A bargaining set for roommate problems

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Abstract

Since stable matchings may not exist, we propose a weaker notion of stability based on the credibility of blocking pairs. We adopt the weak stability notion of Klijn and Massó (2003) for the marriage problem and we extend it to the roommate problem. We first show that although stable matchings may not exist, a weakly stable matching always exists in a roommate problem. Then, we adopt a solution concept based on the credibility of the deviations for the roommate problem: the bargaining set. We show that weak stability is not sufficient for a matching to be in the bargaining set. We generalize the coincidence result for marriage problems of Klijn and Massó (2003) between the bargaining set and the set of weakly stable and weakly efficient matchings to roommate problems. Finally, we prove that the bargaining set for roommate problems is always non-empty by making use of the coincidence result.

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1 Introduction

Gale and Shapley (1962) introduce a two-sided matching model to answer questions such as who marries whom, who gets which school seat, who shares a dormitory with whom. In their seminal paper, they first introduce the marriage problem, in which there are two disjoint sets of agents, say men and women, and each agent has preferences over agents on the other side of the problem with the possibility of remaining single. No agent can be matched with an agent from the same side. Following the marriage problem, they investigate a generalization, the so-called roommate problem. In the roommate problem, there exists a set of agents each endowed with preferences over all agents. Each agent is interested in forming at most one partnership.[1] A matching is said to be stable[2] if there is no agent who prefers being unmatched to her prescribed partner and no pair of agents prefer being matched to each other to their current partners. They show that stable matchings always exist for the marriage problem, whereas their existence is not guaranteed for the roommate problem. This is a reason why the literature often restricts the analysis to solvable roommate problems (i.e. roommate problems with stable matchings) and on conditions to guarantee the existence of stable matchings (see e.g. Tan, 1991; Chung, 2000; Diamantoudi, Miyagawa and Xue, 2004; Klaus and Klijn, 2010).

In this paper, instead of restricting the analysis to solvable roommate problems, we adopt a weaker notion of stability based on the credibility of blocking pairs for solving the roommate problem: weak stability. An individually rational matching is weakly stable if all blocking pairs are weak. A blocking pair is said to be a weak blocking pair if for every blocking pair one of the partners can find a more attractive partner with whom he forms another blocking pair for the original matching. The motivation behind the notion of weakly stable matchings is that the existence of a blocking pair, which undermines the stability of a matching, is not always credible in the sense that one of the partners may form another blocking pair for the original matching with a more preferred partner. In other words, an individually rational matching is weakly stable if every blocking pair is not credible in the sense above. In the marriage problem, the existence of weakly stable matchings is guaranteed, since by definition, all stable matchings satisfy weak stability.[3] However, for the roommate problem, the existence of a weakly stable matching does not follow from the existence of a stable matching since such matching may fail to exist.

Our main results follow. First, we guarantee the existence of weakly stable matchings by constructing such a matching even for unsolvable roommate problems. Moreover, when

[1] The roommate problem is a model with important applications or extensions including coalition formation (Bogomolnaia and Jackson, 2002), network formation (Jackson and Watts, 2002), kidney exchange problem (Roth, Sönmez and Ünver, 2005) among others. Roth and Sotoymayor (1990) and Manlove (2013) provide a comprehensive survey on matching theory.

[2] For one-to-one matching models, this stability notion is equivalent to core stability. A matching is in the core if there is no subset of agents who, by forming only partnerships among themselves, can all obtain a strictly preferred outcome.

[3] See Klijn and Massó (2003).

the core is non-empty, in general, it is a strict subset of the set of weakly stable matchings. Second, we adopt a solution concept based on the credibility of deviations: the bargaining set. Klijn and Massó (2003) adapt a variation of the bargaining set introduced by Zhou (1994) to the marriage problem. They show that the bargaining set coincides with the set of weakly stable and weakly efficient matchings.[4] We show that weak stability is not sufficient for a matching to be in the bargaining set of a roommate problem. Third, we prove that the bargaining set is always non-empty. To prove the non-emptiness of the bargaining set, we generalize the coincidence result for marriage problems of Klijn and Massó (2003) between the bargaining set and the set of weakly stable and weakly efficient matchings to the roommate problem.

Our results are robust with respect to the enforceability notion satisfying coalitional sovereignty. The standard enforceability notion used to define the bargaining set violates the assumption of coalitional sovereignty, the property that an objecting coalition cannot enforce the organization of agents outside the coalition. Coalitional sovereignty requires that nothing changes for the unaffected agents after the deviation of a coalition. Unaffected agents are those agents who are not part of the deviating coalition and were not together with any agent of the deviating coalition in the original matching. For roommate problems, if a coalition deviates, then it is free to form any match between its members; it cannot affect existing matches between agents outside the coalition, and previous matches between coalition and non-coalition members are destroyed.[5] However, our results are not affected if we replace the classical enforceability condition by the enforceability condition that does satisfy coalitional sovereignty.

Other concepts based on a relaxation of the stability notion have been proposed for roommate problems. These include almost stable matchings (Abraham, Biró and Manlove, 2006), P-stable matchings (Iñarra, Larrea and Molis, 2008), absorbing sets (Iñarra, Larrea and Molis, 2013), $\mathcal{Q}$-stable matchings (Biró, Iñarra and Molis, 2016), and SaRD matchings (Hirata, Kasuya and Tomoeda, 2020).

Iñarra, Larrea and Molis (2008) introduce the notion of P-stable matchings based on the stable partitions due to Tan (1991). For solvable roommate problems, the set of stable matchings coincide with the set of P-stable matchings. However, one can verify that none of the P-stable matchings in Example 1 of Iñarra, Larrea and Molis (2008) belongs to the bargaining set of the given problem. Moreover, there exists a matching in

[4]A matching is weakly efficient if there does not exist another matching in which all agents are better off.

[5]Several papers have used notions of enforceability that respect coalitional sovereignty, see Diamantoudi and Xue (2003) for hedonic games, Mauleon, Vannetelbosch and Vergote (2011) for one-to-one matching problems with farsighted agents, Klaus, Klijn and Walzl (2011) for roommate markets with farsighted agents, Echenique and Oviedo (2006) and Konishi and Ünver (2006) for many-to-many matching problems, Mauleon, Molis, Vannetelbosch and Vergote (2014) for one-to-one matching problems and for roommate markets, Herings, Mauleon and Vannetelbosch (2017) for one-to-one matching problems with myopic agents, Herings, Mauleon and Vannetelbosch (2020) for one-to-one matching problems with myopic and farsighted agents, and Ray and Vohra (2015) for non-transferable utility games.

the bargaining set which is not a P-stable matching. Hence, the set of P-stable matchings and the bargaining set are different.

Iñarra, Larrea and Molis (2013) propose the notion of absorbing sets for the roommate problem. In the context of the roommate problem, a subset of matchings is an absorbing set if every matching in an absorbing set is dominated by any other matching in the same set and no matching outside the set can directly dominate any matching in the absorbing set. The bargaining set for the roommate problem is a unique set-wise solution concept whereas there may exist more than one absorbing set for a given roommate problem. For solvable problems, Iñarra, Larrea and Molis (2013) show that a set of matchings is an absorbing set if and only if it is a singleton set containing a stable matching. Hence, the union of all absorbing sets in a solvable roommate problem coincides with the core. On the other hand, for the marriage problem, Klijn and Massó (2003) observed that the bargaining set can be a strict superset of the core. Since it is a special case of the roommate problem, this result carries over to the roommate problem. Thus, these two notions are different.

Biró, Iñarra and Molis (2016) propose a core consistent solution for roommate problems. A $\mathcal{Q}$-stable matching has the largest set of pairs that are stable within themselves among all matchings and it has the largest number of pairs such that once the pairs are formed they never split. In Example 5 of Biró, Iñarra and Molis (2016), one can verify that not all $\mathcal{Q}$-stable matchings are in the bargaining set whereas a matching in the bargaining set needs not to be a $\mathcal{Q}$-stable matching. Hence, we see that the set of $\mathcal{Q}$-stable matchings and the bargaining set considered in this paper are different.

Recently, Hirata, Kasuya and Tomoeda (2020) introduce a solution concept, the stable against robust deviations (SaRD) matchings for roommate problems. The SaRD matchings is the closest notion to the bargaining set. It is also based on credibility of blocking pairs. A deviation from a matching μ is robust up to depth k, if any of the deviating agents will never end worse-off than at μ after any sequence of at most k subsequent deviations occurs. A matching is SaRD up to depth k, if there is no robust deviation up to depth k. They provide examples to show that the bargaining set and the set of SaRD matchings are different.

The rest of the paper is organized as follows. In Section 2 we introduce the roommate problem and the notion of stability. In Section 3 we extend the notion of weak stability to the roommate problem and we study its structure. In Section 4 we introduce the bargaining set of Zhou (1994), we investigate its relationship with the set of weakly stable matchings and we prove that is always non-empty. In Section 5 we conclude.

2 Roommate problems

A *roommate problem* $(N, \succ)$ consists of a finite set of agents N and a preference profile $\succ = (\succ_l)_{l \in N}$. Each player $l \in N$ has a complete and transitive preference ordering $\succ_l$

over N. Throughout the paper, we assume that the preferences are strict. We write that $j \succ_i k$ if agent i strictly prefers j to k. Since we will consider situations where $j = k$, we write $j \succeq_i k$ when i prefers j at least as well as k. We denote the k^{th} ranked agent in a preference profile of an agent i by $r_k(i)$. An agent j is *acceptable* to another agent i if $j \succ_i i$. A pair of agents $i, j \in N$ are *mutually acceptable* if $j \succ_i i$ and $i \succ_j j$. A pair of agents $i, j \in N$ are *mutually best* if i and j are their respective top choices among their acceptable partners, $r_1(i) = j$ and $r_1(j) = i$.

A *matching* is a one-to-one function $\mu : N \to N$ such that if $\mu(i) = j$, then $\mu(j) = i$. If $\mu(i) = j$, then agents i and j are matched to one another. $\mu(i) = i$ means that agent i is *single* or *unmatched*. Given a roommate problem $(N, \succ)$, we denote the set of all possible matchings by $\mathcal{M}(N, \succ)$. A matching μ is *individually rational* if no agent is matched with an unacceptable partner, that is, $\mu(i) \succeq_i i$ for all $i \in N$. For a given matching μ, a pair of agents (i, j) forms a *blocking pair* if they prefer being matched to each other than to their current partners under matching μ, that is, $j \succ_i \mu(i)$ and $i \succ_j \mu(j)$. For a given matching $\mu \in \mathcal{M}(N, \succ)$, we denote the set of all blocking pairs by $\mathcal{BP}(\mu)$. A matching μ is *stable* if it is individually rational and there are no blocking pairs. Gale and Shapley (1962) show that stable matchings may not exist in the roommate problem. A roommate problem is called *solvable* if the set of stable matchings is non-empty, and is called *unsolvable* otherwise. It is well-known that, in the roommate problem, whenever there exist stable matchings, it coincides with the *core*, in which no subset of agents have incentives to be matched among themselves, possibly by dissolving their current partnerships to obtain a strictly better partner.[6]

Definition 1. Given a roommate problem $(N, \succ)$, a *ring* $\mathcal{A} = (a_1, \ldots, a_k) \subseteq N$ is an ordered subset of agents, $k \geq 3$, such that (subscript modulo k)

$$a_{i+1} \succ_{a_i} a_{i-1} \succ_{a_i} a_i \text{ for all } i \in \{1, \ldots, k\}.$$

A ring $\mathcal{A}$ is an *odd ring* if the number of agents in the ring, $|\mathcal{A}|$, is odd.

Tan (1991) provided a necessary and sufficient condition for a roommate problem with strict preferences to be solvable. He makes use of the notion of *stable partition* to establish a necessary and sufficient condition for a roommate problem to be solvable.

Definition 2. Given a roommate problem $(N, \succ)$, a partition $\mathcal{P}$ of N is stable if

1. for all $S \in \mathcal{P}$, the set S is a ring, a pair of mutually acceptable agents or a singleton, and
2. for all $S_1, S_2 \in \mathcal{P}$ where $S_1 = \{a_1, \ldots, a_k\}$ and $S_2 = \{b_1, \ldots, b_l\}$ with possibility of $S_1 = S_2$, if $b_j \succ_{a_i} a_{i-1}$, then $b_{j-1} \succ_{b_j} a_i$, for all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, l\}$ such that $b_j \neq a_{i+1}$.

[6] When no confusion arises, we simply denote any coalition by its agents, e.g. ij instead of $\{i, j\} = S \subseteq N$.

Condition 1 characterizes the sets of a stable partition and Condition 2 generalizes the notion of stability from matchings to partitions. The stability of a partition is studied by applying Condition 2 among agents from different elements S_1 and S_2 in the partition $\mathcal{P}$ and among agents within the same element S in the partition $\mathcal{P}$.[7]

Tan (1991) proved that for a given unsolvable roommate problem, there always exist stable partitions. Moreover, the following necessary and sufficient condition related to unsolvable roommate problems is established.

Remark 1 (Tan, 1991)**.** (i) A roommate problem $(N, \succ)$ has no stable matchings if and only if there exists a stable partition with an odd ring. (ii) All stable partitions of a roommate problem have exactly the same odd rings and singletons. (iii) All even rings of a stable partition can be broken into pairs of mutually acceptable agents while preserving stability.

The generalization of stability to partitions is an important tool to study roommate problems. Firstly, the literature focusing on proposing stability concepts for unsolvable roommate problems makes use of stable partitions. Secondly, if a stable partition has no odd ring, then we obtain a stable matching. Otherwise, there does not exist any stable matching and we know the reason behind the non-existence. Finally, whenever an agent from each odd ring is left unmatched, we can obtain a matching at which excluded agents from odd rings are single and remaining agents satisfy stability among themselves.

A matching is said to be *weakly efficient* if there is no other matching at which all agents are strictly better off.

Definition 3. Given a roommate problem $(N, \succ)$, a matching $\mu \in \mathcal{M}(N, \succ)$ is *weakly efficient* if there is no matching $\mu' \in \mathcal{M}(N, \succ)$ such that all agents are strictly better off, i.e., $\mu'(i) \succ_i \mu(i)$ for all $i \in N$.

Since stable roommate matchings might not exist, we study the existence of weakly stable matchings in the next section.

3 Weakly stable matchings

Klijn and Massó (2003) introduce the notion of *weak stability* for the marriage problem. We adapt it to the roommate problem. A blocking pair is said to be a *weak blocking pair* if a partner of the blocking pair can form another blocking pair for the original matching with a more preferred partner.

Definition 4. Given a roommate problem $(N, \succ)$, a blocking pair (i, j) for a matching μ is a *weak blocking pair* if there exists another agent $k \in N$ such that $k \succ_i j$ and $i \succ_k \mu(k)$ or $k \succ_j i$ and $j \succ_k \mu(k)$.

[7]For a singleton agent in partition $\mathcal{P}$, $\{a_i\} = S_1 \in \mathcal{P}$, $a_j \succ_{a_i} a_i$ implies $a_j \succ_{a_j} a_i$ whenever $S_2 = \{a_j\} \in \mathcal{P}$, and $a_j \succ_{a_i} a_i$ implies $a_{j-1} \succ_{a_j} a_i$ whenever $a_j \in S_2 \in \mathcal{P}$.

Definition 5. Given a roommate problem $(N, \succ)$, a matching μ is *weakly stable* if it is individually rational and all blocking pairs are weak.

An individually rational matching is weakly stable if every blocking pair is not credible in the sense that one of the partners can find a more attractive partner with whom he forms another blocking pair for the original matching. Since a stable matching is weakly stable by definition, the existence of weakly stable matchings is guaranteed for the marriage problem but not for the roommate problem. Pittel and Irving (1994) remark that the probability of having an unsolvable roommate problem sharply increases as the number of agents increases. Hence, the existence of weakly stable matchings becomes an important issue.

Notice that a pair of agents that are not mutually acceptable cannot block a weakly stable matching since individual rationality is a necessary condition in order for a matching to be weakly stable. Moreover, under a matching that is not individually rational, all agents cannot be strictly better off compared to any individual rational matching. Hence, for expositional simplicity, throughout the paper we consider the preferences restricted to mutually acceptable pairs; i.e., any agent $i \in N$ has a preference ordering $\succ_i$ over agents $j \in N$ such that $j \succ_i i$ and $i \succ_j j$. Let (i, j) be a weak blocking pair for μ. We write $(i, j) \to (i', j)$ if $i' \in N$, $i' \succ_j i$ and (i', j) is a blocking pair for μ.

Next, we construct a weakly stable matching for unsolvable roommate problems. It guarantees the existence of weakly stable matchings for the roommate problem given that every stable matching is also weakly stable for solvable problems.

Theorem 1. *Given a roommate problem $(N, \succ)$, there always exists a weakly stable matching.*

Proof. Since any stable matching is a weakly stable matching, the result straightforwardly follows when the problem is solvable. Hence, it is sufficient to show that there exists a weakly stable matching for unsolvable roommate problems. To do so, we introduce a procedure which returns a weakly stable matching for any given roommate problem.

PROCEDURE: WSMATCH

Phase 0: Let $(N_0, \succ^0)$ be a roommate problem where $N_0 = N$ and $\succ^0 := \succ$. Take the matching at which all agents are single: $\mu_0(i) = i$ for all $i \in N_0$.

Phase $1 \leq t \leq N/2$**:**

Step 1: Fix all mutually best pairs for $(N_{t-1}, \succ^{t-1})$:

$$MB^{t-1}(N_{t-1}, \succ^{t-1}) := \{(i, j) \in N_{t-1} \times N_{t-1} \mid i \text{ and } j \text{ are mutually best according to } \succ^{t-1}\}.$$

Define the set F^{t-1} that consists of agents forming fixed mutually best pairs,

$$F^{t-1} := \{i \in N_{t-1} \mid \exists j \in N_{t-1} \text{ such that } (i, j) \in MB(N_{t-1}, \succ^{t-1})\}.$$

If there is no mutually best pair, $MB(N_{t-1}, \succ^{t-1}) = \emptyset$, go to Phase End. Otherwise go to Step 2.

Step 2: Delete all fixed agents $i \in F^{t-1}$ from the preferences of their all other acceptable partners $j \in N_{t-1}$ such that $i \succ_j^{t-1} j$.
Step 3: Define a reduced problem $(S_t, \succ^t)$ with remaining agents $S_t = N_{t-1} \setminus F^{t-1}$ and rescaled preferences $\succ_l^t := \succ_l^{t-1}{}_{|S_t}$ for all $l \in S_t$.
Step 4: $S_t = N_t$ and go to Phase $t+1$.
Phase End: Return the matching that consists of all fixed pairs together with remaining agents staying single:

$$\begin{aligned} \mu(i) =& j \qquad \text{for all } i \in (i,j) \in \bigcup_{t=0}^{N/2} MB^t(N_t, \succ^t), \\ \mu(i) =& i \qquad \text{for all } i \in N \setminus \bigcup_{t=0}^{N/2} F^t. \end{aligned}$$

First, notice that for a given problem $(N, \succ)$ the maximum number of mutually best pairs is $N/2$, since agents have strict preferences. Let us distinguish two cases to show that Procedure WSMATCH returns a weakly stable matching:

Case 1: There is no mutually best pair.

Since there is no mutually best pair, Procedure WSMATCH returns the matching at which all agents are single. Notice that all acceptable pairs form a blocking pair against the matching at which all agents are single. Since there is no mutually best pair, all blocking pairs are weak. Hence, the matching returned by Procedure WSMATCH is weakly stable.

Case 2: There exists mutually best pairs.

We have to show that Procedure WSMATCH returns a weakly stable matching.

Fixing all mutually best pairs at Step 1 of Phase $1 \leq t \leq N/2$, we guarantee all mutually best pairs are part of the matching obtained by Procedure WSMATCH. If a mutually best pair is part of a matching, no blocking pair can be formed by any partner of the mutually best pair. Hence, by deleting mutually best partners from their all other acceptable partners' preference list at Step 2 and rearranging preferences of remaining agents at Step 3, we restrict the possible blocking pairs in the reduced problem to be formed among the remaining agents.

The procedure returns a stable matching whenever all agents are fixed as a partner of a mutually best pair during the procedure. A stable matching is, by definition, a weakly stable matching. In the second case, after deleting mutually best partners from the preference list of their other acceptable partners, it remains a subset of agents without mutually best pairs. Then, the formation of blocking pairs is restricted to the subset of agents with no mutually best pairs. Since fixed mutually best pairs have been deleted recursively during the Procedure, all blocking pairs formed by the remaining subset of agents are weak. Thus, WSMATCH returns a weakly stable matching.

By means of Procedure WSMATCH, the existence of weakly stable matchings for roommate problems is guaranteed. □

Theorem 1 guarantees the existence of a weakly stable matching even when there is no stable matching. The next examples pinpoint the importance of weakly stable matchings for roommate problems. We provide two examples to show how Procedure WSMATCH runs and returns a weakly stable matching. In the next example, although the core is empty, there exists a unique weakly stable matching. This matching is returned by Procedure WSMATCH.

Example 1. Consider a roommate problem $(N, \succ)$ where $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and the preferences of agents are as follows:

$1 : 2\ 3\ 1$

$2 : 3\ 1\ 2$

$3 : 1\ 2\ 3$

$4 : 7\ 5\ 8\ 4$

$5 : 6\ 4\ 5$

$6 : 7\ 5\ 6$

$7 : 4\ 8\ 6\ 7$

$8 : 4\ 7\ 8.$

Notice that $\mathcal{P} = \{123, 47, 56, 8\}$ is the unique stable partition and this problem is unsolvable. At Phase 0 we start with the matching at which each agent is single: $\mu_0 = \{1, 2, 3, 4, 5, 6, 7, 8\}$. At Step 1 of Phase 1, the pair $(4, 7)$ is fixed since it is the only mutually best pair. Then, at Step 2 we delete agent 4 and agent 7 from their all acceptable partners' preferences. At Step 3, we consider the reduced problem consisting of agents that are still not fixed, $S_1 = \{1, 2, 3, 5, 6, 8\}$ and update their preferences following eliminations at Step 2:

$1 : 2\ 3\ 1$

$2 : 3\ 1\ 2$

$3 : 1\ 2\ 3$

$5 : 6\ 5$

$6 : 5\ 6$

$8 : 8.$

We go to Phase 2 and iterate Step 1-3 for this reduced problem. We fix the pair $(5, 6)$ since they form a mutually best pair. Then, at Step 3 we have the reduced problem with the set of players $S_2 = \{1, 2, 3, 8\}$ and the preferences of agents are as follows:

$1 : 2\ 3\ 1$

$2 : 3\ 1\ 2$

$3 : 1\ 2\ 3$

$8 : 8.$

At Step 1 of Phase 3, since there is no mutually best pair, the Procedure goes to Phase End. It returns the matching formed by the fixed pairs together with remaining agents unmatched: $\mu = \{47, 56, 1, 2, 3, 8\}$. One can easily verify that the constructed matching μ is indeed weakly stable. At the matching $\mu = \{47, 56, 1, 2, 3, 8\}$ there are three blocking pairs, $\mathcal{BP}(\mu) = \{12, 13, 23\}$. All of them are weak: $(1, 2) \to (2, 3) \to (1, 3) \to (1, 2)$.

In Example 1 there is only one weakly stable matching. Nevertheless, given an unsolvable roommate problem, there may exist more than one weakly stable matchings. In Example 2, Procedure $\mathsf{WSMATCH}$ returns only one of the weakly stable matchings.

Example 2. Consider a roommate problem $(N, \succ)$ where $N = \{1, 2, 3, 4\}$ and the preferences of agents are as follows:

$1 : 2\ 3\ 4\ 1$

$2 : 3\ 1\ 4\ 2$

$3 : 1\ 2\ 4\ 3$

$4 : 1\ 2\ 3\ 4.$

Notice first that this is an unsolvable roommate problem at which $\mathcal{P} = \{123, 4\}$ is the unique stable partition. The procedure $\mathsf{WSMATCH}$ starts with the matching at which all agents are single. There is no mutually best pair and hence the Procedure returns the matching at which all agents are single: $\mu = \{1, 2, 3, 4\}$. Since at matching μ all agents are single, all acceptable pair of agents are blocking pairs: $\mathcal{BP}(\mu) = \{12, 13, 14, 23, 24, 34\}$. Since there is no mutually best pair, a partner of any given blocking pair can find a better blocking partner. Thus, all blocking pairs are weak. Hence, the matching $\mu = \{1, 2, 3, 4\}$ obtained by Procedure $\mathsf{WSMATCH}$ is weakly stable.

Although there is no stable matching, there are three other weakly stable matchings than the one returned by Procedure $\mathsf{WSMATCH}$: $\mu_2 = \{14, 2, 3\}$, $\mu_3 = \{24, 1, 3\}$, and $\mu_4 = \{1, 2, 34\}$. At the matching $\mu_2 = \{14, 2, 3\}$ there are three blocking pairs, $\mathcal{BP}(\mu_2) = \{12, 13, 23\}$. All of them are weak: $(1, 2) \to (2, 3) \to (1, 3) \to (1, 2)$. At the matching $\mu_3 = \{24, 1, 3\}$ there are four blocking pairs, $\mathcal{BP}(\mu_3) = \{12, 13, 14, 23\}$. All of them are weak: $(1, 4) \to (1, 2) \to (2, 3) \to (1, 3) \to (1, 2)$. Finally, at the matching $\mu_4 = \{1, 2, 34\}$ there are five blocking pairs, $\mathcal{BP}(\mu_4) = \{12, 13, 23, 14, 24\}$. All of them are weak: $(2, 4) \to (1, 4) \to (1, 2) \to (2, 3) \to (1, 3) \to (1, 2)$.

For the marriage problem, Klijn and Massó (2003) show that the set of weakly stable matchings can be strictly larger than the core (the set of stable matchings). Since the marriage problem is a special case of the roommate problem, their result carries over to the roommate problem.

Corollary 1. *In the roommate problem, unlike the set of stable matchings, the set of weakly stable matchings is always non-empty. Moreover, the set of stable matchings is a (strict) subset of the weakly stable matchings.*

4 Zhou's bargaining set

In this section, following Klijn and Massó (2003), we study a variation of the bargaining set introduced by Zhou (1994).[8] The idea behind the bargaining set is that a matching can be considered plausible (even if it is not in the core) if all objections raised by some agents can be nullified by another subset of agents. Before we define Zhou's (1994) bargaining set for roommate problems, we need to introduce the concepts of enforcement, objection, and counterobjection. Given a matching μ, a coalition $S \subseteq N$ is said to be able to *enforce* a matching μ' over μ if the following conditions hold: for all $i \in S$, if $\mu'(i) \neq \mu(i)$, then $\mu'(i) \in S$. That is, a coalition S can enforce μ' over μ if for each agent in S who has a different partner at μ' than at μ, her new partner at μ' belongs to S too.

Definition 6. An *objection* against a matching μ is a pair (S, μ') where $\emptyset \neq S \subseteq N$ and μ' is a matching that can be enforced over μ by S such that $\mu'(i) \succ_i \mu(i)$ for all $i \in S$.

Definition 7. A *counterobjection* against an objection (S, μ') is a pair (T, μ'') where $T \subseteq N$ with $T \setminus S \neq \emptyset$, $T \cap S \neq \emptyset$, $S \setminus T \neq \emptyset$, and μ'' is a matching that can be enforced over μ by T such that $\mu''(i) \succeq_i \mu(i)$ for all $i \in T \setminus S$ and $\mu''(i) \succeq_i \mu'(i)$ for all $i \in T \cap S$.

An objection is *justified* if there does not exist any counterobjection against it. The counterobjection should satisfy some requirements. There must be at least one agent participating both in the objection and the counterobjection. Otherwise, the counterobjection can be seen as an objection since $S \cap T = \emptyset$. At least one agent involved in the objection should not take part in the coalition T to form a counterobjection. Otherwise, $S \subseteq T$ and the counterobjection can be understood as a reinforcement to the objection. At least one agent in the counterobjection should not be part of the objection. Otherwise, $T \subseteq S$ and the counterobjection can be considered as a refinement to the objection. With the concepts of objection and counterobjection, we adapt Zhou's (1994) notion of bargaining set to the roommate problem.

Definition 8. Given a roommate problem $(N, \succ)$, the *bargaining set* is the set of matchings that have no justified objections:

$$\mathcal{Z}(N, \succ) = \{\mu \in \mathcal{M}(N, \succ) \mid \text{ for every objection at } \mu \text{ there is a counterobjection}\}.$$

Given a roommate problem $(N, \succ)$, let $\mathcal{Z}(N, \succ)$, $\mathcal{WS}(N, \succ)$, and $\mathcal{WE}(N, \succ)$ be the bargaining set, the set of weakly stable matchings, and the set of weakly efficient matchings, respectively. When no confusion arises, we write $\mathcal{Z} = \mathcal{Z}(N, \succ)$, $\mathcal{WS} = \mathcal{WS}(N, \succ)$, and $\mathcal{WE} = \mathcal{WE}(N, \succ)$.

Obviously, if a stable matching exists, then it is in the bargaining set. In contrast to the marriage problem, the non-emptiness of the bargaining set is not guaranteed since a roommate problem needs not have any stable matching. Although, we have shown that

[8] Aumann and Maschler (1964) were first to define the bargaining set for cooperative games.

there always exists a weakly stable matching, it is not sufficient for a matching to be in the bargaining set. The reason behind is that a weakly stable matching needs not satisfy weak efficiency. If a weakly stable matching is not weakly efficient, an objection of the set of agents N cannot be counterobjected. Hence, it is not included in the bargaining set. Next example shows that, for a given roommate problem $(N, \succ)$, there may exist a weakly stable matching that is not weakly efficient.

Example 3. Consider a roommate problem $(N, \succ)$ where $N = \{1, 2, 3, 4\}$ and the preferences of agents are as follows:

$$1 : 2\ 3\ 4\ 1$$
$$2 : 3\ 1\ 4\ 2$$
$$3 : 1\ 2\ 4\ 3$$
$$4 : 1\ 2\ 3\ 4.$$

Note that the only stable partition is $\mathcal{P} = \{123, 4\}$ and there is no stable matching. There are four weakly stable matchings: $\mu_1 = \{14, 2, 3\}$, $\mu_2 = \{1, 2, 3, 4\}$, $\mu_3 = \{24, 1, 3\}$, and $\mu_4 = \{1, 2, 34\}$. Although μ_1, μ_2, μ_3, μ_4 are weakly stable, only μ_1 is weakly efficient. Since agent 4 is matched with her top choice under matching μ_1, $\mu_1(4) = 1$, there is no matching in which all agents can be better off than at μ_1, and hence μ_1 is weakly efficient. Now, consider another matching $\mu' = \{14, 23\}$. All agents are strictly better off at μ' than at $\mu_2 = \{1, 2, 3, 4\}$: $4 \succ_1 1$, $3 \succ_2 2$, $2 \succ_3 3$, and $1 \succ_4 4$. All agents are strictly better off at μ' than at $\mu_3 = \{24, 1, 3\}$: $4 \succ_1 1$, $3 \succ_2 4$, $2 \succ_3 3$, and $1 \succ_4 2$. All agents are strictly better off at μ' than at $\mu_4 = \{1, 2, 34\}$: $4 \succ_1 1$, $3 \succ_2 2$, $2 \succ_3 4$, and $1 \succ_4 3$. Hence, μ_2, μ_3 and μ_4 are weakly stable but not weakly efficient.

Example 3 shows that weak stability is not sufficient for a matching to be in the bargaining set. The matchings μ_2, μ_3, μ_4 are weakly stable but all agents are better off at the matching μ'. That is to say, $S = N$ with the matching μ' constitutes a justified objection against the matchings μ_2, μ_3, μ_4. Hence, neither μ_2 nor μ_3 nor μ_4 are in the bargaining set.

Nevertheless, given a roommate problem $(N, \succ)$, we can construct a matching that lies in the bargaining set. Hence, for any given roommate problem $(N, \succ)$, the bargaining set is always non-empty. To do so, we generalize the characterization of the bargaining set by Klijn and Massó (2003). They show that the set of weakly stable and weakly efficient matchings coincides with the bargaining set for the marriage problem. Next theorem shows that, in the roommate problem, the bargaining set also coincides with the set of weakly stable and weakly efficient matchings.

Theorem 2. *Given a roommate problem $(N, \succ)$, the bargaining set coincides with the set of weakly stable and weakly efficient matchings.*

Since it is a straightforward generalization, we omit the formal proof.[9] Figure 1 depicts the relation among weak stability, weak efficiency, and the bargaining set by means of the matchings considered in Example 3.

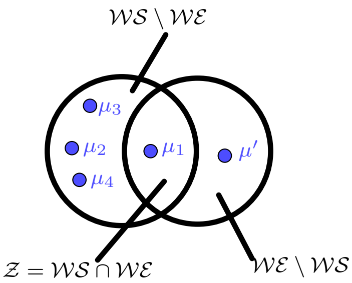

Figure 1: Relation between $\mathcal{WE}$, $\mathcal{WS}$, and $\mathcal{Z}$ in Example 3.

In our main result, for a given roommate problem, we will construct a matching that is weakly stable and weakly efficient. Together with Theorem 2, it will guarantee the non-emptiness of the bargaining set.

Theorem 3. *Given a roommate problem* $(N, \succ)$*, the bargaining set* $\mathcal{Z}(N, \succ)$ *is non-empty.*

Proof. First, notice that, if the problem is solvable, the result straightforwardly follows from the fact that stable matchings always exist and they are in the bargaining set. For unsolvable problems, we will construct a matching that lies in the bargaining set. To do so, we will show that for a given roommate problem, there always exists a matching that is weakly stable and weakly efficient. Together with the characterization of the bargaining set for roommate problems (Theorem 2), the existence of matchings that are both weakly stable and weakly efficient guarantees the non-emptiness of the bargaining set. We distinguish several cases.

Case 1: Given an unsolvable roommate problem $(N, \succ)$, there exist mutually best pairs. From Theorem 1, we know that the matching returned by the Procedure WSMATCH is weakly stable. Moreover, mutually best pairs are matched among themselves and hence they cannot be strictly better off under any other matching. Thus, the matching returned by the Procedure WSMATCH is also weakly efficient.

Case 2: Given an unsolvable roommate problem $(N, \succ)$, there is no mutually best pair.
Case 2.1: There is an odd number of agents at the problem $(N, \succ)$.
Since there is no mutually best pairs, WSMATCH returns the matching at which all agents are unmatched, $\mu(i) = i$ for all $i \in N$, which is weakly stable. Moreover, it is also weakly efficient, since an odd number of agents cannot be strictly better off simultaneously by forming partnerships.

[9]See Atay, Mauleon and Vannetelbosch (2019) for the formal proof.

Case 2.2: There is an even number of agents at the problem $(N, \succ)$.
Note first that all agents would be strictly better off under any individually rational perfect matching, whenever it exists, than the matching returned by WSMATCH at which all agents are unmatched. Then, the matching returned by WSMATCH would not satisfy weak efficiency. From Remark 1 we know that there is a stable partition with odd rings for a given unsolvable roommate problem. We will construct a matching that is both weakly efficient and weakly stable.

Notice first that since the total number of agents is even, at a given stable partition of $\mathcal{P}$ for $(N, \succ)$, if there is an odd number of odd rings, there may be some pairs of mutually acceptable agents and there must be an odd number of singletons. We will consider possible stable partitions $\mathcal{P}$ for $(N, \succ)$. Given a stable partition $\mathcal{P}$ for $(N, \succ)$, we define N_A, N_P, and N_S to be, respectively, the set of ring agents, the set of pair agents, and the set of singleton agents. That is, $N_A = \{a_i \in N : a_i \in \mathcal{A} \in \mathcal{P}\}$, $N_P = \{a_j \in N : \{a_j, a_{j-1}\} \in \mathcal{P}\}$, and $N_S = \{a_k \in N : \{a_k\} \in \mathcal{P}\}$.
Case 2.2.1: $\mathcal{P}$ contains some pair agents.

Given the stable partition $\mathcal{P}$ for $(N, \succ)$, since the set of pair agents N_P is non-empty, we match all the pairs in the partition $\mathcal{P}$: $\mu(a_j) = a_{j-1}$ for all $a_j \in \{a_j, a_{j-1}\} \in \mathcal{P}$ and all other agents remain unmatched: $\mu(l) = l$ otherwise. We first show that the matching μ is weakly stable.

Suppose that $(a_j, a_l) \in \mathcal{BP}(\mu)$. If $\{a_j\} \in \mathcal{P}$ with $\mu(a_j) = a_j$ then, from the stability of $\mathcal{P}$, we have $\mu(a_l) = a_l \succ_{a_l} a_j$ whenever $\{a_l\} \in \mathcal{P}$, and $a_{l-1} = \mu(a_l) \succ_{a_l} a_j$ whenever $a_l \in \{a_l, a_{l-1}\} \in \mathcal{P}$, which contradicts the assumption that $(a_j, a_l) \in \mathcal{BP}(\mu)$. If both $\{a_j, a_l\} \in N_P$ and $a_l \succ_{a_j} a_{j-1} = \mu(a_j)$, then, from the stability of $\mathcal{P}$, we have that $\mu(a_l) = a_{l-1} \succ_{a_l} a_j$, which contradicts the assumption that $(a_j, a_l) \in \mathcal{BP}(\mu)$. Hence, for a pair of agents to block μ, at least one of them should be a ring agent.

Then, let $(a_j, a_l) \in \mathcal{BP}(\mu)$ where $a_j \in N_A$ and consider possible preference relations among a_j and a_l.

1. $a_{j-1} \succeq_{a_j} a_l$.
   Since $a_j \in N_A$, $a_{j+1} \succ_{a_j} a_{j-1}$ and hence $a_{j+1} \succ_{a_j} a_{j-1} \succeq_{a_j} a_l \succ_{a_j} a_j = \mu(a_j)$. From the construction of the matching μ, $\mu(a_{j+1}) = a_{j+1}$ and hence $(a_{j+1}, a_j) \in \mathcal{BP}(\mu)$. Together with $a_{j+1} \succ_{a_j} a_l$, $(a_{j+1}, a_j) \in \mathcal{BP}(\mu)$ implies that (a_j, a_l) is a weak blocking pair.

2. $a_l \succ_{a_j} a_{j-1}$.
   Notice first that if $a_l \in N_P$ and $a_l \succ_{a_j} a_{j-1}$, from the stability of $\mathcal{P}$ we have that $\mu(a_l) = a_{l-1} \succ_{a_l} a_j$, which contradicts the assumption that $(a_j, a_l) \in \mathcal{BP}(\mu)$. Also, if $a_l \in N_S$ and $a_l \succ_{a_j} a_{j-1}$, from the stability of $\mathcal{P}$ we have that $a_l \succ_{a_l} a_j$, which contradicts the assumption that $(a_j, a_l) \in \mathcal{BP}(\mu)$. Thus, $a_l \in N_A$. Since $\mathcal{P}$ is stable, $a_l \succ_{a_j} a_{j-1}$ implies $a_{l-1} \succeq_{a_l} a_j$. This is equivalent to the scenario 1 of the current case and hence $(a_l, a_{l+1}) \in \mathcal{BP}(\mu)$. Together with $a_{l+1} \succ_{a_l} a_l$, $(a_l, a_{l+1}) \in \mathcal{BP}(\mu)$ implies that (a_j, a_l) is a weak blocking pair.

Then, the matching μ is weakly stable. If there is no matching μ' such that $\mu'(i) \succ_i \mu(i)$ for all $i \in N$, then the constructed matching μ is also weakly efficient and we are done.

Suppose that there is another matching μ' such that $\mu'(i) \succ_i \mu(i)$ for all $i \in N$. First, note that if a pair agent $a_j \in N_P$ is matched with another pair agent $a_l \in N_P$ under μ', then from the stability of $\mathcal{P}$, $\mu'(a_j) = a_l \succ_{a_j} a_{j-1} = \mu(a_j)$ implies $\mu(a_l) = a_{l-1} \succ_{a_l} a_j = \mu'(a_j)$, contradicting the assumption that all agents prefer the matching μ' to μ. Also, if a pair agent $a_j \in N_P$ is matched with a singleton agent $a_l \in N_S$ under μ', then from the stability of $\mathcal{P}$, $\mu'(a_j) = a_l \succ_{a_j} a_{j-1} = \mu(a_j)$ implies $\mu(a_l) = a_l \succ_{a_l} a_j = \mu'(a_l)$, contradicting the assumption that all agents prefer the matching μ' to μ. Then, each pair agent should be matched under μ' with a ring agent. That is, the set of mappings

$$\mathcal{N} := \{\text{injective } \nu : N_P \to N_A \mid \text{ for all } a_j \in N_P, \nu(a_j) \succ_{a_j} a_{j-1} = \mu(a_j)\}$$

is non-empty. Since $\mathcal{N}$ is finite, there is $\widehat{\nu} \in \mathcal{N}$ that is weakly efficient for N_P.

We construct the matching $\widehat{\mu}$ such that $\widehat{\mu}(a_j) = \widehat{\nu}(a_j)$ for all $a_j \in N_P$ and $\widehat{\mu}(l) = l$ for all $l \notin (N_P \cup \widehat{\nu}(N_P))$. Notice that $\widehat{\mu}$ must satisfy weak efficiency because if another matching μ^* is better than $\widehat{\mu}$ for all the agents, then its restriction $\mu^*|_{N_P}$ would be a member of $\mathcal{N}$ contradicting the definition of $\widehat{\nu}$. Furthermore, $\widehat{\mu}$ is weakly stable:

Suppose that $(a_j, a_l) \in \mathcal{BP}(\widehat{\mu})$. If $\{a_j\} \in \mathcal{P}$ with $\widehat{\mu}(a_j) = a_j$ then, from the stability of $\mathcal{P}$, we have either $\widehat{\mu}(a_l) = a_l \succ_{a_l} a_j$ whenever $a_l \in N_S$, or $\widehat{\mu}(a_l) \succ_{a_l} a_{l-1} \succ_{a_l} a_j$ whenever $a_l \in \{a_l, a_{l-1}\} \in \mathcal{P}$, which contradicts the assumption that $(a_j, a_l) \in \mathcal{BP}(\widehat{\mu})$. If both $\{a_j, a_l\} \in N_P$ and $a_l \succ_{a_j} \widehat{\mu}(a_j) \succ_{a_j} a_{j-1}$ then, from the stability of $\mathcal{P}$, we have that $\widehat{\mu}(a_l) \succ_{a_l} a_{l-1} \succ_{a_l} a_j$, which contradicts the assumption that $(a_j, a_l) \in \mathcal{BP}(\widehat{\mu})$. Hence, for a pair of agents to block $\widehat{\mu}$, at least one of them should be a ring agent.

Then, let $(a_j, a_l) \in \mathcal{BP}(\widehat{\mu})$ where $a_j \in N_A$ and consider possible preference relations among a_{j-1} and a_l.

1. $a_{j-1} \succeq_{a_j} a_l$.
   Since $a_j \in N_A$, $a_{j+1} \succ_{a_j} a_{j-1}$ and hence $a_{j+1} \succ_{a_j} a_{j-1} \succeq_{a_j} a_l \succ_{a_j} \widehat{\mu}(a_j)$. From the construction of the matching $\widehat{\mu}$, it holds $\widehat{\mu}(a_{j+1}) = a_{j+1}$ or $\widehat{\mu}(a_{j+1}) \in N_P$. Whenever $\widehat{\mu}(a_{j+1}) = a_p \in N_P$ and $a_{j+1} \succ_{a_p} a_{p-1}$, from the stability of $\mathcal{P}$, we have that $a_j \succ_{a_{j+1}} \widehat{\mu}(a_{j+1}) = a_p$. Whenever $\widehat{\mu}(a_{j+1}) = a_{j+1}$ with $a_{j+1} \in N_A$, it also holds that $a_j \succ_{a_{j+1}} \widehat{\mu}(a_{j+1}) = a_{j+1}$. Hence $(a_{j+1}, a_j) \in \mathcal{BP}(\widehat{\mu})$. Together with $a_{j+1} \succ_{a_j} a_l$, $(a_{j+1}, a_j) \in \mathcal{BP}(\widehat{\mu})$ implies that (a_j, a_l) is a weak blocking pair.

2. $a_l \succ_{a_j} a_{j-1}$.
   Notice first that if $a_l \in N_P$ and $a_l \succ_{a_j} a_{j-1}$, from the stability of $\mathcal{P}$, we have that $\widehat{\mu}(a_l) \succ_{a_l} a_{l-1} \succ_{a_l} a_j$, which contradicts the assumption that $(a_j, a_l) \in \mathcal{BP}(\widehat{\mu})$. Also, if $a_l \in N_S$ and $a_l \succ_{a_j} a_{j-1}$, from the stability of $\mathcal{P}$ we have that $a_l \succ_{a_l} a_j$, which contradicts the assumption that $(a_j, a_l) \in \mathcal{BP}(\widehat{\mu})$. Thus, $a_l \in \mathcal{A} \in \mathcal{P}$. Since $\mathcal{P}$ is stable, $a_l \succ_{a_j} a_{j-1}$ implies $a_{l-1} \succeq_{a_l} a_j$. This is equivalent to the scenario 1 of the current case and hence $(a_l, a_{l+1}) \in \mathcal{BP}(\mu)$. Together with $a_{l+1} \succ_{a_l} a_l$, $(a_l, a_{l+1}) \in \mathcal{BP}(\mu)$ implies that (a_j, a_l) is a weak blocking pair.

Notice that if the stable partition $\mathcal{P}$ does not contain singleton agents, the proof immediately follows by simply ignoring the blocking pairs containing a singleton agent.
Case 2.2.2: $\mathcal{P}$ is formed by odd rings and singletons.
Case 2.2.2.1: There exists $\{a_k\} \in \mathcal{P}$ with $r_1(a_k) = a_k$.
The matching at which all agents are unmatched, $\mu(i) = i$ for all $i \in N$, is weakly efficient since $r_1(a_k) = a_k = \mu(a_k)$. From Theorem 1, we know that this matching μ is weakly stable. Hence, the Procedure WSMATCH returns a weakly stable and weakly efficient matching.
Case 2.2.2.2: There does not exist $\{a_k\} \in \mathcal{P}$ such that $r_1(a_k) = a_k$.
Notice that all singletons $\{a_k\} \in \mathcal{P}$ have a top choice that is a ring agent: $r_1(a_k) = a_i$ where $a_i \in \mathcal{A} \in \mathcal{P}$. Then, match each a_k with her top choice if there is no $\{a_l\} \in \mathcal{P}$ such that $r_1(a_l) = a_i$ and $a_{i-1} \succ_{a_i} a_l \succ_{a_i} a_k$: $\mu(a_k) = a_i = r_1(a_k)$, and leave all remaining agents single: $\mu(l) = l$ otherwise. First, note that the matching μ is weakly efficient since for some $\{a_k\} \in \mathcal{P}$, $\mu(a_k) = a_i = r_1(a_k)$.

Next, let us show that the matching μ is also weakly stable. Note that if $\{a_k\} \in \mathcal{P}$ is matched under μ, then $\mu(a_k) = r_1(a_k)$, and hence a matched singleton agent a_k cannot be part of a blocking pair. Whenever $\{a_j\} \in \mathcal{P}$ with $\mu(a_j) = a_j$ and $\{a_l\} \in \mathcal{P}$ with $\mu(a_l) = a_l$, then a_j and a_l cannot form a blocking pair: if $a_l \succ_{a_j} a_j = \mu(a_j)$, then the stability of $\mathcal{P}$ implies $\mu(a_l) = a_l \succ_{a_l} a_j$ and if $a_j \succ_{a_l} a_l = \mu(a_l)$, then the stability of $\mathcal{P}$ implies $a_j = \mu(a_j) \succ_{a_j} a_l$. Hence, for a pair of agents to block μ, at least one of them should be a ring agent.

Note first that for all $a_j \in N_A$, $\mu(a_j) = a_j$ or $\mu(a_j) = a_k$ where $\{a_k\} \in \mathcal{P}$. Whenever $\mu(a_j) = a_j$, $a_{j+1} \succ_{a_j} a_{j-1} \succ_{a_j} a_j = \mu(a_j)$. Whenever $\mu(a_j) = a_k$, we have $r_1(a_k) = a_j$ and then, the stability of $\mathcal{P}$ implies that $a_{j-1} \succ_{a_j} a_k = \mu(a_j)$. Hence, $a_{j+1} \succ_{a_j} a_{j-1} \succ_{a_j} \mu(a_j)$ for all $a_j \in N_A$. Thus, $(a_j, a_{j+1}) \to (a_{j+1}, a_{j+2}) \to \ldots \to (a_{j-1}, a_j) \to (a_j, a_{j+1})$, and any pair of consecutive agents forms a blocking pair.

Suppose $(a_j, a_l) \in \mathcal{BP}(\mu)$ where $a_j \in N_A$. Whenever $\{a_l\} \in \mathcal{P}$, we have that $\mu(a_l) = a_l$ since a matched singleton agent cannot be part of a blocking pair. Since $\mu(a_l) = a_l$ and $\mathcal{P}$ is stable, $a_j \succ_{a_l} a_l$ implies that $a_{j-1} \succ_{a_j} a_l$. Then, together with $(a_{j-1}, a_j) \in \mathcal{BP}(\mu)$, it implies that (a_j, a_l) is a weak blocking pair. Whenever $a_l \in N_A$, consider the possible preference relations among a_j and a_l for $(a_j, a_l) \in \mathcal{BP}(\mu)$.

1. $a_{j-1} \succeq_{a_j} a_l$.
   Note that $(a_j, a_{j+1}) \in \mathcal{BP}(\mu)$. Together with $a_{j+1} \succ_{a_j} a_l$, it implies that (a_j, a_l) is a weak blocking pair.

2. $a_l \succ_{a_j} a_{j-1}$.
   Since $\mathcal{P}$ is stable, $a_l \succ_{a_j} a_{j-1}$ implies that $a_{l-1} \succ_{a_l} a_j$. Together with $(a_l, a_{l-1}) \in \mathcal{BP}(\mu)$, it implies that (a_j, a_l) is a weak blocking pair.

Case 2.2.3: $\mathcal{P}$ is formed by an even number of odd rings.
For notational convenience, let $\mathcal{P} = \{\mathcal{A}_1, \ldots, \mathcal{A}_T\}$, where T is even.

Case 2.2.3.1: There is some agent $a_i \in \mathcal{A}_1 \in \mathcal{P}$ such that there exists a ring $\mathcal{A}_s \in \mathcal{P}$, $s \neq 1$, and $a_j \succ_{a_{i-1}} a_{i-2}$ with $a_j \in \mathcal{A}_s \in \mathcal{P}$.

1. There exists a unique agent $a_i \in \mathcal{A}_1 \in \mathcal{P}$ such that there exists a ring $\mathcal{A}_s \in \mathcal{P}$, $s \neq 1$, and $a_j \succ_{a_{i-1}} a_{i-2}$ with $a_j \in \mathcal{A}_s \in \mathcal{P}$.
   We construct the matching μ at which a_i is matched with her top choice $r_1(a_i)$: $\mu(a_i) = r_1(a_i)$ and all other agents remain unmatched: $\mu(l) = l$ otherwise. Since $\mu(a_i) = r_1(a_i)$, the matching μ is weakly efficient. Let us next show that the matching μ also satisfies weak stability.

   First, suppose $\mathcal{A}_1 = \{a_{i-2}, a_{i-1}, a_i\}$. Notice that $r_1(a_i) = a_{i-2}$ since a_i is the unique agent satisfying the condition $a_j \succ_{a_{i-1}} a_{i-2}$ in $\mathcal{A}_1$. Then, $\mu(a_i) = a_{i-2} = r_1(a_i)$ and hence $(a_i, a_{i-1}) \notin \mathcal{BP}(\mu)$. Note also that $(a_{i-1}, a_{i-2}) \in \mathcal{BP}(\mu)$ since $a_{i-2} \succ_{a_{i-1}} a_{i-1} = \mu(a_{i-1})$ and $a_{i-1} \succ_{a_{i-2}} a_i = \mu(a_{i-2})$. Then, (a_{i-1}, a_{i-2}) is a weak blocking pair since $a_j \succ_{a_{i-1}} a_{i-2}$ and $(a_{i-1}, a_j) \in \mathcal{BP}(\mu)$. For all other blocking pairs $(a_k, a_l) \in \mathcal{BP}(\mu)$ where both $\{a_k, a_l\} \in N_A$, it holds either $(a_k, a_l) \to (a_k, a_{k+1})$ or $(a_k, a_l) \to (a_l, a_{l+1})$ since $\mathcal{P}$ is stable and under the matching μ, $\mu(a_i) = r_1(a_i)$ and all other agents are unmatched.

   Hence, all blocking pairs are weak and the matching μ is also weakly stable.

   Now, let $|\mathcal{A}_1| \geq 5$. Recall that a_i is the unique agent such that $a_j \succ_{a_{i-1}} a_{i-2}$ with $a_j \in \mathcal{A}_s \in \mathcal{P}$, $s \neq 1$. Thus, $r_1(a_i) \in \mathcal{A}_1$.

   Note that $a_{i-1} \neq r_1(a_i) = a_t \in \mathcal{A}_1 \in \mathcal{P}$ since $a_{i+1} \succ_{a_i} a_{i-1}$. Then, $\mu(a_i) = a_t$ and all other agents remain unmatched. Since $\mu(a_i) = r_1(a_i)$, $(a_{i-1}, a_i) \notin \mathcal{BP}(\mu)$. Then, we need to show that (a_{i-1}, a_{i-2}) is a weak blocking pair. From $a_j \succ_{a_{i-1}} a_{i-2} \succ_{a_{i-1}} a_{i-1} = \mu(a_{i-1})$ and $\mu(a_j) = a_j$, $(a_j, a_{i-1}) \in \mathcal{BP}(\mu)$. Together with $a_j \succ_{a_{i-1}} a_{i-2}$, it implies that (a_{i-1}, a_{i-2}) is a weak blocking pair. For all other blocking pairs $(a_k, a_l) \in \mathcal{BP}(\mu)$ where both $\{a_k, a_l\} \in N_A$, it holds either $(a_k, a_l) \to (a_k, a_{k+1})$ or $(a_k, a_l) \to (a_l, a_{l+1})$ since $\mathcal{P}$ is stable and under the matching μ, $\mu(a_i) = r_1(a_i)$ and all other agents are unmatched.

   Hence, all blocking pairs are weak and the matching μ is also weakly stable.

2. There exist several ring agents $\{a_i, a_{i'}\} \in \mathcal{A}_1 \in \mathcal{P}$ such that there exists a ring $\mathcal{A}_s \in \mathcal{P}$, $s \neq 1$, and $a_j \succ_{a_{i-1}} a_{i-2}$, $a_j \succ_{a_{i'-1}} a_{i'-2}$ with $a_j \in \mathcal{A}_s \in \mathcal{P}$.

   Note that there can be different agents $\left\{a_j, a_{j'}\right\} \in \mathcal{A}_s \in \mathcal{P}$ with $s \neq 1$ such that $a_j \succ_{a_{i-1}} a_{i-2}$, $a_j \succ_{a_{i'-1}} a_{i'-2}$, and $a_{j'} \succ_{a_{i-1}} a_{i-2}$, $a_{j'} \succ_{a_{i'-1}} a_{i'-2}$ where $\{a_i, a_{i'}\} \in \mathcal{A}_1 \in \mathcal{P}$. In that case, we fix one of those agents $a_{j^*} \in \mathcal{A}_s \in \mathcal{P}$ with $s \neq 1$ for which there exist several ring agents $\{a_i, a_{i'}\} \in \mathcal{A}_1 \in \mathcal{P}$ such that $a_{j^*} \succ_{a_{i-1}} a_{i-2}$, $a_{j^*} \succ_{a_{i'-1}} a_{i'-2}$.

   Given $a_{j^*} \in \mathcal{A}_s \in \mathcal{P}$ with $s \neq 1$, we distinguish between two cases.

(a) Agents $\{a_i, a_{i'}\} \in \mathcal{A}_1 \in \mathcal{P}$ satisfying the condition are not consecutive agents in $\mathcal{A}_1 \in \mathcal{P}$.

Notice that since agents satisfying the condition are not consecutive, whenever $|\mathcal{A}_1| = 3$ there exists a unique agent satisfying the condition which contradicts the statement. Hence, we consider $|\mathcal{A}_1| \geq 5$.

Given $a_{j^*} \in \mathcal{A}_s \in \mathcal{P}$ with $s \neq 1$, we fix a_{i^*} such that $a_{i-1} \succ_{a_{j^*}} a_{i^*-1}$ among all ring agents $a_i \in \mathcal{A}_1 \in \mathcal{P}$ with $a_{j^*} \succ_{a_{i-1}} a_{i-2}$. That is, a_{i^*} is the agent that satisfies the condition $a_{j^*} \succ_{a_{i^*-1}} a_{i^*-2}$ and a_{i^*-1} is the least preferred by the agent a_{j^*} among all ring agents that prefer a_{j^*} more than her immediate predecessor. Then, we match a_{i^*} with her top choice $r_1(a_{i^*})$: $\mu(a_{i^*}) = r_1(a_{i^*})$ and all other agents remain unmatched: $\mu(l) = l$ otherwise. Notice that, since the agents that satisfy the condition are not consecutive agents, a_{i^*+1} does not satisfy the condition. Hence, $a_j \succ_{a_{i^*}} a_{i^*-1}$ does not hold. Then, $r_1(a_{i^*}) \in \mathcal{A}_1$. Since a_{i^*} is matched with her top choice, the matching μ is weakly efficient. The argument to prove weak stability of matching μ will be identical to scenario 1 of the current case (when $|\mathcal{A}_1| \geq 5$) except that we need to replace i with i^*.

(b) Agents $a_i \in \mathcal{A}_1 \in \mathcal{P}$ satisfying the condition are consecutive agents in $\mathcal{A}_1 \in \mathcal{P}$. If all agents $a_i \in \mathcal{A}_1$ satisfy the condition $a_{j^*} \succ_{a_{i-1}} a_{i-2}$ with $a_{j^*} \in \mathcal{A}_s \in \mathcal{P}$, $s \neq 1$, we fix $a_{i^*} \in \mathcal{A}_1$ such that $a_{i'} \succ_{a_{j^*}} a_{i^*}$ among all ring agents $\{a_{i'}, a_{i^*}\} \in \mathcal{A}_1 \in \mathcal{P}$. Then, we match a_{i^*} with her top choice $r_1(a_{i^*})$: $\mu(a_{i^*}) = r_1(a_{i^*})$ and all other agents remain unmatched: $\mu(l) = l$ otherwise. Since a_{i^*} is matched with her top choice, the matching μ is weakly efficient. Next, let us show that the matching μ is also weakly stable.

Since all agents in the odd ring $\mathcal{A}_1$ satisfy the above condition and we fix a_{i^*} such that $a_{i'} \succ_{a_{j^*}} a_{i^*}$ among all ring agents $\{a_{i'}, a_{i^*}\} \in \mathcal{A}_1 \in \mathcal{P}$, $a_{i^*-1} \succ_{a_{j^*}} a_{i^*}$. Then, $(a_{j^*}, a_{i^*-1}) \in \mathcal{BP}(\mu)$. Together with $a_{j^*} \succ_{a_{i^*-1}} a_{i^*-2}$, it implies that (a_{i^*-1}, a_{i^*-2}) is a weak blocking pair. For all other blocking pairs $(a_k, a_l) \in \mathcal{BP}(\mu)$ where both $\{a_k, a_l\} \in N_A$, it holds either $(a_k, a_l) \to (a_k, a_{k+1})$ or $(a_k, a_l) \to (a_l, a_{l+1})$ since $\mathcal{P}$ is stable and under the matching μ, $\mu(a_{i^*}) = r_1(a_{i^*})$ and all other agents are unmatched. Hence, all blocking pairs are weak and the matching μ is also weakly stable.

If a subset of consecutive agents $\{a_i, a_{i+1}, \ldots\} \subsetneq \mathcal{A}_1 \in \mathcal{P}$ satisfy the condition $a_{j^*} \succ_{a_{i-1}} a_{i-2}$ with $a_{j^*} \in \mathcal{A}_s \in \mathcal{P}$, $s \neq 1$, then there is at least one agent $a_{i+1} \in \mathcal{A}_1$ such that it does not hold $a_{j^*} \succ_{a_i} a_{i-1}$. Then, we fix one agent $a_{i^*} \in \mathcal{A}_1$ such that $a_{j^*} \succ_{a_{i^*-1}} a_{i^*-2}$ and a_{i^*+1} does not satisfy the condition $a_{j^*} \succ_{a_{i^*}} a_{i^*-1}$. Then, we match a_{i^*} with her top choice $r_1(a_{i^*})$: $\mu(a_{i^*}) = r_1(a_{i^*})$ and all other agents remain single: $\mu(l) = l$ otherwise. Note that since $a_{j^*} \succ_{a_{i^*}} a_{i^*-1}$ does not hold, $r_1(a_{i^*}) \in \mathcal{A}_1$. Since a_{i^*} is matched with her top choice, the matching μ is weakly efficient. Next, we show that the matching μ is also weakly stable.

First, let $\mathcal{A}_1 = \{a_{i^*-2}, a_{i^*-1}, a_{i^*}\}$. Recall that $r_1(a_{i^*}) \in \mathcal{A}_1$. Moreover, since there are consecutive agents satisfying the condition in $\mathcal{A}_1$, $a_{j^*} \succ_{a_{i^*-1}} a_{i^*-2}$ and $a_{j^*} \succ_{a_{i^*-2}} a_{i^*}$. Notice first that $r_1(a_{i^*}) \neq a_{i^*-1}$ since $\mathcal{P}$ is stable. Then, it holds that $r_1(a_{i^*}) = a_{i^*-2}$, and hence $\mu(a_{i^*}) = a_{i^*-2}$ and all other agents remain single. Since $\mathcal{A}_1 = \{a_{i^*-2}, a_{i^*-1}, a_{i^*}\}$, $a_{i^*-1} \succ_{a_{i^*-2}} a_{i^*} = \mu(a_{i^*-2})$. Moreover, $(a_{j^*}, a_{i^*-1}) \in \mathcal{BP}(\mu)$ since $\mu(a_{j^*}) = a_{j^*}$, $\mu(a_{i^*-1}) = a_{i^*-1}$ and they are mutually acceptable as $a_{j^*} \succ_{a_{i^*-1}} a_{i^*-2}$ holds. Together with $(a_{j^*}, a_{i^*-1}) \in \mathcal{BP}(\mu)$, $a_{j^*} \succ_{a_{i^*-1}} a_{i^*-2}$ implies that (a_{i^*-1}, a_{i^*-2}) is a weak blocking pair. For all other blocking pairs $(a_k, a_l) \in \mathcal{BP}(\mu)$ where both $\{a_k, a_l\} \in N_A$, it holds either $(a_k, a_l) \to (a_k, a_{k+1})$ or $(a_k, a_l) \to (a_l, a_{l+1})$ since $\mathcal{P}$ is stable and under the matching μ, $\mu(a_{i^*}) = r_1(a_{i^*})$ and all other agents are unmatched. Hence, all blocking pairs are weak and the matching μ is also weakly stable.

Second, let $|\mathcal{A}_1| \geq 5$. The argument to prove weak stability of matching μ will be identical to scenario 1 of the current case (when $|\mathcal{A}_1| \geq 5$) except that we need to replace i with i^*.

Case 2.2.3.2: There does not exist $a_i \in \mathcal{A}_1 \in \mathcal{P}$ such that there exists a ring $\mathcal{A}_s \in \mathcal{P}$, $s \neq 1$, and $a_j \succ_{a_{i-1}} a_{i-2}$ with $a_j \in \mathcal{A}_s \in \mathcal{P}$.
Case 2.2.3.2.1: There exist some agent $a_i \in \mathcal{A}_1 \in \mathcal{P}$ such that there exist a ring $\mathcal{A}_s \in \mathcal{P}$, $s \neq 1$, and $a_{i-1} \succ_{a_i} a_j \succ_{a_i} a_i$, $a_{j-1} \succ_{a_j} a_i \succ_{a_j} a_j$ with $a_j \in \mathcal{A}_s \in \mathcal{P}$.

Fix an arbitrary odd ring $\mathcal{A}^* \in \mathcal{P}$. We construct a matching μ by running an analogue of the deferred acceptance algorithm[10] between $\mathcal{A}^*$ and $N \setminus \mathcal{A}^*$ as follows:
Step 1: Each $a_i \in \mathcal{A}^*$ proposes to the best acceptable agent among $N \setminus \mathcal{A}^*$, if any. Each $a_j \in N \setminus \mathcal{A}^*$ tentatively accepts the best proposal and rejects all others.
Step $t \geq 2$: Each a_i who is not currently engaged proposes to the best acceptable agent among $N \setminus \mathcal{A}^*$ that a_i has not yet proposed to. Then, each $a_j \in N \setminus \mathcal{A}^*$ chooses from the proposals made at the current step and her current tentative match (if any), and tentatively accepts the best one.
Since there are finitely many agents, if no new proposal is made at some *Step T*, then all the tentative matches are finalized.

Let us show that the constructed matching μ is weakly stable. Suppose $(a_j, a_l) \in \mathcal{BP}(\mu)$.

First note that, since $\mu(a_l) = a_l$ or $a_{l-1} \succ_{a_l} \mu(a_l) \succ_{a_l} a_l$ for all $a_l \in N$, for any odd ring $\mathcal{A} \in \mathcal{P}$ (including $\mathcal{A}^*$), any two consecutive agents $\{a_j, a_{j+1}\}$ in $\mathcal{A}$ form a blocking pair for the matching μ. Then, $(a_j, a_{j+1}) \to (a_{j+1}, a_{j+2}) \to \ldots \to (a_{j-1}, a_j) \to (a_j, a_{j+1})$. Next, consider possible preference relation among a_j and a_l for $(a_j, a_l) \in \mathcal{BP}(\mu)$.

1. $a_{j-1} \succeq_{a_j} a_l$.
   Note that $(a_{j+1}, a_j) \in \mathcal{BP}(\mu)$. Together with $a_{j+1} \succ_{a_j} a_l$, it implies that (a_j, a_l) is a weak blocking pair.

[10] We thank one of the Reviewers for suggesting us this algorithm in order to costruct the matching μ.

2. $a_l \succ_{a_j} a_{j-1}$.
   Since $\mathcal{P}$ is stable, $a_l \succ_{a_j} a_{j-1}$ implies $a_{l-1} \succ_{a_l} a_j$. Since any two consecutive agents form a blocking pair, $(a_{l-1}, a_l) \in \mathcal{BP}(\mu)$. Together with $a_{l-1} \succ_{a_l} a_j$, $(a_{l-1}, a_l) \in \mathcal{BP}(\mu)$ implies that (a_j, a_l) is a weak blocking pair.

We still need to show that this matching is weakly efficient. Suppose on the contrary that there exists another matching μ' such that $\mu'(a_l) \succ_{a_l} \mu(a_l)$ for all $a_l \in N$. Since $\mathcal{A}^*$ is an odd ring, there should be at least one $a_i \in \mathcal{A}^*$ such that $\mu'(a_i) \notin \mathcal{A}^*$. Then, however, a_i must have proposed to $\mu'(a_i)$ at some step t of the above algorithm. The only possible reason that a_i is not matched with $\mu'(a_i)$ at μ is that $\mu'(a_i)$ chooses some $a_l \in \mathcal{A}^*$ over a_i. This implies that $\mu'(a_i)$ prefers $\mu(\mu'(a_i))$ to $a_i = \mu'(\mu'(a_i))$, which is a contradiction to the assumption that $\mu'(a_l) \succ_{a_l} \mu(a_l)$ for all $a_l \in N$. Hence, the matching μ is also weakly efficient.

Case 2.2.3.2.2: There does not exist $a_i \in \mathcal{A}_1 \in \mathcal{P}$ such that there exists a ring $\mathcal{A}_s \in \mathcal{P}$, $s \neq 1$, and $a_{i-1} \succ_{a_i} a_j \succ_{a_i} a_i$, $a_{j-1} \succ_{a_j} a_i \succ_{a_j} a_j$ with $a_j \in \mathcal{A}_s \in \mathcal{P}$.
That is to say, all agents have preferences only over agents from the same odd ring. Then, the weakly stable matching returned by WSMATCH, $\mu(i) = i$ for all $i \in N$, is also weakly efficient since all agents from an odd ring cannot be strictly better off simultaneously as their preferences contain only agents from the same odd ring.

We have shown that there always exists a matching that satisfies both weak stability and weak efficiency. Together with Theorem 2, it follows that given a roommate problem $(N, \succ)$, the bargaining set $\mathcal{Z}(N, \succ)$ is always non-empty. $\square$

5 Conclusion

Since stable matchings may not exist in the roommate problem, we have considered a weaker notion of stability based on the credibility of blocking pairs. We have extended the weak stability notion of Klijn and Massó (2003) for marriage problems to roommate problems. First, we have shown that although stable matchings may not exist, a weakly stable matching always exists in a roommate problem. Second, we have adopted a solution concept based on the credibility of the deviations for the roommate problem: the bargaining set. We have shown that weak stability is not sufficient for a matching to be in the bargaining set. Third, we have generalized the coincidence result for marriage problems of Klijn and Massó (2003) between the bargaining set and the set of weakly stable and weakly efficient matchings to roommate problems. Finally, we have proved that the bargaining set for roommate problems is always non-empty by making use of the coincidence result.

An interesting direction for future research is to study the robustness of the bargaining set for matching problems. The bargaining set checks the credibility of an objection at a given matching. Only objections which have no counterobjections are justified, but counterobjections are not required to be justified. Dutta, Ray, Sengupta and Vohra (1989) propose a notion of a consistent bargaining set in which objections and counterobjections need to be justified that could be investigated in a future in roommate problems.

A possible direction for further research is to study computational questions related to the bargaining set. We have provided a procedure (Theorem 1) to obtain a weakly stable matching for a given roommate problem. First, a question related to the cardinality of the returned matching by WSMATCH can be studied. Second, an algorithm to find all weak stable matchings and its computational complexity are other questions that we leave for future research.

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