The Meaning of Tonk

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1 Introduction

The present article follows on a line of research proposed by Ripley in his 2015 article 'Anything goes', where he proposes a conception of logical consequence bearing the peculiar characteristic of discarding the cut rule entirely, and transitivity with it. This is, of course, a bold step to take, given transitivity's usefulness. It still bears some fruitful advantages for an inferential theory of meaning nonetheless, as it allows entrance into the inferential realm of meaning to many previously problematic entities -such as connectives like Prior's infamous tonk. As the title of Ripley's article suggests, it seems that in this interpretation of the turnstile (almost) anything goes. But how exactly does it go, we may ask ourselves. And indeed, this shall be the question we will try to sketch an answer for presently. What exactly could the meaning of tonk be in a framework which accounts for it, i.e. Ripley's? We believe Ripley's intuition to be mainly right, and that connectives like tonk do indeed possess a meaning, so we will try to delve deeper into what the meaning of these faulty connectives could be about. We will also briefly consider a side issue, which has to do with the fact that, even if everything goes, it does not seem to be the case that everything goes in the same way. Indeed, thanks to the many responses to Prior's article numerous differences have been spotted between regular, 'healthy' connectives like conjunction and problematic ones like tonk. Within Ripley's turnstile these can now be regarded as differences in meaning, and so we shall briefly analyse how these connectives articulate themselves differently in a cut-free environment. In order to do this we shall first lay out the framework at play, by introducing inferentialism and sequent calculus, as well as Ripley's definition of the turnstile with its rejection of cut. Then we will provide some philosophical considerations about the meaning of a conservative tonk, centering specially around its peculiar relation with transitivity. And finally, we will compare these conclusions with those available for 'healthy connectives'.

2 Radical Substructural Logics

2.1 Inferentialism and substructural logics

Inferentialism is the idea that the meaning of the expressions of a language has its origin in the inferential relations between the propositions surrounding them. An inferentialist take on logic considers that the meaning of logical expressions (such as the connectives '... or...' [disjunction] and '... and...' [conjunction]) is determined by the inferential relations between the propositions involved. This kind of inferentialism is specially plausible, since it is possible to specify inferential relations with great precision through rules within proof systems. For example, through the rules of introduction and elimination of the conjunction connective ($\wedge$) in Gentzen-style natural deduction systems:

$$(\wedge I)\frac{A \qquad B}{A \wedge B}$$

$$(\wedge E_1)\frac{A \wedge B}{A}$$

$$(\wedge E_2)\frac{A \wedge B}{B}$$

To all accounts, rules of use quite accurately determine the inferential relations between propositions involving the connective: which propositions we can infer from propositions containing the connective (rules of elimination) and from which propositions we can infer propositions containing the connective (rules of introduction).

Along with natural deduction systems, Gentzen (1935) introduced sequent calculi, in which the rules allow "sequents" (statements of the form " " where and are lists of formulas) to be derived from sequents. The equivalent in sequent calculus of classical logic is called LK. Unlike natural deduction, the rules of sequent calculi are, in general, rules of introduction. In sequent calculi, elimination rules from natural deduction correspond to introduction rules to the left of the sequent arrow (the "antecedent" of the sequent), and introduction rules from natural deduction correspond to introduction to the right of the sequent arrow (the "consequent" (succedent) of the sequent). For example, conjunction rules in sequent calculi have the following form:

$$(\wedge R)\frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$(\wedge L_1)\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}$$

$$(\wedge L_2)\frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}$$

An interesting aspect of sequent calculi is that some structural properties of proofs are made explicit by the distinction between operational rules (rules regarding the left or right introduction of a connective, as shown in the example above) and structural rules (rules in which no connective is introduced). The idea is that, while the operational rules determine the meaning of the connectives in question, the structural rules determine some property of the consequence relation itself, a structural property. Paradigmatic examples of structural rules are identity, contraction, weakening and cut:

$$\frac{}{A \vdash A}\text{ Identity}$$

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\text{ Weakening}$$

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\text{ Contraction}$$

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Pi \vdash \Sigma}{\Gamma, \Pi \vdash \Delta, \Sigma}\text{ Cut}$$

Gentzen himself notes that the difference between classical logic and intuitionistic logic, within the context of sequent calculi, is a structural difference: sequent calculi for intuitionistic logic can be obtained from the calculi for classical logic by restricting the number of formulas that can appear in the consequent of the sequent (that is, the difference between classical and intuitionistic logic, does not have to do, in sequent calculi, with any operational rule; it is, therefore, structural). Many other logics have structural differences with classical logic. When these differences imply a restriction or rejection of some structural rule (such as identity, contraction, weakening or cut) such logics are said to be substructural.

Historically, the development of substructural logics followed the line of restriction or abandonment of weakening rules, by the so-called relevant, paraconsistent and non-monotonic logics. There was also some work on the rules of contraction, by logics that rule out the principle of excluded middle such as intuitionist or multivalued logics, as well as linear logic. The rules of identity and cut are apparently too central to our conception of logical consequence, so their position remained undisputed for a long time. It has only been recently that literature on paradoxes is showing an increasing interest in radically sub-structural logics, that is, logics in which the identity or the cut rule may fail.

2.2 Tonk and restrictions in the defining rules

A problem every kind of inferentialism faces is the one raised by Arthur Prior in his short 1960 article 'The Runabout Inference Ticket'. In said article, he proposes a logical connective named 'tonk', whose introduction and elimination rules are those of disjunction and conjunction, respectively. Thus, from A we can infer A tonk B, and from A tonk B we can infer B. That is, from anything (since A and B are random) we can infer anything else. This fatal consequence is of course a dire problem for inferentialism, as according to this theory the meaning of an expression depends on its rules of use, yet tonk apparently fails to mean anything coherent even when its rules of use are clear. As we said, its rules are:

$$\text{(tonk I)} \quad \frac{A}{AtonkB}$$

$$\text{(tonk E)} \quad \frac{AtonkB}{B}$$

Since Prior's challenge was issued, multiple responses have been issued from the inferentialist side. One of the first and perhaps most influential ones was Belnap's response (1962). This author proposed the conservative extension criterion to identify whether a set of rules for a new connective does provide a good definition. The general idea is the following. An L language may contain a number of connectives and rules that govern their operations. When we add a connective with the corresponding rules we are expanding the language (let's call it L), allowing for new formulas that contain the new connective. We are also expanding the number of proofs we can obtain in the new language. It certainly sounds reasonable to make sure that this new language and set of proofs do not distort the prior one, by allowing us to demonstrate new claims about the old language. In other words, L is a conservative extension on L when any provable statement in L involving only sentences of L, could already be proved in in L. It is easy to see how adding Prior's tonk connective generally generates a non-conservative extension, as it allows for new demonstrations while the conclusion remains in the terms of the old language.

The conservative extension criterion, however, does not provide a categorical answer to the question of whether a set of rules is adequate to define a concept. The reason is that the notion of conservative extension is always relative: whether the addition of tonk results in a conservative extension or not will depend on the starting language (it will almost certainly result in a non-conservative extension, but still, the criterion does not depend exclusively on the rules but also on the source language). Exactly the same happens with Belnap's other criterion, that of unique definition, as it depends on the constrains of a language.

For this reason, inferentialists have sought what has been dubbed 'harmony': some property intrinsic to the system of rules of a connective that guarantees that these rules define a concept. This search was largely inspired by some general statements by Gentzen himself about a certain 'coherence' that must exist between the rules of introduction and elimination of a connective. To put it simply, it would seem that a connective cannot have very simple rules of elimination if the rules of introduction are equally simple, for example. This is precisely the case with tonk, but there is also the possibility that both the introduction and elimination rules are very demanding. In both cases there would be no balance and therefore no meaning. The problem is that this concept is quite difficult to define and so there are many different approaches, although they all agree on the motivation: to prevent cases like tonk from constituting meaning. However, it is difficult to see how any account of harmony can avoid depending on the underlying conception of logical consequence (what Belnap called 'context of deducibility').

2.3 Inferentialism in the context of radical sub-structurality

Recently, radically sub-structural logics, in which the cut rule or the identity rule may fail to hold, have gained popularity for their ability to resolve inconsistencies around vagueness and semantic paradoxes (see Zardini (2008) and Cobreros, Egré, Ripley and van Rooij (2012) for the case of vagueness, Cobreros, Egré, Ripley and van Rooij (2013), for semantic paradoxes).

The debate about the defining capacity of certain rules is particularly affected by radically substructural logics. While, within transitive logics in general, the existence of a tonk-like connective is capable of trivializing the theory, logics that restrict or abandon the cut rule are capable of incorporating tonk-like connectives. In his 2015 paper, 'Anything goes' David Ripley argues that, indeed, once we give up unrestricted use of the cut rule, tonk-style inference rules -within some general type of sequent calculus- do meet the property of conservativity and are therefore as legitimate as any other pair of more conventional rules for connectives. Ripley's bold position opens new questions about inferentialism and the nature of logic that determine the objectives of this paper. In line with his conclusions, we will hold that tonk does in fact possess a coherent meaning. Consequently, the questions we will try to solve are the following: what meaning can a connective like tonk have? What differences are there between concepts like tonk, conventional connectives, or truth predicates?

3 Ripley's framework

Let us start by approaching the problem side by side with Ripley and see what his proposal to solve Prior's tonk problematic is. Ripley starts by showing how, in order to be able to use the conjunction and define its rules of use, we need to be handling already many structural rules, such as weakening, contraction and reflexivity. This is necessary, for example, if we want phenomena such as explosion to hold, as it does in classical logic when we allow for negation and conjunction (we shall be considering, along Ripley, an LK framework, although, as he indicates, his considerations may also perhaps work in other environments). Hence, it would seem that we are already handling many rules when asserting any logical consequence. Following with this reasoning, this rules must come from the turnstile: they are structural rules. Crucially, the cut rule is not among them, we shall see where and how it should be introduced.

Now, in order to provide an understanding on what philosophical definition of the turnstile he is handling, Ripley borrows from the one used by Restall (2005). In section 3 of his article, 'A conception of consequence', he defines the turnstile he will be considering as following:

> What it is for a bunch of premises to entail a bunch of conclusions is that if you assert the premises and deny the conclusions, then you're out of bounds. That's the story I want to work with. It's a story that starts with three moving parts: assertion, denial, and out of bounds. Using those three notions, it gives an understanding of consequence. (Ripley, 2015, p. 28)

He does not provide a characterisation of negation and consequence, which he considers most generally, as long as they are necessary for sustaining any theory. He does stop, on the other hand, to clarify what the 'out of bounds' requirement entails. According to him, it plays the role of a 'constraint on what kinds of things people can get away with in the conversational positions that they adopt' (íbid). That is, something is out of bounds when it cannot be asserted by a person without being called out, for example when she asserts something and its negation.

This simple example shows that reflexivity is already part of the definition of such a turnstile, and what Ripley then proceeds to do is showing how other rules, such as weakening and contraction, are also required. We saw how they were required in order to make explosion valid like in classical logic. Ripley, however, does a complete halt, as we have seen, in one particular structural rule which apparently does not fit in this conception of logical consequence. This rule is the cut rule, which for Ripley is not contained in said definition of the turnstile much unlike the other structural rules. His way of dealing with it is simple enough: he considers the rules required by Restall's definition of the turnstile necessary, but very swiftly proceeds to discard the cut rule as a necessary structural rule, providing a 'cut-free sequent calculus for classical propositional logic' (Ripley, 2015, p. 29).

This step is quite bold, as it entails getting rid of transitivity. What does cut entail in Restall's definition of the turnstile that can justify such a move?

Well, according to Ripley:

> Cut says that if you've done some things that rule out your denying A, and you've done some other things that rule out your asserting A, then it's already too late. What you've already done doesn't fit together. You've got to leave yourself a path open to take on A, either leave open the option of asserting it or the option of denying it, *for any A always.* (Ripley, 2015, p. 30)

Ripley's problem with this is that there are things that 'the problem is with *them*'. That is, things that cannot be possibly asserted nor denied, but whose existence should not make me be out of bounds, for they have nothing to do with me. And yet it seems true that, according to Ripley, 'nothing in the nature of assertion and denial rules them out', that is, there is nothing in the definition of assertion and denial that rules out things that cannot be accepted nor denied). If the existence of something both unassertable and undeniable is possible then cut cannot be universal, not applicable '*for any A always*', for I can perfectly be in bounds when both my asserting and denying that pinoccio's nose would grow if he said 'I am lying' have been ruled out. Cut, in consequence, is not required to assert or deny, it only serves the purpose of keeping the things we are considering 'reasonably well behaved' - and Pinoccio's mouth safely shut in matters of lying.

The idea that cut can be put into brackets is not entirely new, as it has widely been considered to be an admissible rule, one that can be resorted to in order to make our reasonings more straightforward and smooth, but which is not actually required to obtain the conclusions. It just serves the purpose of avoiding spiky, complex arguments, but in the end, everything provable with cut is provable without cut. It is, in Ripley's words, a 'useful shortcut' (Ripley, 2015, p. 30). Ripley points out that the fact that it can be removed without affecting the provable conclusions is already a sign that it should not be applied 'come what may', as it is only justified for practical reasons.

Ripley brings out the swap rule to provide an analogy with cut. Swap, while admissible for a bare calculus devoid of operational rules (with only the structural rules of weakening, contraction and reflexivity, as well as maybe even some specific operational rules such as those for negation), it ceases to be admissible when conjunction comes into play. The key point here is that there are no doubts that the conjunction is a decent connective generally accepted, and quite basic to any logic. This indicates that the problem is in this case definitively not with the conjunction, a good connective, but rather with swap. This rule would be then just contingent, admissible in some cases but not a necessary structural rule of this particular conception of logical consequence. Indeed, in case we would consider swap to be a structural rule, then conjunction would trivialise the system, as we could infer p q from p.

$$\frac{\Gamma \vdash \Delta}{\Delta \vdash \Gamma}\,\text{Swap}$$

$$\frac{p \land q \vdash p}{p \vdash p \land q}\,Swap + \land E$$

Notice how this is very similar to what happens with cut and tonk, as if we believe cut to be a structural rule then tonk trivialises the language: we can infer B from any A. However, tonk is not as generally accepted as the conjunction connective, and so the blame has typically gone to tonk. The problem lies in that this bears dire consequences not only for the weird, on-the-nose little tonk connective, but aims to dismantle a whole theory of meaning such as inferentialism. This would justify Ripley's attempt to provide an alternative explanation, not unlike what we already did with the swap rule: making cut non-structural, and then only admissible. Inferentialism, as well as tonk's meaning, are safe.

And yet, an argument could be made here against Ripley that swap and cut are really not that similar, as they operate in very different ways. Cut allows for pieces of vocabulary to disappear which is why it conflicts with conservative extension, while swap allows for new demonstrations which mirror the ones already valid in the language. While the intention of this paper is not to dwell on these differences, it seems important to note them. Nonetheless, the analogy works just fine in showing how some apparently structural rules become problematic when paired with some connectives, whilst other apparently do not clash with operational rules and hence are considered by Ripley truly structural. Cut and swap belong to the first group, and the fact that they possess counterexamples does seem to point out to their contingency.

Another substantial difference between cut and swap, one which motivates that we easily discard the latter but stick to the former as much as we can. The reason we resist giving cut up is that, as interesting as swap is, it is not as useful as cut at all. Indeed, we build systems based on cut-admissibility because they allow for plenty of complex demonstrations, only made hassle-free thanks to transitivity. And yet this still is not, as Ripley indicates, a valid justification -only a practical one. This has consequences for any consideration about conservativeness and harmony, two interesting proposed solutions to the tonk-problematic.

Belnap's conservative extension requirement (first sketched in his 1962 article 'Tonk, plonk and plink') asks that there must be no new rules added to a prior language which makes it possible to prove new things in the new language with vocabulary from the old. Ripley's interpretation of the turnstile is conservative because it does not allow for the only rule which may endanger this requirement: cut. Only cut can eliminate old pieces of vocabulary, so only cut can illegitimately bare the conclusion of intermediate newly added vocabulary. The conclusion would then be expressed in terms of the old language, but it could have never been reached if we only operated in it. This is precisely what tonk does thanks to cut, and the reason why Belnap accused tonk of being non conservative. Ripley notes that if we remove cut there cannot be anything in the premises that does not show up in the conclusion, hence making his conception of logical consequence inherently conservative, with the side consequence that connectives such as tonk are then perfectly fine, as the conclusion of ay premises containing tonk will also contain it.

The fact that Ripley's turnstile is conservative is important because conservativeness is deeply linked with inferentialism. In his 2002 manual on substructural logics, Paoli shows how if we want a truly inferential theory of meaning that relies only on the rules for each connective and does not appeal to a holistic theory of meaning, conservativity is necessary in that it does not allow for formulae to be removed from the reasoning (Paoli, 2002, p. 20). Let us say, for example, that we prove B resorting to the conjunction but then remove any instance of the conjunction. Can we truly say that we have the meaning of that proven thing independently of the conjunction? No. The so-called 'subformula property' (the fact that every subformula of a reasoning is present in the conclusion) is necessary for a truly inferential theory of meaning because if not present then the meanings of logical constants are not independent. They do not have separate rules, such as the introduction and elimination rules of the conjunction, but are rather dependent on other constants. Hence, Ripley's turnstile, being conservative, packs a good punch for inferentialism.

But let us not forget the aim of this paper: given that tonk is no longer trivial, what exactly is its meaning? Ripley raises this issue when discussing the matter of harmony. Harmony is the idea that there is something inherent to the rules that makes cases like tonk as disharmonious. There most be some sort of relation between introduction and elimination rules, and this relation is in the case of tonk asymmetric. This is done in order to show that tonk has no meaning, but Ripley disagrees. Provided we possess conservative extension, which is the case of his particular usage of the turnstile, tonk does not trivialise. Furthermore, his idea of meaning is quite permissive: To have a meaning is to be able to be 'introduced properly' (left and right) and so be usable as both premise and conclusion. Wanng's one-place connectives like super-tonk (immune to cut) do not have a meaning, but tonk does. What could this meaning be? Ripley's only clue is that it is 'at best, pretty useless' (Ripley, 2015, p.33).

The thing is that Ripley himself is more interested with truth predicates and what approaching them in such a framework can say about paradoxes. However, there are some interesting claims made about cut that may help defining tonk's meaning. He states that:

Tonk and truth predicates are very different, since although both lead to failures of cut in the present setting, they do so for very different reasons—truth because attempts to eliminate cut in its presence never end (see eg [Tennant, 1982]), tonk because attempts to eliminate cut in its presence can't get started. But this is just to point to some of the diversity we rule out when we insist on cut. Truth and tonk are indeed very different, in just this way. Yet insisting on cut prevents us from engaging fully with either. (Vague predicates are more closely related to tonk; see [Ripley, 2013b].) (Ripley, 2015, p. 34).

Let us now turn to engage a little bit with tonk, and what connectives like it may actually mean.

4 The meaning of tonk

As futile and uninteresting as tonk's applications may be, we want to show a little bit of love to this quirky connective and analyse what its meaning could be from an inferential point of view. Let us take a peek again at tonk's rules to see if there is something we could do with them:

$$(\text{tonk R}) \quad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, AtonkB}$$

$$(\text{tonk L}) \quad \frac{B, \Gamma \vdash \Delta}{AtonkB, \Gamma \vdash \Delta}$$

This are Ripley's translation of Prior's natural deduction rules for tonk to a sequent cualculus setting. As he crucially points out, there already seems to be some meaning to tonk, as these rules can be used in some way, and they are legitimate in that they are left and right introduction rules (and hence, in Ripley's interpretation, usable as premises and conclusion). But obviously, there is a problem, and that is that, in the presence of cut and reflexivity, tonk yields the following result:

$$\text{Cut}\ \dfrac{\text{tonk R}\ \dfrac{\text{Identity}\ \dfrac{}{A \vdash A}}{A \vdash AtonkB} \qquad \dfrac{\dfrac{}{B \vdash B}\ \text{Identity}}{AtonkB \vdash B}\ \text{tonk L}}{A \vdash B}$$

The problem is easy to see, as for any A and B, tonk allows us to infer B from A. This trivialises the language, as it entails that everything can be inferred from anything else. However, if we remove cut, even tonk becomes conservative extending, as said connective (newly introduced to the previous language) cannot be removed from the conclusion and hence nothing new is proven in the old language vocabulary:

$$\dfrac{\text{tonk R}\ \dfrac{\text{Identity}\ \dfrac{}{A \vdash A}}{A \vdash AtonkB} \qquad \dfrac{\dfrac{}{B \vdash B}\ \text{Identity}}{AtonkB \vdash B}\ \text{tonk L}}{AtonkB, A \vdash B, AtonkB}$$

Now we can quite clearly see why Ripley deemed any possible meaning of tonk 'useless'. We certainly advance little in our knowledge by using a reasoning of this sort. However, as we said, we are crucially doing something here. There is some meaning to tonk, and that is enough for us to start divagating: what could this fantastically useless connective be saying to us? From an inferential point of view, it is simple enough: tonk means what the rules tell us. Tonk is the connective that, in its presence, let us extract any B from any given A.

Now, there are various philosophical ways to interpret this. One first consideration is that tonk's meaning would apparently be special in many ways, after all we are talking of a connective which, unlike many others, fails to conservatively extend a language in the presence of cut, and which leaves truth values undetermined or over defines them, as Stevenson's model theory approach pointed at. It seems cut and tonk are natural adversaries, and indeed Ripley points out that tonk is a source of counterexamples to said rule (Ripley, 2015, p. 33). Tonk and cut bear hence some peculiar sort of relation, and thus a tentative approach to define said connective's meaning may stem from this. By considering tonk's deep connection to the cut rule, we can aim to characterise it with the help of what we already know about it.

4.1 Tonk vs Transitivity

What is the meaning of the cut rule? This rule, as Ripley puts it, is 'a particular sort of generalization of transitivity' (Ripley, 2015, p. 30). Even fairly weak versions of it, such as precisely the one Ripley considers, make tonk trivial. It seems then that tonk's problem with cut is rather a problem with transitivity itself, so let us analyse how tonk could be characterised in its relation to this property. It is no secret that transitivity is one of the most precious reasoning tools we possess, and apparently one of the most basic ones to most cognitive beings.

The argument for transitive reasoning's practicality is pretty powerful, as it's usefulness is free from any doubt. Let us provide an empirical example. An evolutive theory of rationality would point out how wonderful an evolutive tool it is, as it removes quite a cognitive load from its employer while providing accurate information, hence making it quite fit in the race for survival. It is no wonder how it has been demonstrated in many animals, specially those animals with hierarchical social structures. For instance, pinyon jays have been demonstrated to infer that they are superior to a bird C, from the fact that they beat bird B and that bird B beat bird C in a fight for dominance, even though they never beat actually beat C (Paz y Miño et. al., 2004, through Fitch, 2010, p. 130). This example shows very clearly how transitivity is admissible only, however, as it is not required to reach that conclusion: bird A could fight bird C and see for itself. This is already a symptom, as we saw with Ripley, that the cut rule is but a shortcut, quite useful but not mandatory. However, it also shows that transitivity is a pretty basic cognitive tool, perhaps even necessary in order to develop complex rational systems.

And yet, these are practical issues. There is nothing in transitivity that a priori assures us the conclusion, no non-practical justification of it in Ripley's words. In fact, transitivity as a cognitive resort may fail in complex settings, for example, in case bird C's fighting style was bird A's style's perfect counter. What non ideal settings show is the lack of an actual justification for transitive judgements, the lack of a tie between premises and conclusions. Asking for it is already asking for more than transitivity. Paoli cites Bolzano as one of the first proponents of non-transitive relations, and more recently, Smiley and others 'have introduced systems of logic whose derivability relations are not unrestrictedly transitive. For B to be deducible from A, they argue, there has to be a meaning connexion. But the relation of meaning connexion is not transitive' (Paoli, 2002, p. 18). Perhaps tonk encapsulates this idea: given no connection of meaning, anything can be entailed by anything else, hence for transitivity to hold there must be some sort of further connection.. The fact that it asks for conservativity may be a clue: everything in the road must be taken into account in the conclusion, there must be a justification from A to B. In this sense, tonk would be a symptom of the overpowered nature of transitivity, and go to show that if it is handled indiscriminately there are counterexamples to be found.

If we take tonk to show how transitivity can fail, showing us that our transitive inferences ultimately can lead to infer everything from everything, we need to fight against a deeply rooted intuition. Of course, going against transitivity is highly counterintuitive: how could transitivity not work? If B is derived from A and C is derived from B, it seems simply mandatory to say that C can be derived from A, isn't it? Well, not exactly. In fact, in many cases this will not be, such as in the pinyon jays case. There are two problems here. The first is a matter of context. In order for transitivity to hold, the connections between A and B, and between B and C must be similar or *a priori*. In any other case, the difference in setting could deny transitivity. For example, if B followed from A in context X, while C followed from B in context Y but not in context X. The second problem was already hinted at by the birds' fighting simulation. Even in the same context, bird A would win against B and loose against C (who previously won against B). That is because C and A bear some sort of relation (or lack of it) which conditions their relationship of entailment.

Let us propose a mental experiment. Imagine a very big chain, potentially infinite, of transitive relations so that from A you can infer B, from B you can infer C, from C D and so on. It seems we would be more reluctant to say that from A you can derive, say, Z, because there are many cut steps. Hence we are removing a lot of content, which should definitely be safe given what we consider to be consequence relations, but which nonetheless seems a little bit dangerous. Do all of these relation really hold together? The possibility of the context or some internal features to interfere with the causal transitivity relation increases exponentially with the amount of steps, so that we are left wondering: can we really skip so much content? Our deeply rooted intuition for transitivity betrays us quite logically in this broken phone situation. Could this be a sign that we may be oversimplificating the matter at hand, and that transitivity is more delicate than we thought?

These issues are solved when we propose an ideal transitivity, one in which both entailments are necessary, hence ensuring that we operate in the same *a priori* context and with no interferences. Indeed, if B is derived necessarily from A and C is derived necessarily from B, C seems doomed to be necessarily derived from A. The thing is that we may want to keep explicit the idea that this is a simplification, as, even if it is necessary for some kinds of logic, it comes with its own deal of problems. Avoiding cut would not let us incur in this useful simplification, but it allows us to accommodate other intuitions such as this oversimplificating nature of ideal conditions, justified as not only it is sometimes not even possible *a priori*, such as with paradoxes, but it also is continuously proven wrong in empirical, complex settings. As tonk attempts against a fully ideal account of transitivity, maybe its duty is to point outside of its constraints, accommodating all those intuitions against it.

In any case, the fear of this exchange is certainly legitimate, as we would seemingly be loosing much productive power. Ripley has to say, however, that with his interpretation of the turnstile we can actually develop productive logical systems even without cut, but that is another discussion. Let us now delve deeper into tonk meaning by considering its role inside Ripley's conception of logical consequence.

4.2 General Tonk

We have seen how Ripley's turnstile is all about being in or out of bounds. What could the meaning of tonk be in such a turnstile? Now, provided we discard cut, there is a way in which we can say that tonk is indeed very reasonable. Not only that, perhaps it may be all too reasonable -maybe even obvious. Let us try to provide an explanation. We know that if by asserting A we are in bounds, then we are also in bounds when we say A tonk B. This is the right introduction rule, whereas the left introduction rule says that if we are in bounds when asserting A tonk B if we are also in bounds whenever we say B.

However, according to his conservative conception of the turnstile, tonk's rules say that in order to say B we need to say it comes, via tonk, from A. Hence, this could be taken as a most general way to show that, provided that we possess the necessary relation between A and B, then asserting that B comes from A is in bounds. Tonk would then stand as 'the necessary relation for asserting that $A \vdash B$ is in bounds'. This of course would sit well with the considerations given in the previous section, and proves to be a very shallow and perhaps useless meaning, but a meaning at that.

Also, this happens to quite merrily align with what Smiley et. al. had to say about a connection of meaning: a relationship holds only if there is some sort of connection. So in Ripley's framework, tonk could be an obvious and most general way of asserting that, from A to extract B, these two should bear some sort of relation. This would be something taken for granted whenever we say that B comes from A: tonk would only be giving us this very general, useless intuition in an explicit shape. What good comes from this? Almost nothing, only making explicit what was already obvious, but it does have some deep practical impact: after all, it is thanks to this call for something more than transitivity that we are having this discussion in the first place. Tonk would be like a car horn, pretty vague, simple, and general, but alarming enough to stop us colliding with stuff like paradoxes in our transitivity-induced dream.

In essence, tonk would be little more than a placeholder relation: it only entails that for extracting anything from other thing there should be a connection between them. This connection is not defined or further constrained in any way, which makes it explanatorily somewhat useless. The reason tonk becomes trivial with cut is because this relation is neglected in non-conservative settings, and it just so happens that all tonk means is that there is a relation. Whenever this relation gets taken out, tonk would most reasonably loose its meaning, which may be very general, but certainly constitutes a meaning. The fact that everything is kept in the conclusion entails that we must conclude B but also A tonk B, the conclusion is conditioned to include a relation.

Hence, the inferential meaning of tonk is clear enough. We can use it as a conclusion extracting A (as cut is not allowed) and A tonk B from A, and as a premise using A tonk B to extract A tonk B and B. That is, if asserting A and A tonk B is in bounds, then asserting B is also in bounds (crucially, only along A tonk B). If we use this interpretation of the turnstile, then it is easy to see how tonk is obvious. Provided we are in bounds saying A and we are in bounds saying that there is a relation ('tonk') that says that B is also in bounds when A and A tonk B are, then B is in bounds. Quite useless indeed, and yet somehow meaningful: it calls for conservativity.

4.3 Tonk vs Other Connectives

If we are to define tonk's meaning like this, it is easy to see why it would differ from the meaning for any other connective. Let us compare the rules for conjunction with those for tonk.:

$$(\wedge I)\frac{A \qquad B}{A \wedge B}$$

$$(\wedge E_1)\frac{A \wedge B}{A}$$

$$(\wedge E_2)\frac{A \wedge B}{B}$$

$$(\text{tonk I}) \quad \frac{A}{AtonkB}$$

$$(\text{tonk E}) \quad \frac{AtonkB}{B}$$

Conjunction right introduction rule says that if you are in bounds by asserting A and by asserting B then you are in bounds by asserting A and B. Similarly, the left introduction states that if you are in bounds in asserting A and B, then you are also in bounds when asserting A or asserting B. Notice how the second rule, that of introduction to the left, is the same as tonk's left rule. No need to elaborate on that one then. The other one, on the other side, is more strict: you need to be in bounds by asserting both A and B. In Tonk you only needed A.

Similarly, disjunction has stricter rules on one side, but this time the opposite. This are the disjunction introduction and elimination rules in natural deduction:

$$(\vee I_1)\frac{A}{A \vee B}$$

$$(\vee I_2)\frac{B}{A \vee B}$$

$$(\vee E)\quad \frac{A \vee B \qquad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}$$

Right introduction rules say that if you are in bounds by asserting A you are in bounds also when asserting A or B. And the elimination rule is in this case the stricter: if you are in bounds when you assert A or B, then you can only get rid of it when you are in bounds asserting both A and B.

This has been used by harmony theorists to propose as a solution a balance between left and right rules, so that if I rules are strict, for example, E rules should be more lax. Many accounts of harmony can be found, of which Restall (to come out, p. 166) cites Steinberger (2011), according to whom 'The fact that tonkE allows us to infer B from A tonk B where tonk I did not give B as one of the premises from which we could infer A tonk B means that the tonk's E-principles are unduly permissive (relative to its I-principles)'. However, as Restall points out, this and the other accounts of harmony take for granted what Belnap called the 'context of deducibility', as the the answer to any question about rules being harmonious 'is always and only relative to the structural rules of the underlying proof system' (Restall, to come out, p. 167). Hence when the structural rules change, such as when we remove cut with Ripley, so does harmony.

We see that Tonk is disharmonious when cut is present but it is not when it leaves: the presence of cut at both sides of the turnstile seems to suffice to make its rules harmonious. How is this? Well, because, as we have seen, conservativeness allows for its conclusions (fruit of very little restricted I and E rules) to be maximally general without trivialising. It has a meaning, which is just less intense and more extensive than the meanings of disjunction and conjunction. In fact, it is to be expected that any other connective will be more restrictive in its meaning. Tonk would possess the maximally general meaning out of the connectives. This relation constrains on the general relation tonk would go to mean and hence would be more limited.

Before we move to the last part and the conclusions, let us see some quick considerations about other tonk-like connectives. What would the meaning of a connective whose rules are strict in both sides be? In this case, the problem seems to be the opposite, so perhaps a tentative attempt at characterising such a meaning would go through maximal intension instead of extension, in a similar but opposite way than tonk's.

4.4 Tonk vs Reality

Let us offer some last, crazier considerations following the intuition that reality always consists of complex settings. Maybe we could interpret tonk's meaning in a more terrible, more fantastical meaning. Perhaps what it accuses is not that transitivity is flawed, but rather that reality itself is, due to its complexity. If we understand its requirement as not a conditional but rather as an assertion, what tonk means that ultimately everything can be proven from everything else if we possess enough information about certain aspects of a complex setting.

So it would not be:

$\forall A | AtonkB \vdash B$

But rather:

$\forall A \exists AtonkB \ni B$

Provided tonk meant 'in considering some very specific justification', then B could follow from any A truthfully. To put a simple example, that butterflies fly could be derived by the fact that you are wearing socks right now in the sense that the fact you wearing socks entails that it is cold, which could trigger a longing for summer in a particularly melancholic bystander, which would in turn to an idealised memory of those times with butterflies flying over a golden field. Hence 'that butterflies fly' would be derived in some sense from the fact that you wear socks: butterflies flying in a person's head is a logical consequence of you wearing socks. Of course it is a highly localised sense, not quite satisfactory in many regards, but it is in at least one regard a valid sense.

This way, again, tonk would be logically kind of 'useless', but not at all meaningless. It serves little purpose in logic, namely, it points to the fact that the world is complex and interrelated and that there are possibly infinite relations between realities. It calls for the fact that there is a connection between A and B in reality. We do not consider many of these peculiar connections in our regular reasonings, yet that is not to say that they do not exist, but rather that they are useless and uncalled for. In fact, the name 'tonk' could be said to have been brilliantly chosen for these fantastically odd, undesired and yet meaningful relations. Asking for conservativity would in this interpretation be but a way of reminding that the context is to be borne in mind: perhaps any B tonks from any A, yes, but 'tonking' could be any sort of relation -most probably, it will be a negligible one. And yet it would still be meaningful because in such peculiar, most of the time negligible sense, it works, it means. Reality is indeed somewhat trivial if we do not strongly standardise contexts, and yet this standardisation is always a blunt step in the first place. In this interpretation, tonk would possess an appalling meaning indeed.

Furthermore, it would also remain being useless because it really does not say much of what such a relation can be, in fact it says nothing but that it exists. Hence, if we are to give tonk a meaning, tonk is the idea that ultimately we can find a connection between everything. From we can always find a relation between A and B, no matter how remote or specific -how absolutely 'tonkers' it is.

5 Conclusion

We can now provide some sort of explanation to the problems registered with tonk regarding truth values, conservative extension and harmony. In all of these, what is accused is a lack of determination: tonk is an underdetermined connective. But this does not mean it does not possess any meaning, what it means is that tonk has a very general meaning, perhaps the most general. Tonk is any relationship linking two realities A and B. It is a given. If you have A and B together then there is some link between them. It is useless in a way because we already take it for granted prior to any logical characterization, but it is a priceless tool to show that not 2every logical relation needs to be transitive, even if transitivity is a sign of a manageable logical relation. Further research into tonk could go through its connection with paradoxes: instead of approaching them trough truth predicates delve into how tonk allows to bypass unassertable realities.

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