UNIVERSITAT DE BARCELONA

Facultat de Matemàtiques i Informàtica

**GRAU DE MATEMÀTIQUES**

**Treball final de grau**

# TOPOLOGICAL STRUCTURES IN COMPLEX DYNAMICS

**Autora: Ayla Ayadi Antón**

**Directora:** Dra. Núria Fagella Rabionet

**Realitzat a:** Departament de Matemàtiques i Informàtica

**Barcelona, 24 de gener de 2023**

# Abstract

Many sets in the field of planar topology are considered "exotic" and not frequently encountered in everyday life. However, these sets possess unique and intriguing topological properties, and quite often also a visually appealing aesthetic. In recent years, thanks to the resurgence of complex dynamics, many of these exotic sets have been found to be Julia sets for complex analytic functions. In this work, we delve into the world of planar topology, provide an overview of the basics of complex dynamics, and present four examples of such sets: dendrites, Cantor sets, Sierpiński curves, and Cantor bouquets. To conclude, we also explain how these sets arise through specific families of complex maps, such as the quadratic family, the complex exponential family, and a certain type of singularly perturbed rational maps.

## Acknowledgements

Para empezar, me gustaría dedicar este trabajo a Núria, mi tutora. Ella ha sido el faro que ha iluminado cada uno de mis pasos a lo largo de este trabajo, a la vez que un ejemplo a seguir durante estos años de formación, no solo como matemática, sino también como profesora y como mujer.

Si tuviera que nombrar a las personas a las que les debo esta carrera empezaría por mi madre, Adela, mi padre, Hamid, y mi abuela, Pepa. Ellos han vivido cada alegría y cada noche en vela como si fuesen suyas. Gracias por darme el regalo más grande: la vida, y por el apoyo incondicional que me brindáis diariamente.

Todo esto tampoco hubiera sido posible sin Rubén, ya no solo como compañero de viaje, sino como amigo y confidente.

Gracias también a Joan, por todo lo vivido y todo lo que juntos conseguimos más allá de graduarnos, que no es poco.

Gracias a Clàudia y a Laura, a Gigi y a Giselle. A todos aquellos con los que he compartido estos años, desde una asignatura hasta un descanso en la biblioteca. Gracias por el apoyo, solidaridad y compañerismo que se vive en esta facultad.

Gracias a Seb por tanto que ni cabe. Por todo el amor, apoyo y alegría de vivir. Ta présence me calme et ton regard m'apaise, quand je suis auprés de toi je sais que je peux être moi-même. Eres casa.

Por último, gracias a mí misma. Por la perseverancia, el esfuerzo y el sentido común. Por convertirme en la persona que soy y por haber llegado hasta aquí. Que aunque esto suene a final, en realidad solo es el principio.

# Contents

## Introduction

As society becomes more specialized, individuals within each field tend to believe that they only need to be knowledgeable in their own area. This kind of narrow-mindedness hinders our ability to see the connections and understand the bigger picture of our world. As Neri Oxman once said, the relation between science, engineering, design, and art is similar to a clock, where one is constantly moving back and forth between different domains, and what is input in one area becomes output in another. We could also apply this concept for different areas of mathematics. The momentum appears at "12 o clock", when science meets art, just as when complex dynamics meets topology.

Let's begin by delving into the following image:

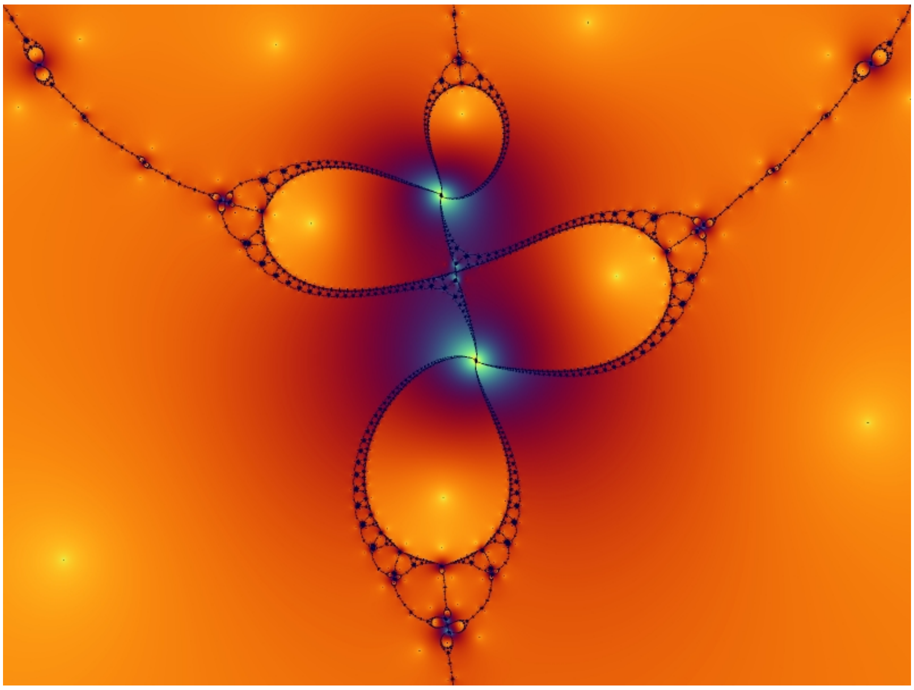

Figure 1: Fatou domains.

We may wonder what secrets are hidden beneath these artistic outcomes and how this beautiful result could ever be linked to scientific fields. However, it is no secret that mathematics can explain just about every line, colour, and motif of these artworks.

Despite being totally distinct areas, mathematics and fine arts have had a long-standing and intertwining relationship. Many artists throughout history, such as Hilma as Klint, Emma Kunz, and Agnes Martin, have used mathematical concepts and principles in their work, such as the use of geometric shapes and patterns (fractals), the exploration of symmetry and balance, and the manipulation of perspective and proportion. Others, like M.C. Escher, even explored the concept of infinity through his drawings.

On the other hand, mathematicians have also been inspired by art, using it not only as a source of inspiration for their concepts and theories, but also as a tool to represent them. In summary, these two fields have a long interlaced history, with each one of them influencing and inspiring the other in many ways.

Similarly, topology and complex dynamics are two branches of mathematics that are closely related. Topology is concerned with the properties of objects that remain unchanged when they are stretched or bent, but not torn or glued, while complex dynamics examines the behavior of complex functions when they are iterated, such as polynomials and rational functions.

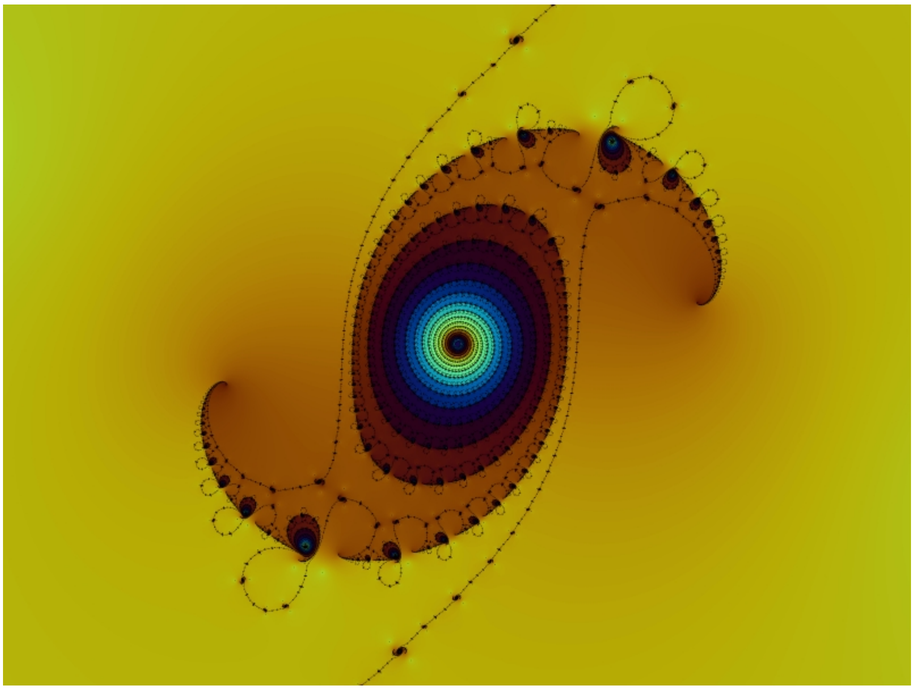

Figure 2: Fatou domains.

One of their key connections is the study of Julia sets (as well as its complementaries, the Fatou sets), and Mandelbrot sets. The Julia set of a polynomial is the boundary of the set of points that do not escape to infinity under iteration of the function. Not only do these sets have a fractal structure, but they also possess a wide range of fascinating topological properties (see [7], [19]).

In fact, for nearly a century, topologists have been captivated by the exceptional characteristics of intriguing objects like indecomposable continua, Sierpiński curves, and Cantor bouquets. However, it is only recently that complex dynamicists have begun to appreciate the complexity and elegance of these objects known as Julia sets. Recent advancements have shown that many of these spaces from planar topology can also be found in Julia sets, bringing the two fields closer together ([9], [12], [10], [11]). In this work we will discuss a few examples of this overlap between topology and complex dynamics.

To summarize, complex dynamics has a strong connection with both topology and art. On one hand, topology offers geometric and topological methods for studying the behavior of intricate sets. On the other hand, the study of complex dynamics leads to striking visuals related to dynamical sets, linked to fundamental mathematical concepts such as complex analysis or set theory, as well as leading to new topological questions and results.

Finding these beautiful exotic topological structures in unexpected environments by surprise is like finding a needle in a haystack without even looking for it. And, through this kind of discoveries, we acknowledge the legacy of mathematicians like Cantor, who devoted his life to the study of set theory. Despite making groundbreaking contributions, he faced rejection and criticism from his peers, and struggled with mental health issues. Even figures like Leopold Kronecker said that "Cantor's set theory is a disease from which one has to recover".

So here we are, not only to "recover" from these discoveries, but to show how relevant they have become, since even without "inventing" them, they just magically appear while iterating functions on the complex plane. As a matter of fact, despite facing harsh criticism, Cantor's work was later widely recognized and celebrated. In 1904, the Royal Society presented him with its highest award for mathematical achievement, the Sylvester Medal, and figures as David Hilbert defended Cantor's ideas from its detractors by saying, "No one shall expel us from the paradise that Cantor has created".

This work is divided into three chapters. The initial chapter covers the basic principles of planar topology, and provides the definition and certain characteristics of fractals, which will give us the foundation we need to comprehend various concepts of dimension. Lastly, we will explore examples of the exotic topological models mentioned before, such as dendrites, Cantor sets, Sierpiński carpets, and Cantor bouquets, and examine their characteristics.

In chapter 2, we will establish the foundations of complex dynamics. We will begin by providing a brief overview of iteration on the Riemann sphere to ease understanding of the concepts of normality and the Montel's theorem, which are essential in the Fatou-Julia sets theory. We also provide a formal definition of the Fatou and Julia sets, and demonstrate some key properties that will be necessary, presented along a comprehensive examination of dynamics, covering various types of fixed points and orbits. After giving the basics of local, semilocal, and global theory, we will delve into the five types of periodic Fatou components using the Classification Theorem. In addition, we will also include a section on singular values and polynomial dynamics, which will be highly beneficial for the next chapter.

To conclude, Chapter 3 will demonstrate how the sets presented in Chapter 1 appear as the Julia set of particular families of complex maps, such as the quadratic family, the complex exponential family, and a specific class of singularly perturbed rational maps, by using the tools discussed in the previous chapters.

2020 Mathematics Subject Classification. 37B02, 37B45, 37F10, 37F12

# Chapter 1

# Planar Topology

The main goal of this chapter is to provide an introduction to the basics of planar topology, including key concepts and definitions such as connectedness, compactness, etc. Additionally, we will explore some of the most relevant properties of the well known geometrical objects called fractals, such as the fractal dimension. To solidify the concepts covered in the chapter, we will also present examples of topological models and their most pertinent characteristics.

## 1.1 General Concepts

In this section we review some of the basic topological ideas associated with the topological models we will see further in this chapter. The main references for this section are [20] and [21].

Broadly speaking, a topological space is a mathematical structure that allows the formal definition of concepts such as connectivity, continuity, and compactness. The most widely used definition is that in terms of open sets:

**Definition 1.1. (Topological space)** A *topological space* $(X, \mathcal{T})$ is a set $X$ together with a collection of open subsets of $X$, $\mathcal{T}$, that satisfies the four following conditions:

1. The empty set $\emptyset$ is in $\mathcal{T}$.
2. $X$ is in $\mathcal{T}$.
3. The intersection of a finite number of sets in $\mathcal{T}$ is also in $\mathcal{T}$.
4. The union of an arbitrary number of sets in $\mathcal{T}$ is also in $\mathcal{T}$.

**Definition 1.2. (Subspace topology)** Given a topological space *(X ,$\mathcal{T}$)* and a subset $X' \subset X$, the *subspace* (or *relative*) *topology* on $X'$ is defined by

$$\mathcal{T}_x = \{T \cap X' \mid T \in \mathcal{T}\}.$$

With this topology, *(X ,$\mathcal{T}_x$)* is a topological space in its own right.

In this work, we will focus on *planar sets*, which are sets $X$ such that $X \subset \mathbb{R}^2$ where, for our purposes, $\mathbb{R}^2$ will be identified with the complex plane $\mathbb{C}$.

The usual topology for the complex plane $\mathbb{C}$ is the topology induced by the metric

$$d(x, y) := |x - y|$$

for $x, y \in X$ where $|\cdot|$ is the complex modulus. It is clear that the above topology coincides with topology induced by the Euclidean metric on $\mathbb{R}^2$ .

When taking into account subsets $X \subset \mathbb{C}$, we will always consider them equipped with the subset topology. Hence, $X' \subset X$ is open (respectively closed) if $X' = A \cap X$, where $A \subset \mathbb{C}$ is open (respectively closed). We will now introduce some basic topological concepts for $\mathbb{C}$.

**Definition 1.3. (Neighborhood and open ball)** A *neighborhood* of a point $z_0 \in \mathbb{C}$ is an open set that contains $z_0$ in its interior. An *open ball* or *open disk* of a complex number $z_0$, consists of all points $z$ lying inside but not on a circle centred at $z_0$ and with radius $r > 0$ and it is expressed by

$$B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

**Definition 1.4. (Limit point)** A point $x$ is a *limit point* or *accumulation point* of a subset $X$ of $\mathbb{C}$ if every neighborhood of $x$ contains at least one point of $X$ different from $x$ itself. A limit point of a set $X$ does not itself have to be an element of $X$.

**Definition 1.5. (Closed set)** A *closed set* is a set which contains all its limit points.

Closed sets also provide a useful characterization of compactness:

**Definition 1.6. (Compact spaces)** For any subset $X$ of $\mathbb{C}$, $X$ is *compact* if and only if it is closed and bounded.

**Definition 1.7. (Connected space)** A *connected space* is a space $X \subseteq \mathbb{C}$ that cannot be expressed as the union of two or more disjoint nonempty open subsets. In other words, it cannot be expressed as the disjoint union of $A \cap X$ and $B \cap X$ where $A$ and $B$ are open sets of $\mathbb{C}$.

Connectedness is a major topological property used to distinguish topological spaces. Another associated concept is local connectivity, which does not imply or derive from connectivity:

**Definition 1.8. (Locally connected space)** We say that a space $X$ is *locally connected at $x$* if every neighborhood of $x$ contains a connected open neighborhood of $x$, i.e. the point $x$ admits a neighborhood basis consisting entirely of open, connected sets. A *locally connected* is a space which is locally connected at every point.

In the case of the *arc-connectivity* what is asked is to be able to join any two points with a path within $X$. By a *path*, we mean the image of $[0,1]$ under a continuous injective map.

**Definition 1.9. (Arc-connected space)** A space $X$ is said to be *arc-connected* or *arcwise connected* if for every pair of points $p$ and $q$ of $X$ there is a path in $X$ joining $p$ and $q$.

Disconnected sets can always be expressed as the disjoint union of smaller sets which are connected. This is the concept of *connected component*.

**Definition 1.10. (Connected component)** Consider a set of points $X$. Every point $x \in X$ is contained in a unique maximal connected subset $C$ of $X$, called *connected component* of $x$.

Note that every point of a set $X$ lies in a unique connected component of $X$, which is the union of all the connected sets containing the point.

We can easily see that $C$ is closed since if $C$ is a connected component of a space $X$, $\overline{C}$ is a connected subset of the space meeting $C$, and is therefore contained in $C$. It follows that the components of any set $X$ are closed in $X$.

**Definition 1.11. (Totally disconnected space)** A a space $X \subseteq \mathbb{C}$ is *totally disconnected* if the connected components in $X$ are single points.

Numerous natural objects in topology and complex dynamics are best studied in the context of continuum theory, which is the study of compact, connected, metric spaces.

**Definition 1.12. (Continuum)** A *continuum* (plural:"*continua*") is a nonempty, compact and connected metric space. A continuum that contains more than one point is called *nondegenerate*.

**Definition 1.13. (Cut Point)** A point $x$ of a continuum $X$ is a *cut point* of $X$ if $X - \{x\}$ is not connected.

**Definition 1.14. (Negligible)** A *negligible* set is a set that is small enough that it can be ignored for some purpose.

**Definition 1.15. (Cardinality)** The *cardinality* of a set is defined as the number of elements of the set. The cardinality of a set $X$ is usually denoted $|X|$ and, if $X \subset \mathbb{C}$, is the number of points of the set.

**Definition 1.16. (Perfect Set)** A *perfect* set is a closed set such that every single point in the set is a limit point of the set.

## 1.2 Fractals and fractal dimension

During the late 19th century, mathematicians such as Weierstrass, Koch, Levy, and Cantor began exploring abstract objects that defied traditional geometric principles. These objects, today known as fractals, challenge our intuition and provide a middle ground between the orderly geometry of Euclid and the chaotic roughness of fragmentation. At first, it may seem that creating complex forms would require complex rules, but fractal geometry shows us that even seemingly chaotic shapes can be perfectly ordered.

Before we can define the concept of fractal, it is essential to have a clear understanding of both the topological dimension and the fractal dimension of a space. There are several versions of *topological dimension*, each with its own properties. While they all agree for metric spaces, they can be vastly different for other topological spaces. For the purpose of this explanation, we will focus on the version known as *small inductive dimension*.

By definition, the topological dimension is always an integer value.

*Reminder:* a metrizable space is a topological space that is homeomorphic to a metric space.

**Definition 1.17. (Topological dimension)** The *topological dimension* of $X$, denoted $dim_{\mathcal{T}}(X)$, is defined inductively as follows:

1. $dim_{\mathcal{T}}(X) = -1 \Leftrightarrow X = \emptyset$

2. $dim_{\mathcal{T}}(X) \leq n$ if for every point $x \in X$, $x$ has arbitrarily small neighborhoods $U$ with $dim_{\mathcal{T}}(\partial U) \leq n-1$, where $\partial U$ is the boundary of $U$.

3. $dim_{\mathcal{T}}(X) = n$ if (2) is true for $n$, but false for $n-1$.

4. $dim_{\mathcal{T}}(X) = \infty$ if, for every $n$, $dim_{\mathcal{T}}(X) \leq n-1$ is false.

*Fractal dimension* only has meaning when applied to a metric space and is related to how many sets of a given size are needed to cover the set being studied. In fractal geometry, fractal dimension is a ratio that provides a statistical index of complexity by comparing the level of detail in a pattern as the scale at which it is measured changes. Importantly, a fractal dimension does not have to be a integer number.

There are various methods of defining fractal dimension, and not all are equivalent. Two commonly used definitions are the *Hausdorff dimension* and the *Minkowski dimension*. The Hausdorff dimension is defined for any set, but can be challenging to calculate. On the other hand, determining the Minkowski dimension is more practical and easier to calculate, but it does not always exist for a given set. For the purpose of this work, we will only consider the Minkowski dimension for its ease of calculation.

**Definition 1.18. (Totally bounded)** A subset $K$ of a metric space is called *totally bounded* if for any $\epsilon > 0$, it can be covered by a finite number of balls of diameter $\epsilon$. In the Euclidean space, this is the same as being a *bounded* set.

For a totally bounded set $K$, let $N(K,\epsilon)$ denote the minimal number of sets of diameter at most $\epsilon$ needed to cover $K$.

**Definition 1.19. (Minkowski dimension)** We define the *upper Minkowski dimension* as

$$\overline{\dim}_{\mathcal{M}}(K) = \limsup_{\epsilon \to 0} \frac{\log N(K,\epsilon)}{\log 1/\epsilon}$$

and the *lower Minkowski dimension* as

$$\underline{\dim}_{\mathcal{M}}(K) = \liminf_{\epsilon \to 0} \frac{\log N(K,\epsilon)}{\log 1/\epsilon}$$

If the two values exist and agree, the common value is simply called the *Minkowski dimension* (or *box-counting dimension*) of $K$ and denoted by $\dim_{\mathcal{M}}(K)$.

**Definition 1.20. (Fractal)** A *fractal* is a subset of $\mathbb{R}^n$ whose fractal dimension strictly exceeds its topological dimension.

## 1.3 Topological Models

In planar topology, we come across objects that may seem peculiar to those who are not familiar with the field. These objects not only possess intriguing topological characteristics but also have a certain allure, since they are powerfully and mysteriously attractive and fascinating. This section will provide an overview of some examples of these exotic topological models. For more information, readers can refer to references [1], [20], and [23].

### 1.3.1 Dendrites

Dendrites are typically recognized as the sensitive endings of neurons that act as receptors in the field of neuroscience. However, in mathematics, dendrites are a subject of continuum theory and have numerous applications in the realm of complex dynamics.

**Definition 1.21. (Dendrite)** A *dendrite* is a locally connected continuum that contains no simple closed curves.

In addition to the topological concepts inherent in their definition, dendrites may also exhibit characteristics related to other branches of mathematics. Hereunder we will mention a few results without providing proofs, but these can be found in reference [20].

**Corollary 1.22** ([20, Corollary 10.6])**.** *Every subcontinuum of a dendrite is a dendrite.*

We now introduce the definition of an *arc hedgehog* space that will enable us to present some examples of dendrites.

**Example 1.23.** The *arc hedgehog* space $ah(\omega)$ is a one point union of a shrinking sequence of arcs of length $1/2^n$. It is easy to see that $ah(\omega)$ is uniquely arc-wise connected and is therefore a dendrite.

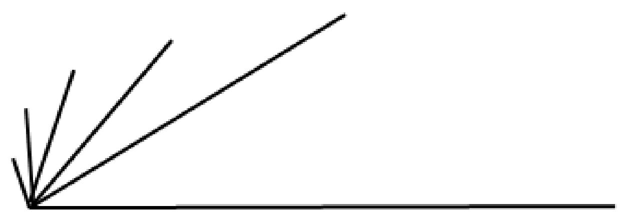

Figure 1.1: The arc hedgehog dendrite.

**Example 1.24.** Let's start with $ah(\omega)$, the arc hedgehog dendrite. If at the midpoint $m$ of a segment of length $1/2^n$ we attach a copy of $ah(\omega)$ scaled to have diameter $1/2^{n+1}$, and we continue the process inductively in a dense pattern, we obtain the following dendrite called *Wazewski's universal dendrite*.

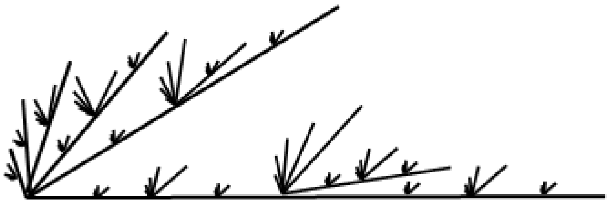

Figure 1.2: The Wazewski's universal dendrite.

Notice there are no open sets at the Wazewski's universal dendrite homeomorphic to an open interval. In fact, this dendrite contains a homeomorphic copy of every dendrite as a retract. Hence, as far as dendrites go, this is as complicated as they get.

### 1.3.2 Cantor Set

The set known today as the *ternary Cantor set* is a set of points in the line first presented by the German mathematician Georg Cantor in 1883. It has various definitions and constructions, but despite being just a subset of the real numbers, it possesses several unique properties.

The study of this set by Cantor and others played a significant role in the development of modern point-set topology.

**Definition 1.25. (Cantor set)** A *Cantor set* is a closed, totally disconnected, and perfect subset of $\mathbb{R}^2$.

The most common and accessible construction is the ternary Cantor set. Let us first describe its construction and its formula, and then show some interesting properties of it.

Figure 1.3: The first six steps of the process of the ternary Cantor Set.

**Example 1.26. (Middle third Cantor set)** The *middle third* (or *ternary) Cantor Set* is created by iteratively deleting the open middle third from a set of line segments.

Consider the interval $I_0$ = [0,1]. Divide $I_0$ into three equal length intervals with end points $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$. We start removing the open middle third $(\frac{1}{3}, \frac{2}{3})$ from the interval $I_0$, leaving two line segments: $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1] = I_1$. Then, remove the middle third of each of these remaining segments, leaving four line segments $[0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1] = I_2$, and continue that process to infinity. Inductively, we construct a sequence of sets $I_n$ which is a union of $2^n$ disjoint closed intervals each of them with diameter $\frac{1}{3^n}$.

The set $\mathcal{C} = \bigcap_{n=1}^{\infty} I_n$ is called the middle third Cantor set. This is therefore the set of points in the interval [0,1] which are not removed at any stage of this endless process shown above in Figure 1.3.

Now, we will see that, indeed, $\mathcal{C}$ is a Cantor set. That is to say that it is a closed, totally disconnected, and perfect subset of [0,1].

**Theorem 1.27.** *$\mathcal{C}$ is a Cantor set.*

*Proof.* (i) $\mathcal{C}$ **is a closed subset in [0, 1]:** Through out the construction of the middle third Cantor set, we note that $I_n$ is a finite union of closed intervals, so it is closed in $I$ = [0, 1]. Now, $\mathcal{C} = \bigcap_{n=1}^{\infty} I_n$ is the intersection of nested closed sets, so it is closed in [0, 1] and nonempty.

(ii) $\mathcal{C}$ **is totally disconnected:** Let $x, y \in \mathcal{C}$ be distinct. Then, $x, y \in I_k$ for all $k \in \mathbb{N}$. Now, since $x$ and $y$ are distinct, we can find $N \in \mathbb{N}$ such that $\frac{1}{3^N} < |x - y|$. Hence, $x$ and $y$ belong to different intervals of $I_N$. By the construction of the middle third Cantor set, there must be at least one interval between $x$ and $y$ which does not belong to $I_N$, and so does not belong to $\mathcal{C}$. Select one such interval. Choosing any point $z$ in this interval satisfies that $z$ lies between $x$ and $y$ and $z \notin \mathcal{C}$. Therefore, $\mathcal{C}$ is totally disconnected.

(iii) $\mathcal{C}$ **is a perfect space:** Let $\epsilon > 0$ be given and consider $B(x, \epsilon)$ for any $x \in \mathcal{C}$. Let $J_k$ denote the interval to which $x$ belongs in $I_k$. We can find $N \in \mathbb{N}$ such that $J_N \subset B(x, \epsilon)$. Now, this interval must have two endpoints $a_N$ and $b_N$ (one of which could possibly be equal to $x$). By the construction of the middle third Cantor set, we know that the endpoints of any interval are never removed, and so $a_N, b_N \in \mathcal{C}$. Furthermore, we have that $a_N, b_N \in J_N \subset B(x, \epsilon)$). Therefore, $x$ is not isolated.

□

From a topological perspective, $\mathcal{C}$ it is a very interesting set due to its unique properties that defy intuition and distinguish it.

**Proposition 1.28.** *(a) $\mathcal{C}$ is **negligible**. In other words, its Lebesgue measure is 0.*

*(b) $\mathcal{C}$ has no **interior points**, that is, it is nowhere **dense**.*

*(c) The cardinality of $\mathcal{C}$ is the **continiuum**. That is, $\mathcal{C}$ has the same cardinality as the interval [0,1].*

*(d) $\mathcal{C}$ is an **uncountable** set.*

*(e) $\mathcal{C}$ is **bounded**.*

*(f) $\mathcal{C}$ is **self-similar** since is exactly or approximately similar to a part of itself.*

*Proof.* Let's prove some of the more interesting properties:

(a) $\mathcal{C}$ is obtained by successively removing intervals. We will measure these intervals removed. At each step the number of intervals doubles and their length decreases by 3. Total length/measure of intervals removed:

$$\frac{1}{3} + 2 \cdot \frac{1}{3^2} + 2^2 \cdot \frac{1}{3^3} + ... = \sum_{n=0}^{\infty} 2^n \cdot \frac{1}{3^{n+1}} = ... = 1$$

"Length"/measure of $\mathcal{C}$ = 1 - 1 = 0.

(b) Since its length is 0, it contains no intervals.

(c) We can represent real numbers in any base. We will use "$_{(n)}$" as notation to specify the number's base, $n$. In this particular case, we will use ternary representation, since the Cantor set has a special representation in base 3: a number is in Cantor's set if and only if its ternary representation contains only the digits 0 and 2.

$$\mathcal{C} = \{x \in [0,1] : x = 0.c_1c_2c_3 \dots c_n \dots_{(3)} \ \textit{where } c_n = 0 \textit{ or } 2\}$$

The function $f : \mathcal{C} \to$ [0,1] defined by:

$$f(0.c_1c_2c_3 \dots c_n \dots_{(3)}) := 0.\frac{c_1}{2}\frac{c_2}{2}\frac{c_3}{2} \dots \frac{c_n}{2} \dots_{(2)}$$

is surjective, so $card(\mathcal{C}) \leq card([0,1])$. But, clearly, $card(\mathcal{C}) \geq card([0,1])$. Hence, $card(\mathcal{C}) = card([0,1])$, as we wanted to prove.

(d) Direct consequence of (c).

$\square$

We can observe the Cantor set as a key example to understanding fractal dimensions. The Cantor set has topological dimension of zero, but yet it has the same cardinality as the real line, in that sense we would expect its dimension to be one. On the other hand, the Cantor set has no interval in it, and in that sense we'd expect its dimension to be zero. The answer then, lies somewhere in the middle. The Cantor set should have a dimension greater than zero, but smaller than one.

To compute the Minkowski dimension of the Cantor set, we cover it with smaller and smaller boxes, taking the box scaling based on the natural size structure of the fractal. That is, we use boxes of side length $1/3, 1/3^2, 1/3^3$, etc. Then, we have that $N(1/3) = 2$, $N(1/9) = N((1/3)^2) = 4 = 2^2$, $N(1/27) = N((1/3)^3) = 8 = 2^3$, *ad infinitum*. From the relation $N((1/3)^n) = 2^n$ we see:

$$\dim_{\mathcal{M}}(C) = \lim_{n\to\infty} \frac{\log N((1/3)^n)}{\log(1/((1/3)^n))} = \lim_{n\to\infty} \frac{\log(2^n)}{\log(3^n)} =$$

$$= \lim_{n\to\infty} \frac{n\log(2)}{n\log(3)} = \frac{\log(2)}{\log(3)} \approx 0.63$$

### 1.3.3 Sierpiński Carpet

The *Sierpiński carpet* is one of the best known planar, compact and connected sets, first described by Wacław Sierpiński in 1916. Sierpiński's work in set theory and topology was extensive. He spent much effort on giving a topological characterization of the continuum (the set of real numbers) and in this way discovered many examples of topological spaces with unexpected properties, of which the Sierpiński carpet is the one of the most famous.

**Definition 1.29. (Sierpińsky carpet)** The *Sierpiński carpet* is a plane fractal, a generalization of the Cantor set to two dimensions, formed by repeated subdivision of a square.

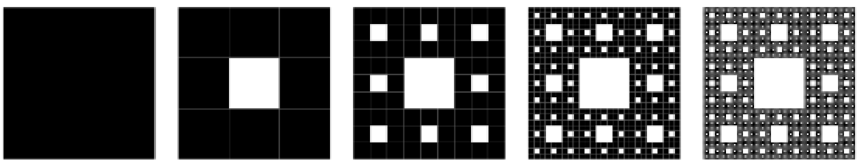

Figure 1.4: The first five steps of the process of the Sierpiński Carpet.

The construction of the Sierpiński carpet starts with the unit square. This square is divided into 9 congruent sub-squares in a grid of 3 by 3, and the central sub-square is removed. The same procedure is then recursively applied to the remaining 8 sub-squares, *ad infinitum*.

Note that all of the open squares removed during the construction of the Sierpinński carpet have boundaries that are pairwise disjoint simple closed curves. Indeed, the lines $x = 1/2$ and $y = 1/2$ meet the Sierpiński carpet in a middle third Cantor set, with the endpoints of this Cantor set providing the intersections of the boundaries of removed squares.

In 1958, Gordon Whyburn [23] uniquely characterized the Sierpińsky carpet as follows:

**Theorem 1.30** ([23, Theorem 3])**. (Whyburn's Theorem)** *Any nonempty planar set that is compact, connected, locally connected, nowhere dense, and has the property that any two complementary domains are bounded by disjoint simple closed curves is homeomorphic to the Sierpiński carpet.*

While this set may at first look rather tame, its topology is actually quite rich: the Sierpiński carpet contains a homeomorphic copy of any compact, connected one (topological) dimensional planar set, no matter how complicated that set is. Basically, any compact planar curve can be homeomorphically manipulated so that it fits inside the carpet. As an example, the eccentric curve in Figure 1.5 fits neatly within the carpet.

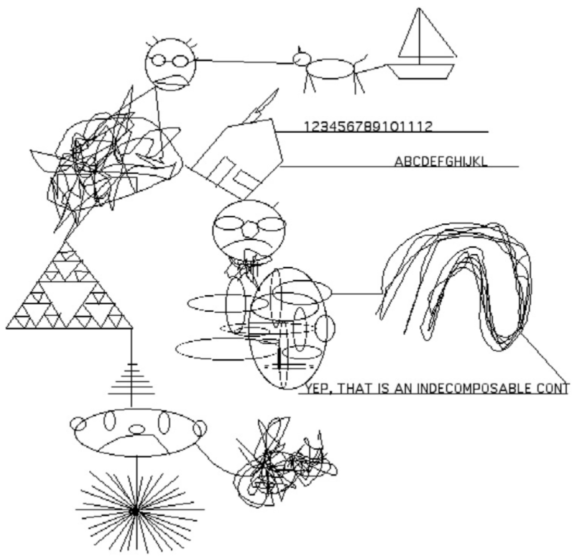


Figure 1.5: Curve that fits inside the Sierpiński carpet [10].

In this sense, the Sierpiński curve is a *universal planar continuum*. Sets with this property are known as *Sierpiński curves*.

By *curve* we are referring to a one dimension topological object.

**Definition 1.31. (Sierpińsky curve)** A *Sierpiński curve* is a planar set that is homeomorphic to the renowned Sierpiński carpet fractal.

It is easy to check that the carpet is compact, connected, locally connected, and nowhere dense in the plane. Other noteworthy characteristics include the following.

**Proposition 1.32.** *(a) The area of the carpet is zero (in standard Lebesgue measure).*

*(b) The interior of the carpet is empty.*

*Proof.* (a) Denote as $a_i$ the area of iteration $i$. Then $a_{i+1} = \frac{8}{9}a_i$. So $a_i = (\frac{8}{9})i$, which tends to 0 as $i$ goes to infinity.

(b) Suppose by contradiction that there is a point $p$ in the interior of the carpet. Then there is a square centered at $p$ which is entirely contained in the carpet. This square contains a smaller square whose coordinates are multiples of $\frac{1}{3^k}$ for some $k$. But, if this square has not been previously removed, it must have been holed in iteration $k+1$, so it cannot be contained in the carpet - a contradiction.

□

We will now determine the Minkowski dimension of the Sierpiński carpet, $S$, using the same method as we did for the Cantor set. Note that, in this case, at the $n^{th}$ step of the process, the side of the tiling squares is $3^{-n}$ but there are $8^n$ tiles in $S_n$, which means that $N((1/3)^n) = 8^n$. Therefore,

$$\dim_{\mathcal{M}}(S) = \lim_{n\to\infty} \frac{\log N((1/3)^n)}{\log(1/((1/3)^n))} = \lim_{n\to\infty} \frac{\log(8^n)}{\log(3^n)} = $$
$$= \lim_{n\to\infty} \frac{n\log(8)}{n\log(3)} = \frac{\log(8)}{\log(3)} \approx 1.892789$$

### 1.3.4 Cantor Bouquet and Straight Brush

In this last section, we introduce a structure consisting of uncountably many pairwise disjoint curves, known as the *Cantor bouquet*. To describe the structure of a Cantor bouquet, we need to introduce the notion of a *straight brush* since, following Aarts and Oversteegen [1], a Cantor bouquet is any planar set that is homeomorphic to a straight brush.

**Definition 1.33. (Straight brush)** Let $\mathcal{B}$ be a subset of $[0, \infty) \times \mathcal{I}$ where $\mathcal{I}$ is a dense subset of the irrational numbers. The set $\mathcal{B}$ is a *straight brush* if it has the following three properties:

1. *Hairiness*. For each point $(x, \alpha) \in \mathcal{B}$, there is a $t_\alpha \in [0, \infty)$ such that $\{t \mid (t, \alpha) \in \mathcal{B}\} = [t_\alpha, \infty)$. The point $(t_\alpha, \alpha)$ is the *endpoint* of the *hair* given by $[t_\alpha, \infty) \times \{\alpha\}$.

2. *Endpoint density*. For each $(x, \alpha) \in \mathcal{B}$, there exists a pair of sequences $\{\beta_n\}$ and $\{\gamma_n\}$ in $\mathcal{I}$ converging to $\alpha$ from both above and below and such that the corresponding sequences of endpoints $t_{\beta_n}$ and $t_{\gamma_n}$ converge to $x$.

3. *Closed*. The set $\mathcal{B}$ is a closed subset of $\mathbb{R}^2$.

Aarts and Oversteegen [1] have also shown that any two straight brushes are ambiently homeomorphic, i.e., there is a homeomorphism of $\mathbb{R}^2$ taking one brush onto the other. This leads to a formal definition of a *Cantor bouquet*.

**Definition 1.34. (Cantor bouquet)** A *Cantor bouquet* is any subset of the plane homeomorphic to a *straight brush*.

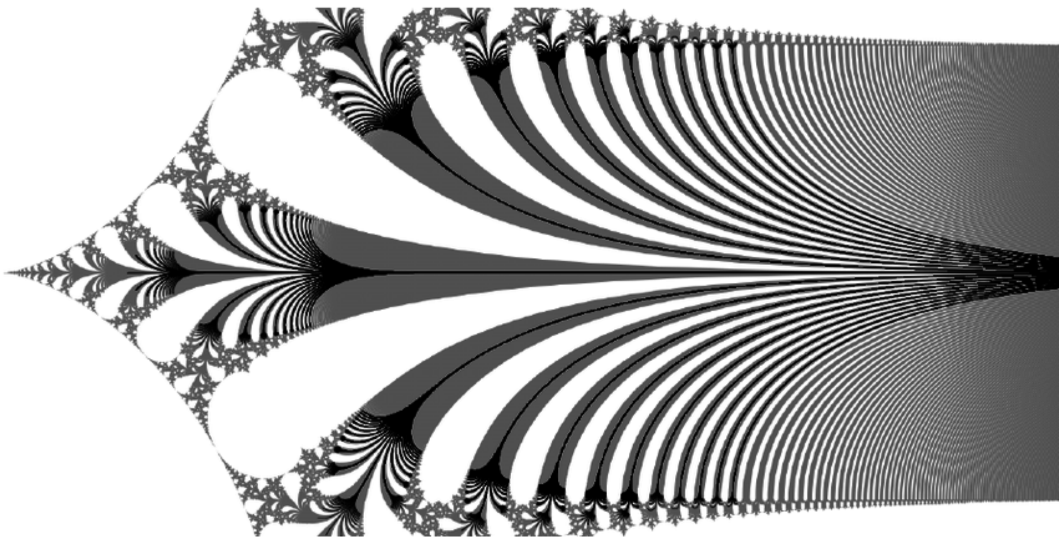

Figure 1.6: Cantor bouquets (in black).

Broadly speaking, a Cantor bouquet is an uncountable collection of disjointed continuous curves tending to $\infty$ in a certain direction in the plane, each of which has a distinct endpoint.

**Definition 1.35. (Crown)** We call the set of endpoints of a Cantor bouquet the *crown*.

Since a Cantor bouquet is homeomorphic to a straight brush with the points at $\infty$ coinciding, it follows that any Cantor bouquet has the amazing connectedness property that the crown together with $\infty$ is connected, but the crown alone is totally disconnected [11].

# Chapter 2

# Complex dynamics: the dynamical partition

The fundamental purpose of the theory of dynamical systems is to understand the eventual or asymptotic behaviour of a process that evolves with time, which may be continuous or discrete.

If this process is a differential equation whose independent variable is time, the theory tries to predict the final behavior of the solutions of the equation in the distant future ($t \to \infty$) or in the distant past ($t \to -\infty$).

If the process is a discrete process, like iterating a function, then the theory hopes to understand the eventual behavior of the points $x, f(x), f^2(x), \dots, f^n(x) = f \circ \overset{n)}{\dots} \circ f(x)$ as $n$ gets big. In other words, dynamical systems make us wonder where points are going and what they're doing when they get there.

In this chapter, we will try to answer this question at least for dynamical systems of one complex variable. The functions that determine dynamical systems are also called *mappings*, or *maps*, for the sake of brevity. These terms connote the geometrical process of taking one point to another. We will use all these terms synonymously.

## 2.1 Preliminaries

This section contains a series of preliminary results and concepts from complex analysis in one dimension, necessary for the development of the core themes of this project. Most of the content in this chapter can be found in [7] and [19], where the reader can go for further details. Some of the proofs will not be included because they fall outside the scope of the work.

In complex analysis, a *holomorphic function* is, broadly speaking, a complex differentiable function. The complex differentiability condition is very strong, and leads to a particularly elegant computing theory for these functions.

**Definition 2.1. (Holomorphic function)** Let $f : \Omega \to \mathbb{C}$ where $\Omega \subseteq \mathbb{C}$ is an open set. The function $f$ is said to be *(complex) differentiable* at $z_0 \in \Omega$ if the following limit exists

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

In that case, the limit is called the *derivative* of $f$ at $z_0$ and is denoted by $f'(z_0)$. We also say that $f$ is *holomorphic* in $\Omega$ if it is differentiable at every point of $\Omega$.

**Definition 2.2. (Conformal mapping)** A function $g : \mathbb{R}^n \to \mathbb{R}^n$ is called *conformal* at $z_0$ if it preserves angles at this point.

Holomorphic maps are conformal at all points $z_0$ such that $f'(z_0) \neq 0$. Conversely, one can show that conformal maps are holomorphic. From a global point of view, given $U, V \subseteq \mathbb{C}$ two open sets, we say that $f : U \to V$ is conformal if $f$ is holomorphic and bijective (hence, $f'(z) \neq 0$ for all $z \in U$).

We will now present *Riemann's Theorem*, which gives us, under certain conditions, the existence (although it cannot always be found explicitly) of a bijective function between a certain subset of the complex plane and the open ball of radius 1. This result of existence does not mean that an explicit function can be found for each subset. In fact, few sets exist in which Riemann's function can be expressed in terms of elementary functions. Nevertheless, existence is assured for all open simply connected sets, which is quite remarkable since they can be very complicated.

**Theorem 2.3. (Riemann's Mapping Theorem)** *Let $\Omega \subset \mathbb{C}$ be a simply connected region $\Omega \neq \mathbb{C}$. Then there exists a bijective conformal map $f : \Omega \to \mathbb{D}$, where $\mathbb{D}$ is the open unit disk. Furthermore, for any fixed $z_0 \in \Omega$, we can find $f$ such that $f(z_0) = 0$ and $f'(z_0) > 0$. With such specification, $f$ is unique.*

Before stating a definition of the Fatou set and the Julia set of a rational map, we must present some basic ideas in order to broaden the concept of continuity of function families. We will now introduce *normal families*, a notion formulated by P. Montel in 1911, and later used in complex iteration theory works of Fatou and Julia during the same decade. To conclude, we will present *Montel's Theorem*.

**Definition 2.4. (Equicontinuous family)** We say that a family $\mathcal{F}$ of functions on $\Omega \subset \mathbb{C}$ is *equicontinuous* at $z_0 \in \Omega$ if for any $\epsilon > 0$, there is $\delta > 0$ such that if $z \in \Omega$ satisfies $|z - z_0| < \delta$, then $|f(z) - f(z_0)| < \epsilon$ for all $f \in \mathcal{F}$.

A special property of holomorphic dynamical systems is the splitting of the phase space induced by the concept of normal families:

**Definition 2.5. (Normal family)** Let $\Omega \subset \overline{\mathbb{C}}$ be a domain and let $\mathcal{F}$ be a family of holomorphic maps from $\Omega$ to $\overline{\mathbb{C}}$. $\mathcal{F}$ is a *normal family* in $\Omega$ if for any infinite sequence of elements, $f_n \in \mathcal{F}$, $f_n$ has a subsequence converging uniformly on compact sets of $\Omega$ to some limit map.

Note that local uniform convergence on $\Omega$ (with respect the spherical metric) means uniform convergence on compact subsets of $\Omega$, so it is sufficient to check normality on open discs on $\Omega$. Moreover, by the well-know Weierstrass Theorem, the limit function of the convergent subsequence is a holomorphic map.

In fact, the condition of normality can be phrased in terms of equicontinuity by the Arzeltà-Ascoli theorem:

**Theorem 2.6. (Arzeltà-Ascoli Theorem)** *Let $\Omega \subset \mathbb{C}$ be compact set and $\mathcal{F}$ be a family of continuous functions on $\Omega$ that is uniformly bounded. Then the following statements are equivalent:*

*(i) $\mathcal{F}$ is equicontinuous at each point of $\Omega$.*

*(ii) Each sequence of functions in $\mathcal{F}$ has a subsequence that converges uniformly on $\Omega$.*

An easy way to check normality is to apply Montel's Theorem.

**Theorem 2.7. (Montel's Theorem)** *Let $\Omega \subset \overline{\mathbb{C}}$ be a domain, and let $\mathcal{F}$ be a family of holomorphic functions from $\Omega$ to $\overline{\mathbb{C}}$. If there are three points $a, b, c \in \overline{\mathbb{C}}$ that are omitted by every $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family in $\Omega$.*

In particular, if the family $\mathcal{F}$ of holomorphic functions from $\Omega$ to $\overline{\mathbb{C}}$ is uniformly bounded, then $\mathcal{F}$ is a normal family.

Normally we will consider maps defined on domains, but this can be generalized to Riemann surfaces, concept that we will introduce in the upcoming section.

## 2.2 Iteration. The Riemann Sphere

The goal of this section is to introduce iteration of holomorphic maps in one complex variable. Often, when working with rational maps, $\infty$ can be considered as any other point of the plane. To this end, we intend to extend the complex plane $\mathbb{C}$ to the *Riemann sphere* by adding the point at infinity, i.e.

$$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

That being said, we can treat infinity as an extra point of the plane and operate with it. Considering the whole as a sphere we may find ourselves with functions which are perfectly tamed and well behaved everywhere, for example, dividing by infinity, or expressions such as $\frac{1}{0} = \infty$ are properly defined within the sphere.

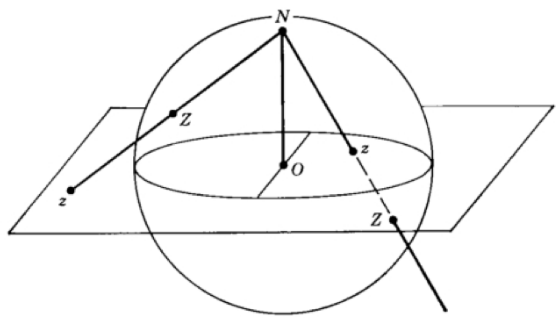


Figure 2.1: Riemann's Sphere.

This insight may be clarified: let's think of the Euclidean sphere

$$S_2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

and the complex plane, $\mathbb{C}$, holding its equator. For any point $z$ on this equatorial plane, we can trace a straight line that connects it with the north pole of the sphere, i.e. the point $N = (0, 0, 1)$, and this line will eventually cross the sphere. If $z$ is external to the sphere (points outside the unit disc $\mathbb{D} \in \mathbb{C}$), it will cross its northern hemisphere. If $z$ lies within the sphere (points inside $\mathbb{D}$), the line will cross its southern hemisphere. And if $z$ lies on the sphere, then he is particularly on the equator, and he himself will be the point of intersection. This method of relating exactly one point on the sphere to each point of the plane is called *stereographic projection*, $\pi$, and at the points $\pi(z)$ can be defined in Cartesian coordinates as

$$\pi(z) = \left( \frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right)$$

where $x = Re(z)$, and $y = Im(z)$, are the real and imaginary part of $z \in \mathbb{C}$, respectively. With this Riemann's model, $\infty$ is close to the numbers with a very large module, whereas the 0 point is close to the numbers of very small module.

The Riemann sphere, as a differentiable variety, is a compactification of the complex plane by the addition of the point of infinity. Being a compact surface, sometimes makes it better or simply easier to work on than $\mathbb{C}$.

When working in the Riemann sphere we shall use the *spherical metric* defined as the following:

**Definition 2.8. (Spherical length)** Suppose $\gamma : [0,1] \to \overline{\mathbb{C}}$ is a path in $\overline{\mathbb{C}}$. The *spherical length* of $\gamma$ is defined as

$$l(\gamma) := 2\int_{\gamma} \frac{|dz|}{1+|z|^2} = 2\int_0^1 \frac{|\gamma'(t)|}{1+|\gamma'(t)|^2}dt.$$

**Definition 2.9. (Distance)** Let $z_1, z_2 \in \overline{\mathbb{C}}$, and let $\Gamma$ be the set of all paths in $\overline{\mathbb{C}}$ from $z_1$ to $z_2$. Then, the distance from $z_1$ to $z_2$ in the *spherical metric* is defined as

$$\sigma(z_1, z_2) := \inf_{\gamma \in \Gamma} l(\gamma).$$

More intuitively, this is the shortest distance to travel from $z_1$ to $z_2$ if we think of these points as being on the Riemann sphere, and we can only travel on the Riemann sphere itself. We cannot "drill" a straight line from $z_1$ to $z_2$, since this process would correspond to the *cordal metric* (which is another possible metric).

All in all, if we have a rational map $f : \mathbb{C} \to \mathbb{C}$, with $f(z) = P(z)/Q(z)$ where $P$ and $Q$ are complex polynomials with no common factors, and the degree of $f$ is

$$d = max(deg(P), deg(Q)) \geq 2,$$

we can extend $f$ to $\overline{\mathbb{C}}$ by defining $f(p_i) = \infty$ (where $p_i$ are the zeros of $Q$), and

$$f(\infty) = \lim_{z\to\infty} \frac{P(z)}{Q(z)}.$$

By Morera's theorem, the extension $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is holomorphic. Conversely, one can show that every holomorphic map of $\overline{\mathbb{C}}$ must be a rational function [5].

Observe that maps with an essential singularity like $e^z$ cannot be extended continously to $\infty$.

In general, in this work we will only consider rational maps $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ (where polynomials are the special case for which $f^{-1}(\infty) = \{\infty\}$), or entire transcendental functions $f : \mathbb{C} \to \mathbb{C}$ (i.e. functions with an essencial singularity at $\infty$). To unify this notation, from now on we will consider dynamical systems generated by the iteration of holomorphic mappings $f : S \to S$, where $S \in \{\mathbb{C}, \overline{\mathbb{C}}\}$.

**Definition 2.10. (Orbit)** Given a map $f : S \to S$, the (*forward*) *orbit* of a point $z_0$ under $f$ is the sequence of iterates

$$O^+(z_0) = \{z_n = f^n(z_0)\}_{n\in\mathbb{N}}.$$

The *backward orbit* of $z_0$ is the set

$$O^-(z_0) = \{z : f^n(z) = z_0\}_{n\in\mathbb{N}}.$$

**Definition 2.11. (Grand orbit)** Given a set $f : S \to S$ and a point $z \in S$, its *grand orbit* consists of all points in $S$ which are related forwards or backwards with $z$ under iteration of $f$. More precisely,

$$GO(z) = \{w \in \overline{\mathbb{C}} \mid f^p(z) = f^q(w) \text{ for some } p,\ q \in \mathbb{N}\}.$$

Understanding the dynamical system generated by the iterates of $f$ means understanding the fate of all orbits in terms of their initial condition, i.e. their asymptotic behaviour when the time $n$ tends to $\infty$.

## 2.3 The dynamical parition: Julia and Fatou sets

The purpose of this section is to show a few classic results regarding the properties of Julia and Fatou sets, as well as to give a clear definition of these two new concepts. Although we will prove almost all the lemmas listed in this section, their proofs may also be found in [3] or [19], among other interesting results related to the subject.

Let $f : S \to S$ be a holomorphic mapping where $S \in \{\mathbb{C},\ \overline{\mathbb{C}}\}$, and let $\{f^n\}_{n\in\mathbb{N}}$ denote the $n^{th}$-iterate. By classifying the points of $S$, our surface is naturally partitioned into two disjointed invariant subsets, one that behaves tamely (the set of normality), and the other that behaves in a chaotic manner (its complement).

**Definition 2.12. (The Fatou and Julia Sets)** We define the *Fatou* set $\mathcal{F}(f)$ of a given map $f : S \to S$, as the set of points $z_0 \in S$ such that $\{f^n\}$ is a normal family in some neighborhood of $z_0$. The complement of the Fatou set is the *Julia* set, $\mathcal{J}(f)$.

Since the map is clear from the context, we will use throughout this chapter the notation $\mathcal{J} = \mathcal{J}(f)$ for Julia's set, and $\mathcal{F} = \mathcal{F}(f)$ for Fatou's set.

**Example 2.13.** An outstanding family of Julia sets is obtained from simple quadratic functions: $f_c = z^2 + c$, where $c$ is a complex number. Julia sets obtained from this function are denoted by $\mathcal{J}_c$.

Let $z_n = Q_c^n(z_0)$, where $z_0$ is an initial condition in $\mathbb{C}$. It can be shown that if $|z_n| > 2$, then the orbit diverges to $\infty$ and the point $z$ does not belong to the Julia set (see Chapter 3). Therefore, it is enough to find a single term of the sequence that verifies $|z_n| > 2$ to be certain that $z_0$ is not in the Julia set.

In the images below, colors give an indication of the speed with which the sequence diverges (its module tends to infinity): in orange, after few calculations it is known that the point is not in the Julia set; and in green, it has taken much longer to prove it.

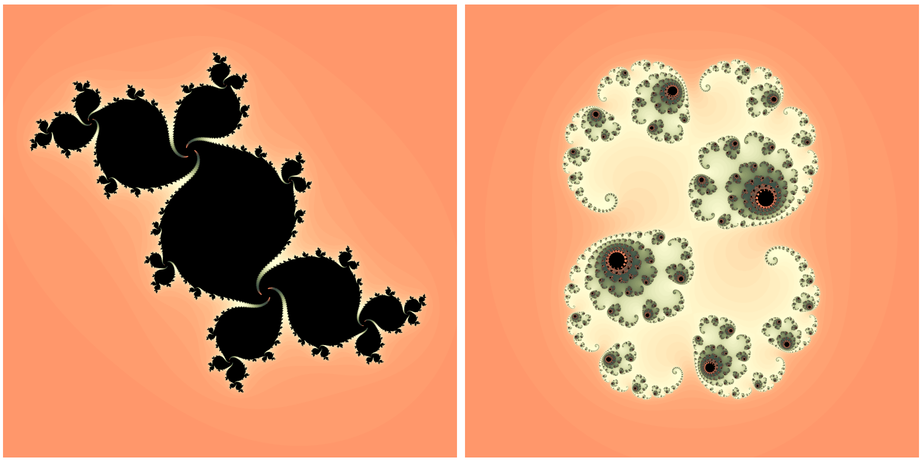

Figure 2.2: Julia sets of $Q(z) = z^2 + (-0.09 + 0.68i)$ and $Q(z) = z^2 + (0.34 + 0.04i)$

Here we have defined the Fatou set as the normality set of the family of iterates of a rational map, but we could also have done as the equicontinuity set of this family. As we already saw, the two concepts are perfectly interchangeable.

A property shared by Fatou and Julia sets is the complete invariance.

**Lemma 2.14. (Invariance Lemma)** *The Julia set $\mathcal{J}$ and the Fatou set $\mathcal{F}$ of a holomorphic map $f : S \to S$ are completely invariant under f. That is, $z \in \mathcal{J}$ if and only if $f(z) \in \mathcal{J}$, and the same holds for $\mathcal{F}$.*

*Proof.* Once we see that Fatou's set is invariant by $f$, the case of Julia's set is automatically proven since $\mathcal{J}(f) = \overline{\mathbb{C}} \backslash \mathcal{F}(f)$.

($\Rightarrow$) Suppose that $z_0 \in \mathcal{F}(f)$. By definition, there is an open set $U \subseteq S$ such that $z_0 \in U$ and $\{f^n\}$ is a normal family in $U$. Then $f(z_0) \in f(U)$, being $f(U)$ an open neighborhood if $f(z_0)$ since $f$ is open.

Let $\{f^{n_k}\}$ be a sequence in $U$. There exists a subsequence $f^{n_{k_j}-1} \rightrightarrows g$ in $k$, which implies that $f^{n_{kj}} \rightrightarrows f \circ g$ in $f(U)$. Subsequently, $\{f^n\}$ is normal in $f(U)$.

($\Leftarrow$) The process would be done backwards in an analogous way. $\square$

**Lemma 2.15.** *The Julia set $\mathcal{J}$ is closed, and the Fatou set $\mathcal{F}$ is open.*

*Proof.* Trivial by the definition of $\mathcal{J}$ and $\mathcal{F}$: the Fatou set is the biggest open set of normality, and therefore the Julia set is closed and compact (since $\overline{\mathbb{C}}$ is compact). $\square$

**Lemma 2.16. (Iteration Lemma)** *For any $k \in \mathbb{N}$, $\mathcal{J}(f^k) = \mathcal{J}(f)$.*

*Proof.* As in the preceding case, we can also work on the lemma with the Fatou set. We want to see that $z \in \mathcal{F}(f^k) \Leftrightarrow z \in \mathcal{F}(f)$. Or, equivalently, that: every partial of $\{f^{kn}\}$ has a convergent partial $\Leftrightarrow$ every partial of $\{f^n\}$ has a convergent partial.

($\Leftarrow$) For any $n_j \to \infty$, we know that $\{f^{n_j}\}$ has a convergent partial, and we want to see that $\{f^{kn_i}\}$ has a convergent partial too. In fact, this is obvious since we can define $kn_i = n_j$, which implies that $\{f^{kn_i}\} = \{f^{n_j}\}$ is partial of $\{f^n\}$, and by hypothesis we know every partial of $\{f^n\}$ has a convergent partial.

($\Rightarrow$) We know that $\{f^{kn_i}\}$ has a convergent partial, and we want to see that every $\{f^{n_j}\}$ has it as well. Given $\{f^{n_j}\}$, we can divide it into $k$ groups, for $k \in \mathbb{N}$, so that $\exists\ i = 0, 1, \ldots, k-1$ such that $n_j = km_j + i$ for infinite indices $j$. Since there are infinite elements, there must be one of these groups that has infinite elements.

If we know that for every infinite $\{f^{kn_i}\}$, there is a convergent partial, it is also true that any $\{f^{kn_i+l}\}$ there is a convergent partial, since:

$$\{f^{kn_i}\} \rightrightarrows g \Leftrightarrow \{f^{kn_i+l}\} \rightrightarrows f^l \circ g$$

Therefore, let $\{f^{km_j+i}\}$ be the group with infinite elements. By hypothesis, we have a convergent partial. But it is also a partial sequence of $\{f^{n_j}\}$, therefore $\{f^{n_j}\}$ has a convergent subsequence.

$\square$

**Lemma 2.17. (Transitivity / Blow-up Property)** *Let $z_0$ be any point in the Julia set $\mathcal{J}(f) \subseteq \overline{\mathbb{C}}$, and let N be an arbitrary neighborhood of $z_0$. Then, the union U formed by all iterates $f^n(N), \forall n \geq 0$, contains the entire Julia set and contains all but at most two points of $\overline{\mathbb{C}}$.*

*Proof.* We will argue through contradiction. Assume that three values have been omitted. Since the family of maps is not normal (due to the fact that z belongs to the Julia set), we use Montel's theorem, and we reach the contradiction.

$\square$

In order to be able to continue stating and proving more properties, we need to characterize certain types of points that have a special behaviour and features.

**Definition 2.18. (Periodic point, periodic orbit, fixed point and multiplier)**

An element t $z_0 \in \mathbb{C}$ is said to be a *periodic point* of $f$ of period $p \geq 1$ if

$$f^p(z_0) = z_0,$$

and, for every $j < p$, $f^j(z_0) \neq z_0$. In this case, the orbit of $z_0$, called *periodic orbit* or *cycle*, is given by the finite set

$$<z_0> := \{z_0, z_1, \ldots, z_{p-1}\}.$$

If $p = 1$, $z_0$ meets the property $f(z_0) = z_0$, and it is called a *fixed point* for $f$.

The number $\lambda = f'(z_0)$ is called the *multiplier* of $f$ at $z_0$.

**Definition 2.19. (Multiplier of a periodic orbit)** We define the *multiplier of the orbit* as the number

$$\lambda = (f^p)'(z_0) = f'(z_0)f'(z_1)\ldots f'(z_{p-1}) = \prod_{i=o}^{p-1} f'(z_i)$$

**Definition 2.20. (Preperiodic point)** An element $z_0 \in \mathbb{C}$ is called a *preperiodic point* of $f$ if $f^p(z_0)$ is periodic for some $p \in \mathbb{N}$ and *strictly preperiodic* if it is preperiodic but not periodic.

Though we only took into account two types of special orbits, in certain cases we can determine how the neighboring points of the periodic point will behave when they are iterated by the same function $f$.

**Definition 2.21. (Classification of fixed points)** We classify the fixed points according to $f'(z_0) = \lambda$, their *multiplier*, as follows:

- ***Attracting*** if $|\lambda| < 1$. In particular, if $\lambda = 0$, we refer to a ***superattracting*** fixed point.
- ***Repelling*** if $|\lambda| > 1$.
- ***Rationally neutral or parabolic*** if $|\lambda| = 1$ and for some integer n, $\lambda^n = 1$.
- ***Irrationally neutral*** if $|\lambda| = 1$ and $\lambda^n$ is never equal to 1.

**Definition 2.22. (Basin of attraction)** We define the *basin of attraction* of an attracting $p$-periodic orbit $<z_0> = \{z_0, z_1, \ldots, z_{p-1}\}$ as the set of points that tend to the orbit under iteration of $f$, i.e.

$$\mathcal{A}(<z_0>) = \{z \in \overline{\mathbb{C}} \mid f^{np} \to z_i \text{ as } n \to \infty, \text{ for some } 0 \leq i \leq p-1\}.$$

The union of the connected components of $\mathcal{A}(<z_0>)$ which contain the cycle is denoted by $\mathcal{A}^*(<z_0>)$ and it's called the *immediate basin of attraction* of $<z_0>$.

**Observation 2.23.** *$\mathcal{A}(<z_0>)$ is open, because it is the union of the backwards iterates $f^{-n}(D(z_0, \epsilon))$, for a given small $\epsilon < 0$.*

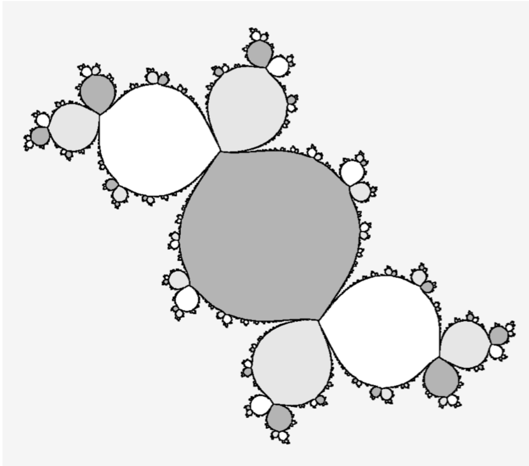

Figure 2.3: Julia set with an attracting cycle.

We present the following theorem as a compendium of basic properties of Julia and Fatou sets.

**Theorem 2.24. (Julia and Fatou Sets Properties)** *Let $f : S \to S$ be a holomorphic map, and $\mathcal{F}(f)$ and $\mathcal{J}(f)$ its Fatou and the Julia sets, respectively. Then:*

*(i) If $z_0$ belongs to $\mathcal{J}(f)$, the set of all preimages of $z_0$ is dense on $\mathcal{J}(f)$.*

*(ii) If $A \subset \overline{\mathbb{C}}$ is the basin of attraction of some attracting periodic orbit, then the topological boundary $\partial A = \overline{A} \backslash A$ is equal to the entire Julia set. Moreover, every connected component of the Fatou set $\overline{\mathbb{C}} \backslash \mathcal{J}(f)$ either coincides with some connected component of this basin $A$ or else is disjoint from $A$.*

*(iii) If the Julia set contains an interior point, then it must be equal to the entire Riemann Sphere $\overline{\mathbb{C}}$.*

*(iv) If $D$ is a union of components of $\mathcal{F}$ that is completely invariant, then $\mathcal{J} = \partial D$.*

*Proof.* (i) This fact is a consequence of the Blow-up Property. Let be $w \in \mathcal{J}(f)$ and $U$ a neighborhood of $w$. We shall prove that $U$ contains any preimage of $z_0$. Given that the images of $U$ should cover all $\overline{\mathbb{C}}$ (except, at most, two points), it must exist $N > 0$ such that $z_0 \in f^N(U)$. This yields that $U$ has some point that, under $N$ iterates, is sent to $z_0$; which is the definition of preimage of $z_0$.

(ii) If $U$ is any neighborhood of a point of the Julia set, then, by Transitivity Lemma, implies that some $f^n(U)$ intersects $A$, hence $U$ itself intersects $A$.

This proves that $\mathcal{J}(f) \subset \overline{A}$. But $\mathcal{J}(f)$ is disjoint from $A$, so it follows that $\mathcal{J}(f) \subset \partial A$.

On the other hand, if $U$ is a neighborhood of a point of $\partial A$, then any limit of iterates $f^n_{|U}$ must have a jump discontinuity between $A$ and $\partial A$, therefore $\partial A \subset \mathcal{J}(f)$.

Finally, one can see that any connected Fatou component which intersects $A$ must coincide with some component of $A$, since it cannot intersect the boundary of $A$.

(iii) If $\mathcal{J}(f)$ has an interior point $z_1$, then choosing a neighborhood $N \subset \mathcal{J}(f)$ of $z_1$, the union $U \subset \mathcal{J}(f)$ of forward images of $N$ is everywhere dense, $U \subset \overline{\mathbb{C}}$. Since $\mathcal{J}(f)$ is a closed set, it follows that $\mathcal{J}(f) = \overline{\mathbb{C}}$.

(iv) This fact is directly a consequence of being $\partial D$ a subset of $\mathcal{J}(f)$ (by Iteration Lemma).

□

**Lemma 2.25.** *The attracting periodic points and their basins of attraction belong to the Fatou set.*

*Proof.* As we mentioned in the last observation, the basin of attraction of any attracting periodic point of period $p \geq 1$ is an open set. Therefore, if $z$ belongs to the basin of attraction of a fixed point $z_0$, for any neighborhood $U$ sufficiently small (inside the basin of attraction), the iterates of $f$ converge on $U$ towards the constant function $g(z) \equiv z_0$. Hence, $z$ is normal and belongs to $\mathcal{F}(f)$.

On the other hand, if $z$ belongs to the basin of attraction of a periodic point $z_0$ with period $p > 1$, we can use the same argument over the function $h = f^p$.

□

**Lemma 2.26** ([19, Lemma 4.6])**.** *The repelling points belong to Julia set. Furthermore, they form a dense set on $\mathcal{J}(f)$. In other words,*

$$\mathcal{J}(f) = \overline{\{z \in \overline{\mathbb{C}} : z \textit{ is a repelling periodic point}\}}.$$

Recall that a periodic point $z_0 = f^p(z_0)$ is said to be *parabolic* if its multiplier $\lambda$ is a root of 1, but no iterate of $f$ is the identity map. Parabolic periodic points also come equipped with a basin of attraction.

**Definition 2.27. (Parabolic Component - Parabolic cycle)** A $p$-periodic component $U$ of the Fatou set $\mathcal{F}$ is called *parabolic* if there is a neutral fixed point $\zeta$ for $f^p$ on its boundary such that all points in $U$ converge to $\zeta$ under iteration by $f^p$. The domains $U, f(U), ..., f^{p-1}(U)$ form a *parabolic cycle* of Fatou components.

**Lemma 2.28.** *Every parabolic periodic point belongs to the Julia set.*

*Proof.* By a suitable change of coordinates, we may assume that the periodic point is $z = 0$. Therefore, some iterate $f^n$ will be written as $z + a_p z^p + H.O.T$, where $p \geq 2$, and $a_p \neq 0$ with $p$-th derivative $(p!)a_p$. It follows that $f^{nk}$ will likewise be written as $z + ka_p z^p + H.O.T$. Thus, the $p$-th derivative of $f^{nk}$ for an arbitrary $k \in \mathbb{N}$ at the origin is $(p!)a_p k$, which diverges to infinity as $k \to \infty$. It follows by Weierstrass Uniform Convergence Theorem [19, Theorem 1.4], that no subsequence $\{f^{nk_j}\}$ can converge locally uniformly as $k_j \to \infty$, so we obtain the non-normality.

□

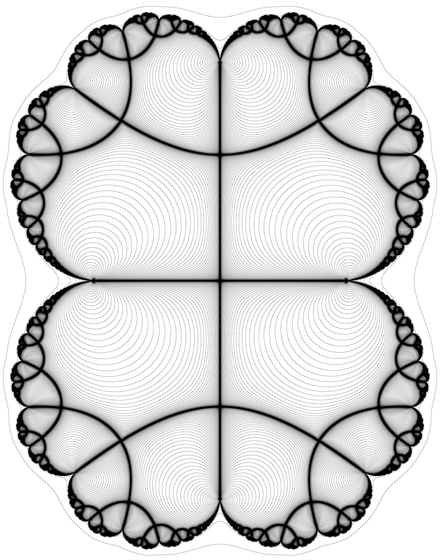

Figure 2.4: Julia set with a parabolic cycle.

**Lemma 2.29. (Not empty)** *If deg$f \geq 2$, $\mathcal{J}$ is not empty.*

*Proof.* For the sake of simplicity, we shall only prove this for the rational case.

Let $f$ be a rational map of degree $d \geq 2$. Arguing by contradiction, suppose that the Julia set is empty, $\mathcal{J} = \emptyset$. Then, $\{f^n\}$ is a normal family on all $\overline{\mathbb{C}}$, so there is a uniformly convergent subsequence $\{n_k\}$, $k \in \mathbb{N}$, such that $f^{n_k} \to g$ for $g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. Since $f^n$ are holomorphic, the limit map $g$ is holomorphic too, so it's either the constant map $\infty$ or else a rational map.

If $g$ is constant, the image of $f^{n_k}$ is eventually contained in a small neighborhood of the constant value $\infty$, which is impossible since $f^{n_k}$ covers the whole $\overline{\mathbb{C}}$.

If $g$ is not constant, eventually $f^{n_k}$ has the same number of zeros as $g$ (this follows from the Argument Principle), which is also impossible, since $d \geq 2$ and

$$deg(f^{n_k}) = d^{n_k} \to \infty,$$

which implies that we have a rational function with not finite degree (contradiction). □

As in certain cases holomorphic functions may have *exceptional points*, it is necessary to define the concept of *exceptional set* of a rational map $f$ in order to prove that the Julia set is perfect.

We always consider functions of degree $\geq 2$. Consequently, the set of preimages of a point is always infinite, with a few exceptions.

**Definition 2.30. (Exceptional points - Exceptional set)** Given a set $f : S \to S$, a point $z \in \overline{\mathbb{C}}$ is *exceptional* under $f$ if its grand orbit, $GO(z) \subseteq \overline{\mathbb{C}}$, is a finite set.

We define the *exceptional set* $\mathcal{E}(f)$ as the set of points with a finite grand orbit, i.e., the set of exceptional points.

One can prove that if $f$ is a rational map of degree two or more, then the set $\mathcal{E}(f)$ of exceptional points can have, at most, two elements. These exceptional points, if they exist, must be critical points of $f$, and they must belong to the Fatou set of $f$ (see [19]). For example, in polynomials, infinity is exceptional since it has no preimage, and it is a superattracting fixed point .

**Lemma 2.31. (Perfect)** *The Julia set $\mathcal{J}$ is perfect, that is it is closed with no isolated points.*

*Proof.* Once again, this will only be proven for the rational case.

Let $z_0 \in \mathcal{J}$ be an arbitrary point and let $V$ be an arbitrary neighborhood for such $z_0$. We must prove that there is another point in $V$ which belongs to $\mathcal{J}$.

We will first study the situation in which $z_0$ is not a periodic point.

As $z_0$ is not periodic, we know that $f(z_0)$ cannot contain $z_0$, and thus we can choose some $z_1 \in f^{-1}(z_0)$ that satisfies $z_1 \neq z_0$. For all $n$, $f^n(z_0) \neq z_0$, so $f^n(z_0) \neq z_1$. It follows that $V$ contains a point $z_2 \in f^{-k}(z_1)$ for some natural number $k$, since backwards iterates of $z_1$ are dense in $\mathcal{J}$. Thus, $z_2 \in \mathcal{J} \cap V$, $z_2 \neq z_0$.

We now suppose $z_0$ is periodic (of period n) and it is the only solution to the equation $f^n(z_0) = z_0$. Then $z_0$ would be a superattracting fixed point for $f^n$, contradicting $z_0 \in \mathcal{J}(f) = \mathcal{J}(f^n)$.

Lastly, suppose there is an alternative solution $z_1 \neq z_0$ with $f^n(z_1) = z_0$. Furthermore $f^j(z_0) \neq z_1$ for all $j$, since otherwise it would hold for some $0 \leq j < n$

(by periodicity) and therefore $z_0 \neq z_1 = f^j(z_0) = f^{n+j}(z_0) = f^n(z_1) = z_0$, contradicting the minimality of $n$. Therefore, we would be back to the first case: $z_1$ must have a preimage in $\mathcal{J} \cap V$ which cannot be $z_0$.

□

## 2.4 Local and semilocal theory

The contents of this chapter can be found in Chapter II and III of [7], where the reader can go for further details.

A good starting point for understanding the dynamics of holomorphic functions is to consider periodic points and the behaviour that takes place in their surroundings. Given a function $f$, to determine the basin of attraction of an attracting periodic point can sometimes be complex, and we could find ourselves barely understanding the dynamics of its neighbouring points. To fix this, we can make use of simpler functions, whose behavior is already known, in order to extract certain information regarding the original function. We do that through conjugations.

**Definition 2.32. (Conformal conjugacy)** We say that a function $f : U \to U$ is *(conformally) conjugate* to a function $g : V \to V$ if and only if there is a conformal one-to-one map $\varphi : U \to V$ such that $g = \varphi \circ f \circ \varphi^{-1}$, i.e.

$$\varphi(f(z)) = g(\varphi(z))$$

This last equality is called Schröder's equation. In other words, the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\varphi"'] & U \arrow[d, "\varphi"] \\ V \arrow[r, "g"] & V \end{tikzcd}$$

This definition also implies that the iterates $f^n$ and $g^n$ are also conjugated by the same map $\varphi$, i.e., $g^n = \varphi \circ f^n \circ \varphi^{-1}$. Indeed, whenever well-defined, the inverses $f^{-1}$ and $g^{-1}$ are also related by $\varphi$, since $g^{-1} = \varphi \circ f^{-1} \circ \varphi^{-1}$.

We can also verify that conjugacies send orbits to orbits, periodic orbits of period $p$ to periodic orbits of period $p$, fixed points to fixed points, attracting points to attracting points, etc. Hence, we consider the dynamics of conjugate maps to be "the same".

Now, our aim is, according to the multiplier of a fixed point, to get the normal form of a function close to the fixed point. In order to do this, suppose $z = 0$ is a fixed point of $f(z)$ with the multiplier $\lambda$. We may ask ourselves if the dynamics in a neighborhood of $z_0$ is conformally conjugate to $z \mapsto \lambda z$ in a neighborhood of 0, i.e. under which conditions there exists a conformal map $\varphi$ such that $\varphi(z_0) = 0$ and $\varphi(f(z)) = \lambda(\varphi(z))$. This is known as the *linearization problem*, which has different solutions according to the kind of fixed point we are considering.

The easiest fixed points to deal with are the attracting fixed points that are not superattracting, i.e., those whose multiplier lies between $0 < |\lambda| < 1$. In this case, the answer is affirmative.

**Theorem 2.33** ([7, Theorem 2.1]). **(Koenigs Linearization Theorem)** *Suppose $f$ has an attracting fixed point at $z_0$, with its multiplier $\lambda$ satisfying $0 < |\lambda| < 1$. Then, there is a conformal map $\zeta = \varphi(z)$ of a neighborhood of $z_0$ onto a neighborhood of 0, which conjugates $f(z)$ to the linear function $g(\zeta) = \lambda\zeta$. Furthermore, the conjugating function is unique, up to multiplication by a nonzero scalar factor.*

The existence of a conjugating map when $z_0$ is a repelling fixed point follows immediately from the attracting case. Suppose that $f(z) = z_0 + \lambda(z - z_0) + \dots$ such that $|\lambda| > 1$. Then, by the Inverse Function Theorem, the local branch of $f^{-1}(z) = z_0 + (z - z_0)/\lambda + \dots$ has an attracting fixed point at $z_0$. Any map conjugating $f^{-1}(z)$ to $\zeta/\lambda$ also conjugates $f(z)$ to $\lambda\zeta$.

In short, for the attracting and repelling cases, $f$ is locally conformally conjugate to its linear part $z \to \lambda z$. This result is from 1884, and on account of Koenigs.

In the case of superattracting fixed points, $\lambda = 0$, one can also demonstrate the existence of a conjugacy, proved for the first time by L.E. Böttcher in 1904.

**Theorem 2.34** ([7, Theorem 4.1]). **(Böttcher's Theorem)** *Suppose $f$ has a superattracting fixed point at $z_0$*

$$f(z) = z_0 + a_p(z - z_0)^p + \dots, \; a_p \neq 0, \; p \geq 2.$$

*Then there is a conformal map $\zeta = \varphi(z)$ of a neighborhood of $z_0$ onto a neighborhood of 0 which conjugates $f(z)$ to $\zeta^p$. Furthermore, the conjugation is unique, up to a multiplication by a $(p-1)$-th root of the unity.*

This theorem is relevant for polynomials and their dynamics, since every polynomial of degree $\geq 2$ can be extended to the Riemann sphere as a rational function such that it has a superattracting point at infinity.

Throughout this chapter we have seen the existence of conjugations for attracting, repelling and superattracting fixed points. Therefore everything that remains to be considered is the case where $|\lambda| = 1$, that is $\lambda = e^{2\pi i\theta}$, where can either have $\theta$ being rational (parabolic case) or $\theta$ being irrational (irrational case).

To describe the dynamics around the parabolic point (rational case), the following definition of *petals* is required.

**Definition 2.35. (Attracting and repelling petals)** Let $f$ be defined and univalent in a neighborhood $U$ of the origin. An open set $P \subseteq U$ is called an *attracting petal* for $f$ at the origin if:

$$f(\overline{P}) \subseteq P \cup \{0\} \quad \textit{and} \quad \bigcap_{k \geq 1} f^k(\overline{P}) = \{0\}.$$

An open set $P \subseteq U$ is called a *repelling petal* for $f$ at the origin if it is an attracting petal for $f^{-1}$, where $f^{-1}$ is well defined in a neighborhood of 0, mapping 0 to 0.

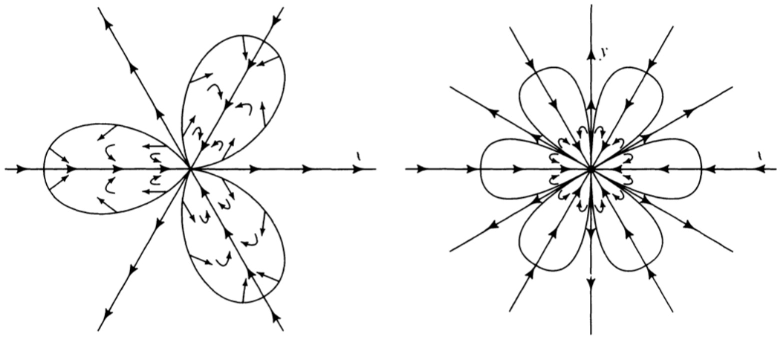

Figure 2.5: Pattern of atracting petals for $z + z^4$ and $-z + z^4$.

**Theorem 2.36** ([19, Theorem 10.7])**. (Leau-Fatou Flower Theorem)** *Let*

$$f(z) = z + az^{m+1} + (H.O.T.) \quad \textit{with } a \neq 0,\ n \geq 1,$$

*be holomorphic in some neighborhood of the origin, then there exist* $2n$ *petals* $P_j$*, where* $P_j$ *is either repelling or attracting depending on whether* $j$ *is even or odd. Furthermore, these petals can be chosen so that the union*

$$\{0\} \cup P_0 \cup \ldots \cup P_{2n-1}$$

*is an open neighborhood of* $z = 0$*. When* $n > 1$*, each* $P_j$ *intersects each of its two immediate neighborhoods in a simply connected region* $P_j \cup P_{j\pm1}$*, but is disjoint from the remaining* $P_k$ *(we consider* $j$ *module* $2n$*).*

From this local description, it is clear that $f$ is not conjugate to its linear part (the identity) in any neighborhood of the fixed point.

For the case $\lambda = e^{2\pi i\theta}$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$, a fixed point is considered to be a *Siegel point* if the local linearization $\varphi(f(z)) = \lambda(\varphi(z))$ is possible. The remaining cases are referred to as *Cremer points* and the dynamics are much more sophisticated. Further details can be found in [7] and [19].

**Definition 2.37. (Siegel Disk)** The maximal neighborhood of the fixed point (*Siegel point*) where the linearization is preserved is called a *Siegel disk*.

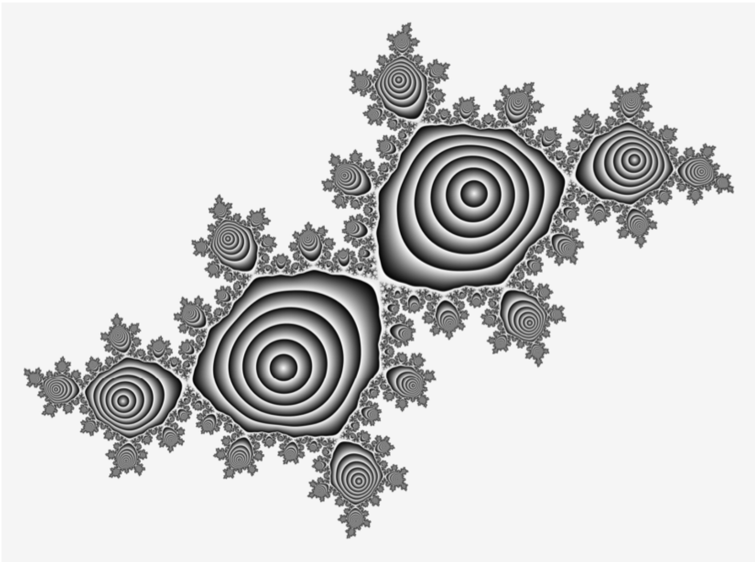

Figure 2.6: Julia set with Siegel disks.

One can see that $p$ is a Siegel point if and only if $p \in \mathcal{F}$ (Montel's Theorem).

Cremer, in 1927, showed that for a generic selection of rotation numbers, linearization is impossible . Only in 1942 did Siegel prove that this was not always the case. In fact, he demonstrated that if $\theta$ is *Diophantine*, then there is a Siegel disk around the fixed point.

**Definition 2.38. (Diophantine number)** A real number $\theta$ is *Diophantine* if it is badly approximable by rational numbers, in the sense that there exists $c > 0$ and $\mu < \infty$ so that, for all $p/q \in \mathbb{Q}$,

$$|\theta - \frac{p}{q}| \geq \frac{c}{q^{\mu}}.$$

If we write $\lambda = e^{2\pi i\theta}$, this condition is equivalent to

$$|\lambda_n - 1| \geq c \cdot n^{1-\mu},\ n \geq 1.$$

The following is the original theorem of Siegel.

**Theorem 2.39** ([7, Theorem 6.4]). **(Siegel)** *If $\theta$ is Diophantine, and if $f$ has a fixed point at $z = 0$ with multiplier $e^{2\pi i\theta}$, then there is a solution to the Schröder equation.*

Since then, the conditions used by Siegel have been improved by other mathematicians such as Herman or Bryuno. For further information see [6] and [14].

## 2.5 Global theory

Once again, let us consider $f : S \to S$, a holomorphic mapping in which $S \in \{\mathbb{C}, \overline{\mathbb{C}}\}$. Our aim is to describe the global structure of the Fatou set. In order to do that, we suppose that $\mathcal{J} \neq S$, so we have an open, nonempty Fatou set $\mathcal{F}$.

**Definition 2.40. (Fatou component)** A *Fatou component* for a nonlinear rational map $f$ is a connected component of $\mathcal{F}$, the Fatou set.

By the Maximum Principle and the total invariance of the Julia set, if $U$ is a Fatou component, $f(U)$ is also a Fatou component. Therefore, we may consider dynamics on the set of Fatou components.

**Definition 2.41. (Types of Fatou components)** Consider a Fatou component $U$. Then:

1. If $f^p(U) = U$ for some minimal $p > 0$, then $U$ is a *p-periodic component* of $\mathcal{F}$. In particular, if $p = 1$, then $f(U) = U$, and we call $U$ a *fixed* (or *invariant) component*.

2. If $f^k(U)$ is periodic for some $k > 0$ but $U$ is not, we call $U$ a *(strictly) preperiodic component*.

3. Otherwise, if all $f^k(U)$ are distinct for every $k$, we call $U$ a *wandering domain*.

One of the most significant theorems which has advanced complex dynamics over the past few years is Sullivan's Theorem, proven in 1985.

**Theorem 2.42** ([7, Theorem 1.3]). **(Sullivan Theorem - No wandering domains)** *Let $f$ be a rational map of degree at least 2. Then, there does not exist any wandering domain.*

Once we have discarded the possibility of wandering domains, we can ensure that every component of the Fatou set is periodic or preperiodic.

Now, let's define a new type of sets that are highly linked to Siegel disks: *Herman Rings*. Broadly speaking, we can see Herman rings as Siegel disks with a hole.

**Definition 2.43.** **(Herman Ring)** A component $U$ of the Fatou set of $f$ is called a *Herman ring* if $U$ is conformally isomorphic to some annulus $A_r = \{\ z;\ 1 < |z| < r\}$, on which the dynamics of $f^p$, for some $p \geq 1$, is conformally conjugate to an irrational rotation of this annulus.

Thus, a Herman ring is a subset of the Fatou set. Michel Herman showed their existence in 1979 [14].

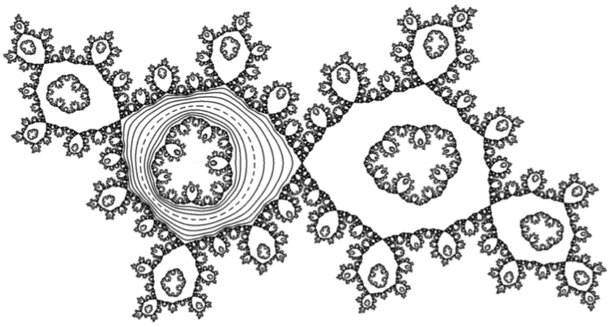

Figure 2.7: Julia set with a Herman ring.

Siegel disks are simply connected components, while Herman rings are doubly connected. Furthermore, Siegel disks and Herman rings are often collectively called *rotation domains*.

The behaviour of repeated iterations of $f$ on periodic components is well understood, and the following widely-known result, originally stated by Fatou, sums up the various possibilities that one can have.

**Theorem 2.44** ([7, Theorem 2.1])**.** **(Classification of periodic Fatou components)** *Let $f : S \to S$ be a holomorphic mapping in which $S \in \{\mathbb{C}, \overline{\mathbb{C}}\}$, of degree at least 2. Let $U$ be a p-periodic Fatou component. Let* <U> = $\{U, f(U), ..., f^{p-1}(U)\}$*. Then, exactly one of the following holds:*

1. *The cycle <U> is called the* ***immediate basin of attraction*** *of the attracting cycle <$z_0$>, i.e.*
$$\exists\ z_0 \in U \ s.t.\ f^{np}(z) \to z_0,\ n \to \infty,\ z \in U.$$

2. *The cycle <U> is called the* ***immediate parabolic basin of attraction*** *of the parabolic cycle <$z_0$>, i.e.*
$$\exists\ z_0 \in \partial U \ s.t.\ f^{np}(z) \to z_0,\ n \to \infty,\ z \in U, \text{ and } (f^p)'(z_0) = 1.$$

3. *The cycle <U> is called a* $p$***-cycle of Siegel Disks****, i.e.*

$$\exists \text{ conformal map } \phi : U \to \mathbb{D} \text{ s.t. } (\phi \circ f^p \circ \phi^{-1})(z) = e^{2\pi i\theta} z,\ \theta \in \mathbb{R} \setminus \mathbb{Q},\ z \in U.$$

4. *The cycle <U> is called a* $p$***-cycle of Herman Rings****, i.e.*

$$\exists r > 1 \text{ and a conf. map } \phi : U \to \{1 < |z| < r\} \text{ s.t. } (\phi \circ f^p \circ \phi^{-1})(z) = e^{2\pi i\theta} z,$$

$$\theta \in \mathbb{R} \setminus \mathbb{Q},\ z \in U.$$

5. *The cycle <U> is called the* $p$***-cycle of Baker Domains*** *and* $z_0$ *is an essential singularity , i.e.*

$$\exists\ z_0 \in \partial U \text{ s.t. } f^{np}(z) \to z_0,\ n \to \infty,\ z \in U, \text{ but } (f^p)(z_0) \text{ is not defined.}$$

**Observation 2.45.** Baker domains, as well as wondering domains, do not exist for rational maps, since any cycle of Baker domains contains an essential singularity on their boundary.

In the case of Herman rings, they do not exist for polynomials due to the Maximum Modulus Principle, nor do they exist for entire transcendental maps, since they require the existence of poles.

## 2.6 Singular values

We shall now examine the relation between singular values and periodic points. For further details on this section, see [7], [5], and [19].

Although a global inverse is never well defined for rational maps of degree $d \geq 2$ or for entire transcendental functions, local inverse branches often are.

**Definition 2.46. (Singular value - Singular orbit)** A point $w \in S$ is called *regular* if all possible inverse branches of $f$ are well defined in some neighborhood of $w$. Alternatively $w$ is known as a *singular value* of $f$. Its orbit is referred to as a *singular orbit*.

The set of singular values, denoted by $S(f)$, known as the *singular set* of $f$, may contain three types of points:

- *Critical values*. Defined as the images of the critical points
- *Asymptotic values*. A point a $w \in S$ is an asymptotic value of $f$ if $w$ is an essential singularity, and there is an unbounded curve

$$\gamma(t) \xrightarrow[t\to\infty]{} \infty \text{ such that } f(\gamma(t)) \xrightarrow[t\to\infty]{} w.$$

Morally, asymptotic values have some of their preimages at infinity. For instance, an example is $w = 0$ for the exponential map, with $\gamma(t)$ being any path whose real part tends to $-\infty$.

- Limits of the above.

It is cleat from the definition that rational maps do not have asymptotic values.

In holomorphic dynamics, singularities of the inverse map and their orbits play a significant role. This is due to the fact that every cycle of Fatou components and every non-repelling cycle is associated to the orbit of some singular value.

Now we are going to present one of the most outstanding theorems of complex dynamics, which will be of great use on the third chapter of this work:

**Theorem 2.47** ([7, Theorem 2.2 and 2.3]). *If $z_0$ is an attracting or parabolic periodic point, then the immediate basin of attraction $A(z_0)^*$ contains at least one critical point or an asymptotic value. Moreover, the orbit of the critical point is infinite.*

*Proof.* We will prove it for the attracting case, and, for the sake of simplicity, only for rational maps.

If $z_0$ is an attracting fixed point, we may assume that, its multiplier, $\lambda$, satisfies $0 < |\lambda| < 1$. Let $U_0 = \Delta(0, \epsilon))$ be a small disk, invariant under $f$, on which the analytic branch, $g$, of $f^{-1}$ satisfying $g(z_0) = z_0$ is defined. The branch $g$ maps $U_0$ into $A(z_0)^*$, and is one-to-one. Thus, $U_1 = g(U_0)$ is simply connected, and $U_0 \subset U_1$, if $U_0$ is appropriately chosen. In case we don't find any critical point, we continue doing this process, constructing $U_{n+1} = g(U_n), U_n \subset g(U_n)$, and extending $g$ analytically to $U_{n+1}$. If this process does not end, we obtain a sequence $g^n : U_0 \to U_n$ of analytic functions on $U_0$ which omits $J$, and is therefore normal on $U_0$. But this is impossible, since $z_0 \in U_0$ is a repelling fixed point for $g$. Then, eventually, we reach a $U_n$ to which we can not extend $g$. Then there is a critical point $p \in A(z_0)^*$ such that $f(p) \in U_n$.

If $z_0$ is periodic with period $n > 1$ and $|(f^n)'(z_0)| < 1$, this argument shows that each component of $A(z_0)^*$ contains a critical point of $f^n$. Since $(f^n)'(z) = \prod f'(f^j(z))$, $A(z_0)^*$ must also contain a critical point of $f$.

The same result is true for a parabolic basin, and can be proved using a similar argument. □

**Definition 2.48. (Postsingular and postcritical set)** The *postsingular* set $P(f)$ (also called the *postcritical* set if $f$ is rational), is the closure of the union of forward iterates of the singular set. i.e.

$$P(f) = \overline{\bigcup_{w \in S(f)} \bigcup_{n \geq 0} f^n(w)}.$$

Following the steps of the previous sections, rotation domains are related to the presence of critical points. Is clear to see that Siegel disks and Herman rings do not contain any critical points. However, in a sense, they can be linked to them:

**Theorem 2.49** ([7, Theorem 1.1]). *If $U$ is a Siegel disk or a Herman ring, then the boundary of $U$ is contained in the postcritical set of $f$, i.e.*

$$\partial U \subset P(f).$$

*In particular, the postcritical set must be infinite.*

In other words, orbits of critical points must accumulate on $\partial U$. Moreover, if $z_0$ is a Cremer point, then $z_0 \in P(f)$, i.e. it is also accumulated by orbits of critical points.

The following is one of the very remarkable connections between complex dynamics and topology. In this occasion, we see how the dynamics of the singular values determine a pure topological property as in local connectivity.

**Theorem 2.50** ([19, Theorem 19.6, Theorem 19.7]). *Let $f$ be rational map. If $\mathcal{J}(f)$ is connected, and every critical orbit is either finite or converges to an attracting periodic orbit, then $\mathcal{J}(f)$ is locally connected.*

## 2.7 Polynomial dynamics

The general, theory explained in the above sections takes on a very particular form when discussing polynomial dynamics. The contents of this section can be found in [19], where the reader can go for more details.

As mentioned above, we will now focus on the specific case of polynomials.

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a polynomial of degree $d \geq 2$, that is

$$f(z) = a_d \cdot z^d + \ldots + a_1 \cdot z + a_0$$

with $a_d \neq 0$. Note that $f(\infty) = f^{-1}(\infty) = \{\infty\}$. Therefore, $f$ has a fixed point at infinity. Using the change of variables $1/z$, one can see that $\infty$ is in fact a superattracting fixed point. We will refer to its basin of attraction (the set of points whose orbit converges to $\infty$) as the *basin of infinity*, and denote it by

$$A_f(\infty) = \{z \in \mathbb{C} \mid f^n(z) \to \infty, \text{ when } n \to \infty\}.$$

It is always connected since $\infty$ has no preimages (see [19, Lemma 9.4]).

In particular, we can always find a constant $k_f$ so that every point $z$ in the neighborhood $|z| > k_f$ belongs to $A_f(\infty)$.

**Definition 2.51. (Filled Julia Set)** The complement of $A_f(\infty)$

$$K_f = \mathbb{C} \backslash A_f(\infty) = \{z \in \mathbb{C} \mid f^n(z) \text{ is bounded}\}$$

is called the *filled Julia set* of $f$.

**Observation 2.52.** The filled Julia set is completely invariant and compact. It is also full, which means that its complement, $\mathbb{C} \backslash K_f = A_f(\infty)$, is connected, or equivalently, that all bounded Fatou components are simply connected [5]. This is a consequence of the Maximum Modulus Principle.

In fact, all three sets, $\mathcal{J}(f)$, $int(K_f)$, and $A_f(\infty)$, follow the property of being totally invariant, that is, if a point belongs to one of them, so does its entire orbit, both positive and negative.

**Lemma 2.53** ([19, Lemma 9.4]). *For any polynomial $f$ of degree at least 2, the filled Julia set $K_f \subset \mathbb{C}$ is compact.*

The basin of infinity, as any basin of attraction, belongs to the Fatou set. But so do all points in the interior of the filled Julia set (if it is nonempty), just by Montel's theorem. Instead, points in the common boundary of $A_f(\infty)$ and $K_f$ must belong to the Julia set since any neighborhood contains points whose orbit converges to $\infty$, but also points with bounded orbit. In other words, $A_f(\infty)$ and $K_f$ have a common topological boundary, which is equal to the Julia set

$$\mathcal{J}(f) = \partial A_f(\infty) = \partial K_f.$$

The union of the connected component $A_f(\infty)$ and all connected components of the interior of $K_f$, if there is any, form the Fatou set

$$\mathcal{F}(f) = A_f(\infty) \cup int(K_f).$$

Since $f$ is polynomial, the Fatou components will be either Siegel disks, attracting basins or parabolic basins.

**Theorem 2.54** ([19, Theorem 9.5]). **( $K_f$ connected $\Leftrightarrow$ Bounded Critical Orbits)** *Let $f$ be a polynomial of degree $d \geq 2$. If the filled Julia set $K_f$ contains all of the finite critical points of $f$, then both $K_f$ and $\mathcal{J}(f) = \partial K_f$ are connected.*

Although we can find the full proof in [19], a glimpse of the demonstration is that, if the basin of infinity of $f$ has no critical points, then we can expand the Böttcher's coordinates to the whole basin, and we will always obtain Jordan curves, i.e. we can always describe $K_f$ as the intersection of closed connected nested subsets.

# Chapter 3

# Topological structures and Julia sets

In the field of planar topology, there are sets that are considered "exotic" and not commonly found in everyday life. These sets have unique and interesting topological properties, and often have a visually pleasing appearance. Recently, with the resurgence of complex dynamics, many of these exotic sets have been discovered to be Julia sets for complex analytic functions. This chapter will explain how four specific examples of these sets, dendrites, Cantor sets, Sierpiński curves, and Cantor bouquets, arise from particular families of complex maps, including the quadratic family, the complex exponential family, and a type of singularly perturbed rational maps. The content in this chapter can mostly be found in references [4], [13], [9], [10], [11], and [12], for those who want more detailed information.

## 3.1 Dendrites

As discussed in previous chapters, a fundamental example in the theory of complex variable iteration is the *quadratic family* defined as

$$Q_c(z) = z^2 + c$$

where $c \in \mathbb{C}$ is a parameter. This family of functions hides, under an evident simplicity, an extraordinary dynamic richness, and has been the subject of intense mathematical research during the last decades of the 20th century, continuing nowadays.

More particularly, $z = 0$ is the only critical point for all polynomials in the quadratic family. Thus, the only critical value of the polynomial $Q_c$ is $z = c$.

In the following sections, we will focus exclusively on the quadratic family, unless otherwise specified. To refer to its respective basin of infinity and filled Julia set, we will use the following notation: $A_c(\infty)$ and $K_c$.

Before presenting the key theorem of this section, we will first introduce a proposition that will be used in its proof. Given that $Q_c(z)$ is polynomial, **Observation 2.52** states that $K_c$ is full.

**Proposition 3.1.** *If $K_c = \mathcal{J}_c$, then $\mathcal{J}_c$ contains no simple closed curves.*

*Proof.* Suppose that it does contain a simple closed curve, $\gamma$. If $\text{int}(\gamma) \not\subset K_c$, then $\gamma$ disconnects $A_c(\infty)$, which is a contradiction. If $\text{int}(\gamma) \subset K_c$, then $\text{int}(\gamma) \subset \mathcal{F}_c$, but $\mathcal{F}_c = A_c(\infty)$ (given that $K_c = \mathcal{J}_c$), which is also contradiction. Therefore, $\mathcal{J}_c$ contains no simple closed curves.

□

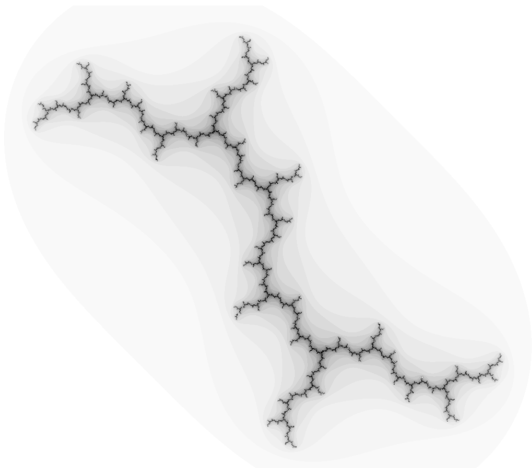

Figure 3.1: Julia set of $Q(z) = z^2 + i$ — a dendrite.

The main theorem in this section reads as follows:

**Theorem 3.2.** *Consider $Q_c(z) = z^2 + c$. Let's assume that $z = 0$ is not a periodic point but $Q_c^k(0)$ for some $k > 0$ is. Then, $K_c = \mathcal{J}_c$ (i.e. $int(K_c) = \emptyset$), and $\mathcal{J}_c$ is a dendrite.*

*Proof.* To prove that $\mathcal{J}_c$ is a dendrite, we must see that it is a locally connected continuum that contains no simple closed curves.

By **Theorem 2.54**, we know that $K_c$ is connected if and only if the orbit $Q_c^k(0)$ does not tend to $\infty$. Hence, we have that $K_c$ is connected. We can also affirm that $K_c$ is closed, since it is always compact (**Lemma 2.53**).

In order to see that its interior is empty, we have to study the orbit of its only critical point $z = 0$, which is not periodic, but strictly preperiodic. By **Theorem 2.47**, we know that it's not possible to have either a parabolic or attracting basin of attraction because the orbit of 0 falls on a cycle, which is finite. This orbit being finite, as stated by **Theorem 2.49**, means that we cannot have a Siegel disk. Additionally, the map being entire precludes the possibility of a Herman ring, and the absence of any essential singularities means that a Baker domain cannot exist either (**Observation 2.45**). All things considered, there is no Fatou component other than $A_c(\infty)$. Therefore, the interior of $K_c$ must be empty, and $K_c = \mathcal{J}_c$.

We know, by **Lemma 2.29**, that $\mathcal{J}_c$ is nonempty and, since $K_c = \mathcal{J}_c$, $\mathcal{J}_c$ is compact and connected (as we already saw). As a result, it can be seen that on one hand, $\mathcal{J}_c$ is a continuum (by definition), and on the other hand, according to **Theorem 2.50**, it is also locally connected.

Since $K_c = \mathcal{J}_c$, by **Proposition 3.1**, $\mathcal{J}_c$ contains no simple closed curves.

Based on the fact that $\mathcal{J}_c$ is a locally connected continuum that contains no simple closed curves, it can be deduced that it is a dendrite. □

## 3.2 Cantor Set

The Julia sets of certain complex polynomials, particularly those in the quadratic family with a sufficiently large value of $|c|$, can take the shape of Cantor sets. These fractal sets possess distinctive and unique geometric characteristics.

Figure 3.2: The Julia set for $Q_c$, $c = 0, 1 + i$ is a Cantor set (although $|c| < 2$).

In order to understand these Julia sets, we must first present the following result:

**Proposition 3.3.** *Let $Q_c(z) = z^2 + c$ be the quadratic family, and let $R = max\{2, |c|\}$. If $|z| > R$, then*

$$\lim_{n\to\infty} Q_c^k(z) = \infty.$$

*Proof.* See that if $|z| > R$ then $|z| > 2$ and $|z| > |c|$. Therefore,

$$\frac{|z^2 - 2|}{|z|} \geq |z| - \frac{|c|}{|z|} \geq |z| - 1 \geq 1,$$

so $|Q_c(z)| > |z|$ and $|Q_c^k(z)| \to \infty$. □

Now, we can see that following the previous proposition, we are able to locate $K_c$ in the complex plane.

**Corollary 3.4.** *The filled Julia set $K_c$ of the polynomial $Q_c$ is contained in the disk with center 0 and radius $R$, where $R = max\{2, |c|\}$.*

We will now provide a lemma that will assist in proving the primary outcome of this section.

**Lemma 3.5.** *Let $f : U \to \mathbb{C}$ be a holomophic function, $U \subset \mathbb{C}$ convex, such that $|f'(z)| \leq \lambda < 1$, for any $z \in U$. Let $K \subset U$, then $diam(f(K)) \leq \lambda \cdot diam(K)$. In particular, if $f : U \to U$, then $diam(f^n(K)) \leq \lambda^n \cdot diam(K)$, so $diam(f^n(K)) \to 0$.*

Finally, we can present the main result of this section:

**Proposition 3.6.** *If $|c| > 2$, the orbit of $z = 0$ tends to infinity (i.e. $z = 0$ belongs to the basin of attraction of infinity). Consequently, the Julia set $\mathcal{J}_c$ of $Q_c$ is a Cantor set.*

*Proof.* Although we will give an idea of the prove for $|c|$ sufficiently large, the statement is true for all $|c| > 2$.

First, we show that

$$Q_c^k(0) \xrightarrow[k\to\infty]{} \infty.$$

Observe that $Q_c^3(0) = c^2 + c$, and $|c^2 + c| > |c|$. Recall that using the previous corollary, all points with modulus greater than $R = max\{2, |c|\}$ have an unbounded orbit, and we observe that if |c| > 2, this is exactly the complement of the disk of radius $|c|$. Since $|c^2 + c| > |c|$, then the orbit of $z = 0$ tends to infinity.

Let us denote the closure of the disk of radius $|c|$ by $D$ . Although we want to characterize the points whose orbit never leaves D, we will study its complement. That is, we will see which are the points of D that fall outside of D after an iteration of $Q_c$. In order to do this, we calculate the preimage of the border of D, which is a figure in the shape of an eight that passes through point 0, as shown in Figure 3.3.

Indeed, all points of the circle have two preimages of opposite sign except $c$, which has only one, $z = 0$. If we denote the two parts of the eight figure by $D_0$ and $D_1$ respectively, we observe that $Q_c$ sends each of them bijectively to the disk $D$. However, all points in $D$ that are not in $D_0$ or $D_1$ are sent out of $D$. Hence, the Julia set lies in the interior of $D_0 \cup D_1$.

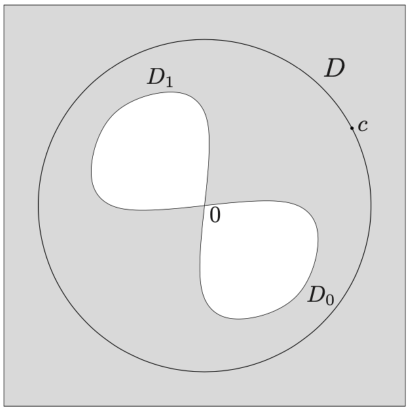


Figure 3.3: Preimage of $D$ through $Q_c$, which is an eight [13].

Next, we are going to study which points inside $D_0 \cup D_1$ remain there after one or more iterations. Since $D_0$ and $D_1$ are sent bijectively to $D$ by $Q_c$, the preimage of the eight figure will consist of two smaller eight figures, one inside each of the lobes:

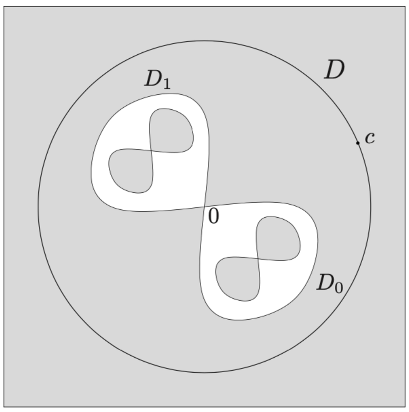


Figure 3.4: Second preimage of disk D, which consists of two eight figures, one inside each of the two lobes $D_0$ and $D_1$ of the first eight figure [13].

Each one of these eights is bijectively sent to $D_0 \cup D_1$. Thus, the Julia set is contained within the lobes of the four constructed eight figures.

This process can be repeated *ad infinitum*, so that, after $k$ steps, the Julia set is contained in a $2^k$ eight figures, each of them being a bijective preimage of $D_0 \cup D_1$ under $Q_c^k$. Therefore, points that never escape to $\infty$, must lie within one of these eight figures, for all $k$.

The key to finally see that is a Cantor set is to show that each of the intersections of these closed nested eight figures, as $k$ tends to infinity, consists of a unique point. In other words, that the diameter of these figures tends to zero.

Let $B$ denote the disk of radius $1/2$ centered at the origin. Then, $Q_c(B)$ is the disk of radius $1/4$ centered at $c$. Let us assume that $Q_c(B) \cap (D_0 \cup D_1) = \emptyset$. Note that if $|Q_c'(z)| \leq 1$, then $z \in B$. Thus, our assumption implies that any point with derivative less than one is mapped out of $D_0 \cup D_1$. For $c$ big enough, if $|z| > 3/4$ (or just $|z| > 1/2 + \epsilon$), then $|Q_c'(z)| > 3/2 > 1$. Hence, for any branch $g$ of $Q_c^{-1}((D_0 \cup D_1))$, we will have $|g'(z)| < 2/3 < 1$. Therefore, any inverse iteration of $Q_c^{-1}$ that we take in a smaller eight figure will have a diameter at least $2/3$ times smaller than the previous eight figure. If we do this *ad infinitum*, by **Lema 3.5**, we have that the diameter goes to zero, which implies that we have a point out of any sequence of inverse branches. Thus, the Julia set is totally disconnected.

We know by **Lemma 2.15** that the Julia set is always closed. Since the orbit of 0 goes to infinity, we cannot have any Fatou component besides $A_c(\infty)$. Therefore, once again, we have that $K_c = \mathcal{J}_c$. Also, by **Lemma 2.31**, the Julia set is perfect, because $\mathcal{J}_c$ never has isolated points.

Given that we have a closed, totally disconnected, and perfect subset of $\mathbb{R}^2$, the Julia set is a Cantor set. $\square$

## 3.3 Sierpiński Carpet

In this section we consider the family of rational maps of the complex plane given by

$$F_\lambda(z) = z^2 + \frac{\lambda}{z^2}$$

with $\lambda \neq 0$. This represents a more approachable collection of the functions found in 1993 by Milnor and Tan Lei, where Sierpiński curves emerge as their Julia sets.

As we already know, the Julia set of $F_\lambda(z)$ is the complement of the set of points $z \in \mathbb{C}$ such that $\{F_\lambda^n\}$ is a normal family in some neighborhood of $z$.

The case $\lambda = 0$ is analog to considering the quadratic family $Q_0 = z^2 = F_0$, whose dynamics is quite simple: the Julia set is the unit circle, and all orbits in

$|z| < 1$ tend to the attracting fixed point at the origin, while all orbits in $|z| > 1$ tend to $\infty$. On the other hand, when $\lambda \neq 0$, the map is a degree-four rational map, and we will say that $F_\lambda$ has undergone a singular perturbation.

To be more precise, we will narrow down our attention to the particular case of $\lambda = -1/16$. We will use the notation $F = F_{-1/16}$. Therefore, the main theorem of our section is the following.

**Theorem 3.7.** *For $\lambda = -1/16$, the Julia set of $F_\lambda$ is a Sierpiński curve.*

Although this is the case we will be proving, it is important for the reader to know the fact that in any neighborhood of the origin in the complex $\lambda$-plane, there are infinitely many open sets of parameters for which the Julia sets of the corresponding maps $F_\lambda$ are Sierpiński curves (see [4]).

Moreover, any two such Julia sets are homeomorphic since, as we already saw in Chapter 1, any planar set that is compact, connected, locally connected, nowhere dense, and that has the property that any two complementary domains are bounded by simple closed curves that are disjoint, by Wyburn's theorem (**Theorem 1.30**), is homeomorphic to the Sierpiński carpet, and is therefore a Sierpiński curve. That is to say that, topologically speaking, they are all the same. By contrast, in terms of dynamical systems, most of them are not topologically conjugate, meaning that the dynamics on these Julia sets are rather different.

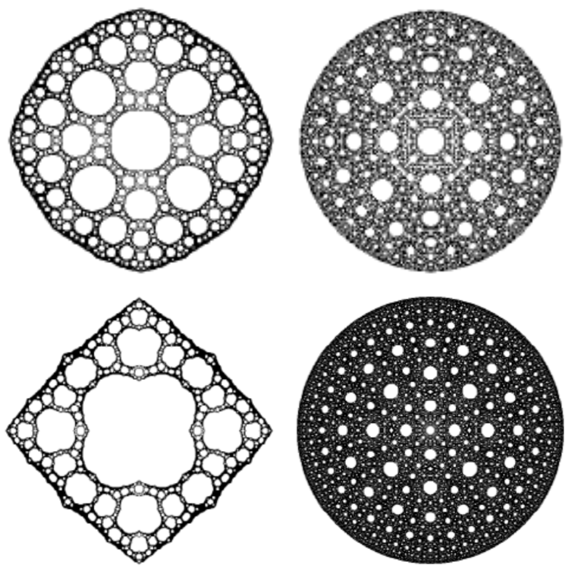

Figure 3.5: Julia sets for $\lambda = -1/16$, $\lambda = -0.01$, $\lambda = -1/4$, $\lambda = -0.001$, respectively.

Our objective for the remainder of this section is to establish **Theorem 3.7** as true. In order to accomplish that, we first need to get a general idea about how these Julia sets are constructed. We see that $|\lambda/z^2|$ can be quite small for a reasonably large $|z|$. Therefore, it is fair to say that, close to infinity, $F_\lambda$ is mainly given by $z \to z^2$. As a result, any orbit sufficiently far from the origin simply tends to $\infty$.

Considering the smallness of $|\lambda|$, the boundary of the basin of infinity is a simple closed curve surrounding the origin, just as in the case of $z^2$. However, the dynamical behavior within this curve is quite complicated.

It is clear that $F$ has a pole of order two at 0, together with four pre-poles at the points $\pm 1/2$ and $\pm i/2$. Also, its four critical points lie at $w/2$ where $w$ is a fourth root of $-1$. Thus, the two critical values are $F(w/2) = \pm i/2$ and, since $F^2(w/2) = F(\pm i/2) = 0$, the second iterate of each of the critical points lands on the pole at the origin, making this case very special.

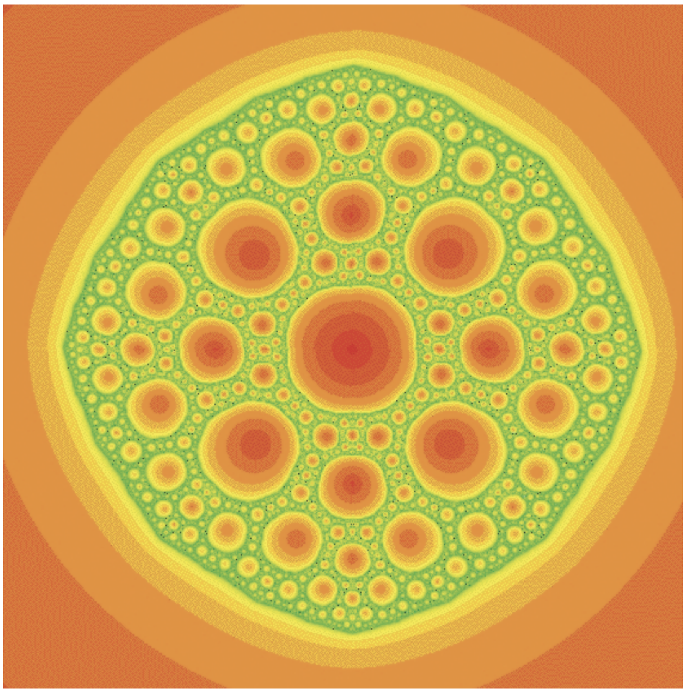

Figure 3.6: The Julia set for $F_\lambda(z)$, $\lambda = -1/16$ is a Sierpiński curve [10].

The preimage of $\mathbb{R}$ under $F_\lambda$ consists of the real and imaginary axes while the preimage of the imaginary axis consists of two sets: the four rays $\theta = \pm\pi/4$, $\pm 3\pi/4$ and the circle of radius $1/2$ centered at the origin. Note that all four critical points as well as the four pre-poles lie on this circle. For this reason, we call the circle $r = 1/2$ the *critical circle*.

Now, let $B$ denote the basin of attraction of $\infty$, which, as we already mention, is bounded by a simple closed curve, $\gamma$, on which $F$ is conjugate to $z \to z^2$. Note that $F$ is two-to-one on the immediate basin of $\infty$, $B$. Given that $F$ is conjugate to $z^2$ on $\gamma$, there is a unique fixed point on $\gamma$. This must be the fixed point $p \in \mathbb{R}$, since we know that this point lies on the boundary of $B$.

Let $T$ denote the component of the preimage of $B$ that contains the origin. We call $T$ the *trap door*, since any orbit that enters it falls through the trapdoor and then tends to $\infty$. The function $F$ maps $T$ in two-to-one fashion (except at the pole at the origin) onto $B$.

Therefore, since the pole has order two, $F^{-1}(B)$ is exclusively $T$. The boundary of T, $\tau$ is mapped in two-to-one fashion onto $\gamma$. Note that $\tau$ and $\gamma$ are disjoint. This follows from the fact that the circle of radius $3/4$ about the origin is mapped strictly inside itself. Continuing in this manner yields the set of points whose orbits eventually enter $B$, and each of these preimages is bounded by a simple closed curve which is disjoint from those previously constructed, as well as from the boundaries of $T$ and $B$. These are the analogues of the removed open squares in the Sierpiński carpet. It is known that the union of these sets forms the Fatou set of $F$.

In order to see that the Julia set of $F$ is homeomorphic to the Sierpiński carpet, we will use Wyburn's theorem (**Theorem 1.30**). By **Lemma 2.29**, $\mathcal{J}_c$ is always nonempty. Given that the orbits of all critical points are finite, there cannot be any Fatou component besides $A_c(\infty)$.

Although $F$ is a rational function and we defined the filled Julia set $K_c$ only for polynomials, given that $\infty$ is attracting, it makes sense to talk about $K_c$, which is the set of points with bounded orbit. The difference is that now $A_c(\infty)$ is not formed by a unique component, but by many components, since it contains the immediate basin of attraction, the trapdoor, and all its preimages. Since the only Fatou component is $A_c(\infty)$, we could say that $K_c = \mathcal{J}_c$. Therefore, $\mathcal{J}_c$ is compact since $\mathcal{J}_c = K_c$ and, by **Lemma 2.53**, $K_c$ is compact.

Given that there is a basin if infinity, the Julia set it can't be the entire $\overline{\mathbb{C}}$. By **Lemma 2.24** (iii), if the Julia set contains an interior point, then it must be equal to the entire Riemann Sphere. Therefore, $\mathcal{J}_c$ has empty interior, i.e. is nowhere dense. It can also be inferred that $\mathcal{J}_c$ is connected as we have seen by the previous construction of the Fatou set that the preimages of the basin of infinity are simply connected. Since the Julia set is connected and every critical orbit is finite, $\mathcal{J}_c$ is also locally connected (**Theorem 2.50**).

Last, we have seen that each of the preimages of T is bounded by a simple closed curve which is disjoint from those previously constructed, as well as from the boundaries of $T$ and $B$.

In conclusion, since the Julia set is nonempty, compact, connected, locally connected, nowhere dense, and has the property that any two complementary domains are bounded by disjoint simple closed curves, then, by Wyburn's Theorem (**Theorem 1.30**), $\mathcal{J}_c$ is homeomorphic to the Sierpiński carpet.

## 3.4 Cantor Bouquet - Straight Brush

To see how Cantor bouquets arise from Julia sets, we should henceforth consider the complex exponential family

$$E_\lambda = \lambda e^z.$$

Proving the homemorphism of this Julia set to a straight brush requires more advanced mathematics. Alternatively, instead of providing a well-crafted and detailed construction, we will briefly describe what it entails and share some interesting information about these unique structures.

In this work, only real and positive values of $\lambda$ will be considered, primarily because all the relevant phenomena present for other complex $\lambda$-values are present in this situation as well.

As we already saw in previous sections, the orbit of 0 (the critical point) plays an essential role in the dynamics of $Q_c(z)$. With the exponential family, it is no different. But, in this case, 0 is an asymptotic value of all members of the complex exponential family $E_\lambda$, rather than a critical point, of which there are none.

Once again, we will denote the Julia set of $E_\lambda$ by $\mathcal{J}(E_\lambda)$, and it is the set of points at which the family of iterates $\{E_\lambda^n\}$ is not normal.

In order for Cantor bouquets to take place, we will consider $\lambda$ such that $0 < \lambda \leq 1/e$.

Here is an overview of how a Cantor bouquet is constructed. For $0 < \lambda < 1/e$, the graph of $E_\lambda$ intersects the diagonal $y = x$ twice, at an attracting fixed point $a_\lambda$, and at a repelling one $r_\lambda$, as we can see in Figure 3.8. Note that $E_\lambda(v_\lambda) = E_\lambda(-\log(\lambda)) = 1$, so that $a_\lambda < -\log(\lambda) < r_\lambda$.

If we consider in $\mathbb{C}$ the vertical line $Re(z) = -\log(\lambda)$, we can see that if $x_0 \in \mathbb{R}$ and $x_0 < -\log(\lambda)$, then $E^n(x_0)$ tends to the fixed point at $a_\lambda$. Thus, the vertical line $\mathrm{Re}(z) = -\log(\lambda)$ is mapped around a circle centered at the origin and lying to the left of $x = -\log(\lambda)$, since $E_\lambda(-\log(\lambda)) = 1 < -\log(\lambda)$.

All points to the left of this line are therefore contracted inside this circle. Consequently, by the Contraction Mapping Principle, all orbits originating in the half plane $H = \{z \mid \mathrm{Re}(z) < -\log(\lambda)\}$ must tend to the attracting fixed point $a_\lambda$.

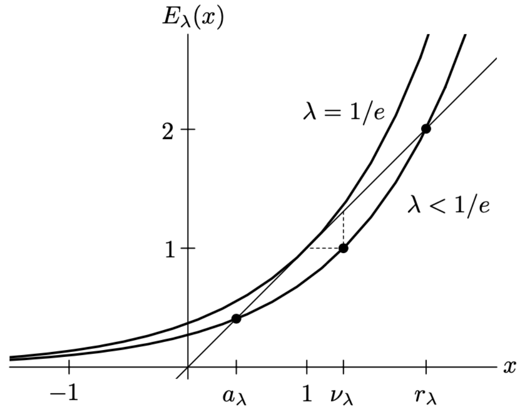


Figure 3.7: The graphs of $E_\lambda$ for $\lambda = 1/e$ and $\lambda < 1/e$ [11].

Hence this half plane lies in the stable set, i.e., in the Fatou set. We will try to paint the picture of the Julia set of $E_\lambda$ by painting rather his complement. Since the basin of attraction of $q_\lambda$ is completely invariant under $E_\lambda$, we can obtain the entire stable set by considering all preimages of this half plane $H$ under $E_\lambda$.

Any point lying on a horizontal line of the form $\text{Im}(z) = (2k+1)\pi$, for each integer $k$, is mapped by $E_\lambda$ to the negative real axis, so these points lie in the basin. Hence, there are open neighborhoods of each of these lines that lie in the stable set too. Then, there is an open set about these lines to the right of $H$ that is shaped like a finger pointing to $\infty$. See Figure 3.8.

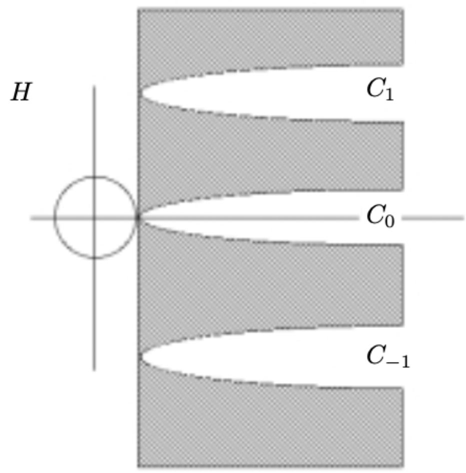


Figure 3.8: The preimage of $H$ for $\lambda = 1/e$ consists of $H$ and the shaded region [11].

The complement of these open sets consists of infinitely many closed "C"-shaped regions extending to $\infty$ in the right half plane. Each of these regions is contained within the strip of two horizontal lines given by $-\frac{\pi}{2} + 2j\pi \leq \mathrm{Im}(z) \leq \frac{\pi}{2} + 2j\pi$, and each is mapped in one-to-one fashion onto the half plane forming the complement of $H$ in $\mathbb{C}$.

Continuing in this fashion inductively, we remove infinitely many subfingers at each iteration of $E_\lambda$. In the limit, the set of points which do not fall into $H$ after some iterate of $E_\lambda$ is the Julia set of $E_\lambda$, $\mathcal{J}(E_\lambda)$, and it is known that this set consists of infinitely many curves, each with a distinguished endpoint and a "stem", i.e., the portion of the curve that extends from the endpoint to $\infty$ in the right half plane. This is the Cantor bouquet.

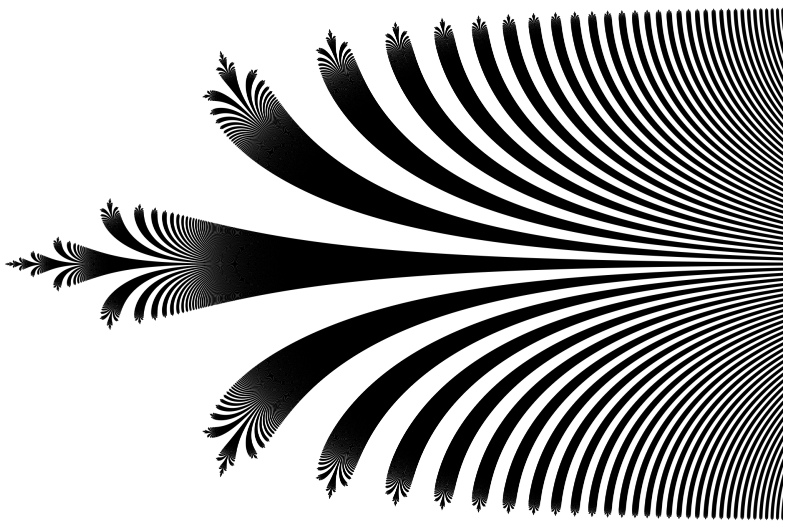

Figure 3.9: The tip of the Cantor bouquet for $E_\lambda$ with $\lambda = 1/e$.

Despite being outside the scope of this work, we still want to conclude by presenting some surprising findings related to Cantor bouquets.

On one hand, the endpoints of these curves have a very special characteristic. If we take the set of endpoints together with infinity, then we have a connected set. However, if we take it without infinity, we have a set that is totally disconnected [11, Theorem 3.2].

Another interesting result is the Karpińska paradox. This mathematical concept was proposed by Polish mathematician Bożena Karpińska [15], and states that the set of curves without endpoints has Hausdorff dimension 1; but the set of endpoints has Hausdorff dimension 2.

# Conclusions

In summary, this work has provided, in first place, a comprehensive look into the fundamentals of planar topology and complex dynamics.

Just as we aimed, we have covered the basic principles of planar topology as well as defined fractal and fractal dimension. We have also examined examples of exotic topological models such as dendrites, Cantor sets, Sierpiński carpets, and Cantor bouquets, and explored their characteristics from a topological perspective.

Thereafter, we have established the foundations of complex dynamics by providing an overview of iteration on the Riemann sphere, and concepts of normality and Montel's theorem. We have also given formal definitions of the Fatou and Julia sets, and discussed their key properties.

After introducing the basics of local, semilocal, and global theory, we have delved into the five types of periodic Fatou components using the Classification Theorem. Furthermore, we have talked about singular values and polynomial dynamics, which was highly beneficial for the final chapter.

Lastly, we have demonstrated how the sets presented in Chapter 1 appear as the Julia sets of specific families of complex maps such as the quadratic family, the complex exponential family, and a particular class of singularly perturbed rational maps using the tools discussed in previous chapters.

Overall, this essay has aimed to highlight the beauty and complexity of planar topology and how complex dynamics adds a new dimension to its exploration. We have explored the ways in which different topological structures naturally arise as the Julia sets of some complex maps, which was our main goal. These sets are truly mesmerizing, and the fact that they can exhibit such rich topological properties despite their simple geometric structure is quite remarkable.

An additional avenue for further research in this field, besides delving deeper into case where Cantor Bouquets arise as Julia sets of $E_\lambda$, could be to examine the emergence of the Knaster continuum. These new exotic topological structures, in a similar way, naturally appear as Julia sets of $E_\lambda$ too.

It is evident that there is a lot of unexplored territory in this field, and I hope that this work will inspire others to continue exploring this captivating area of study.

# Bibliography


[1] J. M. Aarts and L. G. Oversteegen, *The geometry of Julia sets*, Trans. Amer. Math. Soc. 338, 1993, 897-918.

[2] M. Barnsley, *Fractals everywhere*, Academic Press INC, 1988.

[3] A. F. Beardon, *Iteration of Rational Functions*, Springer, 1991.

[4] P. Blanchard, R. L. Devaney, D. M. Look, P. Seal, and Y. Shapiro, *Sierpiński curve Julia sets and singular perturbations of complex polynomials*, Ergod. Th. & Dynam. Sys. 25, 2005, 1047-1055.

[5] B. Branner and N. Fagella, *Quasiconformal Surgery in Holomorphic Dynamics*, Cambridge University Press, 2014.

[6] A. D. Bryuno, *Convergence of transformations of differential equations to normal forms*, Dokl. Akad. Nauk USSR 165, 1965.

[7] L. Carleson and T. W. Gamelin, *Complex dynamics*, Springer-Verlag, 1992.

[8] S. Curry, *One-dimensional nonseparating plane continua with disjoint $\epsilon$-dense subcontinua*, Topology Appl. 39 , 1991, 145-151.

[9] R.L. Devaney, *An introduction to Chaotic Dynamical Systems*, Benajmin / Cummings, 1986.

[10] R. L. Devaney, *Exotic topology in complex dynamics*, Indagationes Mathematicae, Volume 27, Issue 5, 2016, 1116-1126.

[11] R. L. Devaney, *Cantor bouquets, explosions, and knaster continua: dynamics of complex exponential*, Publicacions Matemàtiques, Volume 43, 1999, 27-54.

[12] R. L. Devaney and X. Jarque, *Indecomposable continua in exponential dynamics*, Conform. Geom. Dyn. 6, 2002, 1-12.

[13] N. Fagella and X. Jarque, *Iteración compleja y fractales*, Vicens Vives, 2007.

[14] M. Herman, *Conjugaison quasi-symétrique des difféphismes du cercels et applications aux disques singuliers de Siegel* (unpublished manuscript), 1986.

[15] B. Karpińska, *Hausdorff dimension of the hairs without endpoints for λexp(z)*, C. R. Acad. Sei. Paris Sér. I Math, 328, 1999. 1039-1044.

[16] B. Knaster, *Un continu dont tout sous-continu est indécomposable*, Fund. Math. 3, 1922, 247-286.

[17] J. Lee. *Introduction to smooth manifolds*, Springer-Verlag, 2012.

[18] B. Mandelbrot, *The fractal geometry of nature*, W.H. Freeman and Company, New York, 1977.

[19] J. Milnor, *Dynamics in One Complex Variable*, Third Edition, Princeton University Press, 2006.

[20] S. Nadler, *Continuum theory: an introduction*, CRC Press, 1992.

[21] M. H. A. Newman, *Elements of the topology of plane sets of points*, Praeger, 1939.

[22] D. Sullivan, *Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou - Julia problem on wandering domains*, Annals of Mathematics, Second Series, Vol. 122, No. 2, 1985, 401-418.

[23] G. T. Whyburn, *Topological characterization of the Sierpiński curve*, Fund. Math. 45, 1958, 320-324.