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# Introduction to complex geometry and Calabi-Yau manifolds motivated by physics

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## Abstract

In this work, we give an introduction to complex geometry and Calabi-Yau manifolds. We begin by recalling the necessary background of differential geometry, as well as defining the de Rahm cohomology and the Ricci curvature. Then we turn to complex geometry, giving some precise examples, extending differential forms to the complex case and defining Dolbeault cohomology, Chern classes and holonomy. We then focus on Kähler manifolds, which are the previous steps to define Calabi-Yau manifolds, whose properties will be briefly studied. We close the work with a few ideas of a basic string theory model.

## Resum

En aquest treball, introduim la geometria complexa i les varietats de Calabi-Yau. Comencem recordant la part de geometria diferencial necessària, defininint també altres conceptes com la cohomologia de de Rahm o la curvatura de Ricci. Passem seguidament a la geomtria complexa, donant alguns exemples precisos i extenent les formes diferencials al cas complex, com també definint la cohomologia de Dolbeault, les classes de Chern o l'holonomia. Ens fixem després en les varietats de Kähler, que seran el pas intermedi per definir les varietats de Calabi-Yau, les propietats de les quals estudiarem breument. Tanquem el treball amb algunes idees d'un model bàsic de teoria de cordes.

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Seguidament, gràcies infinites als meus pares, que de fet són pares i amics. Han estat i són suport i exemple per a mi. No vull tampoc oblidar-me dels meus germans i els meus padrins, que no he pogut veure tant com hauria volgut; de l'avi, que malauradament ja no hi és; i de la iaia, sobretot de la iaia, que m'ha aguantat i cuidat tots aquests anys. I que me n'haurà d'aguantar encara alguns més: iaia, ens haurem de fer la punyeta encara una mica.

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# Contents

# Introduction

Since the 19th century, physics has had a huge progress. However, three main problems have needed to be faced from then, and the last one has not been solved yet. During the second part of that century, Maxwell developed his *Maxwell equations*, which describe the electromagnetic field. However, a first problem quickly emerged. Mixing Maxwell equations with Newton's laws of motion implied that an observer could, for example, be as quick as light, and hence be in a reference frame in which light would not move. It is well known that Einstein gave the solution to this problem by introducing his theory of *Special Relativity* (SR) in 1905. Recall that before Einstein, space $S$ and time $T$ were absolute, with

$$S = \mathbb{R}^3, \qquad T = \mathbb{R}$$

and both are equipped with an euclidean metric. SR had some important implications in our conception of space and time: time (as space) would not anymore be absolute, but it depended on the motion of the observer. We could not continue considering space and time as separate, different things of Universe. We, from then on, must deal with spacetime, a four-dimensional space with three spatial dimensions and another temporal, but not anymore independent. This spacetime was called the Minkowski space

$$\mathcal{M}_4 = (T \times S, g)$$

where $g$ is a metric which is not euclidean, but lorentzian. But a second problem emerged: since light always moves at $c$, i.e. it mantains in every reference frame the same speed, and no thing can move faster than light, gravity seemed to contradict it, as it was considered to act instantaneously. Again, Newton gave response to this problem by publishing in 1915 his theory of *General Relativity* (GR).

GR implications went even further on our conception of spacetime, since not only the observer's motion, but even mass and energy could modify the spacetime. In fact, mass and energy would curve the spacetime, and the action of gravity would be explained by these deformations on spacetime. However, locally we could continue considering the space as a Minkowski space. Hence, everything seemed to be solved. But again, a new problem was found. A new field of physics was being developed exponentially during the first half of the 20th century: *Quantum Mechanics* (QM). It had a totally different approach and goals, since it firstly studied little things as atoms and fundamental particles. QM and GR worked well: in fact, all experimental tests seemed to prove the validity of both theories. However, going through a theoretical analysis of these theories considered together, they were proved to be incompatible: QM and GR, although giving predictions of reality with and outstanding accurancy both, could not be valid at the same time. This is the third main problem, known as the *central problem of modern physics* [10].

Back on time, Kaluza had suggested in 1919 that the Universe could have more than four dimensions. Precisely, he proposed that the spatial dimensions might not be three, but four, giving therefore rise to a five-dimensional spacetime. The reason behind this bizarre idea was the following: he realised that adding this new spatial dimension, it was much more simple to put General relativity and the Maxwell equations of the electromagnetic field together. In fact, since he had an extra dimension, he got a few more equations than the ones Einstein had gotten before. Equations relating the three common dimensions were exactly the same of GR. But surprisingly the extra ones were the Maxwell equations. He had unified gravity and electromagnetism just by adding a new spatial dimension. Although melting space and time was a revolutionary idea and in some sense strange, in the end we ended up with the same dimensions we started with. But how can we even imagine that there is an extra spatial dimension if we can not see it [6]?

The answer to this apparently difficult question was given by the same Kaluza together with Klein in 1926. Basically, they postulated that spatial dimensions can be either large, as the three known dimensions which are infinite, or curled up on themselves: so little and curled up that we have not been able to find them. Further analysis suggested that the characteristic length of these extra dimensions could be around the Planck length, which is approximately $10^{-35}$ m. Hence, if this dimension is enough small we might not be able to see it, although it actually exists. Imagine looking at an electricity wire from a big distance: we would see a line, i.e. a one-dimensional space, even if in fact there are more dimensions, which we may discover getting closer to it. The space could then be described, according to Kaluza and Klein, as

$$M_5 = \mathcal{M}_4 \times M_1$$

where $M_1$ would be the new dimension. They both proposed this new extra dimension to be circular, hence $M_1 \cong \mathbb{S}^1$. Therefore, an object at a point of the space would have four degrees of freedom of movement, although the one curled up would not affect the classical motion of the bodies, at least at first glance. What remained proved was that one could not reject the existence of extra dimensions if these were enough small. These ideas were put under the name of *Kaluza-Klein theory*. Anyway, further reasearch proved contradiction when introducing the electron, since the mass-charge ratio was rather different than the empirical, which at that time was already very accurate. Hence, the interest in Kaluza-Klein theory reduced much, also because from the 1930s to the 1960s many fundamental particles were discovered and much progress was made in quantum mechanics. There was no need of extra dimensions to continue progressing [6].

By the end of the 1960s the main structure of the *Standard Model* was finished, all predictions were generally proved right by experimental methods but it still remained the problem of incompatibility between GR and QM. In that situation, Kaluza-Klein theory was rediscovered: considering that Kaluza started his studies on the topic in the late 1910s little was known about nuclear forces, it was thought that Kaluza might have been too conservative by only adding an extra dimension. Since more forces were known, more dimensions might be needed. So physicists added more new extra dimensions, imposing at the same time the validity of supersymmetry, which roughly speaking was a continuation of the classical study of symmetries, but adding modern variables as the spin [9, 13, 10].

And finally *string theory* arrived, which seemed to solve the incompatibility of GR with QM. It postulates that the fundamental constituents of matter are not particles -point-like, without an spatial structure- but tiny strings of a characteristic length of the Planck length. Hence, fundamental objects were thought to be two-dimensional objects whose mass, charge and other intern properties would depend on their vibration mode. Mixing string theory with supersymmetry and fundamental forces unification resulted in the need of a ten-dimensional spacetime. Therefore, six extra dimensions were needed, and they would have to be curled up, tiny. This is one of the basic postulates of string theory, although further reaserch has suggested eleven-dimensional spacetimes or even more.

Therefore six extra curled up dimensions were from then on needed. The local structure of the new spacetime considered could be written as

$$M_{10} = \mathcal{M}_4 \times M_6$$

where $\mathcal{M}_4$ is the local Minkowski space. Physicists started asking themselves about the shape of these dimensions, about what was $M_6$. If $M_6$ was considered as complex more supersymmetry emerged, and it was rather convenient for these dimensions to be Ricci-flat in the abscence of masses around, so a kind of complex manifolds were proposed, the Calabi-Yau manifolds [9, 4, 16, 13].

## About this work

In order to deeply understand the problem of extra dimensions of the Universe, and hence at least be capable of introducing Calabi-Yau manifolds, a strong background on complex geometry is needed. Therefore, our goal in this work is introducing complex geometry and the necessary objects to arrive to Calabi-Yau manifolds.

After a quickly review of smooth manifold, i.e. the real case, we give an introduction to complex geometry and its basic concepts and objects. Complex geometry is the branch of mathematics that studies complex manifolds, which are topological spaces that locally resemble $\mathbb{C}^n$, for a certain $n$. These manifolds are defined by the transition functions between charts being holomorphic functions, which can be defined as differentiable functions that satisfy the Cauchy-Riemann equations. The main object of study in complex geometry are the complex structures of these manifolds, which are defined as a choice of an almost complex structure, which is a smooth endomorphism of the tangent bundle, that is integrable: we will later discover what does integrable mean. Smooth manifolds that admit an integrable complex structure are called complex manifolds. Actually, we will see that there exist two different and equivalent constructions of complex manifolds [11].

An intermediate step between complex manifolds and Calabi-Yau manifolds is Kähler manifolds, which are complex manifolds equipped with a metric compatible with the complex structure and whose fundamental form is closed. We will give some examples and characterization of Kähler, going then quickly to Calabi-Yau manifold's definition.

Calabi-Yau manifolds were named after Eugenio Calabi and Shing-Tung Yau, who first introduced them in the 1970s. They are defined as a class of compact Kähler manifolds with a Ricci-flat metric, although there are plenty of equivalent definitions in the literature. As Kähler manifolds, they are also equipped with a Kähler metric. The most important property of Calabi-Yau manifolds is the existence of a Ricci-flat Kähler metric. This means that their scalar curvature is zero and that their complex structure is parallel with respect to the Levi-Civita connection. Calabi-Yau manifolds have many interesting geometric properties, and they have been extensively studied in mathematics, particularly in algebraic geometry and differential geometry. Their properties have also made them useful in physics, particularly in theories such as string theory and supersymmetry, where they are used as internal spaces for extra dimensions, as already said [16, 4].

Finally, although not being the scope of this work but the motivation, we will close the work by giving some ideas on string theory. Precisely, we will focus on a basic model of bosonic string theory [9, 13].

# Chapter 1

# Preliminaries

*"The Beauty of the House is immesurable. Its Kindness infinite."*

- Susanna Clarke, *Piranesi*

In this chapter we aim to introduce manifolds with an infinite-differentiable structure and the related objects that we will later use in the next chapters. To do so we mainly follow [8, 5]. We first recall what a topological manifold is:

**Definition 1.1.** An *n-dimensional topological manifold* is a topological space $M$ satisfying that:

1. $M$ is second countable, i.e. its topology admits a countable basis.
2. $M$ is Hausdorff, i.e. for all pairs $x, y \in M$ there exist disjoint open subsets $U, V \subseteq M$ such that $x \in U$ and $y \in V$.
3. $M$ is locally homeomorphic to $\mathbb{R}^n$, i.e. for all $x \in M$ there exists an open neighbourhood $U \subseteq M$ which admits an homeomorphism $U \cong \mathbb{R}^n$.

## 1.1 Smooth manifolds

**Definition 1.2.** A *smooth atlas* on a topological manifold $M$ of dimension $n$ is a set of tuples $\{(U_i, \varphi_i)\}$ such that $M = \bigcup_i U_i$ and $\varphi_i : U_i \to \mathbb{R}^n$ are the coordinate functions, which induce homeomorphisms $\varphi_i : U_i \cong \varphi(U_i) \subseteq \mathbb{R}^n$ and such that the transition functions

$$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

are smooth, i.e. $\mathcal{C}^\infty$.
Any pair $(U, \varphi)$ is called a *smooth chart*. Two smooth atlases $\{(U_i, \varphi_i)\}$, $\{(V_j, \psi_j)\}$

are called *equivalent* if $\forall i, j$ the map

$$\varphi_i \circ \psi_j^{-1} : \psi_j(U_i \cap V_j) \to \varphi_i(U_i \cap V_j)$$

is smooth.

We might see that the equivalence between smooth atlases is a relation of equivalence, which can be easily proved by checking the three properties of relations of equivalence.

**Definition 1.3.** An *n-dimensional smooth manifold* $(M, \mathcal{A})$ is a topological manifold $M$ endowed with a smooth atlas $\mathcal{A}$.

**Definition 1.4.** A *smooth structure* on a topological manifold $M$ is a class of equivalence of smooth atlases. For a smooth atlas $\mathcal{A}$, the smooth structure is denoted $[\mathcal{A}]$.

**Remark 1.5.** Given two smooth manifolds with equivalent atlases, the smooth manifolds are usually said to be equivalent. Actually, it is common defining a smooth manifold as a topological manifold endowed with a smooth structure.

**Definition 1.6.** A smooth map $f : M \to N$ of smooth manifolds $(M, \{(U_i, \varphi_i)\})$ and $(N, \{(V_j, \psi_j)\})$ is a map for which for all $i, j$ the functions

$$\psi_j \circ f \circ \varphi_i^{-1} : \varphi_i(U) \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

are smooth.

**Example 1.7.**
1. (Euclidean spaces). For $n \geq 1$, $\mathbb{R}^n$ is a smooth manifold when endowed with the atlas $\{(\mathbb{R}^n, id)\}$.

2. ($n$-spheres) Define the $n$-sphere to be the set

$$\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : x_1^2 + \ldots x_{n+1}^2 = 1\}$$

Let $\mathbb{S}^n$ be endowed with the atlas $\{(\mathbb{S}^n \setminus \{x\}, \phi_x), (\mathbb{S}^n \setminus \{y\}, \phi_y)\}$, where $x, y \in \mathbb{S}^n, x \neq y$ and $\phi_x, \phi_y$ are the stereographic projections from the sphere to $\mathbb{R}^n$ from the basepoints $x, y$ respectively. Then the $n$-sphere is a smooth manifold.

3. The cartesian product of smooth manifolds is a smooth manifold as well. The dimension of the product manifold is the sum of the dimensions.

4. Any open subset $N$ of a smooth manifold $M$, the latter with smooth atlas $\{(U_i, \varphi_i)\}$, is itself a smooth manifold with induced smooth atlas $\{(Y \cap U_i, \varphi_i|_Y)\}$.

## 1.2 Tangent spaces

Let $M$ be a smooth manifold. We denote $\mathcal{F}(M)$ the set of all maps $f : M \to \mathbb{R}$ such that for every chart $(U, \varphi)$, the composition

$$f \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \mathbb{R}$$

is smooth.

**Remark 1.8.** Note that the triple $(\mathcal{F}(M), +, \cdot)$ has the structure of an $\mathbb{R}$-vector space, with the operations defined as

$$(f+g)(p) := f(p)+g(p), \qquad (\lambda f)(p) := \lambda f(p), \qquad \forall f, g \in \mathcal{F}(M),\ \lambda \in \mathbb{R}, p \in M$$

Moreover, it is a ring with the product

$$(fg)(p) := f(p)g(p)$$

**Definition 1.9.** An $\mathbb{R}$-*derivation on an n-dimensional smooth manifold $M$ at $p \in M$* is an $\mathbb{R}$-linear map $D : \mathcal{F}(M) \to \mathbb{R}$ satisfying the Leibniz product rule, i.e.

$$D(fg) = D(f)g(p) + f(p)D(g), \quad \forall f, g \in \mathcal{F}(M)$$

**Definition 1.10.** The set of all derivations at $p$ is called *the tangent space of $M$ at $p$* and is denoted $T_pM$.

**Remark 1.11.** Observe that $(T_pM, +, \cdot)$ is an $\mathbb{R}$-vector space with the operations naturally defined as

$$(D_1 + D_2)f = D_1 f + D_2 f, \quad (\lambda \cdot D)f = \lambda D f, \qquad D_1, D_2, D \in T_pM,\ \lambda \in \mathbb{R}$$

The following result is proved using elementary linear algebra. For a detailed proof, see §2.1 of [5].

**Theorem 1.12.** *$T_pM$ is a n-dimensional $\mathbb{R}$-vector space.*

For a chart $(U, \varphi)$ of $M$ with $\varphi = (x_1, \ldots, x_n)$ local coordinates, for $p \in U$ the ordered set of derivations

$$\left\{ \left( \frac{\partial}{\partial x_1} \right)_p, \ldots, \left( \frac{\partial}{\partial x_n} \right)_p \right\}$$

forms a basis of $T_pM$. Each derivation of the basis acts on a smooth function $f : U \subseteq M \to \mathbb{R}$ as

$$\left( \frac{\partial}{\partial x_i} \right)_p (f) = \frac{\partial (f \circ \varphi^{-1})}{\partial x_i} (\varphi(p))$$

**Definition 1.13.** Let $f : U \subseteq M \to \mathbb{R}$ be a smooth function and $p \in M$. We define and denote the *differential of the function $f$ at the point $p \in M$* as

$$\begin{aligned}(df)_p : T_pM &\longrightarrow \mathbb{R}\\ D &\mapsto D(f)(p)\end{aligned}$$

**Definition 1.14.** The *cotangent space* $T_p^*M$ is the dual vector space $(T_pM)^*$, i.e. the space of $\mathbb{R}$-linear maps $\omega : T_pM \to \mathbb{R}$. The elements of the cotangent space are called 1-*forms*.

**Remark 1.15.** Observe that in the way that we defined the differential of an application at a point of the manifold, $(df)_p$ is a 1-form.

For a chart $(U, \varphi)$ of $M$ with $\varphi = (x_1, \ldots, x_n)$ the local coordinates, we have that the ordered set of differentials

$$\{(dx_1)_p, \ldots, (dx_n)\}$$

is the dual basis of $\left\{ \left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p \right\}$, therefore a basis for $T_p^*M$. Each differential of the basis acts on a basis derivation as

$$(dx_i)_p \left(\frac{\partial}{\partial x_j}\right)_p = \frac{\partial x_i}{\partial x_j}(\varphi(p)) = \delta_{ij}$$

Smooth maps between manifolds induce maps between their tangent spaces:

**Definition 1.16.** Let $f : M \to N$ be a smooth map between manifolds. The map $f_* : T_pM \to T_{f(p)}N$ defined by

$$f_*(v)(h) := v(h \circ f), \qquad h : N \to \mathbb{R} \text{ a smooth function}$$

is called the *pushforward of $v$ by $f$*.

**Remark 1.17.** $f_*$ is also called the *differential of $f$* and denoted $df$, as it sends derivations on $M$ to derivations on $N$.

## 1.3 Tangent bundles

**Definition 1.18.** A *fibre bundle on a topological space $M$ with fibre $F$* is a surjective continuous map $\pi : E \to M$ such that for every point $p \in M$, there exists an open neighbourhood $U$, so that there is a homeomorphism $h : \pi^{-1}(U) \cong U \times F$, where $F$ is a topological space. Then, $U$ is called a *trivializing neighbourhood of $p$* and $h$ a *local trivializing map*. A fibre bundle is called *trivial* if $M$ is a trivializing neighbourhood of every point $p \in M$ and hence $E \cong M \times F$. $E_p := \pi^{-1}(p)$ is said to be the *fibre over $p \in M$*.

From now on we will consider $M$ as an $n$-dimensional smooth manifold again and the fibre bundle $\pi : E \to M$ to be smooth.

**Definition 1.19.** Let $\pi : E \to M$ be a fibre bundle and $M$ a smooth manifold. The fibre bundle is said to be a *real vector bundle of rank $k$* if

1. The fibre of every point has the structure of a $k$-dimensional real vector space.

2. The local trivialization maps $h$ are diffeomorphisms.

3. $h(p, \cdot) : E_p \to \{p\} \times \mathbb{R}^k$ is an isomorphism of vector spaces.

**Definition 1.20.** For $U \subseteq M$ an open subset, a *local section of a real vector bundle* $\pi : E \to M$ is a smooth map $s : U \to E$ such that $\pi \circ s = id_U$. A section is called *global* if it is defined on the entire manifold $M$.

We might take the disjoint union of tangent spaces on every point $p \in M$ to form a vector bundle, which will be called the *tangent bundle of $M$* and denoted $TM$:

$$TM := \bigsqcup_{p \in M} T_pM = \{(p, v) : p \in M,\ v \in T_pM\}$$

where the projection function $\pi$ is defined as

$$\pi : TM \longrightarrow M, \qquad \pi(p, v) = p$$

Thus, the fibre at each point $p \in M$ will be its tangent space, i.e. $\pi^{-1}(p) = T_pM$.

**Remark 1.21.** The tangent bundle $TM$ of $M$ is itself a smooth manifold of dimension $2n$.

For a chart $(U, \varphi)$ of $M$ we define the map

$$\begin{aligned} \Psi_U : \pi^{-1}(U) &\longrightarrow \mathbb{R}^{2n} \\ (p, v) &\longmapsto (p_1, \dots, p_n, \lambda_1, \dots, \lambda_n) \end{aligned}$$

with $\varphi(p) = (p_1, \dots, p_n)$ and $\lambda_i$ the components of $v$ in the canonical basis of $T_pU$. Then $TM$ can be given a topology by the preimages of the open sets of $\mathbb{R}^{2n}$ endowed with the Euclidean topology. An atlas is then given by $\{(\pi^{-1}(U_i), \Psi_{U_i})\}$. We now introduce the notion of vector field on a manifold:

**Definition 1.22.** A section of the tangent bundle $s : M \to TM$ is called a *vector field on $M$*. We denote $\mathcal{X}(M)$ the set of all vector fields on $M$.

**Remark 1.23.** Note that a vector field is of the form $X := (p, X_p)$, where $X_p$ is a derivation at $p$. Hence $\mathcal{X}(M)$ acts canonically on $\mathcal{F}(M)$ as

$$\begin{aligned} X : \mathcal{F}(M) &\longrightarrow \mathcal{F} \\ f &\longmapsto Xf : M \longrightarrow \mathbb{R} \\ &\qquad\qquad p \longmapsto X_p f(p) \end{aligned}$$

We now introduce an important object called the *Lie bracket*. Although it seems that its derivatives are of order higher than 1, we will see that in fact it defines a vector field on $M$:

**Definition 1.24.** Let $X, Y \in \mathcal{X}(M)$ be vector fields. We define the *Lie bracket* $[X, Y]$ of $X$ and $Y$ by

$$[X, Y]f := X(Yf) - Y(Xf)$$

where $f : M \to \mathbb{R}$ is any smooth function.

**Proposition 1.25.** *The Lie bracket* $[X, Y]$ *of* $X$ *and* $Y$ *is a vector field.*

*Proof.* For any chart $(U, \varphi)$ of $M$, with $\varphi = (x_1, \ldots, x_n)$, locally we can express the vector fields as

$$X = \sum_i a_i \frac{\partial}{\partial x_i}, \qquad y = \sum_i b_i \frac{\partial}{\partial x_i}$$

where $a_i, b_j$ are smooth. Therefore

$$[X, Y] = XY - YX = \sum_i \sum_j a_i \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j} \frac{\partial}{\partial x_i}$$

which is clearly a vector field. □

**Corollary 1.26.** *The Lie bracket is antisymmetric.*

## 1.4 Differential $k$-forms

We now introduce the *cotangent bundle*

$$\pi : T^*M \longrightarrow M$$

which is defined similarly as we have done with the tangent bundle. It is also given the structure of a smooth manifold as we have done before with the tangent bundle.

**Definition 1.27.** A *differential* 1-*form on* $M$ is a section of the cotangent bundle $s : M \to T^*M$, which assigns to any point $p \mapsto (p, w_p)$, where $w_p \in T_p^*M$ is a 1-form. We denote by $\mathcal{A}^1(M)$ the set of all differential 1-forms.

Note that locally on a chart $(U, \varphi)$, with $\varphi = (x_1, \ldots, x_n)$, the 1-form can be given by

$$\omega := \sum_{i=1}^{n} \omega_i dx_i$$

where the functions $\omega_i : U \to \mathbb{R}$ are smooth.

**Example 1.28.** As we have already seen, the local expression for the differential of a function $f \in \mathcal{F}(M)$ is

$$df = \sum_{i=1}^{m} \frac{\partial (f \circ \varphi^{-1})}{\partial x_i} dx_i$$

We will now introduce the generalization of forms, i.e. the $k$-forms. They are sections of the $k$-th exterior power of the cotangent bundle, which for a smooth manifold $M$ we will denote

$$\bigwedge^k M := \bigwedge^k (TM)^*$$

**Remark 1.29.** This object is indeed a vector bundle and it is given a smooth structure again in an analogue way as we have already done more than once for bundles.

**Definition 1.30.** Let $k \geq 0$. A *differential $k$-form of $M$* is a section of $\bigwedge^k M$, i.e. a map

$$\begin{aligned} \omega : M &\longrightarrow \bigwedge^k M \\ p &\longmapsto (p, \omega_p) \end{aligned}$$

We denote by $\mathcal{A}^k(M)$ the set of all differential $k$-forms on $M$.

**Remark 1.31.** The differential 0-forms are the functions $f \in \mathcal{F}(M)$.

Locally the differential $k$-forms can be expressed as

$$\omega_p := \sum_{i_1 < \ldots < i_k} \omega_{i_1, \ldots, i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

with functions $w_{i_1, \ldots, i_k}$ being smooth.

**Definition 1.32.** The *differential $d\omega$ of a differential k-form $\omega$* is the $k+1$-form defined as

$$d\omega(X_0,\ldots,X_k) := \sum_i (-1)^i X_i(\omega(X_0,\ldots,\hat{X}_i,\ldots,X_k)) + \\ + \sum_{i<j} (-1)^{i+j} \omega([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k)$$

where $X_i \in \mathcal{X}(M)$ and the terms with a hat above are omitted.

Although the global definition might not be always handy, the differential $d\omega$ of a $k$-form $\omega$ is locally expressed as

$$d\omega := \sum_j \sum_{i_1<\ldots<i_k} \frac{\partial \omega_{i_1,\ldots,i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

which is quite more useful. In addition, we get a map for $k \geq 0$ on the set of differential $k$-forms

$$d : \mathcal{A}^k(M) \longrightarrow \mathcal{A}^{k+1}(M)$$

**Definition 1.33.** Forms that are image of other forms under $d$ are called *exact*, while forms whose image under $d$ is 0 are called *closed*.

**Proposition 1.34.** *The composition of $d$ with itself is the zero map, i.e. $d \circ d = 0$.*

*Proof.* Take a $k$-form $\omega$ and get its differential $d\omega$. Locally it will be

$$d\omega := \sum_j \sum_{i_1<\ldots<i_k} \frac{\partial \omega_{i_1,\ldots,i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

and we might now apply again $d$:

$$d^2\omega := \sum_k \sum_j \sum_{i_1<\ldots<i_k} \frac{\partial^2 \omega_{i_1,\ldots,i_k}}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

By the smoothness of $\omega_{i_1,\ldots,i_k}$, the crossed partial derivatives satisfy $\frac{\partial^2 \omega_{i_1,\ldots,i_k}}{\partial x_k \partial x_j} = \frac{\partial^2 \omega_{i_1,\ldots,i_k}}{\partial x_j \partial x_k}$ and considering that the wedge product is antisymmetric we get the result. □

**Remark 1.35.** The relation $d \circ d = 0$ states that exact forms are closed.

**Definition 1.36.** The *exterior wedge product of differential forms* is defined as

$$\begin{aligned} \wedge : \mathcal{A}^k(M) \times \mathcal{A}^l(M) &\longrightarrow \mathcal{A}^{k+l}(M) \\ (\alpha,\beta) &\longmapsto \alpha \wedge \beta \end{aligned}$$

The set of all forms $\mathcal{A}_{dR}(M) := \bigcup_{k\geq 0} \mathcal{A}^k(M)$ has the structure of an algebra with the product defined as in Definition 1.36. This algebra is usually called the *de Rham algebra* and for this reason it is denoted with the subscript $dR$.

**Lemma 1.37.** *Let $\alpha$ be a k-form and $\beta$ a l-form. Then*

1. $\beta \wedge \alpha = (-1)^{kl} \alpha \wedge \beta$
2. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$

*Proof.* Since the wedge product is antisymmetric and we need to take $kl$ permutations to invert it, we get the factor $(-1)^{kl}$ and hence the first statement. Then, considering the general Leibniz product and that the exterior derivative acts on a form as in Definition 1.32, one can see that $k$ permutations are needed to take the product of each differential of the basis with $\beta$, getting as a consequence the factor $(-1)^k$ of the second statement. □

Finally, given a smooth map between manifolds $f : M \to N$, there exists an induced function $f^* : \mathcal{A}_{dR}(N) \to \mathcal{A}_{dR}(M)$ called the *pullback map* which locally is defined as

$$f^*(\omega) = \sum_{i_1 < \ldots < i_k} (\omega_{i_1,\ldots,i_k} \circ f) df_{i_1} \wedge \ldots \wedge df_{i_k}$$

with $f_i$ being the $i$-th component of $f$ in $N$.

## 1.5 De Rham cohomology

We now introduce de Rham cohomology, which will provide us a topological invariant. Let $(M, J)$ be an $n$-dimensional complex manifold. Since $d$ satisfies $d \circ d \equiv 0$ we can define a cochain complex

$$0 \to \mathcal{A}^0(M) \to \mathcal{A}^1(M) \to \ldots \to \mathcal{A}^k(M) \to \ldots \mathcal{A}^n(M) \to 0$$

called the *de Rham complex*.

**Remark 1.38.** Note that since the cross product cancels for dimension $> n$, the de Rham complex terminates.

Related to the de Rham complex there is the de Rham cohomology.

**Definition 1.39.** Let $M$ be a smooth manifold. The de *Rham cohomology group of M* is defined as

$$H^k_{dR}(M, \mathbb{R}) := \frac{\mathrm{Ker}(d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M))}{\mathrm{Im}(d : \mathcal{A}^{k-1}(M) \to \mathcal{A}^k(M))}$$

**Example 1.40.** For any manifold $M$ composed of $k$ disconnected components, each one of them connected, $H^0_{dR}(M, \mathbb{R}) = \mathbb{R}^k$. The kernel considered represents the set of functions on $M$ with derivative 0 everywhere, so these functions must be constant on each connected component. Hence we see that the dimension of $H^0(M, \mathbb{R})$ is actually the number of disconnected components of $M$.

It turns out that the de Rham cohomology groups only depend on the topological structure of the manifold, i.e. there is not a dependence on the differential structure at all and hence any two homeomorphic manifolds have the same cohomology groups [7, 4]. Therefore de Rham cohomology is a topological invariant and so does the dimension of the cohomology groups. We then define the following invariant:

**Definition 1.41.** Let $M$ be an $n$-dimenional smooth manifold. The *Betti numbers* $b^k(M)$ are defined as

$$b^k(M) := \dim_{\mathbb{R}} H^k_{dR}(M, \mathbb{R}), \qquad k = 0, \dots, n$$

**Remark 1.42.** Note that for compact $n$-dimensional smooth manifolds $b^k < \infty$, for all $k = 0, \dots, n$.

Related to Betti numbers of a compact manifold we can define the Euler characteristic, which will be a topological invariant of $M$ [7]:

**Definition 1.43.** Let $M$ be a compact $n$-dimenional smooth manifold. The *Euler characteristic* is defined as

$$\chi(M) = \sum_{k=0}^{n} (-1)^k b^k(M)$$

## 1.6 Riemannian manifolds and Ricci curvature

**Definition 1.44.** Let $M$ be a smooth manifold. A *Riemannian metric on $M$* is a family of positive-definite inner products

$$g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$$

such that for any vector fields $X, Y \in \mathcal{X}(M)$ the assignement

$$\begin{aligned} g : M &\longrightarrow \mathbb{R} \\ p &\longmapsto g_p(X|_p, Y|_p) \end{aligned}$$

is set to be smooth. A smooth manifold endowed with a Riemannian metric is called a *Riemannian manifold* $(M, g)$.

**Remark 1.45.** A Riemannian metric provides the manifold a norm $|\cdot|_M$ by setting

$$\begin{aligned} |\cdot|_p : T_pM &\longrightarrow \mathbb{R}_+ \\ v &\longmapsto g_p(v,v) \geq 0 \end{aligned}$$

**Remark 1.46.** We could relax the conditions on the function $g$ by non imposing the function to be positive-definite. Then, we would say $g$ is a *pseudo-Riemannian metric* and hence a manifold $M$ equipped with $g$ a *pseudo-Riemannian manifold*. It is a more general notion than Riemannian manifolds.

We now define the concepts we need to be able to define the Ricci curvature, which will be needed in the following chapters. Since there is no natural way to define the directional derivative of a vector field on a manifold, we introduce its equivalent, the connections [11, 1]:

**Definition 1.47.** Let $M$ be a smooth manifold. A *connection on $TM$* is an $\mathbb{R}$-bilinear map $\mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ such that for $X, Y \in \mathcal{X}(M)$ and $f \in \mathcal{C}^\infty(M)$,

$$D_{fX}Y = fD_XY \qquad \text{and} \qquad D_X(fY) = (Xf)Y + fD_XY$$

**Definition 1.48.** For any connection $D$ on $TM$, the *torsion of the connection $D$* is defined as

$$T : (X,Y) \mapsto D_XY - D_YX - [X,Y]$$

A connection $D$ on $TM$ is called *torsion-free* if $T \equiv 0$, i.e. for $X, Y \in \mathcal{X}(M)$,

$$D_XY - D_YX = [X,Y]$$

**Theorem 1.49.** *There exists on any Riemannian manifold $(M,g)$ a unique torsion-free connection consistent with the metric, i.e. sucht that if $X, Y, Z \in \mathcal{X}(M)$*

$$Xg(Y,Z) = g(D_XY, Z) + g(Y, D_XZ)$$

See §§2.B of [8] for proof.

**Definition 1.50.** The torsion-free connection consistent with the metric is called the *Levi-Civita connection*.

**Example 1.51.** The Levi-Civita connection of the Euclidean space is simply $D_XY = dY(X)$.

Since we will need it in following chapters, we now define parallel transport of vectors along curves on manifolds. We need first a result whose proof can be found as well in §§2.B of [8].

**Theorem 1.52.** *Let $(M, g)$ be a Riemannian manifold, $D$ be its Levi-Civita connection and $c : I \subseteq \mathbb{R} \to M$ be a curve on $M$. There exists a unique operator, denoted $\frac{D}{dt}$, defined on the vector space of of vector fields along the curve $c$ which satisfies*

1. *For any real function $f : I \to \mathbb{R}$ ,*

$$\frac{D}{dt}(fY)(t) = f'(t)Y(t) + \frac{D}{dt}Y(t)$$

2. *if there exists a neighbourhood of $t_0 \in I$ such that $Y$ is the restriction to $c$ of a vector field $X$ defined on a neighbourhood of $c(t_0)$ in $M$, then*

$$\frac{D}{dt}Y(t_0) = (D_{c'(t_0)}X)_{c(t_0)}$$

**Definition 1.53.** A vector field along a curve $c$ is called *parallel* if $\frac{D}{dt}X = 0$.

**Definition 1.54.** The *parallel transport from $c(0)$ to $c(t)$ along a curve $c$* in $(M, g)$ is the linear map $P_t$ from $T_{c(0)}M$ to $T_{c(t)}M$, which associates to $v \in T_{c(0)}M$ the vector $X_v(t)$, where $X_v$ is the parallel vector field along $c$ such that $X_v(0) = v$.

We finally define the notions of curvature and Ricci curvature:

**Definition 1.55.** The *curvature tensor* of a Riemannian manifold $(M, g)$ is the tensor defined by

$$R_p(x, y)z = (D^2_{Y,X}Z - D^2_{X,Y}Z)_p = (D_Y(D_XZ) - D_X(D_YZ) + D_{[X,Y]}Z)_p$$

where $X, Y, Z \in \mathcal{X}(M)$ and $x = X_p, y = Y_p, z = Z_p$.

We can now define the Ricci curvature and, moreover, what is understood for a Ricci-flat manifold. It will be an important concept to understand Calabi-Yau manifolds.

**Definition 1.56.** The *Ricci curvature of a Riemannian manifold* $(M, g)$ is the trace of the endomorphism of $T_pM$ given by $v \mapsto R_p(x, v)y$. We shall denote the Ricci curvature by *Ric*.

**Definition 1.57.** Riemannian manifolds whose Ricci curvature is proportional to the metric, i.e. $Ric = \lambda g$, are called *Einstein manifolds*. Riemannian manifolds whose Ricci curvature is zero are called *Ricci-flat*.

# Chapter 2

# Complex manifolds

*"SÓC.-Vamos a ver si de algún modo nos ponemos de acuerdo, Crátilo. ¿No dirías tú que una cosa es el nombre y otra aquello de lo que es nombre?"*

- Platón, *Crátilo*

We introduce in this chapter what a complex manifold is, give some examples and see the differences between complex and smooth manifolds. We then define Dolbeault homology, Chern classes and holonomy, which will be topological and complex structure invariants that will allow us to characterize and to distinguish the different complex structures. To do so, we generally follow [11], but also [4, 1, 17].

## 2.1 Holomorphic functions

Due to the reiterate use that we will do of the *holomorphic* property of complex functions of several variables, we first recall what a holomorphic function of several variables $f : U \subseteq \mathbb{C}^n \to \mathbb{C}^m$ is
We shall define the general case in three natural steps.

**Definition 2.1.** Let $U \subseteq \mathbb{C}$ be an open subset. A function $f : U \to \mathbb{C}$ is *holomorphic* if for all $z_0 \in U$ the limit

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. Then $f'(z)$ is called the *derivative of* $f(z)$ *in* $U$.

**Definition 2.2.** Let $f_i : U \subseteq \mathbb{C}^n \to \mathbb{C}$ be a a complex function of $n$ complex variables. We say that $f_i$ is *holomorphic* if it is holomorphic in each variable when fixing the others.

Now we just need to extend the Definition 2.2 to the general case

**Definition 2.3.** Let $f : U \subseteq \mathbb{C}^n \to \mathbb{C}^m$ be a complex function of $n$ complex variables to $m$ values in $\mathbb{C}$. We say that $f := (f_1, \dots, f_m)$ is *holomorphic* if $\forall i = 1, \dots, m$ the function $f_i$ is holomorphic.

**Remark 2.4.** Let be $f : U \subseteq \mathbb{C}^n \to \mathbb{C}^m$ and denote $z_j = x_j + iy_j$ and $f(z_1, \dots, z_n) = (P_1 + iQ_1, \dots, P_m + iQ_m)$, where $x_j, y_j \in \mathbb{R}$ and $P_i, Q_i : U \subseteq \mathbb{R}^{2n} \to \mathbb{R}$. Note that a function will be holomorphic if and only if $P_i(z_j), Q_i(z_j)$ are differentiable in $\mathbb{R}^{2n}$ and the pairs of Cauchy-Riemann relations hold, i.e.

$$\frac{\partial P_i}{\partial x_j} = \frac{\partial Q_i}{\partial y_j}, \qquad \frac{\partial P_i}{\partial y_j} = -\frac{\partial Q_i}{\partial x_j}$$

## 2.2 Complex manifolds

**Definition 2.5.** A *holomorphic atlas* on a smooth manifold of dimension $n$ is an atlas $\{(U_i, \varphi_i)\}$ such that the coordinate functions induce a homeomorphism $\varphi_i : U_i \cong \varphi(U_i) \subseteq \mathbb{C}^n$ and such that the transition functions

$$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \mathbb{C}^n$$

are holomorphic.
Any pair $(U, \varphi)$ is called a *holomorphic chart*. Two holomorphic atlases $\{(U_i, \varphi_i)\}$, $\{(V_j, \psi_j)\}$ are said to be *equivalent* if $\forall i, j$ the map

$$\varphi_i \circ \psi_j^{-1} : \psi_j(U_i \cap V_j) \to \varphi_i(U_i \cap V_j)$$

is holomorphic.

Note that, as in the smooth case, the equivalence of holomorphic atlases is a relation of equivalence. The definition of complex manifold is analogue to the real case, too:

**Definition 2.6.** An *n-dimensional complex manifold* $(M, \mathcal{A})$ is a smooth manifold $M$ of real dimension $2n$ endowed with a holomorphic atlas $\mathcal{A}$. We will often refer to the complex manifold simply as $M$.

We will see that there is an equivalent treatment of complex manifolds based on smooth manifolds having an endormorphism on their tangent bundle which satisfies some properties. However, we first see some examples of complex manifolds:

**Example 2.7.** 1. $\mathbb{C}^n$ is a complex manifold for all $n \geq 1$ with holomorphic atlas $\{(\mathbb{C}^n, Id)\}$.

2. Any open subset $Y$ of a complex manifold $X$, the latter with holomorphic atlas $\{(U_i, \varphi_i)\}$, is itself a complex manifold with induced holomorphic atlas $\{(Y \cap U_i, \varphi_i|_Y)\}$.

3. The cartesian product of complex manifolds is a complex manifold.

4. (Complex Lie groups) Let $G$ be a complex manifold and a group. $G$ is called a complex Lie group if the group operations $\cdot : G \to G$ and $^{-1} : G \to G$ are holomorphic. Examples of Lie groups are $GL(n, \mathbb{C}), SL(n, \mathbb{C})$. Also $\mathbb{C}^n$ can be seen as a Lie group.

5. (Complex projective spaces) Let $P\mathbb{C}^n$ be the $n$-dimensional complex projective space, i.e. $P\mathbb{C}^n$ is the set of lines $[z]$ in $\mathbb{C}^{n+1}$. Equivalently it can be seen as
$$P\mathbb{C}^n := (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*,$$
where here $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1} \setminus \{0\}$ as a complex scalar multiplication. The points $[z] \in P\mathbb{C}^n$ have homogeneous coordinates $[z_0 : \ldots : z_n]$.

   It can be given an open covering by the sets
$$U_i := \{[z] \in P\mathbb{C}^n : z_i \neq 0\}$$
which we assure to be open giving $P\mathbb{C}^n$ the quotient topology. Considering the bijective maps
$$\begin{aligned} \varphi : U_i &\longrightarrow \mathbb{C}^n \\ [z_0 : \ldots : z_n] &\longmapsto (\frac{z_0}{z_i}, \ldots, \hat{z}_i, \ldots, \frac{z_n}{z_i}) \end{aligned}$$
we get an atlas $\{U_i, \varphi_i\}$ which is holomorphic.

As in the real case, we can define functions between complex manifolds:

**Definition 2.8.** A *holomorphic map* $f : M \to N$ between complex manifolds $(M, \{(U_i, \varphi_i)\})$ and $(N, \{(V_j, \psi_j)\})$ is a smooth map for which the functions
$$\psi_j \circ f \circ \varphi_i^{-1} : \varphi_i(U) \to \mathbb{C}^n$$
are holomorphic.

**Definition 2.9.** A *biholomorphic map* $f : X \to Y$ is a bijective holomorphic map whose inverse is also holomorphic. Two complex manifolds $X$ and $Y$ are said to be *biholomorphic* if there exists a biholomorphic map between them.

Although complex and smooth manifolds seem similar, the rigidity of holomorphic functions compared to the smooth ones gives them really different behaviours. We might see a first difference between complex and smooth manifolds, regarding functions on manifolds having the property of compactness:

**Proposition 2.10.** *Let $X$ be a compact connected complex manifold. Any holomorphic function $f : X \to \mathbb{C}$ is constant.*

*Proof.* Since $X$ is compact, there is a point $x \in X$ in which $|f(x)|$ is maximum. Given a local chart of $x$, $(U, \varphi)$, i.e for which $x \in U$, the map $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ is defined on an open subset $\varphi(U) \subseteq \mathbb{C}^n$, therefore due to the maxiumum principle on $\varphi(U)$ is locally constant. Since $X$ is connected, $f$ must be constant. □

However, it is a property that clearly doesn't apply to the real case, as we see in the following example:

**Example 2.11.** Consider the smooth manifold $\mathbb{S}^1$ with the differential atlas $\{\mathbb{S}^1, i : \mathbb{S}^1 \hookrightarrow \mathbb{R}^2\}$. Clearly $\mathbb{S}^1$ is a compact and connected manifold. However, we can easily see that the function $\pi_x : \mathbb{S}^1 \to \mathbb{R}$ defined by $\pi_x(x, y) = x$ is differentiable and non-constant.

In the last example we have given $\mathbb{S}^1$ as a submanifold of the smooth manifold $\mathbb{R}^2$. We give now the precise definition for complex submanifolds:

**Definition 2.12.** Let $X$ be a complex manifold of complex dimension $m$ and let $Y \subseteq X$ be a smooth submanifold of real dimension $2k$. $Y$ is a *complex submanifold of complex dimension $k$* if there is an atlas $\{(U_i, \varphi_i)\}$ of $X$ satisfying $U_i \cap Y \cong \varphi_i(U_i) \cap \mathbb{C}^k$, where we embed $\mathbb{C}^k$ into $\mathbb{C}^m$ via $(z_1, \dots, z_k) \mapsto (z_1, \dots, z_k, 0, \dots, 0)$.

We have as a consequence of Proposition 2.10 the following corollary

**Corollary 2.13.** *There are no compact complex submanifolds $M$ of $\mathbb{C}^m$ of positive dimension*

*Proof.* If we had a compact complex submanifold $M$ of $\mathbb{C}^m$, the chart maps of $\mathbb{C}^m$ would restrict to non-constant functions on the compact $M$, contradicting Proposition 2.10. □

Again, in Example 2.11 we have a case in which the compact smooth manifold $\mathbb{S}^1 \subseteq \mathbb{R}^2$ is indeed a submanifold of $\mathbb{R}^2$, providing us a way of seeing this huge difference between complex and smooth manifolds.

## 2.3 Almost complex manifolds

We give in this section an equivalent definition of complex manifolds, using the fact that smooth manifolds equipped with an structure satisfying some particular conditions are indeed complex manifolds and there is an equivalence among them. This alternative definition facilitates the extension of differential $k$-forms to complex manifolds.
Let $U \subseteq \mathbb{C}^n$ be an open subset, which in particular is a $2n$-dimensional smooth manifold. Hence, for each $p \in U$, the tangent space $T_pU$ has dimension $2n$. The canonical basis for $T_pU$ is

$$\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \right\}$$

where $z_i = x_i + iy_i$ are the canonical coordinates of $\mathbb{C}^n$. The vector space $T_pU$ admits an endomorphism $J$ given by

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \qquad J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}$$

which can extended to a vector bundle endomorphism on the tangent bundle $J : TM \to TM$.

**Definition 2.14.** An *almost complex structure* on a smooth manifold $M$ is a linear endomorphism $J : TM \to TM$ such that $J^2 = -1$. The couple $(M, J)$ is called an *almost complex manifold*.

We see how $J^2 = -1$ is similar to the imaginary unit $i^2 = -1$. Assuming we are in $\mathbb{R}^{2n}$, the almost complex structure provides us a Cauchy-Riemann relation being given the other. Hence, $J$ allows us to endow $T_pM$, and therefore $TM$, with the structure of a complex vector space whose scalar multiplication is given by $(x + iy)v := xv + yJ(v)$.

**Proposition 2.15.** *Let $(M, J)$ be an almost complex structure. Then $M$ is even-dimensional.*

*Proof.* Since $J^2 = -1$, we have that $\det(J^2) = (-1)^n$, where $n := \dim M$. Given that $M$ is smooth and therefore a real manifold, $\det(J)$ must be real. Hence $n$ must be even. □

Not every even-dimensional smooth manifold admits an almost complex structure. The following theorem states which $n$-spheres admit an almost complex structure.

**Theorem 2.16.** ***(Borel & Serre)*** *The sphere* $\mathbb{S}^n$ *admits an almost complex structure if and only if* $n = 2$ *or* $n = 6$.

The reader eventually interested in its proof, which is beyond the scope of this work, can check the original paper [3].

**Definition 2.17.** A *map of almost complex manifolds* is a smooth map $f : M \to N$ such that the map $f_* : TM \to TN$ satisfies $f_* \circ J_M = J_N \circ f_*$, where the subscript of $J$ indicates the manifold associated to the bundle considered.

**Theorem 2.18.** *Any complex manifold* $X$ *admits a natural almost complex structure, i.e* $X$ *has an underlying almost complex manifold.*

*Proof.* Take the holomorphic charts $(U, \varphi)$, $\varphi = (z_1, \ldots, z_n)$ on $X$ and for $z_i = x_i + iy_i$ define the linear morphism $\partial/\partial x_i \mapsto \partial/\partial y_i$, $\partial/\partial y_i \mapsto -\partial/\partial x_i$. See that it does not depend on the chart. $\square$

We now give a result that gives us the actual equivalence between complex manifolds and almost complex manifolds under a particular condition. When a certain tensor related to the almost complex structure is identically zero for any pair of vector fields, an almost complex manifold will have an underlying complex manifold and conversly, the smooth manifold underlying a complex manifold can be added an almost complex structure. As it is again beyond our goal, the interested reader might check [17].

**Theorem 2.19.** ***(Newlander-Nirenberg integrability theorem)*** *An almost complex structure* $J$ *on a smooth manifold* $M$ *arises from a complex manifold if and only if* $N_J \equiv 0$, *where* $N_J : TM \times TM \to TM$ *is the* Nuijenhuis tensor, *defined by*

$$N_J(X, Y) := [X, Y] + J[X, JY] + J[JX, Y] - [JX, JY]$$

**Remark 2.20.** Since the *Lie bracket* $[X, Y]$ is antisymmetric, the Nuijenhuis tensor is antisymmetric as well.

**Definition 2.21.** Let $J$ be an almost complex structure. If $N_J \equiv 0$, $J$ is called an *integrable almost complex structure* or a *complex structure*.

We finally give an alternative definition of complex manifold, which is equivalent to Definition 2.6 by Theorem 2.19 and often given as the main definition in many references.

**Definition 2.22. (Alternative)** Let $(M, J)$ be an almost complex manifold. If $J$ is integrable, we say that $(M, J)$ is a *complex manifold*.

It is an open problem whether it exists an almost complex manifold not admitting a complex manifold for dimension $n \geq 6$. In particular, it is not known if $\mathbb{S}^6$ is or is not a complex manifold. The following result is conjectured by Yau:

**Conjecture 2.23.** ***(Yau).*** *Any almost complex manifold of dimesion $n \geq 6$ is a complex manifold.*

## 2.4 Complexified differential forms

We fix $(M, J)$ to be an almost complex manifold of dimension $2n$ in this section. Consider now the complexification of $TM$, i.e. the cross product $T_{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C}$. Then $J$ can be extended to $T_{\mathbb{C}}M$ by $J(v \otimes z) := J(v) \otimes z$. The eigenvalues of $J$ are $\pm i$. We can consider the $\pm i$-eigenspaces of $J$ on $T_{\mathbb{C}}M$:

$$T^{1,0}M := \{v \in T_{\mathbb{C}}M : J(v) = +iv\} \qquad T^{0,1}M := \{v \in T_{\mathbb{C}}M : J(v) = -iv\}$$

**Remark 2.24.** $T_{\mathbb{C}}M, T^{1,0}M$ and $T^{0,1}M$ are actually tangent bundles in $\mathbb{C}$, i.e. complex vector bundles.

However, even for $M$ being an complex manifold $T_{\mathbb{C}}M, T^{1,0}M$ and $T^{0,1}M$ have not *a priori* a holomorphic structure, i.e. they are not generally holomorphic vector bundles.

**Definition 2.25.** The bundles $T^{1,0}M$ and $T^{0,1}M$ are called respectively *holomorphic tangent bundle* and *antiholomorphic tangent bundle* of the almost complex manifold $M$.

Natural bases of the bundles $T^{1,0}M$ and $T^{0,1}M$ are, respectively,

$$\left\{\frac{\partial}{\partial z_i} := \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right)\right\} \qquad \left\{\frac{\partial}{\partial \bar{z}_i} := \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right)\right\}$$

**Remark 2.26.** Note that $J$ restricted to the holomorphic tangent bundle corresponds to the multiplication by $i$, while $J$ restricted to the antiholomorphic tangent bundle corresponds to multiplication by $-i$.

**Proposition 2.27.** *Let $M$ be an almost complex manifold. Then we get a direct sum decomposition*

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$$

*Proof.* It follows from the direct sum decomposition on all fibres, i.e. the tangent spaces for all $p \in M$. □

This direct sum decomposition induce decompositions on the dual tangent bundle as well as on the exterior powers of these objects.
We now introduce the complexified cotangent bundle similarly as we have already done with the tangent bundle. We have

$$T^*_{\mathbb{C}}M := T^*M \otimes_{\mathbb{R}} \mathbb{C}$$

We can introduce the generalization of differential $k$-forms, the complexified differential $k$-forms or $\mathbb{C}$-differential forms. They will be sections of the complexified $k$-th exterior power of the cotangent bundle:

**Definition 2.28.** For an almost complex manifold $M$ one defines the vector bundles

$$\bigwedge^k M := \bigwedge^k (T_{\mathbb{C}}M)^* \qquad \bigwedge^{p,q} M := \bigwedge^p (T^{1,0}M)^* \oplus_{\mathbb{C}} \bigwedge^q (T^{0,1}M)^*$$

These complexified objects are indeed vector bundles and they are given a smooth manifold structure again in an analogue way as we did for the tangent bundle. Note that we are using the fact that we got a decomposition of the complexified (co)tangent bundle.

**Definition 2.29.** Let $k \geq 0$. The $\mathbb{C}$*-differential k-forms of M* are the sections of $\bigwedge^k M$, i.e. maps

$$\begin{aligned} \omega : M &\longrightarrow \bigwedge^k M \\ p &\longmapsto (p, \omega_p) \end{aligned}$$

We denote by $\mathcal{A}^k_{\mathbb{C}}(M)$ the set of all $\mathbb{C}$-differential $k$-forms in $\mathbb{C}$.

**Remark 2.30.** In the following we will often refer to $\mathbb{C}$-differential $k$-forms simply as differential $k$-forms; the context will tell us whether the forms are complexified or not. The $\mathbb{C}$-differential $k$-forms we have defined are in fact defined on the complexified bundles.

The map $d$ is often called the *exterior derivative*. We may now use the decomposition of the complexified exterior power of the cotangent bundle to actually decompose the differential $k$-forms.

$$\mathcal{A}^k_{\mathbb{C}}(M) := \mathcal{A}^k_{dR}(M) \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(M)$$

where $\mathcal{A}^{p,q}(M)$ is the space of sections of $\bigwedge^{p,q} M$. Again, the almost complex structure permits us to refine the mathematical objects we are considering, in this case the $k$-forms. Using the same fact we can refine the action of the map $d$:

**Definition 2.31.** We define the *operators* $\partial$ *and* $\bar{\partial}$ as

$$\partial := \Pi^{p+1,q} \circ d : \mathcal{A}^k(M) \longrightarrow \mathcal{A}^{p+1,q}(M), \qquad \bar{\partial} := \Pi^{p,q+1} \circ d : \mathcal{A}^k(M) \longrightarrow \mathcal{A}^{p,q+1}(M)$$

where given $k = p + q$, the *projection operators of k-forms* are defined as

$$\Pi^{p,q} : \mathcal{A}^k(M) \longrightarrow \mathcal{A}^{p,q}(M)$$

We finally want to give a characterization theorem of complex manifolds. As it will be more clear, we will state the particular results in a few lemmas and then collect them, already proved, in a general theorem.

**Lemma 2.32.** *Let* $(M, J)$ *be an almost complex manifold.* $N_J \equiv 0$ *if and only if* $[T^{0,1}M, T^{0,1}M] \subseteq T^{0,1}M$.

*Proof.* ($\Rightarrow$) We know by Theorem 2.19 that $N_J \equiv 0$ equives to the statement that $(M, J)$ is a complex manifold. Then consider that $M$ is a complex manifold and a char $(U, \varphi)$, with $z_j = x_i + iy_j$ its $j$-th component. Given $\{e_1, \dots, e_{2n}\}$ the standard basis of $\mathbb{R}^{2n}$, by definition

$$\frac{\partial}{\partial x_j} = \varphi_*^{-1}(e_j), \qquad \frac{\partial}{\partial y_j} = \varphi_*^{-1}(e_{j+n}), \qquad j = 1, \dots, n$$

where moreover the structure $J$ is compatible with the induced map $\varphi_*$. Using the local basis $\{\frac{\partial}{\partial \bar{z}_j}\}$ of $T^{0,1}M$ we may write two local vector fields $X, Y$ in $T^{0,1}M$ as

$$X = \sum_i X_i \frac{\partial}{\partial \bar{z}_i}, \qquad Y = \sum_i Y_i \frac{\partial}{\partial \bar{z}_i}$$
$$[X, Y] = \sum_{i,j} X_i \frac{\partial Y_j}{\partial \bar{z}_i} \frac{\partial}{\partial \bar{z}_i} - \sum_{i,j} Y_i \frac{\partial X_j}{\partial \bar{z}_i} \frac{\partial}{\partial \bar{z}_i}$$

and therefore, since $[X, Y]$ is also a local vector field in $T^{0,1}M$, we see that $N_J = 0 \Longrightarrow [T^{0,1}M, T^{0,1}M] \subseteq T^{0,1}M$.
($\Leftarrow$)Conversely, for $X, Y$ vector fields of $T^{0,1}M$, defining the vector field $Z := [X + iJX, Y + iJY]$ one can see that

$$Z - iJZ = N_J(X, Y) - iJN(X, Y)$$

and hence $Z$ is a vector field of $T^{0,1}M$ only if $N_J \equiv 0$. $\square$

**Remark 2.33.** Note that $[T^{0,1}M, T^{0,1}M] \subseteq T^{0,1}M$ and $[T^{1,0}M, T^{1,0}M] \subseteq T^{1,0}M$ are equivalent.

**Lemma 2.34.** *Let $(M, J)$ be an almost complex manifold. Then $d = \partial + \bar{\partial}$ if and only if for all forms in $\mathcal{A}^{1,0}(M)$ is satisfied $\Pi^{0,2} \circ d = 0$.*

*Proof.* ($\Rightarrow$) The first implication is trivial since $d = \partial + \bar{\partial}$ clearly implies $\Pi^{0,2} \circ d = 0$ for all forms in $\mathcal{A}^{1,0}(M)$.
($\Leftarrow$) The converse holds true as well, since $d = \partial + \bar{\partial}$ holds on $\mathcal{A}^{p,q}(M)$ if and only if $d\alpha \in \mathcal{A}^{p+1,q}(M) \oplus \mathcal{A}^{p,q+1}(M)$. Then, locally one has that $\alpha \in \mathcal{A}^{p,q}(M)$ can be expressed as

$$\alpha = \sum f \alpha_{i_1} \wedge \ldots \wedge \alpha_{i_k} \wedge \bar{\beta}_{j_1} \wedge \ldots \wedge \bar{\beta}_{j_l}$$

where $\alpha_i \in \mathcal{A}^{1,0}(M)$ and $\bar{\beta}_j \in \mathcal{A}^{0,1}(M)$. Now, using the Leibniz rule, $df \in \mathcal{A}^{1,0}(M) \oplus \mathcal{A}^{0,1}(M)$ trivially. $d\alpha_i \in \mathcal{A}^{2,0}(M) \oplus \mathcal{A}^{1,1}(M)$ by assumption. Then, complex conjugating the hypothesis we get $\Pi^{2,0} \circ d = 0$ on $\mathcal{A}^{0,1}(M)$ and hence $d\bar{\beta}_j \in \mathcal{A}^{1,1}(M) \oplus \mathcal{A}^{0,2}(M)$, getting then the result that states $d = \partial + \bar{\partial}$. □

**Lemma 2.35.** *Let $(M, J)$ be an almost complex manifold. $\Pi^{0,2} \circ d\alpha = 0$ for all forms in $\mathcal{A}^{1,0}(M)$ if and only if $[T^{0,1}M, T^{0,1}M] \subseteq T^{0,1}M$.*

*Proof.* Let $\alpha$ be a $(1,0)$-form and $X, Y$ sections of $T^{0,1}M$. Then, it can be seen that differentiating using standard formulae one gets

$$(d\alpha)(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]) = -\alpha([X, Y])$$

and hence $\alpha$ satisfies $(\Pi^{0,2} \circ d)\alpha = 0$ if and only if $[X, Y]$ is always of type $(0,1)$. □

We can finally collect the last results in a general theorem, which is basically already proved.

**Theorem 2.36.** *Let $(M, J)$ be an almost complex manifold. Then the following are equivalent:*

1. *$M$ is a complex manifold.*
2. $N_J \equiv 0$
3. $d = \partial + \bar{\partial}$
4. $[T^{0,1}M, T^{0,1}M] \subseteq T^{0,1}M$
5. *On $\mathcal{A}^{1,0}(M)$ one has* $\Pi^{0,2} \circ d = 0$
6. $\bar{\partial}^2 = 0$

*Proof.* It is a consequence of Lemmas 2.32, 2.34, 2.35. □

**Corollary 2.37.** *On a complex manifold we get*

$$\partial^2 = \bar{\partial}^2 = 0, \qquad \partial\bar{\partial} + \bar{\partial}\partial = 0$$

*Proof.* It is direct from the fact that on a complex manifold $d = \partial + \bar{\partial}$ and $d \circ d \equiv 0$. □

## 2.5 Cohomology on complex manifolds

The de Rahm cohomology groups that were defined in the previous chapter can be adapted to the case of $\mathbb{C}$-differential forms on complex manifold. We have

$$H^k_{dR}(M,\mathbb{C}) := \frac{\mathrm{Ker}(d : \mathcal{A}^k_{\mathbb{C}}(M) \to \mathcal{A}^{k+1}_{\mathbb{C}}(M))}{\mathrm{Im}(d : \mathcal{A}^{k-1}_{\mathbb{C}}(M) \to \mathcal{A}^k_{\mathbb{C}}(M))} = H^k_{dR}(M,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$$

But we might also use the fact that the bundles of complex manifolds decompose in holomorphic and antiholomorphic bundles. Since $\bar{\partial}$ satisfies $\bar{\partial} \circ \bar{\partial} \equiv 0$ we can define another cochain complex

$$0 \to \mathcal{A}^{p,0}(M) \to \mathcal{A}^{p,1}(M) \to \ldots \to \mathcal{A}^{p,k}(M) \to \ldots \mathcal{A}^{p,n}(M) \to 0$$

called the Dolbeault complex.

**Definition 2.38.** In an analogue way, the *Dolbeault cohomology groups* are defined as

$$H^{p,q}_{\bar{\partial}}(M) := \frac{\mathrm{Ker}(\bar{\partial} : \mathcal{A}^{p,q}_M \to \mathcal{A}^{p,q+1}_M)}{\mathrm{Im}(\bar{\partial} : \mathcal{A}^{p,q-1}_M \to \mathcal{A}^{p,q}_M)}$$

**Remark 2.39.** Since $\partial$ and $\bar{\partial}$ are complex conjugate, and hence $H^{p,q}_{\bar{\partial}}(M) = H^{q,p}_{\partial}(M)$, we might only calculate the Dolbeault cohomology, instead of calcultating as well the *antiDolbeault cohomology*, which is related to the $\partial$-operator.

**Example 2.40.** Consider the complex projective space $\mathbb{C}P^n$, $n \geq 1$. Its Dolbeault cohomology groups are [11]

$$H^{p,q}_{\bar{\partial}}(\mathbb{C}P^n) \cong \begin{cases} \mathbb{C} & p = q,\ p \leq n \\ 0 & p \neq q \end{cases}$$

Dolbeault cohomology is an invariant of complex structures. Moreover, it is related to de Rham cohomology, so it is useful to compute the latter in many cases. For compact complex manifolds the dimension of both de Rham and Dolbeault cohomologies are finite. We define an analogue of Betti numbers for Dolbeault cohomology groups:

**Definition 2.41.** Let $(M, J)$ be an $n$-dimensional complex manifold. The *Hodge numbers of M* are defined as

$$h^{p,q}(M) := \dim_{\mathbb{C}} H^{p,q}_{\bar{\partial}}(M), \qquad 0 \leq p, q \leq n$$

Betti and Hodge numbers are not independent. In fact there are many relations, with these depending on the kind of complex manifold we are looking at. In the most general case it can be seen the following [11]:

**Proposition 2.42.** *Let M be an n-dimensional complex manifold, $n < \infty$. It is satisfied*

$$\begin{aligned} h^{p,q} &= h^{n-p,n-q} \qquad \textit{(Serre duality)}, \\ b^k &\leq \sum_{p+q=k} h^{p,q} \end{aligned}$$

It is common to organize the Hodge numbers into a diamond-shape diagram, the so-called Hodge diamond. As an example, for a 3-dimensional complex manifold $M$ we would write

$$\begin{array}{cccccccc} b^6 & & & & h^{3,3} & & & \\ b^5 & & & h^{3,2} & & h^{2,3} & & \\ b^4 & & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ b^3 & h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ b^2 & & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ b^1 & & & h^{1,0} & & h^{0,1} & & \\ b^0 & & & & h^{0,0} & & & \end{array}$$

**Remark 2.43.** Note that for all $n$-dimensional connected compact complex manifolds $h^{0,0}(M) = 1$, since forms $f \in \mathcal{A}^{0,0}(M)$ with $\bar{\partial} = 0$ are the holomorphic functions, which globally can only be constant, i.e. $H^{0,0}_{\bar{\partial}}(M) \cong \mathbb{C}$. Therefore, by Serre duality, $h^{n,n}(M) = 1$.

**Example 2.44.** Consider the complex projective space $\mathbb{C}P^2$. Its Hodge diamond is [11, 9]

$$\begin{array}{cccccc} 1 & & & 1 & & \\ 0 & & 0 & & 0 & \\ 1 & 0 & & 1 & & 0 \\ 0 & & 0 & & 0 & \\ 1 & & & 1 & & \end{array}$$

**Example 2.45.** Consider the Hopf surface $\mathcal{M} \cong \mathbb{S}^1 \times \mathbb{S}^3$. Its Hodge diamond is [11, 9]

$$
\begin{array}{ccccccccc}
1 & & & & 1 & & & & \\
1 & & & 1 & & 0 & & & \\
0 & & 0 & & 0 & & 0 & & \\
1 & & & 0 & & 1 & & & \\
1 & & & & 1 & & & &
\end{array}
$$

## 2.6 Chern classes

Given a fiber $F$, a structure group $G$ and a base manifold $M$ we may construct many fibre bundles over $M$. It is interesting to classify these bundles and see how much they differ from the trivial bundle $M \times F$. To do so, we might use characteristic classes, which are subsets of cohomology classes of the base space, i.e. $M$. Therefore they measure the non-triviality of the bundle and represent obstructions which prevent the bundle from being trivial. Precisely, we are interested in Chern classes:

**Definition 2.46.** Let $E$ be a complex vector bundle over a complex manifold $M$. Let $F = dA + A \wedge A$ be the curvature two-form of a connection $A$ on $E$. We define the *total Chern class of $E$* as

$$c(E) := \det(1 + \frac{i}{2\pi} F)$$

**Remark 2.47.** Since $F$ is a two-form, $c(E)$ is a direct sum of forms of even degrees. Moreover, since $E$ is $n$-dimensional, for $2j > n$ we get that $c_j(E) \equiv 0$ and as a consequence of the $k$-rank of the bundle, for $j > k$ Chern classes vanish as well.

**Definition 2.48.** We define the *Chern classes* $c_j(E) \in H^{2j}_{dR}(M)$ by the expansion

$$c(E) = 1 + c_1(E) + c_2(E) + \dots$$

**Definition 2.49.** The *Chern classes of a complex manifold $M$* are

$$c_j(M) := c_j(T^{1,0}M) \in H^{2j}_{dR}(M)$$

**Remark 2.50.** Since the definition of Chern classes depends on a connection $A$ on the bundle, one might think that Chern classes depend on the choice of $A$. However this is not the case and Chern classes are invariants. For further information see [4].

We might give some useful formulae to compute Chern classes. For the deduction of these formulae, see [15].

$$
\begin{aligned}
c_0(E) =& [1] \\
c_1(E) =& \left[\frac{i}{2\pi} TrF\right] \\
c_2(E) =& \left[\frac{1}{2}\left(\frac{i}{2\pi}\right)^2 (TrF \wedge TrF - Tr(F \wedge F))\right] \\
&\vdots \\
c_k(E) =& \left[\left(\frac{i}{2\pi}\right)^k \det F\right]
\end{aligned}
$$

## 2.7 Holonomy

There is another concept relating the complex geometry of a Riemannian manifold $(M, g)$ which is holonomy. Since to the metric $g$ there is a unique Levi-Civita connection $D$, so it is possible to define the parallel transport of vectors along a path $\gamma$ on $M$, we can obtain plenty of isomorphisms

$$P_\gamma : T_pM \cong T_qM$$

Since the only scope of this section is introducing holonomy in order to make little use of it when defining Calabi-Yau manifolds, we are not giving proves. See §§4.A of [11] for detailed deductions.

**Lemma 2.51.** *$P_\gamma$ is an isometry.*

**Corollary 2.52.** *If $\gamma$ is a closed path with $\gamma(0) = \gamma(1) = p$ then $P_\gamma \in O(T_pM, g_p) \cong O(n)$.*

**Definition 2.53.** Let $(M, g)$ be a Riemanninan manifold. For any point $p \in M$, the *holonomy group at p $Hol_p(M, g) \subseteq O(T_pM)$* is the group of all parallel transports $P_\gamma$ along closed paths $\gamma : [0, 1] \to M$ with $\gamma(0) = \gamma(1) = p$.

**Definition 2.54.** Let $(M, g)$ be a Riemanninan manifold. For any point $p \in M$, the *restricted holonomy group at p $Hol_p^\circ(M, g) \subseteq Hol_p(M, g)$* is the group of all parallel transports $P_\gamma$ along contractible paths $\gamma$, i.e. with $1 = [\gamma]$.

**Proposition 2.55.** *Let $(M, g)$ be a simply connected Riemannian manifold. Then $Hol_p^\circ(M, g) \cong Hol_p(M, g)$.*

*Proof.* Given the fact that $M$ is simply connected, all paths $\gamma$ satisfy $[\gamma] = 1 \in \pi_1(M)$. Hence we get we get the non trivial inclusion $Hol_p(M, g) \subseteq Hol_p^\circ(M, g)$ and therefore the isomorphism. □

**Proposition 2.56.** *If two points $p, q \in M$ can be connected by a path $\gamma : [0, 1] \to M$ such that $\gamma(0) = p, \gamma(1) = q$, then $Hol_p(M, g) \cong Hol_q(M, g)$.*

*Proof.* Since there is a path $\gamma$ connecting $p$ and $q$, we can define the parallel transport along $\gamma$, $P_\gamma$. Hence we get $Hol_q(M, g) = P_\gamma \circ Hol_p(M, g) \circ P_\gamma^{-1}$, which leads to the isomorphism. □

Given the isomorphism between holonomy groups at points which are path-connected, we can finally define the holonomy group of a simply connected manifold $M$:

**Definition 2.57.** Let $(M, g)$ be a simply connected Riemannian manifold. We define its *holonomy group* $Hol(M, g)$ as the holonomy group at any point $p \in M$.

## 2.8 Holomorphic bundles

We will only introduce holomorphic vector bundles in order to be able to define the canonical bundle of a complex manifold. However, we should notice that a holomorphic vector bundle is not the same as a complex vector bundle: the latter is simply a differentiable vector bundle whose fibers are complex.

**Definition 2.58.** Let $M$ be a complex manifold. A *holomorphic vector bundle of rank $r$ on $M$* is a complex manifold $E$ together with a holomorphic projection map $\pi : E \to M$ and the structure of an $r$-dimensional complex vector space on any fibre $E(x) := \pi^{-1}(x)$ satisfying that there exists an open covering $M = \bigcup_i U_i$ and biholomorphic maps $\psi_i(U_i) \cong U_i \times \mathbb{C}^r$ commuting with the projections to $U_i$ such that the induced map $\pi^{-1}(x) \cong \mathbb{C}^r$ is complex linear.

**Remark 2.59.** A holomorphic vector bundle is also a complex vector bundle.

**Example 2.60.** The holomorphic tangent bundle $T^{1,0}M$ of a complex manifold $M$ and its dual are both holomorphic vector bundles.

The following result, although tagged as theorem, is a meta-theorem. There is an analogue for real (smooth) vector bundles.

**Theorem 2.61** (Meta-theorem)**.** *Any canonical construction in linear algebra gives rise to a geometric version for holomorphic vector bundles.*

Let $M$ be a complex manifold. We will now denote by $\Omega_M$ the holomorphic cotangent bundle $(T^{1,0}M)^*$ and by $\mathcal{T}_M$ the holomorphic tangent bundle $T^{1,0}M$.

**Definition 2.62.** The *bundle of holomorphic p-forms* is defined $\Omega_M^p := \bigwedge^p \Omega_M$ for $0 \leq p \leq n$ and $K_M := \det(\Omega_M) = \Omega_M^n$ is called the *canonical bundle of M*.

It can be proved [11, 1] that the definition is independent of the open covering and the maps $\varphi_i$. Hence, for different choices with get isomorphic vector bundles and thus $\mathcal{T}_M, \Omega_M$ and $K_M$ are invariants of the complex manifold $M$.

**Definition 2.63.** The simplest holomorphic vector bundle over $M$ of rank $k$, i.e. $M \times \mathbb{C}^k$, is called *trivial*.

# Chapter 3

# Kähler manifolds

*"No sé per on camino, sé només*
*que vaig cap al no-res. [...]*
*No podré veure mai*
*la veritable mar al fons."*

- Joan Vinyoli, *Vent d'aram*

## 3.1 Hermitian structures

The first step in order to introduce Kähler manifolds is defining what does actually mean for a Riemannian metric and an almost complex structure to be compatible. In the case they will be compatible, we will be able to define the notion of hermitian structure, which is the previous step to Kähler. In this section we mainly follow [11, 1]

**Definition 3.1.** Let $(M, J)$ be an almost complex manifold. A Riemannian metric $g$ on $M$ is said to be *compatible* with the complex structure $J$ if for all $p \in M$ one has

$$g_p(v, w) = g_p(Jv, Jw), \quad \forall v, w \in T_pM$$

**Definition 3.2.** A Riemannian metric $g$ on $M$ is an *hermitian structure on $M$* if it is compatible with the complex structure $J$. The induced form $\omega := g(J(\cdot), \cdot)$ is called the fundamental form.

**Proposition 3.3.** *Every almost complex manifold admits a hermitian structure.*

*Proof.* Given a Riemannian metric $g$ on $M$, we can define a new metric

$$h(X,Y) := g(X,Y) + g(JX, JY)$$

that satisfies the requirements of Riemannian metrics and using the fact that $g$ is positive-definite it satisfies $h(X,Y) = h(JX, JY)$. $\square$

**Proposition 3.4.** *The fundamental form is a real* $(1,1)$*-form.*

*Proof.* Since

$$g_p(J(v), w) = g_p(J(J(v)), J(w)) = -g_p(v, J(w))$$

we see that $\omega$ is alternating and hence $\omega \in \bigwedge^2 M$. Now one finds applying $J$ to $\omega$ that

$$J(\omega)(v,w) = \omega(J(v), J(w)) = g_p(J^2 v, Jw) = \omega(v,w)$$

and hence, since $J\omega = \omega$, we get $\omega \in \bigwedge^{1,1} M$. $\square$

**Definition 3.5.** A complex manifold $(M, J)$ endowed with an hermitian structure is called an *hermitian manifold*.

Roughly speaking, we have just taken a step forward the riemannian structures imposing our Riemannian metric to be compatible with the almost complex structure. Note then that the set of Hermitian manifolds is a subset of Riemannian manifolds.

**Remark 3.6.** The hermitian structure $g$ is uniquely determined by the fundamental form $\omega$ together with the almost complex structure $J$. Actually, $g(\cdot,\cdot) = \omega(\cdot, J(\cdot))$.

Locally the fundamental form $\omega$ is of the form

$$\omega = \frac{i}{2} \sum_{i,j=1}^{n} h_{ij} dz_i \wedge d\bar{z}_j$$

where $(h_{ij}(p))$ is a positive-definite hermitian matrix for any $p \in M$.

**Remark 3.7.** Note that until now the definitions are equally valid for almost complex structures, either integrable or not. However, from now on we are using the differential decomposition $d = \partial + \bar{\partial}$, which implies the almost complex structure to be integrable and hence the manifold to be complex.

## 3.2 Kähler manifolds

We now consider the almost complex structure $J$ integrable, i.e. $J$ is actually a complex structure. We add a restriction to the set of hermitian manifolds to get Kähler manifolds defined. This restriction will be on the fundamental form of the hermitian manifold.

**Definition 3.8.** A *Kähler structure* is an hermitian structure $g$ whose associated fundamental form $\omega$ is closed, i.e. $d\omega = 0$. A complex manifold endowed with a Kähler structure is called a *Kähler manifold*. The fundamental form $\omega$ is then known as the *Kähler form*.

**Remark 3.9.** Although hermitian structures exist on any complex manifold, Kähler structures do not always exist. In Example 3.18 we see that the Hopf surface $\mathcal{M} = \mathbb{S}^1 \times \mathbb{S}^3$ is not Kähler.

$d\omega = 0$ implies that the hermitian metric tensor is defined by a unique function $u$ called the Kähler potential:

$$h_{\alpha\bar{\beta}} = \frac{\partial^2 u}{\partial z_\alpha \partial \bar{z}_\beta}$$

Actually, the condition $d\omega = 0$ is equivalent to have the hermitian structure $g$ osculating in any point to order two to the standard metric, i.e. $(h_{ij})$ being of the form

$$(h_{ij}) = id + O(|z^2|)$$

**Example 3.10.** 1. $\mathbb{C}^n$ is Kähler with the standard flat metric

$$h_{ij} = h\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right) = \frac{1}{2}\delta_{ij}$$

with a fundamental form

$$\omega = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j = \frac{i}{2}\partial\bar{\partial}|z|^2$$

and hence with Käher potential $u(z) = \frac{1}{2}|z|^2$.

2. Products of Kähler manifolds are Kähler.

3. (Complex projective spaces) Getting a hermitian structure fixed on $\mathbb{C}^{n+1}$, the Fubini-Study metric is a canonical Kähler metric on the projective space $P\mathbb{C}^n$. Being $U_i$ the standard open covering, one defines

$$\omega_i := \frac{i}{2\pi}\partial\hat{\partial}\log\left(\sum_{j=1}^{n}\frac{|z_l|^2}{|z_i|^2}\right) \in \Omega^{1,1}(U_i)$$

4. A Riemann surface, i.e. a compact complex 1-dimensional manifold with $h$ hermitian metric is always Kähler, since $\omega$ is always closed by dimensional reasons, since there are no 3-forms. Therefore, any $U \subseteq \mathbb{C}$ endowed with a compatible metric is Kähler.

**Proposition 3.11.** *Any complex submanifold of a Kähler manifold is again Kähler.*

*Proof.* Let $g$ be a Kähler metric on a complex manifold $X = (M, J)$ and consider the restriction to the submanifold $Y \subseteq X$, $g|_Y$, which is clearly again riemannian. Since $T_pY \subseteq T_pX$, $g|_Y$ is invariant under $J$ for any $p \in Y$ and the restriction to $T_pY$ is an almost complex structure $J_Y$ on $Y$, $g|_Y$ is compatible with $J_Y$ and therefore $g|_Y$ defines an hermitian structure on $Y$. Then by definition $\omega_Y = g|_Y(J_Y(\cdot), \cdot) = g(J(\cdot), \cdot)|_Y = \omega|_Y$ and therefore we get $d_Y\omega_Y = d_Y(\omega|_Y) = (d_X\omega)|_Y = 0$. $\square$

**Corollary 3.12.** *Any projective manifold is Kähler.*

*Proof.* By definition we can embed any projective space into $P\mathbb{C}^n$ for a certain $n$. By restriction of the Fubini-Study metric we get the Kähler structure. $\square$

However it should be clear that the inverse is not true. Therefore:

**Remark 3.13.** Not every Kähler manifold is projective, as we have seen that $\mathbb{C}^n$, which is not projective, is Kähler.

However, we might ask ourselves whether compact Kähler manifolds either are or not always projective spaces, which is much more stronger than the last remark. See [18] as a reference, where it is shown that for dimension greater than or equal to 4, there exist compact Kaehler manifolds which do not have the homotopy type of projective complex manifolds. Therefore:

**Proposition 3.14.** *Not every compact Kähler manifold is projective.*

We can now find properties and restrictions of Kähler manifolds related to Betti and Hodge numbers, i.e. using cohomology. This set of restrictions is commonly known as the *Kähler package*, being actually a powerful set of restrictions on the cohomology of compact complex manifolds. It will be mainly useful to easily identify which manifolds are not Kähler just looking at their Hodge diamonds. In the following we will consider compact complex manifolds.

**Proposition 3.15.** *Let M be a compact n-dimensional Kähler manifold.* $b^{2k}(M) > 0$ *for* $k = 0, \ldots, n$.

*Proof.* Assume $b^{2k}(M) = 0$. Since $d\omega^k = 0$ we get $\omega^k = d\alpha$. Using Stokes theorem we would find

$$\int_M \omega^n = \int_M (\omega^{n-k} \wedge d\alpha) = \int_M d(\omega^{n-k} \wedge \alpha) = 0$$

which cannot be because $\omega^n$ is a volume form and hence never zero. □

The following theorem is basic for the topological characterisation of Kähler manifolds. See [11, 1] for proof.

**Theorem 3.16.** ***(Hodge decomposition)*** *Let M be a compact Kähler manifold. We have an isomorphism*

$$H^k_{dR}(M) \otimes \mathbb{C} \cong \bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(M)$$

*and* $H^{p,q}_{\bar\partial}(M) = \bar{H}^{q,p}_{\bar\partial}(M)$.

Therefore, Theorem 3.16 imposes conditions on Hodge numbers and hence symmetries on the Hodge diamond. We list them in the following corollary:

**Corollary 3.17.** *Let M be a compact Kähler manifold. Then*

$$b^k = \sum_{p+q=k} h^{p,q}, \qquad h^{p,q} = h^{q,p}, \qquad b^{2k-1} \in 2\mathbb{Z}$$

*Proof.* The first and second relations follows directly from dimensional analysis of the theorem results. For the third one we have, for $k \in \{1, \ldots, n\}$,

$$b^{2k-1} = \sum_{\substack{p+q=2k-1 \\ 0 \le p,q \le n}} h^{p,q} = \sum_{\substack{p+q=2k-1 \\ 0 \le p<q \le n}} (h^{p,q} + h^{q,p}) = \sum_{\substack{p+q=2k-1 \\ 0 \le p<q \le n}} 2h^{p,q} \in 2\mathbb{Z}$$

that actually proves what we wanted to. □

**Example 3.18.** The Hopf surface $\mathcal{M} = \mathbb{S}^1 \times \mathbb{S}^3$ is not Kähler, as $b^1(\mathcal{M}) = 1 \notin 2\mathbb{Z}$ [4, 1].

The next two propositions gives us characterisations of Kähler manifolds which will be useful in the next chapter, when we will study the properties of Calabi-Yau manifolds. For detailed proof see [1].

**Proposition 3.19.** *A connected Riemannian manifold of real dimension* $2n$ *is a Kähler manifold if and only if its holonomy group is contained in* $U(n)$.

**Proposition 3.20.** *A connected Riemannian manifold of real dimension* $2n$ *is a Ricci-flat Kähler manifold if and only if its reduced holonomy group is contained in* $SU(n)$.

# Chapter 4

# Calabi-Yau manifolds

*"Tu ho has volgut,*
*que et trenes com un fil amb el seu fil*
*i retorces la corda."*

- Gabriel Ferrater, *Teoria dels cossos*

We now introduce the concept of Calabi-Yau manifolds, which are a particular case of Kähler manifolds. As we will see, there are plenty of different definitions of what a Calabi-Yau manifold is. We enumerate some of them for then seeing the eventual equivalences that there exist. References for this section are [14, 4, 19, 16].

## 4.1 Definition and equivalences

**Definition 4.1.** A *Calabi-Yau manifold of real dimension* $2n$ is a compact $n$-dimensional Kähler manifold $(M, J, g)$ such that $g$ is a Ricci-flat metric.

Given a first definition of Calabi-Yau manifolds, we aim to compare it to the different definitions that one can find out in the literature. We now see that a Ricci-flat Kähler manifold equives to a Kähler manifold with vanishing first Chern class:

**Proposition 4.2.** *Let $(M, J, g)$ be a compact Ricci-flat Kähler manifold. Then $c_1(M) = 0$.*

*Proof.* We have seen that for a complex vector bundle $E$ over $M$, $c_1(E) = \left[\frac{i}{2\pi}\mathrm{Tr}F\right]$, where $F$ is the curvature two-form of a connection $A$. Recall also that the Chern classes of a manifold $M$ are defined as the Chern classes of their holomorphic tangent bundles. In this case it can be seen that $F = -iR$, where $R$ is the Ricci two-form. Then

$$c_1(M) = \left[\frac{1}{2\pi}\mathrm{Tr}R\right]$$

and if $R \equiv 0$ then $c_1(M) = 0$. □

However the opposite is much more complicated. It was conjectured by Calabi that a compact Kähler manifold with vanishing first Chern class admits a unique Ricci-flat metric. The uniqueness was proved by Calabi, while the existence by Yau much more later [12]. Hence

**Theorem 4.3.** *Let $(M, J, g)$ be a compact Kähler manifold with $c_1(M) = 0$. Then there exists a unique metric $g'$ being Ricci-flat.*

Given the equivalence, we get that a compact Kähler manifold is a Calabi-Yau manifold if and only if its first Chern class vanishes. Actually, in many references a Calabi-Yau manifold is defined by imposing the first Chern class of a compact Kähler manifold to vanish. Actually, we will now see three more results which will lead to an equivalence between five different conditions, all of them equivalent for defining Calabi-Yau manifolds.

**Remark 4.4.** Recall that by Proposition 3.20 we have that for $(M, J, g)$ a compact Kähler manifold, $M$ is Ricci-flat if and only if the holonomy group satisfies $\mathrm{Hol}(g) \subseteq \mathrm{SU}(n)$.

**Proposition 4.5.** *Let $(M, J, g)$ a compact Kähler manifold. The canonical bundle $K_M$ is trivial if and only if there exists a globally defined, non-vanishing $(n, 0)$-form.*

*Proof.* ($\Rightarrow$) Recall that $K_M = \bigwedge^{n,0} M$, so its sections are the $(n, 0)$-forms. Since $K_M$ is trivial, it is isomorphic to $M \times \mathbb{C}$. Therefore, there is a globally-defined differential $(n, 0)$-form $\alpha$, which is a section of $M \times \{1\}$, which is non-vanishing.
($\Leftarrow$) The existence of a globally-defined and non-vanishing differential $(n, 0)$-form $\alpha$ implies that the canonical bundle is trivial directly. □

**Remark 4.6.** Using the last proof, it is clear that any globally defined $(n, 0)$-form on $M$ can be written $f\alpha$ for some function $f$. Since $f\alpha$ is wanted to be holomorphic and $M$ is compact, $f$ must be holomorphic and hence, by the maximum principle, $f$ has to be constant. We get in this case $h^{3,0}(M) = 1$.

It can be proved [4] that $c_1(E) = 0$ equives to $\bigwedge^k E$ being trivial, where $k$ is the rank of $E$. Therefore we see this last result:

**Proposition 4.7.** *Let $(M, J, g)$ a compact Kähler manifold. $c_1(M) = 0$ if and only if the canonical bundle $K_M$ is trivial.*

*Proof.* The canonical bundle is the determinant line bundle of the holomorphic cotangent bundle. Therefore $K_M = \bigwedge^n (T^{1,0}M)^*$ is trivial if and only if $c_1((T^{1,0}M)^*) = -c_1(T^{1,0}M) = -c_1(M) = 0$. □

Seen the equivalences above, we can give a more general definition of Calabi-Yau manifolds, which indeed contain inside five different definitions which can be found in the literature.

**Definition 4.8. (Alternative)** A *Calabi-Yau manifold of real dimension* $2n$ is a compact $n$-dimensional Kähler manifold $(M, J, g)$ satisfying either one of the following conditions.

1. $M$ is Ricci-flat,
2. $c_1(M) = 0$,
3. $\mathrm{Hol}(g) \subseteq \mathrm{SU}(n)$,
4. the canonical bundle $K_M$ is trivial,
5. there exists a globally defined, non-vanishing $(n, 0)$-form.

**Remark 4.9.** It is possible to generalize the definition of Calabi-Yau manifolds to noncompact manifolds, hence admitting the *noncompact Calabi-Yau manifolds*, also known as *local Calabi-Yau manifolds*, in the sense that they are open neighbourhoods in Calabi-Yau manifolds. The simplest local Calabi-Yau manifold is $\mathbb{C}^n$.

We are mostly interested in low dimensional Calabi-Yau manifolds, specially in Calabi-Yau manifolds of complex dimension $n = 3$. This motivates the following definition:

**Definition 4.10.** Let $M$ be a Calabi-Yau manifold of complex dimension $n$. Then we say that $M$ is

- a *Calabi-Yau elliptic curve*, if $n = 1$.
- a *K*3 *surface*, if $n = 2$.
- a *Calabi-Yau threefold*, if $n = 3$.

**Remark 4.11.** Note that the real dimension of Calabi-Yau threefolds is 6.

## 4.2 Cohomology

We restrict the study of cohomology to complex dimension 3, i.e. to Calabi-Yau threefolds, as we have already said that these are the Calabi-Yau manifolds of physical interest. However many results extend to higher dimensions.

**Remark 4.12.** Recall that for Kähler manifolds, the Hodge numbers satisfy, by complex conjugation and Serre duality, $h^{p,q} = h^{q,p} = h^{n-p,n-q} = h^{n-q,n-p}$.

For Calabi-Yau manifolds there is further duality, i.e. more symmetry in the Hodge diamond, which is often called *holomorphic duality*.

We have seen in Remark 4.6 that there is a unique holomorphic volume form and hence $h^{3,0}(M) = 1$. Moreover, by using Stokes theorem one can see [4] that given a $(0,q)$ cohomology class, there is a unique $(0,3-q)$ cohomology class related, so $h^{0,q}(M) = h^{0,3-q}(M)$. Using both facts we get

$$h^{3,3} = h^{0,0} = h^{3,0} = h^{0,3} = 1$$

Still, in it is proved [4] that $h^{1,0}(M) = 1$. This directly implies that

$$h^{1,0} = h^{0,1} = h^{3,2} = h^{2,3} = h^{2,0} = h^{0,2} = h^{3,1} = h^{1,3} = 1$$

and therefore we get a Hodge diamond very simplified, with only two Hodge numbers to determine left:

$$\begin{array}{c|ccccccc}
1 & & & & 1 & & & \\
0 & & & 0 & & 0 & & \\
b^4 & & 0 & & h^{1,1} & & 0 & \\
b^3 & 1 & & h^{2,1} & & h^{2,1} & & 1 \\
b^2 & & 0 & & h^{1,1} & & 0 & \\
0 & & & 0 & & 0 & & \\
1 & & & & 1 & & &
\end{array}$$

We can still use the Euler characteristic two get a constraint for the two Hodge numbers left. Computing it we get

$$\chi(M) = \sum_{k=0}^{6} (-1)^k b^k = 2(h^{1,1} - h^{2,1})$$

It actually implies that if we can compute the Euler characteristic, then we only need to compute one of the two independent Hodge numbers to get all the topological information of the manifold.

In [4] it is seen that $\chi(M)$ can be obtained by the integral over $M$ of the top Chern class of the manifold, in this case $c_3(M)$. We hence get another constraint

$$\chi(M) = \int_M c_3(M) = 2(h^{1,1} - h^{2,1})$$

An interesting fact, specially in the field of theorethical physics, is that Calabi-Yau threefolds are conjectured to come in mirror pairs $(M, W)$, where $M, W$ are Calabi-Yau manifolds such that $H^{1,1}_{\bar{\partial}}(M) \cong H^{2,1}_{\bar{\partial}}(W)$ and $H^{2,1}_{\bar{\partial}}(M) \cong H^{1,1}_{\bar{\partial}}(W)$. This is the idea behind *mirror symmetry* and *supersymmetry* [19].

## 4.3 The quintic $Q$ in $P\mathbb{C}^4$

This final section briefly gives an example of a type of Calabi-Yau manifolds in the complex projective space $P\mathbb{C}^4$. We give final results. The reader might check [9, 4, 2].

The quintic $Q$ is given by a polynomial of degree 5 in the coordinates of $P\mathbb{C}^4$, which are obviously homogeneous. In [4] it is proved that the total Chern class of $Q$ is

$$c(Q) = 1 + 10x^2 - 40x^3$$

A first remark is that $Q$ is a submanifold of $P\mathbb{C}^4$, which is Kähler, so $Q$ is Kähler by Proposition 3.11 and Corollary 3.12. Accepting that the total Chern class of $Q$ is as stated, we can see how the first order term is 0, i.e. $c_1(Q) = 0$ and $Q$ is a Calabi-Yau manifold.

Then, we know that integrating $c_3(Q) = -40x^3$ over $Q$ we get the Euler characteristic $\chi(Q)$. It can be done using *Poincaré duality* [4]. Computing it one gets

$$\chi(Q) = \int_Q c_3(Q) = -200$$

Finally, determining one out of the two Hodge numbers left we get directly the other. The idea for computing $h^{2,1}(Q)$ is that $h^{2,1}$ is related to the number of infinitesimal deformations of the complex structure of the manifold. In the case these manifold is given by a polynomial, that is our case, these deformations are actually the free parameters. Further computing gets to the existence of 101 free parameters for $Q$. Hence

$$h^{2,1}(Q) = 101 \qquad \Longrightarrow \qquad h^{1,1}(Q) = \frac{\chi(Q)}{2} + h^{2,1}(Q) = 1$$

A final remark concerning $h^{1,1}(Q)$ is that it is the number of different Ricci-flat Kähler forms on $Q$. We see then that there is only 1 for $Q$.

To sum up, the Hodge diamond of $Q$ is

$$\begin{array}{cccccccc}
1 & & & & 1 & & & \\
0 & & & 0 & & 0 & & \\
1 & & 0 & & 1 & & 0 & \\
204 & 1 & & 101 & & 101 & & 1 \\
1 & & 0 & & 1 & & 0 & \\
0 & & & 0 & & 0 & & \\
1 & & & & 1 & & &
\end{array}$$

However, we recall that Calabi-Yau manifolds come in mirror pairs. Therefore, there exists a Calabi-Yau manifold $\tilde{Q}$ with just 1 infinitesimal deformation of the complex structure and 101 different Ricci-flat Kähler forms, which is the mirror couple of $Q$.

# Chapter 5

# Basic string theory model

*"Se vogliamo che tutto rimanga come è, bisogna che tutto cambi"*

- Tomasi di Lampedusa, *Il Gattopardo*

In this chapter we mainly follow Chapters 11 and 12 of [13] to introduce a basic model of string theory. Consider a manifold $X$, which we do not characterize yet. The manifold $X$ will be our spacetime, often a product of a certain $n$-dimensional space $M$ and time $\mathbb{R}$. We want to define *field theories on the manifold* $X$. The field theory will be called a $(n+1)$*-dimensional field theory*, where $n$ is the dimension of the spatial part of $X$.

**Definition 5.1.** Let $X$ be a manifold. We say that $f$ is a *field* on $X$ if $f$ can be locally expressed in local coordinates on $X$.

Generally, the fields of interest will be sections of vector bundles on $X$ and maps from $X$ to another manifold. Let now $\mathcal{F}$ be a set of fields on $X$. A physical theory is determined by giving an *action*

$$S : \mathcal{F} \longrightarrow \mathbb{R}$$

which is a functor over the fields in $\mathcal{F}$.

**Example 5.2.** In classical physics, the solutions to the equations of motion can be found by considering a $(0+1)$-dimensional field theory, i.e. with $X = \mathbb{R}$ being the time. Then, the position $x(t)$of a particle of mass $m$ is found by minimizing the action $S$ over $\mathcal{F} := \{f : X \to \mathbb{R}^3\}$, in this case considering $\mathbb{R}^3$ as the space in which the particle can move. In this case the action is

$$S(x(t)) = \int \left( \frac{m}{2} \left( \frac{dx}{dt} \right)^2 - V(x(t)) \right) dt$$

where $V$ is a potential energy depending on the position. Clearly in this case minimizing the action is equivalent to solving a differential equation on $x(t)$. In the physician's jargon, minimizing $S$ is written $\delta S[x(t)] = 0$.

**Definition 5.3.** A field is called *bosonic* if it commutes with any other field. The fields which are not bosonic are called *fermionic*.

**Example 5.4.** A typical fermionic field can be given considering 1-forms on a manifold and definint the product of these forms as the wedge product. Then, locally we can express

$$\psi(x) = \sum f_i(x)dx_i, \qquad \varphi(x) = \sum g_i dx_i, \quad \Rightarrow \quad \psi(x) \wedge \varphi(x) = -\varphi(x) \wedge \psi(x)$$

We now introduce an elementary bosonic string model. Among the bosonic fields in the Euclidean space $\mathbb{R}^n$ there are the smooth maps

$$f : \Sigma \longrightarrow \mathbb{R}^n$$

where $\Sigma$ is a compact 2-dimensional smooth manifold. We assume $\Sigma$ to be orientable.

**Remark 5.5.** Note that the smooth maps $f : \Sigma \longrightarrow \mathbb{R}^n$ clearly satisfy the condition to be bosonic fields, as with the typical product $f(p)g(q) = g(q)f(p)$, for $f, g$ smooth maps and $p, q \in \Sigma$.

This is called a *string theory* in the sense that $\Sigma$ can be sliced into circles, which are the strings of the model. The idea is that some slices can contain more than a string, which would mean that there is more than one "particle", and even in some slices circles can eventually intersect, which would represent the string interaction.

We need an additional field on our bosonic model to determine the action on it. This additional field is a Riemannian metric on $\Sigma$:

**Definition 5.6.** A *Riemannian metric on* $\Sigma$ is a smooth section of $T^*\Sigma \otimes T^*\Sigma$ which is positive definite at every point of $\Sigma$.

Recall that in a chart $U$ with local coordinates $(x_1, x_2)$ we can express the metric as

$$\sum_{i=1}^{2} \sum_{j=1}^{2} g_{ij}(x) dx_i \otimes dx_j$$

where $g_{ij}(x)$ are smooth real-valued functions at $x \in U$.

It can be shown that the bosonic string action [13], which is a functional of the field $f$ and the metric $g$ can be written as

$$S(f,g) = \frac{-1}{2\pi\alpha}\int_{\Sigma}\sum_{i=1}^{n}\sum_{j,k=1}^{2}\left(g^{jk}(x)\frac{\partial f_i}{\partial x_j}\frac{\partial f_i}{\partial x_k}\right)\Phi$$

where $\alpha$ is a constant with units of area, $g^{ij}(x)$ the $(i,j)$-component of the inverse of the matric $(g_{ij})$ and $\Phi = \left(\sqrt{\det g(x)}\right) dx_1 \wedge dx_2$ is the area form.

It happens that studying string theory in $\mathbb{C}^n$ rather than in $\mathbb{R}^n$ increases the number of *supersymmetry transformations*. Since it is not the scope of this work to introduce supersymmetry, we will say, roughly speaking, that supersymmetry transformations are changes of variables which do not affect the action $S$ and that transform bosonic fields to fermionic ones and vice versa. However, the reader can check Chapter 11 of [13] for an introduction, or even [9, 2] to go further.

In the complex case, following the notation we have used in prior chapters for complex geometry, the bosonic string action for a field $f : \Sigma \to \mathbb{C}^n$, locally expressed $\phi_1(z), \dots, \phi_n(z)$, becomes

$$S(f,g) = \frac{-i}{2\pi\alpha}\int_{\Sigma}\sum_{j=1}^{n}\left(\frac{\partial\phi_j}{\partial z}\frac{\partial\bar{\phi}_j}{\partial\bar{z}} + \frac{\partial\bar{\phi}_j}{\partial z}\frac{\partial\phi_j}{\partial\bar{z}}\right) dz \wedge d\bar{z}$$

Still, turning $X$ into a complex manifold is not enough to get relevant supersymmetry transformations of the action $S$. Actually, it is needed that the metric satisfies $d\omega = 0$, i.e. the manifold to be Kähler.

Although we will end here, considering the string theory that is attempting to describe our Universe and which needs, as already said, 6 extra dimensions, it might seem reasonable imposing them to be Ricci-flat, as it could for exemple exist some symmetry with GR. In the end, GR is proved to work well and in the abscence of masses we get a flat spacetime. So more or less string theory has to be "compatible" with GR, although obviously it should improve it and eventually introduce corrections on GR. In that case, Calabi-Yau threefolds would directly be the candidates to give a shape to these extra dimensions, as they are indeed.

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