{1}{(1-|z|^2)^{\alpha-1/p}}, \quad z \in \mathbb{D}.$$

PROOF. Suppose $f \in H^p$ and let h be the least harmonic majorant of $|f(z)|^p$. More explicitly,

$$h(z) = \frac{1}{2\pi}\int_{\mathbb{D}} P(e^{it}, z)|f(e^{it})|^p dt, \quad z \in \mathbb{D},$$

where $P(e^{it}, z)$ is the Poisson kernel at $z \in \mathbb{D}$. By Fatou's lemma and the definition of $\mathcal{B}^p_\alpha$-inner functions,

$$\int_{\mathbb{D}} |\varphi(z)|^P h(z) dB^p_\alpha(z) \leq \liminf_{r\to 1^-} \int_{\mathbb{D}} |\varphi(z)|^p h_r(z) dB^p_\alpha(z) = h(0),$$

where $h_r(z) = h(rz)$ for $z \in \mathbb{D}$. It follows that

$$\int_{\mathbb{D}} |\varphi(z)f(z)|^p dB^p_\alpha(z) \leq \int_{\mathbb{D}} |\varphi(z)|^p h(z) dB^p_\alpha(z) \leq h(0) = \|f\|^p_{H^p},$$

so that φ is a contractive multiplier from H^p into $\mathcal{B}^p_\alpha$.

for any $z \in \mathbb{D}$, consider the function

$$f_z(w) = \left(\frac{1-|z|^2}{(1-\overline{z}w)^2}\right)^{1/p}, \quad w \in \mathbb{D}.$$

Then f_z is a unit vector in H^p, and so φf_z has norm less than or equal to 1 in $\mathcal{B}^p_\alpha$. Moreover, by the estimation (2) of Chapter 2, we know that

$$|\varphi(z) f_z(z)| \leq \frac{1}{(1-|z|^2)^\alpha}. \tag{16}$$

It follows that

$$|\varphi(z)| \leq \frac{1}{(1-|z|^2)^{\alpha-1/p}}, \quad z \in \mathbb{D},$$

as claimed. ⊠

The following proposition will exhibit a close relation between $\mathcal{B}^p_\alpha$-inner functions and invariant subspaces of $\mathcal{B}^p_\alpha$. In particular, such proposition will give us more examples of $\mathcal{B}^p_\alpha$-inner functions.

We will say that a closed subspace I of $\mathcal{B}^p_\alpha$ is invariant if $zf \in I$ whenever $f \in I$. It is easy to see that a closed subspace I is invariant if and only if it is closed under multiplication by bounded analytic functions.

A convenient example of invariant subspace is the following one. Consider $A = \{a_n\}_n$ a $\mathcal{B}^p_\alpha$-zero set, if I_A consists of all functions in $\mathcal{B}^p_\alpha$ whose zero set contain A (counting multiplicities), then I_A is an invariant subspace of $\mathcal{B}^p_\alpha$. These subspaces are called zero-based invariant subspaces.

Proposition 5.24 *Let $I_A \subset \mathcal{B}^p_\alpha$ be the zero-based invariant subspaces as above. Suppose that G is any function that solves the extremal problem*

$$\sup\left\{Ref^{(n)}(0) : f \in I_A, \quad \|f\|_{\mathcal{B}^p_\alpha} \leq 1\right\},$$

where n is the smallest nonnegative integer such that there exists a function $f \in I_A$ with $f^{(n)}(0) \neq 0$. Then G is a $\mathcal{B}^p_\alpha$-inner function.

PROOF. It is obvious that G is a unit vector. We will prove the proposition by a variational argument.

Fix a positive integer k, and set

$$re^{i\theta} = \int_{\mathbb{D}} |G(z)|^p z^k dB^p_\alpha(z),$$

where $0 < r < 1$ and $-\pi < \theta \leq \pi$. For any complex number λ, we consider the function

$$f_\lambda(z) = \frac{(1+\lambda z^k)G(z)}{\|(1+\lambda z^k)G\|_{\mathcal{B}^p_\alpha}}.$$

Since f_λ is a unit vector in I_A, the extremal property of G gives

$$\text{Re } f^{(n)}(0) \leq \text{Re } G^{(n)}(0).$$

This implies that

$$1 \leq \int_{\mathbb{D}} |G(z)|^p |1+\lambda z^k|^p dB^p_\alpha(z)$$

for all $\lambda \in \mathbb{C}$, so that

$$1 \leq 1 + p\text{Re}\left[\lambda \int_{\mathbb{D}} |G(z)|^p z^k dB^p_\alpha(z)\right] + O(|\lambda|^2).$$

Put $\lambda = -\varepsilon e^{i\theta}$, where $\varepsilon > 0$ is small and θ is as above. We obtain that

$$0 \leq -r + O(\varepsilon).$$

Letting $\varepsilon \to 0$, we see that $r = 0$, and so G is $\mathcal{B}^p_\alpha$-inner. ⊠

We are almost prepared to our main theorems, we only need to do some remarks that will be useful. First, by definition we have that for $0 < p < +\infty$ and $1/p < \alpha < +\infty$,

$$f \in \mathcal{B}^p_\alpha \quad \Leftrightarrow \quad \int_{\mathbb{D}} \left(|f(z)|(1-|z|^2)^\alpha\right)^p \frac{dA(z)}{(1-|z|^2)^2} < +\infty, \tag{17}$$

and for any $0 < \beta$ we have that

$$f \in \mathcal{A}^{-\beta} \quad \Leftrightarrow \quad (1-|z|^2)^\beta |f(z)| \leq C, \quad z \in \mathbb{D}, \tag{18}$$

where $C > 0$ is a constant that depends on f. Hence it follows from (17) and (18) that, (*i*) If $f \in \mathcal{A}^{-\alpha} \Rightarrow f \in \mathcal{B}^p_{\alpha+1/p+\epsilon}$, for all $\epsilon > 0$.

(*ii*) If $f \in \mathcal{B}^p_{\alpha+1/p} \Rightarrow$ there exists $g \in \mathcal{A}^{-\alpha}$ such that the zero set of g contains the zero set of f, counting multiplicities.

We are prepared to derive some very sharp conditions that are necessary or sufficient for a sequence A to be a zero set of a function in $\mathcal{B}^p_\alpha$. For $0 < p < +\infty$ and $1/p < \alpha < +\infty$, let

$$\mathcal{B}^p_{\alpha-} = \bigcup_{\beta:\ \beta<\alpha} \mathcal{B}^p_\beta,$$

and

$$\mathcal{B}^p_{\alpha+} = \bigcap_{\beta:\ \alpha<\beta} \mathcal{B}^p_\beta.$$

Notice that $\mathcal{B}^p_{\alpha-} \subseteq \mathcal{B}^p_\alpha \subseteq \mathcal{B}^p_{+\alpha}$ (see Theorem 2.5).

Theorem 5.25 *Suppose* $0 < p < +\infty$, $1/p < \alpha < +\infty$, *and that* A *is a sequence in* $\mathbb{D}$. *Then* A *is a zero for* $\mathcal{B}^p_{\alpha+}$ *if and only if* $D^+(A) \leq \alpha - 1/p$.

PROOF. If $D^+(A) \leq \alpha - 1/p$, then by Theorem 5.2, A is a zero set of a function in $\mathcal{A}_+^{-\beta+1/p}$. Since $\mathcal{A}_+^{-\beta+1/p} \subseteq \mathcal{B}^p_{\alpha+}$, we conclude that A is a zero set of some function in $\mathcal{B}^p_{\alpha+}$.

Conversely, if A is a zero set for $\mathcal{B}^p_{\alpha+}$, then A is a zero set for some function in $\mathcal{B}^p_\beta$, for all $\beta > \alpha$. Let G_β be an extremal function for the zero-based invariant subspace I_A of $\mathcal{B}^p_\beta$. Hence, by Proposition 5.24 we know that G_β is a $\mathcal{B}^p_\beta$-inner function and by Lemma 5.23 we have that $G_\beta \in \mathcal{A}^{-\beta+1/p}$. Let A_β be the zero set of G_β. Then $A \subset A_\beta$, and hence by Theorem 5.2,

$$D^+(A) \leq D^+(A_\beta) \leq \beta - 1/p.$$

Letting $\beta \to \alpha^-$, we arrive at $D^+(A) \leq \alpha - 1/p$. ⊠

Theorem 5.26 *Suppose* $0 < p < +\infty$, $1/p < \alpha < +\infty$, *and* A *is a sequence in* $\mathbb{D}$. *Then* A *is a zero set for* $\mathcal{B}^p_{\alpha-}$ *if and only if* $D^+(A) < \alpha - 1/p$.

PROOF. If $D^+(A) < \alpha - 1/p$, then by Corollary 5.21 A is a zero set for a function in $\mathcal{A}^{-\alpha+1/p+\varepsilon}$, for all $0 < \varepsilon$ small enough. Since $A^{-\alpha+1/p+\varepsilon} \subset \mathcal{B}^p_{\alpha-}$, we conclude that A is a zero set of a function in $\mathcal{B}^p_{\alpha-}$.

Now, assume that A is a zero set of a function in $\mathcal{B}^p_{\alpha-}$, then there exists $1/p < \beta < \alpha$ such that A is a zero set for a function in $\mathcal{B}^p_\beta \subset \mathcal{B}^p_{\alpha-}$. Let G_β be the extremal function for the zero-based invariant subspace I_A of $\mathcal{B}^p_\beta$, then by Proposition 5.24 we know that G_β is a $\mathcal{B}^p_\beta$-inner function and by Lemma 5.23 we have that $G_\beta \in \mathcal{A}^{-\beta+1/p}$. Let A_β be the zero set of G_β, hence, $A \subset A_\beta$ and by Theorem 5.2,

$$D^+(A) \leq D^+(A_\beta) \leq \beta - 1/p < \alpha - 1/p.$$

⊠

Note that the result above simply state that the condition $D^+(A) \leq \alpha - 1/p$ is a necessary condition and the condition $D^+(A) < \alpha - 1/p$ is a sufficient condition for A to be a zero set of a function in $\mathcal{B}^p_\alpha$.

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