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CIRCLE MAPS AND THE ARNOLD FAMILY

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Abstract

As a dynamical system, the Arnold Family of circle maps is itself a source of many curious and particular results and phenomena.

The main goal of this work is to explain and classify the possible dynamics of the family while encountering these special occurrences. In order to view and deeply understand the origin behind all the dynamical events that are occurring, we begin by setting a complete background on circle homeomorphisms. We later make use of the complexification tool and complex dynamics so we not only complete our discussion, but we extend it. Actually, we see the dynamics in the whole complex plane, and with the aid of some programs we provide a visual representation of it.

2020 Mathematics Subject Classification. 30D05, 30D30, 37E10, 37F10, 37F50.

Contents

Introduction

The Arnold or Standard Family of functions is a 2-parameter family of the unit circle given by

$$f_{w,\epsilon}(\theta) = \theta + w + \frac{\epsilon}{2\pi} sin(2\pi\theta)\ (mod\, 1), \quad w,\, \epsilon \in \mathbb{R}.$$

It is named after V. Arnold[2], who studied it for the first time [Arn91]. Throughout this project we consider the case $\epsilon \in [0, 1)$, which makes it a family of circle diffeomorphisms. The Standard Family is the central pillar of this thesis and it will be used as the guiding thread which connects circle maps with complex dynamics, via complexification.

In a global view, maps of the circle are those functions which are self-maps of $S^1 := \{(x, y) \in \mathbb{R}^2 |\ x^2 + y^2 = 1\}$. They are particularly useful due to the topological properties of S^1, which makes them ideal for the study of cycles and phases of an oscillator. Any circle map can be lifted to a map of the real line via the projection or covering map $\Pi(x) := e^{2\pi i x}$, which sends the real line to the complex unit circle. So we can think of circle maps as a subset of the real maps which satisfy a certain almost-periodicity condition.

For the purpose of this work we are mostly interested in homeomorphisms of the circle. A surprising and important property which only holds for homeomorphisms is that all points of S^1 rotate on average the same angle. In other words, for n sufficiently large, $f^n(\theta)$ turns around the unit circle the same number of times independently of the chosen $\theta \in S^1$. We will see that whether the *rotation number*, defined as the average rotation of a point under a given homeomorphism (see definition 2.7), is rational or irrational leads to completely different dynamics.

The simplest example of circle homeomorphisms is given by the family of rigid rotations: $t_w(\theta) = \theta + w\ (mod\, 1)$, for a fixed $w \in [0, 1)$. When the rotation number is rational, all orbits are periodic; whereas when it is irrational, then all orbits are dense in S^1 because of the famous Jacobi's theorem. One of the central questions we eventually aim to give an answer to is whether a given homeomorphism is conjugate to a rigid rotation. This is known as the *linearization problem* and it is closely related to the arithmetic properties of the rotation number.

In that context, the Arnold or Standard Family is an excellent choice to explore the

[2]Vladímir Arnold. Ucranian mathematician, 1937-2010.

properties of generic circle homeomorphisms. Indeed, it is nothing but a periodically perturbed rigid rotation: $f_{w,\epsilon}(\theta) = \theta + w + \frac{\epsilon}{2\pi} sin(2\pi\theta) \ (mod\,1)$; therefore it is not a trivial family. But, for small ϵ, it neither falls far apart from rigid rotations. Outside pure mathematics, it also serves to describe a wide range of natural phenomena involving oscillating quantities. Among its applications, the *phase-locked* phenomenon (Figure 1) or the *Arnold tongues* regions (Figure 2) have been found in areas such as biological processes or cardiac electric waves. We encounter them while achieving one of our goals, namely to make a description of the dynamics of the Standard Family according to the real parameters.

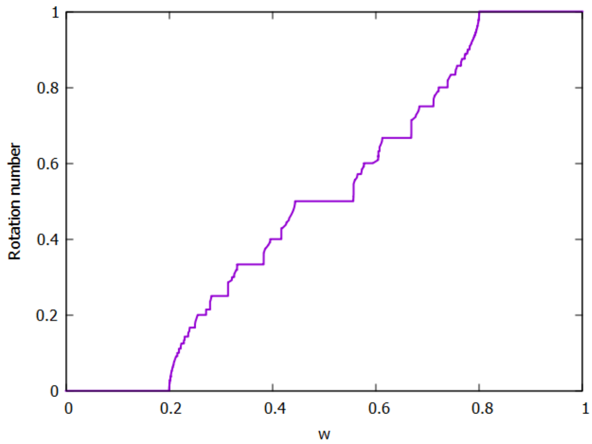


Figure 1: Devil staircase graph which results from plotting the rotation number as a function of w for the Standard Family.

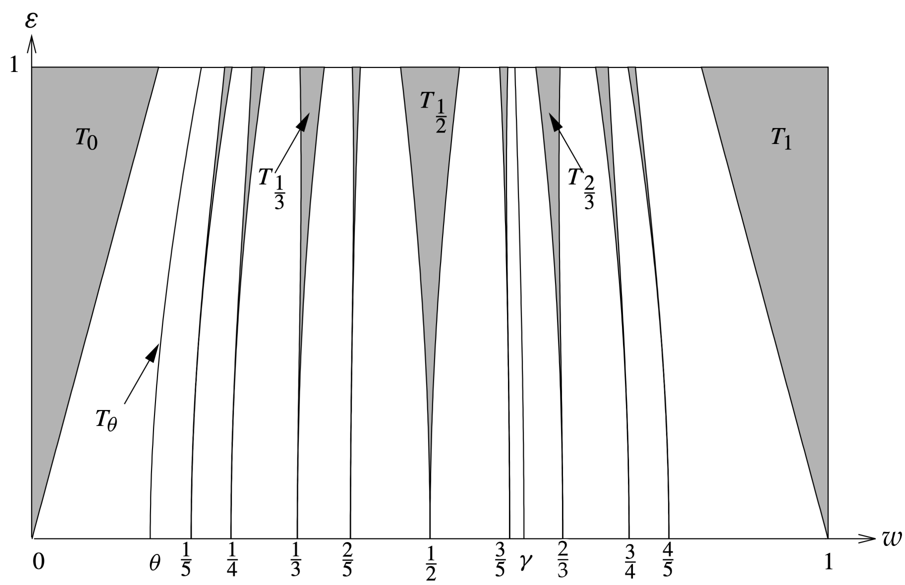


Figure 2: Arnold tongues of the Standard Family: level sets T_ρ of the rotation number ρ, for $w, \epsilon \in [0, 1]$. Tongues of every rational number are connected and have nonempty interior.

The connection of the Standard Family (or in fact any real analytic 1-dimensional dynamical system) with complex dynamics comes from its complexification. Although complex functions are more complicated a priori, the complex space gives us the complete picture of what happens because it is an algebraically closed space. A successful example of the complexification tool is the quadratic family $Q_c(x) := x^2 + c,\ x,\ c \in \mathbb{R}$. When we consider $z,\ c \in \mathbb{C}$, the famous Mandelbrot set, named after B. Mandelbrot[3] and first seen in 1978,

$$M := \{c \in \mathbb{C} |\ Q_c^n(0) \not\to_n \infty\},$$

actually gives us a complete description of the dynamics in terms of $c \in \mathbb{C}$. This was not only a historical milestone in complex dynamics which captures the usefulness of complexification as a tool, but it also uncovers a whole world of new mathematics, questions and connections to other areas.

To address the complexification of maps of S^1, the first problem which must be solved is when and how a circle map can be extended to a neighbourhood of S^1. Form a general point of view, the domain of real analytic maps reaches (at least) as long as the radius of convergence of the series at every point. Hence, any real analytic function can be complexified in a neighbourhood of the real line, and the resulting function is analytic and therefore holomorphic. The case of circle maps is similar. Each analytic circle map, as those in the Standard Family, can be complexified in a neighbourhood of S^1, and potentially into $\mathbb{C}$, with $\{0, \infty\}$ being essential singularities. Indeed, the complexification of the Standard Family

$$f_{w,\epsilon}(z) = ze^{2\pi i w} e^{\frac{\epsilon}{2}(z - \frac{1}{z})}$$

is a holomorphic function defined in $\mathbb{C} \setminus \{0\}$ with essential singularities at 0 and ∞.

Once we know the Standard Family can be complexified, we need the instruments from complex dynamics to understand its dynamics in the complex domain. Complex dynamics is the branch of mathematics which studies the dynamics of complex functions. It is a relatively new area of research, which did not gain consistency until the early 20^{th} century, when Gaston Julia[4] and Pierre Fatou[5] developed the global theory. Since then, plenty of powerful results which allow us to understand the dynamics of complex functions have been proven, some of them due to renowned mathematicians like D. Sullivan[6], J. Milnor[7], J. C. Yoccoz[8] or M. R. Herman[9]. In a general mark, complex dynamics is closely related to complex analysis and dynamical systems, but it also has deep connections with other areas such as number theory or topology. In our complexification of the Standard Family it will be interesting to see how the different rotation numbers give rise to radically different dynamics in the complex plane (see Figure 3).

[3] Benoit Mandelbrot. Polish-born French and American mathematician, 1924 - 2010.

[4] Gaston Maurice Julia. French mathematician, 1893 - 1978.

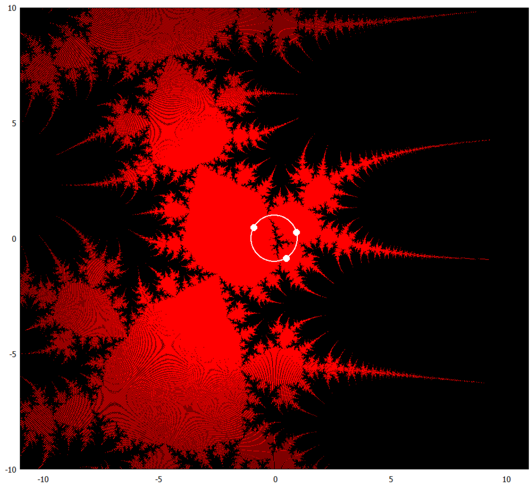
[5] Pierre Joseph Louis Fatou. French mathematician and astronomer, 1878 - 1929.

[6] Dennis Parnell Sullivan. USA mathematician, 1941 - present.

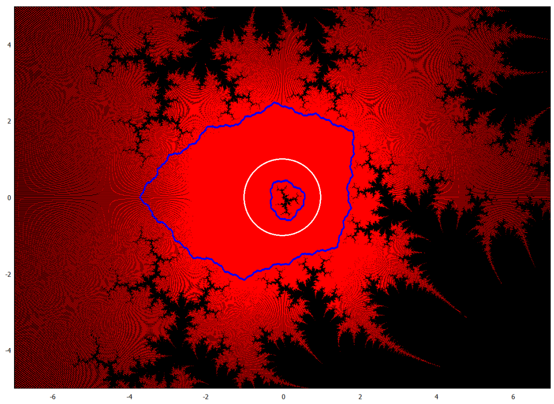
[7] John Milnor. USA mathematician, 1931-present.

[8] Jean-Christophe Yoccoz. French mathematician, 1957 - 2016.

[9] Michael Robert Herman. French American mathematician, 1942 - 2000.



(a) The complex dynamical plane of the Standard Family for parameters in the $T_{1/3}$ tongue. The points coloured in red converge to the attracting 3-cycle (in white) on the unit circle.



(b) The complex dynamical plane of the Standard Family for parameters in a irrational tongue. The orbits of the points inside the 2-connected domain (its boundary in blue) are conjugate to an irrational rotation.

Figure 3: The complex dynamical plane of the Standard Family for different parameters.

In order to achieve the goals of this project we structure the contents as follows. In chapter 1, we introduce some preliminary concepts that will be used afterwards. In chapter 2, we treat the theory of circle maps putting emphasis on the linearization problem. Moreover, we view how the Standard Family behaves on the unit circle according to the rotation number. In chapter 3, we introduce the basic notions of complex dynamics for rational maps, but also for transcendental ones, since they will be relevant afterwards. Finally, in chapter 4, we analytically extend the Standard Family to $\mathbb{C} \setminus \{0\}$ and we use the results from chapters 2 and 3 to discuss the dynamics of the complex Standard Family.

Chapter 1

Preliminaries

1.1 Arithmetic

The contents of this section can be found in [BH] and [PM97]. We list subsets of the irrational numbers that have special properties. The classification criteria is mainly the velocity of the approximation of an irrational number by rational numbers.

Definition 1.1. (Diophantine numbers) $x \in \mathbb{R} \setminus \mathbb{Q}$ is a Diophantine number of order $k \geq 2$ if there exists $\epsilon > 0$ such that

$$|x - \frac{p}{q}| > \frac{\epsilon}{q^k}$$

for all rational numbers $\frac{p}{q}$. We denote $\mathcal{D}(k)$ the set of all Diophantine numbers of order k, and $\mathcal{D} := \cup_{k \geq 2} \mathcal{D}(k)$ the set of all Diophantine numbers.

Remark 1.2. The sequence $\{\mathcal{D}(k)\}_k$ is a sequence of nested sets. where $\mathcal{D}(k) \subset \mathcal{D}(l)$ if and only if $k \leq l$. It can be proved that the set of Diophantine numbers has full Lebesgue measure in $[0, 1)$. In fact, the set of Diophantine numbers of order greater than 2 has full measure whereas $\mathcal{D}(2)$ has zero measure.

We define the continued fraction expansion of $x \in \mathbb{R} \setminus \mathbb{Q}$ as $[a_1, a_2, a_3, ...]_x$, where

$$x = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + ...}}}.$$

We write $\frac{p_n}{q_n} := [a_1, a_2, ..., a_n]$, which is the best approximation to x by fractions with denominator at most q_n.

Consider the continued fraction expansion of $x \in \mathbb{R} \setminus \mathbb{Q}$. Then

$$\left(\frac{q_{n+1}}{q_n^{k-1}} \right)_n < C \in \mathbb{R} \iff x \in \mathcal{D}(k).$$

A superset of the Diophantine numbers which is relevant in dynamical systems is the set of Bryuno numbers.

Definition 1.3. (Bryuno numbers) $x \in \mathbb{R} \setminus \mathbb{Q}$ is said to be a Bryuno number if

$$\sum_n \frac{log(q_{n+1})}{q_n} < \infty,$$

where q_n is as above.

We will also mention the set of **Herman numbers**. Its definition is omitted for it is quite complicated and the precise characterization of such set lays out of our interest, yet we recognise its existence and see they have some special properties when it comes to the linearization problem.

The respective order of such sets of numbers is given by the set inclusions

$$\mathcal{D}(2) \subset \mathcal{D} \subset \mathcal{H} \subset \mathcal{B}.$$

The set of Liouville numbers is defined as $(\mathbb{R} \setminus \mathbb{Q}) \setminus \mathcal{D}$, and it has null measure.

1.2 Real and complex analysis

The contents of this section can be found in [MH99].

Theorem 1.4. ***(Countable Nested Intervals Theorem)*** *Let $(I_n)_n \subset \mathbb{R}$ be a nested sequence of closed and bounded intervals ($I_{n+1} \subset I_n \ \forall n$). Then $\cap_n I_n$ is non-empty.*

Theorem 1.5. ***(Uncountable Nested Intervals Theorem)*** *The Countable Nested Intervals Theorem holds for an uncountable intersection $(I_\alpha)_\alpha \subset \mathbb{R}$.*

Proof. We write $I_\alpha = [a_\alpha, b_\alpha]$. Then

$$[\cap_\alpha I_\alpha]^c = [\cup_\alpha I_\alpha^c] = \cup_\alpha \left[(-\infty, a_\alpha) \cup (b_\alpha, \infty)\right].$$

Suppose $\cup_\alpha [(-\infty, a_\alpha) \cup (b_\alpha, \infty)] = \mathbb{R}$ and observe that $\cup_\alpha [(-\infty, a_\alpha) \cup (b_\alpha, \infty)] = [\cup_\alpha(-\infty, a_\alpha)] \cup [\cup_\alpha(b_\alpha, \infty)]$. Besides, recall that $a_\alpha < b_\alpha \ \forall \alpha$. Now, the fact that $\mathbb{R}$ is a union of two disjoint open sets is a contradiction with the connectedness of $\mathbb{R}$. Hence it is proved that $[\cap_\alpha I_\alpha]^c \neq \mathbb{R}$, so $\cap_\alpha I_\alpha \neq \emptyset$. □

Definition 1.6. (C^0 proximity) Let (X_1, d_1) and (X_2, d_2) be metric spaces and $f, g : X_1 \to X_2$ functions. We say f and g are $C^0 - \delta$ close if $\forall x \in X_1, d_2(f(x), g(x)) < \delta$.

Definition 1.7. (Uniform convergence) Let $(f_n : (X_1, d_1) \to (X_2, d_2))_n$ be a sequence of functions and let $f : (X_1, d_1) \to (X_2, d_2)$ be a function, where (X_1, d_1) and (X_2, d_2) are metric spaces. The sequence $(f_n)_n$ converges uniformly to f if

$$\forall \epsilon > 0, \exists n_0 \textit{ such that } \forall n > n_0, \ d(f_n(z), f(z)) < \epsilon \ \forall z \in U.$$

We denote it by $(f_n)_n \rightrightarrows f$. Note that uniform convergence is stronger than pointwise convergence since the ϵ value is required to be the same for all points.

We say that a sequence converges uniformly on compact subsets (u.c.c.) of U if the sequence converges uniformly on every compact subset of U.

We denote by $\mathcal{F}$ a family of functions from one metric space to another.

Definition 1.8. (Equicontinuity) Let (X_1, d_1) and (X_2, d_2) be metric spaces and $\mathcal{F} = \{f_\alpha : X_1 \to X_2\}_\alpha$ a family of functions. We say the family is equicontinuous at $p \in X_1$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that for all $x \in X_1$ and all f_α,

$$d_1(p, x) < \delta \implies d_2(f_\alpha(p), f_\alpha(x)) < \epsilon.$$

Definition 1.9. (Bounded variation function) Let (X, d) be a compact metric space. We say that $f : X \to \mathbb{R}$ is of bounded variation if

$$V = Var(f) := sup\{\sum_{k=1}^{n} d(g(x_k), g(x_{k-1}))\} < \infty,$$

where the supreme is taken over all possible partitions $\{x_0, \dots, x_n\}$ of X.

Remark 1.10. If f is Lipschitz and K is a bound of (X, d), then

$$Var(f) \leq \sum_{k=1}^{n} C\, d(x_k, x_{k-1}) \leq CKn$$

, hence f is of bounded variation.

Now we consider a complex function f and our topological space is the complex plane or the Riemann sphere.

Definition 1.11. (Riemann sphere) We define the Riemann sphere as $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. One can define the chordal metric

$$d(z_1, z_2) := \frac{2|z_1 - z_2|}{\sqrt{1 + |z_1|^2}\sqrt{1 + |z_2|^2}},$$

defined in $\hat{\mathbb{C}}$. Together they form a metric space.

Throughout the rest of the chapter we assume $U \in \hat{\mathbb{C}}$ is an open subset.

Definition 1.12. (Holomorphic function) Let $f : U \to \hat{\mathbb{C}}$ be a complex function. Then f is holomorphic at $z \in U$ if

$$lim_{h\to 0}\frac{f(z+h) - f(z)}{h}$$

exists. Then we write $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$. We say f is holomorphic on U if it is holomorphic $\forall z \in U$ and we denote $f \in H(U)$. An entire function is a function that is holomorphic on $\mathbb{C}$.

Definition 1.13. (Meromorphic function) Let $f : U \to \hat{\mathbb{C}}$ be a complex function. f is said to be meromorphic in U if f is holomorphic in $U \setminus E$, where $E \subset U$ only contains isolated points called singularities of f. Moreover, the singularities of f are the poles of f; i.e. $lim_{z\to z_0}|f(z)| = \infty\ \forall z_0 \in E$.

Definition 1.14. (Conformal map) Let $U, V \in \hat{\mathbb{C}}$. A conformal map $f : U \to V$ is an holomorphic and bijective map. It is also called biholomorphic.

Definition 1.15. (Rational map) A rational map $R = \frac{P}{Q}$ is a quotient of polynomials (with no common roots). The degree of R is $deg(R) := max(deg(P), deg(Q))$. It coincides with the topological degree of f, i.e. the number of preimages (counting multiplicity) of an arbitrary point.

Definition 1.16. (Normal family) Let $\mathcal{F} \subset H(U)$ be a family of holomorphic functions. $\mathcal{F}$ is a normal family on U if every sequence $(f_n)_n \subset \mathcal{F}$ contains a subsequence which converges u.c.c. on U.

Remark 1.17. By Hurwitz's theorem all possible limit functions must be analytic functions of U, or the subsequence converges u.c.c to ∞.

Theorem 1.18. ***(Arzela-Ascoli)*** *A family $\mathcal{F}$ of holomorphic functions is normal in $U \subset S$ if and only if $\mathcal{F}$ is equicontinuous on every compact subset of U with respect to the spherical metric.*

Theorem 1.19. ***(Montel's Theorem)*** *Let $\mathcal{F} = \{f_\alpha : U \to \hat{\mathbb{C}}\}_{\alpha \in I}$ be a holomorphic family. If there exist $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ such that $\cup_\alpha f_\alpha(U) \subset \hat{\mathbb{C}} \setminus \{z_1, z_2, z_3\}$, then $\mathcal{F}$ is normal in U.*

The next results are well-known theorems of complex analysis.

Theorem 1.20. ***(Analytic continuation principle)*** *Let $U \subset \mathbb{C}$ be open and simply connected and let f be holomorphic in U. If the set $\{z|\ f(z) = 0\}$ has a limit point in U, then f is identically 0.*

Theorem 1.21. ***(Maximum modulus principle)*** *Let $U \subset \mathbb{C}$ be open and simply connected and let f be holomorphic in U. If $|f|$ has a local maximum in U, i.e. it exists z_0 such that $|f(z_0)| \geq |f(z)|$ for all z in a neighbourhood of z_0, then f is constant.*

Theorem 1.22. ***(Open mapping theorem)*** *Let $U \subset \mathbb{C}$ be open and simply connected and let f be a non-constant holomorphic function in U. Then f sends open subsets of U to open subsets of $\mathbb{C}$.*

1.3 Dynamics and conjugacies

Let X be a topological space and consider $f : X \to X$ continuous. We consider the iterations $f(x), f^2(x), f^3(x), ...$ of $x \in X$ under f. The following definitions enable us to describe and classify points according on the properties of the iterations.

Definition 1.23. (Orbit) We define the orbit of $x \in X$ as $O(x) := \{f^n(x), n \in \mathbb{Z}\}$. And the respective forward and backward orbit as $O^+(x) := \{f^n(x), n \in \mathbb{N}\}$ and $O^-(x) := \{f^{-n}(x), n \in \mathbb{N}\}$.

Definition 1.24. (Periodic point) We say x_1 is fixed if $f(x_1) = x_1$. The point x_1 is p-periodic if $f^p(x_1) = x_1$ for some $p \in \mathbb{N}_{>0}$ and $f^q(x_1) \neq x_1$ for $0 < q < p$. We denote by $< x_1 >= \{x_1, ..., x_p\}$ the p-cycle obtained by the orbit of x_1. Finally, x is pre-periodic if some image of x is periodic.

In the case that x is not periodic its orbit is infinite, so $O(x)$ might have limit points.

Definition 1.25. (Invariant set) $U \subset X$ is (forward) invariant under f if $f(U) \subset U$ and backward invariant if $f^{-1}(U) \subset U$. U is totally invariant if it is forward and backward invariant.

Definition 1.26. (w-limit set) The $w-$limit set of $x \in X$, $w(x)$, is the set of all limit points of the forward orbit of x.

It is easy to check that w-limit sets are closed and invariant.

Definition 1.27. (Wandering domain) Let $f : X \to X$ be an homeomorphism. Then $W \subset X$ is a wandering domain if $f^n(W) \cap f^m(W) = \emptyset$ for all $n, m \in \mathbb{Z}$.

Definition 1.28. (Conjugacy, analytic conjugacy and semiconjugacy) Let X, Y be topological spaces and let $f : X \to X$ and $g : Y \to Y$ be continuous functions. We say g and f are conjugate or topologically conjugate if there exists an homeomorphism $h : X \to Y$ such that

$$h \circ f = g \circ h.$$

Depending on the properties of the conjugacy function h, we have particular types of conjugacies. We say f and g are analytically conjugate if h is analytic. We say g is semi conjugate to f if h is only continuous and surjective.

Remark 1.29. Observe that the relation of being semi conjugate is not reciprocal. So there might exist a semi conjugacy h such that $h \circ f = g \circ h$, but there is no semiconjugacy h' such that $h' \circ g = f \circ h'$.

Conjugacies are most useful when studying dynamical properties of functions. Observe that if $g = hfh^{-1}$ is conjugate to f, then g^n is conjugate to f^n. Therefore orbits are preserved under conjugacies meaning that fixed points are sent to fixed points under h, periodic orbits to periodic orbits, wandering intervals to wandering intervals and invariant sets to invariant sets.

Proof. Suppose $f^p(x) = x$, then $g^p(h(x)) = hf^p(h^{-1}(h(x))) = hf^p(x) = h(x)$. Suppose $f^n(W) \cap f^m(W) = \emptyset \implies hf^n(W) \cap hf^m(W) = \emptyset$, then $g^n(h(W)) = hf^n h^{-1}(h(W)) = hf^n(W)$ and $g^m(h(W))$ are disjoint. Suppose $f^n(U) \subset U$, then $g^n(h(U)) = hf^n(h^{-1}(h(U))) = hf^n(U) \subset h(U)$. $\square$

Chapter 2

Homeomorphisms of the circle

The aim of this section is to describe properties and types of orientation-preserving homeomorphisms $f : S^1 \to S^1$, that is homeomorphisms of the circle that preserve the order of every pair of points. The case of orientation-reserving homeomorphisms does not add much difficulty and will also be discussed in less detail. First, we define the tools we work with and derive important properties. Then we see when an orientation-preserving homeomorphism behaves similar to rigid rotations, known as the linearization problem. We end the chapter discussing the Standard Family, which is a good model for analytic circle maps.

2.1 Lifts and rotation number

In this section we follow Section 1.14 of [Dev03]. We begin defining lifts of circle maps, since such functions allow us to view circle maps in the real line and to define the rotation number.

Definition 2.1. **(Lift)** Let $f : S^1 \to S^1$ be at least continuous. We call $F : \mathbb{R} \to \mathbb{R}$ a lift of f if

$$\Pi \circ F = f \circ \Pi,$$

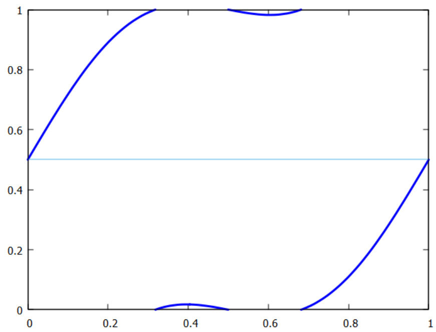
where $\Pi : \mathbb{R} \to S^1$ is defined as $\Pi(x) := exp(2\pi i x)$.

Note that we have the following diagram:

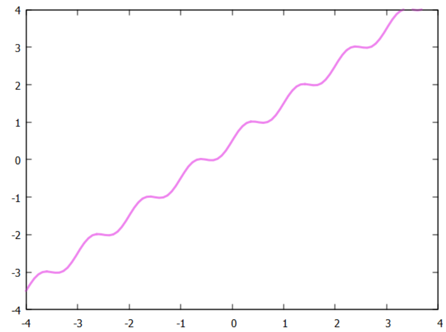
$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R} \\ \Pi \downarrow & & \downarrow \Pi \\ S^1 & \xrightarrow{f} & S^1 \end{array}$$

Remark 2.2. Any lift F of f satisfies that F is semi conjugate to f by Π, but it is not a topological conjugacy.

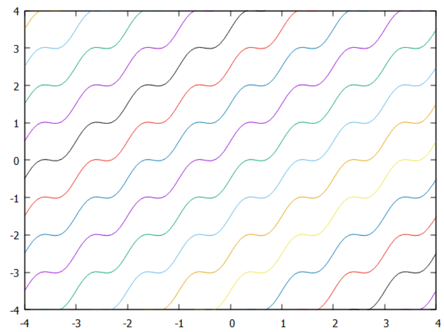
Remark 2.3. For convenience we will usually refer to a point of S^1 by $\theta \in [0, 1)$. In this case, the above commutative diagram would be the same, but with $\Pi(x) := x - [x]$ and $\frac{\mathbb{R}}{\mathbb{Z}}$ instead of S^1. Recall that $S^1 \cong \frac{\mathbb{R}}{\mathbb{Z}}$. Having said that, we will refer by f to both maps $f : \frac{\mathbb{R}}{\mathbb{Z}} \to \frac{\mathbb{R}}{\mathbb{Z}}$ and $f : S^1 \to S^1$.



(a) Surjective circle map f(x).



(b) Lift F(x) of f(x) such that $F(0) \in [0, 1)$.



(c) Set of lifts, $\{F(x) + k, k \in \mathbb{Z}\}$.

Figure 2.1: The function represented is $f(x) = x + 0.5 + 0.2sin(2\pi x)$.

Notice that the next properties hold for continuous circle maps, not necessarily homeomorphisms.

Given a circle map f, one can wonder if f has a lift. What's more, we do not know yet whether lifts exist. The answer is affirmative and it is a rather algebraic property of circle maps. The proof can be found in [May99].

Proposition 2.4. ***(Existence of a lift)*** *Let $f : S^1 \to S^1$ be a continuous map, then f has a lift.*

Proof. Let $I \subset \mathbb{R}$ be an interval with length less than 1, then $\Pi_{|I} : I \to S^1$ is an homeomorphism. Let J be a proper closed subset of S^1, then the set $\Pi^{-1}(J) = \{L_i\} \subset \mathbb{R}$ is a countable union of disjoint closed intervals with length less than 1. Therefore, for all $j \in J$, each L_i has exactly one point belonging to $\Pi^{-1}(\{j\})$. Now if f is continuous then it is possible to divide $\mathbb{R}$ into a countable family $\mathcal{I}$ of closed intervals such that $f(\Pi(I_i)) =: J_i$ is a closed proper interval of S^1 $\forall I_i \in \mathcal{I}$. We build F by construction as follows. Name $I_0 = [a, b]$ an interval of $\mathcal{I}$ containing 0 and choose $p_0 \in \Pi^{-1}(\{f(0)\})$. We define $F(0) := p_0$. Name $L_0 \subset \mathbb{R}$ the interval homeomorphic to J_0, so $\Pi_{|L_0} : L_0 \to J_0$, which contains p_0. We define F on I_0 as $F := (\Pi_{|L_0})^{-1} \circ f \circ \Pi$. It is clear that $\Pi \circ F = f \circ \Pi$. Now we proceed to do the same with the endpoints a, b; with the only caution of choosing the correct p_a and p_b so that they belong to the same interval of the family of intervals $\Pi^{-1}(f(I_0))$ as p_0. That way we assure there is no jump between L_0 and L_a since p_a belongs to both of them. This recurrent construction determines F. $\square$

Proposition 2.5. ***Properties of lifts.*** *Let f, g be continuous maps of the circle and F, G their respective lifts. Then*

1. *Lifts are not unique. In particular, F_2 is a lift of f if and only if $F_2 - F = k, k \in \mathbb{Z}$.*
2. *$F(x+1) = F(x) + d$ for some $d \in \mathbb{Z}$. If f is a homeomorphism, then $F(x+1) = F(x) \pm 1$. The number $d \in \mathbb{Z}$ is known as the degree of f.*
3. *If F is surjective, then f is surjective.*
4. *f is one to one $\iff$ F is one to one.*
5. *F^n is a lift of f^n, for $n \in \mathbb{N}$.*
6. *Suppose f is a homeomorphism, then F is an homoeomorphism and F^{-1} is a lift of f^{-1}.*
7. *Suppose g is a homeomorphism, then $G^{-1} \circ F \circ G$ is a lift of $g^{-1} \circ f \circ g$.*
8. *f is orientation-preserving $\iff$ F is increasing.*
9. *$f^n(\Pi(p)) = \Pi(p) \iff F^n(p) = p + k,\ k \in \mathbb{Z}$.*
10. *If f has degree d, then $F - dId$ is periodic with period 1. As a result the image of $F - dId$ is bounded.*
11. *if $|x - y| < 1$, then $|F(x) - F(y)| < d$.*

Proof.

1. $F_2 = F + k \implies \Pi \circ F_2 = \Pi \circ F$. In the other direction, let F_2 be a lift of f, then $\Pi \circ F_2 = \Pi \circ F \implies F_2 = F + k$.
2. Since $\pi(x+1) = \pi(x)$, directly from the definition of a lift we must have $\pi(F(x+1)) = \pi(F(x))$. It follows $F(x+1) = F(x) + d, d \in \mathbb{Z}$. Imposing f to be a homeomorphism we get $d = \pm 1$, depending on the orientation.
3. If F is surjective, then $\Pi F = f\Pi$ is surjective, and then f is surjective.
4. It is easy to prove that both statements are equivalent to say that given a real x, the only points to have the same image under both $\Pi \circ F$ and $f \circ \Pi$ are of the form $y = x + n$ for $n \in \mathbb{Z}$.
5. $\Pi \circ F = f \circ \Pi \implies \Pi \circ F^n = f \circ \Pi \circ F^{n-1} \implies f^2 \circ \Pi \circ F^{n-2} \implies \dots \implies \Pi \circ F^n = f^n \circ \Pi$.
6. If f is a homeomorphism then it is either strictly increasing or strictly decreasing, so it is F, then F is an homeomorphism. Now $\Pi \circ F = f \circ \Pi \implies f^{-1} \circ \Pi \circ F = \Pi \implies f^{-1} \circ \Pi = \Pi \circ F^{-1}$.
7. $\Pi G^{-1} F G = g^{-1} \Pi F G = g^{-1} f \Pi G = g^{-1} f g \Pi$.
8. Suppose F is increasing. Let $\Pi(x) \in (\Pi(y), \Pi(z))$, we pick $z - y < 1$, $x \in (y, z)$, and suppose $f\Pi(x) \notin (f\Pi(y), f\Pi(z))$. Observe that by continuity there must exist an x' such that it exits the interval but it is as close to it as we want to, suppose $f\Pi(x') = f(\Pi(z))\Pi(\epsilon)$, $\epsilon > 0$, and x' and z are close enough so that

$f\Pi(x')$ does not wrap the circle completely one more time than $f(\Pi(z))$. Then $\Pi F(x') = \Pi(F(z))\Pi(\epsilon) = \Pi(F(z)+\epsilon) \implies F(x') > F(z)$, which is a contradiction. Hence, it must be $f\Pi(x) \in (f\Pi(y), f\Pi(z))$. Similarly, if F is decreasing f must be orientation-reversing.

9. $\Pi(p)$ is fixed by $f \iff p$ is fixed by $\Pi F \iff F(p) = p + k$, $k \in \mathbb{Z}$.

10. The periodicity follows from (2). A periodic and continuous function is bounded on the real line. We can compute precise bounds. Suppose f is orientation-preserving and let $x = M$ and $x = m$ be the maximum and minimum of $F - Id$ in a period [y, y+1]. Suppose $m < M$. Recalling that $(F - dId)(x)$ is periodic with period 1 and F is monotonically increasing, then $(F - dId)(M) < (F - dId)(y) + d - d(M - y)$ and $(F - dId)(m) > (F - dId)(y) - d(m - y)$. So $(F - dId)(M) - (F - dId)(m) < d - d(M - m) < d$, hence the image of $F - dId$ must be bounded within a length $|d|$ interval.

11. The arguments done in the proof of (10) hold taking y and x instead of the maximum and minimum. Then $F(x) - F(y) = (F - dId)(x) - (F - dId)(y) + dx - dy < d - d(x - y) + dx - dy = d$.

$\square$

Remark 2.6. The opposite is also true, if a continuous function F is defined on an interval $[a, a+1]$ under the condition $F(a+1) = F(a) + d$ for some $d \in \mathbb{Z}$; that defines a lift of some circle map f. Indeed, F can be defined recurrently on $\mathbb{R}$ by such property and it is globally continuous, and then $f(\Pi(x)) := \Pi(F(x))$ is a well defined continuous map of the circle.

From now on we assume f is an orientation-preserving homeomorphism of the circle. A very important topological invariant of circle maps is the rotation number. We will see that it plays a crucial role in determining the properties of the orbits, as well as whether f is semiconjugate or not to a rigid rotation.

Definition 2.7. (Rotation number) Let F be a lift of an orientation-preserving homeomorphism $f : S^1 \to S^1$. We define the rotation number of f as

$$\rho(f) = \lim_{n\to\infty} \frac{F^n(x)}{n} \quad (mod\, \mathbb{Z}) \ \in [0,1)$$

The rotation number gives us an idea of the average rotation of a point under f, which happens to be the same for all points in S^1. Sometimes it may be useful to consider the equivalent definition

$$\rho(f) := \lim_{n\to\infty} \frac{F^n(x) - x}{n} \quad (mod\, \mathbb{Z}).$$

Proposition 2.8. ***(The rotation number is well-defined)***

Proof. Uniqueness. First note that $\rho(f)$ does not depend on the lift chosen since it is the fractional part by definition. We need to prove that it does not depend on x. Recall that if $|y-x|<1$, then $|F^n(y)-F^n(x)|<1$ because of proposition 2.5 (11). Otherwise, let $[x]$ denote the integer part of x. Then,

$$|F^n(x)-F^n(y)| = |F^n(x)-F^n(y+[x-y])+[x-y]| \le 1+|[x-y]| < \infty,$$

and therefore

$$\lim_{n\to\infty}\frac{(F^n(x)-F^n(y))}{n}=0.$$

To prove that the limit exists we will distinguish two cases: the case when f has at least one periodic point or the case when it does not.

(a) Suppose $z=\Pi(x)$ has period m, then $f^m(z)=z$ and $F^m(x)=x+k$, for some $k\in\mathbb{Z}$. Writing $n=jm+r$, $0\le r<m$ and $j=j(n)\in\mathbb{N}$, we have:

$$\lim_{n\to\infty}\frac{F^n(x)}{n}\le(\ge)\lim_{n\to\infty}\frac{F^{jm}(x)}{n}+\lim_{n\to\infty}\frac{F^n(x)-F^{jm}(x)}{n}=\frac{k}{m}\text{ given that}$$

$$\lim_{n\to\infty}\frac{F^{jm}(x)}{n}=\lim_{j\to\infty}\frac{F^{jm}(x)}{jm+r}=\lim_{j\to\infty}\frac{x}{jm+r}+\lim_{j\to\infty}\frac{k}{m+r/j}=0+\frac{k}{m}\text{ and}$$

$$\lim_{n\to\infty}\frac{|F^n(x)-F^{jm}(x)|}{n}=\lim_{n\to\infty}\frac{|F^r(F^{jm}(x))-F^{jm}(x)|}{n}=0$$

because F^r-Id is bounded for any r.

(b) Now $F^n(x)-x\notin\mathbb{Z}\ \forall x\ \forall n$. Then, $\exists k_n\in\mathbb{Z}$ s.t. $k_n<F^n(x)-x<k_n+1\ \forall x$.

Taking $x=0, F^n(0),\dots,F^{nm}(0)$ we obtain $k_n<F^n(0)<k_n+1$, $k_n<F^{2n}(0)-F^n(0)<k_n+1$, ... , $k_n<F^{mn}(0)-F^{n(m-1)}(0)<k_n+1$. Adding all the inequalities and dividing by mn it yields $\frac{k_n}{n}<\frac{F^{mn}(0)}{mn}<\frac{k_n+1}{n}$, which means that $|\frac{F^{mn}(0)}{mn}-\frac{F^n(0)}{n}|<\frac{1}{n}$. It follows that $|\frac{F^m(0)}{m}-\frac{F^n(0)}{n}|<\frac{1}{n}+\frac{1}{m}$, meaning that $\frac{F^n(0)}{n}$ is a Cauchy sequence and therefore it converges.

□

The next important corollaries follow from the proof above and will be used later.

Corollary 2.9. *Let F be a lift of an orientation-preserving homeomorphisim of the circle. If $k_1<(F-Id)(x)<k_2\ \forall x\in\mathbb{R}$, then $nk_1<(F^n-Id)(x)<nk_2\ \forall n$.*

Remark 2.10. Observe that the non-dependence on x of the rotation number is assured only for homeomorphisms.

Once we have seen the rotation number is well-defined, we proceed to focus on some of the most important results about it.

Proposition 2.11. *Let $f:S^1\to S^1$ be an orientation-preserving homoemorphism of the circle, then $\rho(f^m)=m\,\rho(f)\pmod{\mathbb{Z}}$.*

Proof. $\lim_{n\to\infty} \frac{F^{mn}(x)}{n} = m \lim_{n\to\infty} \frac{F^{mn}(x)}{mn} = m \lim_{n\to\infty} \frac{F^n(x)}{n}$. □

We will now see that the rotation number is a very robust quantity that depends continuously on the circle maps.

Proposition 2.12. ***(Continuity of the rotation number)*** *The rotation number depends continuously on f. Let $f : S^1 \to S^1$ and $g : S^1 \to S^1$ be orientation-preserving homoemorphisms of the circle. Then, for all $\epsilon > 0$, there exists δ such that if f and g are $C^0 - \delta$ close, then $|\rho(f) - \rho(g)| < \epsilon$.*

Proof. Let F and G be lifts of f and g respectively. First notice that F^m and G^m are as close as we want to if F and G are close enough. Similarly, F and G are as close as we want to if f and g are close enough. It follows that $|F^m - G^m| < \epsilon'$ if taking a proper δ. Because of that and recalling $F - Id$ image is bounded, there exist lifts F and G such that

$$r - M < F^m - Id, G^m - Id < r + M, \quad r \in \mathbb{Z}, M \in \mathbb{R}.$$

Using 2.9 it yields

$$k(r - M) < F^{mk}(0), G^{mk}(0) < k(r + M),$$

and therefore $|\frac{F^{mk}(0)}{mk} - \frac{G^{mk}(0)}{mk}| < \frac{2M}{m}$. Hence we have proved $\lim_{mk\to\infty} \frac{|F^{mk}(0) - G^{mk}(0)|}{mk} = 0$, which implies $\lim_{n\to\infty} \frac{|F^n(0) - G^n(0)|}{n} = 0$. □

Proposition 2.13. ***(Rationality of ρ and periodic points)*** *Let $f : S^1 \to S^1$ be an orientation-preserving homoemorphism. Then $\rho(f)$ is irrational if and only if f has no periodic points.*

Proof. $\Longrightarrow$ From the case (a) of the proof of proposition 2.8 it follows that if f has a periodic point x_0, then $\rho(f)$ is rational. What's more, $\rho(f) = p/q$ are the integers such that $F^q(x_0) = x_0 + p$.

$\Longleftarrow$ Suppose $\rho(f)$ is rational, we need to prove f has periodic points. Observe that if $\rho(f) = p/q$, then $\rho(f^q) = p \ (mod \mathbb{Z}) = 0$. Hence we may assume $\rho(f) = 0$ and proceed to study the existence of fixed points, without loss of generality. Name F the lift of f for which the rotation number is directly 0, that is $\lim_{n\to\infty} \frac{F^n(x)}{n} = 0$. If $F(p) = p$ for some p we are done. Then we may assume it is either $F(x) > x$ or $F(x) < x$ for all x. We suppose $F(x) > x$ since the other case is handled similarly.

Now, if there exists k s.t $F^k(0) > 1$, using 2.9 it yields $F^{km}(0) > m$. Hence $\lim_{m\to\infty} \frac{F^{mk}(x)}{m} > 1$ implying $\lim_{n\to\infty} \frac{F^n(x)}{n} > \frac{1}{k} > 0$, which is a contradiction.

The other case is $F^n(0) < 1 \ \forall n$. Since $F^n(0)$ is increasing and bounded it must converge. Name y its limit, then $F(y) = F(\lim_{n\to\infty} F^n(0)) = \lim_{n\to\infty} F^{n+1}(0) = y$, which proves that y is a fixed point. □

Corollary 2.14. *What's more, we proved that if $\rho(f) = p/q$, then $\Pi(\lim_{n\to\infty} F^n(0))$ has period q.*

Finally, we show that the rotation number is a dynamical invariant since it does not change under topological conjugacies.

Proposition 2.15. ***(Invariance of*** ρ***)*** *Let f and g be orientation-preserving homoemorphisms of the circle such that g is semi conjugate to f, then $\rho(g) = \rho(f)$*

Proof. Let F be an arbitrary lift of f. First, recall $F - Id$ is bounded within a length $d_f \in \mathbb{Z}$. Call h the semi conjugacy between f and g, so $h \circ g = f \circ h$. In the case h is not injective we define $h^{-1}(f(h(x))) := g(x)$ and we also get that $(H^{-1} - Id)(F(H(x)))$ is bounded since $(H^{-1} - Id)(F(H(x))) := G(x) - F(H(x)) = G(x) - H(G(x))$ and $H - Id$ is bounded. Now, we pick x, $H(x)$ respectively since the rotation number does not depend on the number chosen.

$$|\rho(g) - \rho(f)| = |\lim_{n\to\infty} \tfrac{H^{-1}F^nH(x)}{n} - \lim_{n\to\infty} \tfrac{F^n(x)}{n}| =$$
$$|\lim_{n\to\infty} \tfrac{H^{-1}F^nH(x)}{n} - \lim_{n\to\infty} \tfrac{F^n(H(x))}{n}| = |\lim_{n\to\infty} \tfrac{H^{-1}F^nH(x) - F^n(H(x))}{n}| \leq$$
$$\lim_{n\to\infty} \tfrac{d_h}{n} = 0.$$

□

We have seen that orentation-preserving homeomorphisms of the circle can have either irrational or rational rotation number. Therefore, they may or may not have fixed and periodic points. Recall that there exists some $\Pi(x) \in S^1$ with period q by f if and only if $F^q(x) = x + p$, for some $p \in \mathbb{Z}$, if and only if $\rho(f) = p/q$. We end this section with a result that tells us that the behaviour is quite different when the homeomorphism is orientation-reversing. Actually, these homeomorphims must have zero rotation number.

Proposition 2.16. ***(Orientation-reversing homeomorphisms)*** *Let $f : S^1 \to S^1$ be an orientation-reversing homeomorphism. Then f has exactly 2 fixed points, and hence $\rho(f) = 0$.*

Proof. Observe that a lift F of an orientation-reversing homeomorphism is decreasing and $F(x+1) = F(x) - 1$. Consider the lift F s.t. $F(0) \in [0, 1)$ and $G(x) := F(x) + 1$. On the interval $[0, 1)$ both $F - Id$ and $G - Id$ are 0 for some x since

$$(F - Id)(0) = F(0) \geq 0, \quad (F - Id)(1) = F(0) - 2 \leq -1.$$

Name p_1 and p_2 the respective fixed points. Clearly $p_1 \neq p_2$ since $p_1 = F(p_1) \neq G(p_1)$ and therefore $\Pi(p_1) \neq \Pi(p_2)$ because both points lay within [0, 1).

Let θ be a distinct point of S^1, we need to prove θ is not fixed by f. Swapping p_1 for p_2 if needed we have that $\theta \in (\Pi(p_1), \Pi(p_2))$, but then $f(\theta) \in (\Pi(p_2), \Pi(p_1))$ because f must map the arc $(\Pi(p_1), \Pi(p_2))$ onto the arc $(\Pi(p_2), \Pi(p_1))$ given that it is orientation-reversing, so θ is not fixed. □

2.2 Rigid rotations

In this section we introduce a simple yet important orientation-preserving family of circle diffeomorphims, rigid rotations.

Definition 2.17. *The family of rigid rotations is given by*

$$t_w(\theta) = \theta + w \ (mod\, 1),$$

where $w \in [0,1)$ is the angle parameter that defines a point on the unit circle, so the rotation map rotates all points an angle $2\pi w$. Abusing notation, we can say

$$t_w(e^{2\pi i\theta}) = e^{2\pi i(\theta+w)}.$$

Observe that the family

$$T_w(x) = x + w$$

is a lift of the rigid rotation satisfying $T_w(0) \in [0,\, 1)$ and $T_w^n(x) = x + wn$.

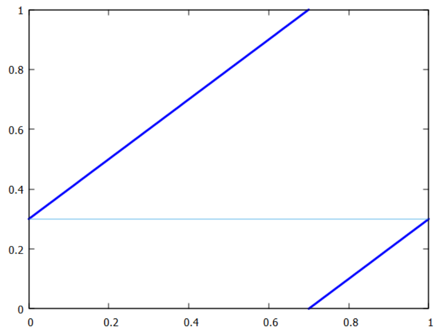


Figure 2.2: $t_{0.3}(\theta)$

It follows directly from the rotation number definition that $\rho(t_w) = w$. One can clearly see that if $w = \frac{p}{q}$ then all points have period q.

However, studying the behaviour of the orbits of t_w with irrational rotation number is not so obvious. The following theorem tells us what orbits look like in this case. The prove is found in [Dev03].

Theorem 2.18. ***(Jacobi's theorem)*** *Let w be irrational and consider the rigid rotation $t_w(\theta)$. Then every orbit of t_w is dense in $[0,\, 1) \cong S^1$*

Proof. Let us consider the orbit of an arbitrary θ. All points of the orbit are distinct, since if $t_w^n(\theta) = t_w^m(\theta) \implies 2\pi n w = 2\pi m w \implies (n-m)w \implies w \in \mathbb{Z}$, which is a contradiction. Now, the orbit of θ is an infinite set of distinct points, therefore there must be a limit point. Otherwise, $\exists\, \epsilon$ s.t. $|t_w^n(\theta) - t_w^m(\theta)| > \epsilon\ \forall n, m$, meaning that there are at most $\frac{2\pi}{\epsilon}$ distinct points on the orbit! Hence, for any $\epsilon > 0$, there are n, m for which

$$|t_w^n(\theta) - t_w^m(\theta)| < \epsilon, \implies |t_w^r(\phi) - \phi| < \epsilon,\ r := n - m,\ \phi := t_w^m(\theta).$$

It follows that the sequence $\phi, t_w^r(\phi), t_w^{2r}(\phi), ...$ is an arbitrary small partition of S^1 as t_w preserves the length of intervals. $\square$

2.3 The linearization problem

Our goal in this section is to get conditions which determine when an orientation-preserving homeomorphism f is conjugate to a rigid rotation. That tells us a lot since the dynamics of a rigid rotation is well-known to us. First we see that the problematic case is when $\rho(f)$ is irrational, since the rational case is trivial. We proceed to prove Poincare's theorem, regarding semiconjugacy, and which also gives us a sufficient condition for f to be conjugate to a rigid rotation: have no wandering intervals. Lastly we prove Denjoy's theorem, which gives f being C^2 another sufficient condition for f to be conjugate to a rigid rotation. Throughout this section we follow [KH95] and [Tur19].

Definition 2.19. (Linearization) A continuous map $f : S^1 \to S^1$ is linearizable if it is conjugate to a rigid rotation.

First we see that if ρ is rational the only possible conjugacy is the identity, so no function with rational rotation number can be linearizable unless it is a rigid rotation already.

Proposition 2.20. *Let $f : S^1 \to S^1$ be continuous and with rational rotation number. Then, either f is a rigid rotation or it is non linearizable.*

Proof. Suppose f is linearizable and $\rho(f) = p/q$ is rational. Then f is h-conjugate to $t_{p/q}$. This implies that all points are periodic by f with period p/q, since $f^q(x) = h^{-1}(t^q_{p/q}(h(x))) = h^{-1}(h(x)) = x$ and $t^n_{p/q}(y) \neq y$ for $0 < n < q$. Then $F^q(x) = x + p$ and therefore $F(x) = x + p/q$, so f is indeed the rigid rotation map and $h = Id$. □

It remains to study the factors that determine whether f is linearizable when $\rho(f)$ is irrational. The procedure is more complicated, first we prove that every orientation-preserving homeomorphism is at least semi conjugate to a rigid rotation.

Theorem 2.21. ***(Poincare's theorem)*** *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism with irrational rotation number ρ. Then f is semi-conjugate to t_ρ via an orientation-preserving function.*

Before proving it, we prove a list of preliminary propositions that enable us to get to the main result. This is important since the proof of this theorem is closely tied to the existence of wandering intervals, which determines the properties of the conjugacy h.

Proposition 2.22. *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism with irrational rotation number, $y \in S^1$ and $m, n \in \mathbb{Z}$, $m \neq n$. Then, for all $x \in S^1$, both the positive and negative semiorbit of x meet the interval $I = [f^n(y), f^m(y)]$.*

Proof. The proof for positive semiorbits is equivalent to prove that $S^1 = \cup_{k\in\mathbb{N}} f^{-k}(I)$.

We denote $I_k := f^{-k(n-m)}(I) = [f^{n-kn+km}(y), f^{-kn+m+km}(y)]$. We have that $I_{k-1} := f^{-(k-1)(n-m)}(I) = [f^{-kn+km}(y), f^{-(k-1)n+km}(y)]$, so I_k and I_{k-1} are contiguous. This implies that either $\cup_{k\in\mathbb{N}} f^{-k}(I) = S^1$ or both endpoints of $\{I_k\}_k$ converge. Suppose the endpoints tend to z, then

$$z = \lim_{k\to\infty} f^{-k(n-m)}(f^m(y)) = \lim_{k\to\infty} f^{(-k+1)(n-m)}(f^m(y)) =$$
$$\lim_{k\to\infty} f^{-k(n-m)} f^{n-m}(f^m(y)) = f^{n-m} \lim_{k\to\infty} f^{-k(n-m)}(f^m(y)) = f^{n-m}(z),$$

which is a contradiction given that f has no periodic points. Hence $S^1 = \cup_{k\in\mathbb{N}} f^{-k}(I)$. The proof for negative semiorbits follows from the same argument but considering $k \in \mathbb{Z}_{\leq 0}$. □

Proposition 2.23. ***(Uniqueness of the w-limit set)*** *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism with irrational rotation number and $x, y \in S^1$ arbitrary points. Then the w-limit set of x and y coincide. We call E the w-limit set of all points of S^1.*

Proof. Let $z \in w(x)$. By definition of the w-set, it $\exists (l_n)_n$ such that $f^{l_n}(x) \to z$. Define $I_k := [f^{l_k}(x), f^{l_{k+1}}(x)]$. Because of proposition 2.22, it yields that for all I_k there exists n_k such that $f^{n_k}(y) \in I_k$. It follows that the sequence $f^{n_k}(y)$ tends to z, then $z \in w(y)$. That proves $w(x) \subset w(y)$, and swapping x for y we get the other inclusion. □

Now we can consider the w-limit set of a given orientation-preserving homeomorphism, since it does not depend on the point. The following definition characterizes the E set in some cases.

Definition 2.24. (Cantor set) Let X be a topological space. A set of such space is a Cantor set if it is non-empty, perfect and nowhere dense. A set is perfect if all points are limit points and it is nowhere dense if it has no interior.

Proposition 2.25. *Let E be the w-limit set of an orientation-preserving homeomorphism with irrational rotation number. Then either $E = S^1$ or E is closed, perfect and totally disconnected, i.e. a Cantor set.*

Proof. $E = w(x)$ is closed since it is an w-limit set. We claim it is the minimal non-empty closed set that is f-invariant. Indeed, let A be a non-empty closed f-invariant set and $y \in A$. By invariance, $f^n(y) \in A\ \forall n \in \mathbb{Z}$, then $E = w(x) = w(y) \subset (f^n(y))_{n\in\mathbb{Z}} \subset A$.

The boundary of any set is closed, we claim that it is also f-invariant. Therefore $\partial E \subset E$ is either E or $\emptyset$. In the first case E is nowhere dense. In the second $E = S^1$. We want to prove that when $\partial E = E$, then all points of E are limit points. Let $y \in E = w(y)$, then there exists a sequence k_n such that $f^{k_n}(y) \to y$ and $f^{k_n}(y) \neq y\ \forall k_n$, so y is an accumulation point of E.

It remains to be proven that ∂E is f-invariant. Let $z \in \partial E$, then there exists $\epsilon > 0$ such that the set $\{f^n(z), n \in \mathbb{N}\}$ does not intersect $(z, z+\epsilon)$ (or $(z-\epsilon, z)$). Then, $\{f^n(z), n \in \mathbb{N}\}$ does not intersect $(f(z), f(z+\epsilon))$ since f is orientation-preserving. Hence, $f(z)$ belongs to ∂E. Arguing similarly, if $z \in Int(E)$, then $f(z) \in Int(E)$. □

Finally, we want to prove that the semiconjugacy is orientation-preserving, the following result gives us a condition for it to be strictly increasing at least in some points.

Proposition 2.26. *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism with irrational rotation number ρ and F the lift of f which gives directly ρ without integer part for $x = 0$. Then, for $n_1, n_2, m_1, m_2 \in \mathbb{Z}$ and $x \in \mathbb{R}$, we have that:*

$$n_1\rho + m_1 < n_2\rho + m_2 \iff F^{n_1}(x) + m_1 < F^{n_2}(x) + m_2.$$

Proof. We define

$$g(x) := F^{n_1}(x) + m_1 - F^{n_2}(x) - m_2,$$

observe that if $g(x) = 0 \implies \Pi(x)$ is periodic by f which is a contradiction, so it is enough to consider the second statement of the proposition for one x, we prove it for $x = 0$.

$\Longleftarrow$: Suppose $F^{n_1}(x) + m_1 < F^{n_2}(x) + m_2$, $\forall x$.

$$F^{n_1}(x) + m_1 < F^{n_2}(x) + m_2, \forall x \iff F^{n_1 - n_2}(y) - y < m_2 - m_1, \forall y$$

since F is an homeomorphism and $y := F^{n_2}(x)$. In particular, $F^{2(n_1-n_2)}(0) - F^{n_1-n_2}(0) < m_2 - m_1$ and $F^{n_1-n_2}(0) < m_2 - m_1$. Combining both inequalities we get $F^{2(n_1-n_2)}(0) < 2(m_2 - m_1)$. And inductively, $F^{n(n_1-n_2)}(0) < n(m_2 - m_1)$. Now

$$\rho := \lim_{n\to\infty} \frac{F^{n(n_1-n_2)}(0)}{n(n_1 - n_2)} \leq \lim_{n\to\infty} \frac{n(m_2 - m_1)}{n(n_1 - n_2)} = \frac{m_2 - m_1}{n_1 - n_2},$$

but the equality does not hold because ρ is irrational. That completes the implication.

$\implies$: By contraposition. We can argue the same but swapping $<$ for $>$ which leads to $\rho > \frac{m_2 - m_1}{n_1 - n_2}$. Recall the case of equality in the second statement needs not to be considered. □

Proof of Poincare's Theorem 2.21. $T(x) = x + \rho$ is a lift of t_ρ and let F be a lift of f. We pick $x \in \Pi^{-1}(E)$ and define $B := \{F^n(x) + m |\ n, m \in \mathbb{Z}\}$, which is the union of all orbits of x under any lift of f. Therefore $\Pi(B)$ is the orbit of $\Pi(x)$ under f. Besides, observe that $\Pi(\bar{B}) \subset E$. Indeed, if $y \in \bar{B} \implies \exists\{n_i\}, \{m_i\}$ such that $F^{n_i}(x) + m_i \to y$ and therefore $f^{n_i}(\Pi(x)) \to \Pi(y)$, so $\Pi(y) \in E$.

We will construct the real semi conjugacy H explicitly so that it satisfies the lift property. We begin by defining H in B, then we extend it continuously in $\bar{B}$ and eventually in $\mathbb{R}$. That defines the semiconjugacy h in S^1 we are looking for. We define H_B as $H_B : B \to \mathbb{R}$ as $H_B(F^n(x) + m) := n\rho + m$, which is strictly increasing because of 2.26 and it is continuous. We can extend H_B to $\bar{B}$ as follows, let $y \in \bar{B}$ and $(y_n)_n \to y$ a sequence in B tending to y, then we claim that $H_{\bar{B}}(y) := \lim_{n\to\infty} H_B(y_n)$ is well-defined. To prove the existence of the limit consider a sequence $(y_n)_n$ which converges to y monotonously, then $H_B(y_n)$ is also monotone since H_B is strictly increasing. Besides, let $z \in B$ greater than y if the sequence is increasing (or smaller if the sequence is decreasing), such a z always exists since B is not bounded. It follows $H_B(z)$ is a bound of $(H_B(y_n))$. That proves convergence since it is a bounded and monotonous sequence. Now let $(y'_n)_n$ be another sequence converging to y in B. $\forall\epsilon > 0$, Jacobi's Theorem 2.18 implies $H_B(B) := \{n\rho + m |\, n, m \in \mathbb{Z};\ \rho \in \mathbb{R} \setminus \mathbb{Q}\}$ is dense in $\mathbb{R}$ and therefore $\exists s_\epsilon, t_\epsilon \in B,\ s_\epsilon < t_\epsilon$ such that

$$\lim_{n\to\infty} H_B(y_n) - \epsilon < H_B(s_\epsilon) < \lim_{n\to\infty} H_B(y_n) < H_B(t_\epsilon) < \lim_{n\to\infty} H_B(y_n) + \epsilon.$$

Now we pick n_ϵ large enough such that y'_{n_ϵ} lays within (s_ϵ, t_ϵ). Hence because of monotonousness of H_B

$$\lim_{n\to\infty} H_B(y_n) - \epsilon < H_B(s_\epsilon) < H_B(y'_{n_\epsilon}) < H_B(t_\epsilon) < \lim_{n\to\infty} H_B(y_n) + \epsilon \quad \forall\epsilon > 0,$$

which proves that both limits are equal so we can write $\lim_{n\to\infty} H_B(y_n) = H_{\bar{B}}(y)$, then

$$H_{\bar{B}}(y) - \epsilon < H_B(y'_{n_\epsilon}) < H_{\bar{B}}(y) + \epsilon \quad \forall\epsilon > 0 \tag{2.1}$$

Finally suppose $y \in B$, we can pick the sequence $((y_n) := y)_n$ which clearly converges to y, therefore we get $H_{\bar{B}}(y) := \lim_{n\to\infty} H_B(y_n) := \lim_{n\to\infty} H_B(y) = H_B(y)$. That means $H_{\bar{B}} = H_B$ in B. What's more, we have proved continuity of H_B by sequences.

The next step is proving $H_{\bar{B}}$ is continuous and increasing. First, we prove continuity by sequences, we already know continuity for sequences in B. Let $(y_n)_n \subset \bar{B}$ be a sequence converging to $y \in \bar{B}$ then we need to prove $\lim_{n\to\infty} H_{\bar{B}}(y_n) = \lim_{n\to\infty} H_{\bar{B}}(y'_n)$, where $(y'_n)_n \subset B$ is a sequence converging to y. It is sufficient to pick $y'_n \in B$ such that $|y'_n - y_n| < \frac{1}{n}$ and $|H_{\bar{B}}(y'_n) - H_B(y_n)| < \frac{1}{n}$ for all n, observe that such y'_n exists because of equation 2.1. That proves continuity in all $\bar{B}$. Now let $y, z \in \bar{B}$, $y < z$ then we can pick sequences $(y_n)_n$ and $(z_n)_n$ in B converging to y and z respectively such that $y_n < z_n$ for all n. Therefore $H(y_n) < H(z_n)$ and $\lim_{n\to\infty} H(y_n) \leq \lim_{n\to\infty} H(z_n)$. So it is proved that H is strictly increasing in B and increasing in $\bar{B}$.

Now we claim $H_{\bar{B}}(\bar{B}) \to \mathbb{R}$ is surjective. Let $r \in \mathbb{R}$, as $H_B(B)$ is dense in $\mathbb{R}$ there exists an increasing sequence $(r_n)_n \subset B$ converging to r. Take $(b_n := H_{\bar{B}}^{-1}(r_n))_n$ the preimage of the sequence, which is bounded by $H_{\bar{B}}^{-1}(r')$ for any $r' \in H_B(B)$, $r' > r$; and strictly increasing, so $(b_n)_n$ converges to some point p, then $b \in \bar{B}$. Now $H_{\bar{B}}(b) = \lim_{n\to\infty} H_B(b_n) = \lim_{n\to\infty} r_n = r$, so $H_{\bar{B}}(\bar{B}) = \mathbb{R}$.

The last step is to extend $H_{\bar{B}}$ to $\mathbb{R}$. Let $y \in \mathbb{R} \setminus \bar{B}$, then y lays within an interval $(b_1, b_2) \subset \mathbb{R} \setminus \bar{B}$ where $b_1, b_2 \in \bar{B}$. We define $H(y) := H_{\bar{B}}(b_1) = H_{\bar{B}}(b_2)$. We claim

$$H_{\bar{B}}(y) = H_{\bar{B}}(z) \iff (y, z) \cap B \textit{ is either one point or empty.} \tag{2.2}$$

If $H_{\bar{B}}(y) < H_{\bar{B}}(z)$, we have that $H_B(B)$ is dense in $(H_{\bar{B}}(y), H_{\bar{B}}(z))$ and therefore B is dense in (y, z). If $H_{\bar{B}}(y) = H_{\bar{B}}(z)$ then $H_{\bar{B}}$ is constant in $[y, z]$ but also strictly increasing in B, so there is at most one point of B in (y, z). Hence H is well-defined, surjective, continuous by construction, and increasing. We claim $H \circ F = T_\rho \circ H$ in B since

$$H \circ F(F^n(x) + m) = H(F^{n+1}(x) + m) := (n+1)\rho + m = T_\rho(n\rho + m) = T_\rho \circ H(F^n(x) + m).$$

The equality also holds in $\bar{B}$ because of continuity

$$H(F(y)) = \lim_{n\to\infty} H(F(y_n)) = \lim_{n\to\infty} T_\rho(H(y_n)) = T_\rho(H(y))$$

and in $\mathbb{R}$. Let us use the same notation where $r \in \mathbb{R} \setminus \bar{B}$ and $b_1, b_2 \in \bar{B}$ are the end-points described before. Because of equation 2.2 we have $H(b_1) = H(b_2)$, which implies $T_\rho(H(b_1)) = T_\rho(H(b_2))$ and $H(F(b_1)) = H(F(b_2))$. Then $H(F(r)) = H(F(b_1)) = H(F(b_2))$ and $T_\rho(H(r)) = T_\rho(H(b_1)) = T_\rho(H(b_2))$.

H must satisfy one last property so that it is indeed a lift

$$H(F^n(x) + m + 1) := n\rho + m + 1 = H(F^n(x) + m) + 1.$$

Taking sequences and limits the equality holds in $\mathbb{R}$. That guarantees us that $h(\Pi(x)) := \Pi(H(x))$ is an increasing continuous surjective map such that

$$HF = T_\rho H \implies \Pi HF = \Pi T_\rho H \implies h\Pi F = t_\rho \Pi H \implies hf\Pi = t_\rho h\Pi \implies hf = t_\rho h.$$

□

Corollary 2.27. *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism with irrational rotation number ρ. Then f is semi conjugate to $t_\rho \iff S^1 = E$, where E is the w-limit set of f.*

Proof. $\Longleftarrow$ If $E = S^1 = w(\Pi(x))$ then $\overline{\{f^n(\Pi(x))\}} = S^1 \implies \bar{B} = \mathbb{R}$ and therefore h has no constant intervals, so it is strictly increasing and therefore bijective.

$\Longrightarrow$ Suppose f is conjugate to a rotation and let $\theta \in S^1$, then $\{t^n_\rho(\theta)\}$ is dense in S^1 which implies $\{h^{-1}t^n_\rho h(\theta)\}$ is dense too. Indeed let $\phi \in S^1$ be any point, $\forall\theta'$, $\exists n$ such that $|t^n_\rho(\theta') - h(\phi)| < \delta$ which implies $|h^{-1}t^n_\rho(h(\theta)) - \phi| < \epsilon$ by continuity of h^{-1}, where $\theta = h^{-1}(\theta')$. □

Proposition 2.28. ***(Wandering intervals)*** *Let $f : S^1 \to S^1$ be an orientation-preserving homeomophism with irrational rotation number which is not conjugate to a rotation. Then the intervals $I \subset S^1 \setminus E$ are wandering under f.*

Proof. E is a Cantor set because of 2.25, so $S^1 \setminus E$ is a union of intervals. Recall that the semiconjugacy h constructed in the proof of theorem 2.21 maps intervals of $S^1 \setminus E$ into single points. Let I be one of the intervals of $S^1 \setminus E$ and let y be the point such that $\{y\} = h(I)$, then

$$h(f^m(I)) = t^m_\rho(h(I)) = \{y + m\rho\}.$$

Let $n, m \in \mathbb{Z}$, $n \neq m$, then $h(f^m(I)) = \{y + m\rho\}$ and $\{y + n\rho\} = h(f^n(I))$ are disjoint since ρ is irrational, which implies $f^m(I) \cap f^n(I) = \emptyset$. □

Remark 2.29. Observe that the case where H (and therefore h) is not injective, the graph of the semiconjugacy has some particular properties. Then $S^1 \setminus E$ is a Cantor set and there are infinitely many wandering intervals whose union is $(S^1 \setminus E)^c$. It follows that h is constant on infinitely many intervals and strictly increasing in E. We will discuss this type of functions in section 2.5 with more detail.

Observe that Poincare's Theorem gives a clear distinction between orientation-preserving homeomorphisms. On the one hand, there are some with rotation-like behaviour which means orbits are dense - whereas on the other, there are homeomorphisms with wandering intervals.

Next we prove Denjoy's theorem. Observe that from now on we focus particularly on diffeomorphisms.

Theorem 2.30. ***(Denjoy's theorem)*** *Let $f : S^1 \to S^1$ be a C^1 orientation-preserving diffeomorphism with irrational rotation number ρ such that f' is of bounded variation. Then f is conjugate to the rotation t_ρ.*

The proof requires some preliminaries.

Proposition 2.31. *Let $f : S^1 \to S^1$ be a orientation-preserving homeomorphism with irrational rotation number and let $p \in S^1$ be any point. Then there are infinitely many $n \in \mathbb{N}$ such that all intervals $f^k((p, f^{-n}(p)))$ are disjoint for $k = 0, \dots, n-1$.. We name $N_p^f \subset \mathbb{N}$ the infinite set of such naturals.*

Proof. First suppose f is a rotation, name it t, so every orbit is dense. We name $x_k := t^k(p)$, $x_0 = p$ and $I_n := (p, x_{-n})$. Suppose n is such that $x_k \notin I_n$ for $k = -n+1, ..., 0, ..n-1$, then all intervals $t^k((x_0, x_{-n}))$ are disjoint. Indeed, let $0 \leq l < k < n$, we want to prove $t^l((x_0, x_{-n}))$ and $t^k((x_0, x_{-n}))$ are disjoint. We have that x_{k-l-n} and x_{k-l} lay outside I_n by hypothesis, then $t^l(x_{k-l-n}) = x_{k-n}$ and $t^l(x_{k-l}) = x_k$ lay outside $t^l(I_n)$. Since these 2 points are precisely the endpoints of $t^k(I_n)$ there are 2 possibilities, either $t^l(I_n) \subset t^k(I_n)$ or they are disjoint. The first case is false given that $t^l(I_n) = (x_0 + l2\pi\rho, x_{-n} + l2\pi\rho)$ and $t^l(I_n) = (x_0 + k2\pi\rho, x_{-n} + k2\pi\rho)$ so they have the same length. It remains to be proved that there are infinitely many n such that $x_k \notin I_n$ for $k = -n+1, ..., 0, ..n-1$. Since the orbit of p is dense, there is a subsequence of (x_k) converging to $p = x_0$, so we can always pick an n such that the interval I_n is as small as we want, in particular, we need it to be smaller than all the intervals I_k for $k = -n+1, ..., 0, ..n-1$. That completes the prove for rigid rotations.

Now let f be an arbitrary orientation-preserving homeomorphism with irrational rotation number, it follows from Poincare theorem 2.21 that f is semiconjugate to an irrational rotation t by a continuous orientation-preserving function h. Hence

$$h(f^j(p), f^m(p)) = (h(f^j(p)), h(f^m(p))) = (t^j(h(p)), t^m(h(p))),$$

and therefore $t^k((h(p), t^{-n}(h(p)))$ are disjoint $\implies$ $h(f^k(p), f^{k-n}(p))$ are disjoint $\implies$ $(f^k(p), f^{k-n}(p))$ are disjoint. Therefore $N_{h(p)}^t \subset N_p^f$ so N_p^f is also infinite. $\square$

Proposition 2.32. *Let $f : S^1 \to S^1$ be a orientation-preserving diffeomorphism such that f' is of bounded variation. Then $\Phi(x) = log f'(x)$ is a function of bounded variation.*

Proof. Observe that $log(f')$ is well-defined. First we claim that there exists $\lambda > 0$ such that $f'(x) \geq \lambda$. If f' was arbitrary close to 0 then $(f^{-1})'$ would be arbitrary close to infinity, but that can't be since f^{-1} is a continuous function in the compact set S^1. Now let $\{t_0, t_1, ..., t_n\}$ be any partition of S^1. Then

$$\sum_{k=1}^{n} |\Phi(t_k) - \Phi(t_{k-1})| = \sum_{k=1}^{n} |\int_{f'(t_{k-1})}^{f'(t_k)} \frac{1}{x} dx| \leq \sum_{k=1}^{n} |\int_{f'(t_{k-1})}^{f'(t_k)} \frac{1}{\lambda} dx| =$$
$$\sum_{k=1}^{n} \frac{1}{\lambda} |f'(t_k) - f'(t_{k-1})| \leq \frac{Var(f)}{\lambda}.$$

Hence $Var(\Phi) \leq \frac{Var(f)}{\lambda}$. □

Proposition 2.33. *Let $f : S^1 \to S^1$ be a orientation-preserving diffeomorphism with a wandering interval I and such that f' is of bounded variation. Let $V := Var(\Phi) = Var(log(f'))$. Then*

$$e^{-V} \leq (f^n)'(x)(f^{-n})'(x) \leq e^V \quad \forall x \in I,$$

for infinitely many natural numbers n

Proof. First observe that because of proposition 2.32, Φ is of bounded variation. Fix x in I, then for all $n \in N_x^f$ we get that $P_n := \{f^k(p), f^{k-n}(p) | k = 0, 1, ..., n-1\}$ forms a partition of S^1 where every point $f^m(p)$ is contiguous (there is no other point of the partition in between) to $f^{m-n}(p)$.

$$V \geq \sum_{k=0}^{n-1} |\Phi(f^k(x)) - \Phi(f^{k-n}(x))| \geq |\sum_{k=0}^{n-1} \Phi(f^k(x)) - \sum_{k=0}^{n-1} \Phi(f^{k-n}(x))| =$$
$$\left| log \left[\prod_{k=0}^{n-1} f'(f^k(x)) \right] - log \left[\prod_{k=0}^{n-1} (f')(f^{k-n}(x)) \right] \right| =$$
$$\left| log \frac{\prod_{k=0}^{n-1} f'(f^k(x))}{\prod_{k=0}^{n-1} f'(f^{k-n}(x))} \right| = \left| log \frac{(f^n)'(x)}{(f^n)'(f^{-n}(x))} \right|,$$

where the last step follows from the derivative chain rule. Hence we have proved that

$$e^{-V} \leq \frac{(f^n)'(x)}{(f^n)'(f^{-n}(x))} \leq e^V.$$

Next we apply the inverse function derivative theorem to f^n. It yields

$$e^{-V} \leq (f^n)'(x) \cdot (f^{-n})'(x) \leq e^V.$$

It remains to prove that N_x^f does not depend on $x \in I$. Actually, we prove there is an infinite subset of N_x^f that does not depend on x, which is enough. Let $y \in I$, it follows from the construction of the semiconjugacy h in 2.21 that $h(x) = h(y)$. Hence following the notation in 2.31

$$h(f^j(x), f^m(x)) = (h(f^j(x)), h(f^m(x))) = (t^j(h(x)), t^m(h(x))) =$$
$$(t^j(h(y)), t^m(h(y))) = (h(f^j(y)), h(f^m(y))) = h(f^j(y), f^m(y)),$$

so both N_x^f and N_y^f have $N_{h(x)=h(y)}^t$ as a subset. □

We are eventually ready to prove Denjoy's Theorem.

Proof of Denjoy's Theorem 2.30. Suppose f does not conjugate to a rotation and we see it leads to a contradiction. Taking into account 2.28 we get that f has a wandering interval, we claim that all of its images and preimages together do not fit in S^1. The hypothesis is Φ is of bounded variation, then the contradiction comes from the inequaltiy of 2.33. Let n be as 2.33 ($n \in N^{t_\rho}_{h(x)}$), then

$$l(f^n(I)) + l(f^{-n}(I)) := \int_I (f^n)'(x)dx + \int_I (f^{-n})'(x)dx =$$
$$\int_I \left[(f^n)'(x) + (f^{-n})'(x)\right] dx \geq \int_I 2\sqrt{(f^n)'(x) \cdot (f^{-n})'(x)}dx \geq$$
$$\int_I 2\sqrt{e^{-V}}dx = 2l(I) \cdot e^{-\frac{V}{2}}.$$

Therefore $l(S^1) \geq \sum_{i=-\infty}^{\infty} l(f^i(I)) \geq \sum_{n \in N^t_{h(x)}} l(I) \cdot e^{-\frac{V}{2}} = \infty$, so the claim is proved. □

Corollary 2.34. *Let f be a C^1 orientation-preserving diffeomorphism of the circle with irrational rotation number such that f' is of bounded variation. Then all orbits under f are dense in S^1.*

Proof. By contradiction. Because of theorem 2.30, $t^n h = hf^n$. Suppose there exists $I = (a, b) \subset S^1$, $a < b$ and $x \in S^1$ such that $f^n(x) \notin I\, \forall n$. Then $hf^n(x) \notin h(I)\ \forall n$, and $h(I)$ is an interval since h is an orientation-preserving homeomorphism by Denjoy's theorem 2.30. Actually, $t^n h(x) = hf^n(x) \notin h(I)\ \forall n$. It follows that $f^n(x) \notin I\, \forall n$, which contradicts Jacobi's theorem. □

2.4 Analytic linearization

In this section we treat analytic linerization of analytical maps of the circle. It is not our objective to prove the results, but to present them and give a state-of-the-art view on the subject. The general idea is we have analytic linearization depending on how fast the rotation number can be approximated with rational numbers, hence depending of the arithmetic properties of ρ. The results are mainly due to Herman and Yoccoz. The following results are found in [BF14], [FG03], and were originally proven in [Her79], [Her] and [Yoc84].

We will see in the following chapter that whether there is analytic linearization or not has a huge impact on the complex dynamics of the respective complexified map.

Definition 2.35. (Analytic linearization) A continuous map $f : S^1 \to S^1$ is analytically linearizable if it is analytically conjugate to a rigid rotation, i.e., the conjugacy function is analytic.

Theorem 2.36. ***(Analytic linearization, [Yoc84])*** *Let $f : S^1 \to S^1$ be an analytic diffeomorphism and $\rho \in (0, 1]$. Then*

- *If $\rho \in \mathcal{H}$, then evert map f with $\rho(f) = \rho$ is analytically linearizable.*
- *If $\rho \notin \mathcal{H}$, then there exists a map f with $\rho(f) = \rho$ such that f is not analytically linearizable.*

- *If $\rho(f) = \rho \in \mathcal{B} \setminus \mathcal{H}$, and f is close enough to t_ρ, then f is analytically linearizable.*

Hence if $\rho(f) \in \mathcal{H}$, f is analytically linearizable. If $\rho(f) \in \mathcal{B}$, there always exist maps which are analytically linearizable and maps that are not. Finally, we comment that it is yet an open question what happens for $\rho(f) \notin \mathcal{B}$, the existence of non analytic linearizable maps is assured. However, it is an open question for which families this Bryuno condition is actually optimal. Recall that the rational rotation number obviously cannot lead to analytic linearization.

We later discuss the analytic linearization problem for the Standard Family.

2.5 The Standard Family

We consider the Standard Family, also called the Arnold Family, of circle maps

$$f := f_{w,\epsilon}(\theta) = \theta + w + \frac{\epsilon}{2\pi} sin(2\pi\theta)\ (mod\ \mathbb{Z}), \quad \theta,\ w,\ \epsilon \in [0, 1),$$

which has the associated lifts

$$F := F_{w,\epsilon}(x) = x + w + \frac{\epsilon}{2\pi} sin(2\pi x),\ x \in \mathbb{R}.$$

Notice that $f_{w,\epsilon}$ is a diffeomorphism as long as $0 \leq \epsilon < 1$, since $f'(\theta) = 1 + \epsilon cos(2\pi\theta) \neq 0\ \forall\theta \in [0, 1)$. We want to discuss how $\rho(f_{w,\epsilon})$ varies with w.

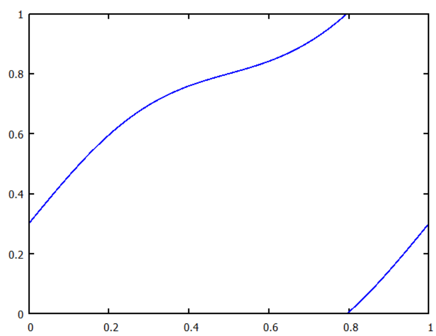


Figure 2.3: $f_{0.3,\pi/5}(x) = x + 0.3 + 0.1sin(2\pi x)$.

Let us focus on the most obvious properties of the family. The case $\epsilon = 0$ is already known to us since $f_{w,0}(\theta) = t_w(\theta)$, so $\rho(f_{w,0}) = w$. Furthermore, $\theta = 0$ is a fixed point for all $f_{0,\epsilon}$ so $\rho(f_{0,\epsilon}) = 0$ for all ϵ. Besides, recall that $\rho(f_{w,\epsilon})$ is continuous on w and ϵ and is increasing on w given that $F_{w,\epsilon}(x)$ is strictly increasing on w for all x and ϵ. It is particularly interesting to study when $\rho(f_{w,\epsilon})$, viewed as a function of w, is strictly increasing and when it is constant. The next results are aimed in this direction.

Since our goal is to discuss the rotation number in terms of w, in the following propositions we consider that $0 < \epsilon < 1$ is fixed and write $f_w := f_{w,\epsilon}$ and $F_w := F_{w,\epsilon}$.

Proposition 2.37. *$\rho(f_w)$ takes all possible values independently of the fixed $\epsilon \in [0, 1)$.*

Proof. Let $w = 1$. Then $F_1^n(0) = n$ and hence $\rho(f_1) = 1$. Moreover, we have already claimed that $\rho(f_0) = 0$. It follows from the continuity of the rotation number on w, see proposition 2.12, that $\rho(f_w)$ takes all possible values from 0 to 1 independently of the fixed $\epsilon \in (0, 1)$. □

The preceding proposition enables us to consider any possible value of the rotation number, for which there exists at least one member of the standard family with such rotation number. First we discuss the case when $\rho(f_w)$ takes rational values. The prove is found in [Dev03].

Theorem 2.38. **($\rho(f_w)$ *is rationally constant*)** *Let* $0 < \epsilon < 1$. *Suppose* w_0 *is such that* $\rho(f_{w_0}) = p/q$ *is rational, then there exists an interval* $\mathcal{I}$ *containing* w_0 *and with non empty interior s.t.* $\rho(f_w) = p/q \ \forall w \in \mathcal{I}$.

Proof. Recall that there exists x_0 s.t. $F_{w_0}^q(x_0) = x_0 + p \iff \rho(f_{w_0}) = p/q$. Define the function

$$g(w, x) := F_w^q(x) - (x + p).$$

We have $g(w_0, x_0) = 0$ and $DG(w_0, x_0) = (\frac{\partial F_{w_0}^q}{\partial w}|_{w=w_0}, \quad \frac{\partial F_{w_0}^q}{\partial x}|_{x=x_0} - 1)$.

If $\frac{\partial F_{w_0}^q}{\partial x}|_{x=x_0} \neq 1$ the Implicit Function Theorem holds. Hence there is a neighbourhood $\mathcal{I}$ of w_0 and a function $x(w)$ defined on a neighbourhood of x_0 such that $g(w, x(w)) = 0$, meaning that $F_w^q(x(w)) = x(w) + p$, so $\rho(f_w) = p/q$ for all w $\in \mathcal{I}$.

If $\frac{\partial F_{w_0}^q}{\partial x}|_{x=x_0} = 1$ the preceding argument does not hold and it is slightly more complicated. We consider the Taylor expansion since $F_{w_0}^q$ is analytic:

$$F_{w_0}^q(x) - (x + p) = x_0 + p + (x - x_0) + \frac{(F_{w_0}^q)^{(j)}(x_0)}{j!}(x - x_0)^j + ... - (x + p) =$$
$$\frac{(F_{w_0}^q)^{(j)}(x_0)}{j!}(x - x_0)^j + ...$$

where $j \geq 2$ is the order of the first non-vanishing derivative, after $j = 1$, which exists. Otherwise it would be $F_{w_0}^q(x) = x + p$ which is the case $\epsilon = 0$. We need to prove $\mathcal{I} = \{w | \ F_w^q(x) - (x + p) = 0 \text{ for some x}\}$ has no null interior.

Suppose j is even and ${F_{w_0}^q}^{(j)}(x_0) > 0$. First notice that $F_w^q(x_0) - (x_0 + p) < F_{w_0}^q(x_0) - (x_0 + p) = 0$ for $w < w_0$. It follows from the hypothesis on ${F_{w_0}^q}^{(j)}(x_0)$ that there exists x_M such that $F_{w_0}^q(x_M) - (x_M + p) = M > 0$ when x_M is close enough to x_0. Because of continuity on w there exists $\delta > 0$ s.t. $F_w^q(x_M) > 0$ for $w \in (w_0 - \delta, w_0 + \delta)$. Hence for $w \in (w_0 - \delta, w_0)$ we have $F_w^q(x) - (x + p) = 0$ for some x laying within x_0 and x_M because of the Intermediate Value Theorem. That proves $\mathcal{I} \supset (w_0 - \delta, w_0]$. Similarly, $\mathcal{I} \supset [w_0, w_0 + \delta)$ if j is even and ${F_{w_0}^q}^{(j)}(x_0) < 0$ and $\mathcal{I} \supset (w_0 - \delta, w_0 + \delta)$ if j is odd. □

So $\rho(f_w)$ must be increasing only when it takes values in $\mathbb{R} \setminus \mathbb{Q}$, the following theorem tells it is actually increasing when $\rho(f_w)$ is irrational.

Theorem 2.39. **($\rho(f_w)$ *is irrationally increasing*)** *Fix* $\epsilon \in (0, 1)$. *For each irrational* ρ *in* $[0, 1)$ *there is a unique* $w_0 \in [0, 1)$ *such that* $\rho(f_{w_0}) = \rho$. *Equivalently,* $\rho(f_w)$ *is strictly increasing at* w_0.

We follow the proof from [dMvS93], which consists of some preliminary results before the eventual proof of the theorem.

Proposition 2.40. *Let $0 < \epsilon < 1$ and $\alpha > 0$. Then $F^n_{w+\alpha}(x) \geq F^n_w(x) + \alpha \ \forall n \in \mathbb{N}$.*

Proof. By induction. The case $n = 1$ is true as

$$F^n_{w+\alpha}(x) = x + w + \alpha + \frac{\epsilon}{2\pi} sin(2\pi x) = F^n_w(x) + \alpha.$$

Now we suppose the case $n - 1$ holds, then

$$F^n_{w+\alpha}(x) = F_{w+\alpha}(F^{n-1}_{w+\alpha}(x)) \geq F_{w+\alpha}(F^{n-1}_w(x)) = F^n_w(x) + \alpha.$$

□

Corollary 2.41. *Let $0 < \epsilon < 1$ and $\alpha < 0$. Then $F^n_{w+\alpha}(x) \leq F^n_w(x) + \alpha \ \forall n \in \mathbb{N}$.*

Proof. $F^n_{w+\alpha}(x) + (-\alpha) \leq F^n_w(x)$, so 2.40 applies with $\alpha' := -\alpha$ and $w' := w + \alpha$. □

Remark 2.42. Note that f is a C^2 diffeomorphism and therefore it is linearizable when $\rho(f)$ is irrational.

Finally, we enunciate the famous Zorn's Lemma, which will be used in the proof.

Lemma 2.43. ***(Zorn's Lemma)*** *Let P be a partially ordered set for some order relation $(\leq)$. Suppose for every totally ordered subset T of P, then T has an upper bound in P (there exists $M \in P$ such that $x \leq M$ for all $x \in T$). Then P has a maximal element ($k \in P$ is a maximal element of P if there is no greater element than k in P).*

Proof of theorem 2.39. Define $P := \{$closed proper subsets of S^1 invariant by $f_{w_0}\}$ and the order relation $U \geq V \iff U \subset V$. Let T be a totally ordered subset of P, then $U_T := \bigcap_{U \in T} U$ is a superior cote of T. Indeed, T is clearly closed and invariant by f_{w_0}, and it is non-empty because of the Nested Intervals Theorem. It follows from Zorn's Lemma that P has a maximal element, name it K. By construction, every orbit in K is dense in K, otherwise there would be an element greater than K. Besides, $\forall \theta \in K$; $\theta, f_{w_0}(\theta), f^2_{w_0}(\theta), ...$ are all distinct since there are no periodic orbits. Therefore for every θ in K there exists a sequence $\{n_i\}$ of increasing naturals such that $f^{n_i}_{w_0}(\theta) \to \theta$, this is due to f_{w_0} being conjugated to a rotation because of theorem 2.30 so corollary 2.27 implies thT all orbits are dense in S^1. What's more, we can choose subsequences tending to θ strictly from the left and strictly from the right. Therefore, there existS $z_i \in \mathbb{Z}$ such that $\{F^{n_i}_{w_0}(p) - p - z_i\} \to 0$ strictly from the left and strictly from the right where $p := \Pi(\theta)$. We now look at $F_{w_0+\alpha}(p)$ as a function on α. First, we consider the subsequence converging from the left, so

$$F^{n_i}_{w_0}(p) < p + z_i$$

and because of proposition 2.40

$$F^{n_i}_{w_0+(p+z_i-F^{n_i}_{w_0}(p))}(p) \geq F^{n_i}_{w_0}(p) + (p + z_i - F^{n_i}_{w_0}(p)) = p + z_i.$$

Hence the Intermediate Value Theorem states that $\exists \alpha_i \in [0, p + z_i - F^{n_i}_{w_0}(p)]$ such that $F^{n_i}_{w_0+\alpha_i}(p) = p + z_i$ and therefore $\rho(f_{w_0+\alpha_i}) > \rho(f_{w_0})$ because one rotation number is rational and the other is irrational. Observe that $\{\alpha_i\}$ tends to 0 from above, so if $w > w_0$ we can choose α_i s.t. $w_0 < \alpha_i < w$ and therefore $\rho(f_w) \geq \rho(f_{w+\alpha_i}) > \rho(f_{w_0})$. If $w < w_0$ the proof continues by considering a sub sequence $\{n_i\}$ converging from the right so that $\alpha_i < 0$ and tends to 0 and argue the same with help of corollary 2.41. We get $\rho(f_w) \leq \rho(f_{w+\alpha_i}) < \rho(f_{w_0})$. Thus, $\rho(f_w)$ is strictly increasing when it takes irrational values. □

It follows from the previous results that $\rho(w) := \rho(f_{w,\epsilon\in(0,1)})$ is constant on an interval when it takes rational values and increasing when it takes irrational values. This is at least surprising since not only the set of $w's$ such that $\rho(w)$ is irrational (and therefore increasing) is nowhere dense (actually it is a Cantor set), but also its complementary set is everywhere dense, yet $\rho(w)$ manages to increase from 0 to 1 continuously. The graph of $\rho(w)$ is often referred to as a devil's staircase due to the form that it takes, see figure 2.4. Observe that the staircase is symmetric due to the behaviour of the rotation number being the same when considering negative values of w.

Proposition 2.44. *The set $C = \{w |\ \rho(f_w)$ is strictly increasing$\}$ is a Cantor set.*

Proof. It is nowhere dense, for if $w_0 \in C$ and let $(w_0 - \epsilon, w_0 + \epsilon)$ be an arbitrary small environment of w_0, then

$$\rho(f_{w_0+\epsilon/2}) > \rho(f_{w_0}) > \rho(f_{w_0-\epsilon/2}).$$

Hence there is a rational laying within each rotation number because they are dense in [0, 1], implying that there are intervals contained in $(w_0, w_0 + \frac{\epsilon}{2})$ and $(w_0 - \frac{\epsilon}{2}, w_0)$ where $\rho(f_w)$ is constant.

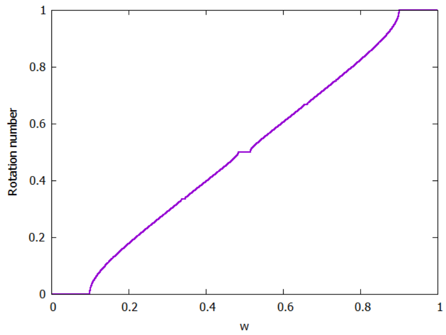
Let w_0 be as before. We want to prove that w_0 is a limit point of C. Suppose it is not, then there exists $\epsilon > 0$ such that $\rho(f_w)$ is constant in $(w_0 - \epsilon, w_0)$ and $(w_0, w_0 + \epsilon)$. Since $\rho(f_{w_0})$ is increasing at w_0 and monotonous in general, again, it is

$$\rho(f_{w_0+\epsilon/2}) = \frac{p}{q} > \rho(f_{w_0}) > \rho(f_{w_0-\epsilon/2}) = \frac{p'}{q'},$$

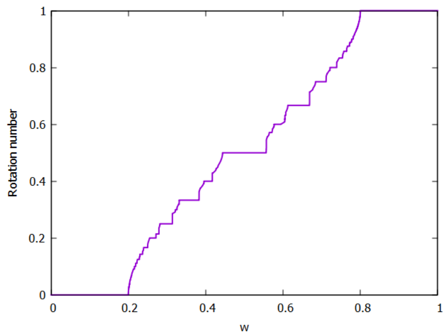
where $\frac{p'}{q'}$, $\frac{p}{q}$ are the constant rational values of $\rho(f_w)$ mentioned before. Therefore, there are infinitely many irrational numbers within $(\frac{p'}{q'}, \frac{p}{q})$, which means that there are infinitely many w in $(w_0 - \epsilon, w_0 + \epsilon)$ belonging to C. That contradicts our assumption and ends the proof. □

It is also interesting to consider the Lebesgue measure of C. Although we do not discuss this problem, it can be proven that C has actually positive Lebesgue measure.

Remark 2.45. Observe the similarities between the rotation number depending on w for the Standard Family and the semiconjugacy constructed in Poincare's theorem 2.21. However, the rotation number case also enables us to consider the rotation number as a function of the 2-dimensional parameter space (w, ϵ), resulting from all devil staircases for different values of ϵ. Also, one can easily compute the rotation numbers for different members of the Standard Family, see the graphs 2.4 and 2.5.



(a) Rotation number of f_w for $\frac{\epsilon}{2\pi} = 0.1$.



(b) Rotation number of f_w for $\frac{\epsilon}{2\pi} = 0.2$ at $x = 0$.

Figure 2.4: Rotation number as a function on w for different members of the Standard Family. The graph is said to be a devil stair-case. Observe that the staircase is more visible for large values of ϵ and it tends to the straight line $y = x$ as $f_{w,\epsilon}$ gets closer to the rotation family t_w. Recall that in (b), since $\epsilon \geq 1$, the rotation number depends on the point.

Let us now discuss the bifurcation diagram of the Standard Family. If we plot the region with $\rho(f_{w,\epsilon}) = p/q$ rational in the parameter space $\epsilon - w$ for $\epsilon \in [0, 1)$, we have that for $\epsilon = 0$ there is a unique $w = p/q$, and as we increase ϵ it gives origin to an area with width equal to the interval of w's given by 2.38, resulting in a tongue-like shape region given by

$$T_\rho := \{(w, \epsilon) \mid \rho(f_{w,\epsilon}) = \rho\}.$$

When $\rho(f_{w,\epsilon})$ is irrational there is a unique w for each ϵ with such a rotation number, so that gives place to curves with no interior. Hence, we refer to T_ρ either as a tongue or as a curve depending on the case. Observe that every point of the parameter space where $f_{w,\epsilon}$ is a diffeomorphism has a unique rotation number associated to it, so any of the regions mentioned do not intersect. Again, this is a striking result for if we consider the horizontal lines $\epsilon = 0$ and $\epsilon > 0$, each $w \in [0, 1) \cap \mathbb{Q}$ expands to an interval of

$w' \in [0, 1)$ defined by $\rho(f_{w',\epsilon}) = \rho(f_{w,0})$, while each $w \in [0, 1) \cup \mathbb{Q}$ identifies bijectively with a unique $w' \in [0, 1)$. But yet the length of the horizontal line remains the same.

The tongue corresponding to $\rho(w, \epsilon) = 0$ is easily calculated. Equivalently, $f_{w,\epsilon}$ must have a fixed point:

$$F_{w,\epsilon}(x) = x \implies \epsilon = -\frac{2\pi w}{sin(2\pi x)}$$

for some x. Therefore, this tongue is bounded by $\epsilon(w) = 2\pi w$ and $\epsilon(w) = -2\pi w$. Since $\rho(f_{w,\epsilon}) = 0$ is an indicator of whether $f_{w,\epsilon}$ has fixed points, the interior of the tongue T_0 contains the subfamily of $f_{w,\epsilon}$ which has 2 fixed points. And its boundary is actually a saddle-node bifurcation, so it contains the sub family with 1 neutral fixed point. Outside this region, $f_{w,\epsilon}$ has no fixed points.

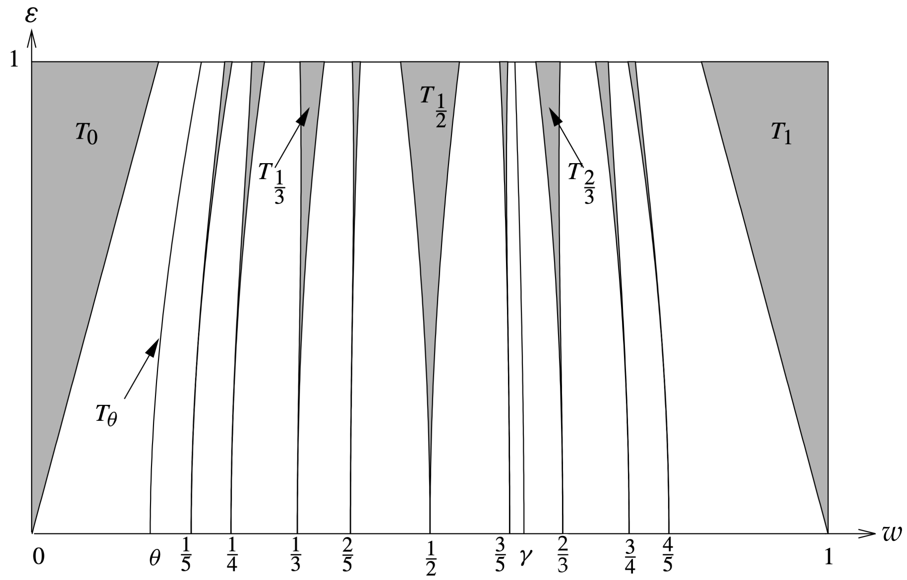


Figure 2.5: Arnold tongues and curves of the Standard Family in the parameter space. Observe that given a fixed ϵ, the width of T_0 for such ϵ is equal to the length of the first step of the devil staircase $\rho(w)_\epsilon$.

Let us discuss the dynamics deeper. Observe the nature of the fixed points of $f_{w,\epsilon} = \{\theta | \, sin(2\pi\theta) = \frac{-2\pi w}{\epsilon}\}$. If $\frac{2\pi|w|}{\epsilon} < 1$ then $f_{w,\epsilon}$ has 1 attractive fixed point and 1 repelling fixed point - since $f'(\theta) = 1 + \epsilon \, cos(2\pi\theta)$. If $\frac{2\pi|w|}{\epsilon} = 1$ it has a non-hyperbolic fixed point at $2\pi\theta = \pi/2$ or $2\pi\theta = -\pi/2$ and lastly if $\frac{2\pi|w|}{\epsilon} > 1$ it does not have any fixed point. Therefore when $-\frac{2\pi w}{\epsilon} = -1$ there is one non-hyperbolic fixed point at $2\pi\theta = \frac{3}{2}\pi$. As $-\frac{2\pi w}{\epsilon}$ goes from -1 to 1 there are 2 fixed points originated from the saddle-node bifurcation $2\pi\theta = \frac{3}{2}\pi$ which race in opposite directions until they meet at $2\pi\theta = \pi/2$ and disappear in a saddle-node bifurcation.

The discussion is quite more complicated for a general rational tongue $T_{p/q}$ since there is no analytic solution of the periodic points equation. However, we see that the behaviour seen for T_0 generalises for all the rational tongues.

Consider $(w, \epsilon) \in T_{p/q}$. We notate x_0 a q-periodic point of $f_{w,\epsilon}$. The following result follows directly from the proof of theorem 2.38.

Corollary 2.46. *We use the notation of theorem 2.38, where $j \geq 2$ is the order of the first non-vanishing derivative of $F^q_{w,\epsilon}$ at x_0:*

- *If $\frac{\partial F^q_{w,\epsilon}}{\partial x}|_{x=x_0} \neq 1$, then (w, ϵ) belongs to the interior of the tongue.*
- *If $\frac{\partial F^q_{w,\epsilon}}{\partial x}|_{x=x_0} = 1$ and j is even, then (w, ϵ) belongs to the boundary of the tongue.*

It remains to see what happens if $\frac{\partial F^q_{w,\epsilon}}{\partial x}|_{x=x_0} = 1$ and j is odd. Actually, we see that we never encounter the mentioned situation.

Proposition 2.47. *If $\frac{\partial F^q_{w,\epsilon}}{\partial x}|_{x=x_0} = 1$, then j is necessarily even. Actually $j = 2$.*

Proof. We omit the parameters in the notation. Observe that:

$$\begin{gathered} F(x) = x + w + \tfrac{\epsilon}{2\pi} sin(2\pi x), \\ F'(x) = 1 + \epsilon cos(2\pi x), \\ F''(x) = -2\pi\epsilon sin(2\pi x), \\ F^{(2n+1)}(x) = K_n\, cos(2\pi x). \end{gathered}$$

On the other hand:

$$\begin{gathered} F^q(x) = x + qw + \tfrac{\epsilon}{2\pi} sin(2\pi x) + ... + \tfrac{\epsilon}{2\pi} sin(2\pi F^{q-1}(x)), \\ (F^q)'(x) = 1 + \epsilon cos(2\pi x) + \epsilon cos(2\pi F(x))F'(x) + ... + \epsilon cos(2\pi F^{q-1}(x))(F^{q-1})'(x), \\ (F^q)'(x) = 1 + \epsilon cos(2\pi x) + \epsilon \textstyle\prod_{i=0}^{0} cos(2\pi F(x))(1 + \epsilon cos(2\pi F^i(x))) + ... + \\ \epsilon \textstyle\prod_{i=0}^{q-2} cos(2\pi F^{q-1}(x))(1 + \epsilon cos(2\pi F^i(x))). \end{gathered}$$

Hence, assume x_0 is such that

$$1 = (F^q)'(x_0) = \prod_{i=0}^{q-1} F'(F^i(x_0)) = \prod_{i=0}^{q-1} (1 + \epsilon cos(2\pi F^i(x_0))).$$

Observe that the derivative calculated by the chain rule and the sum of the derivatives calculated directly are related by:

$$\sum_{i=0}^{q-1} (F^q)'(F^i(x_0)) - (q-1) = \prod_{i=0}^{q-1} (1 + \epsilon cos(2\pi F^i(x_0))) - \prod_{i=0}^{q-1} \epsilon cos(2\pi F^i(x_0)).$$

Therefore:

$$q - (q-1) = 1 - \prod_{i=0}^{q-1} \epsilon cos(2\pi F^i(x_0)),$$

then $\prod_{i=0}^{q-1} \epsilon cos(2\pi F^i(x_0)) = 0$, which implies $cos(2\pi F^k(x_0)) = 0$ for some k. Hence, by the chain rule, $F^{(2n+1)}(x_0) = 0 \iff \exists k$ *such that* $cos(2\pi F^k(x_0)) = 0$. So it is proven that all odd derivatives except the first are 0.

Furthermore, observe that in such case the second derivative does not vanish. Indeed, $F^{(2n)}(x_0) = 0 \iff \exists k$ *such that* $sin(2\pi F^k(x_0)) = 0$. Hence either all even derivatives are 0 or all are distinct than 0. The first case does not happen since then it would be $F^q(x_1) = (x_1 + p) + (x - x_1)$, which is a contradiction. □

Final discussion on the circle dynamics when ρ is rational

Hence, $f_{w,\epsilon}$ has either attracting and repelling cycles or neutral cycles, but it cannot have both types of cycles. Observe that the topological properties of S^1 imply that the number of attracting cycles is equal to the number of repelling cycles. Whenever there exists an attracting (repelling) q-cycle, it must exist a repelling (attracting) q-cycle in between. See figure 2.6.

Until now we have seen the nature of the cycles, but we want to quantify them. A priori, although $\rho = p/q$ assures there is at least one q-cycle, it says nothing about the number of $q-$cycles. We will argue in chapter 4 (using complexification) that every map of the Standard Family has at most one attracting cycle or one neutral cycle. Therefore, dynamics like 2 attracting cycles or 2 neutral cycles are not possible.

To summarize it, there are 2 possibilities, either one, there is exactly one neutral q-cycle, or two, there is exactly one attracting q-cycle and a repelling q-cycle in between. The first case happens at the boundary of the tongue and the second on the interior of the tongue. The neutral cycle is nothing but the attracting and repelling cycle collapsing. So basically the behaviour seen for T_0 generalises for all the rational tongues.

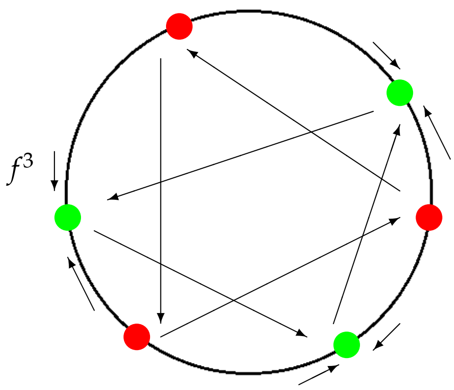


Figure 2.6: Dynamics for $\rho = \frac{1}{3}$. All points of the circle, except the 3 repelling periodic points, are attracted towards an attracting point, in green, under f^3. The inside arrows correspond to f.

Final discussion on the circle dynamics when ρ is irrational

The following discussion is crucial in order to comprehend what the dynamics looks like in a neighbourhood of S^1, when the function is complexified. First of all, recall that because of Poincare's and Denjoy's theorem 2.21 and 2.30, $f_{w,\epsilon}$ is conjugate to a rigid rotation and therefore all orbits are dense in S^1. However, the results to highlight are whether the conjugacy function is analytic or not.

We end the section viewing how theorem 2.36 applies to the Standard Family.

Theorem 2.48. ***(Analytic linearization for the Standard Family, I)*** *Let $\rho \in \mathbb{R} \setminus \mathbb{Q}$ be the rotation number of a member of the Standard Family $f_{w,\epsilon}$. Then*

- *If $\rho \in \mathcal{H}$, then $f_{w,\epsilon}$ is analytically linearizable.*

- *If $\rho \in \mathcal{B} \setminus \mathcal{H}$, then there exists $M(\rho) \in (0, 1]$ such that $f_{w,\epsilon}$ is analytically linearizable for all $0 \leq \epsilon < M(\rho)$.*

- *If $\rho \notin \mathcal{B}$, then $f_{w,\epsilon}$ is not analytically linearizable.*

The next result is due to Herman and a surgical construction in [FG03].

Theorem 2.49. ***(Analytic linearization for the Standard Family, II)*** *Let $\epsilon_0 \in (0, 1)$. Then there exists $\rho \in \mathcal{B} \setminus \mathcal{H}$ such that the maps $\{ f_{w,\epsilon} | (w, \epsilon) \in T_\rho$ and $\epsilon \leq \epsilon_0 \}$ are analytically linearizable and the maps $\{ f_{w,\epsilon} | (w, \epsilon) \in T_\rho$ and $\epsilon > \epsilon_0 \}$ are non-analytically linearizable.*

These theorems are better understood observing the Arnold tongues, see figure 2.5 and 2.7.

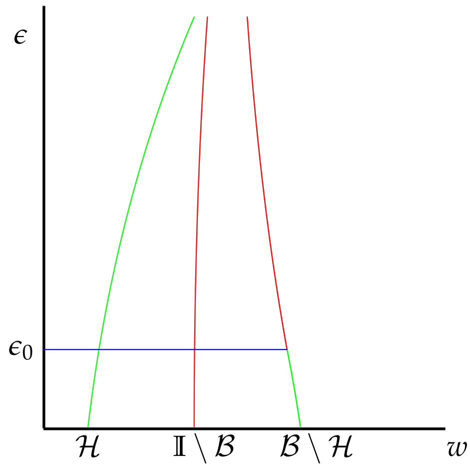


Figure 2.7: Sketch of the analytic linearization for the Standard Family. Green maps are analytically linearizable while red ones are not.

We will see in chapter 4 how those circle maps dynamics generalise to the complex plane.

Chapter 3

Complex Dynamics

Let $S \in \{\mathbb{C}, \hat{\mathbb{C}}\}$, where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere. Holomorphic maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ are rational maps. Otherwise, if $f : \mathbb{C} \to \mathbb{C}$ or $f : \mathbb{C} \to \hat{\mathbb{C}}$ is holomorphic and cannot be extended continuously to infinity, it means that ∞ is an essential singularity, and f is called transcendental. Given a complex function $f : S \to S$, the phase space S splits into two sets: one where the dynamics is well-behaved, there is stability and nearby points have similar orbits, And its compliment, where f is chaotic. Our goal is to characterize such sets. First, we introduce some dynamically important points. Then we see how the function behaves near such points, and we extend such behaviour to a semilocal environment whenever possible. Finally, we discuss the global theory, where we introduce the classification of the stable components theorem and see some of their properties. We warn the reader that several results of this chapter are not proven, since our main goal is to give an overview of the basics of complex dynamics to later apply them to the Standard Family. The general concepts of complex dynamics come from [Mil06], [BH], [CG93] and [Dev03].

3.1 Introduction

Definition 3.1. **(Multiplier)** The multiplier of a p-cycle $\{z_1, z_1, ..., z_p\}$ is defined as $\lambda = f'(z_1)\, f'(z_2) ... f'(z_p) = (f^p)'(z_i)$, which happens to be the same for any $i = 1, ..., n$ because of the chain rule. Note that the multiplier is invariant under conformal conjugacies.

That enables us to classify periodic points.

Definition 3.2. **(Classification of periodic points)** Let $z_1 \in S$ be a periodic point and λ its multiplier, then z_1 is

- attracting if $0 < |\lambda| < 1$.
- superattracting if $|\lambda| = 0$.
- repelling if $|\lambda| > 1$.
- neutral or indifferent if $|\lambda| = 1$. It is rationally indifferent if $\lambda = e^{2\pi i \frac{p}{q}}$, $\frac{p}{q} \in \mathbb{Q}$ and irrationally indifferent otherwise.

Remark 3.3. The point at ∞ is a fixed point by f if and only if 0 is a fixed point by $h \circ f \circ h^{-1}$, where $h(z) := 1/z$. As an example, it is easily proven that $z_1 = \infty$ is a superattracting fixed point for polynomials of degree equal or greater than 2.

Theorem 3.4. ***(About the number of periodic points [Ber93], [Ber02])*** *Any entire transcendental function and meromorphic transcendental function has infinitely many repelling p-periodic points for all $p \geq 2$.*

Other dynamically important points and values are the following ones.

Definition 3.5. (Singular, regular, critical and asymptotic values) Let $f : U \subset S \to \hat{C}$. We say f is regular at v if there exists a neighbourhood V of v such that for all components U of $f^{-1}(V)$, then $f_{|U} : U \to V$ is a homeomorphism. Otherwise, we say f is singular at v.

A point $z \in U$ is said to be a critical point if $f'(z) = 0$. We say its image $f(z)$ is a critical value.

A point v is an asymptotic value for f if there exists a path $z(t)$ terminating at ∞ when $t \to \infty$ such that $lim_{t\to\infty} f(z(t)) = v$.

Remark 3.6. The set of critical values and asymptotic values of f is exactly the set of points where some inverse branch of f cannot be defined. We denote such set by $sing(f^{-1})$.

Example 3.7. Consider the quadratic family $f(z) = z^2 + c$, $c \in \mathbb{C}$. Then $z_1 = 0$ is a critical point, whose critical value is $z_2 = f(0) = c$. Then $z_2 = c$ is a singularity of $f^{-1}(z) = (z-c)^{1/2}$. Indeed, the square root is non well-defined in a neighbourhood of $z = 0$.

Definition 3.8. (Exceptional value) z_1 is an exceptional value for f if its backward orbit is finite.

Example 3.9. $f(z) = e^z$ has asymptotic values at $z = \infty$ (by taking the path through the positive real line terminating at ∞) and $z = 0$ (by taking the path through the negative real line terminating at ∞). These values are also exceptional, since $O^-(\infty) = \{\infty\}$ and $O^-(0) = \emptyset$. Moreover, $z = 0$ is a singular value, since $f_{|U} : U \to V$ cannot be surjective, since $0 \in V$ is omitted. It is well-known that any inverse branch of the exponential in a neighbourhood of $z = 0$ is not well defined.

Theorem 3.10. ***(About the number of exceptional values)*** *Holomorphic and meromorphic functions have at most 2 exceptional values. If f is entire then ∞ is an exceptional value. If f is rational then the exceptional values are in the Fatou set. If f has exactly one pole at z_1, then z_1 and ∞ are the exceptional values.*

Definition 3.11. (Basin of attraction) Let $z_1 \in S$ be an attracting fixed point, its basin of attraction is defined as

$$A_f(z_1) = A(z_1) := \{z \in S |\ f^n(z) \to_n z_1\}.$$

The connected component of $A(z_1)$ which contains z_1 is said to be the immediate basin of attraction of z_1, we note it $A^*(z_1)$. The basin of attraction of the p-cycle $< z_1 >$ is the set

$$A(< z_1 >) := \{z \in S| \ f^{np}(z) \to_n z_i; \ \textit{for some } i = 1, ..., p\}.$$

The immediate basin of attraction is the union of the connected components which contain z_i for $i = 1, ..., p$.

Proposition 3.12. *Let z_1 be a (super) attracting periodic point, then there exists a neighbourhood U of z_1 such that $f^{np}(z) \to_n z_1 \ \forall z \in U$. In particular, $A(z_1) \setminus \{z_1\}$ is not empty.*

Proof. Let λ be the multiplier of $< z_1 >$. The limit definition of the derivative implies that there is a neighbourhood U of z_1 such that $\frac{|f^p(z)-f^p(z_1)|}{|z-z_1|} < |\lambda| + \epsilon$. Taking $\epsilon = 1 - |\lambda| > 0$, it follows that

$$1 > \rho := \frac{|f^p(z) - f^p(z_1)|}{|z - z_1|} = \frac{|f^p(z) - z_1|}{|z - z_1|},$$

then $|f^p(z) - z_1| = |z - z_1|\rho$ and inductively $|f^{np}(z) - z_1| = |z - z_1|\rho^n \to_{n\to\infty} 0, \ \forall z \in U$. □

3.2 Local and semilocal theory of fixed points

Our goal is to describe the behaviour of f near periodic points. We distinguish the attracting or repelling case, the superattracting case, the rationally indifferent case and the irrationally indifferent case. The idea is that there might exist a neighbourhood of the periodic points where f^p is conjugate to a simple map such as the linear map or a monomial. Then we proceed to extend such conjugacy to a larger neighbourhood. We also remark the relevance of critical points in such semilocal neighbourhoods. The indifferent case is somewhat different and linearization is not assured. Actually, the rationally indifferent case is never linearizable.

Throughout this section we assume z_1 is a periodic point by f of period p, that is $f^p(z_1) = z_1$, where f is a holomorphic function at least locally. Observe that then 0 is a p-periodic point by the map $f(z) - z_1$, indeed $f^p(0) - z_1 = z_1 - z_1 = 0$, so we may assume that the periodic point being considered is 0 without loss of generality. Furthermore, since there is an equivalence between the p-periodic points of f and the fixed points of f^p, it is enough to study the local dynamics around fixed points. Taking into consideration these observations, we write

$$f(z) = f(0) + f'(0)z + \frac{(f)^{(2)}(0)}{2}(z-0)^2 + ... = \lambda z + a_2 z^2 + ...$$

as the expansion in series of f in a neighbourhood around the fixed point $z_1 = 0$, where λ is the multiplier of $z_1 = 0$.

Finally, we remind the reader that the detailed discussion of local and semilocal dynamics is found in [BH] and [Mil06].

Attracting and repelling fixed points

We begin discussing attracting and repelling fixed points, i.e. the multiplier $|\lambda| \notin \{0, 1\}$. We see this case thoroughly and then we comment on the others with less detail.

Theorem 3.13. ***(Koenigs' linearization theorem)*** *Let z_1 be a fixed point with $|\lambda| \notin \{0, 1\}$. Then f is conformally conjugate to the linear map $g = \lambda z$, i.e. there exists a conformal map $\Phi(z)$ (from a neighbourhood of z_1 onto a neighbourhood of 0 with $\Phi(z_1) = 0$) such that $\Phi \circ f = g \circ \Phi$. Moreover, Φ is unique up to multiplication.*

Observe that the inverse function exists at least locally around z_1 as long as z_1 is not superattracting, if and only if its derivative does not vanish. Thus, the Inverse function theorem holds and $(f^{-1})'(z_1) = (f^{-1})'(f(z_1)) = \frac{1}{f'(z_1)}$. Then z_1 is attracting (repelling) by $f \iff z_1$ is repelling (attracting) under f^{-1}. What's more, we can consider a neighbourhood U of z_1 where $f_{|U} : U \to f(U)$ is a bijective map between neighbourhoods U and $f(U)$ of z_1. Therefore, we can study the dynamics of a repelling point z_1 by f as an attracting point by $f^{-1} : f(U) \to U$. In particular, there is a local conformal conjugacy between f and $\lambda z \iff f^{-1}$ is local and is conformally conjugate to $\frac{1}{\lambda}z$.

That being said, the proof reduces to the case where z_1 is an attracting fixed point located at $z_1 = 0$.

Proof. **Existence.** Recall we assume $z_1 = 0$ is an attracting fixed point.

We define $\Phi_n(z) := \frac{1}{\lambda^n} f^n(z)$, observe that $\Phi_n \circ f = \frac{1}{\lambda^n} f^{n+1}(z) = \lambda \Phi_{n+1}$. We claim that $(\Phi_n)_n$ converges uniformly to some Φ. Taking limits it yields $\Phi \circ f = g \circ \Phi$. It remains to be proved that Φ exists and is conformal.

Note that

$$f(z) = \lambda z + a_2 z^2 + ... \implies \quad \exists \delta > 0 : |f(z) - \lambda z| \leq C|z|^2, \ |z| < \delta.$$

Then $|f(z)| \leq |\lambda||z| + C|z|^2 \leq (|\lambda| + C|z|)\,|z| < (|\lambda| + C\delta)\,|z|$. And taking $\delta' = min\big(\frac{1-|\lambda|}{C}, \delta\big)$ so that all iterates of z stay inside $B(0, \delta')$, we get inductively that $|f^n(z)| < (|\lambda| + C\delta')^n\,|z|$, for $|z| < \delta'$.

Hence

$$|\Phi_{n+1}(z) - \Phi_n(z)| = \Big|\frac{f(f^n(z)) - \lambda f^n(z)}{\lambda^n}\Big| \leq \frac{C|f^n(z)|^2}{|\lambda|^{n+1}} \leq \frac{C\,(|\lambda| + C\delta')^{2n}\,|z|^2}{|\lambda|^{n+1}}.$$

Now we take $0 < \delta^* < min\big(\frac{\sqrt{|\lambda|} - |\lambda|}{C}, \delta'\big)$ so that $\rho := \frac{(|\lambda| + C\delta^*)^2}{|\lambda|} < 1$. Therefore, let $m > n$, we have

$$|\Phi_m(z) - \Phi_n(z)| < \sum_{i=n}^{m-1} \frac{C|z|^2}{|\lambda|}\rho^i = \frac{C\delta^{*2}}{|\lambda|} \sum_{i=n}^{m-1} \rho^i < \frac{C\delta^{*2}}{|\lambda|}(m-n)\rho^n, \quad |z| < \delta^*,$$

then $(\Phi_n)_n$ is Cauchy uniformly and therefore it converges uniformly.

Finally, observe that Φ is holomorphic since Φ_n is holomorphic $\forall n > 0$, what's more, $\Phi_n' \rightrightarrows \Phi'$. Using that and $f(z) = \lambda z + a_2 z^2 + ... \implies f^n(z) = \lambda^n z + (\lambda^{n-1} a_2 + ...)z^2 +$

... $\implies \Phi_n(z) = z + O(z^2) \implies \Phi'_n(z) = 1 + O(z)$. So the Inverse function theorem assures us Φ is indeed a conformal map in a neighbourhood of 0.

Uniqueness. Let Ψ be another such map, then $\Psi \circ \Phi^{-1}$ is a conformal map, and $(\Psi \circ \Phi^{-1})(0) = 0$, so $(\Psi \circ \Phi^{-1})(z) = b_1 z + b_2 z^2 + b_3 z^3 + ... \neq 0$.

$$\Phi^{-1} \circ g \circ \Phi = f = \Psi^{-1} \circ g \circ \Psi \implies \Psi \circ \Phi^{-1} \circ g = g \circ \Psi \circ \Phi^{-1}.$$

It follows

$$\sum_{i>0} b_i \lambda^i z^i = \sum_{i>0} \lambda b_i z^i \implies b_i = 0,\ i \geq 2\ (\textit{Recall } \lambda \notin \{0, 1\}).$$

Now $(\Psi \circ \Phi^{-1})(z) = b_1 z \implies \Psi = b_1 \Phi, \quad b_1 \neq 0.$ □

Remark 3.14. The method used for the construction of the conjugacy is general. That is, f is conjugate to g if and only if $(\Phi_n)_n := (g^{-n} f^n)_n$ converges uniformly to some Φ. And in that case, Φ is the conjugacy.

Corollary 3.15. ***About the periodic case.*** *Let $< z_1 >$ be a p-cycle with $|\lambda| \in (0, 1)$ and U_i the neighbourhoods of z_i obtained in Koenigs' theorem. Let z be such that $z \in U_1$, $f(z) \in U_2, ..., f^{p-1}(z) \in U_p$, then all iterates of z: $f^{i-1+kp}(z),\ i = 1, ..., p$ lay inside U_i respectively and each is being attracted to the respective periodic point z_i linearly with constant λ after p iterations are completed.*

The next question is whether the conformal conjugacy found in theoem 3.13 can be extended to a more global environment.

Remark 3.16. Extension of the conjugacy to the basin of attraction. We aim to analytically extend the local conjugacy from the neighbourhood U of theorem 3.13 to a semilocal neigbourhood of z_1, namely, to the whole basin of attraction $A(z_1)$. Let $z \in A(z_1)$, and consider $f^n(z)$ the first iterate belonging to the neighbourhood of z_1 defined in Koenig's theorem. We define

$$\Phi^*(z) := \frac{\Phi(f^n(z))}{\lambda^n},$$

which is a composition of analytic (holomorphic) maps. Then $\Phi^*(z) := \frac{\Phi(f^0(z))}{\lambda^0} = \Phi(z)$, so both definitions coincide inside the neighbourhood and

$$\Phi^*(f(z)) := \frac{\Phi(f^n(z))}{\lambda^{n-1}} = \frac{\lambda\Phi\{f^{-1}(f^n(z))\}}{\lambda^{n-1}} = \frac{\lambda\Phi^*(f^{n-1}(z))}{\lambda^{n-1}} := \frac{\lambda\frac{\Phi(f^n(z))}{\lambda}}{\lambda^{n-1}}$$
$$= \frac{\lambda\Phi(f^n(z))}{\lambda^n} =: \lambda\Phi^*(z).$$

So f is analytically conjugate to the linear map λz on $A(z_1)$ by the conjugacy Φ^*.

Observe that Φ^* is not injective in general, though, as there might be points in $A(z_1)$ whose iterates coincide up to certain point. Moreover, for points that are infinitely close to the boundary of $A(z_1)$, it takes infinitely many iterations to lay inside U. Therefore, $\Phi^* : A(z_1) \to \mathbb{C}$ is holomorphic and onto, but not necessarily injective.

Theorem 3.17. ***(Fatou-Julia theorem)*** *The immediate basin of attraction of any attracting fixed point z_1 contains at least one critical point or asymptotic value.*

Proof. We prove it by contradiction. Let $\lambda \in B(0, 1), \lambda \neq 0$ be the multiplier of z_1. Suppose there are no critical or asymptotic values. Let U_0 be a simply connected neighbourhood of z_1 inside the neighbourhood defined by the Koenings' theorem and suppose that it does not contain a critical point or asymptotic value. Then f is regular there, so there exists a bijective inverse branch $g : U_0 \to U_1$ with $U_0 \subset U_1$ and U_1 is simply connected. Inductively, there exists a bijective inverse branch $g_n : U_0 \to U_n$, with $U_0 \subset U_n$ for all $n > 0$ provided that there are no critical points or asymptotic values in U_n for all $n > 0$. Furthermore, observe that by construction U_n always lay inside the immediate basin of attraction and note that f^n is a holomorphic bijection between U_n and U_0, whose inverse is g_n.

We will see later in proposition 3.36 that $A(z_1) \subseteq F_f$ omits infinitely many points, hence, since $\cup_{n>0} g_n(U_0) \subseteq A(z_1)$, $(g_n)_n$ is normal by Montel's theorem. However, this is already a contradiction, since $z_1 \in U_0$ is a repelling point by g, and all functions fail to be normal in a neighbourhood of a repelling point, see proposition 3.36.

□

Superattracting fixed points

Next we see the case of superattracting fixed points, here we omit the proof, nonetheless it is found in [CG93].

Theorem 3.18. ***(Böttcher theorem)*** *Let z_1 be a superattracting fixed point under f (then $f(z) = z_1 + a_p(z - z_1)^p + ...,\ a_p \neq 0,\ p > 1$). Then, f is conformally conjugate to z^p in a neighbourhood of z_1 onto a neighbourhood of 0. Furthermore, the conjugating map is unique up to multiplication by a $(p-1)$th root of unity.*

Remark 3.19. The Böttcher conjugacy Φ is such that it satisfies the functional equation $\Phi(f(z)) = \Phi(z)^p$, which can be extended, as we did with attracting fixed points. The extension of the conjugacy remains analytic except in the set $\{z | f(z) = z_1\}$. Finally, the conjugacy holds on the whole basin of attraction if and only if there are no other critical points apart from z_1, in that case f is conjugate to a monomial from the immediate basin of attraction onto the unit disk. This phenomenon is discussed in [Mil06].

Theorem 3.20. *The immediate basin of a super attracting fixed point z_1 contains at least one critical point, which is z_1 itself.*

Rationally indifferent fixed points

We assume again that the fixed point is $z_1 = 0$, and $\lambda = 1$. Then $f(z) = z + a_n z^n + 0(z^{n+1});\quad a_{n+1} \neq 0$, where n is the multiplicity of the fixed point.

Observe that f is a local bijection from a neighbourhood U of 0 into a neighbourhood $f(U)$ of 0, exactly as what happens with attracting fixed points since $\lambda \neq 0$.

Definition 3.21. (Parabolic point) $z \in S$ is said to be a parabolic point if $f'(z)$ is a root of the unity.

Definition 3.22. (Petal) Let P be an open connected set such that $\bar{P} \in U \cap f(U)$. P is said to be an attracting petal at 0 for f if

$$f(\bar{P}) \subset P \cup \{0\} \quad \textit{and} \quad f^n(z) \to 0 \quad \forall z \in P.$$

We say P is a repelling petal if it is an attracting petal for f^{-1}.

The following theorem tells us what the dynamics looks like around rationally indifferent fixed points.

Theorem 3.23. ***(Leau-Fatou Flower Theorem)*** *Let 0 be a fixed point for f with multiplicity $n \geq 2$, such that $\lambda = 1$. Then there are exactly $(n-1)$ disjoint attracting petals P_i and $(n-1)$ disjoint repelling petals P_i' for f, wrapping 0 such that attracting petals alternate with repelling petals P_1, P_1', P_2, P_2',.... Moreover, $(\cup_i P_i) \cup (\cup_i P_i') \cup \{0\}$ is a neighbourhood of 0.*

Now we discuss the theorem for a general λ. Notice that if λ is a m^{th} root of the unity, then $f^m(z) = z + az^l + ...$ reduces to the $\lambda = 1$ case.

Theorem 3.24. ***(Leau-Fatou Flower Theorem for*** $\lambda \neq 1$***)*** *Let 0 be a fixed point for f such that $\lambda^m = 1$, $\lambda^k \neq 1$ $k < m$. Then f admits $m(l-1)$ attracting petals and $m(l-1)$ repelling petals, in the sense that these petals are not fixed by f, but by f^m.*

Remark 3.25. It is evident that f does not conjugate onto a linear map in any neighbourhood of a rationally indifferent fixed point, since any neighbourhood contains points which are attracted and points which are repelled. Also observe that the rationally indifferent fixed point lays in the boundary of all petals.

Remark 3.26. The attracting petals can be extended to Leau domains, also called parabolic basins. More information about these domains is found in [Ber00].

Theorem 3.27. *Leau domains contain at least one critical point.*

Irrationally indifferent fixed points

We end the local theory of fixed points enunciating some theorems concerning irrationally indifferent fixed points. They are found in [BH] and were introduced in [Sie42], [Bry71] and [Yoc95]. We denote $\lambda = e^{2\pi i\theta}$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and as we might already expect, the arithmetic properties of θ play an important role.

Theorem 3.28. *There exists $\lambda = e^{2\pi i\theta}$ such that all polynomials with an irrationally indifferent fixed point z_1 with multiplier λ have no local conjugacy around z_1.*

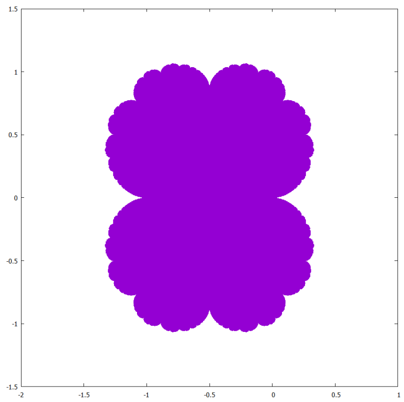


Figure 3.1: Representation of the leau domain for the map $f(z) = z^2 + z$ in the complex plane. Because of the previous discussion, it has exactly 1 attracting and repelling petal around the only fixed point $z_1 = 0$. Observe that the critical point $z = -\frac{1}{2}$ lays inside the attracting petal. Because there are not more critical points, this map does not have attracting basins nor other leau domains. It is easily proven that the positive inverse branch (the map has degree 2), $f^{-1}(z) := \frac{-1+\sqrt{1+4z}}{2}$, attracts a neighbourhood of the positive real line. Hence, the repelling petal of f contains the positive real line, and therefore it is unbounded.

Theorem 3.29. ***(Siegel theorem)*** *Let f have a fixed point at $z_1 = 0$ with multiplier $\lambda = e^{2\pi i\theta}$, then $f(z) = \lambda z + a_2 z^2 + ...$ If $\theta \in \mathcal{D}$, then f is conformally conjugate to the irrational rotation λz in a neighbourhood of 0. We say f is linearizable around z_1.*

Actually the sharp condition of the Siegel theorem is known.

Theorem 3.30. ***(Yoccoz theorem)*** *If $\theta \in \mathcal{B}$, then f is linearizable.*

We comment that in general whether f is linearizable or not is an open problem for $\theta \notin \mathcal{B}$.

Theorem 3.31. *Let f have an irrationally indifferent fixed point at z_1. Then f is linearizable around z_1 if and only if $\{f^n\}_n$ is normal in a neighbourhood of z_1.*

Remark 3.32. Note the similarities between the analytic linearization problem for circle maps discussed in chapter 2 and these results.

Definition 3.33. **(Siegel and Cremer points)** An irrationally indifferent fixed point with multiplier $\lambda = e^{2\pi i\theta}$ is said to be a Siegel point if it is locally conjugate to the irrational rotation λz. In that case, the Siegel disk of such point is the maximal neighbourhood where f is conjugate to λz. Otherwise we say it is a Cremer point.

Observe that whenever there exists linearization, every orbit inside the Siegel disk is dense in a Jordan curve around the fixed point, due to Jacobi's theorem.

Theorem 3.34. *The boundary of a Siegel disk contains a critical point or an asymptotic value.*

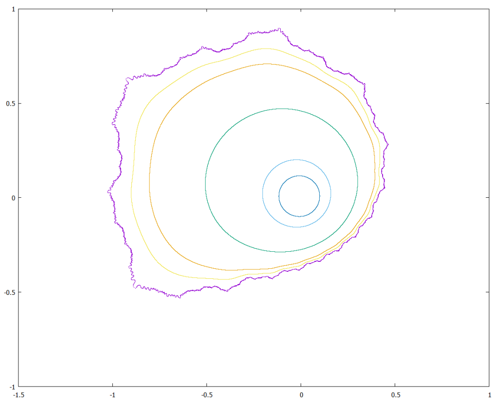


Figure 3.2: Representation of the Siegel disk for the map $f(z) = e^{0.5i}z + \frac{1}{2}z^2$. $z_1 = 0$ is a Siegel fixed point. The domain of conjugacy to a rotation, i.e., the Siegel disk, is limited by the orbit of the only critical point.

3.3 Global theory

Once we have seen the dynamics around periodic points locally, we discuss them from a global point of view. We enunciate some results which connect the local theory done with the Julia and Fatou sets, so that we can enunciate the Classification of Fatou components theorem. We also pay attention to its variations depending on the type of map taken into consideration. We follow [Ber93], in addition to the references mentioned before, for the classification theorem of Fatou components.

Definition 3.35. (Fatou and Julia sets) We define the Fatou set of f as

$$F_f := \{z \in S | \ (f^n)_n \text{ is normal in a neighbourhood of z}\}.$$

Convergence on compact subsets to ∞ is also accepted as normality. We define the Julia set of f as

$$J_f := S \setminus F_f.$$

We list some general properties of the Fatou and Julia set. Some follow immediately from the definition and others connect with the local theory made in the previous section.

Proposition 3.36. ***(Properties of J and F)*** *We denote a holomorphic or meromorphic function by f, a rational function of $deg(f) \geq 2$ by R and a cycle under f by $< z_1 >$.*

1. ***Blow-up property.*** *Let $z \in J_f$ and U be any neighbourhood of z, then $\hat{\mathbb{C}} \setminus \{a, b\} \subseteq \cup_{n\geq 0} f^n(U)$. This property gives an idea of the "chaotic behaviour" in J_f.*

2. *Both F_f and J_f are completely invariant.*

3. *F_f is open and therefore J_f is closed.*

4. *$F_{f^p} = F_f$ (and $J_{f^p} = J_f$) $\forall p > 0$.*

5. *J_f has infinitely many points.*

6. *Attracting basins, attracting leau domains and Siegel disks are contained in* F_f*. What's more,* $J_f = \partial A(< z_1 >)$ *for any attracting cycle.*

7. *If* z_1 *is repelling, parabolic or a Cremer periodic point, then* $z_1 \in J_f$*.*

8. *Either* $Int(J_f) = \emptyset$ *or* $J_f = \hat{\mathbb{C}}$*.*

9. J_f *is a perfect set.*

10. *Repelling periodic points are dense in* J_f*, and thus the closure of repelling periodic points is equal to* J_f*.*

11. *If* $z_1 \in J_f$ *and is not exceptional, then* $J = \overline{O^-(z_1)}$*.*

Proof. We prove some statements.

1. Suppose $\cup_{n\geq 0} f^n(U)$ omits at least three values. Then Montel's theorem implies $(f^n)_n$ is normal in U, which contradicts $z \in J_f$.

4. $F_{f^p} \subset F_f$ is trivial. Let $z \in F_f$, then $\exists n_k$ such that f^{n_k} u.c.c. on a neighbourhood of z to g. If we split n_k into p subsequences each containing only numbers equal to $j = 0, 1, 2 ..., p-1$ $(mod\, p)$, at least one has infinitely many numbers, say $n_{kj} + j$ is that subsequence. Then $(f^p)^{n_{kj}} = f^{pn_{kj}+j-j} = (f^{pn_{kj}+j})(f^{-j})$ u.c.c. to $g \circ f^{-j}$.

6. A bounded attracting domain is bounded, therefore it omits infinitely many points and it is normal by Montel's theorem. Let $z \in J_f$ and U a neighbourhood of z_1, then $\cup_{n\geq 0} f^n(U) \supseteq \hat{\mathbf{C}} \setminus \{a, b\} \implies \exists n,\ f^n(U) \cap A(< z_1 >) \neq \emptyset \implies z \in \partial A(< z_1 >)$. If $z_1 \in \partial A(< z_1 >)$ then every neighbourhood of z_1 is not normal, for if it was then $(f^{n_k}) \to z_1$ in U but $f^{n_k}(z) \not\to z_1$ for all $z \in \partial A(< z_1 >)$. Hence $z_1 \in J_f$.

7. Suppose a repelling fixed point z_1 is normal, then $(f^{n_k})_{n_k} \rightrightarrows g$, and $(f^{n_k})'_{n_k} \rightrightarrows g'$, but that is impossible since $(f^{n_k})'(z_1) = \lambda^{n_k} \to \infty$. If z_1 is a parabolic point then any neighbourhood U of it intersects an attracting petal and a repelling petal. Suppose $f^n \rightrightarrows g$, then $g = z_1$ in the attracting petal, so $g = z_1$ in U, contradicting the existence of a repelling petal. Finally, Cremer points being in J_f is trivial from the definition.

10. We prove it only for rational functions of degree larger than 2. Let $z \in J_f$ and not exceptional, and U a neighbourhood of z. We need to prove that U contains a repelling periodic point. We consider Φ_0, Φ_1 2 distinct well-defined branches of f^{-1}. We define

$$g_n(w) := \frac{(f^n(w) - \Phi_0(w))(w - \Phi_1(w))}{(f^n(w) - \Phi_1(w))(w - \Phi_0(w))}.$$

Observe that $g_n(w) = 0$, $g_n(w) = \infty \iff f^{n+1}(w) = w$ and $g_n(w) = 1 \iff f^n(w) = w$. Hence if U has no periodic points $\implies g_n$ is normal in U (by Montel's theorem) $\implies f^n$ is normal in U, which contradicts $z \in J_f$.

□

Lemma 3.37. *Let* $z \in J_f$*, U be a neighbourhood of z and D be an arbitrary bounded set. Then there is no sub sequence of the iterates* n_k *such that* $\cup_{n_k} f^{n_k}(U) \subset D$*.*

Definition 3.38. (Fatou components) A Fatou component is said to be a maximal connected component of F_f.

Remark 3.39. (About the dynamics of Fatou components) Let U be a non-wandering Fatou component, we assume f is holomorphic in U. Then $f(U)$ lays within another Fatou component V because F_f is completely invariant, and $f(\partial U) \subset J_f$. Because of the open mapping theorem, $\partial f(U) \subset f(\partial U)$ provided that ∂U is bounded. Therefore it is $f(U) = V$

Actually $f(U) = V$ if f is rational. If f is transcendental the equality does not necessarily hold. However, the elements of the set $V \setminus f(U)$ are omitted asymptotic values of f.

Definition 3.40. (Fatou components) Fatou components can can only be

- periodic, if $f^p(U) = U,\ \exists p > 0$.
- preperiodic, if $f^k(U)$ is periodic for some $k > 0$.
- wandering, if $f^n(U) \cap f^m(U) = \emptyset,\ \forall n,\, m > 0,\ n \neq m$.

Furthermore, all components of a cycle of Fatou components are necessarily of the same type, as Fatou components are a generalized version of the local theory seen in the previous section (where we saw that the multiplier of a cycle is the same for all points) and simply connected sets map onto simply connected sets.

Theorem 3.41. ***Classification of Fatou components.*** *Let f be a rational, entire or meromorphic transcendental map and U a periodic Fatou component for $f : \mathbb{C} \to \hat{\mathbb{C}}$. Then*

- *U is an **attracting basin**; i.e., there exists an (super) attracting periodic point $z_1 \in U$ such that $f^{np}(z) \to z_1\ \forall z \in U$.*
- *U is a **parabolic basin**; i.e., there exists a parabolic periodic point $z_1 \in \partial U$ such that $f^{np}(z) \to z_1\ \forall z \in U$.*
- *U is a **Siegel disk**; i.e., there exists an irrational indifferent periodic point $z_1 \in U$ and f^p is conformally conjugate to an irrational rotation $ze^{2\pi i\theta}$ in U.*
- *U is a **Herman ring**; i.e., U is 2-connected and f^p is conformally conjugate to an irrational rotation $ze^{2\pi i\theta}$ of the standard annulus $A_r = A(0, r < 1, 1) := \{z \in \mathbb{C} |\ r < |z| < 1\}$ in U.*
- *U is **Baker domain**; i.e., $\exists z_1 \in \partial U$ such that $f^{np}(z) \to z_1$ for all $z \in U$, but $f^p(z_1)$ is not defined. z_1 must be an exceptional point.*

Some types of Fatou components require some particular properties for f, so not all functions are expected to have all cyclic Fatou components, neither wandering domains. The next results discuss when and when not it is possible to have them. We distinguish between rational, entire transcendental, and meromorphic transcendental maps.

First we see which maps may have or have not wandering domains and/or Baker domains. The following result is due to [Sul82].

Theorem 3.42. ***(Sullivan)*** *A rational map has no wandering domains.*

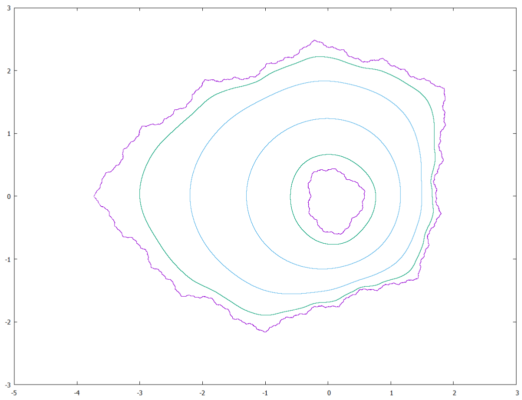


Figure 3.3: Example of a Herman ring for a map of the Standard family.

This last theorem actually extends to other functions.

Theorem 3.43. ***(Extension of Sullivan's theorem)*** *Let $p(z)$ be a polynomial, $r(z)$ a rational map and $\tau \in \mathbb{C}$. We notate $S := \{f \mid sing(f^{-1})$ is a finite set$\}$, $F := \{f(z) = z + r(z)e^{p(z)}\}$, $N := \{f \mid$ the poles of f have finite order and $f'(z) = r(z)e^{p(z)}(f(z) - z)\}$ and $R := \{f'(z) = r(z)(f(z) - z)^2$ or $f'(z) = r(z)(f(z) - z)(f(z) - \tau)\}$. Functions in S, F, N and R do not have wandering domains.*

Remark 3.44. By definition, rational maps cannot have Baker domains. Actually, Baker domains can only be found at $z_1 = \infty$ if f is entire transcendental (and at the poles if it is meromorphic transcendental).

Theorem 3.45. ***(Maps with no Baker domains.)*** *If $f \in S$ as above or f is entire and the set $sing(f^{-1})$ is bounded, then f has no Baker domains.*

Theorem 3.46. *If $f \in S$, then f has at most 2 completely invariant domains.*

Theorem 3.47. ***(About entire functions)*** *If f is a polynomial, then all bounded Fatou components are simply connected. If f is an entire transcendental function and let U be a Fatou component which is not simply connected, then U is wandering.*

Proof. We prove the first statement. In particular, f is an entire rational map. Let γ be a closed curve in a Fatou component U. Because of Sullivan's theorem, U eventually lays in a cycle of Fatou components. We assume all images of U and all components of the eventual cycle are bounded. Hence $\cup_{n>0}|f^n(z)| < M$, $\forall z \in U \implies \cup_{n>0}|f^n(z)| < M$ $\forall z \in \gamma \implies \cup_{n>0}|f^n(z)| < M$ $\forall z \in Int(\gamma)$, where the last implication is a consequence of the maximum modulus principle. Observe that the sum on n is finite since U eventually lays in a cycle. Now Montel's theorem assures $Int(\gamma) \subset F_f$, in particular $Int(\gamma) \subset U$, so U is indeed simply connected.

To prove our assumption, observe that because of the maximum modulus principle f is bounded $\forall z \in U$ since U is bounded and f entire. More concretely, if $\exists z \neq \infty$ such that $f(z) = \infty$, then $f := \infty$. Therefore, $f(U)$ is bounded. $\square$

Corollary 3.48. *If f is entire then no Fatou component of f is a Herman ring.*

In the following theorem we summarize the results commented in [Ber93]. They have also been discussed in the local theory section and they remark the importance of critical points.

Theorem 3.49. ***About singularities and Fatou components.*** *Let f be a meromorphic function and $C = \{U_1, U_2, ..., U_p\}$ a p-cycle of Fatou components. Recall that the set of singularities of f^{-1} is the set of critical and finite asymptotic values of f and their limit points. We notate $sing(f^{-1})$.*

- *If C is a cycle of attracting or parabolic domains, then there exists $j \in \{1, ..., p\}$ such that $U_j \cap sing(f^{-1}) \neq \emptyset$.*
- *If C is a cycle of Siegel disks or Herman rings, then $\partial U_j \subset \overline{O^+(sing(f^{-1}))}$ $\forall j = 1, ..., p$.*
- *Baker domains need no singularities.*

One can find more about the connection between Baker domains and singularities in the reference mentioned.

At this point, one could wonder if these singularities really determine the number of Fatou cycles. It is evident that attracting and parabolic domains cannot share a common singularity. However, the same cannot be said about Herman rings nor Siegel disks a priori. But this is actually the case, the famous **Fatou-Shishikura inequality** [Shi87] resolves that.

Theorem 3.50. ***(Fatou-Shishikura inequality)*** *If $\#sing(f^{-1})$ is finite, then:*

$$\#\{attracting\ cycles\} + \#\{parabolic\ cycles\} + \#\{Cremer\ cycles\} + \#\{Siegel\ disks\} +$$
$$+2\#\{Herman\ rings\} \leq \#sing(f^{-1}).$$

Moreover, if f is rational, then $\#sing(f^{-1}) \leq 2deg(f) - 2$

Corollary 3.51. ***(Corollary of the Fatou-Shishikura inequality proof)*** *If $C = \{U_1, U_2, ..., U_p\}$ is a cycle of Siegel disks and $\partial U_j \subset \overline{O^+(c)}$ $\forall j = 1, ..., p$, for some $c \in sing(f^{-1})$. Then $O^+(c)$ does not accumulate anywhere else.*

We do not prove it. The idea of the proof, however, is that a sufficiently small change in the function transforms the Siegel point into an attracting point without changing the orbits of the critical points. Hence, the orbit of the critical point cannot lay far from the boundary of the Siegel disk. We will use this result later.

Chapter 4

Complexification of the Standard Family

Our goal in this chapter is to analyse the complex dynamics of analytic circle maps $f : S^1 \to S^1$, that is, when extended to holomorphic maps $\tilde{f}$ defined at least on a neighbourhood of S^1, and such that $\tilde{f}_{|S^1} = f$. We will combine the results in Chapter 2 and 3 to give a description of the complex phase space. We will mainly focus on the Standard Family, described in section 2.5, in order to view how the properties of the Arnold Family affect the complex dynamics of the complexified version, and vice versa. Throughout the discussion we follow [Fag99] and [FG03].

4.1 Complexification of analytic circle maps

We denote a circle map by f, a real map by F and a complex map by $\tilde{F}$. First, we discuss the complexification of real functions, namely, functions satisfying the lift condition $F(x+1) = F(x) + k$, and then we apply our discussion to circle maps.

Proposition 4.1. ***(Analytic extension*** $\tilde{F}$ ***of an analytic real function*** F***)*** *Let* F *be an analytic real function, then there exists a complex function* $\tilde{F}$ *which is analytic in a neighbourhood* U *of the real line and* $\tilde{F}_{|\mathbb{R}} = F$.

Proof. Since F is analytic, $F_{x_0}(x) := \sum_{n\geq 0} a_{n,x_0}(x-x_0)^n$ converges in $(x_0 - R_{x_0},\, x_0 + R_{x_0})$, $R_{x_0} > 0$, for all $x_0 \in \mathbb{R}$. Hence, the complex series $\tilde{F}_{x_0}(z) := \sum_{n\geq 0} a_{n,x_0}(z-x_0)^n$ converges in $D(x_0,\, R(x_0))$ for all $x_0 \in \mathbb{R}$. Then, $\tilde{F}(z) := \tilde{F}_{x_0}(z)$ if $z \in D(x_0,\, R(x_0))$ is well defined in $\cup_{x_0\in\mathbb{R}} D(x_0,\, R(x_0))$. This is due to the fact that if $z_0 \in D(x_0,\, R(x_0)) \cap D(x_1,\, R(x_1))$, then $\tilde{F}_{x_0}(z_0) = \tilde{F}_{x_1}(z_0) = \sum_{n\geq 0} a_{n,z_0}(z-z_0)^n$. By definition $\tilde{F}_{|\mathbb{R}} = F$. $\square$

Corollary 4.2. *A map* $\tilde{F}$ *defined in a neighbourhood of* $\mathbb{R}$ *is an analytic complexification of a real function if and only if* $\tilde{F}$ *can be expressed as a series with real coefficients.*

Corollary 4.3. *If* F *is periodic, then the domain of definition of the complexified map* $\tilde{F}$ *contains* $\mathbb{R} \times (-\delta,\, \delta)$, *for some* $\delta > 0$.

Proof. Note that we think of $\mathbb{C}$ as $\mathbb{R}^2$. Let p be the period. First observe that $[0, p] = \cup_{x_0 \in [0,p]} D(x_0, R(x_0)) \cap [0, p] = \cup_{x_0 \in A} D(x_0, R(x_0)) \cap [0, p]$, where A is a finite subset since $[0, p]$ is compact. Hence, using the periodicity

$$\mathbb{R} \subset \cup_{x_0 \in \{A+np,\, n \in \mathbb{Z}\}} D(x_0, R(x_0)) \subset \cup_{x_0 \in \mathbb{R}} D(x_0, R(x_0)).$$

And therefore $inf_{x_0 \in \mathbb{R}}\{R(x_0)\} = inf_{x_0 \in [0,p]}\{R(x_0)\} \geq min_{x_0 \in A}\{R(x_0)\} := \delta > 0$. □

Remark 4.4. If F is a lift of a circle map, then $F - Id$ is periodic. Note that the neighbourhood of the real line where $\tilde{F}$ converges is the same as $\widetilde{F - Id}$ because Id is analytic everywhere. Therefore corollary 4.3 also holds for lifts.

Proposition 4.5. *Let F be a real analytic map and $\tilde{F}$ its complexification. If $F(x+1) = F(x) + d$, then $\tilde{F}(z+1) = \tilde{F}(z) + d$.*

Proof. Consider $G(z) := \tilde{F}(z+1) - \tilde{F}(z) - d$. Then $G(z) = 0$ for all $z \in \mathbb{R}$, and because of the analytic continuation principle, $G(z) = 0$ on all its domain. □

Let us now consider an analytic circle map f and a lift F. We denote by $\tilde{F}$ its complexification. Let $\Pi(z) := e^{2\pi i z}$. Notice the abuse of notation, since Π is actually the complexification of Π defined in $\mathbb{R}$. As discussed previously, $\tilde{F}$ is defined from a neighbourhood of the real line to another neighbourhood of the real line. We denote them by $\mathbb{R} \times (-\delta, \delta)$ and $\mathbb{R} \times (-\delta', \delta')$ respectively.

Remark 4.6. A circle map f is analytic if and only if its lift F is analytic. We consider $\widetilde{\Pi F} = \Pi\tilde{F} : \mathbb{R} \times (-\delta', \delta') \to A'$, where A′ in an annulus containing the unit circle

$$A' := \{z \in \mathbb{C} |\, e^{-2\pi\delta'} < |z| < e^{2\pi\delta'}\}.$$

And we consider $\widetilde{f\Pi} = \tilde{f}\Pi$, the complexification of the real analytic map $f\Pi$. Recall that $\Pi\tilde{F} = \tilde{f}\Pi$ in $\mathbb{R}$, and because of the analytic continuation principle the equality holds in $A := \mathbb{R} \times (-\delta, \delta)$. Thus, $\tilde{f}$ must be defined on $A := \Pi(\mathbb{R} \times (-\delta, \delta))$ and $\tilde{f}_{|S^1} = f$.

So we have just proved the existence of complexifications of analytic circle maps.

Proposition 4.7. ***(Complexification of analytic circle maps)*** *Let f be an analytic circle map and F a lift of f. Let $A := \mathbb{R} \times (-\delta, \delta)$ be the domain of the complexification of the lift $\tilde{F}$. Then, there exists a complexification of f, we denote it by $\tilde{f}$, defined on A, i.e. $\tilde{f}_{|S^1} = f$ and $\tilde{f}$ is holomorphic on A.*

Remark 4.8. Note that if the radius of convergence of the series of F at $x \in \mathbb{R}$ is ∞, then $\tilde{F}$ is entire and $\tilde{f}$ is analytic in $\mathbb{C} \setminus \{0\}$.

Theorem 4.9. ***(Symmetry of $\tilde{f}$)*** *Let f be a circle map and $\tilde{f}$ the complexification of f in $A := \{z \in \mathbb{C} |\, e^{-2\pi\delta} < |z| < e^{2\pi\delta}\}$. Then $\tilde{f}$ is symmetric with respect S^1, that is $\tilde{f} \circ \tau = \tau \circ \tilde{f}$, where $\tau(z) := \frac{1}{\bar{z}}$. In other words, $\tilde{f}(\frac{1}{\bar{z}}) = \frac{1}{\overline{\tilde{f}(z)}}$.*

Proof. We notate $A^+ := \{z \in \mathbb{C} | \, 1 < |z| < e^{2\pi\delta}\}$ and $A^- := \{z \in \mathbb{C} | \, e^{-2\pi\delta} < |z| < 1\}$. It is easy to see that $\tau : A^- \to A^+$ is actually a bijection. And $\tau_{|S^1} = Id$. Then

$$\tilde{f}\tau(e^{2\pi i\theta}) = \tilde{f}(e^{2\pi i\theta}) \quad and \quad \tau\tilde{f}(e^{2\pi i\theta}) = \tilde{f}(e^{2\pi i\theta})$$

because $\tilde{f}(S^1) \subset S^1$. Hence the equality holds in S^1, and because of the analytic continuation principle, it holds in A, too. □

Corollary 4.10. *Having $\tilde{f}$ defined in A^+ completely determines $\tilde{f}$, since $\tilde{f}(\frac{1}{\bar{z}}) = \frac{1}{\overline{\tilde{f}(z)}}$ defines $\tilde{f}$ in A^-, and vice versa. As a consequence, the dynamics of $\tilde{f}$ in A^+ determines the dynamics in A^-, since $f^n(\frac{1}{\bar{z}}) = \frac{1}{\overline{f^n(z)}}$.*

Example 4.11. (Rigid rotation) A rigid rotation of the circle is $f(e^{2\pi i\theta}) := e^{2\pi i(\theta+\rho)} \implies f(z) = ze^{2\pi i\rho}$, $|z| = 1$, $\rho \in [0, 1)$. Hence

$$\tilde{f}(z) = ze^{2\pi i\rho}.$$

Another way of seeing it is to consider the complexification of the lift. In this case, $F(x) = x + \rho$ and hence $\tilde{F}(z) = z + \rho$, which obviously has an infinite radius of convergence. That gives us the commutative diagram $\tilde{f}\Pi = \Pi\tilde{F}$. Hence $\tilde{f}(e^{2\pi iz}) = e^{2\pi i(z+\rho)} = e^{2\pi iz}e^{2\pi i\rho}$ and therefore $\tilde{f}(z) = ze^{2\pi i\rho}$.

Theorem 4.12. ***(Local linearization)*** *Let $\tilde{f} : A \to A$ be the holomorphic extension of f. If f is analytic linearizable, then $\tilde{f}$ is conformally conjugate to $e^{2\pi i\rho}z$ in a neighbourhood of S^1.*

Proof. There exists an analytic circle map ϕ such that $g \circ \phi = \phi \circ f$, where $g(e^{2\pi i\theta}) = e^{2\pi i(\theta+\rho)}$ is a rigid rotation. We consider the holomorphic extension of the rigid rotation, $\tilde{g}(z) := ze^{2\pi\rho} = |z|e^{iArg(z)+2\pi i\rho}$, and the holomorphic extension $\tilde{\phi}$ of ϕ. We claim $\tilde{\phi} \circ \tilde{f} = \tilde{g} \circ \tilde{\phi}$. The equality holds on S^1 and therefore, because of the analytic continuation principle, holds everywhere $\tilde{\phi}$ and $\tilde{f}$ are defined. □

Corollary 4.13. *If f is analytically linearizable, then $\tilde{f}$ has a Herman ring which contains S^1.*

4.2 The Standard Family

Recall that $f = f_{w,\epsilon}(\theta) = \theta + w + \frac{\epsilon}{2\pi} sin(2\pi\theta)$ and $F = F_{w,\epsilon}(\theta) = \theta + w + \frac{\epsilon}{2\pi} sin(2\pi\theta)$. Let us consider the diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F_{w,\epsilon}} & \mathbb{R} \\ {\scriptstyle e^{2\pi i\theta}}\downarrow & & \downarrow{\scriptstyle e^{2\pi i\theta}} \\ S^1 & \xrightarrow{f_{w,\epsilon}} & S^1 \end{array}$$

Therefore

$$f(e^{2\pi i\theta}) = e^{2\pi iF_{w,\epsilon}} = e^{2\pi i\theta}e^{2\pi iw}e^{2\pi i\frac{\epsilon}{2\pi}(sin(2\pi\theta))} = e^{2\pi i\theta}e^{2\pi iw}e^{2\pi i\frac{\epsilon}{2\pi}(\frac{e^{2\pi i\theta}-e^{-2\pi i\theta}}{2i})} =$$
$$e^{2\pi i\theta}e^{2\pi iw}e^{\epsilon(\frac{e^{2\pi i\theta}-e^{-2\pi i\theta}}{2})}.$$

From this we obtain

$$f(z) = ze^{2\pi i w} e^{\frac{\epsilon}{2}(z-\frac{1}{z})}, \ |z| = 1,$$

is another way of expressing the standard family. Hence, the complexification of f must be $\tilde{f}(z) = ze^{2\pi i w} e^{\frac{\epsilon}{2}(z-\frac{1}{z})}$ defined in $\mathbb{C} \setminus \{0\}$. That was totally expectable since the radius of convergence of $sin(x)$ is infinite.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\tilde{F}_{w,\epsilon}} & \mathbb{C} \\ {\scriptstyle e^{2\pi i\theta}}\downarrow & & \downarrow{\scriptstyle e^{2\pi i\theta}} \\ \mathbb{C}\setminus\{0\} & \xrightarrow{\tilde{f}_{w,\epsilon}} & \mathbb{C}\setminus\{0\} \end{array}$$

Proposition 4.14. ***(Generalities of the standard family)***

1. *If $\epsilon = 0$, then $\tilde{f}$ is a rigid rotation in $\mathbb{C}$ and hence the dynamics is trivial.*

2. *If $\epsilon \in (0, 1)$, there are exactly 2 real critical points (symmetric with respect to S^1).*

3. *$\tilde{f}$ has no asymptotic values different from $0, \infty$.*

4. *$\tilde{f}$ has no Baker domains nor wandering domains.*

Proof. 2. The critical points are solutions of the equation $\tilde{f}'(z) = 0 \iff z^2 + \frac{2}{\epsilon}z + 1 = 0$. Hence there are 2 real critical points $c_{+,-} := \frac{1}{\epsilon} \pm \sqrt{\frac{1}{\epsilon^2} - 1}$.

3. Due to the periodicity of the covering map, we consider the asymptotic values of $\tilde{F}(z) = z + w + \frac{\epsilon}{2\pi} sin(2\pi z)$ in $\mathbb{C} \setminus \mathbb{Z}$. Observe that the asymptotic values of $\tilde{F}$ and $\tilde{f}$ are related by $\tilde{f}(\gamma(t)) = \Pi\tilde{F}\Pi^{-1}\gamma(t)$, where γ is the path of an asymptotic value.

Therefore, suppose $v \in \mathbb{C} \setminus \mathbb{Z}$ is an asymptotic value of $\tilde{F}$; i.e., $\exists \gamma : [0, 1) \to \mathbb{C} \setminus \mathbb{Z}$ which:

$$Im\{\gamma(t)\} \to_{t\to 1} \pm\infty, \quad \tilde{F}(\gamma(t)) \to_{t\to 1} v,$$

since the real part of γ lays inside an interval of 2 consecutive integers. Let Γ be such that $\Pi(\Gamma) = \gamma$. We denote $y(t) := Im\{\Gamma(t)\} = -\frac{1}{2\pi}ln(|\gamma(t)|)$, $x(t) := Re\{\Gamma(t)\} = \frac{1}{2\pi}(Arg\{\gamma(t)\} + 2\pi k)$.

Then,

$$\tilde{F}(\Gamma(t)) = x + iy + w + \frac{\epsilon}{2\pi}\frac{e^{2\pi i x}e^{-2\pi y} - e^{-2\pi i x}e^{2\pi y}}{2i}.$$

Observe that $y(t) \to_{t\to 1} -\infty$. The asymptotic behaviour of $x(t)$ is not determined, we distinguish 2 cases.

First, suppose there exists a sequence $(t_n)_n \to 1$ such that $(x(t_n))_n$ is bounded. Then,

$$\tilde{F}(\Gamma(t_n)) \sim x + iy + \frac{\epsilon}{4\pi i}e^{2\pi i x}e^{-2\pi y} \sim iy + \frac{\epsilon}{4\pi i}e^{2\pi i x}e^{-2\pi y} \to_n \infty.$$

Otherwise $x(t) \to \pm\infty$, and we can pick a sequence $(t_n)_n \to 1$ such that $x(t_n) := k_n \in \mathbb{Z}\ \forall n$. Then, using the formula $sin(2\pi(x + iy)) = sin(2\pi x)cosh(2\pi y) + icos(2\pi x)sinh(2\pi y)$, it yields:

$$\tilde{F}(\Gamma(t_n)) = x(t_n) + iy(t_n) + w + i\frac{\epsilon}{2\pi}sinh(2\pi y(t_n)) \sim x(t_n) + i\frac{\epsilon}{2\pi}sinh(2\pi y(t_n)) \to_n \infty.$$

Hence, if the limit exists, it must be:

$$Im\{\tilde{F}(\Gamma(t))\} \to \pm\infty.$$

Now, if the limit exists,

$$|\tilde{f}(\gamma(t))| = |\Pi\tilde{F}\Pi^{-1}\gamma(t)| = |\Pi\tilde{F}\Gamma(t)| = |e^{2\pi i Re\{\tilde{F}\Gamma(t)\}}e^{-2\pi Im\{\tilde{F}\Gamma(t)\}}| = e^{-2\pi Im\{\tilde{F}\Gamma(t)\}},$$

so $|\tilde{f}(\gamma(t))| \to 0$ *or* ∞. Hence, the only possible asymptotic values of $\tilde{f}$ are 0 and ∞.

4. It is a direct consequence of $f \in S$, since $sing(f^{-1})$ is a bounded set. See theorem 3.43 and theorem 3.45.

□

Now we use the notation $\tilde{f}_{w,\epsilon}$ to emphasize the dependence on the parameters. Whenever we talk about rotation numbers we refer to the respective rotation number of $\tilde{f}_{w,\epsilon\,|S^1} = f_{w,\epsilon}$. We discuss how the complex dynamics evolves when the parameters change along the tongues or curves in the parameter space of the Standard Family (see the Arnold tongues picture, Figure 2.5 in section 2.5). The next results are rather a synthesis of what we learned from the Standard Family, and the complex dynamics implications of the arithmetic properties of the tongue considered. Recall $\epsilon \in [0, 1)$, but we actually consider $\epsilon > 0$, since the case $\epsilon = 0$ is trivial. Recall that ρ denotes the rotation number and $T_\rho := \{(w, \epsilon)|\rho(f_{w,\epsilon}) = \rho\}$ is a curve if ρ is irrational and has no null interior if ρ is rational. Abusing notation, we indicate $f_{w,\epsilon} \in T_\rho$ if $(w, \epsilon) \in T_\rho$.

Our goal in what follows is to describe the complex dynamics of $\tilde{f}_{w,\epsilon}$ depending on the rotation number and using what we know about the circle maps.

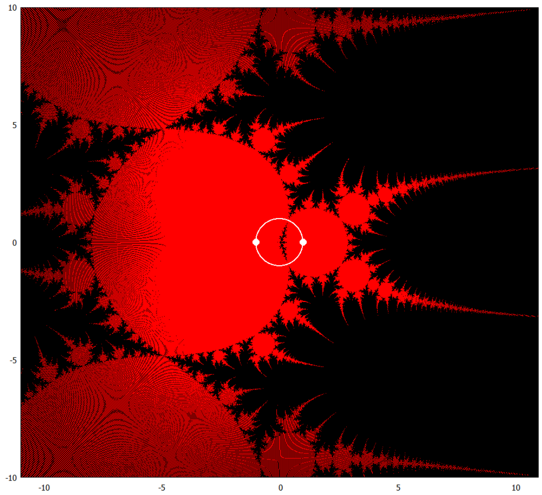
Discussion of rational rotation number and attracting cycles

If $f_{w,\epsilon} \in T_{p/q}$, then $f_{w,\epsilon}$ has an attracting q-cycle and a repelling q-cycle in S^1, then so has $\tilde{f}_{w,\epsilon}$. Hence there exists a q-cycle of attracting basins wrapping around the unit circle, separated by the repelling q-cycle. Moreover, the attracting basins contain all the unit circle except the q repelling periodic points. Fuerthermore, since the 2 critical points are symmetric, they converge under iteration to the attracting cycle. Because of the **Fatou-Shishikura inequality** there are no other periodic Fatou components in the complex plane. See figure 4.1.

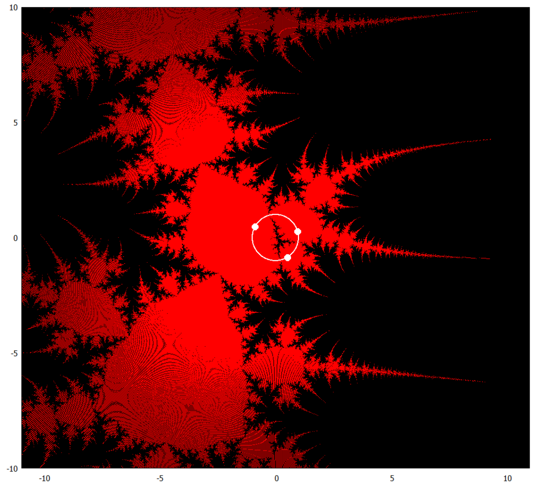
Remark 4.15. If $z_1 \in S^1$ is an attracting fixed point by f^q, then $\tilde{f}^q(z_1) = z_1$ by definition and $|\tilde{f}^{q\prime}(z_1)| = |f^{q\prime}(z_1)| < 1$, taking the unit circle as the path of the complex derivative. That proves attracting periodic points in S^1 are attracting periodic points in $\mathbb{C}$.

The same discussion actually holds as long as we are on the interior of $T_{\rho=p/q}$. On the boundary, each attracting point of the q-cycle collapses with one repelling point of the repelling q-cycle. This results in a q-cycle of leau domains whose respective petals emerge from the q double periodic points in the circle. Again, there is exactly one Fatou cycle, the parabolic one. What's more, since the derivative of $f^q_{w,\epsilon}$ is one at the periodic points and the second derivative does not vanish, there is exactly one attracting petal for each periodic point (see proposition 2.47). See figure 4.2.

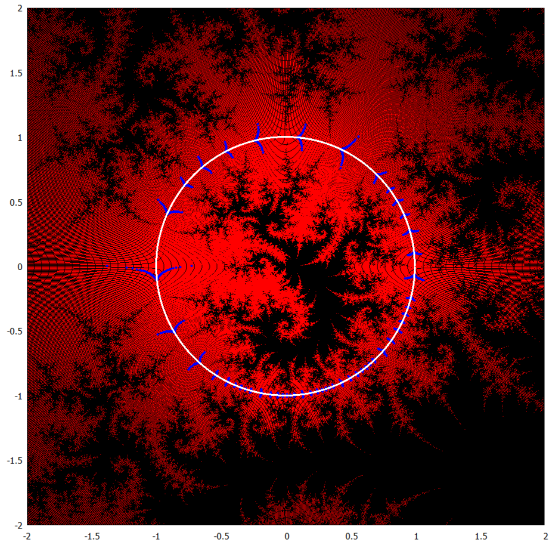
To summarize it, if we set off from the inferior vertex of a rational tongue (where we have a rational rotation), as $\epsilon > 0$ appear the periodic attracting and repelling basins which remain along all the interior of the tongue. The interesting behaviour, though, happens as we move towards the boundary, then the attracting and repelling points collapse and the attracting and repelling cyclic basins become cyclic leau domains.



(a) $w = 0.5,\ \epsilon = 0.8$ ($\rho = 1/2$).

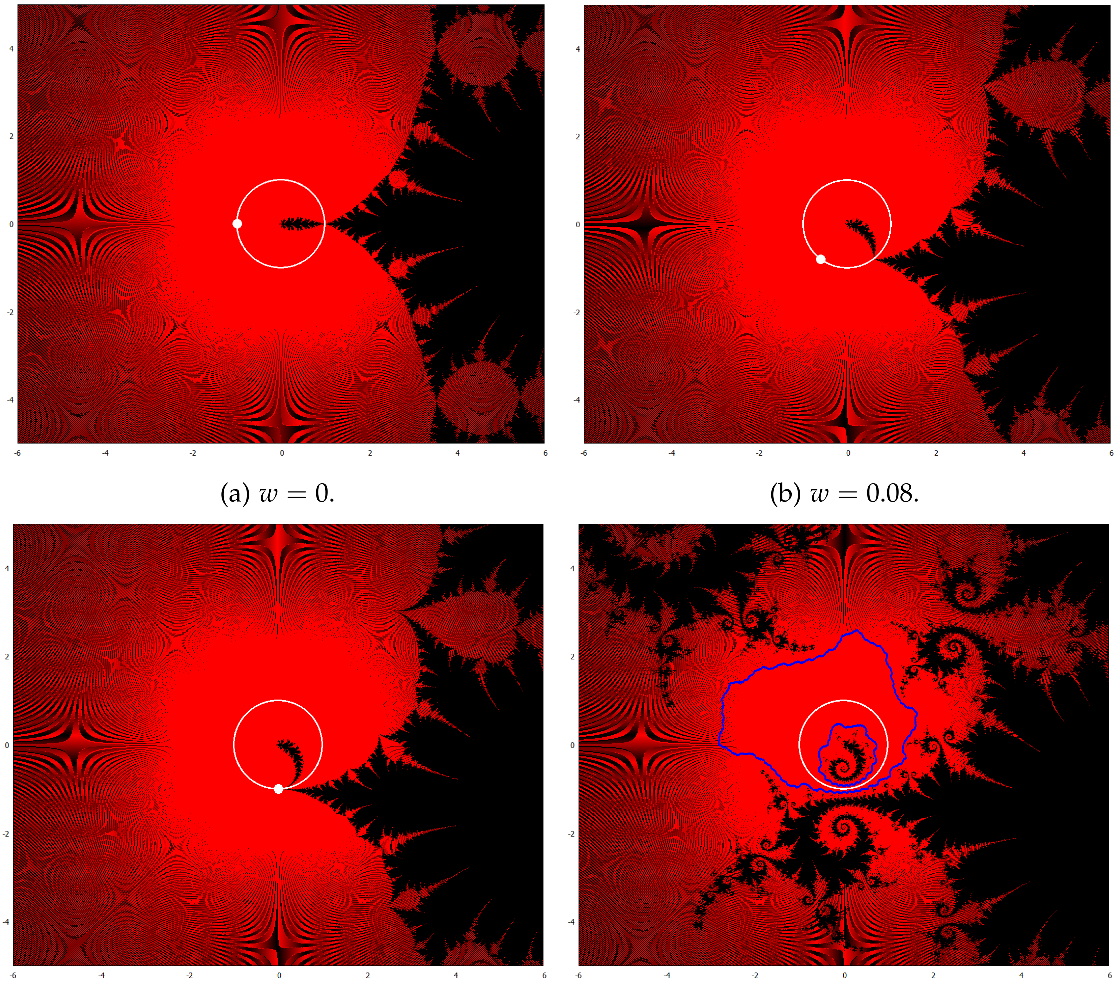


(b) $w = 0.34,\ \epsilon = 0.99$ ($\rho = 1/3$).



(c) $w = 0.2,\ \epsilon = 0.95$ ($\rho = 7/51$).

Figure 4.1: The Fatou set is coloured in red when the points are attracted toward the cycle in the unit circle. The orbits of the points in black eventually escape to infinity or lay in a small neighbourhood of 0, and they either belong to the Julia set or the Fatou set. Function (a) belongs to the central tongue $T_{1/2}$. Therefore it has a 2-cycle of attracting basins, which wraps around the unit circle. The attracting basins contain the whole circle except the 2 repelling points (the 2 immediate basins of attraction are not connected). The attracting points are also indicated in white. (b) belongs to $T_{1/3}$, hence the same happens with a 3-periodic attracting cycle. In picture (c) there is a cycle of period 51, the attracting points are not coloured in white. Instead, the orbits of the critical points are coloured in blue. Recall that they converge to the attracting cycle of period 51.



(a) $w = 0$.

(b) $w = 0.08$.

(c) $w = 0.1$.

(d) $w = 0.12$ ($\rho_{w,\epsilon} = 0.06721085...$).

Figure 4.2: Evolution of the dynamics on the first tongue (T_0) of the family $f_{w,\epsilon=2\pi 0.1}$ for different values of $w \in [0, 0.1]$. See figure 4.3. Hence, for $w = 0$ there is an attracting fixed point at $z_1 = -1$ and a repelling fixed point at $z_2 = 1$. As w is increased the fixed point moves along the unit circle towards $z_3 = -i$. They eventually collapse when $w = 0.1$, giving place to an attracting and a repelling petal flowering at $z_3 = -i$. For $w = 0.12$, the attracting and repelling cycles have disappeared, and a Herman ring (in blue) substitutes them. The map is no more in T_0.

Discussion of the existence of Herman rings when ρ is irrational

Because of theorem 4.12, $\tilde{f}_{w,\epsilon}$ has a Herman ring if and only if $f_{w,\epsilon}$ is analytic linearizable. Note that f is not defined at $z = 0$, therefore linearization on the circle can't lead to Siegel disks. In that case, the Herman ring wraps around $z = 0$, and contains an annulus $A_R := \{z| \frac{1}{R} < |z| < R\}$ for some $R > 1$. Moreover, recall that it is symmetric with respect to S^1. Finally, it is the only cyclic Fatou component of $\tilde{f}_{w,\epsilon}$, because of the **Fatou-Shishikura inequality**. Recall that the orbits of both critical points accumulate onto the boundary of the Herman rings.

We proceed to discuss how maps with Herman rings are distributed along a curve T_ρ of the parameter space of the standard family. If $\rho \in \mathcal{H}$, all maps in T_ρ have a Herman

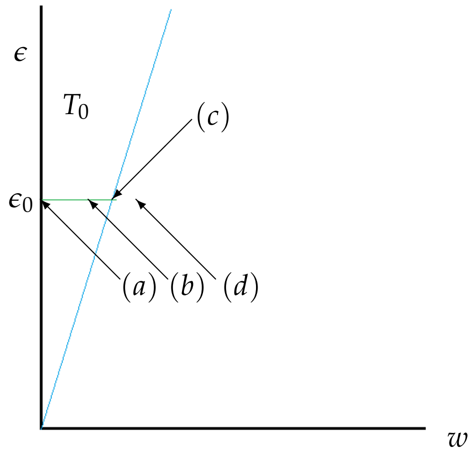


Figure 4.3: Location of the parameters in the Arnold tongues for each case of Figure 4.2.

ring, because there is analytic linearization on the circle (see theorem 2.36). If $\rho \notin \mathcal{B}$, no map in T_ρ has a Herman ring (see theorem 2.36). However, the most interesting case is $\rho \in \mathcal{B} \setminus \mathcal{H}$. The following theorem tells us the modulus of the Herman ring parameterize the curve T_ρ up to a certain limit point where the Herman ring collapses.

Definition 4.16. (Modulus of an annulus) Let U be an annulus (which is therefore conformally conjugate to A_r for some $r < 1$). We define the modulus of U as

$$mod(U) := -\frac{log(r)}{2\pi}.$$

The following theorem can be found in [FG03] and it describes how the modulus of the Herman ring parameterizes the piece of the Arnold curve for which there exists analytic linearization (compare with Figure 2.7).

Theorem 4.17. *Let T_ρ be a curve with $\rho \in \mathcal{B} \setminus \mathcal{H}$. Let $(w', \epsilon') \in T_\rho$ be such that $\tilde{f}_{w',\epsilon'}$ has a Herman ring U. Then, there exists a real analytic map:*

$$\gamma : (0, \infty) \to T_\rho, \quad \gamma(t) := (w(t), \epsilon(t))$$

such that

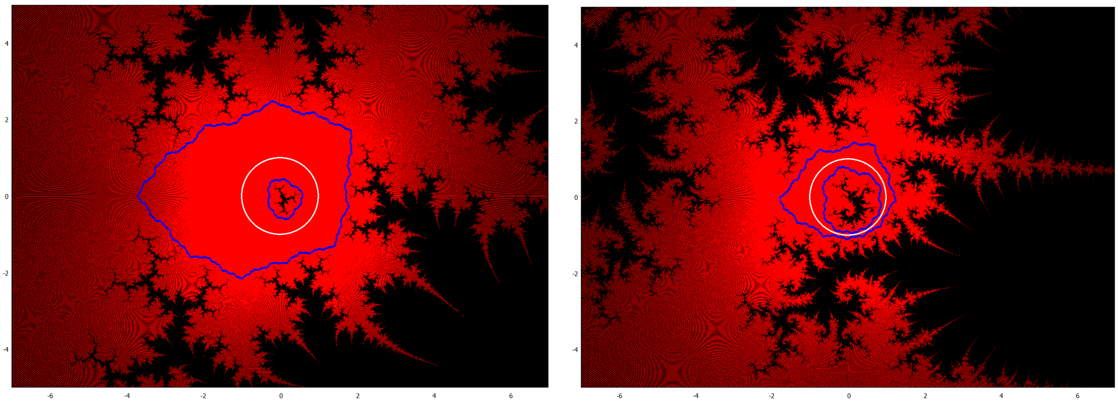
1. *$\gamma(1) = (w', \epsilon')$.*
2. *$\epsilon(t)$ is strictly decreasing, with $\lim_{t\to\infty} \epsilon(t) = 0$ and $\lim_{t\to 0} \epsilon(t) = \epsilon_0 \leq 1$.*
3. *$\tilde{f}_{w,\epsilon}$ has a Herman if and only if $(w, \epsilon) = \gamma(t), \exists t \in (0, \infty)$; if and only if $\epsilon < \epsilon_0$.*
4. *If $\tilde{f}_{w,\epsilon}$ has a Herman ring, its modulus is $t\, mod(U)$.*

Theorem 4.18. *Let $\epsilon_0 \in (0, 1)$. Then there exists $w_0 \in (0, 1)$ such that $(w_0, \epsilon_0) \in T_{\rho \in \mathcal{B}\setminus\mathcal{H}}$, and $\tilde{f}_{w_0,\epsilon_0}$ has no Herman ring. Moreover, if $\epsilon < \epsilon_0$ remaining on the curve, then $\tilde{f}_{w,\epsilon}$ has a Herman ring.*

Remark 4.19. This last theorem tells us that the ϵ_0 in theorem 1 can actually be any $\epsilon \in (0, 1)$.

Remark 4.20. Observe that theorem 4.17 and 4.18 follow from the ideas in the analytic linearization of circle maps (section 2.4) and the analytic linearization for the Standard Family discussion in section 2.5. See theorem 2.48 and 2.49.

Since the domain of the Herman ring is determined by the critical points, observe that $c_{\pm} \to_{\epsilon \to 0} \pm\infty$, and $c_{\pm} \to_{\epsilon \to 1} \pm 1$. Therefore we expect a large Herman ring for small values of ϵ and a small Herman ring for large values of ϵ, whenever it exists.



(a) $w = \frac{1.9}{2\pi}$, $\epsilon = 0.5$ ($\rho_{w,\epsilon} = 0.29526221...$). (b) $w = 0.2$, $\epsilon = 0.85$ ($\rho_{w,\epsilon} = 0.15249090...$).

Figure 4.4: Herman rings for different values of (w, ϵ). Recall that its boundary (in blue) is contained in the closure of the orbits of the 2 critical points. All Herman ring contain S^1 and are symmetric with respect S^1. For large values of ϵ, the Herman ring is smaller.

Discussion of the dynamics when ρ is irrational and there is no Herman ring

Recall that, in opposition to the 2 previous cases, almost no point in the parameter space (w, ϵ) leads to this case - since the set of Diophantine numbers has measure 1. Nevertheless, some things can be said. Actually:

Theorem 4.21. ***(The Julia set is the whole plane)*** *If $\rho(f)$ is irrational and $\tilde{f}$ has no Herman rings, then $J_{\tilde{f}} = \mathbb{C}$.*

The proof of theorem 4.21 is done in several steps. First, we see that S^1 is contained in the Julia set. Then we see that $\tilde{f}$ has no cyclic Fatou components. Hence, it does not have pre periodic components neither. And we already know it does not have wandering domains.

Proposition 4.22. *If $\rho(f)$ is irrational and $\tilde{f}$ has no Herman rings, then $S^1 \subset J_{\tilde{f}}$.*

Proof. Suppose there existed a Fatou component U intersecting S^1. It can only be one Fatou component because of corollary 2.34. But U is not wandering nor pre-periodic because S^1 is invariant. It is not a Herman ring by hipothesis, and therefore it is neither a Siegel disk (there is no analytic linearization). Baker domains do not exist, and it is not an attracting domain nor a parabolic domain because orbits in S^1 do not converge anywhere. Hence $S^1 \subset J_{\tilde{f}}$. □

We already know that Fatou components, if they exist, do not intersect S^1. However, there could be an attracting (parabolic) cycle outside S^1 and the analogous attracting (parabolic) cycle inside S^1, or a similar situation with Siegel disks.

Next we see that the orbits of the 2 critical points eventually accumulate in S^1. Thus, S^1 works as a Herman ring of modulus 0. Observe that this is what actually happens when $\rho \in \mathcal{B} \setminus \mathcal{H}$ and $\epsilon \nearrow \epsilon_0$, when the Herman ring collapses to S^1.

Proposition 4.23. *If $\rho(f)$ is irrational and $\tilde{f}$ has no Herman rings, then $S^1 \subset \overline{O^+(sing(\tilde{f}^{-1}))}$.*

Proof. We notate $P := \overline{O^+(sing(\tilde{f}^{-1}))}$. Let $U := \mathbb{C} \setminus P$. Observe P is closed by definition.

We argue by contradiction. So let $D \subset U$ be an open and bounded set such that it intersect S^1. We see that this implies that $(\tilde{f}^n)_n$ is normal in D, contradicting that $D \cap J_{\tilde{f}} \neq \emptyset$.

We consider the inverse branch g_n of $\tilde{f}^{-n}$ such that $\tilde{f}^{-n}(D)$ contains S^1. Recall that S^1 is invariant, so that is always possible. And the inverse branch is always well-defined since $\cup_n g_n(D)$ never encounters a critical point of $\tilde{f}$. Indeed, if $c \in g_k(D) \cap P$, then $\tilde{f}^k(c) \in D \cap P$, which is a contradiction.

We claim $(g_n)_n$ is normal in D. Indeed, there are infinitely many repelling cycles that do not intersect $g_n(D)$, see theorem 3.4, hence $\cup_n g_n(D)$ omits all those cycles. Now, by Montel's theorem, $(g_n)_n$ is normal in D, therefore, $g_{n_k} \rightrightarrows g$ in D. Moreover, g is analytic since $(g_{n_k})_{n_k}$ is an analytic sequence. Hence, $g(D)$ is either open or it is constant, because of the Open Mapping Theorem.

Suppose $g(z) = z_0 \ \forall z \in D$. In particular, z_0 must be in S^1. We claim that this cannot occur. Let $I := [a, b]$ be a circle interval of $S^1 \cap D$. We denote its length by $l > 0$, $a \neq b$. Then, because both endpoints converge to z_0, for all $\epsilon_1 = l/2$ there exists k_1 such that $|g_{n_k}(a) - g_{n_k}(b)| < \epsilon_1 = l/2 \ \forall k > k_1$. On the other hand, for all $\epsilon_2 = l/2$ there exists k_2 such that $|f^{n_k}(x) - f^{n_k}(y)| < |x - y| + \epsilon_2 \ \forall k > k_2$, as a consequence of the rotation number non-dependence on the initial point. Hence, $\forall k > max(k_1, k_2)$:

$$l = |a - b| = |f^{n_k}(g_{n_k}(a)) - f^{n_k}(g_{n_k}(b))| < |g_{n_k}(a) - g_{n_k}(b)| + \epsilon_2 < l/2 + l/2 = l.$$

Therefore, $g(D)$ is open. We have that $\tilde{f}^{n_k}(g_{n_k}(D)) = D$. By definition, there exists k_0 such that $|g_{n_k}(z) - g(z)| < \epsilon \forall z \in D, \ \forall k > k_0$. Where ϵ is chosen small enough so that $\{z | z \in g(D) \ and \ dist(z, \partial g(D)) > \epsilon\}$ still intersects S^1. Hence,

$$D' := \{z | z \in g(D) \ and \ dist(z, \partial g(D)) > \epsilon\} \subset g_{n_k}(D) \ \forall k > k_0.$$

It follows that $\tilde{f}^{n_k}(D') \subset D \ \forall k > k_0$. Again, by Montel's theorem, $(\tilde{f}^{n_k})_{n_k}$ is a normal family in D'. Therefore, because of Lemma 3.37, $(\tilde{f}^n)_n$ is a normal family in D', contradicting that $D' \cap S^1 \neq \emptyset$.

□

Remark 4.24. Observe that the orbits of actually both critical points eventually accumulate in S^1 because of the symmetry.

Proof of theorem 4.21. We want to prove $J_{\tilde{f}} = \mathbb{C}$. It is sufficient to prove that there are no cyclic Fatou components nor wandering domains. Then there are not pre periodic components. We already knew there are not Baker domains, nor wandering domains. There are no attracting (parabolic) domains since the critical points are not attracted to any cycle. If there was a Herman ring (it cannot intersect S^1), there would be actually 2 because of the symmetry. However, that is impossible due to the **Fatou-Shishikura inequality.** Finally, Siegel disks would require a critical point only dedicated to it, due to corollary 3.51. □

Conclusions

Finally, I would like to give an overview of the project while adding some personal impressions I have had during its course.

I feel it is fair to say that this work separates in two apparently completely different topics: circle maps and complex dynamics, complexification being the tool which connects them. But maybe that has enriched me even more, as well as given me the opportunity to realise the powerful tool compelxification is.

Regarding circle homeomorphisms, I have been fascinated by how much could be said about them. Although it is a seemingly simple topic, the linearization problem gives an idea of the complexity it has, having a large amount of theory behind it. But talking about fascination, nothing compares to the devil staircase and Arnold tongues which result from the Standard Family study. Finding about their particular properties is a reason I am glad I am a mathematician.

On the other hand, the Standard Family has also served as an "excuse" to discover the discipline of complex dynamics, which was totally unknown to me. That first glimpse has been enough to captivate me. I have really enjoyed viewing how some of the most known complex analysis results appear in complex dynamics. Needless to say, it has an intrinsic beauty, too. Finally, I could see how these results apply to the Standard Family. This has allowed me to view how the circle and complex dynamics feed each other, as well as understanding how the circle dynamics generalises to the complex plane.

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