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Treball final de grau

Can one hear the shape of a drum?

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Abstract

In this work we study Mark Kac's classical problem "Can one hear the shape of a drum?" and some of its extensions. They are all inverse problems on characterizing the shape, or at least some geometrical information about the shape, of an Euclidean domain from its Dirichlet spectrum. As to the original problem, we answer it negatively by providing an example of two different shaped planar drums that have the same spectrum of frequencies. As to the extensions, we prove that the spectrum of frequencies of a planar drum characterizes its area. These results are straightforwardly generalized to higher dimensions. Finally, we comment variants of Kac's problem for which there are positive results for the characterization of the shape of a drum from its spectrum.

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Contents

1 Introduction

When we beat a drum its surface vibrates in an oscillatory way. This vibration induces an oscillatory displacement of the surrounding air that propagates through space in a so-called *sound wave*. The frequencies of those oscillations are characterized by the way the surface of the drum vibrates, so they are referred to as the frequencies of the drum. Bass sounds correspond to low frequencies and treble sounds correspond to high frequencies. When the sound wave reaches our ears, their internal structure vibrates according to those frequencies, generating neural impulses that are sent to our brain in order to reconstruct the sound. So we can say that what we hear are the frequencies.

1.1 Fundamental frequencies and eigenvalues of the Laplacian

Since the frequencies of a drum are characterized essentially by the way its surface vibrates, let's set up the mathematical model of vibrations of a membrane.

The shape of the unperturbed surface of the drum (or simply membrane) will be modeled by a domain Ω of $\mathbb{R}^2$, that is, an open, connected proper subset of $\mathbb{R}^2$.

After a beat, the membrane points vibrate vertically in an oscillatory way. Let

$$\begin{array}{rccc} v: & [0,\infty)\times\Omega & \longrightarrow & \mathbb{R} \\ & (t,(x,y)) & \longmapsto & v(t,x,y) \end{array}$$

be the vertical displacement of a point $(x, y) \in \Omega$ at time t.

We would expect that the more tightly curved is the membrane locally at (x, y), the greater the restoring force. On the one hand, by Newton's second law, the restoring force is proportional to the acceleration $\frac{\partial^2 v}{\partial t^2}$; on the other hand, the tightness of the membrane is controlled by its convexity, measured by its Laplacian $\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$, so we expect $\frac{\partial^2 v}{\partial t^2} = \Delta v$ in Ω.

Since the membrane is fixed on its boundary, $v(t, x, y)$ must vanish on all $(x, y) \in \partial\Omega$. If, in addition, the initial distributions of vertical displacements and velocities are known, with patterns $f(x, y)$ and $g(x, y)$ respectively, we deduce that v must be a solution of the wave equation with initial conditions:

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} = \Delta v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \\ v(0,x,y) = f(x,y) & \\ \dfrac{\partial v}{\partial t}(0,x,y) = g(x,y). & \end{cases} \tag{1}$$

Particularly interesting are the so-called *stationary waves*: waves for which the frequency of oscillations is the same for all points and the amplitude of the vertical displacements only depends on the membrane point. Such frequencies are called *fundamental frequencies* of the membrane or *pure tones*.

Stationary waves are modeled by functions of the form $v(t,x,y) = T(t)u(x,y)$. Substituting this in the PDE we have

$$\frac{d^2T(t)}{dt^2}u(x,y) = T(t)\Delta u(x,y),$$

or equivalently

$$\frac{1}{T(t)}\frac{d^2T(t)}{dt^2} = \frac{1}{u(x,y)}\Delta u(x,u).$$

This is an equality between functions of different variables, so both sides have to be equal to a same constant $-\lambda < 0$. The sign must be negative because the force is restoring. For the temporal factor we get the harmonic oscillator differential equation (see [1, Sections 2 and 5]),

$$\frac{d^2T}{dt^2} + \lambda T = 0,$$

which has general solution

$$T(t) = c_1\cos(\sqrt{\lambda}t) + c_2\sin(\sqrt{\lambda}t),\ \ c_1, c_2 \in \mathbb{R}.$$

So $\frac{\sqrt{\lambda}}{2\pi}$ models the frequency of the stationary wave v, that is, $\frac{\sqrt{\lambda}}{2\pi}$ is one of the fundamental frequencies of the membrane.

For the spacial factor we get

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

so u must be an eigenfunction of the Dirichlet Laplacian.

Observe that the eigenvalues λ are in unique correspondence with the fundamental frequencies of the membrane. Therefore, this provides a mathematical model to determine them: the fundamental frequencies of the membrane are the square roots of those positive numbers λ, divided by 2π, for which (2) has a non-trivial solution.

Because of the physical motivation of the problem we will restrict ourselves to functions $u : \Omega \to \mathbb{R}$ continuous in $\overline{\Omega}$ with piecewise continuous derivatives up to second order in $\overline{\Omega}$ (see a precise definition in Section 2).

Definition 1.1. *A non-trivial piecewise $\mathcal{C}^2$ function in Ω solving the Dirichlet problem (2) is called an eigenfunction of the Dirichlet Laplacian, or simply Dirichlet eigenfunction. The set of $\lambda \in \mathbb{R}$ for which there exists a non-trivial solution of (2) is called spectrum of Ω and will be denoted Spec(Ω). Their elements are called eigenvalues of the Dirichlet Laplacian, or simply Dirichlet eigenvalues, and will be denoted $\lambda(\Omega)$, or simply λ if clear by the context.*

Equations (1) and (2) have been deeply studied and many of their properties are standard. The ones that we will need in this work are presented below, with references where to find the proofs.

1.1.1 Classical results for the Laplace operator

The following result, a proof of which can be found in [7, Section 2.4.3], ensures that the wave equation (1) has a unique solution, according to the physical notion that the sound of a drum, given the initial conditions, is unique.

Theorem 1.2. *Let Ω be a domain of $\mathbb{R}^2$. Then, there exists at most one function $v : [0, \infty) \times \Omega \to \mathbb{R}$ solving (1).*

Thanks to the following result, due to Pockels [20], we can tag the eigenvalues of the Dirichlet Laplacian with natural numbers. In this work, natural numbers do not include zero.

Theorem 1.3. *All the eigenvalues of the Dirichlet Laplacian form a non-decreasing sequence of positive numbers tending towards infinity, that is, Spec$(\Omega) = \{\lambda_n\}_{n\in\mathbb{N}}$ with $0 < \lambda_1 \leq \lambda_2 \leq \cdots \nearrow \infty$.*

In general, one should not expect to have closed formulas for the values λ_n, because equation (2) is in general not solvable analytically. In Section 2.2 we will see some particular domains for which we can compute explicitly the eigenvalues.

A key property throughout this work is that the normalized eigenfunctions of the Dirichlet Laplacian form an orthonormal basis of the Lebesgue space $L^2(\Omega)$ equipped with the usual scalar product

$$\langle f, g\rangle := \int_\Omega f(x)\overline{g(x)}dx.$$

Theorem 1.4. *Let Ω be a domain of $\mathbb{R}^2$ with spectrum Spec$(\Omega) = \{\lambda_n\}_{n\in\mathbb{N}}$, where $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$. For all $j \geq 1$, let $u_j \in L^2(\Omega)$ be the eigenfunction of the Dirichlet Laplacian with eigenvalue λ_j and normalized so that $\|u_j\| = 1$. Then the set $\{u_j\}_{j\geq 1}$ is an orthonormal basis of $L^2(\Omega)$.*

This result follows from the Spectral Theorem on compact self-adjoint operators applied to the inverse of the Laplacian (see [2, Theorems 9.8 and 9.9]).

Observe that this theorem provides a method to solving the wave equation (1): assume that the initial position f and velocity g are in $L^2(\Omega)$ and define the solution of the initial wave equation (1) as the following superposition of stationary waves

$$v(t, x, y) = \sum_{j=1}^{\infty} \left(\langle f, u_j\rangle \cos(\sqrt{\lambda_j}t) + \frac{1}{\sqrt{\lambda_j}}\langle g, u_j\rangle \sin(\sqrt{\lambda_j}t) \right) u_j(x, y), \tag{3}$$

where u_j are the eigenfunctions of the Dirichlet Laplacian of eigenvalue λ_j, for all $j \geq 1$.

By construction, each addend is a solution of the PDE with boundary condition in (1). Thus, formally, interchanging derivatives and summation, (3) satisfies the PDE and boundary condition of (1). Moreover, since by Theorem 1.4 the functions

u_j form an orthonormal base of $L^2(\Omega)$, the initial conditions of (1) are also satisfied. Thus, (3) is a solution of the wave equation.

Note that, since the solution of the wave equation (1) is unique, from (3) we deduce that the initial conditions characterize the solution of (1). Then all the possible ways in which a drum can vibrate are characterized by their initial positions and velocities.

More specifically, all the possible ways in which a drum can vibrate can be expressed as a superposition of stationary waves, the coefficients of which are characterized by the initial conditions. This way, any sound emerging from the drum can be decomposed into a superposition of fundamental frequencies. Thus, the problem of characterizing the sound of a drum reduces to finding its fundamental frequencies and, as we have already discussed, this is reduced in turn to finding the eigenvalues of the Dirichlet Laplacian.

1.2 Can one hear the shape of a drum?

We are familiar with the fact that the frequencies of a drum depend on the shape of the drum: big drums generate lower sounds than small drums. In fact, the shape of a drum characterizes its frequencies.

In 1966 Kac [14] went a step further and asked if the converse holds, that is, if the set of fundamental frequencies of a drum characterizes its shape. Note that if the answer were affirmative, from the set of fundamental frequencies we could, theoretically, uniquely reconstruct the shape of the drum, in analogy with what our brain does when reconstructing the sound of the drum from its frequencies. With this analogy in mind, Kac titled his article [14] with the poetic question "Can one hear the shape of a drum?"

More specifically, Kac asked if two isospectral domains Ω_1 and Ω_2, that is, domains with the same Dirichlet spectrum $Spec(\Omega_1) = Spec(\Omega_2)$, are necessarily isometric, that is, that one can be obtained from the other through translations, rotations or reflections.

Although Kac's question can be mathematically generalized to arbitrary dimensions, and even to manifolds, it was posed for domains in $\mathbb{R}^2$. That was appropriate since Kac was aware of Milnor's work [17] (done just a couple of years before), who constructed two non-congruent sixteen dimensional tori whose Dirichlet spectra are identical.

On the other hand, Kac was also aware of Weyl's law [25] (which we will study in detail in Section 4), an asymptotic formula which gives the area of Ω in terms of the eigenvalues of the Dirichlet Laplacian.

Theorem 1.5. *Let Ω be a Jordan measurable domain with area $|\Omega|$ and $Spec(\Omega) = \{\lambda_n\}_{n\in\mathbb{N}}$. Then*

$$\lim_{r\to\infty} \frac{\#\{\lambda_n \in Spec(\Omega) : \ \lambda_n < r\}}{r} = \frac{|\Omega|}{4\pi}. \tag{4}$$

As we shall see in Section 4, the condition for a domain to be Jordan measurable is not quite restrictive and corresponds to domains that can be well approximated by finite unions of rectangles.

In his article [14] Kac studied the asymptotic behaviour of the eigenvalues of the Dirichlet Laplacian from a physical point of view, what lead him to conjecture the following expansion formula.

Conjecture 1.6. *Let Ω be a domain of $\mathbb{R}^2$ with area $|\Omega|$, perimeter $\ell(\partial\Omega)$ and h number of holes. Let $Spec(\Omega) = \{\lambda_n\}_{n\in\mathbb{N}}$. Then* [(2)]

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} = \frac{|\Omega|}{4\pi t} + \frac{\ell(\partial\Omega)}{8\sqrt{\pi t}} + \frac{1}{6}(1-h) + O(t), \quad t \to 0^+. \tag{5}$$

As we will prove in Section 2.1, when they exist, the limit in (4) and

$$\lim_{t\to 0^+} t\sum_{n=1}^{\infty} e^{-\lambda_n t}$$

always have the same value, so the conjectured expansion (5) was consistent with the already known result of Weyl.

It was therefore reasonable to think that the expansion (5) with more terms would involve more geometrical parameters of Ω; so conjecturing that $Spec(\Omega)$ could fully characterize Ω was reasonable.

In Section 3 we will show that the answer to Kac's question is negative by providing an example of two non-isometric domains that are isospectral. The construction of such domains is inspired by an idea from group theory concerning permutation representations, that we shall describe. Having the domains in hand, checking that they are isospectral will be done by inspection, namely, showing that for each Dirichlet eigenfunction of one domain there is another Dirichlet eigenfunction of the other domain with the same eigenvalue. In conclusion, in Section 3 we will prove the following result:

Theorem 1.7. *There exist two non-isometric domains of $\mathbb{R}^2$ that are isospectral.*

Despite this example closes Kac's question answering it negatively, several geometrical properties of the domain can be recovered from its spectrum.

In Section 4 we will prove Weyl's law (4) by following the next steps: first, we will check it for rectangles; then, for unions of rectangles (domains looking like a grid); finally, we will prove it for domains that can be well approximated by unions of rectangles, which are called Jordan measurable domains. The way in which we will be able to execute the last two steps will be through characterizing the Dirichlet eigenvalues as extremes of variational problems. To do so, we will need the structure of Hilbert space of $L^2(\Omega)$ and the orthogonality of the eigenfunctions, guaranteed by Theorem 1.4. From these extreme problems will follow a monotonicity order of

[(2)] In Kac's original work [14] some factors of (5) are different due to a different normalization of equation (2).

the eigenvalues of nested domains. This result will allow to relate the eigenvalues of a Jordan measurable domain with the ones of unions of rectangles approximating it from inside and from outside.

Finally, in Section 5 we consider Kac's problem in arbitrary dimensions and in compact, smooth Riemannian manifolds. We will prove that only in $\mathbb{R}^1$ a "drum" is characterized by its spectrum, and that the example constructed in Section 3 allows to construct non-isometric drums of $\mathbb{R}^d$ that are isospectral, for $d > 2$. We will also present some generalizations of Weyl's law and Kac's conjectured expansion formula and comment variants of Kac's problem (with restricted hypothesis) that allow to characterize the shape of a drum from its spectrum.

2 Mathematical setup and some examples

The Dirichlet problem (2) is well-defined for $\mathcal{C}^2$ functions in Ω. Nevertheless, because of the physical motivation of the problem, we will work with the larger class of functions defined below.

Definition 2.1. *Let $\mathcal{C}^2_*(\Omega)$ be the set of functions $u : \Omega \to \mathbb{R}$ such that u is continuous in $\overline{\Omega}$ with piecewise continuous derivatives up to second order in $\overline{\Omega}$.*

In this two dimensional study, by being piecewise continuous we mean the following:

Definition 2.2. *A function $w : \Omega \to \mathbb{R}$ is said to be piecewise continuous if*

i) there exist domains $\Omega_1, \dots, \Omega_k \subseteq \Omega$ such that $\overline{\Omega} = \overline{\Omega}_1 \cup \cdots \cup \overline{\Omega}_k$,

ii) the domains $\Omega_1, \dots, \Omega_k$ satisfy that $\overline{\Omega}_i \cap \overline{\Omega}_j = \partial\Omega_i \cap \partial\Omega_j$ $\forall i \neq j$, and

iii) w is continuous in Ω_i, $\forall i = 1, \dots, k$.

Remark 2.3. Observe that the derivatives of a $\mathcal{C}^2_*(\Omega)$ function may be discontinuous, but only in a set of Lebesgue measure zero, namely, on the internal boundaries of the domains $\Omega_1, \dots, \Omega_k \subset \Omega$.

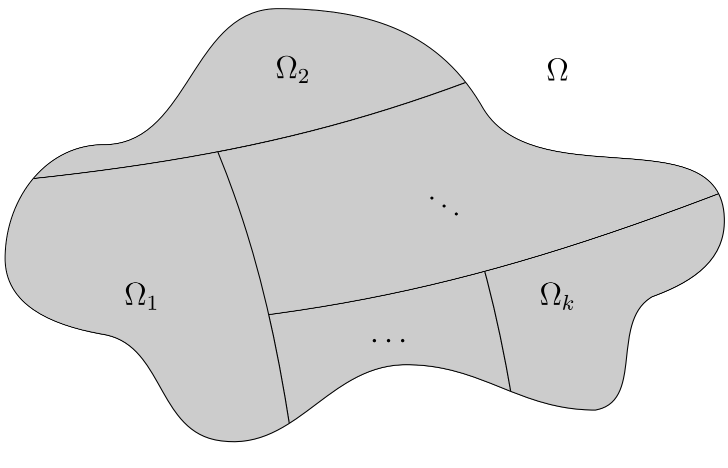


Figure 1: Example of the domain Ω of a piecewise continuous function: in each subdomain Ω_j the function is continuous.

Note that for the class of functions $\mathcal{C}^2_*(\Omega)$, the PDE of (2) may not be well-defined. We understand the equation $\Delta u + \lambda u = 0$ to hold everywhere except on the internal boundaries where the derivatives of u are discontinuous, that is, *almost everywhere* in Ω (in the Lebesgue sense).

Remark 2.4. It turns out, as proved in [7, Section 6.3.1], that a function $u \in \mathcal{C}^2_*(\Omega)$ being a solution of $\Delta u + \lambda u = 0$ has more regularity than just being $\mathcal{C}^2_*(\Omega)$, namely, it is $\mathcal{C}^\infty(\Omega)$ (a standard result known as "interior regularity"). Thus, for convenience we will work in the space of $\mathcal{C}^2_*(\Omega)$ functions, but in practice all the regularity needed for the problem to make sense is guaranteed.

In the proof of properties of the Dirichlet eigenvalues, the eigenvalues of a related problem will be an important tool. The problem at hand is the Neumann Laplacian, although, as far as we know, its Neumann eigenvalues don't have any physical interpretation with drums:

$$\begin{cases} \Delta \tilde{u} + \tilde{\lambda} \tilde{u} = 0 & \text{in } \Omega \\ \dfrac{\partial \tilde{u}}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

Here n is the normal vector on the boundary $\partial\Omega$ pointing outside the domain Ω.

Definition 2.5. *A solution $\tilde{u} \in \mathcal{C}^2_*(\Omega) \setminus \{0\}$ of (6) is called an eigenfunction of the Neumann Laplacian, or simply Neumann eigenfunction. The $\tilde{\lambda} \in \mathbb{R}$ for which there exists a non-trivial solution of (6) are called eigenvalues of the Neumann Laplacian, or simply Neumann eigenvalues, and will be denoted $\tilde{\lambda}(\Omega)$, or simply $\tilde{\lambda}$ if clear by the context.*

Whenever we talk about the spectrum of Ω we will refer to the eigenvalues of the Dirichlet Laplacian, not to the Neumann ones.

The results of Section 1.1.1 hold analogously for the Neumann problem (6). Namely, all the Neumann eigenvalues form a non-decreasing sequence of nonnegative numbers tending towards infinity, $0 = \tilde{\lambda}_1 < \tilde{\lambda}_2 \leq \cdots \nearrow \infty$, and the normalized Neumann eigenfunctions form an orthonormal basis of $L^2(\Omega)$.

2.1 Equivalence of Weyl's law and Kac's first term

To support Kac's conjectured expansion formula (Conjecture 1.6), let's start checking that it is consistent with Weyl's law (Theorem 1.5). The following result implies that Weyl's law and the first term of Kac's expansion formula are equivalent limits.

Proposition 2.6. *Let $\{\lambda_n\}_{n\geq 1}$ be a sequence of non-decreasing positive numbers tending towards infinity. Then the following limits are equivalent:*

i) $\displaystyle\lim_{r\to\infty} \frac{\#\{\lambda_n < r\}}{r} = L$

ii) $\displaystyle\lim_{t\to 0^+} t \sum_{n=1}^{\infty} e^{-\lambda_n t} = L$

Proof. $\boxed{i) \implies ii)}$ Let μ be the measure

$$\mu = \sum_{n=1}^{\infty} \delta_{\lambda_n},$$

where δ_{λ_n} is Dirac's delta at λ_n. Observe that

$$\mu(\{\lambda < r\}) = \#\{\lambda_n < r\}. \tag{7}$$

By *i)*, for all $\varepsilon > 0$ there exists $M > 0$ such that for all $r > M$

$$L - \varepsilon < \frac{\mu(\{\lambda < r\})}{r} < L + \varepsilon. \tag{8}$$

Using the distribution function we have

$$\sum_{n=1}^{\infty} te^{-\lambda_n t} = \int_0^{\infty} te^{-\lambda t} d\mu(\lambda) = \int_0^1 t\mu\left(\{e^{-\lambda t} > s\}\right) ds = \int_0^1 t\mu\left(\left\{\lambda < \frac{-\ln s}{t}\right\}\right) ds.$$

After the substitution $r = \frac{-\ln s}{t}$ we obtain

$$t\sum_{n=1}^{\infty} e^{-\lambda_n t} = \int_0^{\infty} rt^2 \frac{\mu(\{\lambda < r\})}{r} e^{-rt} dr.$$

Then, fixed $\varepsilon > 0$, we can write this as

$$t\sum_{n=1}^{\infty} e^{-\lambda_n t} = \int_0^{M} rt^2 \frac{\mu(\{\lambda < r\})}{r} e^{-rt} dr + \int_M^{\infty} rt^2 \frac{\mu(\{\lambda < r\})}{r} e^{-rt} dr. \tag{9}$$

In the limit $t \to 0^+$ the first integral will vanish, so let's see which is the behaviour of the second integral. By (8) we can bound from above

$$\int_M^{\infty} rt^2 \frac{\mu(\{\lambda < r\})}{r} e^{-rt} dr < \int_M^{\infty} rt^2 \left(L + \varepsilon\right) e^{-rt} dr = \left(L + \varepsilon\right) \int_{tM}^{\infty} xe^{-x} dx,$$

where the substitution $x = rt$ has been applied in the last step. Similarly, we can bound from below

$$\int_M^{\infty} rt^2 \frac{\mu(\{\lambda < r\})}{r} e^{-rt} dr > \left(L - \varepsilon\right) \int_{tM}^{\infty} xe^{-x} dx.$$

Since $\lim_{t\to 0^+} \int_{tM}^{\infty} xe^{-x} dx = \Gamma(2) = 1$, we finally deduce from (9) that

$$L - \varepsilon < \lim_{t\to 0^+} t\sum_{n=1}^{\infty} e^{-\lambda_n t} < L + \varepsilon.$$

$\boxed{ii) \implies i)}$ This follows from a theorem due to Karamata [15], which we prove below, following [22, Theorem 10.3].

Theorem 2.7 (Karamata's Tauberian theorem). *Let μ be a (positive) Borel measure on $[0, \infty)$. Suppose that $\int_0^{\infty} e^{-tx} d\mu(x) < \infty$ for all $t > 0$ and that for some $\gamma > 0,\ D > 0$:*

$$\lim_{t\to 0^+} t^{\gamma} \int_0^{\infty} e^{-tx} d\mu(x) = D.$$

Then

$$\lim_{r\to\infty} \frac{\mu([0, r))}{r^{\gamma}} = \frac{D}{\Gamma(\gamma + 1)}. \tag{10}$$

Assuming this, if the limit $ii)$ holds then

$$L = \lim_{t \to 0^+} t \sum_{n=1}^{\infty} e^{-\lambda_n t} = \lim_{t \to 0^+} t \int_0^{\infty} e^{-tx} d\mu(x),$$

so by Karamata's Tauberian theorem with $\gamma = 1$ and (7) we obtain the limit $i)$.

It therefore only remains to prove Karamata's Tauberian theorem: fix $\gamma, t > 0$ and let μ_t be the measure in $[0, \infty)$ given by

$$\mu_t(A) = t^{\gamma} \mu\left(\left\{t^{-1}a : a \in A\right\}\right),$$

for $A \subset [0, \infty)$. Let $d\nu = x^{\gamma-1}dx$. Since

$$\frac{D}{\Gamma(\gamma)} \nu([0,1)) = \frac{D}{\Gamma(\gamma)} \int_0^1 x^{\gamma-1} dx = \frac{D}{\Gamma(\gamma+1)},$$

and under the substitution $t = 1/r$ we have

$$\lim_{t \to 0^+} \mu_t([0,1)) = \lim_{t \to 0^+} t^{\gamma} \mu\left(\left[0, t^{-1}\right)\right) = \lim_{r \to \infty} \frac{\mu([0,r))}{r^{\gamma}},$$

equation (10) can be rewritten as

$$\lim_{t \to 0^+} \mu_t([0,1)) = \frac{D}{\Gamma(\gamma)} \nu([0,1)). \tag{11}$$

We want to see this as an equality between measures of the same set. Thanks to the following lemma, which is quite standard in measure theory but that for the sake of completeness we will prove later, it suffices to show that

$$\lim_{t \to 0^+} \int_0^{\infty} f(x) d\mu_t(x) = \frac{D}{\Gamma(\gamma+1)} \int_0^{\infty} f(x) d\nu(x) \tag{12}$$

for all $f \in \mathcal{C}_c([0, \infty))$, that is, for all continuous real valued functions with compact support in $[0, \infty)$.

Lemma 2.8. *If* (12) *holds for all* $f \in \mathcal{C}_c([0, \infty))$*, then* (11) *holds.*

On the one hand, by hypothesis we have

$$\lim_{t \to 0^+} \int_0^{\infty} e^{-x} d\mu_t(x) = \lim_{t \to 0^+} t^{\gamma} \int_0^{\infty} e^{-tx} d\mu(x) = D = \frac{D}{\Gamma(\gamma+1)} \int_0^{\infty} e^{-x} d\nu(x),$$

so the measures $e^{-x} d\mu_t(x)$ are uniformly bounded for all $t > 0$ and (12) holds for functions of the form $f(x) = e^{-nx}$ with $n \in \mathbb{N}$ (under the substitution $x \mapsto nx$). Then, by linearity, (12) also holds for all polynomials in the variable e^{-x} with independent term equal to zero.

On the other hand, by the Stone-Weierstrass theorem we have the following result, which we shall prove later.

Lemma 2.9. *The polynomials in the variable e^{-x} with independent term equal to zero are dense, with the norm $\|\cdot\|_\infty$, in the set of continuous real valued functions on $[0,\infty)$ going to zero at infinity.*

Thus, the functions $f \in \mathcal{C}_c([0,\infty))$ can be approximated by such polynomials in the variable e^{-x}. This proves (12). □

It only remains to check that it is enough to show (12) in order to prove (11), and that every continuous real valued function on $[0,\infty)$ going to zero at infinity can be approximated by polynomials in the variable e^{-x} with independent term equal to zero.

Proof of Lemma 2.9. Let $f : [0,\infty) \to \mathbb{R}$ be such that $\lim_{x\to\infty} f(x) = 0$. Define

$$\begin{array}{rccl} g: & [0,1] & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \begin{cases} f(-\ln x) & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0. \end{cases} \end{array}$$

Clearly, g is continuous in the compact $[0,1]$. By the Stone-Weierstrass theorem, for all $\varepsilon > 0$ there exists a polynomial $\tilde{p}(x)$ such that $\|g-\tilde{p}\|_\infty < \varepsilon/2$. In particular,

$$|\tilde{p}(0)| = |g(0) - \tilde{p}(0)| < \varepsilon/2$$

Define the polynomial $p(x) := \tilde{p}(x) - \tilde{p}(0)$. Note that the independent term of p is equal to zero and that, by the triangle inequality,

$$\|g - p\|_\infty \le \|g - \tilde{p}\|_\infty + \|\tilde{p} - p\|_\infty < \varepsilon.$$

Moreover, since for all $t \ge 0$ we have that $x = e^{-t} \in [0,1]$ and $g(e^{-t}) = f(t)$, we get, for all $t \ge 0$,

$$\left|f(t) - p\left(e^{-t}\right)\right| < \varepsilon,$$

which implies that $\|f(x) - p\left(e^{-x}\right)\|_\infty < \varepsilon$. □

Proof of Lemma 2.8. Let $\chi_{[0,1)}(x)$ be the characteristic function on $[0,1)$, that is,

$$\chi_{[0,1)}(x) = \begin{cases} 1 & \text{if } x \in [0,1) \\ 0 & \text{if } x \notin [0,1). \end{cases}$$

Our goal is to show that

$$\lim_{t\to 0^+} \int_0^\infty \chi_{[0,1)}(x)d\mu_t(x) = \frac{D}{\Gamma(\gamma+1)} \int_0^\infty \chi_{[0,1)}(x)d\nu(x). \tag{13}$$

Consider the decreasing sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{C}_c([0,\infty))$ defined by

$$f_n(x) := \begin{cases} 1 & \text{if } x \in [0,1) \\ -nx + 1 + n & \text{if } x \in [1, 1+\frac{1}{n}) \\ 0 & \text{if } x \in [1+\frac{1}{n}, \infty). \end{cases}$$

Clearly f_n converge pointwise to $\chi_{[0,1)}$, so by the monotone convergence theorem we have

$$\lim_{n\to\infty} \int_0^\infty f_n(x) d\mu_t(x) = \int_0^\infty \chi_{[0,1)}(x) d\mu_t(x),$$

for all $t > 0$, and

$$\lim_{n\to\infty} \int_0^\infty f_n(x) d\nu(x) = \int_0^\infty \chi_{[0,1)}(x) d\nu(x).$$

By hypothesis (12) holds for each f_n. Therefore, for all $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $0 < t < \delta_\varepsilon$ then

$$-\varepsilon + \frac{D}{\Gamma(\gamma+1)} \int_0^\infty f_n(x) d\nu(x) < \int_0^\infty f_n(x) d\mu_t(x) < \varepsilon + \frac{D}{\Gamma(\gamma+1)} \int_0^\infty f_n(x) d\nu(x),$$

for all $n \in \mathbb{N}$. Taking the limit $n \to \infty$ we get

$$-\varepsilon + \frac{D}{\Gamma(\gamma+1)} \int_0^\infty \chi_{[0,1)}(x) d\nu(x) \leq \int_0^\infty \chi_{[0,1)} d\mu_t(x) \leq \varepsilon + \frac{D}{\Gamma(\gamma+1)} \int_0^\infty \chi_{[0,1)} d\nu(x),$$

for all $0 < t < \delta_\varepsilon$. Taking now the limit $\varepsilon \to 0^+$ we get (13). $\square$

2.2 Examples: the rectangle and the right isosceles triangle

Given an arbitrary domain Ω, hardly ever one can find closed formulas for the eigenfunctions and the eigenvalues of the Dirichlet and Neumann Laplacians. Only in few cases in which the domain is highly symmetric it is possible to give the spectrum explicitly. This is the case for a rectangle or a right isosceles triangle, as we shall see now.

Through these examples we will see how the geometry of Ω characterizes its spectrum. Moreover, the computations will be useful for further developments.

2.2.1 Rectangle

Let $R = \{(x, y) \in \mathbb{R}^2 : 0 < x < a,\ 0 < y < b\}$ be a rectangle of sides a and b, as shown in Figure 2.

In order to find the eigenfunctions and the eigenvalues of the Dirichlet Laplacian, let's solve the Dirichlet problem (2) by separation of variables: a function of the form $u(x, y) = X(x)Y(y)$ is a solution of (2) if and only if

$$\begin{cases} X''(x)Y(y) + X(x)Y''(y) + \lambda X(x)Y(y) = 0 & \text{in } R \\ X(a)Y(y) = 0 = X(0)Y(y) & y \in [0, b] \\ X(x)Y(b) = 0 = X(x)Y(0) & x \in [0, a]. \end{cases}$$

After dividing the differential equation by $X(x)Y(y)$, this is

$$\begin{cases} \dfrac{X''(x)}{X(x)} + \dfrac{Y''(y)}{Y(y)} + \lambda = 0 & \text{in } R \\ X(a) = 0 = X(0) \\ Y(b) = 0 = Y(0). \end{cases}$$

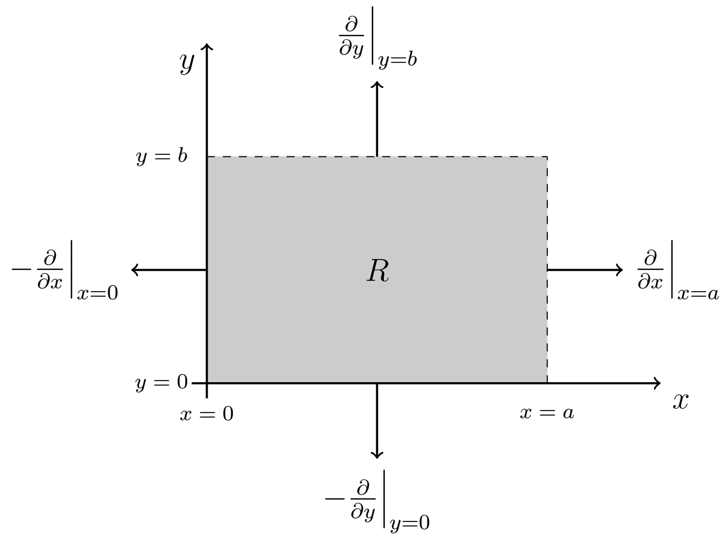


Figure 2: Rectangle of sides a and b with the normal vectors on its boundary.

Since the resulting differential equation is a linear combination of functions of different variables whose sum is constant we must have, for some $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta = \lambda$,

$$\begin{cases} \dfrac{X''(x)}{X(x)} = -\alpha & x \in [0,a] \\ X(a) = 0 = X(0), \end{cases} \qquad \begin{cases} \dfrac{Y''(y)}{Y(y)} = -\beta & y \in [0,b] \\ Y(b) = 0 = Y(0). \end{cases}$$

From standard ODE theory (see [1, Section 2]) this implies that α and β must be of the form

$$\alpha = \frac{\ell^2}{a^2}\pi^2, \qquad \beta = \frac{m^2}{b^2}\pi^2 \qquad \text{with } \ell, m \in \mathbb{N}$$

and that $X(x)$ and $Y(y)$ must be proportional, respectively, to

$$X_\ell(x) = \sin\left(\frac{\ell \pi x}{a}\right), \qquad Y_m(y) = \sin\left(\frac{m\pi y}{b}\right) \qquad \text{with } \ell, m \in \mathbb{N}.$$

Therefore we see that the functions

$$u_{\ell,m}(x,y) = k_{\ell,m} \sin\left(\frac{\ell \pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \qquad (\ell, m \in \mathbb{N},\ k_{\ell,m} \in \mathbb{R} \setminus \{0\})$$

are all eigenfunctions of the Dirichlet Laplacian, with eigenvalues

$$\lambda_{\ell,m} = \left[\left(\frac{\ell}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right]\pi^2 \qquad (\ell, m \in \mathbb{N}). \tag{14}$$

Moreover, up to a normalizing constant, the set $\{u_{\ell,m}\}_{\ell,m\in\mathbb{N}}$ is the Fourier basis of the space of $L^2(R)$ functions vanishing on ∂R, which is well-known to be orthonormal and complete. Therefore, if it existed another eigenfunction $u_\lambda(x,y)$ of the Dirichlet Laplacian of eigenvalue λ different to all $\lambda_{\ell,m}$ of (14), by completeness we should have $u_\lambda(x,y) = \sum\limits_{\ell,m} c_{\ell,m} u_{\ell,m}(x,y)$ and this would contradict Theorem 1.4.

In summary, we have proved the following result:

Proposition 2.10. *Let R be a rectangle of sides a and b. Then the set of eigenvalues of the Dirichlet Laplacian is*

$$Spec(R) = \left\{ \lambda_{\ell,m} = \left[\left(\frac{\ell}{a}\right)^2 + \left(\frac{m}{b}\right)^2 \right] \pi^2 : \ \ell, m \in \mathbb{N} \right\}$$

and an orthogonal basis of eigenfunctions is

$$\left\{ u_{\ell,m}(x,y) = \sin\left(\frac{\ell\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) : \ \ell, m \in \mathbb{N} \right\}.$$

Analogously we obtain, for the eigenvalues and eigenfunctions of the Neumann Laplacian, the following result:

Proposition 2.11. *Let R be a rectangle of sides a and b. Then the set of eigenvalues of the Neumann Laplacian is*

$$\left\{ \tilde{\lambda}_{\ell,m} = \left[\left(\frac{\ell}{a}\right)^2 + \left(\frac{m}{b}\right)^2 \right] \pi^2 : \ \ell, m \in \mathbb{N} \cup \{0\} \right\}$$

and an orthogonal basis of eigenfunctions is

$$\left\{ \tilde{u}_{\ell,m}(x,y) = \cos\left(\frac{\ell\pi x}{a}\right) \cos\left(\frac{m\pi x}{b}\right) : \ \ell, m \in \mathbb{N} \cup \{0\} \right\}.$$

2.2.2 Right isosceles triangle

Let $T = \{(x,y) \in \mathbb{R}^2 : 0 < y < x < c\}$ be a right isosceles triangle of side c and let $R = \{(x,y) \in \mathbb{R}^2 : 0 < x < c, \ 0 < y < c\}$ be the square resulting from adding to T its reflection along the hypotenuse, as shown in Figure 3. The eigenvalues and eigenfunctions of the Dirichlet Laplacian in T can be obtained from the already known solutions in R.

Note that if $u_R(x,y)$ is an eigenfunction of the Dirichlet Laplacian in R of eigenvalue $\lambda(R)$ such that $u_R(x,x) = 0$ for all $x \in [0,c]$, then its restriction to T,

$$u_R(x,y)|_T,$$

is an eigenfunction of the Dirichlet Laplacian in T of eigenvalue $\lambda(R)$; conversely, if $u_T(x,y)$ is an eigenfunction of the Dirichlet Laplacian in T of eigenvalue $\lambda(T)$, then its reflection extension

$$u(x,y) = \begin{cases} u_T(x,y) & \text{if } (x,y) \in T \\ -u_T(y,x) & \text{if } (x,y) \in R \setminus T \end{cases}$$

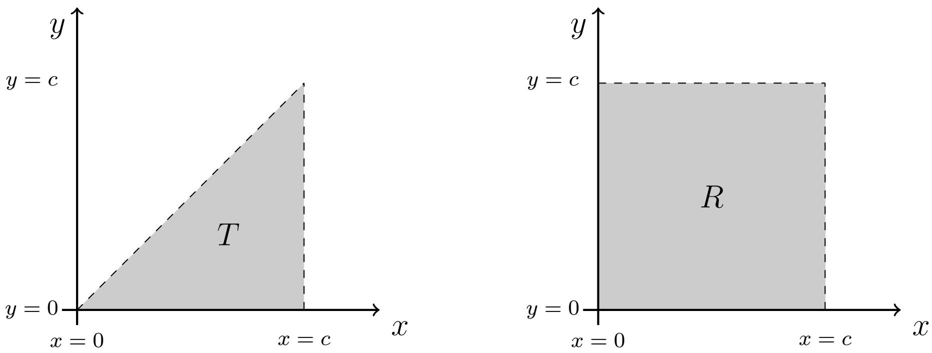


Figure 3: Right isosceles triangle of side c (left) and the square resulting from its mirror reflection along the hypotenuse (right).

is an eigenfunction of the Dirichlet Laplacian in R of eigenvalue $\lambda(T)$.

Following the same notation as in Proposition 2.10 we deduce that all eigenfunctions of T, $u_T(x,y)$, can be obtained as combinations of eigenfunctions of R of the same eigenvalue $\lambda_{\ell,m}$ and vanishing on the hypotenuse. These are of the form:

$$u_T(x,y) = \sum_{i,j\in\{\ell,m\}} k_{i,j} \sin\left(\frac{i\pi x}{c}\right) \sin\left(\frac{j\pi y}{c}\right)$$

with $k_{i,j} \in \mathbb{R}$ (not all zero) such that $u_T(x,x) = 0$. This implies that, up to a normalizing constant,

$$u_T(x,y) = \begin{cases} \sin\left(\frac{\ell\pi x}{c}\right)\sin\left(\frac{m\pi y}{c}\right) - \sin\left(\frac{m\pi x}{c}\right)\sin\left(\frac{\ell\pi y}{c}\right) & \text{if } i=\ell, j=m \\ \sin\left(\frac{m\pi x}{c}\right)\sin\left(\frac{\ell\pi y}{c}\right) - \sin\left(\frac{\ell\pi x}{c}\right)\sin\left(\frac{m\pi y}{c}\right) & \text{if } i=m, j=\ell \\ 0 & \text{if } i=j\in\{\ell,m\} \text{ (trivial).} \end{cases}$$

Observe that the second entry is, up to a sign, equal to the first one. Therefore we have proved the following result:

Proposition 2.12. *Let T be a right isosceles triangle of side c. Then the set of eigenvalues of the Dirichlet Laplacian is*

$$Spec(T) = \left\{ \lambda_{i,j} = \left[\left(\frac{i}{c}\right)^2 + \left(\frac{j}{c}\right)^2\right]\pi^2 : \ i,j\in\mathbb{N},\ i>j \right\}$$

and an orthogonal basis of eigenfunctions is

$$\left\{ u_{i,j}(x,y) = \sin\left(\frac{i\pi x}{c}\right)\sin\left(\frac{j\pi y}{c}\right) - \sin\left(\frac{j\pi x}{c}\right)\sin\left(\frac{i\pi y}{c}\right) : \ i,j\in\mathbb{N},\ i>j \right\}.$$

2.3 Trivial answer for disconnected regions

Kac's question is posed for connected regions because of the physical nature in which it is inspired; if we allow disconnected regions, then it is easy to construct non-isometric regions with the same spectra, thus answering negatively to Kac's question.

To construct such regions we only need the previous computations and the fact that for a disconnected region Ω with connected components $\Omega_1, \dots, \Omega_k$,

$$Spec(\Omega) = Spec(\Omega_1) \cup \dots \cup Spec(\Omega_k).$$

Indeed, if u_j is an eigenfunction of eigenvalue $\lambda(\Omega_j)$ in Ω_j, then the function defined everywhere in Ω as

$$u(x,y) = \begin{cases} u_j(x,y) & \text{if } (x,y) \in \Omega_j \\ 0 & \text{if } (x,y) \in \Omega \setminus \Omega_j \end{cases}$$

is also an eigenfunction of eigenvalue $\lambda(\Omega_j)$ in Ω; conversely, if u is an eigenfunction of eigenvalue $\lambda(\Omega)$ in Ω then it is different from 0 in some component Ω_j, so its restriction $u_j = u|_{\Omega_j}$ is an eigenfunction of eigenvalue $\lambda(\Omega)$ in Ω_j.

Following [4], we consider a region Ω_1 consisting of the disconnected union of a square of side 1 and a right isosceles triangle of side 2, and a region Ω_2 consisting of the disconnected union of a rectangle of sides 1 and 2 and a right isosceles triangle of side $\sqrt{2}$, as shown in Figure 4.

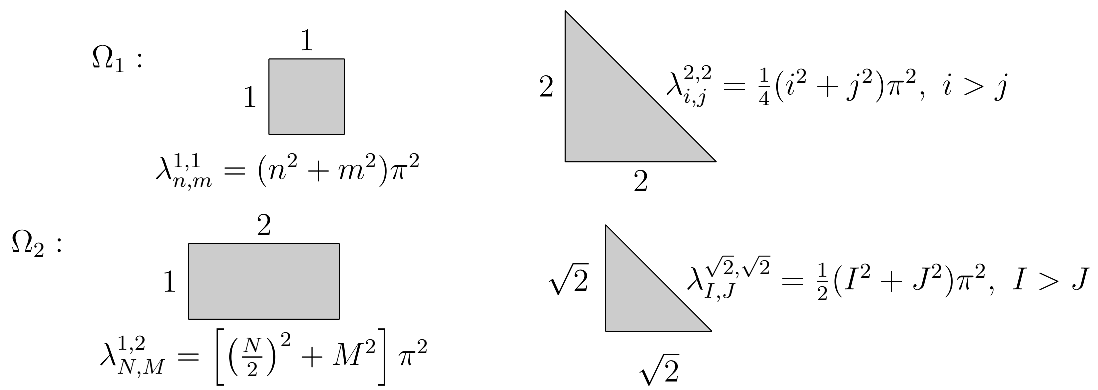


Figure 4: Regions Ω_1 and Ω_2 with the eigenvalues of each connected component, given by Proposition 2.10 and Proposition 2.12.

Proposition 2.13. *The regions Ω_1 and Ω_2 are not isometric but are isospectral, that is,*

$$Spec(\Omega_1) = Spec(\Omega_2).$$

Remark 2.14. Both domains have the same area and the same perimeter, in accordance with Weyl's law and Kac's expansion formula (5).

Proof. That Ω_1 and Ω_2 are not isometric follows from the fact that none of the connected components of Ω_1 is isometric to none of the connected components of Ω_2. That Ω_1 and Ω_2 are isospectral is proved by inspection: note that by the previous argument

$$Spec(\Omega_1) = \left\{(n^2+m^2)\pi^2\right\}_{n,m\geq 1} \cup \left\{\frac{1}{4}(i^2+j^2)\pi^2\right\}_{i>j\geq 1},$$
$$Spec(\Omega_2) = \left\{\left[\left(\frac{N}{2}\right)^2 + M^2\right]\pi^2\right\}_{N,M\geq 1} \cup \left\{\frac{1}{2}(I^2+J^2)\pi^2\right\}_{I>J\geq 1}.$$

This can be written as

$$Spec(\Omega_1) = \left\{(n^2+m^2)\pi^2\right\}_{n,m\geq 1} \cup \left\{\frac{1}{4}(i^2+j^2)\pi^2\right\}_{i>j\geq 1}^{i+j \text{ odd}} \cup \left\{\frac{1}{4}(i^2+j^2)\pi^2\right\}_{i>j\geq 1}^{i+j \text{ even}},$$
$$Spec(\Omega_2) = \left\{\left[\left(\frac{N}{2}\right)^2 + M^2\right]\pi^2\right\}_{N,M\geq 1}^{N \text{ even}} \cup \left\{\left[\left(\frac{N}{2}\right)^2 + M^2\right]\pi^2\right\}_{N,M\geq 1}^{N \text{ odd}}$$
$$\cup \left\{\frac{1}{2}(I^2+J^2)\pi^2\right\}_{I>J\geq 1}.$$

From here it is straightforward to check that $Spec(\Omega_1) = Spec(\Omega_2)$: it suffices to see that each of the three subsets of $Spec(\Omega_1)$ above is equal to one of the three subsets of $Spec(\Omega_2)$.

For example, for all $\lambda_{i,j}^{2,2} \in \left\{\frac{1}{4}(i^2+j^2)\pi^2\right\}_{i>j\geq 1}^{i+j \text{ odd}}$, if i is odd then j is even, and setting $N = i$ and $M = \frac{j}{2}$ we get $\lambda_{i,j}^{2,2} = \lambda_{N,M}^{1,2} \in \left\{\left[\left(\frac{N}{2}\right)^2 + M^2\right]\pi^2\right\}_{N,M\geq 1}^{N \text{ odd}}$; if i is even then j is odd, and setting $N = j$ and $M = \frac{i}{2}$ we get $\lambda_{i,j}^{2,2} = \lambda_{N,M}^{1,2} \in \left\{\left[\left(\frac{N}{2}\right)^2 + M^2\right]\pi^2\right\}_{N,M\geq 1}^{N \text{ odd}}$.

Conversely, for all $\lambda_{N,M}^{1,2} \in \left\{\left[\left(\frac{N}{2}\right)^2 + M^2\right]\pi^2\right\}_{N,M\geq 1}^{N \text{ odd}}$, setting $i = \max\{N, 2M\}$ and $j = \min\{N, 2M\}$ we have that $\lambda_{N,M}^{1,2} = \lambda_{i,j}^{2,2} \in \left\{\frac{1}{4}(i^2+j^2)\pi^2\right\}_{i>j\geq 1}^{i+j \text{ odd}}$. $\square$

3 Construction of two different isospectral domains

In 1992 Gordon, Webb and Wolpert [9, 10] found two non-isometric domains with the same spectrum, thus answering negatively Kac's question. Their construction is inspired by a theorem in group theory due to Sunada [10, Theorem 2.1] (see also Sunada's original work [24]).

In this section we provide an easier example of two isospectral but not isometric domains that Gordon and Webb published later in [11]. The construction is inspired by the same ideas as in [10], but with less algebraic tools.

Without going into details, Sunada's Theorem gives sufficient conditions for isospectral manifolds to exist, and its proof explicitly constructs such manifolds. These conditions require the existence of two almost conjugate subgroups.

Definition 3.1. *Let G be a finite group and let F, H be two subgroups of G. Then F and H are said to be almost conjugate (or Gassmann equivalent) if there exists a bijection from F to H carrying each element $x \in F$ to an element of the form $gxg^{-1} \in H$, where $g \in G$ may depend on x.*

The idea of Gordon, Webb and Wolpert is to adapt Sunada's construction for manifolds to Euclidean domains of the plane. For such an adaptation they need the algebraic tool of *permutation representations*.

Definition 3.2. *Let G be a group. Let X be a finite set and $Perm(X)$ the group of permutations of the elements of X. A permutation representation of G on X is a map $S : G \to Perm(X)$ that assigns to each $g \in G$ a permutation $\overline{g}$ of elements of X that preserves the group structure, that is, such that for all $g, h \in G$, if $\overline{g}, \overline{h}$ and $\overline{gh}$ are the permutations assigned to g, h and gh, respectively, then $\overline{g}\overline{h} = \overline{gh}$. Abusing the notation, we will identify a permutation representation S with its image $S(G)$.*

Remark 3.3. It may happen that two different permutation representations of a group G on a set X, S and S', are almost conjugate. Roughly speaking, if this happens then S and S' look different but encode the same information.

With the previous remark in mind, we sketch the idea to construct two non-isometric domains with same spectra:

From group theory it is known an example of two different permutation representations of the free group G of 3 letters on a set X of 7 elements, S and S', that are almost conjugate. We take X to be 7 right isosceles triangles and we take the three letters of G to be the labels of their edges. We use S and S' to construct two different domains Ω and Ω' by joining the triangles of X through their edges.

Roughly speaking, by inheritance of the properties of S and S', the domains Ω and Ω' look different, so they are candidates to be non-isometric, but they encode similar information, so they are candidates to be isospectral. Finally, we explicitly check that indeed Ω and Ω' are isospectral but not isometric.

Let's carefully follow the steps of the previous sketch to provide a rigorous proof.

3.1 Two permutation representations which are almost conjugate

From [12] we have an indication on where to find almost conjugate permutation representations. Let $G = \langle \alpha, \beta, \gamma \rangle$ be the free group of 3 letters and let $X = \{1, 2, 3, 4, 5, 6, 7\}$. We consider two different permutation representations of G on X, denoted S and S', given by the Cayley graphs of Figure 5.

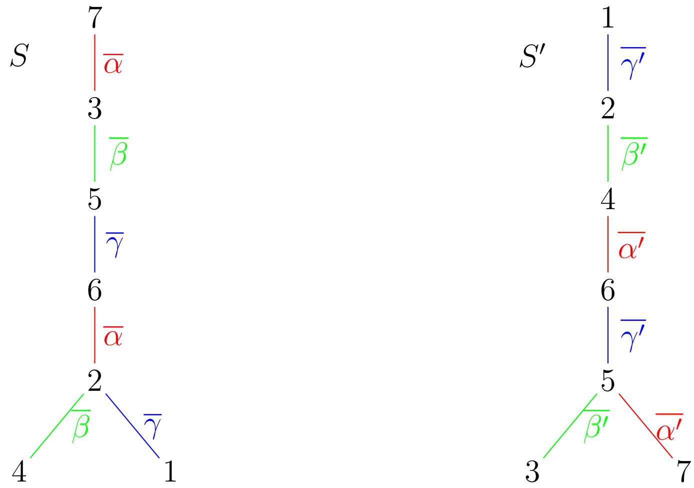


Figure 5: Two different Cayley graphs describing two different permutation representations of G in X, S on the left and S' on the right. Colors are irrelevant now but will be helpful later.

The Cayley graph encodes all the information about the permutation representation. For example, in the graph on the left of Figure 5, the fact that nodes 3 and 7 are joined by an edge $\overline{\alpha}$, expresses the fact that 3 and 7 are permuted by the permutation assigned to α. Thereby, in transposition product notation,

$$\overline{\alpha} = (37)(26) \qquad \overline{\beta} = (35)(24) \qquad \overline{\gamma} = (56)(12)$$

are the generators of the permutation representation S of G on X. Analogously, for the Cayley graph on the right of Figure 5,

$$\overline{\alpha'} = (46)(57) \qquad \overline{\beta'} = (24)(35) \qquad \overline{\gamma'} = (12)(56)$$

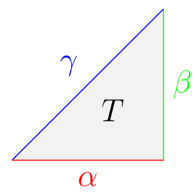
are the generators of the permutation representation S' of G on X.

In [19, Section 3] it is proved that if two subgroups of the symmetric group S_n have the same order and contain the same number of elements of order k, for all divisors k of n, then such subgroups are almost conjugate.

It is straightforward to check that $S = \left\langle \overline{\alpha}, \overline{\beta}, \overline{\gamma} \right\rangle$ and $S = \left\langle \overline{\alpha'}, \overline{\beta'}, \overline{\gamma'} \right\rangle$ are two groups of order $n = 168$ with the same number of elements of order k, for any divisor k of n. So this implies that S and S' are almost conjugate.

3.2 Construction of the domains

We take a model right isosceles triangle T with labels on its edges α, β, γ. Then we choose X to be 7 copies of that model triangle, $X = \{T_1, \dots, T_7\}$ and consider the free group $G = \langle \alpha, \beta, \gamma \rangle$. From X, S and S' we will construct the domains Ω, Ω'.



We use the permutation representation S to construct a domain Ω invariant under the action of S following the steps below.

In the Cayley graph of Figure 5, 7 is joined to 3 by an edge labeled $\overline{\alpha}$, so we reflect the triangle T_7 through its α edge and label the resulting triangle T_3. Since 3 is joined to 5 by an edge labeled $\overline{\beta}$, we reflect the triangle T_3 through its β edge and label the resulting triangle T_5. We continue the process and obtain the domain of Figure 6.

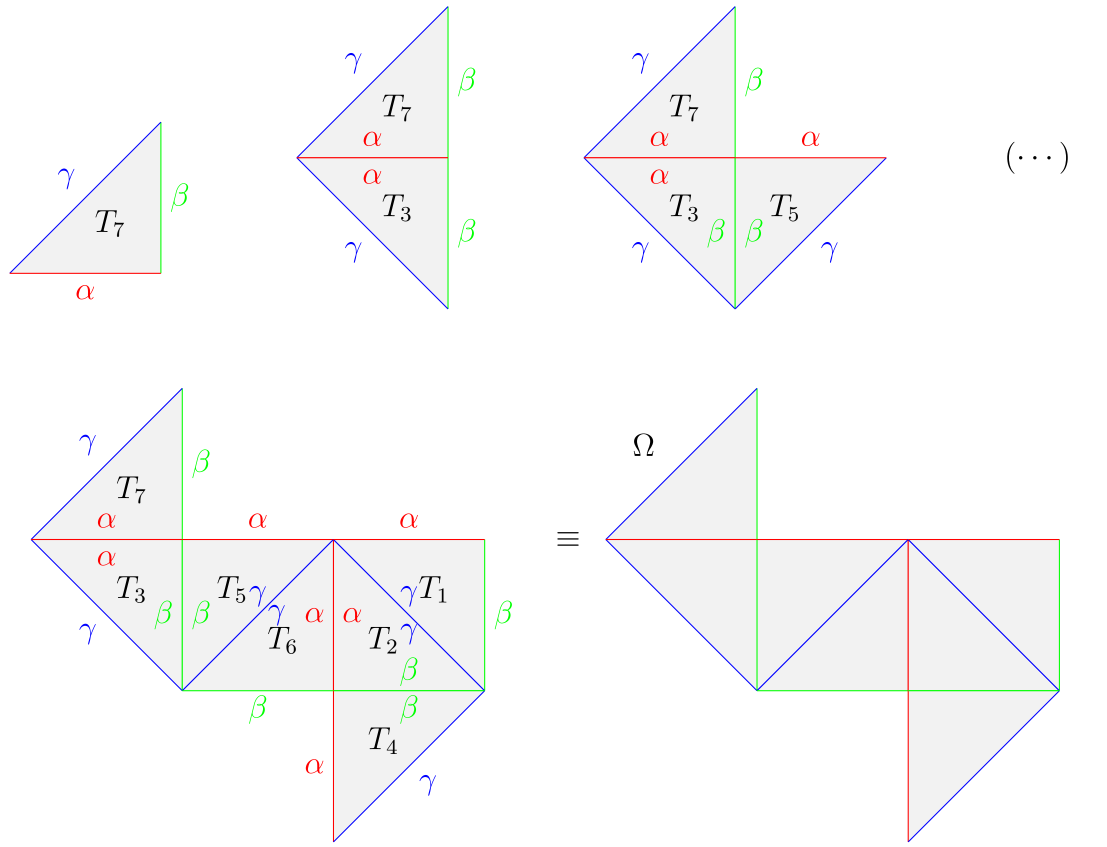


Figure 6: Picture of the first steps to construct the domain Ω, following the previous instructions (above) and the resulting domain (below). Colors are irrelevant but help visualizing reflection symmetries.

Analogously, we use the permutation representation S' to construct the domain Ω' of Figure 7, which is also invariant under the action of S'.

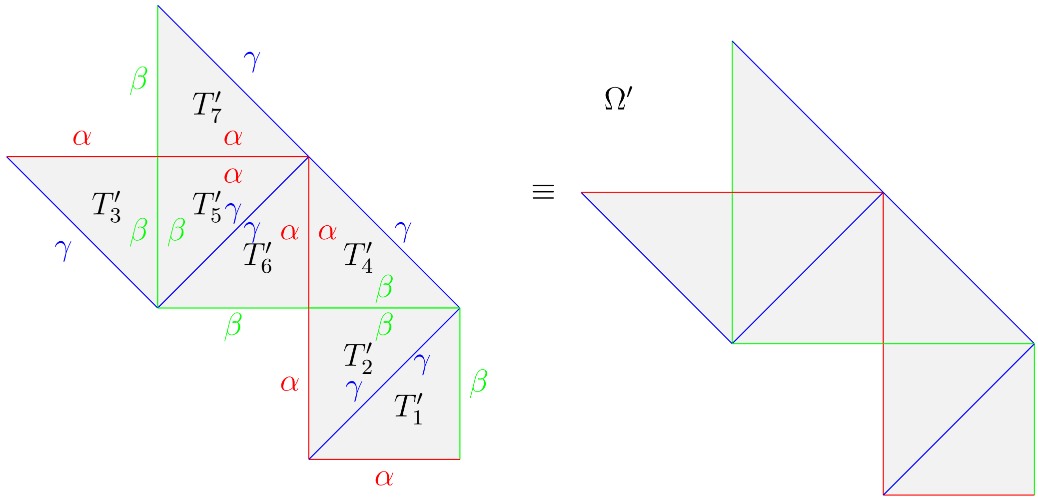


Figure 7: Picture of the domain Ω' constructed from S' following the previous instructions. Colors are irrelevant but help visualize reflection symmetries.

It is straightforward to check that Ω and Ω' are not isometric.

3.3 The domains are isospectral

Our purpose is to construct, from a given eigenfunction ψ of the Dirichlet Laplacian in Ω of eigenvalue λ, an eigenfunction of the Dirichlet Laplacian in Ω' of the same eigenvalue λ. This will ensure that $Spec(\Omega) \subseteq Spec(\Omega')$. The process we will follow to see this inclusion will make clear how to prove the other inclusion.

We will use the following immediate properties of the Dirichlet equation (2).

- **Superposition principle:** a linear combination of solutions of (2) is also a solution of (2).
- **Reflection principle:** let X be a domain whose boundary contains a straight segment L, let $m(X)$ be the mirror reflection of X along that segment and suppose that $\overline{X} \cap \overline{m(X)} = L$. Define the domain $X' = X \cup m(X) \cup L$. Then an eigenfunction ϕ in X can be extended to an eigenfunction ϕ' in X' as the negative of its mirror reflection through the segment L, that is

$$\phi'(x,y) = \begin{cases} \phi(x,y) & \text{if } (x,y) \in X \\ -\phi(m(x,y)) & \text{if } (x,y) \in m(X) \\ 0 & \text{if } (x,y) \in L, \end{cases}$$

  where $m(x,y)$ is the mirror reflection point through the segment L of (x,y).

Observe that the reflection principle is a generalization of the procedure we have followed in Section 2.2.2 to compute the eigenvalues of a right isosceles triangle from the ones of a square.

Let ψ be an eigenfunction of the Dirichlet Laplacian in Ω of eigenvalue λ. For each triangle T_j ($j = 1, \ldots, 7$) of Ω, we denote the restriction of ψ in T_j as ψ_j (with the understanding that it is zero outside the triangle T_j), as shown in Figure 8.

Observe that since $\psi \in \mathcal{C}^2_*(\Omega)$ we have that $\psi_7 = \psi_3$ on the α (red) edge (similarly for the other internal edges), and since ψ is a solution of (2) we also have that $\psi_7 = 0$ on the β (green) and γ (blue) edges (similarly for the other external edges).

By the reflection principle one can define the function ψ_j of triangle T_j in any other triangle T_k, a method that is known as transplantation. For example, ψ_3 can be defined in T_2: first, by reflection along the β edge transplant ψ_3 in T_5 as $-\psi_3$; next, by reflection along the γ edge transplant $-\psi_3$ in T_6 as ψ_3; finally, by reflection along the α edge transplant ψ_3 in T_2 as $-\psi_3$.

We now consider a function ψ' defined in Ω' piecewise on its triangles $T'_1, \ldots, T'_7$, as shown in Figure 8. There, the function $-\psi_3 + \psi_2 - \psi_1$ defined in triangle T'_4 is understood to be the superposition of the transplanted functions ψ_3, ψ_2 and ψ_1 in T'_4 (similarly for the other triangles).

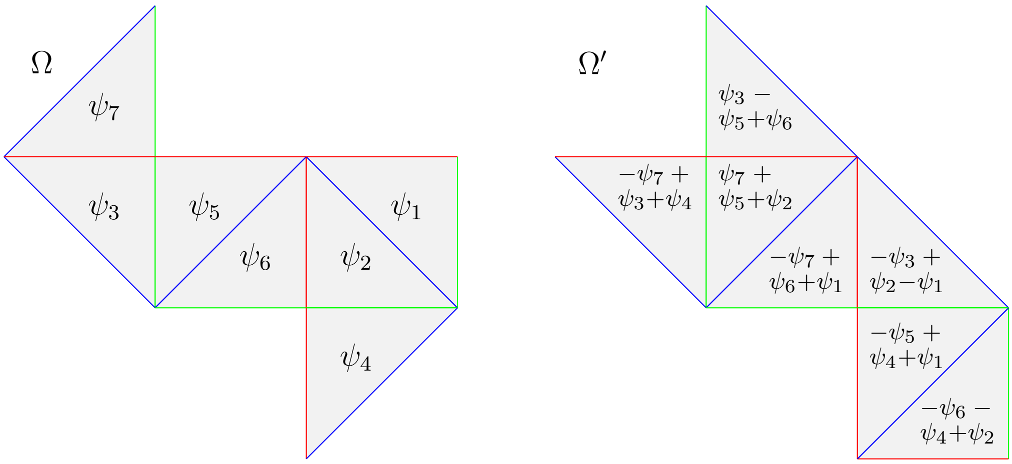


Figure 8: Original eigenfunction in Ω and the transplanted function in Ω'.

We claim that the function ψ' given in Figure 8 is an eigenfunction of the Dirichlet Laplacian in Ω' of eigenvalue λ. Indeed, by linearity, the equation $\Delta\psi' + \lambda\psi' = 0$ is satisfied in each triangle $T'_1, \ldots, T'_7$, because each ψ_j is a solution of the same equation. Also, by linearity, ψ' has piecewise continuous derivatives up to second order. Moreover, it is straightforward to check that ψ' is continuous in Ω', so that $\psi' \in \mathcal{C}^2_*(\Omega')$, and that $\psi' = 0$ on $\partial\Omega'$.

Let's check, as an example, the restriction of ψ' in T'_7. Since $\psi_3 = \psi_5$ on the β edge (green) of $T_3, T_5 \subset \Omega$ and $\psi_6 = 0$ on the β edge (green) of $T_6 \subset \Omega$, we have that $\psi_3 - \psi_5 + \psi_6 = 0$ on the boundary β edge (green) of $T'_7 \subset \Omega'$. Similarly, since $\psi_3 = 0$ on the γ edge (blue) of $T_3 \subset \Omega$ and $\psi_5 = \psi_6$ on the γ edge (blue) of $T_5, T_6 \subset \Omega$, we have that $\psi_3 - \psi_5 + \psi_6 = 0$ on the boundary γ edge (blue) of $T'_7 \subset \Omega'$. So ψ' vanishes on the external boundary of T'_7. Analogously, since $\psi_3 = \psi_7$ on the α edge (red) of $T_3, T_7 \subset \Omega$, one has $\psi_5 = 0$ on the α edge (red) of $T_5 \subset \Omega$ and $\psi_6 = \psi_2$ on the α edge (red) of $T_6, T_2 \subset \Omega$, we have that $\psi_3 - \psi_5 + \psi_6 = \psi_7 + \psi_5 + \psi_2$ on the α edge (red) of $T'_7, T'_5 \subset \Omega'$. Thus, ψ' is continuous through that internal edge of Ω'.

For the sake of completeness, we show in Figure 9 how an eigenfunction ψ' of Ω' is transplanted into an eigenfunction of Ω of the same eigenvalue.

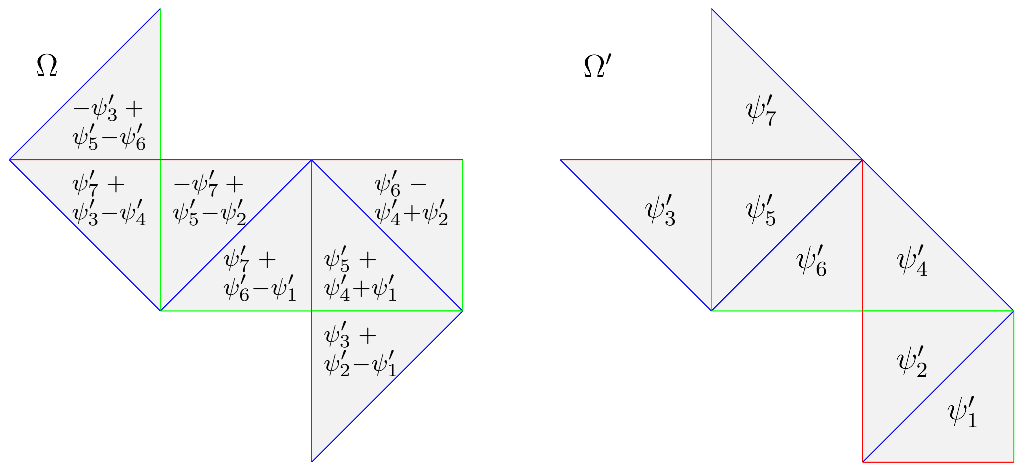


Figure 9: Original eigenfunction in Ω' and the transplanted eigenfunction in Ω.

Following the principles of this construction, several other counterexamples were later constructed; in [4] some of them are shown.

4 We can hear the area of a drum

Even though the domains of Section 3 answer negatively Kac's question, in general some geometrical information of a domain can be inferred from its spectrum, as we explained in Section 1. In this section we prove Theorem 1.5 in the Introduction, which shows that the area of a domain Ω is determined by its spectrum. To prove it, we ask our domain to have a minimum regularity, which is to be Jordan measurable. As we shall prove in Section 4.5, Jordan measurable domains are those domains such that their boundary has zero Lebesgue measure.

Theorem 4.1 (Weyl's law). *Let Ω be a Jordan measurable domain with Jordan measure (or simply area) $|\Omega|$ and $Spec(\Omega) = \{\lambda_n\}_{n\in\mathbb{N}}$. Let $r > 0$ and denote*

$$N_\Omega(r) = \#\{\lambda_n \in Spec(\Omega) : \lambda_n < r\}.$$

Then

$$\lim_{r\to\infty} \frac{N_\Omega(r)}{r} = \frac{|\Omega|}{4\pi}.$$

The proof will need several results, so let us sketch it.

First, we will check it for rectangles. This will allow to state an analogous result for the Neumann eigenvalues. As discussed earlier, the Neumann eigenvalues will be an important tool to prove properties of the Dirichlet eigenvalues.

From these results we will show that Weyl's law holds for finite unions of adjacent rectangles (domains looking like a grid). To see this, we will additionally need the so-called maximin principles, which we will prove using calculus of variations. From these principles we will establish an order relationship between the Dirichlet and Neumann eigenvalues and an order relationship between the eigenvalues of nested domains. This will allow to prove the theorem for grids from Weyl's law for rectangles.

From here it seems natural to prove Theorem 4.1 for "arbitrary" domains by approximating them by finite unions of rectangles. It is to ensure that such a construction can be done that we require our "arbitrary" domains to be Jordan measurable.

In Theorem 4.1 we have implicitly defined the counting function of Dirichlet eigenvalues in the domain Ω, denoted by $N_\Omega(r)$. We can similarly define the counting function of Neumann eigenvalues in the domain Ω as

$$M_\Omega(r) = \#\{\tilde{\lambda} \text{ Neumann eigenvalue in } \Omega :\ \tilde{\lambda} < r\}.$$

4.1 Weyl's law for rectangles

Let R be a rectangle of sides a and b. We have seen in Section 2.2.1 that

$$Spec(R) = \left\{ \lambda_{\ell,m} = \left[\left(\frac{\ell}{a}\right)^2 + \left(\frac{m}{b}\right)^2 \right] \pi^2 :\ \ell, m \in \mathbb{N} \right\}.$$

Geometrically, $N_R(\lambda)$ is the number of points $(\ell, m) \in \mathbb{N} \times \mathbb{N}$ lying inside the ellipse of semi axes $\frac{a\sqrt{r}}{\pi}$ and $\frac{b\sqrt{r}}{\pi}$,

$$\frac{x^2}{\left(\frac{a\sqrt{r}}{\pi}\right)^2} + \frac{y^2}{\left(\frac{b\sqrt{r}}{\pi}\right)^2} = 1. \tag{15}$$

This is a hard arithmetic problem that we will not consider in full detail. Otherwise, we will look for a first-order approximation of the value $N_R(\lambda)$.

Recall that for an ellipse $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$ of semi axes α and β its area is $\pi\alpha\beta$ and its perimeter is $2\pi\sqrt{\frac{\alpha^2+\beta^2}{2}}$.

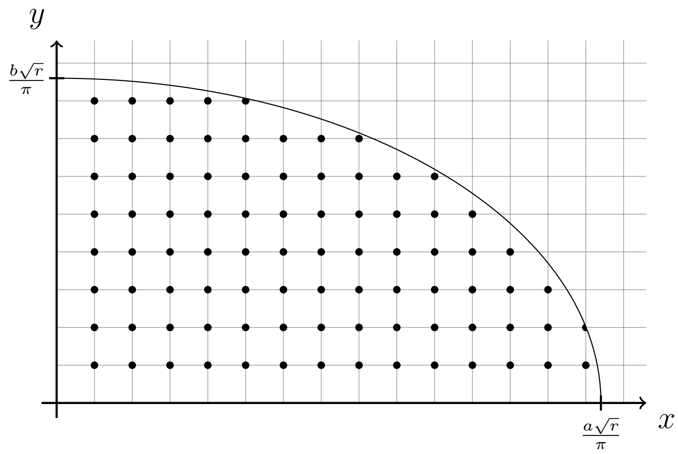


Figure 10: Points of the form $(\ell, m) \in \mathbb{N} \times \mathbb{N}$ lying inside the quarter of ellipse of semi axes $\frac{a\sqrt{r}}{\pi}$ and $\frac{b\sqrt{r}}{\pi}$.

This way, each point (ℓ, m) can be viewed as the right upper corner of a square of side 1 of the grid $\mathbb{N} \times \mathbb{N}$. This square is entirely contained in the first quadrant and lies inside the ellipse (15). Then $N_R(r)$ coincides with the number of such squares, and since they have area 1 we deduce that $N_R(r)$ is at most the area of this quarter ellipse,

$$N_R(r) \leq \frac{1}{4}\pi \frac{a\sqrt{r}}{\pi}\frac{b\sqrt{r}}{\pi} = \frac{ab}{4\pi} r.$$

As we are about to see, for large r the discrepancy between $N_R(r)$ and $\frac{ab}{4\pi}r$ is of order $r^{1/2}$, that is

$$\frac{ab}{4\pi} r - N_R(r) = O(r^{1/2}). \tag{16}$$

As soon as this is proved, there exists a constant $C > 0$ such that for r large

$$\frac{rab}{4\pi} - Cr^{1/2} \leq N_R(\lambda) \leq \frac{rab}{4\pi},$$

which implies the statement:

$$\lim_{r\to\infty} \frac{N_R(r)}{r} = \frac{ab}{4\pi} = \frac{|R|}{4\pi}.$$

It only remains to check that (16) holds.

Let n be the number of unit squares contained in the first quadrant and intersecting the boundary of the ellipse (15). Observe that $N_R(r) - \frac{rab}{4\pi}$ is bounded by n, so it is enough to see that $n = O(r^{1/2})$.

Let $n_x = \left\lfloor \frac{a\sqrt{r}}{\pi} \right\rfloor + 1$ and $n_y = \left\lfloor \frac{b\sqrt{r}}{\pi} \right\rfloor + 1$. Geometrically, n_x is the number of columns of the grid that intersect the boundary of the quarter ellipse. Similarly, n_y is the number of rows of the grid that intersect the boundary of the quarter ellipse.

Since n_x and $n_x + n_y$ are of order $O(r^{1/2})$ it is enough to see that $n_x \leq n \leq n_x + n_y$.

Take r big enough so that $n_x, n_y > 1$.

The inequality $n_x \leq n$ follows from the fact that the boundary of the ellipse touches one or more squares per column of the grid. Let's see now the inequality $n \leq n_x + n_y$.

We can view the boundary of the quarter ellipse as the graph of the continuous and strictly decreasing curve

$$\gamma(x) = \frac{b\sqrt{r}}{\pi}\sqrt{1 - \frac{x^2}{\left(\frac{a\sqrt{r}}{\pi}\right)^2}}, \qquad 0 \leq x \leq \frac{a\sqrt{r}}{\pi}.$$

Therefore, this curve can intersect a square of the grid in five possible ways, shown in Figure 11:

i) crossing its left and right sides,

ii) crossing its upper and lower sides,

iii) crossing its left and lower sides,

iv) crossing its upper and right sides, or

v) crossing one of its vertices (ignorable case).

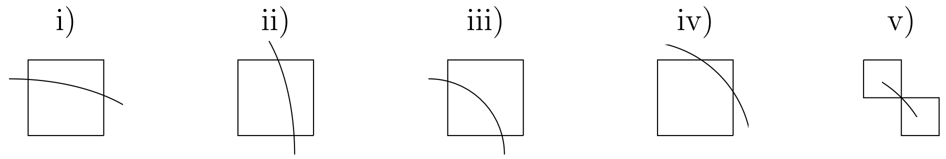


Figure 11: Possible ways in which a continuous and strictly decreasing curve can intersect the squares of a grid.

Assume that case *v)* does not happen. Then, the region defined by the squares of the grid that intersect the boundary of the ellipse is locally as in Figure 12 (like a descending ladder), where $m_1 \geq 0$ and $m_3 \geq 0$ denote the number of adjacent squares of type *i)*, $m_2 \geq 0$ and $m_4 \geq 0$ denote the number of adjacent squares of type *ii)*, and the shaded squares are those of types *iii)* and *iv)*.

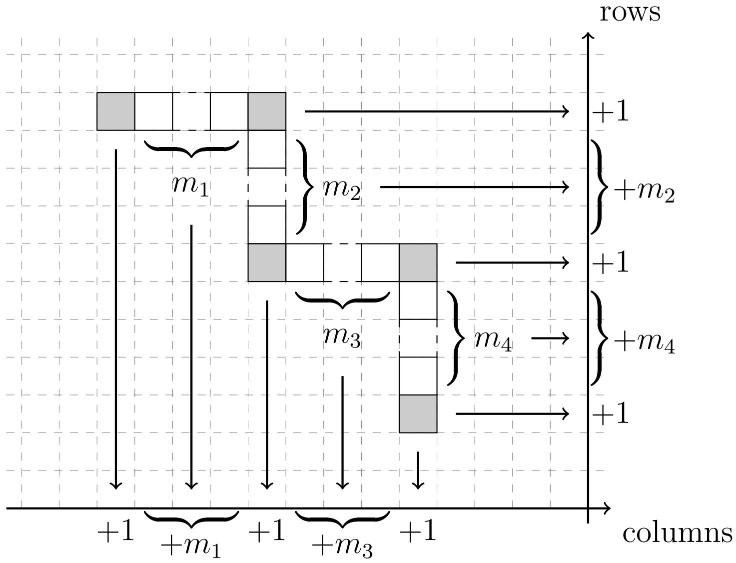


Figure 12: Number of rows and columns of the grid $\mathbb{N} \times \mathbb{N}$ intersecting a local part of the boundary of an ellipse.

As it is clear in Figure 12, locally the number of squares intersecting the boundary of the ellipse is $m_1 + m_2 + m_3 + m_4 + 5$, whereas locally the number of columns that intersect the boundary of the ellipse is $m_1 + m_3 + 3$, and locally the number of rows that intersect the boundary of the ellipse is $m_2 + m_4 + 3$, with

$$m_1 + m_2 + m_3 + m_4 + 5 \leq (m_1 + m_3 + 3) + (m_2 + m_4 + 3)\,.$$

Moreover, the region defined by the squares of the grid that intersect the boundary of the ellipse consists in concatenations of pieces like in Figure 12, so from the previous inequality follows that $n \leq n_x + n_y$.

If case *v)* happened in a common vertex of two squares, add an extra square adjacent to both of them. If the error was bounded by n, it is also bounded by $n+1$, and now the region that define these $n+1$ squares is as before, that is, locally as in Figure 12.

Remark 4.2. Recall that the Neumann eigenvalues of the rectangle R are $\tilde{\lambda}_{\ell,m} = \left[\left(\frac{\ell}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right]\pi^2$ with $\ell, m \in \mathbb{N} \cup \{0\}$. So, denoting

$$M_R(r) = \#\{\tilde{\lambda}_{\ell,m} \text{ such that } \tilde{\lambda}_{\ell,m} < r\},$$

we analogously see that

$$\lim_{r\to\infty} \frac{M_R(r)}{r} = \frac{|R|}{4\pi}.$$

4.2 Characterization of the eigenvalues by maximin principles

We will characterize the Dirichlet and Neumann eigenvalues by means of minimization problems. This will allow to show order relations between them, to compare eigenvalues of nested domains and to show the relation in between the eigenvalues of a domain and the ones of a disconnected subset of it.

Let $\nabla := \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ denote the gradient operator. We denote the norm of $L^2(\Omega)$ by $\|\cdot\|_\Omega$, or simply by $\|\cdot\|$ if Ω is clear by the context. Recall that it is the norm induced by the scalar product, $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$.

Throughout this section we invoke Green's identities (see [23, Sections 7.1, 7.2] for a proof).

Theorem 4.3 (**Green's identities**). *Let $u, v \in \mathcal{C}^2_*(\Omega)$. Then,*

$$\int_{\partial\Omega} v\frac{\partial u}{\partial n} = \int_\Omega \nabla v \cdot \nabla u + \int_\Omega v\Delta u \qquad \text{(Green's first identity)}$$

$$\int_\Omega u\Delta v - v\Delta u = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \quad \text{(Green's second identity)}$$

The first eigenvalues λ_1 and $\tilde{\lambda}_1$ have to be treated separately, so let's start with them.

Theorem 4.4. *Let λ_1 be the smallest eigenvalue of the Dirichlet Laplacian in Ω. Then*

$$\lambda_1 = \min\left\{\frac{\|\nabla w\|^2}{\|w\|^2} : \; w \in \mathcal{C}^2_*(\Omega) \setminus \{0\}, \; w|_{\partial\Omega} = 0\right\}.$$

Moreover, a function w that attains the minimum is an eigenfunction of the Dirichlet Laplacian of eigenvalue λ_1.

Remark 4.5. The minimum of Theorem 4.4, and all the minima that will appear later, should, a priory, be defined as infima, but, as shown in [7, Section 8.2], these infima are attained.

Proof. Let us refer to the functions $w \in \mathcal{C}^2_*(\Omega) \setminus \{0\}$ with $w|_{\partial\Omega} = 0$ as admissible functions.

Let m be the minimum of the statement and let u be an admissible function attaining the minimum. Then, for all admissible functions w

$$m = \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2} \leq \frac{\int_\Omega |\nabla w|^2}{\int_\Omega |w|^2}.$$

In particular this holds for admissible functions of the form $w(x) = u(x) + \varepsilon v(x)$, where $\varepsilon > 0$ and $v(x)$ is also admissible. Then the function

$$f(\varepsilon) := \frac{\int_\Omega |\nabla(u+\varepsilon v)|^2}{\int_\Omega |u+\varepsilon v|^2} = \frac{\int_\Omega |\nabla u|^2 + 2\varepsilon\nabla u \cdot \nabla v + \varepsilon^2|\nabla v|^2}{\int_\Omega u^2 + 2\varepsilon uv + \varepsilon^2 v^2}$$

has a minimum at $\varepsilon = 0$. Therefore,

$$0 = f'(0) = \frac{\left(\int_\Omega u^2\right)\left(2\int_\Omega \nabla u \cdot \nabla v\right) - \left(\int_\Omega |\nabla u|^2\right)\left(2\int_\Omega uv\right)}{\left(\int_\Omega u^2\right)^2}.$$

This implies that

$$\int_\Omega \nabla u \cdot \nabla v = \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} \int_\Omega uv = m \int_\Omega uv. \tag{17}$$

With this, together with Green's first identity, we can write, for all $v \in \mathcal{C}^2_*(\Omega)$ with $v|_{\partial\Omega} = 0$,

$$\int_\Omega (\Delta u + mu)v = \int_\Omega (\Delta u)v + m\int_\Omega uv = -\int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} v\frac{\partial u}{\partial n} + \int_\Omega \nabla u \cdot \nabla v = 0.$$

In particular, this must hold for all $v \in \mathcal{C}^\infty_c(\Omega)$, which implies

$$\Delta u + mu = 0 \text{ in } \Omega.$$

This proves that u is an eigenfunction of the Dirichlet Laplacian of eigenvalue m. It only remains to see that m is the smallest eigenvalue.

Let λ be an arbitrary eigenvalue of the Dirichlet Laplacian, with associated eigenfunction v. Then, by Green's first identity,

$$m \leq \frac{\int_\Omega |\nabla v|^2}{\int_\Omega |v|^2} = \frac{\int_\Omega (-\Delta v)v}{\int_\Omega v^2} = \frac{\int_\Omega (\lambda v)v}{\int_\Omega v^2} = \lambda,$$

as desired. □

A similar minimization problem characterizes the first eigenvalue of the Neumann Laplacian.

Theorem 4.6. *Let $\tilde\lambda_1$ be the smallest eigenvalue of the Neumann Laplacian in Ω. Then*

$$\tilde\lambda_1 = \min\left\{\frac{\|\nabla w\|^2}{\|w\|^2} \;:\; w \in \mathcal{C}^2_*(\Omega)\setminus\{0\}\right\}.$$

Moreover, a function w that attains the minimum is an eigenfunction of the Neumann Laplacian of eigenvalue $\tilde\lambda_1$.

Proof. Let $\tilde m$ be the minimum of the statement and let $\tilde u \in \mathcal{C}^2_*(\Omega)\setminus\{0\}$ be a function that attains the minimum. The argument of the previous proof is valid until equation (17). Now Green's first identity gives

$$\int_\Omega (\Delta\tilde u + \tilde m\tilde u)v = \int_{\partial\Omega} v\frac{\partial \tilde u}{\partial n} \quad \forall v \in \mathcal{C}^2_*(\Omega). \tag{18}$$

In particular, for all $v \in \mathcal{C}^\infty_c(\Omega)$ we must have

$$\int_\Omega (\Delta\tilde u + \tilde m\tilde u)v = 0,$$

which implies that $\Delta\tilde u + \tilde m\tilde u = 0$ in Ω. Then (18) writes as

$$0 = \int_{\partial\Omega} v\frac{\partial\tilde u}{\partial n} \quad \forall v \in \mathcal{C}^2_*(\Omega).$$

In particular, taking $v = \frac{\partial\tilde u}{\partial n}\big|_{\partial\Omega}$, we see that $\frac{\partial\tilde u}{\partial n}\big|_{\partial\Omega} \equiv 0$. This shows that $\tilde u$ is an eigenfunction of the Neumann Laplacian of eigenvalue $\tilde m$.

Similarly we see that $\tilde m$ is the lowest eigenvalue of the Neumann Laplacian. □

The other eigenvalues λ_n, $\tilde{\lambda}_n$ of both the Dirichlet and Neumann Laplacians are characterized by related minimization problems.

Lemma 4.7. *Let $n \geq 2$. Suppose that the first $n-1$ eigenfunctions of the Dirichlet Laplacian $v_1, \ldots, v_{n-1}$ are known, with respective eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$. Then the $n-$th eigenvalue of the Dirichlet Laplacian is*

$$\lambda_n = \min \left\{ \frac{\|\nabla w\|^2}{\|w\|^2} \ : \ w \in \mathcal{C}^2_*(\Omega) \setminus \{0\}, \ w|_{\partial\Omega} = 0, \ w \textit{ orthogonal to } v_1, \ldots, v_{n-1} \right\}.$$

Proof. Let us refer to the functions $w \in \mathcal{C}^2_*(\Omega) \setminus \{0\}$ with $w|_{\partial\Omega} = 0$ and orthogonal to $v_1, \ldots, v_{n-1}$ as admissible functions.

Let m be the minimum of the statement and let u be an admissible function that attains the minimum. Following the same argument as in the proof of Theorem 4.4, we get

$$\int_\Omega (\Delta u + mu)v = 0 \tag{19}$$

for all admissible functions v.

In order to deduce from this that $\Delta u + mu = 0$, we need (19) to hold for all functions $v \in \mathcal{C}^2_*(\Omega) \setminus \{0\}$ with $v|_{\partial\Omega} = 0$, not just for the subspace orthogonal to $v_1, \ldots, v_{n-1}$. Let's see that this is indeed the case.

By Green's second identity we can write, for $j = 1, \ldots, n-1$,

$$\int_\Omega (\Delta u + mu)v_j = \int_\Omega u(\Delta v_j + mv_j) = (m - \lambda_j) \int_\Omega uv_j = 0. \tag{20}$$

Let now $h \in \mathcal{C}^2_*(\Omega) \setminus \{0\}$ with $h|_{\partial\Omega} = 0$ be arbitrary, and define the function

$$v(x) = h(x) - \sum_{k=1}^{n-1} c_k v_k(x), \quad c_k = \frac{\langle h, v_k \rangle}{\langle v_k, v_k \rangle}.$$

Then v is admissible, so by (19)

$$\begin{aligned} 0 = \int_\Omega (\Delta u + mu)v &= \int_\Omega (\Delta u + mu) \left(h - \sum_{k=1}^{n-1} c_k v_k \right) \\ &= \int_\Omega (\Delta u + mu)h - \sum_{k=1}^{n-1} c_k \int_\Omega (\Delta u + mu)v_k. \end{aligned}$$

By (20) we deduce that, for all $h \in \mathcal{C}^2_*(\Omega) \setminus \{0\}$ with $h|_{\partial\Omega} = 0$,

$$\int_\Omega (\Delta u + mu)h = 0.$$

We have already seen that this implies that $\Delta u + mu = 0$, so m is an eigenvalue of the Dirichlet Laplacian. Let's finally see that m is the $n-$th eigenvalue.

Any other eigenfunction v_j of eigenvalue λ_j with $j \geq n$ is an admissible function, so by Green's first identity

$$m \leq \frac{\int_\Omega |\nabla v_j|^2}{\int_\Omega |v_j|^2} = \frac{\int_\Omega (-\Delta v_j) v_j}{\int_\Omega |v_j|^2} = \frac{\int_\Omega (\lambda_j v_j) v_j}{\int_\Omega |v_j|^2} = \lambda_j.$$

Moreover, m is not one of the λ_k, $k = 1, \dots, n-1$, because then u and v_k would be eigenfunctions of the same eigenvalue, which would contradict the orthogonality of u with respect to $v_1, \dots, v_{n-1}$. Therefore $m = \lambda_n$. $\square$

Adapting the previous proof, with the techniques of the proof of Theorem 4.6, an analogue for the Neumann eigenvalues is the following.

Lemma 4.8. *Let $n \geq 2$. Suppose that the first $n-1$ eigenfunctions of the Neumann Laplacian $\tilde{v}_1, \dots, \tilde{v}_{n-1}$ are known, with respective eigenvalues $\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-1}$. Then the $n-$th eigenvalue of the Dirichlet Laplacian is*

$$\tilde{\lambda}_n = \min \left\{ \frac{\|\nabla w\|^2}{\|w\|^2} : \ w \in \mathcal{C}^2_*(\Omega) \setminus \{0\}, \ w \text{ orthogonal to } \tilde{v}_1, \dots, \tilde{v}_{n-1} \right\}.$$

The proof is analogous to the previous one and we skip it.

Remark 4.9. The minimization problems given by the previous lemmas require knowledge on the eigenfunctions. We would like to characterize the eigenvalues without the help of the eigenfunctions, and this is what the maximin principles will achieve.

Theorem 4.10 (Dirichlet's maximin principle). *Let $n \geq 2$ and let $y_1, \dots, y_{n-1}$ be $n-1$ arbitrary piecewise continuous functions in Ω. Define*

$$\Lambda_n(y_1, \dots, y_{n-1}) := \min \left\{ \frac{\|\nabla w\|^2}{\|w\|^2} : \ w \in \mathcal{C}^2_*(\Omega) \setminus \{0\}, \ w|_{\partial\Omega} = 0, \ w \text{ orthogonal to } y_1, \dots, y_{n-1} \right\}.$$

Then the $n-$th eigenvalue of the Dirichlet Laplacian is

$$\lambda_n = \max_{y_1, \dots, y_{n-1}} \Lambda_n(y_1, \dots, y_{n-1}).$$

Proof. Fix $y_1, \dots, y_{n-1}$ piecewise continuous. Let $v_1, \dots, v_n$ be the first n normalized eigenfunctions of the Dirichlet Laplacian and consider

$$w(x) = \sum_{j=1}^{n} c_j v_j(x), \quad c_1, \dots, c_n \in \mathbb{R} \text{ (not all zero)}$$

such that w is orthogonal to $y_1, \dots, y_{n-1}$. This is achievable, since $c_1, \dots, c_n$ are n unknowns and we have $n-1$ linear equations of the form

$$0 = \langle w, y_k \rangle = \sum_{j=1}^{n} \langle v_j, y_k \rangle c_j, \qquad k = 1, \dots, n-1.$$

By definition of Λ_n and the orthogonality of the v_j we have

$$\Lambda_n \leq \frac{\|\nabla w\|^2}{\|w\|^2} = \frac{\left\langle \sum_{j=1}^{n} c_j \nabla v_j, \sum_{l=1}^{n} c_l \nabla v_l \right\rangle}{\left\langle \sum_{j=1}^{n} c_j v_j, \sum_{l=1}^{n} c_l v_l \right\rangle} = \frac{\sum_{j=1}^{n} \sum_{l=1}^{n} c_j c_l \int_\Omega \nabla v_j \cdot \nabla v_l}{\sum_{j=1}^{n} c_j^2}.$$

By Green's first identity, since $v_k|_{\delta\Omega} = 0$ for $k = 1, \dots, n$, we get

$$\Lambda_n \leq \frac{\sum_{j=1}^{n} \sum_{l=1}^{n} c_j c_l \int_\Omega (-\Delta v_j) v_l}{\sum_{j=1}^{n} c_j^2} = \frac{\sum_{j=1}^{n} \sum_{l=1}^{n} c_j c_l \int_\Omega (\lambda_j v_j) v_l}{\sum_{j=1}^{n} c_j^2} = \frac{\sum_{j=1}^{n} c_j^2 \lambda_j}{\sum_{j=1}^{n} c_j^2} \leq \lambda_n,$$

where we have used the orthogonality of the v_j and, in the last step, that $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$.

Therefore $\Lambda_n(y_1, \dots, y_{n-1}) \leq \lambda_n$ for all choices of the $y_1, \dots, y_{n-1}$, hence

$$\max_{y_1, \dots, y_{n-1}} \Lambda_n(y_1, \dots, y_{n-1}) \leq \lambda_n.$$

By Lemma 4.7 we have that $\lambda_n = \Lambda_n(v_1, \dots, v_{n-1})$. Then we have the chain of inequalities

$$\lambda_n = \Lambda_n(v_1, \dots, v_{n-1}) \leq \max_{y_1, \dots, y_{n-1}} \Lambda_n(y_1, \dots, y_{n-1}) \leq \lambda_n,$$

which proves the theorem. □

Adapting the previous proof with the techniques of the proof of Theorem 4.6 we get an analogous result for the Neumann eigenvalues.

Theorem 4.11 (Neumann's maximin principle). *Let $n \geq 2$ and let $\tilde{y}_1, \dots, \tilde{y}_{n-1}$ be $n-1$ arbitrary piecewise continuous functions in Ω. Define*

$$\tilde{\Lambda}_n(\tilde{y}_1, \dots, \tilde{y}_{n-1}) := \min \left\{ \frac{\|\nabla w\|^2}{\|w\|^2} : \; w \in \mathcal{C}^2_*(\Omega) \setminus \{0\}, \; w \text{ orthogonal to } \tilde{y}_1, \dots, \tilde{y}_{n-1} \right\}.$$

Then the $n-$th eigenvalue of the Neumann Laplacian is

$$\tilde{\lambda}_n = \max_{\tilde{y}_1, \dots, \tilde{y}_{n-1}} \tilde{\Lambda}_n(\tilde{y}_1, \dots, \tilde{y}_{n-1}).$$

4.3 Consequences of the maximin principles

The maximin principles are useful to see order relations between the eigenvalues of the Dirichlet and Neumann Laplacians in various domains, because they are stated in terms of minimums over a set. If a set is included in another set, then the minimum over the latter will be lower than the minimum over the former.

This allows to prove that the Neumann eigenvalues are never greater than the Dirichlet ones.

Theorem 4.12. *Let λ_j and $\tilde{\lambda}_j$ be the $j-$th eigenvalues of the Dirichlet and Neumann Laplacians, respectively. Then for all $j \geq 1$,*

$$\tilde{\lambda}_j \leq \lambda_j.$$

Proof. We have to treat separately the eigenvalues λ_1, given by the minimum of Theorem 4.4, and $\tilde{\lambda}_1$, given by the minimum of Theorem 4.6: all functions over which the minimum is taken for λ_1 are also taken to compute the minimum for $\tilde{\lambda}_1$, but not conversely. Hence $\tilde{\lambda}_1 \leq \lambda_1$.

Similarly, fixed $y_1, \dots, y_{n-1}$ piecewise continuous functions in Ω, all functions over which the minimum $\Lambda_n(y_1, \dots, y_{n-1})$ of Dirichlet's maximin principle is taken are also taken to compute the minimum $\tilde{\Lambda}_n(y_1, \dots, y_{n-1})$ of Neumann's maximin principle, but not conversely. Then $\tilde{\Lambda}_n(y_1, \dots, y_{n-1}) \leq \Lambda_n(y_1, \dots, y_{n-1})$, which implies that

$$\tilde{\lambda}_n = \max_{\tilde{y}_1, \cdots, \tilde{y}_{n-1}} \tilde{\Lambda}_n := \tilde{\Lambda}_n(y_1, \dots, y_{n-1}) \leq \Lambda_n(y_1, \dots, y_{n-1}) \leq \max_{y_1, \cdots, y_{n-1}} \Lambda_n = \lambda_n.$$

□

The following consequence of the maximin principles is related to the result that we have seen in Section 2.3, and that we remind now as a lemma.

Lemma 4.13. *Let $G_1, \dots, G_m$ be pairwise disjoint domains, and let G be the disconnected region with connected components $G_1, \dots, G_m$. Then*

i) The set of Dirichlet eigenvalues of G consists in the union of the Dirichlet eigenvalues of $G_1, \dots, G_m$: Spec$(G) =$ Spec$(G_1) \cup \dots \cup$ Spec(G_m).

ii) The set of Dirichlet eigenfunctions in G consists in the union of the Dirichlet eigenfunctions in $G_1, \dots, G_m$ (with the understanding that they vanish everywhere except in one G_k).

Corollary 4.14. *Let Ω be a domain. Consider a partition of Ω in disjoint domains $G_1, \dots, G_m$, and denote by G the disconnected region with connected components $G_1, \dots, G_m$. Let $n \geq 2$ and let $y_1, \dots, y_{n-1}$ be arbitrary piecewise continuous functions in Ω. Then*

i) The union of all Dirichlet eigenvalues of all the separated domains $G_1, \dots, G_m$ forms an increasing sequence $\lambda_1(G) \leq \lambda_2(G) \leq \dots \leq \lambda_n(G) \leq \cdots$ such that

$$\lambda_1(G) = \min \left\{ \frac{\|\nabla w\|^2}{\|w\|^2} : \ w \in \mathcal{C}^2_*(\Omega) \setminus \{0\} \ \textit{and} \ w = 0 \ \textit{on} \ \bigcup_{j=1}^m \partial G_j \right\},$$

and, for $n \geq 2$,

$$\lambda_n(G) = \max_{y_1, \cdots, y_{n-1}} \left(\min \left\{ \frac{\|\nabla w\|^2}{\|w\|^2} : w \in \mathcal{C}^2_*(\Omega) \setminus \{0\}, \ w = 0 \ \textit{on} \ \bigcup_{j=1}^m \partial G_j \ \textit{and} \ w \ \textit{is orthogonal to} \ y_1, \dots, y_{n-1} \right\} \right).$$

ii) The union of all Neumann eigenvalues of all the separated domains $G_1, \dots, G_m$ forms an increasing sequence $\tilde{\lambda}_1(G) \le \tilde{\lambda}_2(G) \le \cdots \le \tilde{\lambda}_n(G) \le \cdots$ such that

$$\tilde{\lambda}_1(G) = \min \left\{ \frac{\|\nabla w\|^2}{\|w\|^2} : \ w \in \mathcal{C}_*^2 \left(\Omega \setminus \bigcup_{j=1}^{m} \partial G_j \right) \setminus \{0\} \right\},$$

and, for $n \ge 2$,

$$\tilde{\lambda}_n(G) = \max_{y_1,\dots,y_{n-1}} \left(\min \left\{ \frac{\|\nabla w\|^2}{\|w\|^2} : w \in \mathcal{C}_*^2 \left(\Omega \setminus \bigcup_{j=1}^{m} \partial G_j \right) \setminus \{0\} \text{ and } w \text{ is orthogonal to } y_1, \dots, y_{n-1} \right\} \right).$$

Proof. The region G can be written as

$$G = \Omega \setminus \bigcup_{j=1}^{m} \partial G_j,$$

and its boundary is

$$\partial G = \bigcup_{j=1}^{m} \partial G_j.$$

Then, functions w continuous in G vanishing on ∂G are continuous functions in Ω, and piecewise continuity of the derivatives in Ω is inherited from piecewise continuity in G. Thus, Theorem 4.4, Theorem 4.6 and Dirichlet's and Neumann's maximin principles in G write as the statements. □

From this result we can order the Dirichlet and Neumann eigenvalues of both a domain and its partitioned components, as we see below.

Theorem 4.15. *Let Ω be a domain and consider a partition of Ω into disjoint domains $\Omega_1, \dots, \Omega_m$. Then, for all $n \ge 1$,*

$$\tilde{\lambda}_n(\Omega_1 \cup \cdots \cup \Omega_m) \le \tilde{\lambda}_n(\Omega) \le \lambda_n(\Omega) \le \lambda_n(\Omega_1 \cup \cdots \cup \Omega_m).$$

Proof. Theorem 4.12 gives the inequality $\tilde{\lambda}_n(\Omega) \le \lambda_n(\Omega)$. Let's now see the inequality $\lambda_n(\Omega) \le \lambda_n(\Omega_1 \cup \cdots \cup \Omega_m)$.

By the previous corollary and Dirichlet's maximin principle (or Theorem 4.4 for $n = 1$), the functions over which the minimum for $\lambda_n(\Omega_1 \cup \cdots \cup \Omega_m)$ is taken must satisfy the extra condition, with respect to the ones of the minimum for $\lambda_n(\Omega)$, to vanish on the internal boundaries. Thus $\lambda_n(\Omega) \le \lambda_n(\Omega_1 \cup \cdots \cup \Omega_m)$.

Similarly we prove that $\tilde{\lambda}_n(\Omega_1 \cup \cdots \cup \Omega_m) \le \tilde{\lambda}_n(\Omega)$.

By the previous corollary and Neumann's maximin principle (or Theorem 4.6 for $n = 1$), the functions over which the minimum for $\tilde{\lambda}_n(\Omega)$ is taken must satisfy the extra condition, with respect to the ones of the minimum for $\tilde{\lambda}_n(\Omega_1 \cup \cdots \cup \Omega_m)$, to be continuous on the internal boundaries. Thus $\tilde{\lambda}_n(\Omega_1 \cup \cdots \cup \Omega_m) \le \tilde{\lambda}_n(\Omega)$. □

Corollary 4.16. *Let Ω be a domain and consider a partition of Ω into disjoint domains $\Omega_1, \dots, \Omega_m$. Then, for all $r > 0$,*

$$N_{\Omega_1}(r) + \cdots + N_{\Omega_m}(r) \leq N_{\Omega}(r) \leq M_{\Omega}(r) \leq M_{\Omega_1}(r) + \cdots + M_{\Omega_m}(r).$$

Theorem 4.17. *Let Ω and Ω' be two nested domains, that is, such that $\Omega \subset \Omega'$. Then for all $n \geq 1$,*

$$\lambda_n(\Omega) \geq \lambda_n(\Omega').$$

Proof. Again we make use of Dirichlet's maximin principle. For such purpose, we assume that the functions $y_1, \dots, y_{n-1}$ over which the maximum is taken can be defined both in Ω and Ω'.

Fix $n \geq 2$. Observe that every function $w \in \mathcal{C}^2_*(\Omega) \setminus \{0\}$ with $w|_{\partial\Omega} = 0$ and orthogonal to $y_1, \dots, y_{n-1}$ can be extended to a function $w' \in \mathcal{C}^2_*(\Omega') \setminus \{0\}$ with $w'|_{\partial\Omega'} = 0$ and orthogonal to $y_1, \dots, y_{n-1}$ if we define w' to be

$$w'(x) = \begin{cases} w(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \Omega' \setminus \Omega. \end{cases}$$

Moreover $\|\nabla w\|^2_{\Omega} = \|\nabla w'\|^2_{\Omega'}$ and $\|w\|^2_{\Omega} = \|w'\|^2_{\Omega'}$.

So $\Lambda'_n(y_1, \dots, y_{n-1}) \leq \Lambda_n(y_1, \dots, y_{n-1})$, and we have already seen that this implies that $\lambda_n(\Omega) \geq \lambda_n(\Omega')$.

The result is clear for $\lambda_1(\Omega)$ and $\lambda_1(\Omega')$ by Theorems 4.4 and 4.6. □

4.4 Weyl's law for grids

Let Ω be the union of a finite number of adjacent rectangles $R_1, \dots, R_m$. Observe that $|\Omega| = |R_1| + \cdots + |R_m|$.

Let $r > 0$. Then we get, by Corollary 4.16,

$$\frac{N_{R_1}(r) + \cdots + N_{R_m}(r)}{r} \leq \frac{N_{\Omega}(r)}{r} \leq \frac{M_{R_1}(r) + \cdots + M_{R_m}(r)}{r}.$$

By the computations of Section 4.1, Weyl's law holds for a single rectangle, thus, for $j = 1, \dots, m$

$$\lim_{r\to\infty} \frac{N_{R_j}(r)}{r} = \frac{|R_j|}{4\pi} = \lim_{r\to\infty} \frac{M_{R_j}(r)}{r}.$$

From this we obtain

$$\lim_{r\to\infty} \frac{N_{R_1}(r) + \cdots + N_{R_m}(r)}{r} = \frac{|\Omega|}{4\pi} = \lim_{r\to\infty} \frac{M_{R_1}(r) + \cdots + M_{R_m}(r)}{r}. \tag{21}$$

By the sandwich theorem, this implies Weyl's law for grids.

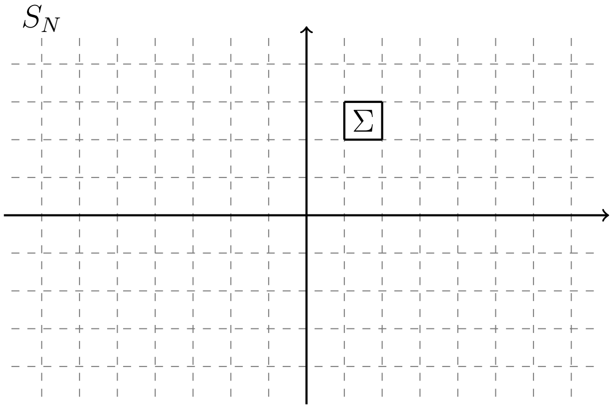
It seems now very natural to prove Weyl's law for an "arbitrary" domain Ω approximating it by two grids G_i and G_o, such that $G_i \subset \Omega \subset G_o$. For such purpose, we would want the two grids to have areas arbitrarily close to the area of Ω. Such construction may not be guaranteed for domains with very irregular boundaries, so we need to impose some restrictions.

4.5 Jordan measurable domains

Fix $N \in \mathbb{N} \cup \{0\}$ and let $h = 2^{-N}$. Consider the lines

$$x = jh,\ j \in \mathbb{Z} \qquad y = kh,\ k \in \mathbb{Z},$$

that define a dyadic lattice S_N that divides $\mathbb{R}^2$ into squares Σ of side length h, parallel to the Cartesian axes.



Observe that when passing from the lattice S_N to S_{N+1} each of the squares of S_N splits into four squares of S_{N+1}.

Let Ω be an arbitrary open bounded nonempty subset of $\mathbb{R}^2$. We consider the domain consisting of all the squares of the lattice S_N that are entirely contained in Ω:

$$\underline{\omega_N}(\Omega) = \{\Sigma \in S_N :\ \Sigma \subset \Omega\}.$$

Similarly, we consider the domain consisting in all the squares of the lattice S_N that intersect Ω:

$$\overline{\omega_N}(\Omega) = \{\Sigma \in S_N :\ \Sigma \cap \Omega \neq \varnothing\}.$$

Clearly the following properties are satisfied, for all $N \geq 0$:

i) If $A \subset B$ then $\underline{\omega_N}(A) \subset \underline{\omega_N}(B)$ and therefore $|\underline{\omega_N}(A)| \leq |\underline{\omega_N}(B)|$.

ii) If $A \subset B$ then $\overline{\omega_N}(A) \subset \overline{\omega_N}(B)$ and therefore $|\overline{\omega_N}(A)| \leq |\overline{\omega_N}(B)|$.

iii) $\underline{\omega_N}(\Omega) \subset \underline{\omega_{N+1}}(\Omega)$ and therefore $|\underline{\omega_N}(\Omega)| \leq |\underline{\omega_{N+1}}(\Omega)|$.

iv) $\overline{\omega_N}(\Omega) \supset \overline{\omega_{N+1}}(\Omega)$ and therefore $|\overline{\omega_N}(\Omega)| \geq |\overline{\omega_{N+1}}(\Omega)|$.

v) $\underline{\omega_N}(\Omega) \subset \Omega \subset \overline{\omega_N}(\Omega)$ and therefore $|\underline{\omega_N}(\Omega)| \leq |\Omega| \leq |\overline{\omega_N}(\Omega)|$.

Therefore, the area sequences $\left\{|\underline{\omega_N}(\Omega)|\right\}_{N\geq 0}$ and $\{|\overline{\omega_N}(\Omega)|\}_{N\geq 0}$ are bounded and monotone, which implies that they have a limit. It may happen that $\underline{\omega_N}(\Omega) = \varnothing$, and in such case we write $|\underline{\omega_N}(\Omega)| = 0$.

Definition 4.18. *Let Ω be an arbitrary open bounded nonempty subset of $\mathbb{R}^2$. The inner Jordan measure of Ω is*

$$m_i(\Omega) = \lim_{N\to\infty} |\underline{\omega_N}(\Omega)| = \sup_{N\geq 0} |\underline{\omega_N}(\Omega)|,$$

and the outer Jordan measure of Ω is

$$m_o(\Omega) = \lim_{N\to\infty} |\overline{\omega_N}(\Omega)| = \inf_{N\geq 0} |\overline{\omega_N}(\Omega)|.$$

The set Ω is said to be Jordan measurable if and only if $m_i(\Omega) = m_o(\Omega)$, and in such case its Jordan measure is $m(\Omega) = m_i(\Omega) = m_o(\Omega)$. The quantity $m(\Omega)$ is also called area of Ω.

Remark 4.19. As we shall see later, Jordan measurable domains Ω are always Lebesgue measurable, with $m(\Omega) = |\Omega|$, but not necessarily the other way around. Actually, although $m(\Omega)$ is called Jordan measure, it does not define a measure: countable unions of Jordan measurable sets need not be Jordan measurable. More precisely, we will see that Jordan measurable domains are those such that their boundaries have Lebesgue measure equal to zero.

Let's now see some properties of Jordan measurable sets.

Definition 4.20. *An elementary figure is a subset σ of $\mathbb{R}^2$ that can be represented as a finite union of rectangles with edges parallel to the Cartesian axes, any two of which either do not intersect or intersect only along some parts of their boundaries.*

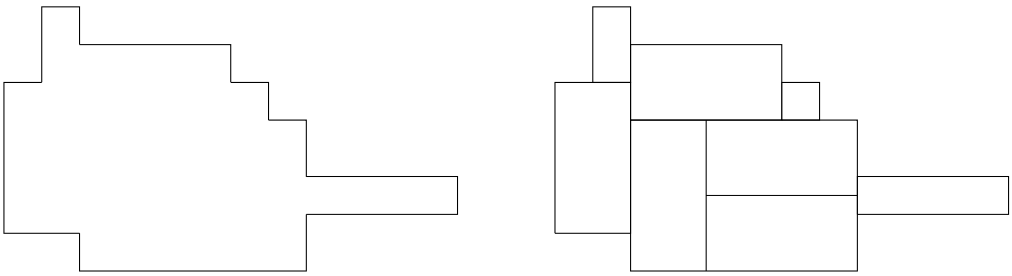


Figure 13: Example of an elementary figure (left) and one of its representations as a finite union of rectangles (right).

Some immediate properties of the inner, outer and Jordan measures are the following. For any Ω open bounded nonempty subset of $\mathbb{R}^2$:

i) $0 \leq m_i(\Omega) \leq m_o(\Omega)$.

ii) If $m_o(\Omega) = 0$ then Ω is Jordan measurable with Jordan measure equal to 0.

iii) For all $N \geq 0$, $\underline{\omega_N}(\Omega)$ and $\overline{\omega_N}(\Omega)$ are elementary figures.

iv) Elementary figures are Jordan measurable sets with Jordan measure equal to the sum of the areas of the rectangles of one of its representations.

Proposition 4.21. *Let Ω be an arbitrary bounded nonempty subset of $\mathbb{R}^2$ and let σ be an elementary figure. Then*

i) $m_i(\Omega) = \sup_{\sigma \subset \Omega} |\sigma|$.

ii) $m_o(\Omega) = \inf_{\sigma \supset \Omega} |\sigma|$.

Proof. We prove *i)*. Part *ii)* is done analogously.

Observe that $\omega_N(\Omega)$ is an elementary figure contained in Ω for all $N \geq 0$, so clearly the inequality

$$\sup_{N\geq 0} |\underline{\omega_N}(\Omega)| \leq \sup_{\sigma\subset\Omega} |\sigma|$$

holds and we only have to prove the reverse inequality.

Let σ be an arbitrary elementary figure contained in Ω. By the monotonicity of $\underline{\omega_N}(\Omega)$, for all $\varepsilon > 0$ exists $n \in \mathbb{N}$ such that

$$|\sigma| < |\underline{\omega_n}(\sigma)| + \varepsilon \leq |\underline{\omega_n}(\Omega)| + \varepsilon \leq m_i(\Omega) + \varepsilon.$$

From this we deduce that for all $\varepsilon > 0$

$$\sup_{\sigma\subset\Omega} |\sigma| \leq m_i(\Omega) + \varepsilon,$$

which implies that

$$\sup_{\sigma\subset\Omega} |\sigma| \leq m_i(\Omega),$$

concluding the proof. □

The following result will help us to prove Weyl's law for Jordan measurable domains.

Proposition 4.22. *Let Ω be an arbitrary bounded nonempty subset of $\mathbb{R}^2$. Then Ω is Jordan measurable if and only if for all $\varepsilon > 0$ there exist elementary figures $\underline{\sigma}$ and $\overline{\sigma}$ such that $\underline{\sigma} \subset \Omega \subset \overline{\sigma}$ and $|\overline{\sigma}| - |\underline{\sigma}| < \varepsilon$.*

Proof. Suppose that Ω is Jordan measurable and fix $\varepsilon > 0$. Then there exists $n \in \mathbb{N}$ such that

$$|\overline{\omega}_n(\Omega)| - \frac{\varepsilon}{2} \leq m(\Omega) \leq |\underline{\omega}_n(\Omega)| + \frac{\varepsilon}{2}.$$

Choosing $\underline{\sigma} = \underline{\omega_n}(\Omega)$ and $\overline{\sigma} = \overline{\omega_n}(\Omega)$ we have $\underline{\sigma} \subset \Omega \subset \overline{\sigma}$ and $|\overline{\sigma}| - |\underline{\sigma}| < \varepsilon$.

Reciprocally, suppose that for all $\varepsilon > 0$ there exist elementary figures $\underline{\sigma}$ and $\overline{\sigma}$ such that $\underline{\sigma} \subset \Omega \subset \overline{\sigma}$ and $|\overline{\sigma}| - |\underline{\sigma}| < \varepsilon$. Then, by Proposition 4.21 we have

$$|\underline{\sigma}| \leq \sup_{\sigma\subset\Omega} |\sigma| = m_i(\Omega) \leq m_o(\Omega) = \inf_{\sigma\supset\Omega} |\sigma| \leq |\overline{\sigma}|.$$

This implies that $m_o(\Omega) - m_i(\Omega) < \varepsilon$, so Ω is Jordan measurable. □

This characterization of Jordan measurable sets seems impractical when it comes to checking whether a set is Jordan measurable. To find a more checkable condition, let's characterize the condition of being Jordan measurable with the more common Lebesgue measure of its boundary.

Lemma 4.23. *Let Ω be an arbitrary bounded nonempty subset of $\mathbb{R}^2$. Then Ω is Jordan measurable if and only if for all $\varepsilon > 0$ there exists an elementary figure σ such that $\partial\Omega \subset \sigma$ and $|\sigma| < \varepsilon$. In such case, $m_o(\partial\Omega) = 0$.*

Proof. Suppose that Ω is Jordan measurable. Then by Proposition 4.22, for all $\varepsilon > 0$ there exist elementary figures σ' and σ'' such that $\sigma' \subset \Omega \subset \sigma''$ and $|\sigma''| - |\sigma'| < \varepsilon$. We can assume that $\partial\Omega \cap \partial\sigma' = \varnothing = \partial\Omega \cap \partial\sigma''$, since otherwise we could enlarge σ'' and shrink σ' by adding and taking away, respectively, rectangles covering $\partial\Omega \cap \partial\sigma' \cap \partial\sigma''$. Then

$$\partial\Omega \subset \sigma'' \setminus \sigma' \subset \overline{\sigma'' \setminus \sigma'} := \sigma.$$

Observe that σ is an elementary figure with $|\sigma| = |\sigma''| - |\sigma'| < \varepsilon$.

Reciprocally, suppose that for all $\varepsilon > 0$ exists an elementary figure σ such that $\partial\Omega \subset \sigma$ and $|\sigma| < \varepsilon$. We can assume that $\partial\Omega \cap \partial\sigma = \varnothing$, since otherwise we could enlarge σ by adding a square of area less than $\frac{1}{2}(\varepsilon - |\sigma|)$ covering a point in $\partial\Omega \cap \partial\sigma$. Consider the elementary figures

$$\sigma'' = \Omega \cup \sigma, \qquad \sigma' = \overline{\Omega \setminus \sigma}.$$

Clearly $\sigma' \subset \Omega \subset \sigma''$, $\overline{\sigma'' \setminus \sigma'} = \sigma$ and $|\sigma''| - |\sigma'| = |\sigma| < \varepsilon$. By Proposition 4.22, this implies that Ω is Jordan measurable. $\square$

Let us denote the inner Lebesgue measure of Ω as $L_i(\Omega)$ and the outer Lebesgue measure of Ω as $L_o(\Omega)$. Recall that

$$L_o(\Omega) = \inf \left\{ \sum_{n\in\mathbb{N}} |R_n| : \ \Omega \subset \bigcup_{n\in\mathbb{N}} R_n, \ R_n \text{ open rectangles} \right\}.$$

Therefore, for all Ω the outer Jordan measure is never lower than the outer Lebesgue measure,

$$L_o(\Omega) \le m_o(\Omega),$$

since every elementary figure is, in particular, a countable union of rectangles. We have equality on compact sets, as we see in the following lemma.

Lemma 4.24. *Let $K \subset \mathbb{R}^2$ be a nonempty compact set. Then $L_o(K) = m_o(K)$.*

Proof. It suffices to see that $L_o(K) \ge m_o(K)$. Fix $\varepsilon > 0$ and consider an open cover of K of the form

$$K \subset \bigcup_{n\in\mathbb{N}} R_n \quad \text{with } R_n \text{ rectangles such that } \sum_{n\in\mathbb{N}} |R_n| \le L_o(\Omega) + \varepsilon.$$

Since K is compact, there exists a finite sub-cover,

$$\{R_{n_k}\}_{k=1}^{N},$$

which, in particular, form an elementary figure. Therefore,

$$m_o(\Omega) \leq \sum_{k=1}^{N} |R_{n_k}| \leq \sum_{n=1}^{\infty} |R_n| \leq L_o(\Omega) + \varepsilon.$$

This implies that $m_o(\Omega) \leq L_o(\Omega)$, concluding the proof. $\square$

Recall that a set E that has $L_o(E) = 0$ is Lebesgue measurable with Lebesgue measure equal to zero. Then we have the following characterization of being Jordan measurable in terms of Lebesgue measure.

Proposition 4.25. *A nonempty bounded set Ω is Jordan measurable if and only if $\partial\Omega$ has Lebesgue measure equal to zero.*

Proof. By Lemma 4.23, the set Ω is Jordan measurable if and only if $m_o(\partial\Omega) = 0$, and since $\partial\Omega$ is compact, by Lemma 4.24, this is equivalent to $L_o(\partial\Omega) = 0$, which concludes the proof. $\square$

4.6 Weyl's law for Jordan measurable domains

Jordan measurable domains are precisely the ones for which the scheme of the end of Section 4.4 works.

Let Ω be a Jordan measurable domain and fix $\varepsilon > 0$. By Proposition 4.22, there exist $\underline{\sigma}$ and $\overline{\sigma}$ elementary figures such that $\underline{\sigma} \subset \Omega \subset \overline{\sigma}$, with $|\underline{\sigma}| \leq |\Omega| \leq |\overline{\sigma}|$, and $|\overline{\sigma}| - |\underline{\sigma}| < \varepsilon$.

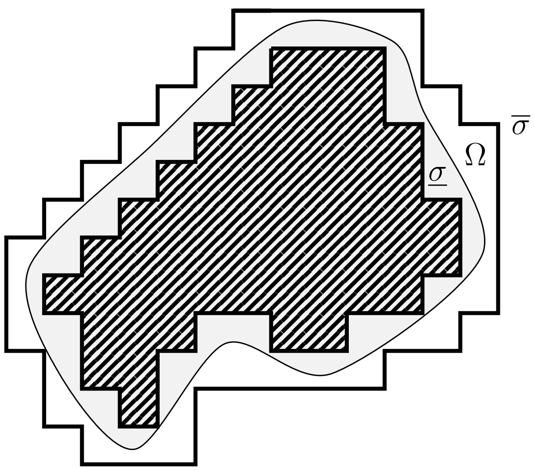


Figure 14: Example of elementary figures approximating the domain Ω from inside and from outside.

Since $\underline{\sigma} \subset \Omega \subset \overline{\sigma}$, by Theorem 4.17, the eigenvalues of $\underline{\sigma}$, $\overline{\sigma}$ and Ω are ordered, for all $n \geq 1$, as

$$\lambda_n(\overline{\sigma}) \leq \lambda_n(\Omega) \leq \lambda_n(\underline{\sigma}),$$

and therefore, for all $r > 0$, we have

$$\frac{N_{\underline{\sigma}}(r)}{r} \leq \frac{N_\Omega(r)}{r} \leq \frac{N_{\overline{\sigma}}(r)}{r}.$$

Since the elementary figures $\underline{\sigma}$ and $\overline{\sigma}$ are finite unions of rectangles, by Section 4.4, we have

$$\frac{|\underline{\sigma}|}{4\pi} = \lim_{r\to\infty} \frac{N_{\underline{\sigma}}(r)}{r} \leq \liminf_{r\to\infty} \frac{N_\Omega(r)}{r} \leq \limsup_{r\to\infty} \frac{N_\Omega(r)}{r} \leq \lim_{r\to\infty} \frac{N_{\overline{\sigma}}(r)}{r} = \frac{|\overline{\sigma}|}{4\pi} < \frac{|\underline{\sigma}| + \varepsilon}{4\pi}.$$

Finally, since $|\underline{\sigma}| \leq |\Omega| < |\underline{\sigma}| + \varepsilon$, in the limit $\varepsilon \to 0^+$ we get

$$\lim_{r\to\infty} \frac{N_\Omega(r)}{r} = \frac{|\Omega|}{4\pi}.$$

4.7 Weyl's law using Karamata's Tauberian theorem

Recall from Section 2.1 that Weyl's law is equivalent, thanks to Karamata's Tauberian theorem, to the limit

$$\lim_{t\to 0^+} t\sum_{n=1}^{\infty} e^{-\lambda_n t} = \frac{|\Omega|}{4\pi}, \tag{22}$$

so an alternative way of proving Weyl's lay is to prove (22). Without going into details, let us present a sketch of this proof.

First, we identify the left hand side of (22) with the trace of the heat kernel, as we shall see below. Thus, the problem reduces to understanding the asymptotic behaviour of the heat kernel.

For all $n \geq 1$, denote by f_n the normalized eigenfunction of the Dirichlet Laplacian in Ω of eigenvalue λ_n.

Denote a point of Ω as $\mathbf{x}$ and consider the heat equation, for $u : [0,\infty) \times \Omega \to \mathbb{R}$,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u(0,\mathbf{x}) = g(\mathbf{x}) & \text{with } g \in L^2(\Omega). \end{cases}$$

On the one hand, the solution, u, of the heat equation can be expressed in terms of the so-called heat kernel,

$$K(t,\mathbf{x},\mathbf{y}) = \sum_{n=1}^{\infty} f_n(\mathbf{x}) f_n(\mathbf{y}) e^{-\lambda_n t},$$

as

$$u(t,\mathbf{x}) = \int_\Omega K(t,\mathbf{x},\mathbf{y})g(\mathbf{y})d\mathbf{y}. \tag{23}$$

Indeed, formally

$$\begin{aligned}
\frac{\partial u(t,\mathbf{x})}{\partial t} &= \int_\Omega \frac{\partial K(t,\mathbf{x},\mathbf{y})}{\partial t} g(\mathbf{y})d\mathbf{y} = \int_\Omega \left(\sum_{n=1}^{\infty} -\lambda_n f_n(\mathbf{x}) f_n(\mathbf{y}) e^{-\lambda_n t}\right) g(\mathbf{y})d\mathbf{y} \\
&= \int_\Omega \left(\sum_{n=1}^{\infty} \Delta f_n(\mathbf{x}) f_n(\mathbf{y}) e^{-\lambda_n t}\right) g(\mathbf{y})d\mathbf{y} = \int_\Omega \Delta K(t,\mathbf{x},\mathbf{y}) g(\mathbf{y})d\mathbf{y} \\
&= \Delta u(t,\mathbf{x}).
\end{aligned}$$

It can be shown (using the monotone and dominated convergence theorems) that every interchange of derivatives, integrals and series is justified. Moreover, (23) satisfies the boundary condition

$$u(t,\mathbf{x})|_{\partial\Omega} = \int_\Omega \left(\sum_{n=1}^{\infty} f_n(\mathbf{x})|_{\partial\Omega} f_n(\mathbf{y}) e^{-\lambda_n t}\right) g(\mathbf{y})d\mathbf{y} = 0,$$

and the initial condition

$$\begin{aligned}
u(0,\mathbf{x}) &= \int_\Omega \left(\sum_{n=1}^{\infty} f_n(\mathbf{x}) f_n(\mathbf{y})\right) g(\mathbf{y})d\mathbf{y} = \sum_{n=1}^{\infty} f_n(\mathbf{x}) \int_\Omega f_n(\mathbf{y}) g(\mathbf{y})d\mathbf{y} \\
&= \sum_{n=1}^{\infty} \langle g, f_n\rangle f_n(\mathbf{x}) = g(\mathbf{x}),
\end{aligned}$$

where in the last equality we have used that, by Theorem 1.4, the functions f_n form a complete orthonormal basis of $L^2(\Omega)$.

On the other hand, the trace of the heat kernel is

$$tr(K) := \int_\Omega K(t,\mathbf{x},\mathbf{x})d\mathbf{x} = \sum_{n=1}^{\infty} e^{-\lambda_n t}. \tag{24}$$

The diagonal of $K(t,\mathbf{x},\mathbf{y})$ has an expansion around $t = 0$ of the form (see [21, Theorem 7.15])

$$K(t,\mathbf{x},\mathbf{x}) \underset{t\to 0^+}{=} \frac{1}{4\pi t}\left(1 + \sum_{j=1}^{\infty} t^j a_j(\mathbf{x})\right), \tag{25}$$

where $a_j(\mathbf{x})$, $j \geq 1$, are smooth functions in Ω. Therefore (22) follows from (24) and (25),

$$t\sum_{n=1}^{\infty} e^{-\lambda_n t} = \int_\Omega tK(t,\mathbf{x},\mathbf{x})d\mathbf{x} = \frac{1}{4\pi}\int_\Omega \left(1 + \sum_{j=1}^{\infty} t^j a_j(\mathbf{x})\right) d\mathbf{x} \underset{t\to 0^+}{\longrightarrow} \frac{|\Omega|}{4\pi}.$$

5 Beyond plane domains

As we have indicated throughout Section 1, Kac's problem can be generalized to arbitrary dimensions, and even to manifolds. In any case, the spectrum of a "drum" does not characterize its shape, except in $\mathbb{R}^1$.

5.1 Drums in $\mathbb{R}$

In one dimension, the sound of a "drum" (or, more appropriately, of a string) characterizes its shape. Let's see how. Let (a, b) be an interval of $\mathbb{R}$ (the analogue of an open, connected proper subset of $\mathbb{R}^2$). As before, the sound of the string is characterized by the eigenvalues of the Dirichlet Laplacian, that writes in one dimension as

$$\begin{cases} \dfrac{d^2u(x)}{dx^2} + \lambda u(x) = 0 & x \in (a,b) \\ u(a) = 0 = u(b) \end{cases} \qquad (\lambda > 0).$$

This is the harmonic oscillator differential equation that we have discussed in Section 1.1, with Dirichlet boundary conditions. From standard ODE theory we know that their solutions are proportional to

$$u_n(x) = \sin\left(\sqrt{\lambda_n}(x-a)\right), \text{ with } \lambda_n = \left(\frac{n\pi}{b-a}\right)^2 \text{ and } n \in \mathbb{N}.$$

Therefore, the spectrum of the "one dimensional drum" is

$$Spec((a,b)) = \left\{ \lambda_n = \left(\frac{n\pi}{b-a}\right)^2 \ : \ n \in \mathbb{N} \right\},$$

and clearly if two "drums" are isospectral then they have the same length, which in $\mathbb{R}$ implies that one is a translation of the other, being therefore characterized.

5.2 Drums in $\mathbb{R}^d$, $d > 2$

Drums in $\mathbb{R}^2$ can naturally be generalized to domains (that is, open connected proper subsets) Ω of $\mathbb{R}^d$, $d > 2$. In higher dimensions, the wave equation writes as (1), where the $d-$th dimensional Laplacian is $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$. Similarly, the fundamental frequencies are in unique correspondence with the eigenvalues of the Dirichlet Laplacian (2) (in d dimensions).

Then, naturally Kac's question can be generalized by asking if two isospectral domains $\Omega_1^{(d)}$ and $\Omega_2^{(d)}$ of $\mathbb{R}^d$, that is, such that $Spec(\Omega_1^{(d)}) = Spec(\Omega_2^{(d)})$, are necessarily isometric, that is, if there exists a bijection $\varphi : \Omega_1^{(d)} \to \Omega_2^{(d)}$ that preserves the Euclidean metric.

It turns out that, as in the two dimensional case, the answer is negative: the construction of the domains in Section 3 can be used to produce examples of two different drums in higher dimensions that sound the same.

Indeed, let Ω and Ω' be the plane domains of Section 3 and consider the domains of $\mathbb{R}^d$ defined as

$$\Omega_1^{(d)} := \Omega \times [0,\pi]^{d-2}, \qquad \Omega_2^{(d)} := \Omega' \times [0,\pi]^{d-2}.$$

As we shall prove below, the spectrum of $\Omega_1^{(d)}$ is characterized by $Spec(\Omega)$ and the spectrum of $\Omega_2^{(d)}$ is characterized by $Spec(\Omega')$ so, by Section 3, the domains $\Omega_1^{(d)}$ and $\Omega_2^{(d)}$ are isospectral but not isometric.

For simplicity we prove it in $\mathbb{R}^3$. By induction on the dimension, by means of adding Cartesian products of the form $[0,\pi]$, the result will be clear in $\mathbb{R}^d$.

Denote the eigenfunctions of the two dimensional Dirichlet Laplacian in Ω by $u_n(x_1,x_2)$, with eigenvalue λ_n. Clearly, these functions induce eigenfunctions of the 3−dimensional Dirichlet Laplacian in $\Omega_1^{(3)}$ given by

$$\psi_{n,k_3}(x_1,x_2,x_3) := u_n(x_1,x_2)\sin(k_3x_3), \quad \text{where } k_3 \in \mathbb{N}.$$

Note that, indeed, each ψ_{n,k_3} is an eigenfunction of the 3−dimensional Dirichlet Laplacian in $\Omega_1^{(3)}$ of eigenvalue $\lambda_n + k_3^2$.

In fact, we claim that

$$\{\psi_{n,k_3}(x_1,x_2,x_3)\}_{n,k_3\in\mathbb{N}} \tag{26}$$

is an orthogonal basis of $L^2(\Omega_1^{(3)})$. That these functions are orthogonal is clear. To prove that they form a base, suppose that there exists $h \in L^2(\Omega_1^{(3)})$ that is orthogonal to all ψ_{n,k_3}, that is, using Fubini's theorem, such that for all $n, k_3 \in \mathbb{N}$

$$\begin{aligned} 0 &= \int_{\Omega\times[0,\pi]} h(x_1,x_2,x_3)u_n(x_1,x_2)\sin(k_3x_3)dx_1dx_2dx_3 \\ &= \int_0^\pi \left(\int_\Omega h(x_1,x_2,x_3)u_n(x_1,x_2)dx_1dx_2\right)\sin(k_3x_3)dx_3. \end{aligned}$$

Since $\{\sin(k_3x_3)\}_{k_3\in\mathbb{N}}$ is the Fourier basis of $L^2([0,\pi])$, this implies that

$$\int_\Omega h(x_1,x_2,x_3)u_n(x_1,x_2)dx_1dx_2 = 0 \ \text{ for almost every } x_3,\ \forall n \in \mathbb{N}.$$

Since $\{u_n\}_{n\in\mathbb{N}}$ is, by Theorem 1.4, an orthogonal basis of $L^2(\Omega)$, this implies that

$$h(x_1,x_2,x_3) = 0 \ \text{ almost everywhere.}$$

This proves that the orthogonal set (26) is complete, and hence, an orthogonal basis of $L^2(\Omega_1^{(3)})$. Thus,

$$Spec(\Omega_1^{(3)}) = \left\{\lambda_n + k_3^2\right\}_{\substack{\lambda_n\in Spec(\Omega)\\ k_3\in\mathbb{N}}}.$$

By induction, for arbitrary $d > 2$ we obtain

$$Spec(\Omega_1^{(d)}) = \left\{ \lambda_n + \sum_{j=3}^{d} k_j^2 \right\}_{\substack{\lambda_n \in Spec(\Omega) \\ k_3,\dots,k_d \in \mathbb{N}}}, \quad Spec(\Omega_2^{(d)}) = \left\{ \lambda_n' + \sum_{j=3}^{d} k_j^2 \right\}_{\substack{\lambda_n' \in Spec(\Omega') \\ k_3,\dots,k_d \in \mathbb{N}}}.$$

Nevertheless, as in the two dimensional case, the Dirichlet eigenvalues characterize several geometrical parameters of the domain. In particular, Weyl's law generalizes in $\mathbb{R}^d$ as

$$\lim_{r\to\infty} \frac{N_\Omega(r)}{r^{d/2}} = \omega_d \frac{|\Omega|}{(2\pi)^d},$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$ and $|\Omega|$ is the $d-$dimensional volume of Ω. In fact, in 1980 Ivrii [13] proved the finer asymptotic formula

$$N_\Omega(r) = \omega_d \frac{|\Omega|}{(2\pi)^d} r^{d/2} - \frac{\omega_{d-1}}{4} \frac{\ell(\partial\Omega)}{(2\pi)^{d-1}} r^{(d-1)/2} + O(r^{(d-1)/2}) \quad \text{as } r \to \infty,$$

where $\ell(\partial\Omega)$ is the $(d-1)-$dimensional volume (surface) of $\partial\Omega$.

5.3 Drums as manifolds

The Dirichlet Laplacian can be defined in any $d-$dimensional compact, smooth Riemannian manifold M with metric tensor $g = (g_{ij})$ as follows.

Let Δ be the Laplace-Beltrami operator,

$$\Delta = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i} g^{ij} \sqrt{\det g} \frac{\partial}{\partial x_j},$$

where $g^{-1} = (g^{ij})$ is the inverse tensor of g. If M does not have a boundary (as, for example, the $n-$dimensional torus), the eigenfunctions of the Laplace-Beltrami operator are the solutions $u : M \to \mathbb{R}$, for $\lambda > 0$, of

$$\Delta u + \lambda u = 0 \quad \text{in } M.$$

If M has a boundary (as, for example, the Möbius strip) the eigenfunctions of the Laplace-Beltrami operator with Dirichlet boundary conditions are the solutions $u : M \to \mathbb{R}$, for $\lambda > 0$, of

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } M \\ u = 0 & \text{on } \partial M, \end{cases}$$

In any case, the set of $\lambda > 0$ for which there exist eigenfunctions is the spectrum of M, denoted by $Spec(M)$.

Then the "Can one hear the shape of a drum?" problem naturally poses for manifolds: the condition of being isospectral is now understood in the previous sense, and the condition of being isometric, which gives the notion that two domains are equivalent, is replaced by the condition of being congruent, in the sense that two manifolds M_1 and M_2 with metric g are congruent if and only if there exists a bijection $\varphi : M_1 \to M_2$ that preserves the metric g.

As mentioned in Section 1.2, in 1964 (two years before Kac posed his famous problem) Milnor [17] constructed two non-congruent sixteen dimensional tori whose spectra are identical.

As in $\mathbb{R}^d$ with the Euclidean metric, several expansion formulas related to the eigenvalues of the Laplace-Beltrami operator have been found. For example, in 1967 McKean and Singer [16] found the following expansion formula: if M is a compact, smooth Riemannian manifold without boundary such that $Spec(M) = \{\lambda_n\}_{n\geq 1}$, $\mathrm{Vol}(M)$ is the Riemannian volume of M and $K(\mathbf{x})$ is the scalar curvature at a point $\mathbf{x} \in M$, then

$$(4\pi t)^{d/2}\sum_{n=1}^{\infty} e^{-\lambda_n t} = \mathrm{Vol}(M) + \frac{t}{3}\int_M K(\mathbf{x})d\mathbf{x} + O(t^2), \quad t \to 0^+.$$

All the results that we have presented are related to the study of the eigenvalues. As one could have expected, the eigenfunctions also encode geometrical information of the domain (or manifold), and their study constitutes a broad field of research. Several results concerning the eigenfunctions can be found in Zelditch's book [27].

5.4 Restricted versions of Kac's problem

The restriction of Kac's problem (on characterizing the shape of a domain through its Dirichlet spectrum) to certain families of domains may lead to positive results. As a motivating example, suppose that we know that Ω is a right isosceles triangle: then we can characterize it from its spectrum. Indeed, just note that a right isosceles triangle is characterized by its side c and, from the computations of Section 2.2.2, this is determined by the lowest eigenvalue:

$$\text{since } \lambda_{\text{lowest}} = \lambda_{2,1} = \left[\left(\frac{2}{c}\right)^2 + \left(\frac{1}{c}\right)^2\right]\pi^2, \text{ necessarily } c = \pi\sqrt{\frac{5}{\lambda_{\text{lowest}}}}.$$

This opens Kac's problem to restricted classes of domains with more regularity than just being open, connected proper subsets of $\mathbb{R}^2$ (and all the natural generalizations to higher dimensions and manifolds).

The first non-trivial result came in Durso's doctoral thesis [6], in 1988. She proved that if T_1 and T_2 are two arbitrary isospectral triangles then they are isometric.

In 2009 Zelditch [26] went a step further and proved an analogous result for the larger class $\mathcal{D}$ of domains whose boundary is piecewise analytic and has reflection symmetry along the $x-$axis. More precisely, a domain Ω is in $\mathcal{D}$ if there exists an analytic function $f : [-a, a] \subset \mathbb{R} \to [0, \infty)$ only vanishing on the boundary, that is, $f(x) = 0$ if and only if $x \in \{-a, a\}$, such that Ω is the region bounded by the graphs of $y = f(x)$ and $y = -f(x)$.

5.5 Can a human hear the area of a drum?

An important fact that one should notice is that to recover the area of a drum by Weyl's law we need all the eigenvalues of the Dirichlet Laplacian. To hear the area of a drum we would need a perfect ear, one that hears all the frequencies of the drum. However, humans only perceive a finite range of frequencies.

Thus, the following question naturally arises: are there domains for which their area, or even their shape, is characterized by a finite quantity of Dirichlet eigenvalues?

In 1989 Chang and Deturk [3] proved that, knowing that Ω is a triangle, it is enough to know a finite number of eigenvalues of $Spec(\Omega)$ to determine Ω. However, their proof does not show how many or which eigenvalues are enough.

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