UNIVERSITAT DE BARCELONA

ADVANCED MATHEMATICS
MASTER'S FINAL PROJECT

QUASICONFORMAL SURGERY IN TRANSCENDENTAL DYNAMICS

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Contents

Abstract

The goal of this thesis is to survey some results on quasiconformal analysis and quasiconformal surgery, today an essential tool for any researcher in the field of Complex Dynamics.

The first part of the project will be on quasiconformal geometry, to understand the necessary tools in analysis that lead to the Measurable Riemann Mapping Theorem, the main tool to perform surgery. These include several definitions of quasiconformal mappings, both analytic and geometric. The second part will consist on several applications to holomorphic dynamics, with emphasis on those in transcendental dynamics, showing the power of this technique.

A third part of the project will be dedicated to some original work on a particular family of meromorphic transcendental maps. More precisely, we study the family of transcendental meromorphic maps

$$f_\lambda(z) = \lambda\left(\frac{e^z}{z+1} - 1\right),$$

and we prove, using quasiconformal surgery, that for certain parameter values the Julia set contains what is known as a Cantor Bouquet.

Resum

L'objectiu d'aquesta tesi es recollir alguns resultats d'anàlisi quasiconforme i cirurgia quasiconforme, que esdevenen avui eines fonamentals per a qualsevol investigador o investigadora en el camp de la Dinàmica Complexa.

La primera part del treball tracta sobre la geometria quasiconforme, per tal d'entendre les eines analítiques necessàries que ens porten al Teorema de Riemann mesurable, l'eina principal per fer cirurgia. Aquestes inclouen diverses definicions de les aplicacions quasiconformes, tant analítiques com geomètriques. La segona part tracta sobre diverses aplicacions d'aquests resultats a la dinàmica holomorfa, fent èmfasi en aquelles per a dinàmica transcendent, mostrant així el potencial d'aquesta tècnica.

La tercera part del treball va dedicada a obtenir resultats originals sobre una família de funcions meromorfes transcendents. Més precisament, estudiem la familia de funcions transcendents meromorfes

$$f_\lambda(z) = \lambda\left(\frac{e^z}{z+1} - 1\right),$$

i demostrem, mitjançant cirurgia quasiconforme, que per a uns determinats paràmetres, el conjunt de Julia conté el que s'anomena un Cantor Bouquet.

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Introduction

Iteration theory appears in daily problems, often from a mathematical model regarded as a dynamical system. In many cases, methods from numerical analysis require iteration, for example the Newton method aims at approximating the solutions, real or complex, of the equation $f(z) = 0$, by considering the iterative function

$$N_f(z) = z - \frac{f(z)}{f'(z)}.$$

Given an initial condition $z \in \mathbb{C}$ we consider the sequence

$$z \longmapsto N_f(z) \longmapsto N_f(N_f(z)) \longmapsto \cdots$$

and we wish to determine under which conditions this sequence converges, and if it does so, whether it converges to a zero of f or not.

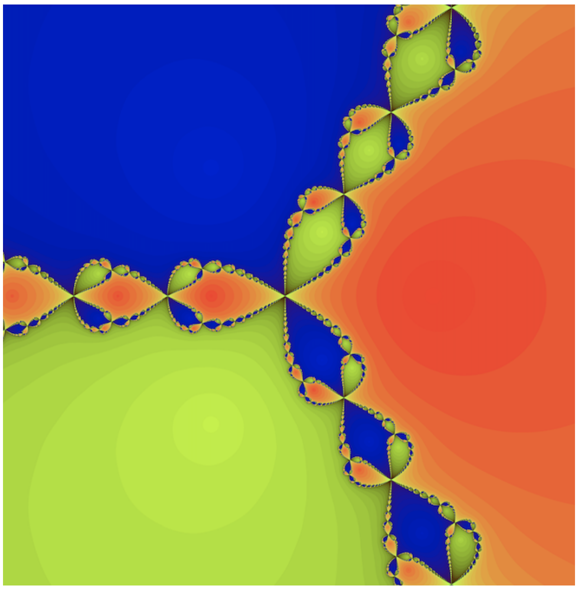

Figure 1: Newton method applied to the cubic polynomial $P(z) = z^3 - 1$. Points of the same color converge to the same root of P under iteration. The boundaries of these basins form the Julia set of this rational map.

The mathematical area that aims to study the iteration of general meromorphic functions of one-complex variable is known as Holomorphic Dynamics. The key ingredient for the study of iteration came in the beginning of the twentieth century with the notion of normal family introduced by Montel, which leads to the definition of the central objects in the field: the Fatou set, where the iterates have Montel's normality (equicontinuity in the spherical metric), and its complement, the Julia set. Julia [Jul] and Fatou [Fat], who set the basis of Holomorphic

Dynamics, based his approach on Montel's theory for the case of rational maps. Fatou, in fact, extended in 1926 some results to the case of transcendental entire functions (entire functions with infinitely many terms in their series expansions). However, he did not consider the more general case of transcendental meromorphic functions, i.e., functions with an essential singularity at ∞, which are allowed to have poles. The interest for these functions is double. The essential singularity, on the one hand, adds a lot of chaos to the dynamical system, mainly because of Picard's Theorem, which states that in each punctured neighborhood of ∞, these functions assume each value of the Riemann Sphere $\mathbb{C}_\infty$, with at most two exceptions, infinitely often. Hence, given a point $z \in \mathbb{C}$, if its orbit

$$\mathcal{O}_f^+(z) = \{f^n(z)\colon n \in \mathbb{N}\}$$

is near ∞ at some moment, after one iteration it can land at almost any place of the plane. On the other hand, the presence of poles allows for more generality, since ∞ is not required to be an omitted value.

The work of Julia and Fatou left many open questions, like Fatou's No-Wandering Domains Conjecture for rational maps, which was proved by Sullivan in 1985 in his famous paper Quasiconformal Homeomorphisms and Dynamics I. Solution of the Fatou-Julia Problem on Wandering Domains (see [Sul]). This paper introduced quasiconformal analysis techniques into holomorphic dynamics, which meant remarkable advances in the field. The notion of quasiconformality is weaker than conformality (see [Ahl, BF, LV]). While the second one preserve angles, the first one distorts them in a bounded way. However, the theory of quasiconformal functions was first introduced with the work of Grötzsch, which was taken into two different directions. On the one hand, Teichmüller showed a connection between quasiconformality and the function theory of Riemann surfaces, and on the other hand, Morrey explored its relation with the solution of PDEs. In this thesis we develop the theory of quasiconformal mappings in Chapter 1.

Quasiconformal surgery is a mechanism to construct holomorphic maps with prescribed dynamics, which is the ultimate goal of Chapter 2. The word surgery emerges because the construction often needs to paste different maps together to obtain a model map (not holomorphic in general), which, together with the Measurable Riemann Mapping Theorem as the essential tool (here we refer to it as the Integrability Theorem), allows to obtain a holomorphic "copy" of the model, i.e. a holomorphic map with the same dynamics. The use of this tool has been effective to prove several important results, like the No-Wandering Domains theorem; or the Straightening Theorem, that explains why we find copies of polynomial Julia sets in the phase plane of rational or transcendental functions. Many surgery applications can be found in [BF].

In this thesis we survey some results on quasiconformal analysis. Which, together with some knowledge of the general theory of Complex Dynamics, lead to some applications, that we present at the end of Chapter 2.

In the last part of the project, Chapter 3, we apply this work to study the family of maps,

$$f_\lambda(z) = \lambda\left(\frac{e^z}{z+1} - 1\right).$$

This family is interesting since it is the simplest meromorphic map with two singularities of the inverse map: $z = 0$, which is a fixed critical point ($f'_\lambda(0) = 0$), and $-\lambda$, which is an asymptotic value ($\lim_{z\to-\infty} f_\lambda(z) = -\lambda$), whose orbit depends on the parameter λ. It has also one single pole, $z = -1$, which is not omitted except for $\lambda = 1$. One can view f_λ as the meromorphic analogue to the well-known Milnor family of cubic polynomials $P_a(z) = z^2(z-a)$ [Mil1] or its entire version $\lambda z^2 e^z$ [FG2].

Opposed to these two cases, the basins of attraction of f_λ are not simply connected and the relation between this topological property and the dynamics of f_λ promises to be a source of interesting problems. In fact, based on previous work in [Rod], we prove here Theorem 3.13 (see

Section 3.3.1), which describes the connectivity properties of the basin of attraction of $z = 0$ in terms of the location of the asymptotic value $-\lambda$.

Nevertheless, our main result in this thesis concerns the Julia sets of the maps f_λ. As mentioned above, an important Theorem in holomorphic dynamics (see [DH]) states that, under certain conditions, one finds copies of polynomial Julia sets (entire) in the dynamical plane of rational or transcendental maps. Inspired by this fact, we shall prove that, for certain parameter values, the Julia set of f_λ contains an invariant copy of the Julia set of the exponential map $g_a(z) = a(e^z - 1)$, a well studied object known as a Cantor Bouquet. More precisely, we apply quasiconformal surgery to prove the following:

Theorem A. *Let $f_\lambda(z) = \lambda(e^z/(z+1) - 1)$ and $g_a(z) = a(e^z - 1)$, for $\lambda, a \in \mathbb{C} \setminus \{0\}$. Denote by $J(f_\lambda)$ and $J(g_a)$ their Julia sets. Then, there exists $\mathcal{C} \subset J(f_{-1/e}))$ such that $f_{-1/e}(\mathcal{C}) = \mathcal{C}$ and $f_{-1/e}$ on $\mathcal{C}$ is quasiconformally conjugate to $g_{1/e}$ on $J(g_{1/e})$. In particular, $\mathcal{C}$ is homeomorphic to $J(g_{1/e})$ and both are homeomorphic to a Cantor Bouquet.*

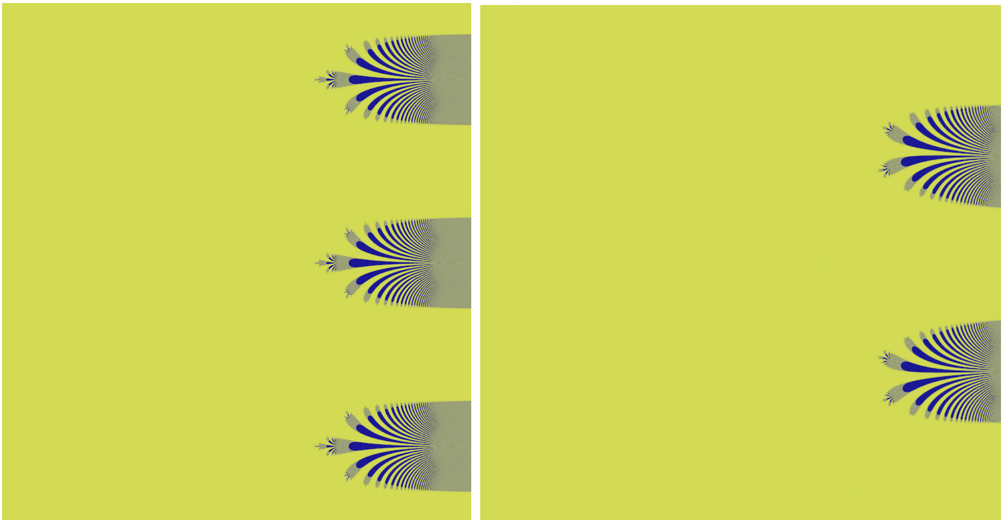

Figure 2: In the left side there is the dynamical plane of $g_{1/e} = (e^z - 1)/e$ and in the right side the dynamical plane of $f_{-1/e}$. The Julia set in both cases is colored in blue and it illustrates the statement of Theorem A.

A Cantor Bouquet is, loosely speaking, a topological structure homeomorphic to a Cantor set cross a one-sided infinite segment (see Definition 3.3 in Section 3.1). Cantor Bouquets appear as Julia sets of a wide variety of transcendental entire functions like $g_a(z)$, the well-known exponential family.

To prove Theorem A we use surgery to convert our meromorphic map f_λ into an entire map, in the same spirit as in the Straightening Theorem (see [DH]). Nevertheless, we must overcome the additional difficulty of dealing with unbounded sets and infinite degree, something that does not occur for polynomials. Because of these challenges, we prove our result for just one value of λ, hoping, however, that this technique can be generalized in the future to a larger set of parameter values and, even more ambitiously, to classes of functions satisfying similar conditions.

Chapter 1

Quasiconformal analysis and geometry

Quasiconformal mappings have recently come to play a very active part in the theory of analytic functions of one complex variable for several reasons. To list some of them, they are a natural generalization of conformal mappings and many theorems relating conformal mappings use only the quasiconformality (hence we are interested in determine in which cases conformality is essential or not). In this thesis we shall see how they can be applied to Holomorphic Dynamics.

1.1 Differentiable quasiconformal mappings

Before we address the general definition of quasiconformal mappings it will be useful for our purpose to work with differentiable maps. After that we will use this part and the a.e. differentiability to obtain properties of more general quasiconformal maps.

1.1.1 Grötzsch's definition

The notion of quasiconformal mapping was introduced by H. Grötzsch back in 1928. If Q is a square and R is a rectangle (which is not a square) then there is no conformal mapping of Q on R which maps vertices on vertices, Grötzsch asked for the most nearly conformal mapping of this kind.

We show now the definition that he gave:

Let $f(z) = w = u + iv$ be a $\mathcal{C}^1$ homeomorphism from one region to another. At a point z_0 it induces a linear mapping of the differentials

$$\begin{aligned} du &= u_x dx + i u_y dy \\ dv &= v_x dx + i v_y dy \end{aligned} \tag{1.1}$$

which yields

$$dw = f_z dz + f_{\bar{z}} d\bar{z}, \tag{1.2}$$

where

$$f_z = \frac{1}{2}(f_x - i f_y) \quad , \quad f_{\bar{z}} = \frac{1}{2}(f_x + i f_y).$$

Note that (1.1) represents an affine transformation from the (dx, dy) plane to the (du, dv) plane. This transformation maps circles about the origin into similar ellipses.

The first goal is to determine the ratio between the axes of this ellipse as well as their direction.

We can write $du^2 + dv^2 = Edx^2 + 2Fdxdy + Gdy^2$ (first fundamental form written in terms of a metric tensor), where $E = u_x^2 + u_y^2, F = u_x u_y + v_x v_y$ and $G = u_y^2 + v_y^2$. Moreover, as something

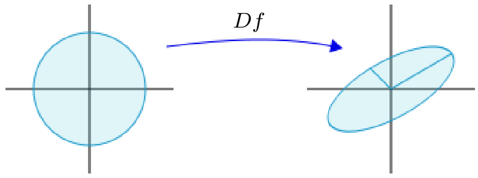

Figure 1.1: Representation of the action of the differential when applied to a circle.

which is intended to be considered as a metric, it can be written as

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix},$$

which has eigenvalues

$$\lambda_{\pm} = \frac{1}{2}\left(E + G \pm \sqrt{(E-G)^2 + 4F^2}\right).$$

Hence, the ratio of the axes is

$$\sqrt{\frac{\lambda_+}{\lambda_-}} = \frac{E + G + \sqrt{(E-G)^2 + 4F^2}}{2\sqrt{EG - F^2}}. \tag{1.3}$$

However, here the complex condition is way more convenient. We have,

$$f_z = \frac{1}{2}(u_x + v_y) + \frac{i}{2}(v_x - u_y) \quad , \quad f_{\bar z} = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y).$$

Hence, $J = |f_z|^2 - |f_{\bar z}|^2 = u_x v_y - u_y v_x$ is the Jacobian, which is positive for orientation preserving maps and negative for orientation reserving maps. To simplify, it will just be considered the orientation preserving case (positive Jacobian).

Note that from (1.2), $(|f_z| - |f_{\bar z}|)|dz| \leq |dw| \leq (|f_z| + |f_{\bar z}|)|dz|$, therefore the ratio of the major axis to the minor axis is

$$D_f = \frac{|f_z| + |f_{\bar z}|}{|f_z| - |f_{\bar z}|} \geq 1 \tag{1.4}$$

and we refer to it as the **dilatation** at the point z. However, it is often more convenient to consider

$$d_f = \frac{|f_{\bar z}|}{|f_z|} < 1 \tag{1.5}$$

which is related to D_f by the identities

$$D_f = \frac{1 + d_f}{1 - d_f} \quad , \quad d_f = \frac{D_f - 1}{D_f + 1}.$$

We define the **complex dilatation** (or **Beltrami coefficient**)

$$\mu_f = \frac{f_{\bar z}}{f_z}. \tag{1.6}$$

and the **second complex dilatation**

$$\nu_f = \frac{f_{\bar z}}{\overline{f_{\bar z}}} = \left(\frac{f_z}{|f_z|}\right)^2 \mu_f.$$

Definition 1.1. *The mapping f is said to be* ***quasiconformal*** *if D_f is bounded. We say that f is K-**quasiconformal** (from now on K-gc) of $D_f \leq K$.*

The restriction to $\mathcal{C}^1$ is unnatural and, as we already said, we want to get rid of it.

1.1.2 Composed mappings

Before we introduce the general definition of quasiconformal mappings, we still need to develop some properties of differentiable quasiconformal mappings, since as we already said, we will see that quasiconformal maps aree differentiable a.e. and the relations proved here will also hold.

We aim to determine the complex derivatives and complex dilatations of a composed mapping $g \circ f$. To tackle a problem related with the notation we introduce an intermediate variable $\xi = f(z)$.

Note that without supposing that we deal with holomorphic functions, to avoid mistakes our functions need to be regarded as of the form $f(z,\bar{z})$, $g(\xi,\bar{\xi})$. Hence,

$$\begin{aligned}\frac{\partial}{\partial z}(g(f(z),\bar{f}(z))) &= \frac{\partial g}{\partial \xi}(f(z),\bar{f}(z))\frac{\partial f}{\partial z}(z) + \frac{\partial g}{\partial \bar{\xi}}(f(z),\bar{f}(z))\frac{\partial \bar{f}}{\partial z}(z) = \\ &= (g_\xi \circ f)f_z + (g_{\bar{\xi}} \circ f)\bar{f}_z, \\ \frac{\partial}{\partial \bar{z}}(g(f(z),\bar{f}(z))) &= \frac{\partial g}{\partial \xi}(f(z),\bar{f}(z))\frac{\partial f}{\partial \bar{z}}(z) + \frac{\partial g}{\partial \bar{\xi}}(f(z),\bar{f}(z))\frac{\partial \bar{f}}{\partial \bar{z}}(z) = \\ &= (g_\xi \circ f)f_{\bar{z}} + (g_{\bar{\xi}} \circ f)\bar{f}_{\bar{z}}.\end{aligned}$$

Thus, if we call J_f the Jacobian of f,

$$\begin{aligned} g_\xi \circ f &= \frac{1}{J_f}\left((g\circ f)_z \bar{f}_{\bar{z}} - (g\circ f)_{\bar{z}}\bar{f}_z\right) \\ g_{\bar{\xi}} \circ f &= \frac{1}{J_f}\left((g\circ f)_{\bar{z}} f_z - (g\circ f)_z f_{\bar{z}}\right)\end{aligned} \tag{1.7}$$

If we apply (1.7) to $g = f^{-1}$, we obtain

$$(f^{-1})_\xi \circ f = \bar{f}_{\bar{z}}/J_f \quad , \quad (f^{-1})_{\bar{\xi}} \circ f = -f_{\bar{z}}/J_f.$$

So,

$$(f^{-1})_\xi = \frac{1}{J_f}\bar{f}_{\bar{z}} \circ f^{-1} \quad , \quad (f^{-1})_{\bar{\xi}} = -\frac{1}{J_f} f_{\bar{z}} \circ f^{-1}.$$

Hence the Beltrami coefficient of f^{-1} is

$$\mu_{f^{-1}} = -\left(\frac{f_{\bar{z}}}{\bar{f}_{\bar{z}}}\right) \circ f^{-1},$$

and since $\overline{(f_z)} = \bar{f}_{\bar{z}}$ and $\overline{f_{\bar{z}}} = \bar{f}_z$, we have

$$d_{f^{-1}} = d_f \circ f^{-1} \tag{1.8}$$

which means that, if the points are properly chosen, a map and its inverse have the same dilatation. Furthermore, from (1.7) we obtain

$$\begin{aligned}\mu_g \circ f &= \frac{(g\circ f)_{\bar{z}} f_z - (g\circ f)_z f_{\bar{z}}}{(g\circ f)_z \bar{f}_{\bar{z}} - (g\circ f)_{\bar{z}}\bar{f}_z} = \frac{f_z}{\bar{f}_{\bar{z}}}\frac{(g\circ f)_{\bar{z}} - (g\circ f)_z \mu_f}{(g\circ f)_z - (g\circ f)_{\bar{z}}(\overline{f_{\bar{z}}}/\overline{f_z})} = \\ &= \frac{f_z}{\bar{f}_{\bar{z}}}\frac{\mu_{g\circ f} - \mu_f}{1 - \overline{\mu_f}\mu_{g\circ f}}.\end{aligned}$$

Observe that:

- If g is conformal, then $\mu_g = 0$, which implies that

$$0 = \frac{f_z}{\overline{f_{\bar{z}}}}(\mu_{g\circ f} - \mu_f),$$

therefore, $\mu_{g\circ f} = \mu_f$.

- If f is conformal, then $\mu_f = 0$ and

$$\mu_g \circ f = \mu_{g\circ f}\frac{f_z f_z}{\overline{f_z f_z}} = \mu_{g\circ f}\left(\frac{f'}{|f'|}\right)^2. \tag{1.9}$$

Moreover,

$$\begin{aligned}\nu_g \circ f &= \frac{g_\xi \circ f}{\bar{g}_{\bar{\xi}}}\frac{g_\xi \circ f}{g_\xi \circ f} = (\mu_g \circ f)\frac{g_\xi \circ f}{\bar{g}_{\bar{\xi}}} = \mu_{g\circ f}\left(\frac{f'}{|f'|}\right)^2 \frac{g_\xi \circ f}{\overline{g_\xi} \circ f}\frac{g_\xi \circ f}{g_\xi \circ f} = \\ &= \mu_{g\circ f}\left(\frac{g_\xi \circ f}{|g_\xi \circ f|}\right)^2\left(\frac{f'}{|f'|}\right)^2.\end{aligned}$$

Hence, $\nu_g \circ f = \nu_{g\circ f}$.

So in both cases, the modulus of the dilatation is invariant with respect to all conformal transformations (in one case the complex dilatation is invariant, and in the other case the second complex dilatation).

1.2 General definition of quasiconformal mappings

As we have already pointed out, the restriction to differentiable mappings is unnecessary. The goal of this section is to present two equivalent definitions of what we will understand as quasiconformal mappings, a notion that generalizes the one in Definition 1.1. They will give different approaches to deal with this maps. The references for this section have been [Ahl, BF, LV].

1.2.1 Geometric approach

Here all mappings ϕ will be **topological**, i.e. orientation preserving homeomorphisms, from a region Ω to a region Ω'.

Definition 1.2. *A **quadrilateral** is a Jordan region Q, $\bar{Q} \subset \Omega$, together with a pair of disjoint closed arcs on the boundary (the b-arcs).*

*Its **module** $M(Q) = a/b$ is determined by conformal mapping on a rectangle (by Riemann's mapping theorem).*

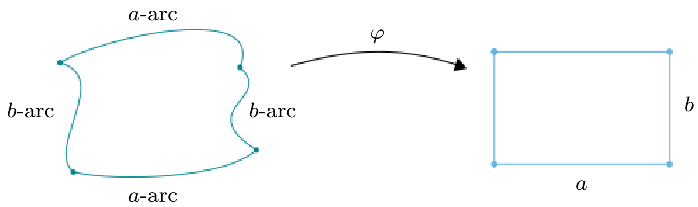

Figure 1.2: Representation of the moduli, here we distinguish between the a-arcs and the b-arcs, which are mapped to the sides

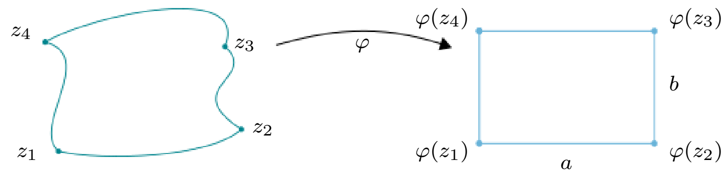

Figure 1.3: Representation of the moduli, computed with the image of the distinguished points of the quadrilateral under the Riemann map φ.

We can also put the emphasis on the vertices by defining a quadrilateral $Q = Q(z_1, z_2, z_3, z_4)$ as a Jordan domain in $\mathbb{C}$ with an ordered sequence of boundary points which agree with the positive orientation of Q. Then by Riemann's mapping theorem, we can map Q to rectangle as in Figure 1.3.

Therefore

$$mod(Q) = M(Q) := \frac{|\varphi(z_1) - \varphi(z_2)|}{|\varphi(z_2) - \varphi(z_3)|}.$$

Definition 1.3. *Given a quadrilateral Q, we define its conjugate Q^*, as Q but identifying in the previous definition the a-arcs.*

Note that by definition, $M(Q^*) = 1/M(Q)$ and that if $Q = Q(z_1, z_2, z_3, z_4)$, then $Q^* = Q(z_2, z_3, z_4, z_1)$.

Definition 1.4 (Geometric definition of quasiconformal mapping)**.** *We say that ϕ is K-qc if the modules of quadrilaterals are K-quasi-invariant, which means that given a quadrilateral Q, $\bar{Q} \subset \Omega$, then*

$$M(\phi(Q)) \leq KM(Q).$$

i.e.

$$K_\phi := \sup_{\bar{Q} \subset \Omega} \frac{M(\phi(Q))}{M(Q)} \leq K.$$

There is also another geometric approach which consists on the invariance of the moduli of annuli under ϕ, for more details see [BF].

Note that if ϕ is K-qc and Q is a quadrilateral, since Q^* is another quadrilateral then

$$1/M(\phi(Q)) = M(\phi(Q^*)) \leq KM(Q^*) = K/M(Q),$$

hence, we have the double inequality $K^{-1}M(Q) \leq M(\phi(Q)) \leq KM(Q)$.

Proposition 1.5 (Properties of qc maps)**.** *Let ϕ be a K-qc mapping.*

(a) If ϕ is $\mathcal{C}^1$, then the definitions agree.

(b) ϕ and ϕ^{-1} are both K-qc.

(c) The class of K-qc mappings is invariant under composition with conformal mappings.

(d) If $\tilde{\phi}$ is $\tilde{K}$-qc, then the composition (whenever it has sense) is $K\tilde{K}$-qc.

Proof. The first one will be proved in more detail when we prove the equivalence between the geometric definition (that we have just presented) and the analytic definition, which will be presented next. The basic idea is that "lengths" and "areas" are just the integrals of somewhat with respect to ϕ_z, $\phi_{\bar{z}}$, which we have under control.

To prove (b), note that we have seen $K^{-1}M(Q) \leq M(\phi(Q))$, hence $M(Q) \leq KM(\phi(Q))$, hence ϕ^{-1} is K-qc.

Note that we define the modulus according to conformal mappings, so it is preserved under composition with conformal maps and hence (c).

Finally, the proof of (d) is straightforward. ∎

1.2.2 Analytic approach

Here we introduce a couple of equivalent definitions that will be regarded as the analytic definition.

We first present some results regarding the absolute continuity of a function and some of its properties. For more information and details see [LV, Roy].

We use the notations $\phi_z = \partial_z \phi$ and $\phi_{\bar{z}} = \partial_{\bar{z}} \phi$, whenever they make sense.

Definition 1.6 (Absolutely continuous function). *A function $f\colon [a,b] \to \mathbb{C}$ is said to be **absolutely continuous** on $[a,b]$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sum_{i=1}^{n} |f(y_i) - f(x_i)| < \varepsilon$$

whenever $\{[x_i, y_i]\colon i = 1, \ldots, n\} \subset [a,b]$ is a finite collection of mutually disjoint subintervals of $[a,b]$ with $\sum_{i=1}^{n} |y_i - x_i| < \delta$.

Remark 1.7. *The following hold:*

- *If f is absolutely continuous on $[a,b]$, then f is uniformly continuous on $[a,b]$.*
- *If f is Lipschitz on $[a,b]$, then f is absolutely continuous on $[a,b]$.*
- *If f is integrable on $[a,b]$, then*

$$F(x) := \int_a^x f(t)dt \ , \ a \leq x \leq b$$

  is absolutely continuous on $[a,b]$.

Proposition 1.8 (Properties of absolutely continuous functions). *Let f be an absolutely continuous function on $[a,b]$. Then the following hold:*

(a) f has bounded variation on $[a,b]$.

(b) f maps sets of measure zero to sets of measure zero.

(c) f has a derivative almost everywhere.

(d) If $f'(x) = 0$ a.e., then f is constant.

(e) f is the indefinite integral of its derivative.

Definition 1.9. *We say that a function $u(x,y) = u(z)$ is **ACL** (absolutely continuous in lines) in the region Ω, if for every closed rectangle $R \subset \Omega$ with sides parallel to the x and the y-axes, $u(x,y)$ is absolutely continuous on a.e. horizontal and a.e. vertical line in R.*

Note that by the properties stated before, an ACL function has partial derivatives u_x, u_y a.e. in Ω.

Definition 1.10 (Analytic definition K-quasiconformal mapping). *A topological mapping ϕ of Ω is said to be K-**quasiconformal** (from now on, K-qc), for $K \geq 1$ a real number, if*

(i) ϕ is ACL (absolutely continuous on lines) on Ω.

(ii) $|\phi_{\bar{z}}| \leq k|\phi_z|$ a.e. where $k := (K-1)/(K+1)$.

We say that a topological mapping ϕ is ***quasiconformal*** *(from now on, qc) if there exists $K \geq 1$ such that ϕ is K-qc.*

Definition 1.11. *We say that ϕ is differentiable in the sense of* ***Darboux*** *if*

$$\phi(z) - \phi(z_0) = \phi_z(z_0)(z - z_0) + \phi_{\bar{z}}(z_0)(\bar{z} - \bar{z}_0) + o(|z - z_0|).$$

Lemma 1.12. *If ϕ is topological and has partial derivatives a.e., then it is differentiable a.e. (in the sense of Darboux).*

The proof is technical and it does not imply any idea that we will need later, For details see [Ahl].

If E is a Borel set in Ω, we define $A(E)$ as the area of the image of E under ϕ, which defines a locally finite additive measure, due to the following theorem:

Theorem 1.13 (Lebesgue,[LV] p.120)**.** *A non-negative, completely additive set function τ, bounded in E, has almost everywhere a finite derivative $\tau'(z)$, which is measurable as a function of z. For every Borel set $B \subset E$ we have*

$$\tau(B) \geq \int_B \tau' d\mu$$

where equality holds for every B, if and only if, τ is locally absolutely continuous in E.

Such a measure has a symmetric derivative a.e., that is

$$J(z) = \lim_{\text{"}Q \to 0\text{"}} \frac{A(Q)}{m(Q)},$$

where Q is a square of center z whose sides tend to zero (see [LV], p. 130). Moreover,

$$\int_E J(z) dm(z) \leq A(E).$$

When ϕ is differentiable at z, then $J(z)$ is the Jacobian, which by the inequality before is locally integrable. Moreover, we know that $J = |\phi_z|^2 - |\phi_{\bar{z}}|^2$, and if $|\phi_{\bar{z}}| \leq k|\phi_z|$ ($k < 1$), then

$$J \geq |\phi_z|^2 - k^2|\phi_z|^2 = |\phi_z|^2(1 - k^2),$$

therefore

$$\frac{J}{1 - k^2} \geq |\phi_z|^2 \geq |\phi_{\bar{z}}|^2,$$

which implies that the partial derivatives are locally square integrable.

If we consider h a test function ($\mathcal{C}^1$ with compact support) and we integrate over horizontal or vertical lines, using Fubini's theorem we can see that

$$\int \phi_x h dx dy = -\int \phi h_x dx dy$$
$$\int \phi_y h dx dy = -\int \phi h_y dx dy$$

therefore ϕ_x and ϕ_y are what we understand as **distributional derivatives** of ϕ. Note that whenever a function is differentiable, the distributional derivatives agree with the derivatives in the usual sense. For more information regarding distributional derivatives see [Roy].

We are interested in the converse statement.

Lemma 1.14. *If ϕ has locally integrable distributional derivatives, then ϕ is ACL.*

Proof. First of all assume the existence of locally integrable functions ϕ_1, ϕ_2 such that

$$\int \phi_1 h dm = -\int \phi h_x dm \quad , \quad \int \phi_2 h dm = -\int \phi h_y dm$$

for every h test function, i.e. $h \in \mathcal{C}_c^\infty(\Omega)$.

Consider a rectangle $R_\nu = \{0 \leq x \leq a, 0 \leq y \leq \nu\}$ and consider $h = h(x)k(y)$ such that its support is contained in $\subseteq R_\nu$. We have,

$$\int_{R_\nu} \phi_1 h(x) dm = -\int_{R_\nu} \phi h'(x) dm,$$

then,

$$\int_0^a \phi_1(x,\nu) h(x) dx = -\int_0^a \phi(x,\nu) h'(x) dx$$

for almost all ν.

Consider now a sequence of test functions $\{h_n\}_n$ such that $0 \leq h_n \leq 1$ and $h_n \equiv 1$ on $(1/n, a-1/n)$, then,

$$\int_0^{1/n} \phi_1(x,\nu) h_n'(x) dx + \int_{1/n}^{a-1/n} \phi_1(x,\nu) dx + \int_{a-1/n}^a \phi_1(x,\nu) h_n(x) dx =$$
$$= -\int_0^{1/n} \phi(x,\nu) h_n'(x) dx - \int_{1/n}^{a-1/n} \phi(x,\nu) \underbrace{h_n'(x)}_{0} dx - \int_{a-1/n}^a \phi(x,\nu) h_n'(x) dx$$

We want to show that when we let n tend to ∞ we have:

$$\int_0^a \phi_1(x,\nu) dx = \phi(a,\nu) - \phi(0,\nu). \tag{1.10}$$

To prove it, we will compute separately

$$\lim_{n\to\infty} \int_0^{1/n} \phi(x,\nu) h_n'(x) dx \quad \text{and} \quad \lim_{n\to\infty} \int_{a-1/n}^a \phi(x,\nu) h_n'(x) dx$$

Note that our test functions resemble (without loss of generality) the ones in Figure 1.4.

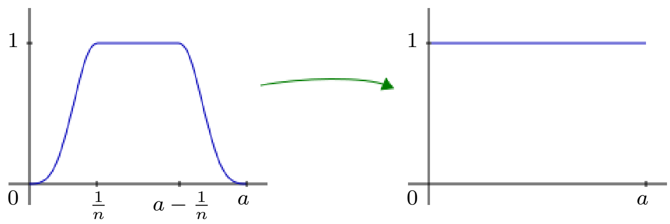

Figure 1.4: Sketch of the test functions in the proof of Lemma 1.14.

So if we consider $h_n'(x)$, then $h_n' \equiv 0$ on $(1/n, a-1/n)$ and $h_n'(0) = h_n'(a) = 0$ (because there exists $\varepsilon_n > 0$ so that $h_n \equiv 0$ on $[0, \varepsilon_n) \cup (a - \varepsilon_n, a]$). Then,

- $h_n'(x) > 0$ on $(0, 1/n)$, so

$$\lim_{n\to\infty} \int_0^{1/n} \phi(x,\nu) h_n'(x) dx \geq \left(\inf_{x\in[0,1/n]} \phi(x,\nu) \right) \int_0^{1/n} h_n'(x) dx =$$
$$= \left(\inf_{x\in[0,1/n]} \phi(x,\nu) \right) (h_n(1/n) - h_n(0)) = \left(\inf_{x\in[0,1/n]} \phi(x,\nu) \right) (1-0) \underset{n}{\to} \phi(0,\nu),$$

and in the other direction,

$$\lim_{n\to\infty}\int_0^{1/n}\phi(x,\nu)h_n'(x)dx \le \left(\sup_{x\in[0,1/n]}\phi(x,\nu)\right)\int_0^{1/n}h_n'(x)dx =$$
$$= \left(\sup_{x\in[0,1/n]}\phi(x,\nu)\right)(h_n(1/n)-h_n(0)) \underset{n}{\to} \phi(0,\nu).$$

Therefore,

$$\lim_{n\to\infty}\int_0^{1/n}\phi(x,\nu)h_n'(x)dx = \phi(0,\nu).$$

- $h_n'(x) < 0$ on $(a-1/n, a)$, so using the same argument as before

$$\lim_{n\to\infty}\int_{a-1/n}^{a}\phi(x,\nu)h_n'(x)dx \le \left(\inf_{x\in[a-1/n,a]}\phi(x,\nu)\right)\int_0^{1/n}h_n'(x)dx =$$
$$= \left(\inf_{x\in[a-1/n,a]}\phi(x,\nu)\right)(h_n(a)-h_n(a-1/n))$$
$$= \left(\inf_{x\in[0,1/n]}\phi(x,\nu)\right)(0-1) \underset{n}{\to} -\phi(a,\nu),$$

and in the other direction,

$$\lim_{n\to\infty}\int_{a-1/n}^{a}\phi(x,\nu)h_n'(x)dx \ge \left(\sup_{x\in[a-1/n,a]}\phi(x,\nu)\right)\int_0^{1/n}h_n'(x)dx =$$
$$= \left(\sup_{x\in[a-1/n,a]}\phi(x,\nu)\right)(h_n(a)-h_n(a-1/n)) \underset{n}{\to} -\phi(a,\nu).$$

Therefore,

$$\lim_{n\to\infty}\int_{a-1/n}^{a}\phi(x,\nu)h_n'(x)dx = -\phi(a,\nu).$$

So we have (1.10) for a.e. ν, hence $\phi(x,\nu)$ is absolutely continuous for a.e. ν.
Furthermore, $\phi_x = \phi_1$ and $\phi_y = \phi_1$ a.e. (because they agree with the distributional derivatives). □

Hence we have proved that:

Theorem 1.15 (Second analytic definition of K-qc mapping)**.** *A topological mapping ϕ of Ω is K-qc, if and only if, ϕ has locally integrable distributional derivatives such that*

$$|\phi_{\bar z}| \le k|\phi_z| \quad \textit{where} \quad k := \frac{K-1}{K+1}.$$

The goal is to prove that these definitions are invariant under composition with a conformal mapping. In fact, we will see that the analytic definition and the geometric definition are equivalent, hence we will obtain this property 'for free'.

1.2.3 The length-area method

In some cases to deal with quasiconformal mappings we will need to use a geometric approach and in some cases an analytic approach. The final goal of this section is proving the equivalence between these two definitions, which will also give us several properties.

The proof will consist on using the length-area method, together with two technical lemmas that will be proved later.

We want to write our conditions in the module of a quadrilateral in terms of an extremal problem so that we can take advantage of the ACL hypothesis.

In order to characterize the module in this setting, consider the canonical mapping of the quadrilateral $Q(z_1, z_2, z_3, z_4)$ onto the rectangle $R = \{u + iv \colon 0 < u < a, 0 < v < b\}$, like in Figure 1.5.

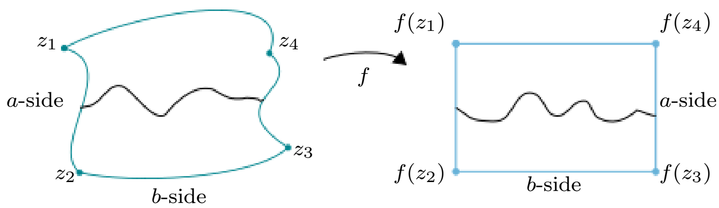

Figure 1.5: The module is defined via the Riemann map. In the right side we have the quadrilateral $Q(z_1, z_2, z_3, z_4)$ and in the left side the rectangle R.

Then
$$\int_Q |f'(z)|^2 dxdy = ab.$$
Let Γ be the family of all locally rectifiable Jordan arcs in Q which join the sides (z_1, z_2) and (z_3, z_4). Then,
$$\int_\gamma |f'(z)||dz| \geq b \qquad \forall \gamma \in \Gamma$$
and we have equality when γ is the inverse of a vertical line segment of R joining its horizontal sides. Thus,
$$M(Q(z_1, z_2, z_3, z_4)) = \frac{\int_Q |f'(z)|^2 dxdy}{\left(\inf_{\gamma\in\Gamma} \int_\gamma |f'(z)||dz|\right)^2}. \tag{1.11}$$
The first step is getting rid of the canonical mapping f. We accomplish this goal by introducing the family $\mathcal{P}$ of non-negative Borel-measurable functions in Q normalized such that for all $\rho \in \mathcal{P}$,
$$\int_\gamma \rho(z)|d(z)| \geq 1 \qquad \forall \gamma \in \Gamma.$$
If we denote
$$m_\rho(Q) := \int_Q \rho^2 dxdy$$
then, we can prove the following characterization of the modulus.

Lemma 1.16. *In the conditions above,* $M(Q(z_1, z_2, z_3, z_4)) = \inf_{\rho\in\mathcal{P}} m_\rho(Q)$.

Proof. For every $\rho \in P$ define ρ_1 in the canonical rectangle R, as in Figure 1.5, by $(\rho_1 \circ f)|f'| = \rho$, i.e. ρ_1 is defined by the change of variables. Then, by the change of variables theorem we see that
$$\int_\gamma (\rho_1 \circ f)|f'|ds = \int_{f(\gamma)} \rho_1 ds \quad , \quad \int_Q (\rho_1 \circ f)^2 |f'|^2 dm(z) = \int_{\underbrace{R}_{f(Q)}} \rho_1^2 dxdy.$$

Then,

$$m_\rho(Q) = \int_R \rho_1^2 dxdy,$$

and using Fubini's theorem,

$$m_\rho(Q) = \int_0^a \int_0^b \rho_1(x+iy)^2 dydx.$$

Note that the correspondence before with ρ and ρ_1 defines a family $\mathcal{P}_1$ of non-negative Borel-measurable functions in $f(Q) = R$ such that for every $\rho_1 \in \mathcal{P}_1$,

$$\int_{f(\gamma)} \rho_1 ds \geq 1 \quad \forall \gamma \in \Gamma.$$

So,

$$1^2 \leq \left(\int_0^b (1 \cdot \rho_1(x,y)) dy \right)^2 \underbrace{\leq}_{\text{Cauchy-Schwarz}} \int_0^b 1^2 dy \cdot \int_0^b \rho_1(x,y)^2 dy = b \int_0^b \rho_1(x,y)^2 dy.$$

Hence,

$$a = \int_0^a 1dx \leq b \int_0^a \int_0^b \rho_1(x,y)^2 dxdy = bm_\rho(Q),$$

which implies that

$$\frac{a}{b} = M(Q(z_1,z_2,z_3,z_4)) \leq m_\rho(Q) \quad \forall \rho \in \mathcal{P}.$$

Finally, if we take $\rho = |f'|/b \in \mathcal{P}$, then $m_\rho(Q) = M(Q(z_1,z_2,z_3,z_4))$. □

One of the advantages of the characterization in Lemma 1.16 is the well-known Rengel's inequality.

Lemma 1.17 (Rengel's inequality)**.** *Let ρ be the euclidean metric, s_1 the euclidean distance of the sides (z_1,z_2) and (z_3,z_4) in Q and m the Euclidean area. Then,*

$$M(Q(z_1,z_2,z_3,z_4)) \leq m(Q)/s_1^2.$$

Proof. If we choose

$$s_1 = \inf_{\gamma \in \Gamma} \int_\gamma |dz|,$$

then $\rho = 1/s_1 \in \mathcal{P}$ and using Lemma 1.16 we see that

$$m_\rho(Q) = \int_Q \frac{1}{s_1^2} dxdy = \frac{m(Q)}{s_1^2} \geq M(Q(z_1,z_2,z_3,z_4))$$

which yields the result. □

Remark 1.18 (Rengel's double inequality)**.** *The following holds: In the conditions in the lemma above, if now*

$$s_2 \coloneqq \inf_{\tilde{\gamma} \in \tilde{\Gamma}} l(\gamma)$$

where $\tilde{\Gamma}$ is the family of all locally rectifiable arcs in Q which join the sides (z_2,z_3) and (z_4,z_1), then

$$\frac{s_2^2}{m(Q)} \leq M(Q) \leq \frac{m(Q)}{s_1^2}.$$

Moreover, we have equality, if and only if, Q is a rectangle.

The double inequality in Remark 1.18 is clear using the conjugate quadrilateral. For more details in the last part see [LV].

1.2.4 Proof of the equivalence between the two definitions

In order to prove the equivalence between the geometric definition and the analytic definition, we need to prove first two technical lemmas, the first of them tells us that the differentiation rules that we had when dealing with differentiable maps also hold in the distributional sense.

Lemma 1.19. *If ω is a $\mathcal{C}^2$ topological mapping and if ϕ has locally integrable distributional derivatives, so does $\phi \circ \omega$ and, if we call $\omega = (\xi, \eta)$, they are given by*

$$(\phi \circ \omega)_x = (\phi_\xi \circ \omega)\frac{\partial \xi}{\partial x} + (\phi_\eta \circ \omega)\frac{\partial \eta}{\partial x}$$
$$(\phi \circ \omega)_y = (\phi_\xi \circ \omega)\frac{\partial \xi}{\partial y} + (\phi_\eta \circ \omega)\frac{\partial \eta}{\partial y}.$$

Proof. We just have to show that these are the distributional derivatives. So we have to consider a test function $h \circ \omega$ and prove:

$$\int [(\phi_\xi \circ \omega)\xi_x + (\phi_\eta \circ \omega)\eta_x](h \circ \omega)dxdy = -\int (\phi \circ \omega)(h \circ \omega)_x dxdy$$
$$\int [(\phi_\xi \circ \omega)\xi_y + (\phi_\eta \circ \omega)\eta_y](h \circ \omega)dxdy = -\int (\phi \circ \omega)(h \circ \omega)_y dxdy.$$

To do it, we will apply a couple of change of variable. Note that we have already computed this expression when everything was differentiable (and we will use it here with the test functions). We have $\omega = (\xi, \eta)$, which is a change of variable from the $(x, y)-$plane to the $(\xi, \eta)-$plane. So we have:

$$\begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix} = \frac{1}{J}\begin{pmatrix} \eta_y & -\xi_x \\ -\eta_x & \xi_x \end{pmatrix}$$

i.e., the following matrices are reciprocal:

$$\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix}.$$

and now, for any test function $h \circ \omega$, we have:

$$\begin{aligned}
&\int [(\phi_\xi \circ \omega)\xi_x + (\phi_\eta \circ \omega)\eta_x](h \circ \omega)dxdy = \\
&\qquad = \int \left[(\phi_\xi \circ \omega)\frac{\xi_x}{J} + (\phi_\eta \circ \omega)\frac{\eta_x}{J}\right](h \circ \omega)Jdxdy = \\
&\qquad = \int (\phi_\xi y_\eta - \phi_\eta y_\xi)hd\xi d\eta = -\int \phi((y_\eta h)_\xi - (y_\xi h)_\eta)d\xi d\eta = \\
&\qquad = \int \phi(-h_\xi y_\eta + h_\eta y_\xi + \overbrace{y_{\eta\xi}h - y_{\xi\eta}h}^{0 \text{ by Schwarz's theorem}})d\xi d\eta = \\
&\qquad = \int (\phi \circ \omega)\left[-(h_\xi \circ \omega)\frac{\xi_x}{J} - (h_\eta \circ \omega)\frac{\eta_x}{J}\right]Jdxdy = \\
&\qquad = \int (\phi \circ \omega)(-(h_\xi \circ \omega)\xi_x - (h_\eta \circ \omega)\eta_x)dxdy = \\
&\qquad = -\int (\phi \circ \omega)(h \circ \omega)_x dxdy.
\end{aligned}$$

The other one goes exactly the same way. $\square$

The next lemma tells us that quasiconformal maps, in the geometric sense, are ACL.

Lemma 1.20. *If ϕ is quasiconformal in the geometric sense, then ϕ is ACL.*

Proof. Let $A(\eta)$ be the image area under the mapping ϕ of the rectangle $\{a \leq x \leq \beta, y_0 \leq y \leq \eta\}$. Since $A(\eta)$ is increasing, then $A'(\eta)$ exists a.e., so we can assume that $A'(0)$ exists. Consider the following figure:

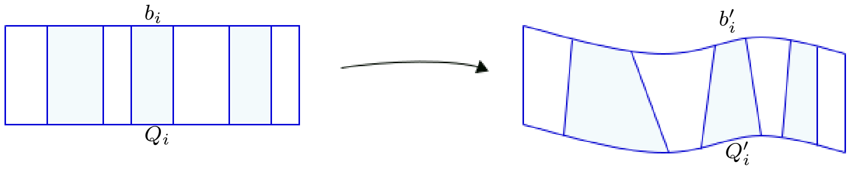

Figure 1.6: Representation of the quadrilaterals in the proof of Lemma 1.20.

where Q_i is the rectangle with heigh η and base b_i (it has to be taken into account that we want to prove that it is ACL, hence we need to take a collection of mutually disjoint squares to apply the quasiconformality). Consider b_i' as the length of the image of b_i. We want to show that if η is small enough, then the length of any curve in Q_i' joining the vertical sides is nearly b_i'.

To do so, consider a polygon such that for $\varepsilon > 0$ (in each Q_i)

$$\sum_{j=1}^{n} |\xi_j - \xi_{j-1}| \geq b_i' - \varepsilon/2.$$

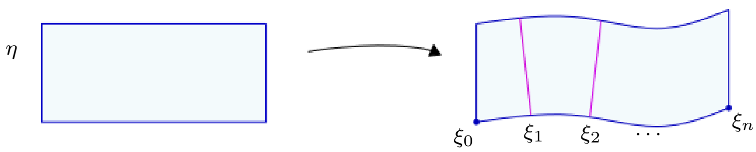

Figure 1.7: Representation of Q_i.

Take η small enough so that the variation of ϕ on vertical segments is less than $\varepsilon/4n$ (which can be achieved thanks to continuity). Now consider the lines through ξ_k that correspond to verticals (as showed in Figure 1.7). Any transversal (to these lines) must intersect all of these lines, hence its length has to be greater or equal than

$$\sum_{j=1}^{n} |\xi_j - \xi_{j-1}| - \varepsilon/2 \geq b_i' - \varepsilon.$$

If $\varepsilon < \min\{b_i'/2\}$, then by Rengel's inequality (Lemma 1.17)

$$m(Q_i') \geq \left(\frac{b_i'}{2}\right)^2 \frac{1}{A_i},$$

where A_i is the area of Q_i'. By the K-quasiconformality we also have $m(Q_i') \leq K b_i/\eta$, therefore,

$$\frac{b_i'^2}{b_i} \leq 4K\frac{A_i}{\eta}. \tag{1.12}$$

Using Cauchy-Schwarz's inequality we see that

$$\left(\sum b_i'\right)^2 \leq \left(\sum b_i'^2\right)\left(\sum 1\right) = \left(\sum b_i'^2 \frac{b_i}{b_i}\right)\left(\sum \frac{b_i}{b_i}\right) \leq \left(\sum \frac{b_i'^2}{b_i}\right)\left(\sum b_i\right)\frac{\max b_i}{\min b_i}.$$

By (1.12) and the previous inequality we obtain

$$\left(\sum b_i'\right)^2 \leq \frac{4K}{\eta}\left(\sum A_i\right)\left(\sum b_i\right)\frac{\max b_i}{\min b_i} = 4K\frac{\max b_i}{\min b_i}\frac{A(\eta)}{\eta}\left(\sum b_i\right).$$

But,

$$\frac{A(\eta)}{\eta} \xrightarrow[\eta\to 0]{} A'(0) < \infty,$$

which shows that if $\sum b_i \to 0$, then $\sum b_i' \to 0$, i.e. ϕ is ACL. □

We can now address the main goal of this section, which is proving the equivalence between the Analytic and Geometric definition, which will give us some useful properties of quasiconformal mappings.

The next theorem proves that the Analytic definition implies the Geometric one. Remember that we defined

$$D_f(w) = \frac{|f_z(w)| + |f_{\bar{z}}(w)|}{|f_z(w)| - |f_{\bar{z}}(w)|}.$$

Theorem 1.21. *Let f be a sense preserving diffeomorphism in a domain Ω such that $D_f(z) \leq K$ for every $z \in \Omega$, then f is K-qc. in the geometric sense.*

Proof. Take a quadrilateral Q compactly contained in Ω, then can take the Riemann maps so that the rectangles are normalized as in Figure 1.8.

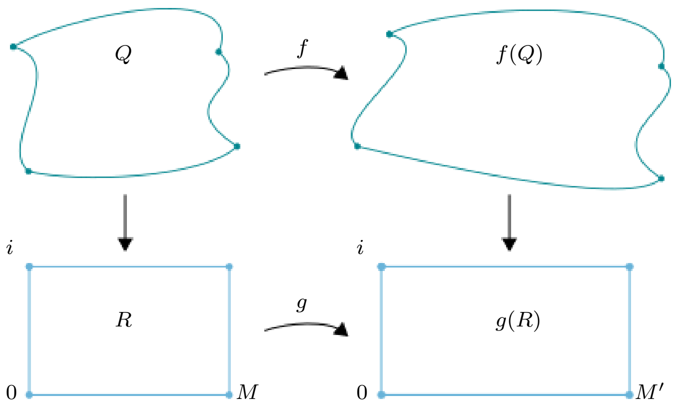

Figure 1.8: If we take a quadrilateral Q, we can consider the induced map g given by the corresponding (normalized) Riemann maps.

Since the dilatation quotient is invariant under conformal maps (by Lemma **??**), then we have $D_f(z) \leq K$, if and only if, $D_g(w) \leq K$.

The inequality $K \geq D_g(z)$ implies that $K(|g_w| - |g_{\bar{w}}|) \geq (|g_w| + |g_{\bar{w}}|)$ and then $KJ_g = K(|g_w| - |g_{\bar{w}}|)(|g_w| + |g_{\bar{w}}|) \geq (|g_w| + |g_{\bar{w}}|)^2 \geq |f_w|^2$.

Note that, by the inequality $|g_w|^2 \leq KJ_g$ we obtain

$$M' = m(g(R)) = \int J_g(w)dm(w) \geq \frac{1}{K}\int_R |g_w(w)|^2 dm(w). \tag{1.13}$$

and we also have

$$M' \leq \int_0^M |g_w(w)|dx,$$

since it is the length of a curve joining the sides $(0,1)$ and $(M', M'+i)$. Then,

$$M'^2 \leq \left(\int_0^M 1\cdot|g_w(w)|\right)^2 \leq \int_0^M 1dx \cdot \int_0^M |g_w(w)|^2 dx,$$

which yields that

$$\frac{M'^2}{M} \le \int_0^M |g_w(w)|^2 dx = \int_0^1 \int_0^M |g_w(w)|^2 dxdy = \int_R |g_w|^2.$$

Together with (1.13), we see that

$$\frac{1}{K}\frac{M'^2}{M} \le M',$$

i.e. $M' \le KM$. □

Note that this result is for the differentiable case. However, Lemma 1.19 allows us to extend it to the general case since there we have differentiability almost everywhere.

Note also that by Lemma 1.20 we have differentiability almost everywhere in quasiconformal mappings in the geometric sense. This allows us to use Lemma 1.19 for the next result, which deals with the converse in Theorem 1.21.

Theorem 1.22. *If ϕ is K-qc in the geometric sense and differentiable at z_0, then $D_\phi(z_0) \le K$.*

Proof. Suppose without loss of generality that $z_0 = 0$, $\phi(0) = 0$, $\phi_z(0) = |\phi_z(0)|$, $\phi_{\bar{z}}(0) = |\phi_{\bar{z}}(0)|$ (which can be assumed after composing with a proper linear transformation in $\mathbb{R}^2$).

Then, in a neighborhood of $z_0 = 0$,

$$\phi(z) = u(z) + iv(z) = |\phi_z(0)|z + |\phi_{\bar{z}}(0)|\bar{z} + o(z). \tag{1.14}$$

We can take $\delta > 0$ small enough so that the rectangle

$$R_\delta = R((-1-i)\delta, (1-i)\delta, (1+i)\delta, (-1+i)\delta)$$

lies in the domain of ϕ and where (1.14) holds. Then, by (1.14), the image of R_δ under ϕ is close to R'_δ as showed in Figure 1.9.

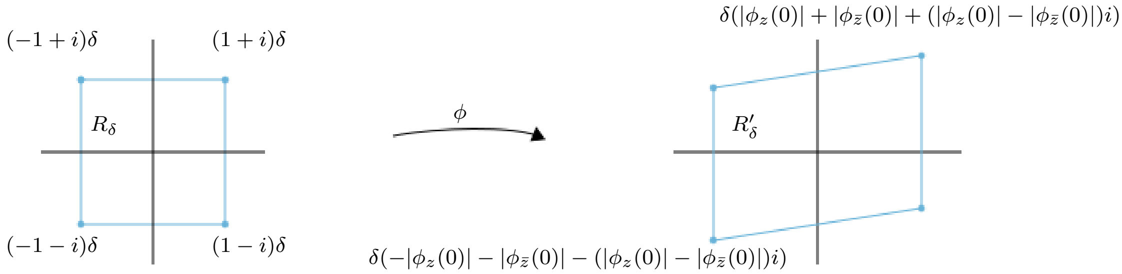

Figure 1.9: The image of R_δ under ϕ as in the proof of Theorem 1.22 is close to R'_δ, the rectangle obtained from taking the image under linear part in equation (1.14).

So the b-sides of $R'_\delta \simeq \phi(R_\delta)$ are at distance $s_b = 2\delta(|\phi_z(0)| + |\phi_{\bar{z}}(0)|) + o(\delta)$ and R'_δ has area, $m(R'_\delta) = 4\delta^2\left(|\phi_z(0)|^2 - |\phi_{\bar{z}}(0)|^2\right) + o(\delta^2)$. By Rengel's inequality (Lemma 1.17),

$$M(R'_\delta) \ge \frac{4\delta^2\left(|\phi_z(0)| + |\phi_{\bar{z}}(0)|\right)^2 + o(\delta^2)}{4\delta^2\left(|\phi_z(0)|^2 - |\phi_{\bar{z}}(0)|^2\right) + o(\delta^2)}, \tag{1.15}$$

but $M(R_\delta) = 1$, so

$$M(R'_\delta) = \frac{M(R'_\delta)}{M(R_\delta)} \le K,$$

which together with equation (1.15), implies that

$$\left(4\delta^2\left(|\phi_z(0)|^2 - |\phi_{\bar{z}}(0)|^2\right) + o(\delta^2)\right)K \ge 4\delta^2\left(|\phi_z(0)| + |\phi_{\bar{z}}(0)|\right)^2 + o(\delta^2).$$

Hence,

$$(|\phi_z(0)|+|\phi_{\bar z}(0)|)^2+\frac{o(\delta)}{\delta}\le K\left((|\phi_z(0)|^2-|\phi_{\bar z}(0)|^2)+\frac{o(\delta)}{\delta}\right).$$

If we make $\delta\to 0$, then

$$(|\phi_z(0)|+|\phi_{\bar z}(0)|)^2+\frac{o(\delta)}{\delta}\le K\left(|\phi_z(0)|+|\phi_{\bar z}(0)|\right)\left(|\phi_z(0)|-|\phi_{\bar z}(0)|\right),$$

i.e.

$$\frac{|\phi_z(0)|+|\phi_{\bar z}(0)|}{|\phi_z(0)|-|\phi_{\bar z}(0)|}=D_\phi(0)\le K.$$

□

Finally, we show some properties that follow from the equivalence between the two definitions.

Proposition 1.23 (Properties of Q.C. mappings)**.** *Let ϕ be a qc map, then the following hold:*

(1) If ϕ is K-qc, then ϕ^{-1} is K-qc.

(2) If ϕ is K-qc, then any composition (either to the left or the right with a conformal mapping) is K-qc.

(3) If ϕ is K-qc and $\tilde\phi$ is $\tilde K$-qc, then $\phi\circ\tilde\phi$ is $K\tilde K$-qc.

(4) ϕ is K-qc, if and only if, ϕ is locally K-qc.

(5) If ϕ is K-qc, then ϕ satisfies the following uniform Hölder condition:

$$|\phi(z_1)-\phi(z_2)|\le M|z_1-z_2|^{1/K}$$

on every compact subset of the domain of definition.

The proof of the last one can be found in [Ahl].

1.3 Quasiconformal geometry

In complex dynamics, one is often interested in constructing maps with prescribed dynamics. However, the rigidity of holomorphic maps does not allow to build models by cutting and pasting. As we shall see, quasiconformal maps are the fundamentals to build models.

Before we explain what we understand as surgery, we need to introduce several results concerning the generic interpretation of the Beltrami coefficient introduced before.

We have already seen how in the case of the differentiable quasiconformal maps case we have the dilatation of the ellipses controlled by the q.c. condition:

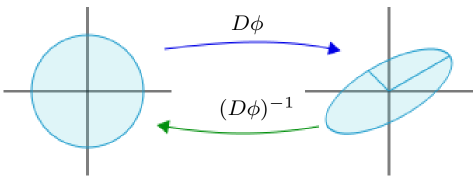

Figure 1.10: Representation of the action of the differential when applied to a circle.

In the general case, since we have differentiability a.e. and a positive, well-defined Jacobian a.e., the same happens. Therefore we can determine the class of K-qc mappings by the maximal dilatation allowed to be applied to an ellipse.

We need to transform the last statement concerning the maximal dilatation intro a precise definition so that we are able to interpret well the Beltrami coefficients, a notion which will end up leading to an important result known as the Integrability theorem (also known as Riemann measurable mapping theorem), which together with Weyl's lemma are the fundamental tools to perform surgery.

Let $U \subset \mathbb{C}$ and $TU = \cup_{u \in U} T_u U$ be the **tangent bundle** over U. We understand an **infinitesimal ellipse** as $E_u \subset T_u U$ defined up to scaling for a.e. $u \in U$. Note that, in the notations of Figure 1.11, the map

$$\begin{aligned} \mu \colon U &\to \mathbb{D} \\ u \mapsto \mu(u) &= \frac{M-m}{M+m} e^{i2\theta}, \end{aligned}$$

completely determines the ellipse E_u (up to scaling). Moreover, $\mu(u)$ is the **Beltrami coefficient** of E_u.

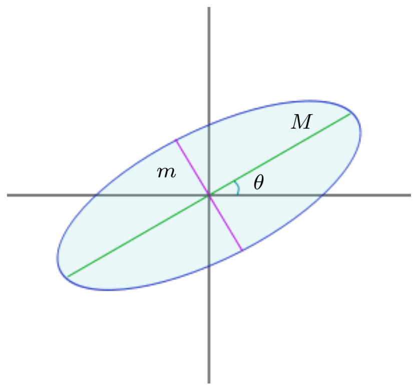

Figure 1.11: Representation of the major axis M, the minor axis m and the angle θ in an ellipse.

Definition 1.24 (Almost complex structure)**.** *An **almost complex structure** (from now on, a.c.s.) σ on U is a measurable field of infinitesimal ellipses $\mathcal{E} \subset TU$. By this we mean an ellipse $E_u \subset T_u U$, defined up to scaling, for a.e. $u \in U$, such that the map $u \in U \mapsto \mu(u) \in \mathbb{D}$, defined as before, is measurable with respect to the Lebesgue measure.*

Note that an almost complex structure on U is completely determined by a measurable function $\mu \colon U \to \mathbb{D}$.

It has already been pointed out that our goal is to stress out the rigidity of conformal maps, the idea behind this notion is that by defining a Beltrami coefficient, which contains all the information about the ellipses (up to scaling), we are setting how much a map is able to distort a circle and therefore we can consider the class of quasiconformal in U that alter the circles in such a way. We are now heading towards determining this class of quasiconformal maps and the result that gives it to us is the Integrability theorem (Theorem 1.27), which is proved in B.

Recall that the **dilatation** of an ellipse E_u is given by

$$K(u) := \frac{1+|\mu(u)|}{1-|\mu(u)|},$$

where $\mu(u)$ is its Beltrami coefficient. We define

$$K(\sigma) := \operatorname*{ess\,sup}_{u \in U} K(u),$$

observe that $K(\sigma) \in [1, \infty]$.

We want to deal now with how to obtain a.c.s. from maps. To accomplish this goal, we need to restrict the types of maps that we consider for technical reasons.

Given $U, V \subset \mathbb{C}$, we understand by $D^+(U,V)$ the class of continuous orientation preserving functions f from U to V, which are $\mathbb{R}$-differentiable a.e. and with a differential

$$D_u f \colon T_u U \to T_{f(u)} V$$

defined a.e. (note that quasiconformal maps satisfy these conditions). So, for this maps we have

$$D_u f = \partial_z f(u) dz + \partial_{\bar{z}} f(u) d\bar{z} \text{ a.e.}$$

and we have already seen how $D_u f$ defines an infinitesimal ellipse in $T_u U$ with Beltrami coefficient

$$\mu_f(u) = \frac{\partial_{\bar{z}} f(u)}{\partial_z f(u)}$$

i.e. with dilatation

$$K_f(u) := K(D_u f) = \frac{1 + |\mu_f(u)|}{1 - |\mu_f(u)|}$$

which defines an almost complex structure on the tangent space.

It is clear that if we do this for all points $u \in U$ for which f is differentiable (which happens a.e.), we obtain a measurable field of ellipses and hence, an almost complex structure σ_f on U, with Beltrami coefficient μ_f.

Definition 1.25 (Pullback of μ_0)**.** *We say that σ_f is the* ***pullback*** *of σ_0 (the field of infinitesimal circles/standard complex structure, i.e. with dilatation* 1 *a.e.) by f, i.e. μ_f is the pullback of $\mu_0 \equiv 0$ by f.*

We write $\mu_f(u) = f^* \mu_0(u)$, or $\sigma_f(u) = f^* \sigma_0(u)$.

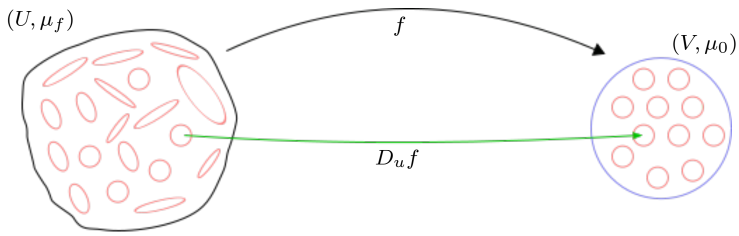

Figure 1.12: Representation of the pullback in Definition 1.25.

Note that we have already defined the dilatation of this almost complex structure, which is

$$K_f := \operatorname*{ess\,sup}_{u \in U} K_f(u).$$

This concept of pullback can be generalized by defining the pullback of any a.c.s. σ under a map f. However, our map f needs to satisfy an extra condition: f has to be absolutely continuous with respect to the Lebesgue measure. Lucky for us, quasiconformal maps satisfy such condition, hence unless we state the opposite, we will suppose that $f \colon U \to V$ is quasiconformal and that μ is the Beltrami coefficient in TV corresponding to an a.c.s. σ in V. Let E_v be the infinitesimal ellipse defined in $T_v V$ for a.e. $v \in V$.

We consider the measurable set of infinitesimal ellipses given by

$$E'_u = (D_u f)^{-1}(E_{f(u)})$$

which is well-defined a.e., in fact, E'_u is defined for all u such that $E_{f(u)}$ is defined and $D_u f$ exists and is non singular.

We write the pullback as $f^*\mu$ and by

$$(U, \mu_1) \xrightarrow{f} (V, \mu_2)$$

we mean that $f\colon U \to V$ and $f^*\mu_2 = \mu_1$.

If now we have that $g\colon V \to W$ is another quasiconformal map and $\mu = \mu_g$, then

$$f^*(\mu_g) = f^*(g^*\mu_0) = (g \circ f)^*\mu_0 = \mu_{g\circ f}.$$

By the computations that we have already done for the differentiable case, we can find explicit formulas for the Beltrami coefficients:

$$f^*\mu_u = \frac{\partial_{\bar{z}} + \mu(f(u))\overline{\partial_z f(u)}}{\partial_z f(u) + \mu(f(u))\overline{\partial_{\bar{z}} f(u)}}.$$

We end up this discussion about pullbacks introducing a key concept that will be of relevance later.

Definition 1.26 (f-invariant a.c.s.)**.** *Let $U \subset \mathbb{C}$ be an open set and f be quasiconformal. Let σ be an a.c.s. on U with Beltrami coefficient μ. We say that μ (or σ) is f-**invariant** if $f^*(\sigma) = \sigma$ a.e. We say that f is holomorphic with respect to μ.*

1.3.1 The Integrability theorem

Remember that we define the **Beltrami coefficient** of a K-qc map ϕ (or the **complex dilatation**) as the measurable function

$$\mu_\phi(z) = \frac{\phi_{\bar{z}}(z)}{\phi_z(z)}.$$

Note that since $\phi_z \neq 0$ a.e. (otherwise we would have $\phi_z = \phi_{\bar{z}} = 0$ a.e., which would imply by the ACL condition that ϕ is constant, contradicting that it is a topological mapping), the Beltrami coefficient is well-defined. Moreover,

$$|\mu_\phi(z)| = \frac{|\phi_{\bar{z}}(z)|}{|\phi_z(z)|} = \frac{D_\phi - 1}{D_\phi + 1} < 1.$$

We can state now the Integrability theorem, which yields that the conditions that we have obtained from the Beltrami coefficient of a K-qc mapping are sufficient conditions so that there exists a quasiconformal mapping with such a Beltrami coefficient. Furthermore, it also states that after imposing some conditions, such quasiconformal mapping is unique.

Theorem 1.27 (Integrability theorem)**.** *Let $U \subset \mathbb{C}$ be an open set such that $U \cong \mathbb{D}$ (resp. $U \cong \mathbb{C}$). Let μ be a Beltrami coefficient on U such that the essential supremum $||\mu||_\infty = k < 1$. Then μ is integrable, i.e. there exists a quasiconformal homeomorphism $\phi\colon U \to \mathbb{D}$ (resp. onto $\mathbb{C}$) which solves the Beltrami equation, i.e. such that*

$$\partial_z\phi(z)\mu(z) = \partial_{\bar{z}}\phi(z)$$

for almost every $z \in U$. Moreover, ϕ is unique up to post-composition with automorphisms of $\mathbb{D}$ (resp. $\mathbb{C}$).

The proof can be found in Appendix B and its relevance to Complex Dynamics will be appreciated through the next chapters.

In order to perform surgery, which is the ultimate goal of this thesis, we will also need Weyl's Lemma, a classical result in Complex Analysis that can be proved using Cauchy-Pompeiu formula.

Theorem 1.28 (Weyl's lemma)**.** *If $f \in \mathcal{C}(\Omega)$, $\Omega \subset \mathbb{C}$ a domain, is such that*

$$\int_{\mathbb{C}} f(z) \frac{\partial \phi}{\partial \bar{z}}(z) dm(z) = 0 \quad \forall \phi \in \mathcal{C}_c^{\infty}(\Omega),$$

then $f \in \mathcal{H}(\Omega)$.

The following is an equivalent statement of Weyl's Lemma, when applied with quasiconformal mappings.

Theorem 1.29 (Weyl's lemma)**.** *If ϕ is 1-quasiconformal, then ϕ is conformal. In other words, if ϕ is quasiconformal and $\partial_{\bar{z}}\phi = 0$ almost everywhere, then ϕ is conformal.*

1.4 Quasiregular mappings

In general, the models we shall deal with will not be injective. In this section we deal with quasiregular mappings, those that are locally quasiconformal at every point except for a discrete number of points.

Definition 1.30 (K-quasiregular map)**.** *Let $U \subset \mathbb{C}$ be an open set and $K < \infty$. A mapping $g\colon U \to \mathbb{C}$ is said to be K-quasiregular (from now on, K-qr), if and only if, g can be expressed as*

$$g = f \circ \phi$$

where $\phi\colon U \to \phi(U)$ is K-qc and $f\colon \phi(U) \to g(U)$ is holomorphic.

Note that g is locally K-qc and it fails to be a homeomorphism at the discrete set of points $\phi^{-1}(Crit(f))$ ($Crit(f)$ is the set of critical points of f), where the map is not injective.

To deal with this new type of functions in the best possible way, we aim to give some equivalent definitions and some properties.

Theorem 1.31 (2nd definition of K-qr)**.** *Let $U \subset \mathbb{C}$ be an open set and $K < \infty$. A continuous mapping $g\colon U \to \mathbb{C}$ is $K-qr$, if and only, g is locally K-qr for except a discrete set of points in U.*

Proof. As we have already pointed out, the first definition implies this one. Let's see the converse.

Consider Ω the discrete set of points for which g is not K-qc in any neighborhood of such a point. Cover $U \setminus \Omega$ by a countable collection of open sets on which g is K-qc (which can be assumed to be countable by Lindelöf's theorem). Then $\partial_z g$ and $\partial_{\bar{z}} g$ are well-defined a.e. in U (a countable union of measure zero sets has measure zero). Moreover, the Beltrami coefficient can also be defined

$$\mu(z) = \partial_{\bar{z}} g(z) / \partial_z g(z)$$

and $||\mu||_\infty \leq k < 1$ in U.

If U is simply connected, we can apply the Integrability theorem, if not, we extend μ to $\tilde{\mu}$ in $\mathbb{C}$ defining $\tilde{\mu}(z) = \mu(z)\mathbb{1}_U$ and then we integrate.

In any case we end up with a K-qc map

$$\phi\colon U \to \phi(U).$$

Then, $f := g \circ \phi^{-1}$ is locally quasiconformal (for except the discrete set $\phi^{-1}(\Omega)$) and

$$f^*\mu_0 = (\phi^{-1})g^*(\mu_0) = (\phi^{-1})^*\mu = \mu_0.$$

By Weyl's lemma, f is locally conformal for except at a discrete set of points where f is continuous, hence f is holomorhic. □

Theorem 1.32 (3rd and 4th definition of K-qr)**.** *Let $U \subset \mathbb{C}$ be an open set and $K < \infty$. Then given a continuous mapping $g\colon U \to \mathbb{C}$, the following are equivalent.*

(a) g is K-qr.

(b) For every $z \in U$, there exist neighborhoods N_z and $N_{g(z)}$ of z and $g(z)$ respectively, a K-qc mapping $\psi\colon N_z \to \mathbb{D}$ and a conformal mapping $\varphi\colon N_{f(z)} \to \mathbb{D}$ such that $(\varphi \circ g \circ \psi^{-1})(z) = z^d$, for some $d \geq 1$.

(c) The partial derivatives ∂g, $\bar{\partial} g$ exist in the distributional sense, belong to L^2_{loc} and satisfy $|\bar{\partial} g| \leq k|\partial g|$,where $k := (K-1)/(K+1)$.

The proof can be found in [BF].

Proposition 1.33 (Properties of quasiregular maps)**.** *Let U, V be open subsets of $\mathbb{C}$.*

(i) If $g_1\colon U \to V$ and $g_2\colon V \to \mathbb{C}$ are K_1 and K_2 quasiregular respectively, then $g_2 \circ g_1$ is K_1K_2-qr.

(ii) $g\colon U \to \mathbb{C}$ is holomorphic, if and only if, g is 1-qr.

(iii) If $f\colon U \to V$ is holomorphic and $\phi\colon V \to \mathbb{C}$ is K-qc, then $g := \phi \circ f$ is K-qr.

(iv) If in Theorem 1.32 (b) the mapping φ to be K'-qc, then g is KK'-qr.

(v) If $g\colon U \to \mathbb{C}$ is quasiconformally conjugate to $f\colon U \to \mathbb{C}$ and f is holomorphic, then g is quasiregular.

(vi) If $g\colon U \to \mathbb{C}$ is quasiregular and $g^*\mu_0 = \mu_0$ a.e., then g is holomorphic.

Proof.

(i) We have $g_j = f_j \circ \phi_j$, $j = 1, 2$, hence

$$g_2 \circ g_1 = f_1 \circ \underbrace{\phi_2 \circ f_1 \circ \phi_1}_{K1K2\text{-qr}}$$

so $g_2 \circ g_1$ is K-qr.

(ii) If g is holomorphic, then g is 1-qr. Conversely, if $g = f \circ \phi$ is 1-qr, then ϕ is 1-qc, hence by Weyl's lemma ϕ is conformal. Therefore g is holomorphic.

(iii) By (ii) f is 1-qr and ϕ is K-qr, hence the claim follows from (i).

(iv) This follows again from (i).

(v) Suppose that there exists ϕ a K-qc map such that $g \circ \phi = \phi \circ f$, then $g = \phi \circ f \circ \phi^{-1}$, therefore g is K^2-qr.

(vi) This is again Weyl's lemma. We have, in fact $g = f \circ \phi$, then $\mu_0 = g^*\mu_0 = \phi^* f^* \mu_0 = \phi^*\mu_0$, which implies that ϕ is conformal, so the claim follows.

□

As it has already been pointed out, we want to take pullbacks of quasiregular maps instead of quasiconformal maps, because they offer us more flexibility. The following result deals with this issue, which is essential in surgery.

Lemma 1.34. *Quasiregular maps and their inverse branches send sets of measure zero to sets of measure zero. Therefore, the pullback of a Beltrami form defined a.e. by a quasiregular map is well-defined a.e..*

Proof. We have $g = f \circ \phi$, hence g is absolutely continuous and the same properties apply to the inverse branches of g (note that we have an inverse branch of g if and only if we have an inverse branch of f). Therefore pullbacks of Beltrami coefficients are defined a.e., we write:

$$(S_1, \mu_1) \xrightarrow{\;f\;} (S_2, \mu_2).$$

□

1.5 The Boundary Value Problem

While we perform surgery, we usually need to interpolate (quasiconformally) between two different maps defined on distinct regions of the plane. It is a well-known fact that we cannot embrace this task with holomorphic maps, since the Analytic Continuation Principle does not allow us to do it. In this section we give some conditions under which we can interpolate in certain domains, i.e. solving the boundary value problem via a quasiconformal map, which is the degree of regularity needed to apply the Integrability Theorem. The proofs of the results in this section can be found in [BF, Pom, LV].

The first step is to define the sets that we will work with.

Definition 1.35 (Curve, closed curve, Jordan arc, Jordan curve)**.** *A **curve** γ in $\mathbb{C}$ is a continuous function $\gamma\colon [a,b] \to \mathbb{C}$ parametrizing the curve. We will also denote its image $\gamma([a,b]) \subset \mathbb{C}$ by γ.*

*The curve is said to be **closed** if $\gamma(a) = \gamma(b)$. It is a **Jordan arc** if it has an injective parametrization and a **Jordan curve** if in addition it is closed.*

Theorem 1.36 (Jordan curve theorem)**.** *Let Γ be a Jordan curve in $\mathbb{C}$. Then $\mathbb{C}\backslash\Gamma$ is disconnected and consists of exactly two connected components, a bounded one, which is known as the interior of the curve and an unbounded one, which is known as the exterior of the curve.*

The proof can be found in [NP].

Definition 1.37 (Jordan domain)**.** *A bounded domain whose boundary is a Jordan curve is called a **Jordan domain**.*

As one can imagine, we need to impose some conditions to our boundary maps in order to be able to find a quasiconformal extension of such map. This yields the definition of quasisymmetric maps, the ones that allows us to accomplish our goal.

Definition 1.38 (1st definition of quasisymmetry)**.** *A map $h\colon \mathbb{S}^1 \to \mathbb{C}$ is **quasisymmetric** if h is injective and, for $z_1, z_2, z_3 \in \mathbb{S}^1$, with $0 \neq |z_1 - z_2| = |z_2 - z_3|$, then we have the following double inequality*

$$\frac{1}{M} \le \frac{|h(z_1) - h(z_2)|}{|h(z_2) - h(z_3)|} \le M$$

*for some $M \ge 1$ (sometimes h is called M-**quasisymmetric**).*

If we write $H(t) := h(e^{2\pi i t})$, the previous inequality reads

$$\frac{1}{M} \leq \frac{|H(x+t)-H(x)|}{|H(x)-H(x-t)|} \leq M \quad \forall x \in \mathbb{R}, t > 0.$$

The following definition of quasisymmetry is equivalent to the previous one, although it is formally stronger (the proof can be found in [Pom, LV]).

Definition 1.39 (2nd definition of quasisymmetry). *A map $h\colon \mathbb{S}^1 \to \mathbb{C}$ is* ***quasisymmetric*** *if h is injective and if there exists a strictly increasing continuous function*

$$\lambda\colon [0,\infty) \to [0,\infty)$$

such that,

$$\frac{1}{\lambda\left(\frac{|z_2-z_3|}{|z_1-z_2|}\right)} \leq \frac{|h(z_1)-h(z_2)|}{|h(z_2)-h(z_3)|} \leq \lambda\left(\frac{|z_1-z_2|}{|z_2-z_3|}\right) \quad \forall z_1, z_2, z_3 \in \mathbb{S}^1$$

Note that in both definitions, if one of the inequalities is satisfied, then the other follows by interchanging the roles of z_1 and z_3.

It is also worth taking into account that from Definition 1.39, since the function λ is strictly increasing and continuous, if $h(\mathbb{S}^1) = \mathbb{S}^1$, then by taking the inverse λ^{-1} of λ, which is a well-defined continuous strictly increasing function, we obtain that h^{-1} is quasisymmetric as well.

In the two equivalent definitions of quasisymmetry introduced before, we are not requiring any further regularity conditions on the function $h\colon \mathbb{S}^1 \to \mathbb{C}$. However, in this document we will deal with maps which are $\mathcal{C}^1$, hence it is worth studying whether we have The next Lemma gives us an affirmative answer.

Lemma 1.40. *If $h\colon \mathbb{S}^1 \to h(\mathbb{S}^1)$ is a $\mathcal{C}^1$-diffeomorphism, then it is quasisymmetric.*

Proof. Define $H(t) = h\left(e^{2\pi i t}\right)$, we aim to prove that H satisfies a bilipschitz condition, $1/K|x-y| \leq |H(x)-H(y)| \leq K|x-y|$, since this will imply that H is K^2-quasisymmetric.

Independently of the $x, t \in \mathbb{S}^1$ that we take, since H' is uniformly continuous on $\mathbb{S}^1$, it is bounded, we have

$$|H(x+t)-H(x)-tH'(x)| = \left|\int_0^t (H'(x+\tau)-H'(x))d\tau\right| \leq |t|,$$

for some $k > 0$.

Therefore, the map $G\colon \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$ defined by

$$G(x,y) := \begin{cases} \dfrac{|H(x)-H(y)|}{|x-y|} & \text{if } x \neq y \\ |H'(x)| & \text{if } x = y \end{cases}$$

is continuous. Moreover, since $\mathbb{S}^1 \times \mathbb{S}^1$ is compact, G attains its maximum and minimum value. From H being injective and the fact that H' does not vanish on $\mathbb{S}^1$ we obtain that this minimum value is non-zero. Hence H satisfies a bilipschitz condition. □

Following the lead of what do we intend to prove in this document, we also need to see that the quasisymmetry is preserved with respect to compositions (after imposing some technical conditions).

Proposition 1.41 (Compositions of quasisymmetric maps). *Let $h_j\colon \mathbb{S}^1 \to \mathbb{C}$, for $j = 1, 2$ be quasisymmetric.*

(a) If $h_1(\mathbb{S}^1) = \mathbb{S}^1$, then $h_2 \circ h_1\colon \mathbb{S}^1 \to \mathbb{C}$ is quasisymmetric.

(b) If $\gamma = h_1(\mathbb{S}^1) = h_2(\mathbb{S}^1)$, then $h_2 \circ h_1\colon \mathbb{S}^1 \to \mathbb{S}^1$ is quasisymmetric.

The proof follows from Definition 1.39 and can be found in [BF], Proposition 2.4.

1.5.1 Extension of quasiconformal maps and boundary maps

We now address the task of stating the interpolation results, which as it has already been pointed out, are the ultimate goal of this section.

The following well-known result, gives necessary and sufficient conditions so that a conformal map from the unit disk $\mathbb{D}$ can be extended continuously to $\overline{\mathbb{D}}$.

Theorem 1.42 (Carathéodory)**.** *Let $f\colon \mathbb{D} \to G$ be a conformal isomorphism and G a bounded domain in $\mathbb{C}$. Then f has a continuous extension $\tilde{f}\colon \overline{\mathbb{D}} \to \overline{G}$, if and only if, ∂G is locally connected. Moreover, f has a continuous and injective extension to $\overline{\mathbb{D}}$, if and only if, ∂G is a Jordan curve.*

Recall that we say that a set $X \subset \mathbb{C}$ is **locally connected** if for every point $x \in X$ and any arbitrarily small $\varepsilon > 0$, the intersection $D(0, \varepsilon) \cap X$ is connected.

We want to prove that Theorem 1.42 remains true for quasiconformal mappings, but first we need to see how a quasiconformal map from $\phi\colon \mathbb{D} \to \mathbb{D}$ can be extended to the whole plane as a quasiconformal map.

Proposition 1.43. *Let $\phi\colon \mathbb{D} \to \mathbb{D}$ be a quasiconformal map, then there exists a quasiconformal map $g\colon \mathbb{C} \to \mathbb{C}$ such that $g_{|\mathbb{D}} = \phi$. In particular, ϕ has a continuous and injective extension to the boundary.*

Proof. Define the Beltrami coefficient in $\mathbb{C}$ given by

$$\mu(z) := \begin{cases} \mu_\phi(z) & \text{if } |z| < 1 \\ 0 & \text{if } |z| = 1 \\ \mu_\phi(1/\bar{z}) & \text{if } |z| > 1 \end{cases}$$

then $||\mu||_\infty = ||\mu_\phi||_\infty$ and by the Integrability theorem there exists a unique $\psi\colon \mathbb{C} \to \mathbb{C}$ quasiconformal such that $\mu = \psi^*(\mu_0)$, $\psi(0) = 0, \psi(1) = 1$ and $\psi(\infty) = \infty$.

We need to prove now that ψ preserves $\mathbb{S}^1$ due to the symmetry in the definition of μ.

Note that $\tau(z) = 1/\bar{z}$ is orientation reserving, we define the pull-back under this map of a Beltrami coefficient by $\tau^\star\mu = \bar{\tau}^*\bar{\mu}$. In our case, due to this symmetry, $\tau^\star\mu = \mu$. Then, if we define $\tilde{\psi} = \tau \circ \psi \circ \tau$, we have $\mu_{\tilde{\psi}} = \tau^\star\mu = \mu, \tilde{\psi}(0) = 0, \tilde{\psi}(1) = 1$ and $\tilde{\psi}(\infty) = \infty$, therefore by the uniqueness in the Integrability theorem, $\tilde{\psi} = \psi$, hence $\psi(\mathbb{S}^1) = \tilde{\psi}(\mathbb{S}^1)$.

Moreover, $\tau \circ \tilde{\psi} = \psi \circ \tau$, then

$$\tilde{\psi}(\mathbb{S}^1) = \psi(\mathbb{S}^1) = (\psi \circ \tau)(\mathbb{S}^1) = (\tau \circ \tilde{\psi})(\mathbb{S}^1),$$

which yields that $\tilde{\psi}(\mathbb{S}^1) = (\tau \circ \tilde{\psi})(\mathbb{S}^1)$. Hence, $\tilde{\psi}(\mathbb{S}^1) = \mathbb{S}^1$, so $\psi(\mathbb{S}^1) = \mathbb{S}^1$.

If we consider $\psi_{\mathbb{D}}$, since it preserves $\mathbb{S}^1$ and $\psi(0) = 0$, we have $\psi_{|\mathbb{D}}\colon \mathbb{D} \to \mathbb{D}$ and $\mu_{\psi_{\mathbb{D}}} = \mu_\phi$, therefore there exists a Möbius transformation M such that $\phi = M \circ \psi$. Note that since $\psi(0) = 0$ and $\phi(0) = 0$, then $M(0) = 0$, hence M is a rotation. Thus $g := M \circ \psi$ is the extension of ϕ to $\mathbb{C}$ as a quasiconformal map. □

Now we can address the proof of the continuous and injective extension to the boundary.

Theorem 1.44. *Let $B \subset \mathbb{C}$ be a bounded domain. A quasiconformal map $\varphi\colon \mathbb{D} \to B$ has a continuous and injective extension to $\overline{\mathbb{D}}$, if and only if, ∂B is a Jordan curve.*

Proof. Consider a Riemann maps $R_B\colon \mathbb{D} \to B$, by Theorem 1.42 it has continuous and injective extensions to $\overline{\mathbb{D}}$, if and only if, ∂B is a Jordan curve. Moreover, we can consider the quasiconformal map $\phi\colon \mathbb{D} \to \mathbb{D}$ given by $\phi := R_B^{-1} \circ \varphi$. By Proposition 1.43, ϕ has a continuous and injective extension to $\mathbb{S}^1$. Note that the extension to the boundary is continuous and injective, therefore φ has a continuous and injective extension to $\overline{A}$ which is given by $R_B \circ \phi$ (maps that can be extended to the respective boundaries). □

Therefore a quasiconformal map from a Jordan domain A onto another Jordan domain B can always be extended to a homeomorphism between the closures of A and B (we just have to use Theorem 1.44 twice).

Now, let $h\colon \partial A \to \partial B$ be a given homeomorphism under which positive orientations of the boundaries with respect to the Jordan domains A and B correspond to each other. The **boundary value problem** is to find necessary and sufficient conditions for h to be the boundary function of a quasiconformal mapping $f\colon A \to B$.

In the case that concerns us, we only need to work with boundary maps

$$h\colon \mathbb{S}^1 \to h(\mathbb{S}^1) \subset \mathbb{C}.$$

Theorem 1.45. *Let $h\colon \mathbb{S}^1 \to \mathbb{S}^1$ is an orientation preserving quasisymmetric homeomorphism, then h can be extended to a homeomorphism $\bar{h}\colon \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ that is real analytic in $\mathbb{D}$ and has the following properties:*

(i) If $\sigma, \tau \in$ Möb$(\mathbb{D})$, then the extension of $\sigma \circ h \circ \tau$ is given by $\sigma \circ \bar{h} \circ \tau$.

(ii) $\tilde{h}$ is quasiconformal in $\mathbb{D}$.

This result was proven independently by Douady-Earle and by Ahlfors-Beuring. The first proof came from Ahlfors and Beuring. However the extension that they constructed does not satisfy (i). The one from Douady and Earle does satisfy (i), and hence it has better properties. The proof can be found in [Pom], Theorem 5.15.

In order to prove the main result of this thesis, we need to interpolate between boundary values in an annulus. Before stating the result we need to introduce the concept of quasidisk.

Definition 1.46 (Quasicircle, quasidisk, quasiannuli)**.** *We say that a curve $\gamma \subset \mathbb{C}$ is a **quasicircle** if, for some $C > 0$, diam$\gamma(z_1, z_2) \leq C|z_1 - z_2|$, for $z_1, z_2 \in \gamma$.*

*We say that a domain $D \subset \mathbb{C}$ is a **quasidisk** if ∂D is a quasicircle. Similarly, an annulus A is said to be a **quasiannulus** if its two boundary components are quasicircles.*

The following result relates the notion of quasicircle and quasisymmetric maps, and show why they are relevant to us and how its geometry is closely related with an analytic condition as in Definition 1.38.

Theorem 1.47 (Quasicircle theorem,[Pom], page 94.)**.** *Let J be a Jordan curve in $\mathbb{C}$ and f map $\mathbb{D}$ conformally onto the inner domain of J. Then the following are equivalent:*

(a) J is a quasicircle.

(b) f is quasisymmetric on $\mathbb{S}^1$.

(c) f has a quasiconformal extension to $\mathbb{C}$.

(d) There exists a quasiconformal exctension of $\mathbb{C}$ to $\mathbb{C}$ that maps $\mathbb{S}^1$ onto J.

We can finally state the main result of this section, which is the one that allows us to interpolate between quasidisks and quasiannuli.

Theorem 1.48 (QC interpolation on quasidisk and quasiannuli,[BF], Propositon 2.30.)**.**

(a) Suppose G_1 and G_2 are quasidiscs bounded by γ_1 and γ_2, and let $f\colon \gamma_1 \to \gamma_2$ be quasisymmetric. Then f extends to a quasiconformal map $\overline{f}\colon \overline{G_1} \to \overline{G_2}$.

(b) For $j = 1, 2$, suppose A_j are open quasiannulus bounded by the quasicircles γ_j^i, γ_j^o (where the i holds for the inner boundary of the annulus and the o holds for the outer boundary of the annulus). Let $f^i\colon \gamma_1^i \to \gamma_2^i$ and $f^o\colon \gamma_1^o \to \gamma_2^o$ be quasisymmetric maps between the inner and outer boundaries respectively. Then there exists a quasiconformal map $f\colon \overline{A_1} \to \overline{A_2}$ extending the boundary maps f^i and f^o.

Chapter 2

Quasiconformal surgery

In this work we focus on some dynamical aspects of transcendental meromorphic functions, i.e., we study the dynamical system given by the iterates of meromorphic functions $f\colon \mathbb{C} \to \mathbb{C}_\infty$ with an essential singularity at ∞. Here the n-th iterate of a point $z \in \mathbb{C}$ is denoted by $f^n(z) = (f \circ \overset{(n)}{\cdots} \circ f)(z)$, and the sequence of iterates $\{f^n(z)\}_{n\in\mathbb{N}}$ is well-defined for all $z \in \mathbb{C}$ except for the countable set of poles and prepols of f of any order.

It is a well-known fact that for transcendental maps, ∞, which is an essential singularity, is in the Julia set, and that we can write the Julia set as the boundary of the escaping set, which are the points that converge to ∞ under iteration.

For a periodic point z_0 of period p (i.e. such that $f^p(z_0) = z_0$), we define its **multiplier** as $\lambda = (f^p)'(z_0)$. Using the chain rule, it can be verified that

$$\lambda = \prod_{n=0}^{p-1} f'(f^n(z_0)) = \prod_{n=0}^{p-1} f'(z_n)$$

and therefore, the multiplier is the same for every periodic point of the orbit. Hence, we regard it as the multiplier of the orbit. Depending on the modulus of the multiplier we say that the orbit is; attracting when $|\lambda| < 1$, repelling when $|\lambda| > 1$ and indifferent when $|\lambda| = 1$.

In dynamical systems we are concerned with classifying the maps according to their dynamics, which leads to the notion of conjugacy.

Definition 2.1 (Conjugacy)**.** *We say that a function $f\colon U \to U$ is* ***conjugate*** *to a function $F\colon V \to V$ if and only if there is a homeomorphism $\varphi\colon U \to V$ such that*

$$\varphi(f(z)) = F(\varphi(z))$$

i.e., the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\varphi"'] & U \arrow[d, "\varphi"] \\ V \arrow[r, "F"] & V \end{tikzcd}$$

Two conjugate functions have the same dynamics. Indeed, the iterates of f are also conjugate by the same map φ since $F^n = \varphi \circ f^n \circ \varphi^{-1}$. The inverses, f^{-1} and F^{-1}, whenever well-defined, are also related by φ.

Depending on the regularity of φ we distinguish different types of conjugacies. The first approach, since we are dealing with meromorphic maps would be using a conformal conjugacy. However, holomorphic maps are too rigid, for example, they preserve the multiplier. Our goal in this work is to stress out this rigidity in order to cover a broad variety of maps by using quasiconformal conjugacies.

The main goal of this thesis is to perform a well-known technique in complex dynamics known as *quasiconformal surgery*, commonly used to construct holomorphic maps with prescribed dynamics. The main idea behind it is that we take advantage of the flexibility of quasiregular maps to build maps with given dynamics by pasting different maps together to obtain what we refer as a **model map**. What we want to determine using the theory of quasiconformal mappings is to verify under which conditions we can find a holomorphic copy, i.e. under which conditions our model map, say f, is quasiconformally conjugate to a holomorphic map F.

It turns out that in order to obtain a holomorphic quasiconformally conjugate map is enough to have a f-invariant Beltrami coefficient.

Lemma 2.2 (Key lemma for surgery)**.** *Suppose $f\colon \mathbb{C} \to \mathbb{C}$ is quasiregular and μ is a Beltrami coefficient ($||\mu||_\infty < 1$) which is invariant under f. Then there exists a quasiconformal map $\varphi\colon \mathbb{C} \to \mathbb{C}$ such that $F = \varphi \circ f \circ \varphi^{-1}$ is holomorphic, i.e., f and F are quasiconformally conjugate.*

Proof. If we use the Integrability theorem we find a quasiconformal map φ such that $\mu = \varphi^*\mu_0$, then

$$F^*\mu_0 = (\varphi \circ f \circ \varphi^{-1})^*\mu_0 = (\varphi^{-1})^* f^* \varphi^* \mu_0 = (\varphi^{-1})^* f^* \mu = (\varphi^{-1})^*\mu = \mu_0.$$

Therefore, by Weyl's lemma f is holomorphic. □

Therefore, in view of Lemma 2.2, we wish to determine, given a model map f, whether we can find a f-invariant Beltrami coefficient or not. Through this chapter we will see a general principle that allows us to find one (Proposition 2.4), and then examples of the two different types of surgery that we distinguish; **soft surgery** where the model map is holomorphic, and **cut and paste surgery**, where the model map is quasiregular.

The references for this chapter are [Ahl, BF].

2.1 General principles of Surgery

The next result, known as Shishikura first principle, tells us a way to make sure that we have an invariant Beltrami coefficient, which is one of the main points in a surgery procedure. This result can be applied in many cut and paste surgery constructions, where we paste together holomorphic and quasiregular maps.

Definition 2.3 (Quasirational, quasientire and quasimeromorphic maps)**.** *A quasiregular map $f\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ is called **quasirational** if it is quasiconformally conjugate to a rational map.*

*We use **quasientire** and **quasimeromorphic** in the same way.*

Proposition 2.4 (First Shishikura principle)**.** *Let $f\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a quasiregular map, $p \geq 1$ and suppose there exist:*

- *$U = U_1 \cup \cdots U_p$, p disjoint open subsets of $\mathbb{C}_\infty$ such that for $1 \leq j < p$,*

$$f(U_j) \subset U_{j+1} \quad \text{and} \quad f(U_p) \subset U_1.$$

- *$\psi\colon U \to \tilde{U}$, $\tilde{U} \subset \mathbb{C}_\infty$ and ψ is quasiconformal (the gluing map).*
- *$H\colon \tilde{U} \to \tilde{U}$ is quasiregular with H^p holomorphic,*

satisfying:

(i) $f_{|U} = \psi^{-1} \circ H \circ \psi$.

(ii) $\partial_{\bar{z}} f = 0$ *a.e. in* $\mathbb{C}_\infty \setminus f^{-N}(U)$ *for some* $N \geq 0$.

Then f *is quasirational.*

Proof. By hypothesis, ψ is K_1-qc, H is K_2-qr and f is K_3-qr, for some $K_1, K_2, K_3 \geq 1$.

The first step is defining a H-invariant Beltrami coefficient $\tilde{\mu}$ a.e. on $\tilde{U}$.

To do so, set $\tilde{U}_j := \psi(U_j)$, for $1 \leq j \leq p$, which are disjoint and

$$H(\tilde{U}_j) \subset \tilde{U}_{j+1} \quad \text{and} \quad H(\tilde{U}_p) \subset \tilde{U}_1.$$

Define a.e. on $\tilde{U}$:

$$\tilde{\mu} := \begin{cases} \mu_0 & \text{on } \tilde{U}_p \\ H^*(\mu_0) & \text{on } \tilde{U}_{p-1} \\ \vdots & \vdots \\ (H^{p-1})^*(\mu_0) & \text{on } \tilde{U}_1 \end{cases}.$$

Since H^p is holomorphic, then $\tilde{\mu}$ is H-invariant and its dilatation $K(\tilde{\mu}) \leq K_2^{p-1}$ a.e. Consider now $\mu = \psi^*(\tilde{\mu})$, then μ is f-invariant a.e. on U (because $f^*(\mu) = (\psi \circ f)^*(\tilde{\mu}) = (H \circ \psi)^*(\tilde{\mu}) = \psi^*(H^*(\tilde{\mu})) = \mu$) and its dilatation $K(\mu) \leq K_1 K_2^{p-1}$ is bounded.

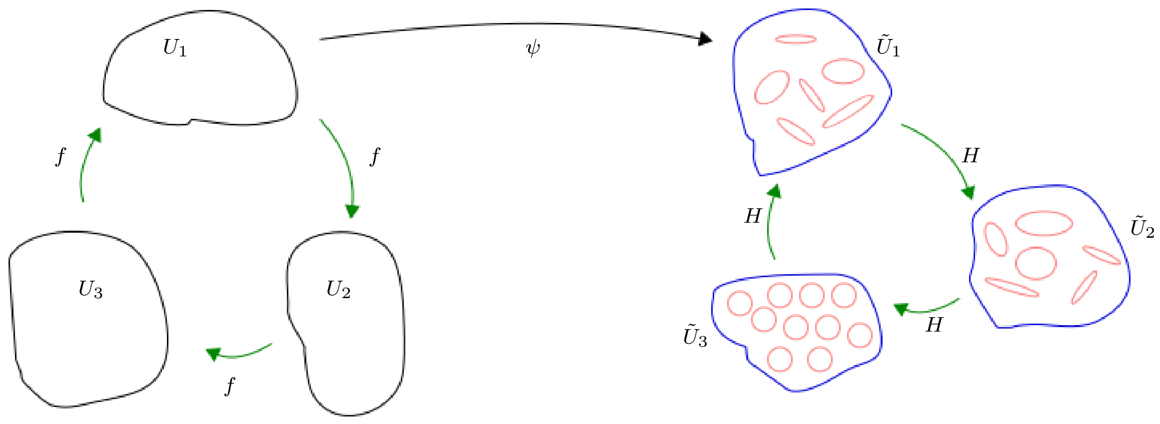

Figure 2.1: Representation of the Beltrami forms defined in the proof of Proposition 2.4.

Now we spread μ recursively by the dynamics of f, i.e. we set:

$$\mu := \begin{cases} f^*(\mu) & \text{on } f^{-1}(U) \\ \vdots & \vdots \\ (f^n)^*(\mu) & \text{on } f^{-n}(U) \\ \vdots & \vdots \\ \mu_0 & \text{on } \mathbb{C}_\infty \setminus \cup U_{j\geq 0} f^{-j}(U) \end{cases}.$$

Since $f(U) \subset U$, then

$$U \subset f^{-1}(U) \subset f^{-2}(U) \subset \cdots \subset f^{-n}(U) \subset \cup_{n\geq 0} f^{-n}(U),$$

hence μ is well-defined a.e. on $\mathbb{C}_\infty$ and f-invariant by construction ($\cup_n f^{-n}(U)$ is completely invariant). Moreover,

$$||\mu||_\infty \leq k = \frac{K-1}{K+1},$$

where $K = K_1 K_2^{p-1} K_3^N$ by (ii) (condition (ii) says that the dilatation is left unchanged after N preimages).

By the Integrability Theorem, we obtain a quasiconformal mapping $\phi\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ such that $\mu = \phi^*(\mu_0)$, and thus

$$F := \phi \circ f \circ \phi^{-1} \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$$

is holomorphic (by Lemma 2.2) and quasiconformally conjugate to f. □

2.2 Examples

Depending on the regularity of the model map and its domain we can classify the different types of surgery that we perform.

We distinguish between **soft surgery** and **cut and paste surgery**. Both of them will be explained through examples in the next sections.

2.2.1 Soft surgery: changing the multiplier

In this type of surgery the model map f is holomorphic. What we produce here is a change of the complex structure. However, we also need to verify that the holomorphic map F that we obtain from Lemma 2.2 and the model map f are different.

The goal now is to show an example about how this surgery can be performed by **changing the multiplier of an attracting cycle**. As we have already said, the multiplier is a conformal invariant. However, two maps near an attracting fixed point (not super-attracting) behave in the same way. Here we prove that, in fact, they are quasiconformally conjugate.

More precisely, the goal is to prove the following:

Theorem 2.5 (Changing the multiplier of an attracting orbit)**.** *Let $f_{\lambda_0}\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational map of degree $d \geq 2$ with an attracting cycle of period p and multiplier $\lambda_0 \in \mathbb{D}^*$. Then there exists a rational map f_λ which is quasiconformally conjugate to f_{λ_0} such that the corresponding p-periodic cycle has multiplier λ.*

But first we need to prove a technical lemma.

Lemma 2.6. *Let*

$$\begin{aligned} M_\lambda \colon \mathbb{D} &\to \mathbb{D} \\ z &\mapsto \lambda z. \end{aligned}$$

Then, whenever $\lambda_0, \lambda \in \mathbb{D}^* = \mathbb{D} \setminus \{0\}$, the maps M_{λ_0} and M_λ are quasiconformally conjugate. Observe that M_λ and M_{λ_0} are not conformally conjugate.

Proof. We want to find a qc map $\varphi\colon \mathbb{D} \to \mathbb{D}$ so that the following diagram commutes.

$$\begin{tikzcd} \mathbb{D} \arrow[r, "M_{\lambda_0}"] \arrow[d, "\varphi"'] & \mathbb{D} \arrow[d, "\varphi"] \\ \mathbb{D} \arrow[r, "M_\lambda"'] & \mathbb{D} \end{tikzcd}$$

We proceed as follows, define $A_\lambda = \{z \in \mathbb{D} \colon |\lambda| \leq |z| < 1\}$.

- Take a quasiconformal map φ_1, so that $A_{\lambda_0} \xrightarrow{\varphi_1} A_\lambda$ and $\varphi_1(\lambda_0) = \lambda$ (see [BF] or [LV] for more details) then, $M_{\lambda_0}^n(A_{\lambda_0}) = \{|\lambda_0|^{n+1} \leq |z| < |\lambda_0|^n\} = A^n$.

- Define $\varphi_{n+1} = M_\lambda^n \circ \varphi_1 \circ M_{\lambda_0}^{-n}$ in A_{n+1}, then
$$M_\lambda \circ \varphi_n = M_\lambda^n \circ \varphi_1 \circ M_{\lambda_0}^{-n+1} \circ M_{\lambda_0}^{-1} \circ M_{\lambda_0} = \varphi_{n+1} \circ M_{\lambda_0}.$$

- Define φ by:
$$\varphi(z) := \begin{cases} 0 & \text{if } z = 0 \\ \varphi_n(z) & \text{if } z \in A_n \end{cases}$$
then, $M_\lambda \circ \varphi = \varphi \circ M_0$.

So M_λ and M_0 are quasiconformally conjugate. □

We can address now the proof of Theorem 2.5.

Proof of Theorem 2.5. The proof goes by considering the previous construction and the fact that f_λ^p is locally conformally conjugate to M_{λ_0} via the linearizing map obtained in Koenig's theorem.

We need to introduce some notation:

- $\alpha = \mathcal{O}(\alpha_0) = \{\alpha_0, \dots, \alpha_{p-1}\}$ denotes the attracting cycle of f_λ, where as usual, $\alpha_{j+1} = f_{\lambda_0}(\alpha_j)$.

- $\mathcal{A}(\alpha)$ denotes the basin of attraction of the orbit α.

- $\mathcal{A}^*(\alpha)$ denotes the connected component of $\mathcal{A}(\alpha)$ containing α, i.e.
$$\mathcal{A}^*(\alpha) = \bigcup_{j=0}^{p-1} \mathcal{A}_j^*(\alpha_j)$$
where $\mathcal{A}_j^*(\alpha_j)$ is the immediate basin of attraction of α_j for $f_{\lambda_0}^p$.

- Let $\Delta_0 \subset \mathcal{A}_0^*$ be a neighborhood of α_0, which is the preimage of $\mathbb{D}$ under a linearizing map $\psi_0 \colon \Delta_0 \to \mathbb{D}$, which conjugates $f_{\lambda_0}^p$ to M_{λ_0} (the map from Koenig's theorem, see [BF, Mil2]), which is conformal. Furthermore, after composing with a Möbius transformation we can suppose that $\mathcal{A}(\alpha)$ does not contain ∞.

- If we consider the map φ as in Lemma 2.6, the one that conjugates M_{λ_0} and M_λ, in order to emphasize that it depends on λ, we write φ_λ.

Note that ψ_0 can be extended to the entire basin of attraction $\mathcal{A}_{f^p}(\alpha_0)$ of α_0 as a fixed point of f^p (see [BF, CG, Mil2]). Then $\psi_0(\mathcal{A}_{f^p}(\alpha_0)) = \mathbb{C}$ and by construction, ψ_0 is a semi-conjugacy (no longer injective) between $f_{\lambda_0}^p$ to M_{λ_0}.

Moreover, we can extend φ_λ to $\mathbb{C}$ by pulling backwards instead of forward in the previous computations. Which yields a quasiconformal map φ_λ defined on $\mathbb{C}$ and we can consider $\mu_\lambda = \varphi_\lambda^* \mu_0$. Since ψ_0 is holomorphic, we can pullback μ_λ under ψ_0 to obtain a $f_{\lambda_0}^p$-invariant Beltrami coefficient $\tilde{\mu}_\lambda$ on $\mathcal{A}_{f^p}(\alpha_0)$ (with the same dilatation as μ_λ).

Everything is summarized in the following commutative diagram.

$$\begin{array}{ccc}
(\mathcal{A}_{f^p}(\alpha_0), \tilde{\mu}_\lambda) & \xrightarrow{f_{\lambda_0}^p} & (\mathcal{A}_{f^p}(\alpha_0), \tilde{\mu}_\lambda) \\
\psi_0 \downarrow & & \downarrow \psi_0 \\
(\mathbb{C}, \mu_\lambda) & \xrightarrow{M_{\lambda_0}} & (\mathbb{C}, \mu_\lambda) \\
\varphi_\lambda \downarrow & & \downarrow \varphi_\lambda \\
(\mathbb{C}, \mu_0) & \xrightarrow{M_\lambda} & (\mathbb{C}, \mu_0)
\end{array}$$

The final step so that we can obtain our holomorphic copy is to extend the Beltrami coefficient $\tilde{\mu}_\lambda$ to the whole basin $\mathcal{A}(\alpha)$ by pulling back $\tilde{\mu}_\lambda$ under f_{λ_0} (as in Proposition 2.4), we will refer to the Beltrami coefficient obtained by $\tilde{\mu}_\lambda$.

Note that, since $\tilde{\mu}_\lambda$ on $\mathcal{A}_{f^p}(\alpha_0)$ is $f^p_{\lambda_0}$-invariant, then the pullback under f_{λ_0} of $\tilde{\mu}_\lambda$ on $\mathcal{A}_{f^p}(\alpha_1)$ agrees with $\tilde{\mu}$ on $\mathcal{A}_{f^p}(\alpha_0)$, and we can keep doing it in order to obtain a Beltrami coefficient defined on $\mathcal{A}(\alpha)$, the whole basin of attraction of the periodic orbit.

We can finally extend $\tilde{\mu}_\lambda$ to the whole Riemann sphere $\mathbb{C}_\infty$ by setting $\tilde{\mu}_\lambda \equiv 0$ in $\mathbb{C}_\infty \setminus \mathcal{A}(\alpha)$. By the Integrability theorem, there exists a quasiconformal map $\phi_\lambda \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, normalized to fix $0, 1$ and ∞ such that $\tilde{\mu}_\lambda = \phi^*(\mu_0)$, hence

$$f_\lambda \coloneqq \phi_\lambda \circ f_{\lambda_0} \circ \phi_\lambda^{-1} \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$$

is holomorphic, i.e. it is rational, of degree d (the same as f_λ), and quasiconformally conjugate to f_{λ_0}.

$$\begin{tikzcd}
(\mathbb{C}_\infty, \mu_0) \arrow[r, "f^p_\lambda"] & (\mathbb{C}_\infty, \mu_0) \\
(\mathbb{C}_\infty) \arrow[u, "\phi_\lambda"] \arrow[r, "f^p_{\lambda_0}"] & (\mathbb{C}_\infty, \tilde{\mu}_\lambda) \arrow[u, "\phi_\lambda"']
\end{tikzcd}$$

It stills remains to check that f_λ fulfills our purpose, i.e. it has an attracting cycle of period p and multiplier λ. The first part is clear since f_λ and f_{λ_0} are topologically conjugate via the integrating map ϕ_λ. To see the second part, note that we have the following commutative diagram:

$$\begin{tikzcd}
(\phi_\lambda(\Delta_0), \mu_0) \arrow[r, "f^p_\lambda"] & (\phi_\lambda(\Delta_0), \mu_0) \\
(\Delta_0, \tilde{\mu}_\lambda) \arrow[u, "\phi_\lambda"] \arrow[r, "f^p_{\lambda_0}"] \arrow[d, "\psi_0"'] & (\Delta_0, \tilde{\mu}_\lambda) \arrow[u, "\phi_\lambda"'] \arrow[d, "\psi_0"] \\
(\mathbb{D}, \mu_\lambda) \arrow[r, "M_{\lambda_0}"] \arrow[d, "\varphi_\lambda"'] & (\mathbb{D}, \mu_\lambda) \arrow[d, "\varphi_\lambda"] \\
(\mathbb{D}, \mu_0) \arrow[r, "M_\lambda"] & (\mathbb{D}, \mu_0)
\end{tikzcd}$$

Therefore, by considering $\psi_\lambda = \varphi_\lambda \circ \psi_0 \circ \phi_\lambda^{-1} \colon \phi_\lambda(\Delta_0) \to \mathbb{D}$, we obtain that $\psi_\lambda^* \mu_0 = \mu_0$, hence by Weyl's lemma ψ_λ is a conformal map which conjugates locally f^p_λ and M_λ. $\square$

2.2.2 Cut and paste surgery: the Straightening theorem

We end this chapter presenting the Straightening theorem, which was one of the earliest applications of quasiconformal mappings in complex dynamics. This result justifies why when we look at the dynamical plane of many non-polynomial families we see polynomial Julia sets.

However, before we can state it we need to introduce some definitions which tell us which are the conditions that need to be imposed in a map so that we can find polynomial Julia sets in the dynamical plane of non-polynomial families. These type of maps are known as polynomial like mappings, and they came from the idea that rational maps may behave locally as a polynomial.

Definition 2.7 (Polynomial-like mapping)**.** *Let $U, V \subset \mathbb{C}$ be bounded, simply connected and bounded by analytic curves such that $\overline{U} \subset V$. The triple $(f; U, V)$ is called a* ***polynomial-like mapping*** *of degree d if $f \colon U \to V$ is holomorphic and of proper degree d*

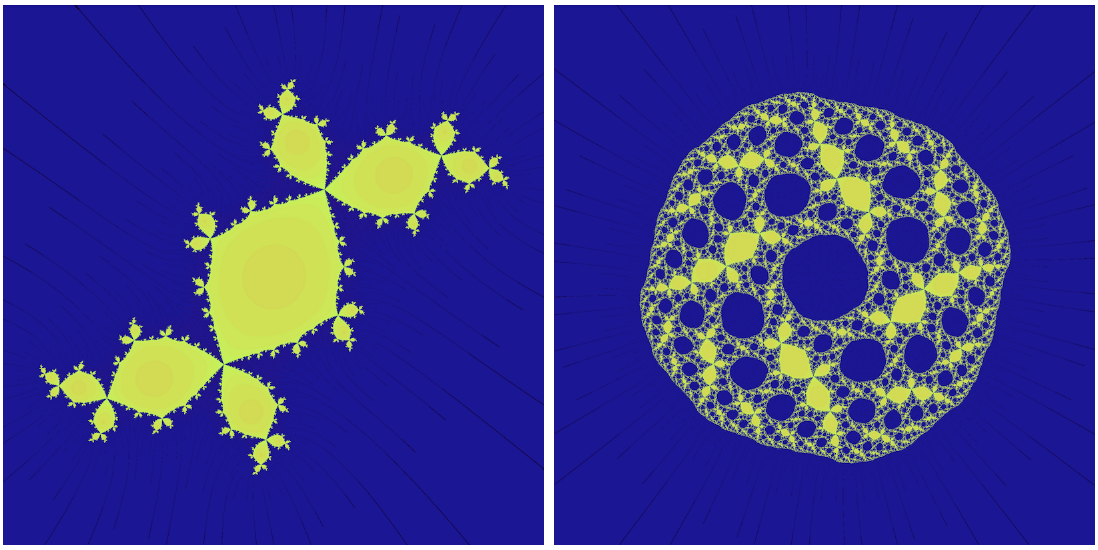

Figure 2.2: Illustration of Theorem 2.8. In the left side we can see the dynamical plane of $Q_c(z) = z^2 + c$ for $c = -0.12256 + 0.74486i$. In the right side we can see the dynamical plane of $R_a(z) = z^2 + a/z^2$, for $a = 0.00848556 + 0.0547416i$, and how some similar patterns can be appreciated.

Remember that the **degree** is the number of inverse images of a point $z \in V$, counted with multiplicity. This number is independent of z. The map is said to be **proper** if the inverse of a compact set is also a compact set.

Note that any polynomial P can be restricted to a domain U so that $(P_{|U}, U, P(U))$ is a polynomial-like mapping of the degree of the polynomial.

Given a polynomial-like mapping $(f; U, V)$, we define its **filled Julia set** $\mathcal{K}_f$ as

$$\mathcal{K}_f := \cap_{n>0} f^{-n}(V),$$

i.e. the points $z \in U$ such that $f^n(z) \in U$ for all $n \geq 0$.

We can finally state and proof the Straightening theorem.

Theorem 2.8 (Straightening theorem)**.** *Every polynomial-like mapping $(f; U, V)$ of degree d is conformally equivalent to a polynomial of degree d. The quasiconformal conjugacy ϕ can be taken so that $\partial_{\bar{z}}\phi = 0$ on $\mathcal{K}_f$.*

Proof. Choose any $\rho > 1$ and let $R\colon \mathbb{C}_\infty \setminus \overline{V} \to \mathbb{C}_\infty \setminus \overline{D(0, \rho^d)}$ be the Riemann map fixing ∞.

Then R extends continuously to the boundary (by Theorem 1.42) as an analytic map, which we call $\psi_1\colon \partial V \to \mathbb{S}^1_{\rho^d}$. Take a lift of this map $\psi_2\colon \partial U \to \mathbb{S}^1_\rho$ (i.e. such that $\psi_1(f(z)) = \psi_2(z)^d$ for $z \in \partial U$).

Let $A_0 = V \setminus \overline{U}$ (an annulus), and define $\mathcal{A}_{\rho,\rho^d} = \{z \in \mathbb{C}\colon \rho < |z| < \rho^d\}$. Since ψ_1, ψ_2 are analytic, we can apply the interpolation result (Theorem 1.48) in order to obtain a map $\psi\colon \overline{A_0} \to \overline{\mathcal{A}_{\rho,\rho^d}}$ which is quasiconformal in A_0.

Define $F\colon \mathbb{C} \to \mathbb{C}$ by

$$F(z) := \begin{cases} f(z) & \text{if } z \in U \\ R^{-1}(\psi(z)^d) & \text{if } z \in V \setminus U \\ R^{-1}(R(z)^d) & \text{if } \mathbb{C} \setminus V \end{cases}.$$

which is quasiregular.

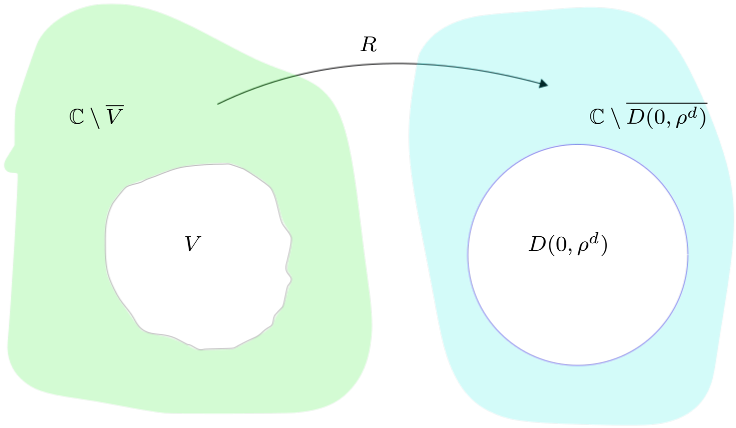

Figure 2.3: Representation the Riemann map $R\colon \mathbb{C}_\infty \setminus \overline{V} \to \mathbb{C}_\infty \setminus \overline{D(0,\rho^d)}$ in the proof of Theorem 2.8.

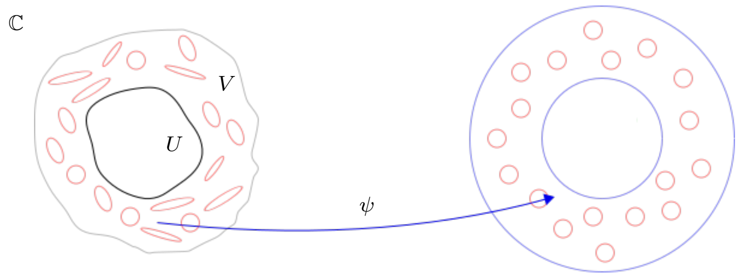

Figure 2.4: Definition of the Beltrami coefficient in the proof of Theorem 2.8

The next step is defining a Beltrami coefficient μ on $\mathbb{C}$, we proceed as follows. Define $A_n = \{z \in U\colon f^n(z) \in A_0\}$, then

$$\mu := \begin{cases} \psi^*(\mu_0) & \text{on } A_0 \\ (f^n)^*(\mu) & \text{on } A_n \\ \mu_0 & \text{elsewhere} \end{cases} .$$

which is F-invariant. That is because points in A_0 are mapped to $\mathbb{C} \setminus V$ and then tend to ∞ under iteration by F and the A_n are disjoint. Since f is holomorphic, the dilatation of μ is the dilatation of $\psi^*(\mu_0)$. By the Integrability theorem, there exists ϕ such that $\mu = \phi^*(\mu_0)$.

$$\begin{array}{ccc} (\mathbb{C},\mu) & \xrightarrow{F} & (\mathbb{C},\mu) \\ \phi\big\downarrow & & \big\downarrow\phi \\ (\mathbb{C},\mu_0) & \xrightarrow{P} & (\mathbb{C},\mu_0) \end{array}$$

Moreover, $\mu = \mu_0$ on $\mathcal{K}_f$ (points that do not tend to ∞), i.e. $\partial_{\bar{z}}\phi = 0$ on $\mathcal{K}_f$. Thus $P := \phi \circ F \circ \phi^{-1}$ is holomorphic (by Lemma 2.2) and near ∞ is $z \mapsto z^d$, i.e. can be extended to a holomorphic map in $\mathbb{C}_\infty$ without poles, therefore it is a polynomial (of degree d). $\square$

Chapter 3

Surgery on a family of meromorphic maps

In this chapter we study the family of transcendental meromorphic maps

$$f_\lambda(z) = \lambda\left(\frac{e^z}{z+1} - 1\right),$$

where $\lambda \in \mathbb{C} \setminus \{0\}$ is a complex parameter.

Maps in this family are the simplest meromorphic maps with two singularities of f^{-1}: $z = 0$ which is a fixed critical point, and $-\lambda$, which is a free asymptotic value. It has also one single pole $z = -1$, which is not omitted except for $\lambda = 1$. Since $z = 0$ is a superattracting fixed point, its basin of attraction $\mathcal{A}_\lambda(0)$ is non-empty for all values of λ.

This family is the meromorphic analogue to the well-known Milnor family of cubic polynomials $P_a(z) = z^2(z-a)$ [Mil1] or its entire version $\lambda z^2 e^z$ [FG2], both having also a superattracting fixed point and a free second singular value, which may or may not be captured by the attracting basin of 0. In both cases all components of the Fatou set are simply connected.

Another well-known result is that functions with only finitely many singular values do not have Wandering nor Baker domains, hence $F(f_\lambda)$ does not have any of these components. Moreover, since any attractive basin or a rotation domain needs a singular value, we can have at most two periodic cycles of Fatou components for every parameter value, one of which is always the basin of $z = 0$. Hence it is to our interest to study the main capture component $\mathcal{C}_0 = \{\lambda \in \mathbb{C}^* \colon -\lambda \in \mathcal{A}^*_\lambda(0)\}$, where $\mathcal{A}^*_\lambda(0)$ is the immediate basin of attraction of $z = 0$. In this case we only have one Fatou component and we can draw an accurate picture of $F(f)$ by considering the points that at a certain iterate land near $z = 0$.

After addressing the study of the dynamical properties of f_λ for $\lambda \in \mathcal{C}_0$, it has already been proved in [Rod] that $\mathcal{A}_\lambda(0)$ is infinitely connected and totally invariant, which is also proved in this document in 3.3.1 (Theorem 3.13), a result that extends the one in [Rod] for more general families of maps.

However, we are more interested in studying the Julia set of f_λ, more precisely we show that the Julia set of $f_{-1/e}$ contains a well-known structure known as a Cantor Bouquet. We prove it in Section 3.1 (Theorem 3.4), as an application of quasiconformal surgery.

3.1 Cantor Bouquet's in the Julia set of transcendental maps

Along the last decades, the study of the dynamics of transcendental entire functions has been of interest. Even the simplest transcendental entire map that one can study, the exponential family $g_a(z) = a(e^z - 1)$, it is a source of interesting problems due to how rich it is the geometry and topology of its Julia set. Moreover, in [BJR] they proved that the Julia set of a subclass of functions in the class $\mathcal{S}$, consisting of all transcendental entire functions with a finite number of singular values, is homeomorphic to the same universal object, a Cantor Bouquet.

In the rational case, the Straightening theorem (Theorem 2.8) tells us why we can see Julia sets of polynomial maps in the Julia set of many rational maps. What we intend to here is a step forward in this direction but in the more general setting of transcendental maps, i.e. we aim to see that we can relate Julia sets of transcendental entire maps in the Julia sets of transcendental meromorphic maps, in the same way that it has been done with polynomial and rational maps.

Due to the similarities of our family of maps with the exponential family g_a, we intend to prove that we can find this structure by performing surgery through a model map obtained by pasting g_a and f_λ. However, the parameters that we use must be properly chosen. Note that when $|a| < 1$, the Fatou set $F(g_a)$ has a unique component, which consists of all the points that converge under iteration to $z = 0$, an attracting fixed point. The Julia set $J(g_a)$ consists of a union of pairwise disjoint arcs tending to ∞, which are called **hairs** (or **dynamical rays**), more precisely:

Definition 3.1 (Hairs, dynamical rays)**.** *Let f be a transcendental meromorphic function. A* ***ray tail*** *of f is an injective curve $\beta\colon [t_0, \infty) \to I(f)$, with $t_0 > 0$ and $I(f)$ the escaping set of f, such that:*

- *For each $n \geq 1$, $t \mapsto f^n(\beta(t))$ is injective with $\lim_{t\to\infty} f^n(\beta(t)) = \infty$.*
- *$f^n(\beta(t)) \to \infty$ uniformly in t as $n \to \infty$.*

A ***hair*** *(or a* ***dynamic ray****) of f is a maximal injective curve $\beta\colon (0, \infty) \to I(f)$ such that the restriction $\beta_{|[t,\infty)}$ is a ray tail for all $t > 0$. We say that β lands at z if $\lim_{t\to 0^+} \beta(t) = z$ and we call z the* ***endpoint*** *of β.*

The union of these hairs and their accumulation has a rich topological structure and it produces what it is known as a **Cantor Bouquet**, which is the object that we are seeking to find in our family of transcendental meromorphic maps f_λ. Therefore, it is to our interest to give a precise definition of this object.

In [AO], there is provided a complete topological description of the Julia sets $J(g_a)$, for $a \in (0, 1/e)$, as well as the Julia sets $J(c\sin(z))$ for $c \in (0, 1)$, and it is proved that these sets are homeomorphic to the same topological object. They proved that by constructing an explicit homeomorphism between those Julia sets and a subset of $\mathbb{R}^2$ called a **straight brush**.

Definition 3.2 (Straight brush)**.** *A subset B of $[0, \infty) \times (\mathbb{R} \setminus \mathbb{Q})$ is called a* ***straight brush*** *if the following properties are satisfied:*

- *B is a closed subset of $\mathbb{R}^2$.*
- *For every $(x, y) \in B$ there exists $t_y \geq 0$ such that $\{x\colon (x, y) \in B\} = [t_y, \infty)$. The set $[t_y, \infty) \times \{y\}$ is called the hair attached at y and the point (t_y, y) is called its endpoint.*
- *The set $\{y\colon (x, y) \in B \text{ for some } x\}$ is dense in $\mathbb{R} \setminus \mathbb{Q}$. Moreover, for every $(x, y) \in B$ there exist two sequences of hairs attached respectively at $\beta_n, \alpha_n \in \mathbb{R} \setminus \mathbb{Q}$ such that $\beta_n < y < \alpha_n$, $\beta_n, \alpha_n \to y$ and $t_{\beta_n}, t_{\alpha_n} \to t_y$ as $n \to \infty$.*

We say that two sets $A, B \subset \mathbb{R}^n$ are **ambiently homeomorphic** if there is a homeomorphism of $\mathbb{R}^n$ to itself that sends A onto B.

In [AO] it is also proved that any two straight brushes are ambiently homeomorphic. This fact, together with the results in [AO] concerning the topological structure of the Julia set mentioned before, motivate the following definition of a **Cantor Bouquet**:

Definition 3.3 (Cantor Bouquet)**.** *A* ***Cantor Bouquet*** *is any subset of the plane that is ambiently homeomorphic to a straight brush.*

The main goal of this chapter is proving Theorem A, which tells us that $J(f_{-1/e})$ contains an invariant Cantor Bouquet.

Theorem 3.4. *Let $f_\lambda(z)$ and $g_a(z) = a(e^z - 1)$, for $\lambda, a \in \mathbb{C} \setminus \{0\}$. Denote by $J(f_\lambda)$ and $J(g_a)$ their Julia sets. Then, there exists $\mathcal{C} \subset J(f_{-1/e})$ such that $f_{-1/e}(\mathcal{C}) = \mathcal{C}$ and $f_{-1/e}$ on $\mathcal{C}$ is quasiconformally conjugate to $g_{1/e}$ on $J(g_{1/e})$. In particular, $\mathcal{C}$ is homeomorphic to $J(g_{1/e})$ and both are homeomorphic to a Cantor Bouquet.*

The idea is to perform surgery in the dynamical plane of $f_{-1/e}$ by pasting together $f_{-1/e}$ and $g_{1/e}$ to remove the pole and convert the map into a holomorphic transcendental map. From the construction we will obtain that this map contains a Cantor Bouquet by the results in [BJR].

3.2 The Surgery Procedure

We have already seen how the interpolation results play a fundamental role in many surgery procedures. However, in each of the cases in the previous chapter the interpolation was performed between bounded curves. In our case, we aim to remove the pole so that we can find an 'entire copy', i.e. an entire map which is quasiconformally conjugate to our map in a region containing a Cantor Bouquet.

Therefore, we need to interpolate in a domain that contains the pole, in which we will paste the dynamics of g_a, i.e. we will redefine the orbit of points in a neighborhood of the pole. Hence we need to determine this domain, which will be determined by the dynamics of f_λ. It has already been proved in [Rod] that one way to reach the pole is by considering a Jordan domain containing $z = 0$, and then by taking preimages of this domain under f_λ (which we call D_n), at some point N we reach the only asymptotic value. Then, the pole is contained in one of the bounded connected components of the complementary of the preimage of D_N. Therefore, the boundary of this domain is unbounded (because it contains the asymptotic value), but our interpolation results only apply for bounded curves. The idea is to go to a bounded setting by composing with Möbius transformations, interpolate and then go back, but this requires to prove by some means that the parametrization of the curves that appear in the bounded setting are quasisymmetric, hence it is clear that the choice of the domain D_0 plays an important role.

Note that in both f_λ and g_a it is easier to take preimages of a disk centered at the asymptotic value, rather than the origin (which in the case that concerns us is attracting for both maps), i.e. the domains that appear when taking preimages of a disk centered at the asymptotic value are simpler. What we intend to do now is to determine this (unbounded) domains that will appear, its boundary and show that when passing to the bounded setting as explained before, the boundary curves, properly parametrized, are quasisymmetric and hence we can obtain an interpolating map ψ, which is necessary to paste f_λ and g_a using quasiconformal surgery.

When we paste maps together we need to make sure that in the boundary the resulting function is continuous, therefore the way we relate the boundaries matters (see Figure 3.1). The only way to do that in our setting is by using the dynamics of both maps. To make this statement more precise, define by Γ the unbounded component of $f_\lambda^{-1}(\partial D_1)$, where D_1 is a disk centered at $-\lambda$, the asymptotic value of f_λ. Define $L = g_a^{-1}(\partial D_2)$, where D_2 is a disk centered at $-a$, the asymptotic value of g_a.

We can consider the Riemann map $\mathcal{R} = bz + c$, $b \in \mathbb{R}^+, c \in \mathbb{C}$ which maps conformally D_1 onto D_2 and does not rotate the disk, then we can relate L and Γ in the following way:

$$L = (g_a^{-1} \circ \mathcal{R} \circ f_\lambda)(\Gamma).$$

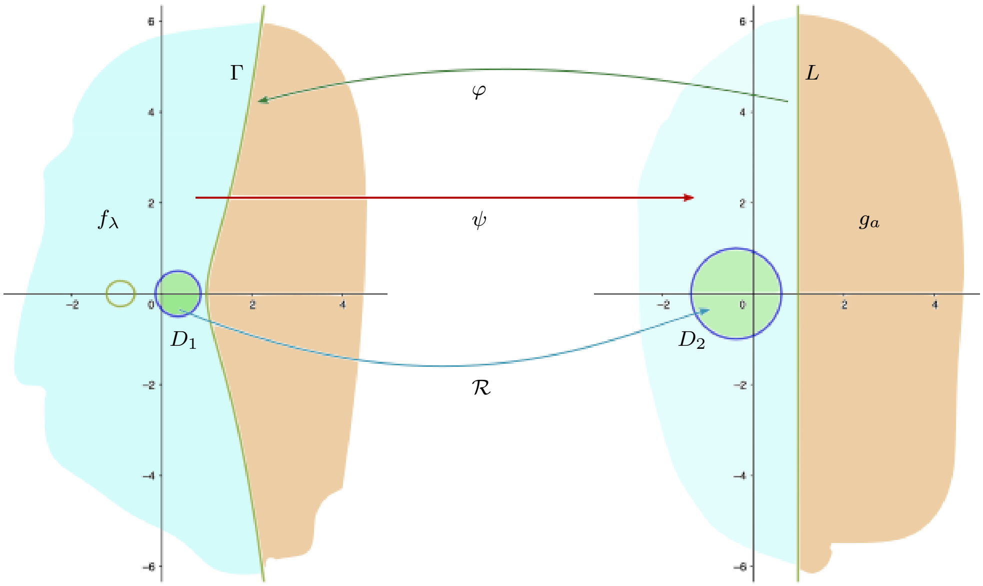

Figure 3.1: Sketch of the parametrizations of the boundary values.

If we parametrize the boundary of D_1 as $w_1 + r_1 e^{it}$, then Γ inherits a parametrization. Therefore, our map φ must be the one that agrees with this parametrization.

Through sections 3.2.1 and 3.2.2 we compute all of these curves and domains, and we prove that we can go to a bounded setting (as explained before) so that the resulting maps are quasisymmetric. Finally, in section 3.2.3, we prove Theorem A.

3.2.1 Study of the family of maps $g_a(z)$

Maps in this family have always $z = 0$ as a fixed point with multiplier $a \in \mathbb{C}$ and they have no critical points ($g_a'(z) = ae^z \neq 0$). Moreover, $z = -a$ is the only asymptotic value, therefore if $|a| < 1$ then

$$g_a^n(-a) \xrightarrow[n\to\infty]{} 0.$$

We intend to use g_a to find a Cantor Bouquet in the dynamical plane of $f_{-1/e}$. Before we address the surgery procedure, we need to find a parameter a (which will be $a = 1/e$) such that $-a$ tends to $z = 0$ under iteration, a disk around the asymptotic value containing $z = 0$ so that its image under g_a is compactly contained inside of itself, and such that the iterates of points in the disk converge uniformly to $z = 0$.

The next lemma gives us the disk mentioned before.

Lemma 3.5. $g_{1/e}(D(-1/e,1)) \subset D(-1/e,1) \subset F(g_{1/e})$.

Proof. If we take $z \in D(-a,r)$, then $z = -a + se^{i\theta} \in D(-a,r)$ for $s < r$ and

$$g_a(z) = ae^{-Re(a)+s\cos(\theta)}e^{i(-Im(a)+s\sin(\theta))} - a,$$

i.e., in order to have $g_a(D(-a,r)) \Subset D(-a,r)$, we need:

$$|a|e^{-Re(a)+r\cos(\theta)} < r \quad \forall\theta \in [0,2\pi).$$

The function $h(\theta) = e^{-Re(a)+s\cos(\theta)}$ attains its maximum at $\theta = 0$, thus we need

$$|a|e^{-Re(a)+r} < r,$$

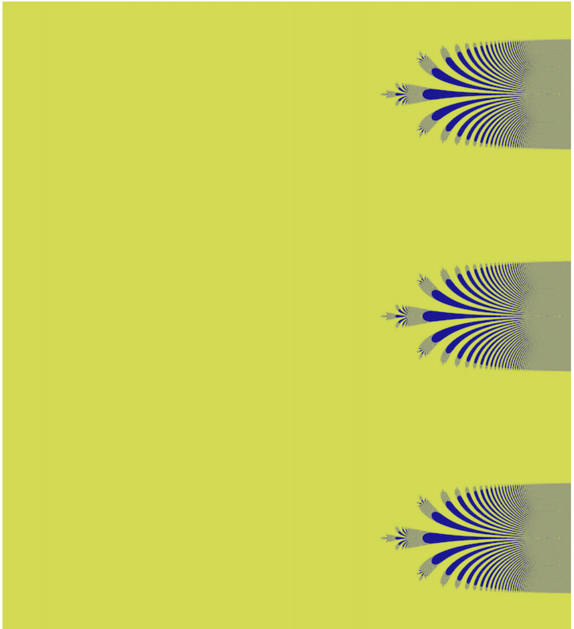

Figure 3.2: $J(g_{1/e})$ is a Cantor Bouquet, which is the set colored in blue. Range $[-9, 8] \times [-9, 9]$.

which is equivalent to $r - \log r < Re(a) - \log |a|$.

For $a = 1/e$, this inequality reads $r - \log r < 1 + 1/e$, thus we can take $r = 1$ (note that $|g'_{1/e}| < 1$ in $\mathbb{D} \cup \{Re(z) \leq 0\}$, which compactly contains $D(-1/e, 1)$).

Hence, $0 \in g_{1/e}(D(-1/e, 1)) \Subset D(-1/e, 1)$. $\square$

Moreover, the preimage of $\partial D(-1/e, 1)$ under $g_{1/e}$ is the line $\{1 + iy \colon y \in \mathbb{R}\}$ and the parametrization α defined as

$$y \in \mathbb{R} \mapsto 1 + iy = \alpha(y)$$

is the one that comes inherited by parametrizing the circle $\partial D(-1/e, 1)$ with $t \mapsto -1/e + e^{it}$ and then taking the preimage under $g_{1/e}$. Note that $g_{1/e}(\alpha(t)) = -1/e + e^{it}$.

If we take the Möbius transformations

$$T(z) = -\frac{2i}{z} \quad \text{and} \quad M(z) = -i\frac{z+1}{z-1},$$

where $M(z)$ maps the unit disk $\mathbb{D}$ conformally onto $\mathbb{H}$

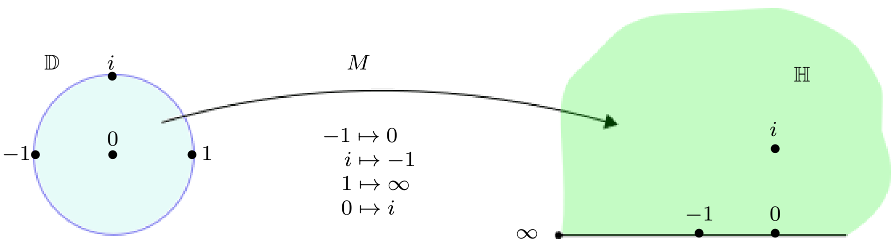

Figure 3.3: The Möbius transformation M, maps $\mathbb{D}$ conformally onto $\mathbb{H}$, the upper half plane.

Then we have,

$$T'(z) = \frac{2i}{z^2} \quad \text{and} \quad M'(z) = \frac{2i}{(z-1)^2}.$$

So, the curve given by $\tilde{\alpha}(s) = (T \circ \alpha \circ M)(e^{is})$ is bounded and we expect it to be differentiable (because the image of a line under a Möbius transformation is a circle).

Lemma 3.6. *The curve $\tilde{\alpha}$ is bounded, $\mathcal{C}^1$ and quasisymmetric.*

Proof. We only need to check that the limits at $s = 0$ and $s = 2\pi$ (points that are mapped to ∞ under $M(e^{is})$) coincide. Moreover,

$$\tilde{\alpha}'(s) = \underbrace{T'(\alpha(M(e^{is})))}_{\frac{2i}{\left(1-i\frac{\sin(s)}{1-\cos(s)}\right)^2}} \underbrace{\alpha'(M(e^{is}))}_{i} \underbrace{M'(e^{is})}_{\frac{2i}{(e^{is}-1)^2}} e^{is} i = \frac{4e^{is}}{\left(\left(1-i\frac{\sin(s)}{1-\cos(s)}\right)(e^{is}-1)\right)^2}.$$

Therefore, we need to compute the limit when $s \to 0$ and $s \to 2\pi$ of

$$\left(1 - i\frac{\sin(s)}{1-\cos(s)}\right)\left(e^{is}-1\right). \tag{3.1}$$

- As $s \to 0$:

  $$e^{is} - 1 \simeq s\left(-\frac{1}{2}s + i\right)$$

  and

  $$-\frac{\sin(s)}{1-\cos(s)} \simeq -\frac{2}{s}.$$

  Therefore,

  $$\lim_{s\to 0}(3.1) = s\left(-\frac{1}{2}s + i\right)\left(1 - i\frac{2}{s}\right) = 2.$$

- As $s \to 2\pi$:

  $$e^{is} - 1 \simeq (2\pi - s)\left(\frac{1}{2}(2\pi - s) + i\right)$$

  and

  $$-\frac{\sin(s)}{1-\cos(s)} \simeq \frac{2}{2\pi - s}.$$

  Therefore,

  $$\lim_{s\to 2\pi}(3.1) = (2\pi - s)\left(\frac{1}{2}(2\pi - s) + i\right)\left(1 + i\frac{2}{2\pi - s}\right) = -2.$$

Hence,

$$\lim_{s\to 0}\tilde{\alpha}'(s) = \lim_{s\to 2\pi}\tilde{\alpha}'(s) = 1,$$

so, $\tilde{\alpha}$ is differentiable and we can apply Lemma 1.40 to obtain that it is quasisymmetric. □

3.2.2 Study of the family of maps $f_\lambda(z)$

The properties of this family of maps have already been explained in this document. We aim to to do the same as in 3.2.1 with f_λ, i.e. we want to take a preimage of a disk centered at the asymptotic value $-\lambda$ and again the only thing that we need to impose is that this disk is compactly mapped inside itself under f_λ and that the iterates converge uniformly to $z = 0$.

The next lemma gives us this disk.

Lemma 3.7. $f_{-1/e}(D(1/e, 1/2)) \subset D(1/e, 1/2) \subset F(f_{1/e})$.

Proof. Given $z = -\lambda + re^{i\theta}$, we need to compute its image. In order to simplify, we can take $\lambda = -x$, for some $x \in \mathbb{R}^+ = \{x \in \mathbb{R} \colon x > 0\}$. Then,

$$f_\lambda(z) = \lambda\frac{e^{-\lambda + r\cos(\theta)}e^{ir\sin(\theta)}}{-\lambda + r\cos(\theta) + ir\sin(\theta) + 1} - \lambda,$$

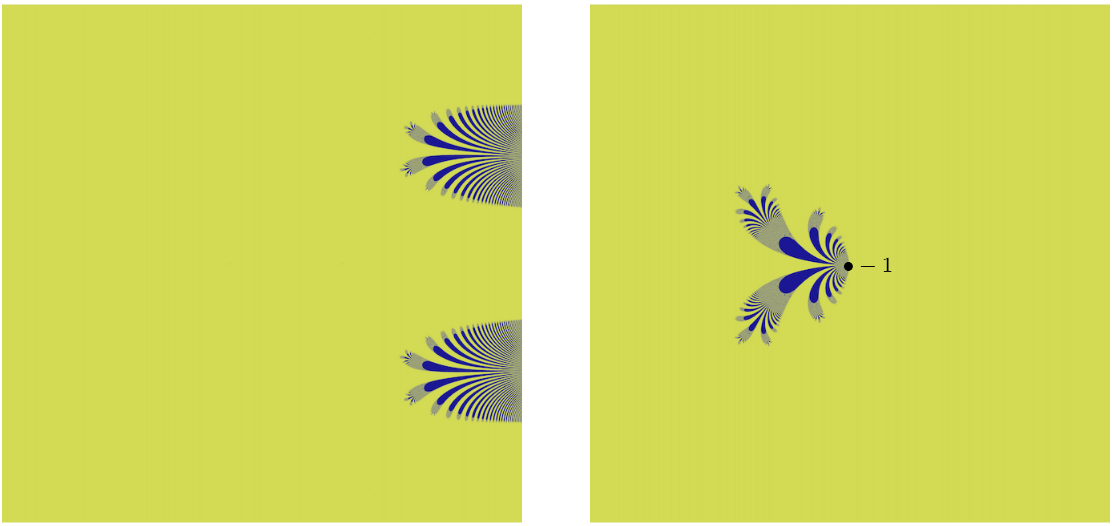

Figure 3.4: Dynamical plane of f_λ for $\lambda = -1/e$, $F(f_{-1/e})$ is colored in green and $J(f_{-1/e})$ is colored in blue. The range of the left side picture is $[-8.5, 8.5] \times [-8.5, 8.5]$ and the range of the right side picture is $[-1.05, -0.95] \times [-0.05, 0.05]$.

which has modulus

$$\frac{xe^{x+r\cos(\theta)}}{\sqrt{(1+x+r\cos(\theta))^2 + r^2\sin^2(\theta)}} = \frac{xe^{x+r\cos(\theta)}}{\sqrt{(1+x)^2 + r^2 + 2(1+x)\cos(\theta)}} = g_x(r,\theta). \tag{3.2}$$

We need $g_x(r,\theta) < r$. Note that, this is equivalent to

$$h_{x,r}(\theta) = \frac{r}{x}\sqrt{(1+x)^2 + r^2 + 2(1+x)r\cos(\theta)} - e^{2(x+r\cos(\theta))} > 0, \forall \theta \in [0, 2\pi).$$

Notice that $h_{x,r}$ has a (global) minimum at $\theta = \pi$ and a (global) maximum at $\theta = 0$, therefore we need $h_{x,r}(\pi) > 0$, which is equivalent to

$$\frac{r}{x}(1+x-r) - e^{x-r} > 0.$$

This equation holds, for example, for $x = 1/e$ and $r = 1/2$.

Now, we need to make sure that $D(x,r) \Subset \mathcal{A}^*_{-x}(0)$ (which is the immediate basin of attraction of $z = 0$ with respect to f_{-x}). To prove it, we can see that f_{-x} is contractive in this disk (with respect to $z = 0$, the super-attracting fixed point), because then we will have normality (all the iterates will tend to $z = 0$).

We have,

$$|f'_{-x}(z)| = \left| -x\frac{ze^z}{(z+1)^2} \right| = \left| \frac{z}{z+1} \right| \left| \frac{xe^z}{z+1} \right| < r \left| \frac{z}{z+1} \right|,$$

where the inequality follows from imposing $g_x(r,\theta) < r$. Moreover, note that $D(x,r) \subset \{z \in \mathbb{C} \colon |z| < |z+1|\} = \{a+ib \in \mathbb{C} \colon a > -1/2\}$, therefore we have $|z| < |z+1|$ and

$$|f'_{-x}(z)| < r < 1.$$

Thus, for every $z \in D(x,r)$,

$$|f_{-x}(z)| = |f_{-x}(z) - f_{-x}(0)| \leq r|z-0| = r|z|,$$

so the map is contractive, i.e. $D(x,r) \subset \mathcal{A}^*_{-x}(0)$ and $0 \in f_{-x}(D(x,r)) \Subset D(x,r)$. $\square$

The next step is parametrizing the outer boundary of $f_{-1/e}^{-1}(\partial D(1/e,1/2))$ by the parametrization inherited from $t \mapsto 1/e + e^{it}/2$. Points, $z \in \mathbb{C}$, in this unbounded curve are such that

$$\frac{e^{it}}{2} = f_{-1/e}(z) - \frac{1}{e} = -\frac{1}{e}\frac{e^z}{z+1} \quad , \quad z = z(t)$$

which yields that we need to solve the equation

$$h(z,t) = -\frac{1}{e}e^z - \frac{1}{2}e^{it}(z+1) = 0. \tag{3.3}$$

Note that these points satisfy $|e^z| = e|z+1|/2$. We have:

Lemma 3.8. *The points in (3.3) can by parametrized by an analytic curve $\gamma\colon \mathbb{R} \to \mathbb{C}$ such that $f_{-1/e}(\gamma(t)) = 1/e + e^{it}/2$. Moreover, $\gamma'(t)$ tends to i as t tends to ∞.*

Proof. We will apply the Implicit Function Theorem to (3.3) in order to accomplish this goal. We have:

$$\frac{\partial h}{\partial z} = -\frac{1}{e}e^z - \frac{1}{2}e^{it},$$

and it is equal to zero, if and only if, $e^z = -ee^{it}/2$, if we write this equality in 3.3, we obtain the condition $e^z z = 0$, hence $\partial h/\partial z \neq 0$ for every point in the curve (because $h(0,t) \neq 0$ for every t). Therefore, by the Implicit Function Theorem, there exists a function $\gamma(t)$ such that $h(z,t) = 0$, if and only if, $z = \gamma(t)$. Note that $\gamma(\infty) = \infty$.

Moreover, by the Implicit Function Theorem, since

$$\frac{\partial h}{\partial t} = -\frac{1}{2}e^{it}(z+1)i,$$

we have

$$\gamma'(t) = -\frac{i}{2}\frac{e^{it}(\gamma(t)+1)}{\frac{1}{e}e^{\gamma(t)} + \frac{1}{2}e^{it}}.$$

This expression can be improved by using (3.3):

$$\frac{\frac{1}{e}e^z + \frac{1}{2}e^{it}}{z+1} = \frac{1}{e}\frac{e^z}{z+1} + \frac{1}{2}\frac{e^{it}}{z+1} \underbrace{=}_{(3.3)} \frac{-1}{e}\frac{e}{2}e^{it} + \frac{1}{2}\frac{e^{it}}{z+1} = \frac{e^{it}}{2}\left(\frac{1}{z+1} - 1\right) = -\frac{e^{it}}{2}\frac{z}{z+1}.$$

Hence,

$$\gamma'(t) = -\frac{ie^{it}}{2}\left(-\frac{2}{e^{it}}\frac{\gamma(t)+1}{\gamma(t)}\right) = i\frac{\gamma(t)+1}{\gamma(t)} \xrightarrow[t\to\pm\infty]{} i. \tag{3.4}$$

□

If we use, again, $M(z)$ (as in Figure 3.3) and $T(z)$ (both as in Lemma 3.8), then

$$M^{-1}(w) = \frac{w-i}{w+i}$$

maps conformally the upper half plane $\mathbb{H}$ to the unit disk $\mathbb{D}$ and

$$(M^{-1})'(w) = \frac{2i}{(w+i)^2}.$$

As before, since the interpolation results that we have apply for bounded curves, we aim to obtain from γ, M and T a bounded differentiable curve. We consider

$$\tilde{\gamma}(s) = T(\gamma(M(e^{is}))) \qquad \tilde{\gamma}\colon \mathbb{S}^1 \to \tilde{\Gamma} \subset \mathbb{C},$$

then it is bounded and smooth at all points for except (at most) $s = 0, 2\pi$, which are the ones that are mapped to ∞ under $M(e^{is})$.

Lemma 3.9. *The curve $\tilde{\gamma}$ is bounded, $\mathcal{C}^1$ and quasisymmetric.*

Proof. We need to compute the limit of $\tilde{\gamma}'(s)$ as $s \to 0$ and $s \to 2\pi$. We have,

$$\tilde{\gamma}'(s) = T'(\gamma(M(e^{is})))\gamma'(M(e^{is})) \underbrace{M'(e^{is})}_{\frac{1}{(M^{-1})'(M(e^{is}))}} e^{is}i =$$

$$= i^2 e^{is}\frac{\gamma(M(e^{is}))+1}{\gamma(M(e^{is}))}\frac{2i}{\gamma(M(e^{is}))^2}\frac{\left(M(e^{is})+i\right)^2}{2i},$$

i.e.

$$\tilde{\gamma}'(s) = -e^{is}\frac{\gamma(M(e^{is}))+1}{\gamma(M(e^{is}))}\left(\frac{M(e^{is})+i}{\gamma(M(e^{is}))}\right)^2. \tag{3.5}$$

It remains to study the last part of equation (3.5), to compute it call $M(e^{is}) = t$, then

$$\frac{t+i}{\gamma(t)} = \frac{1+i/t}{\gamma(t)/t}.$$

Note that we have

$$\frac{\gamma(t)}{t} = \frac{Re(\gamma(t))}{t} + i\frac{Im(\gamma(t))}{t} = \frac{u(t)}{t} + i\frac{v(t)}{t}.$$

Since $\gamma'(t) \xrightarrow[t\to\pm\infty]{} i$, then $u'(t) \xrightarrow[t\to\pm\infty]{} 0$ and $v'(t) \xrightarrow[t\to\pm\infty]{} 1$, and by l'Hôpital's (real) rule,

$$\lim_{t\to\pm\infty}\frac{\gamma(t)}{t} = \lim_{t\to\pm\infty}\frac{u(t)}{t} + i\lim_{t\to\pm\infty}\frac{v(t)}{t} = \lim_{t\to\pm\infty}\frac{u'(t)}{1} + i\lim_{t\to\pm\infty}\frac{v'(t)}{1} = i.$$

Therefore,

$$\lim_{t\to\pm\infty}\frac{t+i}{\gamma(t)} = \frac{1}{i} = -i,$$

and from (3.5),

$$\lim_{s\to 0}\tilde{\gamma}'(s) = \lim_{s\to 2\pi}\tilde{\gamma}'(s) = -1\cdot 1\cdot(-i)^2 = 1.$$

Thus the function $\tilde{\gamma}\colon \mathbb{S}^1 \to \tilde{\Gamma}$ is $\mathcal{C}^1$, and by Lemma 1.40, $\tilde{\gamma}$ is quasisymmetric. □

3.2.3 Proof of Theorem A

The goal now is to address the proof of Theorem 3.4 (or Theorem A). To do so, take $D_1 = D(1/e, 1/2)$ and $D_2 = D(-1/e, 1)$. In one side of the surgery, U_1 (which we recall is the connected component of $f^{-1}_{-1/e}(D_1)$ that contains D_1) is 1-connected and unbounded. In the other side of the surgery, since $D(-1/e, 1)$ contains the asymptotic value of $g_{1/e}$, then U_2 (which we recall is the connected component of $g^{-1}_{1/e}(D_2)$ that contains D_2) is unbounded. We have seen in Section 3.2.1 and Section 3.2.2 how $\Gamma = \gamma(\mathbb{R})$, the outer boundary of U_1, and $\{1+iy\colon y\in\mathbb{R}\} = L$, the boundary of U_2, can be parametrized (Γ by γ and L by $\alpha(t) = 1+it$) such that their projections, with respect to certain Möbius transformations, are quasisymmetric in the sense of Definition 1.38. Each one of this curves and domains is represented in Figure 3.5. The first step to prove Theorem 3.4 is showing that the boundary values after composing with the Möbius transformation $T = -2i/z$ are quasisymmetric.

We need introduce some terminology first. To simplify the notation we keep using the same name for the map obtained by extending a given map to the boundary of its domain of definition.

- $\mathcal{A}_f$ is the annulus that has boundary $\Gamma = \gamma(\mathbb{R})$, where γ is the function obtained in Section 3.2.2 and $\partial D(1/e, 1/2)$. Then $\tilde{\mathcal{A}}_f = T(\mathcal{A}_f)$ is a bounded annulus, with outer boundary $T(\partial D(1/e, 1/2))$ (a circle) and inner boundary $T(\Gamma)$, which parametrized as in Section 3.2.2 is a quasisymmetric function by Lemma 3.9.

- $\mathcal{A}_g$ is the annulus that has boundary $\alpha(\mathbb{R}) = L = \{1+iy \in \mathbb{C}\colon y \in \mathbb{R}\} = g_{1/e}^{-1}(\partial D(-1/e,1))$ and $\partial D(-1/e,1)$. Then $\tilde{\mathcal{A}}_g = T(\mathcal{A}_g)$ is a bounded annulus, with $T(\partial D(-1/e,1))$ (a circle) as its outer boundary and inner boundary $T(L)$ (also a circle), which parametrized as in Section 3.2.1 is a quasisymmetric function by Lemma 3.6

- $\varphi = \gamma \circ \alpha^{-1}\colon L \to \Gamma$. Then after composing with T in both sides we obtain the function $\tilde{\varphi} = T \circ \gamma \circ \alpha^{-1} \circ T^{-1} = (T \circ \gamma \circ M) \circ (M^{-1} \circ \alpha^{-1} \circ T^{-1})$, here $T \circ \gamma \circ M$ by Lemma 3.9 is a quasisymmetric map and the map $M^{-1} \circ \alpha^{-1} \circ T^{-1}$ is the restriction of the conformal map $M^{-1} \circ (z \mapsto 1 + iz) \circ T^{-1}$ to a disk, hence $\tilde{\varphi}$ is quasisymmetric (although it is not a map from $\mathbb{S}^1$, the definition of quasisymmetric map does not depend on the circle from which we take the points).

- $\mathcal{R}(z) = 2z - 3/e$ is the Riemann map between $D(1/e, 1/2)$ and $D(-1/e, 1)$, which extends as a quasisymmetric map between the boundaries and does not rotate the disk. Then, after composing with T in both sides we obtain the map $\tilde{\mathcal{R}}\colon T(\partial D(1/e,1/2)) \to T(\partial D(-1/e,1))$, which is quasisymmetric as well (restriction of a conformal map to a smooth boundary).

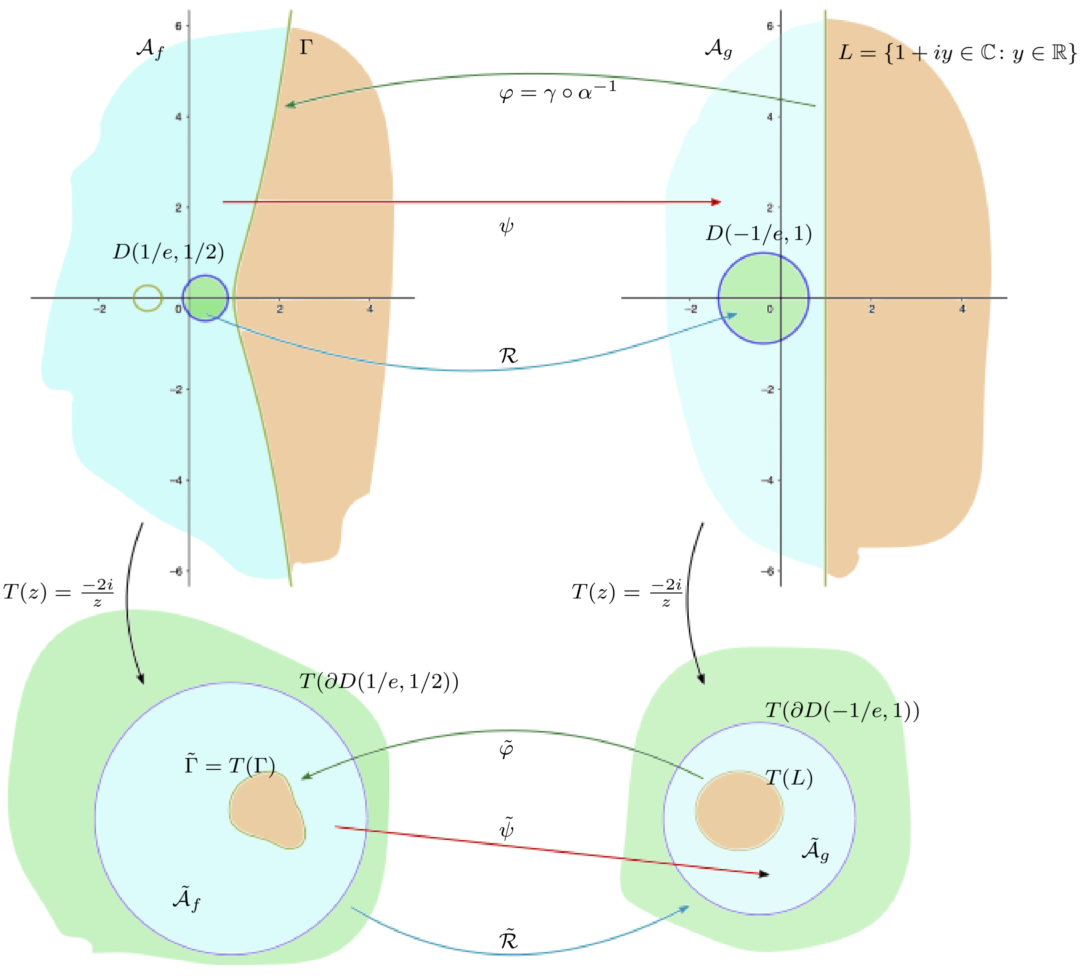

Figure 3.5: Representation of the curves defined before and how they are after composing with the Möbius transformation T. Here $\mathcal{A}_f$ is the annulus that has boundary Γ and $\partial D(1/e, 1/2)$ and, similarly, $\mathcal{A}_g$ is the annulus that has boundary $\{1 + iy \in \mathbb{C}\colon y \in \mathbb{R}\}$ and $\partial D(-1/e, 1)$, they are mapped under T to the (bounded) annulus $\tilde{\mathcal{A}}_f$ and $\tilde{\mathcal{A}}_g$. In the definition of φ, $\alpha(y) = 1 + iy$ and γ is the curve from Lemma 3.8.

Therefore, by Proposition 1.48, we can interpolate between $\tilde{\varphi}^{-1}$ and $\tilde{\mathcal{R}}_{|T(\partial D(1/e,1/2))}$ in order to obtain a quasiconformal map

$$\tilde{\psi}\colon \tilde{\mathcal{A}}_f \to \tilde{\mathcal{A}}_g$$

such that $\tilde{\psi}_{|\tilde{\Gamma}} = \tilde{\varphi}^{-1}$ and $\tilde{\psi}_{|T(\partial D(1/e,1/2))} = \tilde{\mathcal{R}}_{|T(\partial(D(1/e,1/2)))}$. Thus, by considering the map $\psi = T^{-1} \circ \tilde{\psi} \circ T$, which is quasiconformal, we have:

- $\psi_{|\Gamma} = T^{-1} \circ \tilde{\psi} \circ T_{|\Gamma} = T^{-1} \circ T \circ \varphi^{-1} \circ T^{-1} \circ T_{|\Gamma} = \varphi^{-1}$.
- $\psi_{|\partial D(1/e,1/2)} = T^{-1} \circ \tilde{\psi} \circ T_{|\partial D(1/e,1/2)} = T^{-1} \circ T \circ \mathcal{R}_{\partial D(1/e,1/2)} \circ T^{-1} \circ T_{|\partial D(1/e,1/2)} = \mathcal{R}_{|\partial D(1/e,1/2)}$.

i.e. it is an interpolating map between the boundary values φ^{-1} and $\mathcal{R}_{|\partial D(1/e,1/2)}$.

Moreover, we can extend ψ to $\overline{D(1/e,1/2)}$ by setting $\psi_{|D(1/e,1/2)} = \mathcal{R}(z)$, which yields a quasiconformal map $\psi\colon \mathcal{A}_f \cup \overline{D(1/e,1/2)} = U \to \mathcal{A}_g \cup \overline{D(-1/e,1)} = V$.

We have all the tools now to proceed with the surgery, and hence, proving Theorem 3.4. Note that under $f_{-1/e}$ orbits only go through $\mathcal{A}_f$ once.

Define the map

$$F(z) := \begin{cases} (\psi^{-1} \circ g_{1/e} \circ \psi)(z) & \text{if } z \in U \\ f_{-1/e}(z) & \text{if } z \notin U \end{cases}.$$

We claim that this map is quasiregular. To prove it, we only need to see that it is continuous, i.e. that in L the functions $f_{-1/e} \circ \psi^{-1}$ and $\psi^{-1} \circ g_{1/e}$ concide. Take $z = 1 + it \in L$, then

- $\psi^{-1}(1+it) = \gamma(t)$, so $f_{-1/e}(\psi^{-1}(z)) = 1/e + e^{it}/2$.
- $g_{1/e}(1+it) = -1/e + e^{it}$ and ψ^{-1} restricted to $\partial D(-1/e,1)$ is $\mathcal{R}^{-1}$, then $\mathcal{R}^{-1}(-1/e + e^{it}) = 1/e + e^{it}/2$.

So it is continuous and hence quasiregular.

The next step is defining a F-invariant Beltrami form μ, then by Lemma 2.2, we will be able to find a holomorphic copy $G(z)$.

Define $\mu = \psi^*(\mu_0)$, which is represented in Figure 3.6, then

$$F^*\mu = (\psi \circ F)^*(\mu_0) = (\psi \circ \psi^{-1} \circ g_{1/e} \circ \psi)^*(\mu_0) = \psi^* g_{1/e}^*(\mu_0) = \psi^*(\mu_0) = \mu,$$

where we have $g_{1/e}^*(\mu_0) = \mu_0$ because $g_{1/e}$ is holomorphic. Therefore μ is F-invariant.

The next step is spreading μ by the dynamics, together with setting $\mu = 0$ on the Julia set, that is

$$\mu := \begin{cases} \psi^*(\mu_0) & \text{on } U \\ (f_{-1/e}^n)^*(\mu) & \text{on } f_{-1/e}^{-n}(U) \text{ for all } n \geq 0 \\ \mu_0 & \text{on } J(f_{-1/e}) = \mathbb{C} \setminus \cup_{n\geq 0} f^{-n}(U) \end{cases}.$$

Then μ is a F-invariant Beltrami form on $\mathbb{C}$. By the Integrability Theorem (Theorem 1.27), there exists a quasiconformal map $\phi\colon \mathbb{C} \to \mathbb{C}$ such that $\mu = \phi^*(\mu_0)$, $\phi(0) = 0$ and $\phi(1/e) = -1/e$ (i.e. we map the asymptotic value of $f_{-1/e}$ to the asymptotic value of $g_{1/e}$).

$$\begin{tikzcd} (\mathbb{C}, \mu) \arrow[r, "F"] \arrow[d, "\phi"'] & (\mathbb{C}, \mu) \arrow[d, "\phi"] \\ (\mathbb{C}, \mu_0) \arrow[r, "G"] & (\mathbb{C}, \mu_0) \end{tikzcd}$$

By Weyl's lemma, the map $G := \phi \circ F \circ \phi^{-1}\colon \mathbb{C} \to \mathbb{C}$ is holomorphic and quasiconformally conjugate to F.

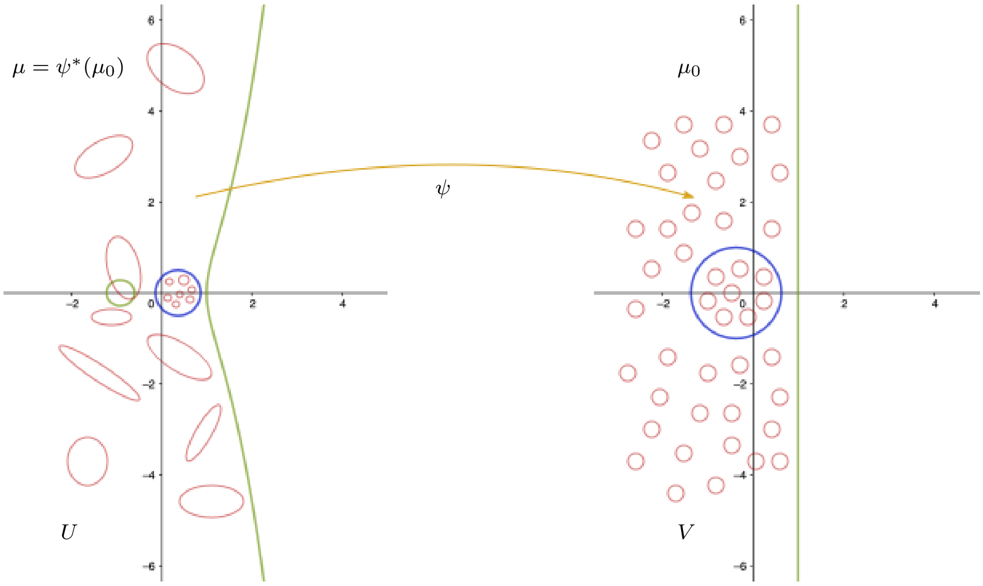

Figure 3.6: Representation of the field of ellipses in the proof of Theorem 3.4. In the right side we put circles and on the left side we take the field of ellipses given by $\psi^*(\mu_0)$, the pullback of μ_0 under ψ.

Moreover, G is an entire function with $z = 0$ as an attracting fixed point and the multiplier of this fixed point is $1/e$ (because $\partial_{\bar{z}}\phi \equiv 0$ in $D(1/e, 1/2)$ and hence the multiplier is preserved), i.e. $G'(0) = 1/e$). Furthermore, it has exactly one asymptotic value (which is $-1/e$ and tends to $z = 0$ under iteration) and has no critical points. So it is one of the maps in [BJR], i.e. $J(G)$ is a Cantor Bouquet, which is contained in $\mathbb{C} \setminus \overline{U}$.

In fact, by Nevanlinna's theorem (see [CJK], Theorem 2.1), after solving a differential equation we obtain

$$G(z) = \frac{1}{ek}\left(e^{kz} - 1\right).$$

If we take $h(z) = kz$, then $(h \circ G \circ h^{-1})(z) = g_{1/e}(z)$, therefore $g_{1/e}$ and G are conformally conjugate.

Finally, note that in $\mathbb{C}\setminus\overline{V}$, G and $f_{-1/e}$ are quasiconformally conjugate, i.e. $g_{1/e}$ and $f_{-1/e}$ are quasiconformally conjugate. In particular, they are quasiconformally conjugate on $J(f_{1/e})$, which is an invariant Cantor Bouquet. Therefore, there exists $\mathcal{C} \subset J(f_{-1/e})$ such that $f_{-1/e}(\mathcal{C}) = \mathcal{C}$.

This completes the proof of Theorem A.

□

Note that attached to each finite preimage of ∞, i.e. the pole $z = -1$, there is a 'copy' of this Cantor Bouquet $\mathcal{C}$, which also appears attached to every point in the backward orbit of $z = -1$.

3.3 Further study: extending Theorem A

We have proved that for a parameter value we can find a Cantor Bouquet, the next question is if we can do the same for other parameter values in the main capture component, i.e. for λ such that $-\lambda \in \mathcal{A}^*_\lambda(0)$.

Our approach is to perform surgery (in the dynamical plane of f_λ) to remove the pole by redefining the orbits of points in a neighborhood of such point, and hence convert the map into an entire transcendental map. As before, we need to find a multiply connected domain $\mathcal{A}$ contained in $\mathcal{A}^*_\lambda(0)$ such that $f_\lambda(\mathcal{A}) \Subset \mathcal{A}$ and with the pole, $z = -1$, contained in one of the bounded components of the complementary of $\mathcal{A}$.

In [Rod] it is proved that we can find such $\mathcal{A}$, but still we need more information about this domain.

We proceed as follows; Consider U_0 a Jordan domain with $0 \in U_0$ and define U_n as the connected component of $f_\lambda^{-n}(U_0)$ that contains U_0. Then $\{U_n\}_{n\geq 0}$ forms a nested sequence of Jordan domains (i.e. with $U_n \subset U_{n+1}$), such that, up to a certain index N, $f_\lambda \colon U_{n+1} \to U_n$ has degree 2 (by the Böttcher coordinates). At some point, these domains must become unbounded, multiply connected, or both.

- **Unbounded:** if $-\lambda \in U_n$, then U_{n+1} is unbounded and $f \colon U_{n+1} \to U_n$ has infinite degree.
- **Multiply connected:** it has been proved in [Rod] that for such parameter values the basin is infinitely connected, so not all U_n's can be simply connected, since an infinite union of nested simply connected sets is simply connected.

Since we need to erase the pole, we need to perform surgery exactly when the sequence is both unbounded and multiply connected, Lemma 3.12 tells us that this happens at the same time, but we need the following lemmas before we prove it.

Lemma 3.10. *Let $U \subset \mathbb{C}_\infty$ be an open set.*

(a) If $f \colon U \to \mathbb{D}$ is a holomorphic covering, then U is simply connected and f is univalent.

(b) If $f \colon U \to \mathbb{D}^* = \mathbb{D} \setminus \{0\}$ is a holomorphic covering, then either:

(i) U is conformally equivalent to $\mathbb{D}^*$ and there exists a biholomorphic mapping $\psi \colon U \to \mathbb{D}^*$, such that $f = (\psi)^d$ for some $d \in \mathbb{N}$, or

(ii) U is simply connected and there exists a biholomorphic mapping $\psi \colon U \to \mathcal{H}_l = \{z \in \mathbb{C} \colon Re(z) < 0\}$ such that $f = \exp \circ \psi$.

The proof can be found in [Zhe].

Now we can prove that:

Lemma 3.11 (Not multiply connected before unbounded)**.** *The sequence $\{U_n\}_n$ cannot become multiply connected before it becomes unbounded.*

Proof. Suppose that U_{k+1} is multiply connected and bounded, and that U_k is simply connected (and bounded). Then $v \notin U_{k+1}$ and $f \colon U_{k+1} \to U_k$ is a proper map. Moreover, $f \colon U_{k+1} \setminus f_\lambda^{-1}(\{0\}) \to U_k \setminus \{0\} \simeq \mathbb{D}^*$ is a holomorphic covering. By Lemma 3.10, $U_{k+1} \setminus f_\lambda^{-1}(\{0\}) \simeq \mathbb{D}^*$, but by hypothesis, U_{k+1} is already multiply connected, so $U_{k+1} \setminus f_\lambda^{-1}(\{0\})$ cannot be conformally equivalent to $\mathbb{D}^*$. □

Lemma 3.12 (Multiply connected and unbounded)**.** *The sequence $\{U_n\}_n$ becomes multiply connected and unbounded at the same time.*

Proof. Suppose that $k = \min\{k \geq 0 \colon -\lambda \in U_k\}$, then U_{k+1} is unbounded and we want to see that it is also multiply connected. Note that ∂U_{k+1} contains a curve unbounded on both ends such that $f_\lambda \colon \partial U_{k+1} \to U_k$ is an infinite degree covering.

Take a disk $D = D(-\lambda, \varepsilon)$ compactly contained in U_k and a smooth curve σ joining ∂D and ∂U_k, then $f_\lambda^{-1}(\sigma)$ contains curves σ_j, $j \in \mathbb{Z}$, which join $f_\lambda^{-1}(\partial D)$ and $f_\lambda^{-1}(\partial U_{k+1})$ (see Figure 3.7). We have that:

- ∂U_{k+1} can be broken in curves $\gamma_{0,j}$ such that $\gamma_{0,j}$ joins $\sigma_j \cap \partial U_{k+1}$ and $\sigma_{j-1} \cap \partial U_{k+1}$. Moreover, each $\gamma_{0,j}$ is mapped in a one-to-one fashion to ∂U_k under f_λ.

- For small ε, $f_\lambda^{-1}(\partial D) = f_\lambda^{-1}(\partial D(-\lambda, \varepsilon))$ is a simple curve which can be broken in curves $\gamma_{1,j}$ such that $\gamma_{1,j}$ joins $\sigma_j \cap f_\lambda^{-1}(\partial D)$ and $\sigma_{j-1} \cap f_\lambda^{-1}(\partial D)$. Moreover, each $\gamma_{1,j}$ is mapped in a one-to-one fashion to ∂D under f_λ.
- The curves $\alpha_j = \gamma_{0,j} \cup \sigma_j \cup \gamma_{1,j} \cup \sigma_{j-1}$ (oriented counter-clockwise) contain in its interior domains F_j such that
  - Each F_j, $j \neq 0$, is mapped in a one-to-one fashion under f_λ to $U_k \setminus D$.
  - Each F_j contains exactly one preimage of $z = 0$.

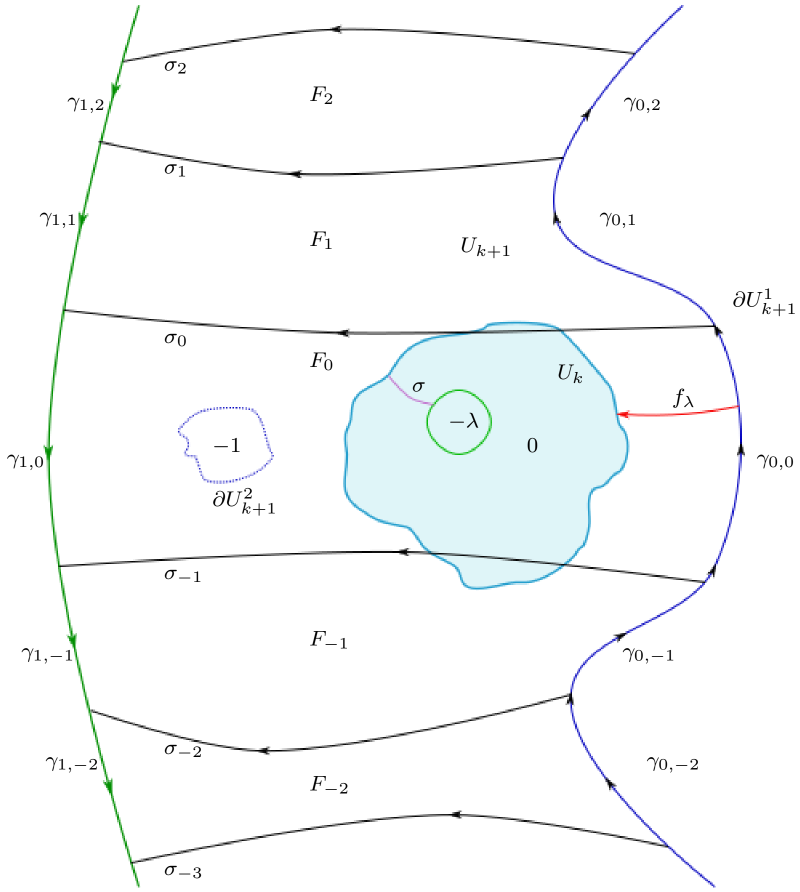

Figure 3.7: Representation of the curves in the proof of Lemma 3.12.

Then, if we take the curve, for $j \in \mathbb{N}$,

$$\beta_j := \left(\cup_{l=-j}^{l=j} \gamma_{0,j}\right) \cup \sigma_j \cup \left(\cup_{l=-j}^{l=j} \gamma_{1,j}\right) \cup \sigma_{-j-1},$$

oriented counter-clockwise, by the Argument Principle,

$$ind(f_\lambda(\beta_j), 0) = \underbrace{\sum_{z \in Z(f_\lambda) \cap int(\beta_j)} m(f_\lambda, z) ind(\beta_j, z)}_{A} - \underbrace{\sum_{z \in P(f_\lambda) \cap int(\beta_j)} m(f_\lambda, z) ind(\beta_j, z)}_{B}. \quad (3.6)$$

Where $ind(\iota, z)$ denotes the winding number of the closed curve ι with respect to $z \notin \iota$, $m(f_\lambda, z)$ represents the multiplicity of a zero or a pole of f_λ, and $Z(f_\lambda)$, $P(f_\lambda)$ represent respectively the set of zeros and poles of f_λ. Note that,

- Since β_j is a Jordan curve oriented counter-clockwise, $ind(\beta_j, z) \in \{0, 1\}$.

- $z = 0$ has multiplicity 2 as a zero of f_λ, while all the other zeros have multiplicity 1.
- Take J such that $\overline{U_k} \subset int(\beta_J)$ (which exists because U_k is bounded), then $ind(f_\lambda(\beta_J), 0) = 2J + 1$.
- In the interior of β_J there are at least $2J$ simple zeros and $z = 0$.

Therefore, equation (3.6) reads $2J+1 = A-B$, where $A \geq 2J+2$. Thus $B \geq 1$ and hence U_{k+1} contains the unique pole of f_λ, thus is multiply connected.

□

However, as we have already pointed out, the interpolation results that we have at our disposal apply for bounded curves, this makes the problem of performing surgery in general quite difficult, because the choice of the sets in the previous construction matter and the number of preimages needed to reach the asymptotic value may also be a problem.

3.3.1 Connectivity of $\mathcal{A}_\lambda(0)$

We end this chapter by studying the connectivity of $\mathcal{A}_\lambda(0)$ in general, which may be used in the future to prove the existence of Cantor Bouquets for more general families of functions.

The conditions under which we can assure some facts about the connectivity of $\mathcal{A}_\lambda(0)$ are summarized in the following result, the proof of whom is the main topic of this section:

Theorem 3.13 (Connectivity of superattracting basins). *Let f be a transcendental meromorphic map with a super-attracting fixed point $z = 0$. Let $\mathcal{A}_f(0)$ denote its basin of attraction and assume that:*

(i) f has at least one pole.

(ii) $z = 0$ is the only critical point in $\mathcal{A}_f(0)$.

(iii) f has at most one asymptotic value in $\mathcal{A}_f(0)$, say v, and if there is one, it is a Picard value with only one asymptotic tract.

Then,

(a) If $v \in \mathcal{A}_f^*(0)$, then $\mathcal{A}_f(0) = \mathcal{A}_f^*(0)$ and it is infinitely connected, where $\mathcal{A}_f^*(0)$ denotes the immediate basin of attraction of $z = 0$.

(b) If $v \notin \mathcal{A}_f^*(0)$, then $A_f(0)$ is a union of simply connected sets.

Proof.

(a) Suppose $v \in \mathcal{A}_f^*(0)$, proving that $\mathcal{A}_f^*(0)$ is multiply connected containing the pole follows from the same construction as in Lemma 3.12, where we do not use any different property from our map rather than the ones that we are supposing in this theorem.

We want to show first that we have $\mathcal{A}_f(0) = \mathcal{A}_f^*(0)$.

Consider $U_0 = D \subset \mathcal{A}_f^*(0)$, a disk centered at $z = 0$ contained in $\mathcal{A}_f^*(0)$ (i.e. as in Lemma 3.12). Now we pull-back U_0 in order to obtain the whole immediate basin $\mathcal{A}_f^*(0)$, that is, for $N > 0$, we define U_N as the connected component of $f^{-1}(U_{N-1})$ that contains U_{N-1}. This recurrence defines a sequence of subsets $\{U_N\}_{N\geq 0}$ such that:

- $U_N \subset \mathcal{A}_f(0)$ for all $N \geq 0$.

- $U_N \subset U_{N+1}$ and $f(U_{N+1}) = U_N$ for all $0 \leq N < M$, where M is the smallest natural number such that $v \in U_M$. Then, for $N \geq M$, $U_N \subset U_{N+1}$ and $f(U_{N+1}) = U_N \setminus \{v\}$ (because the asymptotic value is omitted).
- $\mathcal{A}_f^*(0) = \bigcup_{N \geq 0} U_N$.

Since $v \in \mathcal{A}_f^*(0)$, then we can take the smallest natural number M such that $v \in U_M$ and consider a simple curve $\gamma \subset U_M$ that joins v and 0. Therefore U_{M+1} is unbounded (because v is an asymptotic value), hence the preimage of γ must contain a path that joins 0 and ∞, which is contained in $U_{M+1} \subset \mathcal{A}_f^*(0)$ and thus the only asymptotic tract of v is contained in $\mathcal{A}_f^*(0)$.

Now suppose that $\mathcal{A}_f(0)$ is not connected, then we must have at least two connected components, $\mathcal{A}_f^*(0)$ and U such that

$$f(U) = \mathcal{A}_f^*(0) \setminus \{v\}.$$

So U must contain a tail of an asymptotic path, which contradicts that the only asymptotic tract of v is contained in $\mathcal{A}_f^*(0)$. Therefore $\mathcal{A}_f(0) = \mathcal{A}_f^*(0)$ is connected and totally invariant.

Finally, since $\mathcal{A}_f(0) = \mathcal{A}_f^*(0)$, if we call ∂U_{M+1}^p a connected component of ∂U_{M+1} such that it contains a pole in its interior, then the successive preimages of $\overline{int(\partial U_{M+1}^p)}$ contain points $w \in \mathcal{O}_f^-(\infty) \subset J(f)$, which lie in the interior of a closed curve contained in $\mathcal{A}_f(0)$ (because it is totally invariant). Hence, since the backward orbit of ∞ is an infinite set, $\mathcal{A}_f(0)$ is infinitely connected.

(b) Take the same sequence $\{U_N\}_{N \geq 0}$ that we have constructed before and define the new sequence by $V_0 = U_0 \setminus \{0\}$ and for $N > 0$, $V_N = f^{-N}(V_0) \cap U_N$. Then,

$$f \colon V_{N+1} \to V_N$$

is a holomorphic covering. Moreover, we also know that none of the V_N is simply connected (we have taken out from each U_N at least $z = 0$). We claim now that, in fact, each V_N is conformally equivalent to a punctured disk and we prove it by induction; it is clear that V_0 is conformally equivalent to a punctured disk, that is because U_0 is conformally equivalent to a disk by the Böttcher coordinates map.

Suppose that V_N is conformally equivalent to a punctured disk for $N > 0$, then since

$$f \colon V_{N+1} \to V_N \simeq_c \mathbb{D}^*$$

is a holomorphic covering and, as it has already been pointed out, V_{N+1} is not simply connected, by Lemma 3.10 V_{N+1} is conformally equivalent to a punctured disk.

Therefore, $V_N = U_N \setminus \{0\}$, which implies that,

- $U_N = V_N \cup \{0\}$ is simply connected, so $\{U_N\}_N$ forms a nested sequence of simply connected sets. Hence, $\mathcal{A}_f^*(0)$ is simply connected.
- In $\mathcal{A}_f^*(0)$ there is no other preimage of $z = 0$ rather than $z = 0$ itself.

To finish the proof of the theorem we need to distinguish two different cases,

(i) Case $v \notin \mathcal{A}_f(0)$:
To fill the whole basin $\mathcal{A}_f(0)$ we need to take successive preimages of the immediate basin $\mathcal{A}_f^*(0)$, which we know that is conformally equivalent to a disk. Suppose that U is a component of $f^{-1}(\mathcal{A}_f^*(0))$, then

$$f \colon U \setminus f^{-1}(\{0\}) \to \mathcal{A}_f^*(0) \setminus \{0\}$$

is a holomorphic covering. By Lemma 3.10, $U \setminus f^{-1}(\{0\})$, can either be simply connected or a punctured disk. In the first case we have no preimages of $z = 0$ in U and in the second, we have that $f^{-1}(\{0\}) \cap U$ is a single point and U is conformally equivalent to a punctured disk, thus U is again simply connected.

If we keep taking preimages, now we already have simply connected sets and no critical values, therefore by Lemma 3.10 any component of any preimage is simply connected. So, $\mathcal{A}_f(0)$ is a union of simply connected sets.

(ii) Case $v \in \mathcal{A}_f(0)$:

The same argument as before applies until we reach the asymptotic value. So suppose that in a component A_N of $\mathcal{A}_f(0)$ we have the asymptotic value (A_N is simply connected). If we take any component U of the preimage of A_N under f, then

$$f\colon U \setminus f^{-1}(\{v\}) \to A_N \setminus \{v\} \simeq \mathbb{D}^*$$

is a holomorphic covering and we have no finite preimages of v (because v is a Picard value). By applying again Lemma 3.10, we see that U is simply connected, because otherwise we must have a finite preimage.

As before, since we have already simply connected sets and no more asymptotic values nor critical values, any preimage is simply connected. So, $\mathcal{A}_f(0)$ is a union of simply connected sets.

□

Appendix A

Preliminaries in Complex Dynamics

The goal here is to list the main results in the Complex Dynamics theory needed for this thesis, so that any reader can have at its disposal a quick access to them. For a more detailed introduction, we refer to [Rod].

A.1 Periodic points and local theory

In this work we focus on some dynamical aspects of transcendental meromorphic functions, i.e., we study the dynamical system given by the iterates of meromorphic functions $f\colon \mathbb{C} \to \mathbb{C}_\infty$ with an essential singularity at ∞, where $\mathbb{C}_\infty$ denotes $\mathbb{C} \cup \{\infty\}$ or the Riemann sphere. Here the n-th iterate of a point $z \in \mathbb{C}$ is denoted by $f^n(z) = (f \circ \overset{(n)}{\cdots} \circ f)(z)$, and the sequence of iterates $\{f^n(z)\}_{n\in\mathbb{N}}$ is well-defined for all $z \in \mathbb{C}$ except for the countable set of poles and prepols of f of any order.

The interest for these functions is double; The essential singularity, on the one hand, adds a lot of chaos to the dynamical system, mainly because of Picard's Theorem, which states that in each punctured neighborhood of ∞, these functions assume each value of the Riemann Sphere $\mathbb{C}_\infty$, with at most two exceptions, infinitely often. Hence, given a point $z \in \mathbb{C}$, if its orbit $\mathcal{O}_f^+(z) = \{f^n(z)\colon n \in \mathbb{N}\}$ is near ∞ at some moment, after one iteration it can land at almost any place of the plane. On the other hand, the presence of poles allows for more generality, since ∞ is not required to be an omitted value.

From now on, unless we do not state the opposite, f denotes a non-constant meromorphic function, $f \in \mathcal{M}(\mathbb{C})$, as defined in the introduction.

Definition A.1 (Forward and Backward Orbit, Periodic and Preperiodic Point)**.** *Let $z_0 \in \mathbb{C}$, then*

- *The **forward orbit** of z_0 is the set $\mathcal{O}^+(z_0) = \{z_n = f^n(z_0)\colon n \in \mathbb{N}\}$.*
- *The **backward orbit** of z_0 is the set $\mathcal{O}^-(z_0) = \{z\colon f^n(z) = z_0\ ,\ n \in \mathbb{N}\}$.*
- *z_0 is called **periodic** if exists $n \in \mathbb{N}$ such that $z_n = z_0$, $p = \min\{n \in \mathbb{N}\colon z_n = z_0\}$ is called its **period**. If $p = 1$ we say that z_0 is a **fixed point**.*
- *z_0 is called **preperiodic** if $f^k(z_0)$ is periodic for some $k \in \mathbb{N}$ and **strictly preperiodic** if it is preperiodic but not periodic.*

For a periodic point z_0 of period p, we define its **multiplier** as $\lambda = (f^p)'(z_0)$. Using the chain rule, it can be verified that

$$\lambda = \prod_{n=0}^{p-1} f'(f^n(z_0)) = \prod_{n=0}^{p-1} f'(z_n)$$

and therefore, the multiplier is the same for every periodic point of the orbit. Hence, we regard it as the multiplier of the orbit.

Periodic points can be classified according to their multiplier.

Definition A.2 (Classification of Fixed Points)**.** *Given a periodic point z_0 of period p, the cycle $\mathcal{O}_f^+(z_0)$ is called:*

- ***Attracting*** *iff $0 < |\lambda| < 1$ and **super-attracting** iff $\lambda = 0$.*
- ***Repelling*** *iff $|\lambda| > 1$.*
- ***Indifferent*** *iff $|\lambda| = 1$. We have two possibilities:*
  - ***Rationally indifferent*** *iff $\lambda^m = 1$ for some $m \in \mathbb{N}$, i.e., $\lambda = e^{2\pi i j/m}$ for some $j \in \mathbb{Z}$ (also called a **parabolic** cycle).*
  - ***Irrationally indifferent*** *iff $\lambda = e^{2\pi i\theta}$, for $\theta \in \mathbb{R} \setminus \mathbb{Q}$.*

A.1.1 Normal forms

We are concerned now with the study of the dynamical behavior of a function near a periodic point. Observe first that since periodic points of f are fixed points of f^n for a given n, without loss of generality we may assume that they are fixed points.

To accomplish this goal, we want to represent our function in the simplest possible way, the normal form. To do so, we introduce the concept of conjugacy.

Definition A.3 (Conformal Conjugacy)**.** *We say that a function $f\colon U \to U$ is **(conformally) conjugate** to a function $g\colon V \to V$ if and only if there is a conformal one-to-one map $\varphi\colon U \to V$ such that*

$$\varphi(f(z)) = g(\varphi(z))$$

i.e., the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{f} & U \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ V & \xrightarrow{g} & V \end{array}$$

Two conjugate functions have the same dynamics. Indeed, the iterates of f are also conjugate by the same map φ since $g^n = \varphi \circ f^n \circ \varphi^{-1}$. The inverses, f^{-1} and g^{-1}, whenever well-defined are also related by φ, i.e., $g^{-1} = \varphi \circ f^{-1} \circ \varphi^{-1}$.

It can also be verified that conjugacies send orbits to orbits, fixed points to fixed points, periodic orbits of period p to periodic orbits of period p, attracting points to attracting points, etc.

The following results address the problem of finding the normal form:

Theorem A.4 (Koenigs Linearization Theorem)**.** *Let f be holomorphic in some neighborhood of $z = 0$, a fixed point of f with multiplier λ. If $|\lambda| \neq 0, 1$, then there exists a local conformal change of coordinate $w = \varphi(z)$ with $\varphi(0) = 0$ such that*

$$\varphi \circ f \circ \varphi^{-1}\colon w \mapsto \lambda w$$

for all w in some neighborhood of the origin. Furthermore, φ is unique up to multiplication by a nonzero constant.

Theorem A.5 (Böttcher)**.** *Suppose f has a super-attracting fixed point at $z = 0$, so that in a neighborhood of $z = 0$, the function can be written as*

$$f(z) = a_p z^p + a_{p+1} z^{p+1} + \cdots \qquad \text{for some } p \geq 2.$$

Then there is a conformal map $w = \varphi(z)$ defined in a neighborhood of $z = 0$ onto a neighborhood of $w = 0$ that conjugates $f(z)$ to $w \mapsto w^p$. Furthermore, φ is unique up to multiplication by a $(p-1)$-th root of unity.

The local change of variables given by Theorem A.5 is known as the **Böttcher coordinates** around the super-attracting fixed point.

We analyze here the case where $z = 0$ is a rationally indifferent fixed point of f, i.e., $\lambda = f'(0)$ is a root of unity. Our goal is to characterize the dynamics of f in a neighborhood of the origin. First, we need to introduce some preliminary concepts.

Since $\lambda \neq 0$, then $f'(0) \neq 0$ and f^{-1} is locally well-defined. Hence, we can choose a neighborhood N of $z = 0$ that is small enough so that f maps N conformally onto some neighborhood N_0 of the origin.

Definition A.6 (Attracting and Repelling Petals)**.** *A connected open set U, with compact closure $\bar{U} \subset N \cap N_0$ is called*

- *An **attracting petal** for f at the origin if*

$$f(\bar{U}) \subset U \cup \{0\} \qquad \bigcap_{k \geq 0} f^k(\bar{U}) = \{0\}.$$

- *A **repelling petal** for f at the origin if U is an attracting petal for f^{-1}.*

Definition A.7 (Repulsion and Attraction Vectors)**.** *A vector $v \in \mathbb{C}$ is called a **repulsion vector** for f at the origin if $nav^n = +1$, and an **attraction vector** if $nav^n = -1$.*

Definition A.8 (Nontrivial Convergence)**.** *We say that an orbit $\mathcal{O}_f^+(z_0)$ converges to zero **nontrivially** if $z_k \xrightarrow[k \to \infty]{} 0$ but $z_k \neq 0$.*

Definition A.9 (Directional Convergence)**.** *If an orbit $\mathcal{O}_f^+(z_0)$ under f converges to zero, with $z_k \sim v_j / \sqrt[n]{k}$ (where j is necessarily odd), then we say that this orbit $\{z_k\}_k$ tends to zero **from the direction** v_j.*

Lemma A.10 (Convergence Directions)**.** *If an orbit $\mathcal{O}_f^+(z_0)$ converges to zero nontrivially, then z_k is asymptotic to $v_j / \sqrt[n]{k}$ as $k \to \infty$ for one of the n attraction vectors v_j. In other words, the limit $\lim_k z_k \sqrt[n]{k} = v_j$. Similarly, if an orbit $\mathcal{O}_f^-(z_0')$ under f converges to zero nontrivially, then z_k' is asymptotic to $v_j / \sqrt[n]{k}$, where v_j is one of the n repulsion vectors, with j even.*

A consequence of Lemma A.10 is the following well-known result.

Theorem A.11 (Leau-Fatou Flower Theorem)**.** *Let*

$$f(z) = z + az^{n+1} + (H.O.T.) \quad \text{with} \quad a \neq 0 \ , \ n \geq 1$$

be holomorphic in some neighborhood of the origin, then there exist $2n$ petals $\mathcal{P}_j$, where $\mathcal{P}_j$ is either repelling or attracting depending to whether j is even or odd. Furthermore, we can choose those petals so that

$$\{0\} \cup \mathcal{P}_0 \cup \cdots \mathcal{P}_{2n-1}$$

is an open neighborhood of $z = 0$. When $n > 1$, each $\mathcal{P}_j$ intersects each of its two immediate neighborhoods in a simply connected region $\mathcal{P}_j \cap \mathcal{P}_{j \pm 1}$ but is disjoint from the remaining $\mathcal{P}_k$ (we consider j modulo $2n$).

Theorem A.12 (Parabolic Linearization). *Let*

$$f(z) = \lambda z + az^{n+1} + (H.O.T.) \quad \textit{with} \quad a \neq 0 \ , \ n \geq 1$$

be holomorphic in some neighborhood of the origin, where λ is a primitive q-th root of unity. Then for any attracting or repelling petal $\mathcal{P}$, there is one and, up to composition with a translation of $\mathbb{C}$, only one conformal one-to-one map $\varphi\colon \mathcal{P} \to \mathbb{C}$ such that $\varphi(f(z)) = 1 + \varphi(z)$ for all $z \in \mathcal{P} \cap f^{-1}(\mathcal{P})$.

Finally, we study the remaining case. Consider a map of the form

$$f(z) = \lambda z + \sum_{k \geq 2} a_k z^k$$

which is defined in some neighborhood of $z = 0$, with multiplier:

$$\lambda = e^{2\pi i \theta} \quad , \quad \theta \in \mathbb{R} \setminus \mathbb{Q}.$$

Definition A.13 (Cremer and Siegel Points). *We say that an irrationally indifferent fixed point is a*

- ***Cremer point*** *if a local linearization is not possible.*
- ***Siegel point*** *if a local linearization is possible.*

Theorem A.14 (Siegel). *We say that $\theta \in \mathbb{R}$ is* ***Diophantine*** *if it is badly approximable by rational numbers, in the sense that there exists $k < \infty$ and $\varepsilon > 0$ so that*

$$\left| \theta - \frac{p}{q} \right| \geq \frac{\varepsilon}{q^k} \quad \textit{for all} \quad p/q \in \mathbb{Q}.$$

If θ is Diophantine, and if f has a fixed point at $z = 0$ with multiplier $e^{2\pi i \theta}$, then there exists a solution of the Schröder equation.

A.2 The Julia and Fatou sets

Definition A.15 (Normal Convergence). *We say that a sequence $\{f_k(z)\}_k$ of meromorphic functions on a domain D* ***converges normally*** *to $f(z)$ on D, if the sequence converges uniformly on compact subsets of D to $f(z)$ in the spherical metric.*

Definition A.16 (Normal Family). *A family $\mathcal{F}$ of meromorphic functions on a domain D is a* ***normal family*** *if every sequence in $\mathcal{F}$ has a subsequence that converges normally on D.*

Our applications to transcendental dynamics are based on the following theorem.

Theorem A.17 (Montel).

(a) Suppose that $\mathcal{F}$ is a family of holomorphic functions on a domain D such that $\mathcal{F}$ is uniformly bounded on each compact subset of D. Then $\mathcal{F}$ is a normal family.

(b) A family $\mathcal{F}$ of meromorphic functions on a domain D that omits three values is normal.

From now on f denotes a transcendental function unless we state the opposite.
The dynamical of f plane splits into two sets, the Fatou set and the Julia set.

Definition A.18 (Fatou and Julia Sets). *We define the* ***Fatou set*** *as:*

$$F(f) = \{z \in \mathbb{C}_\infty \colon \{f^n(z) \colon n \in \mathbb{N}\} \textit{ is well-defined and normal in some neighborhood of } z\}$$

and the ***Julia set*** *$J(f) = \mathbb{C}_\infty \setminus F(f)$, where $\mathbb{C}_\infty$ is the Riemann Sphere, i.e., the Alexandroff compactification of the complex plane $\mathbb{C}$.*

Notice that the condition of being "well-defined" is necessary, since orbits are truncated if they land on a pole of f.

A.2.1 Basic properties

The aim of this section is to show some classical results concerning properties of the Julia and Fatou sets.

Definition A.19 (Forward, Backward and Completely Invariant).

- *We say that a set S is **forward invariant** under f if $z \in S$ implies $f(z) \in S$ or $f(z)$ is undefined.*
- *We say that S is **backward invariant** under f if $z \in S$ implies that $w \in S$ for all w such that $f(w) = z$.*
- *We say that S is **completely invariant** if it is both forward and backward invariant under f.*

Theorem A.20 (Basic Properties of the Julia and Fatou sets). *Let f be a transcendental meromorphic function.*

*(a) (**Invariance**) The Julia set and the Fatou set of a transcendental meromorphic function f are completely invariant.*

*(b) (**Iteration**) For any $q \in \mathbb{N}$, $J(f^q) = J(f)$.*

*(c) (**Blow up property**) If $z \in J(f)$ and U is a neighborhood of z, then $\bigcup_{n\in\mathbb{N}} f^n(U)$ covers $\mathbb{C}_\infty$ with at most two exceptions.*

(d) If $z_0 \in J(f)$ is s.t. $\mathcal{O}_f^-(z_0)$ is not finite, then $\overline{\mathcal{O}_f^-(z_0)} = J(f)$.

(e) If $J(f)$ has an interior point, then $J(f) = \mathbb{C}_\infty$.

(f) $J(f) \neq \emptyset$ is closed and does not contain isolated points.

(g) $J(f)$ is the closure of the set of repelling periodic points of f.

Definition A.21 (Basin of Attraction). *If $\mathcal{O}_f$ is an attracting periodic orbit of period m, we define the **basin of attraction** to be the open set $\mathcal{A} \subset \mathbb{C}_\infty$ consisting of all points $z \in \mathbb{C}_\infty$ for which the successive iterates converge to some point of $\mathcal{O}_f$.*

Proposition A.22. *Every attracting periodic orbit is contained in $F(f)$. In fact, the entire basin of attraction $\mathcal{A}$ for an attracting periodic orbit is contained in $F(f)$. However, every repelling periodic orbit is contained in the $J(f)$.*

Proposition A.23. *$F(f)$ contains all Siegel points of f and their linearizing neighborhoods. Instead, $J(f)$ contains all rationally indifferent fixed points and Cremer fixed points.*

A.3 The Classification theorem and the singular values

The aim in this section is to understand the possible limit functions that we can obtain under iteration in the set of normality of the iterates of f. To do so we introduce the components of the Fatou set.

Definition A.24 (Fatou component). *A **component** U of $F(f)$ is a maximal connected domain of normality of the iterates of f.*

Lemma A.25. *Every component of $F(f)$ contains at most one periodic point of f.*

We want to understand now the behavior of f in the different components of the Fatou set. To do so we distinguish two different cases.

Definition A.26 (Preperiodic and Wandering components)**.** *Given a component U of $F(f)$, then by the Invariance Lemma $f^n(U)$ is contained in a component of $F(f)$ that we denote U_n.*

- *U is called **preperiodic** if there exists $n > m \geq 0$ such that $U_n = U_m$. If $m = 0$ we say that U is **periodic** with period n and $\{U, U_1, \dots, U_{n-1}\}$ is called a **cycle** of components. The smallest n with this property is called the **minimal period** of U.*

- *If U is not preperiodic, U is called a **wandering domain**.*

The behavior of the successive iterates of f on periodic components is well understood. The following celebrated result, originally stated by Fatou, summarizes the different possibilities that we can have.

Theorem A.27 (Classification Theorem for periodic components)**.** *Let U be a periodic component of period p. Then we have one of the following possibilities:*

(a) U contains an attracting periodic point z_0 of period p. Then

$$f^{np}(z) \xrightarrow[n\to\infty]{} z_0 \quad \forall z \in U$$

*and U is called the **immediate attractive basin** of z_0.*

(b) ∂U contains a periodic point z_0 of period p and

$$f^{np}(z) \xrightarrow[n\to\infty]{} z_0 \quad \forall z \in U$$

*Then $(f^p)'(z_0) = 1$ if $z_0 \in \mathbb{C}$. U is called a **Leau domain**.*

*(c) Exists $\phi\colon U \to \mathbb{D}$ conformal such that $\phi(f^p(\phi^{-1}(z))) = e^{2\pi i\alpha}z$ for some $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. U is called a **Siegel disk**.*

*(d) Exists $\phi\colon U \to A$ conformal where $A = \{z\colon 1 < |z| < r\}, r > 1$ is an annulus such that $\phi(f^p(\phi^{-1}(z))) = e^{2\pi i\alpha}z$ for some $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. U is called a **Herman ring**.*

(e) Exists $z_0 \in \partial U$ such that

$$f^{np}(z) \xrightarrow[n\to\infty]{} z_0 \quad \forall z \in U$$

*but $f^p(z_0)$ is not defined. In this case U is called a **Baker domain**.*

Note that in the case of a Wandering domain any limit function is constant.

The topological properties of the different Fatou components play a fundamental role in the dynamics of our functions and the results from the classical theory of Complex Analysis are crucial.

Definition A.28 (Winding number)**.** *We denote by $ind(\gamma, a)$ the **index** of a closed curve $\gamma \subset \mathbb{C}$ with respect to a point a (also known as the **winding number** of γ about a).*

The key lemma, which together with Theorem A.27 leads us to our goal is:

Lemma A.29. *Let $f \in E$ and U be a multiply connected component of $F(f)$. Suppose that γ is a Jordan curve that is not contractible in U. Then:*

(a) $f^n \to \infty$ normally on U.

(b) $ind(f^n(\gamma), 0) > 0$ for n large enough.

The points where the inverse function is not well-defined play a fundamental role in the dynamical behavior of our function. In fact, they are closely related with the different Fatou components that we can have.

We consider $f \in \mathcal{M}(\mathbb{C})$ unless we state the opposite.

Definition A.30 (Singular value)**.** *We say that $a \in \mathbb{C}$ is a **singular value** if some branch of f^{-1} is not well-defined (holomorphic and injective) in a neighborhood of $a \in \mathbb{C}$.*

The goal is to analyze the different singular values that we can have.

Let $a \in \mathbb{C}_\infty$ and denote by $D(a, r)$ the disk of radius $r > 0$, in the spherical metric, centered at a. For every $r > 0$, choose a connected component $U(r)$ of $f^{-1}(D(a, r))$ such that $r_1 < r_2$ implies $U(r_1) \subset U(r_2)$. Obtaining a function:

$$U \colon r \mapsto U(r).$$

We have two possibilities:

(i) $\cap_{r>0} U(r) = \{z\}$, $z \in \mathbb{C}$. Then $a = f(z)$. If $a \in \mathbb{C}$ and $f'(z) \neq 0$ or if $a = \infty$ and z is a simple pole, then we say that z is an **ordinary or regular point**.

If $a \in \mathbb{C}$ and $f'(z) = 0$ or if $a = \infty$ and z is a multiple pole of f, then z is called a **critical point** and a is called a **critical value**.

(ii) $\cap_{r>0} U(r) = \emptyset$. Then we say that our choice $r \mapsto U(r)$ defines a (transcendental) **singularity** (for simplicity we just call such U a singularity). For every $r > 0$, the open set $U(r) \subset \mathbb{C}$ is called a **neighborhood of the singularity** U. So if $z_k \in \mathbb{C}$, we say that $z_k \to U$ if for every $\varepsilon > 0$, $\exists k_0$ such that $z_k \in U(\varepsilon)$ for all $k \geq k_0$.

Meanwhile the first type is elementary to determine with this characterization, the second is not, so we are urged to obtain a criterion to decide when we have a singularity.

Definition A.31 (Asymptotic Value)**.** *We say that $a \in \mathbb{C}$ is an **asymptotic value** if there exists a curve γ such that*

$$\gamma(t) \xrightarrow[t\to\infty]{} \infty \quad \textit{and} \quad f(\gamma(t)) \xrightarrow[t\to\infty]{} a.$$

*We call γ an **asymptotic path or curve** of a.*

Proposition A.32. *A point $a \in \mathbb{C}$ is an asymptotic value of f, if and only if, there is a singularity U (as in (ii)).*

Now we want to show why the singularities of the inverse play a fundamental role in the dynamics of a given function. Again, we focus on the case of periodic components with constant limit functions.

Definition A.33 (Singular Set)**.** *We define the set of **singular values***

$$S(f) = \overline{\{\textit{critical and asymptotic values}\}}.$$

*A **postsingular point** is a point on the orbit of a singular value.*

Theorem A.34 (Role of the Singular Values)**.** *Let f be a meromorphic function.*

(a) Suppose that f has an attracting fixed point or cycle. Then there is at least one singular value in the immediate basin of attraction of this point.

(b) *Suppose that f has a parabolic fixed point. Then there is at least one singular value in the immediate basin of attraction of this point.*

Theorem A.35. *Let f be a meromorphic function, and let $C = \{U_0, U_1, \dots, U_{p-1}\}$ be a periodic cycle of components of $F(f)$. If C is a cycle of Siegel disks or Herman rings, then*

$$\partial U_j \subset \overline{\mathcal{O}_f^+(S(f))} \text{ for all } j \in \{0, 1, \dots, p-1\}$$

Theorem A.36. *Let f be a meromorphic function, and let $\{U_0, \dots, U_{p-1}\}$ be a periodic cycle of Baker domains of f. Denote by z_j the limit corresponding to U_j and define $z_p = z_0$. Then*

$$z_j \in \bigcup_{n=0}^{p-1} f^{-n}(\infty) \quad \forall j \in \{0, \dots, p-1\}$$

and $z_j = \infty$ for at least one $j \in \{0, \dots, p-1\}$. If $z_j = \infty$, then z_{j+1} is an asymptotic value of f.

Corollary A.37. *Let f be a meromorphic function and let*

$$\{U_0, \dots, U_{p-1}\}$$

be a periodic cycle of Baker domains of f. Then exists $j \in \{0, \dots, p-1\}$ such that

$$\partial U_j \cap S(f) \neq \emptyset$$

A.3.1 Wandering domains

This Fatou components have been unnoticed during many years due to their absence in rational maps. The no-Wandering Domains Theorem was first proven by Sullivan in 1985 in his famous paper *Quasiconformal Homeomorphisms and Dynamics I. Solution of the Fatou-Julia Problem on Wandering Domains* (see [Sul]). This paper introduced quasiconformal analysis techniques into holomorphic dynamics, which meant remarkable advances in the field.

However, transcendental functions do have Wandering Domains. For example the function

$$g(z) = z + \sin(z) + 2\pi$$

has a Wandering Domain.

They are the least understood Fatou components and still subject of current research. Their analysis is out of the scope of this project. However, some special classes of transcendental maps do not have Wandering Domains.

Definition A.38 (Classes of Meromorphic Functions)**.** *We define the following classes of meromorphic functions:*

- $S = \{f \colon f$ *has only finitely many critical and asymptotic values*$\}$.
- $F = \{f \colon f(z) = z + r(z)e^{p(z)}$, *where $r(z)$ is rational and $p(z)$ a polynomial*$\}$.
- $R = \{f \colon f'(z) = r(z)(f(z) - z)^2$ *or* $f'(z) = r(z)(f(z) - z)(f(z) - \tau)$ *where $r(z)$ is rational and $\tau \in \mathbb{C}$*$\}$.

Theorem A.39. *Functions in S, F and R do not have Wandering Domains.*

Theorem A.40. *Functions in S do not have Baker domains.*

Appendix B

The Integrability theorem

Remember that we define the **Beltrami coefficient** of a K-qc map ϕ (or the **complex dilatation**) as the measurable function

$$\mu_\phi(z) = \frac{\phi_{\bar{z}}(z)}{\phi_z(z)}.$$

Note that since $\phi_z \neq 0$ a.e. (otherwise we would have $\phi_z = \phi_{\bar{z}} = 0$ a.e., which would imply by the ACL condition that ϕ is constant, contradicting that it is a topological mapping), the Beltrami coefficient is well-defined. Moreover,

$$|\mu_\phi(z)| = \frac{|\phi_{\bar{z}}(z)|}{|\phi_z(z)|} = \frac{D_\phi - 1}{D_\phi + 1} < 1.$$

We state now the Integrability theorem, which yields that the conditions that we have obtained on the Beltrami coefficient of a K-qc mapping are sufficient conditions so that there exists a quasiconformal mapping with such a Beltrami coefficient.

Theorem B.1 (Integrability theorem)**.** *Let $U \subset \mathbb{C}$ be an open set such that $U \cong \mathbb{D}$ (resp. $U \cong \mathbb{C}$). Let μ be a Beltrami coefficient on U such that the essential supremum $||\mu||_\infty = k < 1$. Then μ is integrable, i.e. there exists a quasiconformal homeomorphism $\phi\colon U \to \mathbb{D}$ (resp. onto $\mathbb{C}$) which solves the Beltrami equation, i.e. such that*

$$\partial_z\phi(z)\mu(z) = \partial_{\bar{z}}\phi(z)$$

for almost every $z \in U$. Moreover, ϕ is unique up to post-composition with automorphisms of $\mathbb{D}$ (resp. $\mathbb{C}$).

B.1 Tools for the proof

Here we present the results that we need before we address the cornerstone of this project; proving the Integrability theorem.

The first tools are three classical results and a lemma, the proof of them can be found in [Ahl, LV].

Theorem B.2. *A uniform limit of K-qc mappings f_n is also K-q.c*

Theorem B.3 (Weyl's lemma)**.** *If $f \in \mathcal{C}(\Omega)$, $\Omega \subset \mathbb{C}$ a domain, is such that*

$$\int_{\mathbb{C}} f(z)\frac{\partial\phi}{\partial\bar{z}}(z)dm(z) = 0 \quad \forall\phi \in \mathcal{C}_c^\infty(\Omega),$$

then $f \in \mathcal{H}(\Omega)$.

The next generalization of Weyl's lemma will also be needed:

Lemma B.4. *If p and q are continuous and have locally integrable distributional derivatives that satisfy $p_{\bar{z}} = q_z$, then there exists a function $f \in \mathcal{C}^1$ with $f_z = p$, $f_{\bar{z}} = q$.*

aquest resultat es podria demostrar (el de dalt dic, el de sota no).

Theorem B.5 (Calderón-Zygmund inequality)**.** *The operator*

$$Th(\xi) = \lim_{\varepsilon \to 0} -\frac{1}{\pi} \int_{|z-\xi|>\varepsilon} \frac{h(z)}{(z-\xi)^2} dm(z),$$

which is defined for $h \in \mathcal{C}_0^2$, can be extended to L^p, $p \geq 2$, so that

$$||Th||_p \leq C_p ||h||_p$$

for some constant C_p such that $C_p \xrightarrow[p \to 2]{} 1$.

B.1.1 Two integral operators

The solution of the Beltrami equation will be obtained as the fixed point of an operator which is strictly contracting. Hence we are going to introduce the two operators that we need and prove some properties about them. The results here have been obtained from [Ahl].

The first one is P, which acts on functions $h \in L^p$, for $p > 2$ ($L^p = L^p(\mathbb{C})$) and it is defined by

$$Ph(\xi) = -\frac{1}{\pi} \int_{\mathbb{C}} h(z) \left(\frac{1}{z-\xi} - \frac{1}{z} \right) dm(z). \tag{B.1}$$

(we integrate on the whole $\mathbb{C}$.)

Lemma B.6. *Ph is continuous and satisfies a uniform Hölder condition with exponent $1 - 2/p$.*

Proof. Note that

$$(1) = -\frac{1}{\pi} \int_{\mathbb{C}} h(z) \frac{\xi}{z(z-\xi)} dm(z)$$

and $h \in L^p$ (by hypothesis) and $\xi/(z(z-\xi)) \in L^q$ (where q is the conjugate exponent of p, $1 < q < 2$). So by Hölder's inequality we have that (B.1) is convergent. Moreover, by the very same Hölder's inequality on the case $\xi \neq 0$ we obtain

$$|Ph(\xi)| \leq \frac{|\xi|}{\pi} ||h||_p \left|\left| \frac{1}{z(z-\xi)} \right|\right|_q = K_p ||h||_p |\xi|^{1-2/p}, \tag{B.2}$$

where the last equality is obtained using the change of variable $z \mapsto z/\xi$, the constant K_p only depends on p (note that this inequality is satisfied also when $\xi = 0$).

If we take $h_1(z) = h(z + \xi_1)$ we see that

$$\begin{aligned} Ph_1(\xi_2 - \xi_1) &= -\frac{1}{\pi} \int_{\mathbb{C}} h(z+\xi_1) \left(\frac{1}{z + \xi_1 - \xi_2} - \frac{1}{z} \right) \\ &= -\frac{1}{\pi} \int_{\mathbb{C}} h(z) \left(\frac{1}{z-\xi_2} - \frac{1}{z-\xi_1} \right) = Ph(\xi_2) - Ph(\xi_1) \end{aligned}$$

(on the second equality we have just applied the change of variable $z \mapsto z + \xi_1$). So if we apply (B.2) to $Ph_1(\xi_2 - \xi_1)$ we obtain

$$|Ph(\xi_2) - Ph(\xi_1)| \leq K_p ||h||_p |\xi_1 - \xi_2|^{1-2/p}.$$

□

The second operator, which has already been introduced in the Calderón-Zygmund inequality, it is initially defined just for functions in $\mathcal{C}_0^2$ (which is the Cauchy principal value).

$$Th(\xi) = \lim_{\varepsilon \to 0} -\frac{1}{\pi} \int_{|z-\xi|>\varepsilon} \frac{h(z)}{(z-\xi)^2} dm(z). \tag{B.3}$$

Lemma B.7. *For $h \in \mathcal{C}_0^2$, Th is well-defined and of class C^2. Moreover,*

$$(Ph)_{\bar{z}} = h \quad , \quad (Ph)_z = Th \tag{B.4}$$

and

$$\int_{\mathbb{C}} |Th|^2 dm(z) = \int_{\mathbb{C}} |h|^2 dm(z) \tag{B.5}$$

Proof. In order to verify (B.4) we only need to use that $h \in \mathcal{C}_0^1$. Since this is enough condition so that we can use the differentiation under the integral sign theorem.

$$(Ph)_{\bar{\xi}} = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{h_{\bar{z}}}{z-\xi} dm(z) \quad , \quad (Ph)_{\xi} = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{h_z}{z-\xi} dm(z). \tag{B.6}$$

By Cauchy-Pompeiu formula, since h is compactly supported we see that

$$(Ph)_{\bar{\xi}} = h(\xi).$$

If we consider γ_ε as the circle of center ξ and radius ε andwe use Stokes' formula, we obtain

$$\begin{aligned} (Ph)_z &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{dh d\bar{z}}{z-\xi} = \\ &= \lim_{\varepsilon \to 0} \left[-\frac{1}{2\pi i} \int_{\gamma} \frac{h d\bar{z}}{z-\xi} + \frac{1}{2\pi i} \int_{|z-\xi|>\varepsilon} \frac{h dz d\bar{z}}{(z-\xi)^2} \right] = \\ &= Th(\xi) \end{aligned}$$

where the last equality follows from the fact that

$$\lim_{\varepsilon \to 0} \int_{\gamma} \frac{h d\bar{z}}{z-\xi} = 0,$$

and the equality

$$\frac{1}{2\pi i} \int_{|z-\xi|>\varepsilon} \frac{h dz d\bar{z}}{(z-\xi)^2} = -\frac{1}{\pi} \int_{|z-\xi|>\varepsilon} \frac{h}{(z-\xi)^2} dm(z).$$

So (B.4) is proved.

Note that (B.6) yields that

$$P(h_{\bar{z}}) = h - h(0) \quad , \quad P(h_z) = Th - Th(0). \tag{B.7}$$

If we use now that by hypothesis $h \in \mathcal{C}_0^2$ and we apply (B.4) to h_z and we use the second equality from (B.7), we find out that

$$(Th)_{\bar{z}} = P(h_z)_{\bar{z}} = h_z \tag{B.8}$$
$$(Th)_z = P(h_z)_z = T(h_z) = P(h_{zz}) + Th_z(0), \tag{B.9}$$

and this equalities imply that $Th \in \mathcal{C}^1$ and $Ph \in \mathcal{C}^2$.

In order to prove the last part of the theorem (the isometry), note that since h has compact support, as $z \to \infty$ we have $Ph = O(1)$ and $Th = O(|z|^{-2})$. We just have now to calculate:

$$\begin{aligned}\int_{\mathbb{C}} |Th|^2 dm(z) &= -\frac{1}{2i}\int_{\mathbb{C}} (Ph)_z(\overline{Ph})_{\bar{z}} dz d\bar{z} = \\ &= \frac{1}{2i}\int_{\mathbb{C}} Ph(\overline{Ph})_{\bar{z}z} dz d\bar{z} = \frac{1}{2i}\int_{\mathbb{C}} (Ph)\bar{h}_{\bar{z}} dz d\bar{z} = \\ &= -\frac{1}{2i}\int_{\mathbb{C}} \bar{h}(Ph)_{\bar{z}} dz d\bar{z} = \int_{\mathbb{C}} |h|^2 dm(z).\end{aligned}$$

where we have only used the Stokes' formula (which has been also used before) and the definition of distributional derivatives (several times). □

Since the functions of class $\mathcal{C}_0^2$ are dense in L^2, thanks to the isometry (B.4) we are enabled to extend by continuity the operator T to all of L^2. However, P cannot be extended to functions $h \in L^2$. In order to deal with this issue, we use Calderón-Zygmund's inequality, which yields that

$$||Th||_p \leq C_p ||h||_p$$

for any $p > 2$, where C_p can be taken so that $C_p \xrightarrow[p \to 2]{} 1$.

Lemma B.8. *Let $h \in L^p$, $p > 2$, then the relations in (B.4) hold in the distributional sense.*

Proof. We need to prove that

$$\int_{\mathbb{C}} (Ph)\phi_{\bar{z}} = -\int_{\mathbb{C}} \phi h \quad , \quad \int_{\mathbb{C}} (Ph)\phi_z = -\int_{\mathbb{C}} \phi(Th) \tag{B.10}$$

for all test functions $\phi \in \mathcal{C}_0^1$. Note that by the previous theorem we already know that such equalities hold when $h \in \mathcal{C}_0^2$.

Since functions in $\mathcal{C}_0^2$ are dense in L^p, we can consider a sequence $\{h_n\}_n \subset \mathcal{C}_0^2$ such that it converges to h in L^p.

On the right side of both members of (B.10) we just have to use that $||Th - Th_n||_p \leq C_p||h - h_n||_p$. On the left side of both members of (B.10) we just have to apply Lemma B.6, which tells us that

$$|Ph(\xi_2) - Ph(\xi_1)| \leq K_p ||h||_p |\xi_1 - \xi_2|^{1-2/p}.$$

Now the result follows from ϕ having compact support. □

B.2 Proof of the Integrability theorem

The results here have been obtained following mainly [Ahl]. However, [BF] has also been useful.

Lemma B.9 (Uniqueness)**.** *The solution of the Beltrami equation in the conditions of Theorem 1.9 is unique up to post-composition with automorphisms of $\mathbb{D}$ (resp. $\mathbb{C}$).*

Proof. If we have two solutions ϕ_1 and ϕ_2, then $\varphi := \phi_1 \circ \phi_2^{-1}$ is 1-qc, hence by Weyl's lemma φ is conformal, i.e. $\phi_1 = \varphi \circ \phi_2$. □

First we need to address the case where μ has compact support (hence f will be holomorphic at ∞).

We will need to use a fixed exponent $p > 2$ so that $kC_p < 1$, which can be achieved since $k = ||\mu||_\infty < 1$ and $C_p \to 1$. Note that we are also interested in giving enough conditions so that the automorphism of Lemma B.9 is the identity (and hence the uniqueness part of the next result).

Theorem B.10. *If μ has compact support, then there exists a unique solution f of the Beltrami equation such that $f(0) = 0$ and $f_z - 1 \in L^p$.*

Proof. We start with the uniqueness part. Let f be a solution, then since μ has compact support $f_{\bar{z}} = \mu f_z$ is of class L^p, hence $P(f_{\bar{z}})$ is well-defined.

Consider the function $F = f - P(f_{\bar{z}})$, which satisfies $F_{\bar{z}} = 0$ (in the distributional sense), hence by Weyl's lemma F is holomorphic. Moreover, since $f_z - 1 \in L^p$, then $F' - 1 \in L^p$, which is only possible if $F' = 1$, which yields that $F = z + a$, by the normalization we see that $a = 0$, therefore

$$f = P(f_{\bar{z}}) + z \tag{B.11}$$

hence,

$$f_z = T(\mu f_z) + 1 \tag{B.12}$$

So if g is another solution we see that $f_z - g_z = T[\mu(f_z - g_z)]$. By Calderón-Zygmund inequality we obtain $||f_z - g_z||_p \leq kC_p||f_z - g_z||_p$ (remember that $k = ||\mu||_\infty$), hence since $kC_p < 1$ we obtain that $f_z = g_z$, but then by the Beltrami equation we would also have $f_{\bar{z}} = g_{\bar{z}}$. Hence $f - g$ and $\bar{f} - \bar{g}$ are both holomorphic, hence the $f - g$ must be constant and by the normalization condition we see that $f = g$.

The study of the uniqueness also gives us an idea about how to prove the existence. We want to study the equation

$$h = T(\mu h) + T\mu \tag{B.13}$$

Since the linear operator $h \mapsto T(\mu h)$ on L^p has norm $\leq kC_p < 1$ (by the Calderón-Zygmund inequality), the series

$$h = T\mu + T\mu T\mu + \cdots \tag{B.14}$$

converges in L^p (as it has already been proved in the first part of the course) and it is trivially a solution of (13). Furthermore, with h as above the final goal is to prove that

$$f = P[\mu(h+1)] + z \tag{B.15}$$

is the solution of the Beltrami equation. We prove it in three steps:

- Since μ has compact support $\mu(h+1) \in L^p$, hence $P[\mu(h+1)]$ is well-defined and continuous.
- By Lemma B.8 we obtain that

$$f_{\bar{z}} = \mu(h+1) \quad , \quad f_z = T[\mu(h+1)] + 1 = h + 1 \tag{B.16}$$

  where the last equality follows from (B.13).
- $f_z - 1 = h \in L^p$.

□

The function f obtained in the previous theorem will be called the **normal solution** of the Beltrami equation.

The previous result also yields some estimates that we will need later on.

- By (B.13) we see that $||h||_p \leq kC_p||h||_p + C_p||\mu||_p$. Hence,

$$||h||_p \leq \frac{C_p}{1 - kC_p}||\mu||_p. \tag{B.17}$$

- By (B.16):

$$||f_{\bar{z}}||_p \leq \frac{1}{1-kC_p}||\mu||_p. \tag{B.18}$$

- The Hölder condition given by Lemma B.6 together with (B.11):

$$|f(\xi_1) - f(\xi_2)| \leq \frac{K_p}{1-kC_p}||\mu||_p|\xi_1 - \xi_2|^{1-2/p} + |\xi_1 - \xi_2|. \tag{B.19}$$

Consider now another Beltrami coefficient ν which is still bounded by k. Let g be the corresponding normal solution provided by the previous theorem. Then, $f_z - g_z = T(\mu f_z - \nu g_z)$, hence

$$\begin{aligned} ||f_z - g_z||_p &\leq ||T[\nu(f_z - g_z)]||_p + ||T[(\mu - \nu)f_z]||_p \leq \\ &\leq kC_p||f_z - g_z||_p + C_p||(\mu-\nu)f_z||_p \end{aligned}$$

If we suppose that $\nu \to \mu$ a.e. and that the supports are uniformly bounded, then:

Lemma B.11. *In the conditions above we have that $||g_z - f_z||_p \to 0$ and $g \to f$, uniformly on compact sets. Moreover, since $f - g$ is holomorphic at ∞, the convergence is uniform in $\mathbb{C}$.*

Proof. The inequality above together with $kC_p < 1$ shows us that

$$||f_z - g_z||_p \leq \frac{C_p}{1-kC_p}||f_z||_p||(\mu-\nu)||_\infty$$

hence $||f_z - g_z||_p \to 0$. Since they solve the Beltrami equation we also have $||f_{\bar{z}} - g_{\bar{z}}||_p \to 0$, and finally since they are normalized $g \to f$ uniformly on compact sets. □

We want to show now that f has derivatives if μ does:

Lemma B.12. *If μ has a distributional derivative $\mu_z \in L^p$, $p > 2$, then $f \in \mathcal{C}^1$ is a topological mapping.*

Proof. We want to apply Lemma B.4 to try to determine λ so that the system

$$f_z = \lambda \quad , \quad f_{\bar{z}} = \mu\lambda \tag{B.20}$$

has solution. In fact, by Lemma B.4 this will be the case if

$$\lambda_{\bar{z}} = (\mu\lambda)_z = \lambda_z\mu + \lambda\mu_z \tag{B.21}$$

which is equivalent to solving $(\log\lambda)_{\bar{z}} = \mu(\log\lambda)_z + \mu_z$.

Note that the equation $q = T(\mu q) + T\mu_z$ can be solved for q in L^p (because the operator $q \mapsto T(\mu q) + T\mu_z$ is contracting by the Calderón-Zygmund inequality together with $kC_p < 1$, hence it has a unique fixed point), name q the solution and set

$$\sigma = P(\mu q + \mu_z) + ctt,$$

where the constant is such that $\sigma \to 0$ for $z \to \infty$. Hence σ is continuous and it is a simple computation to show that

$$\sigma_z = \mu q + \mu_z \quad , \quad \sigma_z = T(\mu q + \mu_z) = q. \tag{B.22}$$

Hence $\lambda = e^\sigma$ satisfies (B.21), so by B.6, (B.20) can be solved with $f \in \mathcal{C}^1$. If we normalize f by $f(0) = 0$, then it is the normal solution since $\sigma \to 0$, $\lambda \to 1$ and $f_z \to 1$ as $z \to \infty$.

Moreover, the Jacobian $|f_z|^2 - |f_{\bar{z}}|^2 = (1-|\mu|^2)e^{2\sigma} > 0$ (strictly positive), hence f is locally one-to-one. Finally, since $f(z) \to \infty$ as $z \to \infty$ it is a homeomorphism. □

Note that by Lemma B.12 the inverse f^{-1} is also K-qc and has Beltrami coefficient $\mu_{f^{-1}}$, which satisfies $|\mu_{f^{-1}} \circ f| = |\mu|$.

By using f as a change of variables, the fact that solves the Beltrami equation and Hölder, we can see that

$$\begin{aligned}\int_{\mathbb{C}} |\mu_{f^{-1}}|^p dm(w) &= \int_{\mathbb{C}} |\mu|^p (|f_z|^2 - |f_{\bar z}|^2) dm(z) \le \\ &\le \int_{\mathbb{C}} |\mu|^p |f_z|^2 dm(z) = \int_{\mathbb{C}} |\mu|^{p-2} |f_{\bar z}|^2 dm(z) \le \\ &\le \left(\int_{\mathbb{C}} |\mu|^p dm(z)\right)^{\frac{p-2}{p}} \left(\int_{\mathbb{C}} |f_{\bar z}|^p dm(z)\right)^{2/p}\end{aligned}$$

Hence $||\mu_{f^{-1}}||_p \le (1 - kC_p)^{-2} ||\mu||_p$. If we apply (B.19) to f^{-1} we see that

$$|z_1 - z_1| \le K_p (1 - kC_p)^{-1-2/p} ||\mu||_p |f(z_1) - f(z_2)|^{1-2/p} + |f(z_1) - f(z_2)|. \tag{B.23}$$

We are ready to prove:

Theorem B.13. *For any μ with compact support and $||\mu||_\infty \le k < 1$, the normal solution of the Beltrami equation is a qc map with $\mu_f = \mu$.*

Proof. We can approximate μ by functions $\mu_n \in \mathcal{C}^1$ ($\mu_n \to \mu$ a.e.) and $|\mu_n| \le k < 1$ and $\mu_n = 0$ outside a fixed circle. We can apply (B.23) to the normal solutions f_n. Since $f_n \to f$ and $||\mu_{f_n}||_p \to ||\mu||_p$, then f also satisfies (B.23), hence f is injective. By Theorem B.8, since f is a uniform limit of K-qc maps, it is also a K-qc map. Hence it has locally integrable partial derivatives (which are also distributional derivatives). Moreover, as it has been already said, $f_z \neq 0$ a.e., hence μ_f is defined a.e. and it agrees with μ. □

The goal now is to prove it for general Beltrami coefficients μ.

Theorem B.14. *For any measurable μ with $||\mu||_\infty < 1$ there exists a unique normalized qc map f with Beltrami coefficient μ that leaves $0, 1$ and ∞ fixed.*

Proof. The case where μ has compact support has already been proved (we only need to normalize f to obtain the uniqueness part).

Suppose $\mu = 0$ in a neighborhood of 0, then by setting

$$\tilde\mu(z) = \mu\left(\frac{1}{z}\right)\frac{z^2}{\bar z^2}$$

we obtain that $\tilde\mu$ has compact support. If $\tilde f$ is the solution (normalized), then by the uniqueness we have

$$f(z) = 1/\tilde f(1/z).$$

Finally, in the general case we set $\mu = \mu_1 + \mu_2$, where $\mu_1 = \mu \mathbb{1}_{\mathbb{D}}$ (hence it has compact support) and $\mu_2 = \mu \mathbb{1}_{\mathbb{C}\setminus\mathbb{D}}$.

Find f_2 which solves the Beltrami equation (normalized) for μ_2 and define $\tilde\mu := (f_2^{-1})^*\mu$ (the pull-back of the Beltrami coefficient, see [BF]), which is well defined since f_2 is injective and has compact support. Note that on $\mathbb{C} \setminus \mathbb{D}$ we have $\tilde\mu = 0$ (because $\tilde\mu$ is the pull-back under f_2^{-1} of μ, and f_2 solves the Beltrami equation for μ_2), hence we can solve the Beltrami equation (normalized) and find $\tilde f$. Then the map $f := \tilde f \circ f_2$ solves the original Beltrami equation. To see this, call $\mu_0 \equiv 0$, then

$$f^*\mu_0 = f_2^* \tilde f^* \mu_0 = f_2^* \tilde\mu = f_2^*(f_2^{-1})^*\mu = (f_2^{-1} \circ f_2)^*\mu = \mu.$$

Therefore the Theorem is proved. □

This last result completes the proof of the Integrability theorem. □

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