A new Deegan-Packel inspired power index in games with restricted cooperation

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Abstract

We propose a new power index, which we call the *essential coalitions index.* The new index is fit to analyze influence in the formation of stable coalitions to run a government or a company board. Within the field of power indices, it extends the Deegan-Packel power index to situations with restricted cooperation; more specifically, to the class of games introduced by Amer and Carreras in [2]. In general, these are not simple games. We will use the essential coalitions as an analogue to the minimal winning coalitions of a simple game, since they generalize some relevant properties. Similarly to the index that inspires it, we will first define the new index in terms of three reasonable assumptions, resembling those used in [5] for the Deegan-Packel index. Then, we formally characterize the index, using suitable modifications of the properties introduced in [2] to characterize the Shapley value in restricted games. Finally, through numeric examples, we compare the essential coalitions index to the similarly inspired, albeit more constrained, probabilistic Deegan-Packel index. We will see that, in the latter's domain, the two indices only differ in their normalization.

Keywords: Cooperative games, simple games, cooperation index, stable coalitions, government formation.

JEL classification: C71, D71, D72

1 Introduction

Cooperative games have long been used to measure the power agents wield in a range of real-life settings. In Shapley and Shubik's seminal paper ([10]), their namesake index is used to assess the influence a member of a committee has on the final decisions of the group. In the framework of the Shapley-Shubik index, the decisions of the committee are binary in nature (to pass a motion or not, in what is formally called a *simple* game), and the source of power for an agent are so-called *swings*.

A swing for an agent is a coalition that cannot pass a motion, but becomes able to do so if the agent at issue joins it. Thus, power stems from the *decisiveness* of the agent, in the sense that their cooperation is necessary for the members of a swing to pass a motion.

Another approach to measuring power is that of the Deegan-Packel index. In this case, the main idea is that only a certain class of coalitions will actually form in practice. Namely, Deegan and Packel argue in [5] that "only minimal winning coalitions emerge victorious". In the same committee framework we established before, a *minimal winning coalition* is one that is able to pass a motion, but for which this would cease to be the case should any of its members abandon it. In Deegan and Packel's approach, membership in minimal winning coalitions is the source of power for agents.

Regardless of the power measure, ever since the introduction of the Shapley-Shubik index, the literature has mostly focused on axiomatically characterizing these indices. The first of many characterizations of the Shapley value, of which the Shapley-Shubik index is the restriction to simple games, appears in [11], the same paper that introduced this concept; similarly, the Deegan-Packel power index is characterized in [5]. Moreover, as it is pointed out in the latter reference, the two characterizations diverge only slightly, in spite of the apparent difference in their assessment of power.

There is, however, an underlying assumption that hinders the applicability of these indices. Namely, in these initial frameworks it was assumed that agents cooperate with each other without constraint. In such a setting, all coalitions are deemed equally likely to form. Indeed, in particular, Deegan and Packel assume that each minimal winning coalition has the same probability to arise.

It is not difficult to convince oneself that this is unlikely to be the case in real-life situations. For instance, in parliament, committees are bound to have members from different parties, and coalitions will arguably tend to form along ideological lines. This motivates the development of models to restrict cooperation.

The most well studied such models are based on combinatorial structures that capture the relations between the agents. In particular, graphs have been used both to convey feasible channels of communication ([9]) and describe pairwise incompatibilities between agents ([4]). Both the Shapley-Shubik ([9, 3]) and the Deegan-Packel ([1]) indices have been characterized on these restricted cooperation settings.

Arguably though, a more comprehensive restriction model is one that assesses how likely each coalition is to remain united. This is the intent behind *cooperation indices* (first introduced in [2]), which map each coalition to a value in the unit interval. In this same reference, its authors define a restriction to cooperation in TU games via cooperation indices, and characterize the Shapley value on the resulting class of games.

Finally, a similar but distinct approach to that of cooperation indices are *probabilistic* indices. In these, a probability distribution is defined on the set of feasible coalitions. Probabilistic generalizations of the Shapley value and the Deegan-Packel index are discussed in [7] and [6], respectively.

In both cases, though, the probabilistic indices assume probability distributions that are "symmetric with respect to cardinality"; that is, the probability that a coalition forms only depends on its size. This hampers generality. Once again referring to the parliamentary framework, while forming a larger coalition requires more agreements between parties, affecting its probability of occurring, the identities of the parties involved are also likely to have an effect on the probability of formation.

This issue does not arise in the cooperation index approach, which has not been studied as much. In particular, even though the Shapley value in restricted games has been characterized, this is not the case for other power indices.

In what follows, our goal will be to fill this gap. Namely, we shall introduce and characterize a Deegan-Packel inspired index on simple games restricted by a cooperation index. We will then compare the newly introduced index to the probabilistic one. The following section introduces the proper mathematical models needed to describe the aforementioned situations.

In Section 3 we will introduce the restriction models of interest and their potential applications, as well as an analogue to minimal winning coalitions on simple games restricted by a cooperation index. We will use this to motivate the new power index we propose, for which we will provide a characterization in Section 4. Finally, in Section 5 we shall compare the two indices conceptually and via some numerical examples.

2 Preliminaries

A cooperative transferable utility game (henceforth, a TU game, for short) is a pair $(N,\, v)$ where $N = \{1,\, \dots,\, n\}\,,\; n \geqslant 1$, is its set of players, and $v:\; 2^N \to \mathbb{R}$ is its characteristic function. The elements in $2^N = \{S : S \subseteq N\}$ are called *coalitions*, and the only constraint on v is $v\,(\varnothing) = 0$. We denote the set of TU games with player set N by $TU(N)$.

We say $(N, v) \in TU(N)$ is *superadditive* when $v(S \cup T) = v(S) + v(T)$ for any pair of disjoint coalitions $S, T \subseteq N$. The sum of two games $(N, v), (N, w) \in TU(N)$ is defined as a new game $(N, v + w) \in TU(N)$, where $(v + w)(S) = v(S) + w(S)\ \forall S \subseteq N$. A player $i \in N$ in $(N, v) \in TU(N)$ is said to be *null* when $v(S \cup i) = v(S)\ \forall S \subseteq N \setminus i$.[1] A game $(N, v) \in TU(N)$ is null when all of its players are null; equivalently $v(S) = 0\ \forall S \subseteq N$. We say $i, j \in N$ are *symmetric players* if $v(S \cup i) = v(S \cup j)\ \forall S \subseteq N \setminus \{i, j\}$.

A simple game is a TU game such that $v(S) \in \{0, 1\}\ \forall S \subseteq N$, $v(N) = 1$ and $v(S) \leqslant v(T)$ whenever $S \subseteq T$. The latter condition is called *monotonicity*. We denote the set of simple games with player set N by $SI(N)$. The combination of two games $(N, v), (N, w) \in SI(N)$ is a new simple game $(N, v \vee w)$, where $(v \vee w)(S) = 1$ if and only if $v(S) = 1$ or $w(S) = 1$.

If $(N, v) \in SI(N)$, we refer to the coalitions $S \subseteq N$ such that $v(S) = 1$ ($v(S) = 0$, respectively) as winning (losing) coalitions. As such, the game is completely determined by its set of winning coalitions, $\mathcal{W}(N, v) = \{S \subseteq N : v(S) = 1\}$. A simple game is superadditive if and only if no two of its winning coalitions are disjoint; we say such a game is *proper*. We denote the set of proper simple games with player set N by $PS(N)$.

The minimal winning coalitions of a simple game are those winning coalitions that become losing should any subset of its members be removed from them. We denote this set by $\mathcal{W}^m(N, v) = \{S \in \mathcal{W}(N, v) : T \notin \mathcal{W}(N, v)\ \forall T \subsetneq S\}$. Two simple games (N, v) and (N, w) are said to be *mergeable* when for any pair $S \in \mathcal{W}^m(N, v)$, $T \in \mathcal{W}^m(N, w)$ we have $S \not\subseteq T$ and $T \not\subseteq S$.

A *value* defined on a class of games $\mathcal{C}(N) \subseteq TU(N)$ is a function $f : \mathcal{C}(N) \to \mathbb{R}^n$. The i-th component of $f(N, v)$ represents the value of player i according to f. We say a value f is *symmetric* when, for every $(N, v) \in TU(N)$, we have $f_i(N, v) = f_j(N, v)$ whenever i and j are symmetric players. We say $f : TU(N) \to \mathbb{R}^n$ satisfies the null player property when, for every $(N, v) \in TU(N)$, $f_i(N, v) = 0$ if i is a null player.

A value f is *efficient* when for every $(N, v) \in TU(N)$ we have $\sum_{i \in N} f_i(N, v) = v(N)$. We say f satisfies *additivity* when, for any pair $(N, v), (N, w) \in TU(N)$, we have $f_i(N, v + w) = f_i(N, v) + f_i(N, w)\ \forall i \in N$. The following result, due to [11], shows that there is a unique value satisfying all four of the previous properties.

THEOREM 2.1. The only value $f : TU(N) \to \mathbb{R}^n$ satisfying the null player property, symmetry, efficiency and additivity is the *Shapley value*. Given $(N, v) \in TU(N)$, for $i \in N$, the value is defined as

$$\varphi_i(N, v) = \sum_{S \subseteq N \setminus i} \frac{|S|!(n - |S| - 1)!}{n!} \left(v(S \cup i) - v(S)\right).$$

[1]We abuse notation and write $S \cup i$ and $N \setminus i$, instead of $S \cup \{i\}$ and $N \setminus \{i\}$, respectively.

An *index* is a value defined on a subclass of simple games. The Shapley-Shubik index is the restriction of the Shapley value to simple games.

We say an index f satisfies *mergeability* when, for every pair of mergeable games (N, v), $(N, w) \in SI(N)$ and every $i \in N$ we have

$$f_i(N, v \vee w) = \frac{|\mathcal{W}^m(N, v)|\, f_i(N, v) + |\mathcal{W}^m(N, w)|\, f_i(N, w)}{|\mathcal{W}^m(N, v \vee w)|}.$$

As previously discussed in an informal manner, the Deegan-Packel index assesses the power of an agent in a simple game via their membership in minimal winning coalitions. Namely, Deegan and Packel make the following assumptions:

- Only minimal winning coalitions will emerge victorious.
- Each minimal winning coalition has an equal probability of forming.
- Players in a minimal winning coalition divide the "spoils" equally.

From these, Deegan and Packel derive their namesake index, ρ, defined for every $(N, v) \in SI(N)$ and $i \in N$ by

$$\rho_i(N, v) = \frac{1}{|\mathcal{W}^m(N, v)|} \sum_{\substack{S \in \mathcal{W}^m(N, v) \\ i \in S}} \frac{1}{|S|}.$$

The following result, due to [5], provides a set of properties that determine this index.

THEOREM 2.2. The only index $f : SI(N) \to \mathbb{R}^n$ satisfying the null player property, symmetry, efficiency and mergeability is the Deegan-Packel index.

However, as previously suggested, Deegan and Packel's second assumption is debatable. This issue is addressed by these same authors in [6], where a probabilistic generalization of their index is introduced. Let $f : 2^N \to (0, 1)$ be a probability function, mapping each coalition to its probability of forming. It is assumed that f is symmetric with respect to cardinality, that is, if $|S| = |T|$, then $f(S) = f(T)$. Given such f and $(N, v) \in SI(N)$, they define the probability function

$$P^f(S) = \begin{cases} \frac{f(S)}{\alpha^f(N, v)} & \text{if } S \in \mathcal{W}^m(N, v) \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha^f(N, v) = \sum_{S \in \mathcal{W}^m(N, v)} f(S)$.

The following result characterizes the probabilistic version of the Deegan-Packel index.

THEOREM 2.3. Given a symmetric probability function $f : 2^N \to (0, 1)$, there is only one function $\rho^f : SI(N) \to \mathbb{R}^n$ such that

- $\rho_i^f(N, v) = 0$ if and only if $i \in N$ is null in (N, v).
- If $i, j \in N$ are symmetric in (N, v), then $\rho_i^f(N, v) = \rho_j^f(N, v)$.
- $\sum\limits_{i \in N} \rho^f(N, v) = 1 \; \forall (N, v) \in SI(N)$.
- If $(N, v), (N, w) \in SI(N)$ are mergeable, then

$$\rho^f(N, v \vee w) = \frac{\alpha^f(N, v)\, \rho^f(N, v) + \alpha^f(N, w)\, \rho^f(N, w)}{\alpha^f(N, v) + \alpha^f(N, w)}.$$

Moreover, the index is defined as

$$\rho_i^f(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{v(S)}{|S|} P^f(S).$$

3 Games with cooperation indices

We begin this section by summarizing the introduction to cooperation indices in [2]. A cooperation index over player set N is a function $p : 2^N \to [0, 1]$ such that for every $i \in N$ we have $p(\{i\}) = 1$. We will denote the set of cooperation indices over N by $I(N)$.

The notion of the cooperation index is meant to generalize other models that restrict cooperation among agents, in particular, the aforementioned framework of graphs. However, a specific interpretation of the index is not provided in the original paper; instead, a list of options is given.

It is worth pointing out that, since a cooperation index does not add up to 1 in its domain, it does not define a probability function over 2^N. For the purposes of this work, and in order to start motivating applications, we will think of a cooperation index as conveying the *stability* of a coalition. Namely, given that S has formed, we will interpret $p(S)$ as the probability that S will remain united for a standardized period of time.

Condition $p(\{i\}) = 1 \; \forall i \in N$ reinforces this idea, since there is no coalition to break in these cases. Moreover, under this interpretation, we can justify the absence of further structure on a cooperation index. The following example shows a reasonable scenario in which we can define a cooperation index that is non-monotonic with respect to inclusion.

Example 3.1. Consider the set of players $N = \{L, C, R\}$ representing a left-wing, a centrist, and a right-wing party, respectively. Arguably, the most stable non-trivial coalitions in 2^N are $S_1 = \{L, C\}$ and $S_2 = \{C, R\}$, since they only involve two ideologically close agents. Due to its increased size and diversity, it also seems logical for the grand coalition to have a lower cooperation index than both of these.

Finally, we can also support that $S_3 = \{L, R\}$ is less stable than the grand coalition. While $S_3 \subsetneq N$, and both coalitions contain the ideological extremes, one can argue that the presence of the centrist party in N increases communication between the other two.

All in all, it is not unsound to define $p \in I(N)$ so that $p(S_1) > p(N) > p(S_3)$.

Having made these observations, we proceed to establish how a cooperation index serves as a generalization of a graph. In order to be self-complete, let us reproduce here the communication graph restriction model due to Myerson ([9]), which is arguably the most well studied restricted cooperation model. Let $G = (N, E)$ be an undirected graph whose vertices $i \in N$ are players in a simple game (N, v). In this model, an edge $\{i, j\} \in E$ indicates that players i and j can *communicate*.

Recall that two vertices $i, j \in S \subseteq N$ in a graph are said to be *connected* in S when there is a path joining i and j that only passes through vertices in S. We say S is a connected set (in G) if its vertices are pairwise connected in S. The *connected components* of G are its maximal connected sets.

All in all, given a game and a communication graph, in Myerson's model players can only communicate within the connected components of the graph, possibly through intermediaries. Formally, given a simple game (N, v) and a graph $G = (N, E)$, a new game $\left(N, v_G^A\right)$ is defined so that $v_G^A(S) = 1$ if some $T \subseteq S$ is winning in (N, v) and connected in G, and $v_G^A(S) = 0$ otherwise.

Now, let $p \in I(N)$ and consider the following equivalence relation $\sim$ in N. Two players $i, j \in N$ are related if and only if there exist coalitions $S_1, \ldots, S_k$ such that

- $p(S_r) > 0 \; \forall r \in \{1, \ldots, k\}$.
- $S_r \cap S_{r+1} \neq \varnothing \; \forall r \in \{1, \ldots, k-1\}$.
- $i \in S_1, \; j \in S_k$.

The equivalence classes under this relation are called *islands*. Note that the connected components of $G = (N, E)$ coincide with the islands of any cooperation index such that $p(S) = 1$ if $S \in E$ and $p(S) = 0$ whenever S is not connected. Indeed, in mathematical terms, a cooperation index is a *weighted hypergraph*.

The following result shows how, in the words of Amer and Carreras, "islands are the natural unities within which players can negotiate", further validating their role as a generalization of the connected components of a graph.

LEMMA 3.1. Let $n \geqslant 1$, $N = \{1, \ldots, n\}$, $p \in I(N)$. If $p(S) > 0$, then S must be fully contained within an island.

Proof. Since islands are the equivalence classes under $\sim$, it suffices to show that $i \sim j$ for any pair of players $i, j \in S$. This is seen to be true by letting $k = 1$ and $S_1 = S$ in the definition of $\sim$. □

In other words, $p(S) = 0$ whenever S contains players from different islands.

DEFINITION 3.1. A *game with a cooperation index* is a triple (N, v, p) where (N, v) is a TU game and $p \in I(N)$. We denote the set of games with a cooperation index with player set N by $GI(N)$.

Given $(N, v, p) \in GI(N)$, the *p-restriction* of (N, v) (as long as no confusion arises, the restricted game) is a new TU game (N, v_p) with

$$v_p(S) = \max_{\mathcal{P} \in \mathbf{P}^+(S, p)} \sum_{T \in \mathcal{P}} v(T)p(T), \ \forall S \subseteq N,$$

where $\mathbf{P}^+(S, p)$ denotes the set of partitions of S into coalitions with positive index.

Amer and Carreras show that the restricted game above is superadditive and monotonic (the latter as long as v is non-negative). They also provide a characterization for the Shapley value on restricted games. Since our goal is to define a power index similar to the Deegan-Packel index, we will focus on restricted simple games. Unfortunately, the restriction of a simple game is not a simple game in general.

Nonetheless, the expression for the characteristic function of a restricted game is greatly reduced when the original game is a proper simple game. For the sake of notation, let $PI(N)$ denote the set of restricted proper simple games, that is, triples (N, v, p) such that (N, v) is a proper simple game, $p \in I(N)$ and (N, v_p) is not a null game.

LEMMA 3.2. Let $(N, v, p) \in PI(N)$. The p-restriction of such a game satisfies

$$v_p(S) = \max_{T \subseteq S} v(T)p(T), \ \forall S \subseteq N.$$

Proof. Note that, for any $S \subseteq N$ and partition $\mathcal{P} \in \mathbf{P}^+(S, p)$, there is at most one winning coalition $T \in \mathcal{P}$, since we are assuming that (N, v) is a proper game. Thus, as long as no winning $T \subseteq S$ with positive index exists, $v_p(S) = 0 = \max_{T \subseteq S} v(T)p(T)$.

If, on the contrary, such T exists, then $v_p(S) = \max_{T \in \mathcal{W}(N,v) \cap 2^S} p(T) = \max_{T \subseteq S} v(T)p(T)$. □

In particular, if $(N, v, p) \in PI(N)$, then for every $S \subseteq N$, $v_p(S)$ is either zero, or the value of the cooperation index on some coalition $T \subseteq S$. We may think of such a game as modeling the formation of a government or a company board. Namely, the agents in a coalition S seek to organize themselves in the most stable coalition; their payoff $v_p(S)$ is the maximum expected time for a governing coalition within S to remain united.

Be that as it may, Lemma 3.2 motivates the search for a set of coalitions whose cooperation indices capture all non-zero valued coalitions in the restricted game.

Definition 3.2. Given $(N, v, p) \in PI(N)$, we define the set of *essential coalitions* (of the restricted game) as

$$\mathcal{E}\,(N,\, v_p) = \{S \subseteq N :\; v_p(S) > v_p(T)\ \forall T \subsetneq S,\ S \neq \varnothing\}.$$

We will see in that the set $\mathcal{E}\,(N,\, v_p)$ satisfies a property that makes its elements indeed *essential* to define the restricted game. The following characterization of $\mathcal{E}\,(N,\, v_p)$ will aid us in proving the aforementioned result.

Lemma 3.3. The set of essential coalitions satisfies

$$\mathcal{E}\,(N,\, v_p) = \{S \subseteq N :\; v(S)p(S) > v(T)p(T)\ \forall T \subsetneq S,\ S \neq \varnothing\}.$$

Proof. It suffices to show that, given a non-empty $S \subseteq N$, $v_p(S) > v_p(T)\ \forall T \subsetneq S$ if and only if $v(S)p(S) > v(T)p(T)\ \forall T \subsetneq S$. For the "only if" part, using Lemma 3.2,

$$\max_{R \subseteq S} v(R)p(R) = v_p(S) > v_p(T) = \max_{R' \subseteq T} v(R')p(R') \geqslant v(T)p(T),\ \ \forall T \subsetneq S.$$

Thus, it must be the case that $v_p(S) = v(S)p(S)$, and the implication follows.

Conversely, if $v(S)p(S) > v(T)p(T)\ \forall T \subsetneq S$, in particular, $v(S)p(S) > \max_{T \subsetneq S} v(T)p(T)$. Once again, this implies $v_p(S) = v(S)p(S)$. Since $v_p(T) = v(R)p(R)$ for some $R \subseteq T \subsetneq S$, it also follows that $v_p(S) > v_p(T)\ \forall T \subsetneq S$. □

Corollary 3.1. If $E \in \mathcal{E}\,(N,\, v_p)$, then $v_p(E) = p(E) > 0$.

Proof. It was shown in Lemma 3.3 that $v_p(E) = v(E)p(E) > v(T)p(T)\ \forall T \subsetneq E$. In particular, $v(E)p(E) > v(\varnothing)p(\varnothing) = 0$, so $v(E) > 0$, $p(E) > 0$ and $v_p(E) = p(E)$. □

Corollary 3.2. If E is an essential coalition, then it is fully contained in an island.

Proof. This follows from the previous corollary and Lemma 3.1. □

Proposition 3.1. Let $(N, v, p) \in PI(N)$, $\mathcal{C} \subseteq 2^N$. Suppose that for every S such that $v_p(S) > 0$ there is some $T \in \mathcal{C}$ satisfying $v_p(S) = v_p(T)$ and $T \subseteq S$; then, $\mathcal{E}\,(N,\, v_p) \subseteq \mathcal{C}$. Furthermore, the property holds for the set of essential coalitions.

Proof. The first part of the result is equivalent to no $\mathcal{C} \subsetneq \mathcal{E}(N, v_p)$ satisfying the property. To show this is true, let $\mathcal{C} \subseteq 2^N$ and suppose there exists $E \in \mathcal{E}(N, v_p)$ such that $E \notin \mathcal{C}$. By definition, $v_p(E) > v_p(T) \geqslant 0 \ \forall T \subsetneq E$, so the only $T \subseteq E$ for which $v_p(E) = v_p(T)$ is E itself. In particular, $\mathcal{C}$ does not satisfy the property of interest.

For the second part, let $S \subseteq N$ be such that $v_p(S) > 0$. By Lemma 3.2, we know that $v_p(S) = \max_{T \subseteq S} v(T)p(T)$. Let $R \subseteq S$ be the smallest coalition satisfying $v_p(S) = v(R)p(R)$. Hence, $v(R)p(R) > v(T)p(T) \ \forall T \subsetneq R$. By Lemma 3.3, this implies $R \in \mathcal{E}(N, v_p)$ and $v_p(R) = v(R)p(R) = v_p(S)$, which ends the proof. □

It is worth pointing out that Proposition 3.1 does not require that each non-zero valued coalition contains a unique essential coalition with the same value. Indeed, nothing prevents two essential coalitions from having the same cooperation index.

This is not unlike what occurs with minimal winning coalitions of a simple game, which all have the same value. On the other hand, any winning coalition contains *at least* one minimal winning coalition, and no smaller set of coalitions satisfies this property. The connection between both sets is summarized in our definition of essential coalitions, which is evocative of the minimal winning coalitions satisfying

$$\mathcal{W}^m(N, v) = \{S \subseteq N : \ v(S) > v(T) \ \forall T \subsetneq S, \ S \neq \varnothing\}.$$

Now, recall Deegan and Packel's assumption that, in a simple game, only minimal winning coalitions "emerge victorious". They argue in [5] that this is reasonable for players who maximize payoffs. Within the context of government formation, this assumption implies that the agents will not seek more agreements with their peers than those necessary to form a government.

As already argued, in the original framework, the agents exhibit no preference for any coalition. On the other hand, in restricted games, the cooperation index captures the stability of each coalition. Our previous results convey that, if agents in a coalition organize themselves in the most stable winning subcoalition possible, the only coalitions that will form are the essential coalitions.

All in all, the motivation for a Deegan-Packel inspired power index on restricted proper simple games is established. The new index will be defined by the following assumptions on the behavior of the agents in these games:

1) Only essential coalitions will form.

2) Each of these coalitions have an equal probability of emerging.

3) Players in an essential coalition divide the benefits equally.

These assumptions define the *essential coalitions index*, $\mathfrak{e}$, which assigns each game $(N,\, v,\, p) \in PI(N)$ to an n-dimensional vector $\mathfrak{e}\,(N,\, v,\, p)$ whose i-th component is

$$\mathfrak{e}_i\,(N,\, v,\, p) = \frac{1}{|\mathcal{E}\,(N,\, v_p)|} \sum_{\substack{E \in \mathcal{E}(N,\, v_p) \\ i \in E}} \frac{p(E)}{|E|}. \tag{1}$$

Now, we have already discussed at length the first of the three assumptions that define this new index. The other two are merely restatements of those defining the original Deegan-Packel power index, now in terms of essential coalitions. Let us end this section with some observations on them.

Recall that the cooperation index is not meant to represent the probability that a coalition is formed, but that of it not breaking up before a standard period of time. Thus, lacking further information regarding their probability of formation, we argue in favor of still using a uniform probability distribution over the set of essential coalitions.

As for the utility distribution between agents, we once more point out that the value of an essential coalition need not be one. Again using the government formation framework, we may think of the agents in an essential coalition as taking turns in leading the group. Our assumption is that each agent will lead the coalition for the same amount of time.

4 A characterization for the new index

In this section we will identify a set of more formal properties that uniquely define the essential coalitions index, $\mathfrak{e}$. The properties in our proposed characterization are very similar to those used in [2] to characterize the Shapley value on restricted games. With this in mind, and in order to be self-contained, let us recall them.

We say a value $\Psi : GI(N) \to \mathbb{R}^n$ satisfies

- Efficiency (E) when, for every $(N,\, v,\, p) \in GI(N)$ and every island I of p,

$$\sum_{i \in I} \Psi_i\,(N,\, v,\, p) = v_p(I).$$

- Fairness (F) when, for every pair $(N,\, v,\, p_1)\,,\, (N,\, v,\, p_2) \in GI(N)$ such that there is some $R \subseteq N$ for which $p_1(S) = p_2(S)\ \forall S \neq R$, and every $i,\, j \in R$,

$$\Psi_i\,(N,\, v,\, p_1) - \Psi_i\,(N,\, v,\, p_2) = \Psi_j\,(N,\, v,\, p_1) - \Psi_j\,(N,\, v,\, p_2)\,.$$

- Stability (S) when for every pair $(N, v, p_1), (N, v, p_2) \in GI(N)$ such that there is some $R \subseteq N$ for which $p_1(S) = p_2(S)\ \forall S \neq R$ and $p_2(R) = 0$, and every $i \in R$,

$$\Psi_i(N, v, p_2) \leqslant \Psi_i(N, v, p_1).$$

THEOREM 4.1. There is only one value $\Psi : GI(N) \to \mathbb{R}^n$ satisfying (E) and (F), namely, $\Psi(N, v, p) = \varphi(N, v_p)$, that is, the Shapley value of the p-restriction of (N, v). Moreover, such Ψ satisfies (S).

By the end of this section, we will show that our new index satisfies a similar result. Before that, we will need to introduce the relevant properties involved in it.

DEFINITION 4.1. We say a value $\Psi : PI(N) \to \mathbb{R}^n$ satisfies:

- Average Essential Efficiency (AvEE) when for every $(N, v, p) \in PI(N)$ and every island I of p, if $I \notin \mathcal{W}(N, v)$ then $\sum_{i \in I} \Psi_i(N, v, p) = 0$, and if $I \in \mathcal{W}(N, v)$, then

$$\sum_{i \in I} \Psi_i(N, v, p) = \frac{1}{|\mathcal{E}(N, v_p)|} \sum_{\substack{E \in \mathcal{E}(N, v_p) \\ E \subseteq I}} p(E).$$

- Essential Fairness (EF) when for every pair $(N, v, p_1), (N, v, p_2) \in PI(N)$ such that $p_1(S) = p_2(S)\ \forall S \neq R$ for some $R \subseteq N$, and $i, j \in R$,

$$\begin{aligned} &|\mathcal{E}(N, v_{p_1})| \left(\Psi_i(N, v, p_1) - \Psi_j(N, v, p_1)\right) \\ &\qquad = |\mathcal{E}(N, v_{p_2})| \left(\Psi_i(N, v, p_2) - \Psi_j(N, v, p_2)\right). \end{aligned}$$

- Essential Monotonicity (EM) when for every pair $(N, v, p_1), (N, v, p_2) \in PI(N)$,

$$|\mathcal{E}(N, v_{p_2})| \Psi_i(N, v, p_2) \leqslant |\mathcal{E}(N, v_{p_1})| \Psi_i(N, v, p_1).$$

  for every $i \in N$ for which $\mathcal{E}_i(N, v_{p_2}) \subseteq \mathcal{E}_i(N, v_{p_1})$ and $p_2(E) \leqslant p_1(E)$ for every $E \in \mathcal{E}_i(N, v_{p_2})$, where $\mathcal{E}_i(N, v_p)$ denotes the set of essential coalitions of (N, v_p) that contain player i.

Out of these properties, the second one is the normalization of (F) via the number of essential coalitions in each game. In contrast, the term "monotonicity" in (EM) makes the differences between this property and (S) more noticeable. Furthermore, it can be shown that the essential coalitions index does not satisfy "essential stability" defined as a normalization of (S). Regardless, we argue that the properties we propose are reasonable.

If the cooperation index of one coalition is changed it is only *fair* that it produces the same difference in payoffs for every member of that coalition. Moreover, if the new cooperation index of that coalition is zero, we would expect the payoff allocated to its members to decrease. Due to the relevance of essential coalitions in restricted games, it is reasonable to normalize by the number of such coalitions when defining these properties.

It is worth pointing out that both properties resemble axioms that already exist in the literature. Indeed, essential monotonicity is reminiscent of the minimal monotonicity property used to characterize the Deegan-Packel power index in [8]. In [1], minimal fairness is used in the characterization of the Deegan-Packel index in games with cooperation restricted by a communication graph.

Finally, note that our notion of efficiency is well-defined due to Corollary 3.2, which established that any essential coalition is fully contained in an island. On the other hand, as long as the original game is simple and proper, at most one island can be a winning coalition. If no such island exists, then the restricted game is a null game. Any (N, v, p) such that (N, v_p) is null is excluded from $PI(N)$; therefore, the restricted games we shall consider have exactly one winning island.

All in all, (AvEE) states that the agents in the winning island divide a payoff of

$$\frac{1}{|\mathcal{E}(N, v_p)|} \sum_{E \in \mathcal{E}(N, v_p)} p(E).$$

This quantity has a natural interpretation tied to the notion of stability the cooperation index conveys. Namely, if each essential coalition E has an equal probability of forming a government and a probability of $p(E)$ of remaining united after a given amount of time, then the quantity above represents the expected time the governing coalition will stay united. In summary, those in the winning island divide among themselves the average time during which there will be a stable governing coalition.

We will now show that (AvEE) and (EF) characterize $\mathfrak{e}$, which also satisfies (EM). To do so, we will first prove an auxiliary lemma regarding the essential coalitions of two similar restricted games. Then, we will show that the essential coalition index satisfies the three properties at issue; finally, we will argue that no two different mappings can satisfy the first two properties.

LEMMA 4.1. Let $(N, v) \in PS(N)$, $p_1, p_2 \in I(N)$ and $R \subseteq N$ be such that, for every $S \neq R$, $p_1(S) = p_2(S)$. If $R \nsubseteq E$, then $E \in \mathcal{E}(N, v_{p_1})$ if and only if $E \in \mathcal{E}(N, v_{p_2})$.

Proof. The statement we seek to proof is equivalent to every $E \in \mathcal{E}(N, v_{p_1})$ such that $E \notin \mathcal{E}(N, v_{p_2})$ containing R.

By Lemma 3.3, $E \in \mathcal{E}(N, v_{p_1})$ if and only if $v(E)p_1(E) > v(T)p_1(T)\ \forall T \subsetneq E$. Similarly, $E \notin \mathcal{E}(N, v_{p_2})$ if and only if $v(E)p_2(E) \leqslant v(T)p_2(T)$ for some $T \subsetneq E$.

It suffices to observe that, for E to satisfy both inequalities at the same time, it must be the case that $p_1(T) \neq p_2(T)$ for some $T \subseteq E$. Since the only coalition for which this could be true is R, it follows that $R \subseteq E$. □

Proposition 4.1. The map $\mathfrak{e} : PI(N) \to \mathbb{R}^n$ defined by (1) satisfies average essential efficiency, essential fairness and essential monotonicity.

Proof. We start by the first property. By the monotonicity of v, if an island I of p is not winning in (N, v), then neither is any $T \subseteq I$. In particular, by Corollary 3.2, no essential coalitions are contained in such an island, and so $\mathfrak{e}_i(N, v, p) = 0\ \forall i \in I$.

If, on the contrary, $I \in \mathcal{W}(N, v)$, then

$$\begin{aligned}
|\mathcal{E}(N, v_p)| \sum_{i\in I} \mathfrak{e}_i(N, v, p) &= \sum_{i \in I} \sum_{\substack{E \in \mathcal{E}(N, v_p) \\ i \in E}} \frac{p(E)}{|E|} \\
&= \sum_{\substack{E \in \mathcal{E}(N, v_p) \\ E \subseteq I}} \sum_{i \in E} \frac{p(E)}{|E|} \\
&= \sum_{\substack{E \in \mathcal{E}(N, v_p) \\ E \subseteq I}} p(E).
\end{aligned}$$

Moving on to essential monotonicity, let $(N, v, p_1), (N, v, p_2) \in PI(N)$ and $i \in N$ be such that $\mathcal{E}_i(N, v_{p_2}) \subseteq \mathcal{E}_i(N, v_{p_1})$ and $p_2(E) \leqslant p_1(E)\ \forall E \in \mathcal{E}_i(N, v_{p_2})$. Then,

$$\begin{aligned}
|\mathcal{E}(N, v_{p_2})| \mathfrak{e}_i(N, v, p_2) &= \sum_{\substack{E \in \mathcal{E}(N, v_{p_2}) \\ i \in E}} \frac{p_2(E)}{|E|} \\
&\leqslant \sum_{\substack{E \in \mathcal{E}(N, v_{p_2}) \\ i \in E}} \frac{p_1(E)}{|E|} \\
&\leqslant \sum_{\substack{E \in \mathcal{E}(N, v_{p_1}) \\ i \in E}} \frac{p_1(E)}{|E|} \\
&= |\mathcal{E}(N, v_{p_1})| \mathfrak{e}_i(N, v, p_1).
\end{aligned}$$

Finally, for (EF) note that for any pair $(N, v, p_1), (N, v, p_2) \in PI(N)$ and $i, j \in N$,

$$
\begin{aligned}
|\mathcal{E}(N, v_{p_1})| \left(\mathfrak{e}_i(N, v, p_1) - \mathfrak{e}_j(N, v, p_1)\right| &= \sum_{\substack{E \in \mathcal{E}(N, v_{p_1}) \\ i \in E}} \frac{p_1(E)}{|E|} - \sum_{\substack{E \in \mathcal{E}(N, v_{p_1}) \\ j \in E}} \frac{p_1(E)}{|E|} \\
&= \sum_{\substack{E \in \mathcal{E}(N, v_{p_1}) \\ i,j \in E}} \frac{p_1(E)}{|E|} + \sum_{\substack{E \in \mathcal{E}(N, v_{p_1}) \\ i \in E, j \notin E}} \frac{p_1(E)}{|E|} \\
&\quad - \sum_{\substack{E \in \mathcal{E}(N, v_{p_1}) \\ i,j \in E}} \frac{p_1(E)}{|E|} - \sum_{\substack{E \in \mathcal{E}(N, v_{p_1}) \\ i \notin E, j \in E}} \frac{p_1(E)}{|E|} \\
&= \sum_{\substack{E \in \mathcal{E}(N, v_{p_1}) \\ i \in E, j \notin E}} \frac{p_1(E)}{|E|} - \sum_{\substack{E \in \mathcal{E}(N, v_{p_1}) \\ i \notin E, j \in E}} \frac{p_1(E)}{|E|}. \qquad (2)
\end{aligned}
$$

Now, if p_1 and p_2 may only differ on one coalition R, then, by Lemma 4.1, for any $i, j \in R$ the latter expression does not change if the index is p_2 instead of p_1. Hence, for any such i and j,

$$
\begin{aligned}
|\mathcal{E}(N, v_{p_1})| \left(\mathfrak{e}_i(N, v, p_1) - \mathfrak{e}_j(N, v, p_1)\right| &= \sum_{\substack{E \in \mathcal{E}(N, v_{p_2}) \\ i \in E, j \notin E}} \frac{p_2(E)}{|E|} - \sum_{\substack{E \in \mathcal{E}(N, v_{p_2}) \\ i \notin E, j \in E}} \frac{p_2(E)}{|E|} \\
&= |\mathcal{E}(N, v_{p_2})| \left(\mathfrak{e}_i(N, v, p_2) - \mathfrak{e}_j(N, v, p_2)\right|,
\end{aligned}
$$

where the second equality summarizes a chain of equalities analogous to that in (2). All in all, it is shown that $\mathfrak{e}$ satisfies essential fairness, and the proof ends. □

THEOREM 4.2. There exists a unique map $\Psi : PI(N) \to \mathbb{R}^n$ satisfying (AvEE) and (EF). Moreover, such Ψ satisfies (EM).

Proof. Existence was shown in the previous proposition; thus, it only remains to prove that (AvEE) and (EF) uniquely define $\mathfrak{e}$. First of all, note that this is the case for $(N, v, p) \in PI(N)$ such that $p(S) = 1$ if $|S| = 1$ and $p(S) = 0$ otherwise. In such case, the islands of p are the singletons. By our definition of $PI(N)$, exactly one of them is winning; let $\{i\}$ be this island. Then, $\{i\}$ is the only essential coalition of (N, v_p) and so, by (AvEE), it must be the case that $\Psi_i(N, v, p) = 1$, while $\Psi_j(N, v, p) = 0$ if $j \neq i$.

Now, let $\Psi, \Phi : PI(N) \to \mathbb{R}^n$ satisfy (AvEE) and (EF), and let $p \in I(N)$ have the minimum amount of non-zero index coalitions so that there is some $(N, v, p) \in PI(N)$ such that $\Psi(N, v, p) \neq \Phi(N, v, p)$. By our previous discussion, we can find $R \subseteq N$ such that $|R| \geqslant 2$ and $p(R) > 0$. Given such R, define a new index p' so that $p'(S) = p(S)$ for every $S \neq R$ and $p'(R) = 0$. By our choice of p, $\Psi(N, v, p') = \Phi(N, v, p')$.

On the other hand, by essential fairness, if $i, j \in R$,

$$\begin{aligned}|\mathcal{E}(N, v_p)|\left(\Phi_i(N, v, p) - \Phi_j(N, v, p)\right) &= |\mathcal{E}(N, v_{p'})|\left(\Phi_i(N, v, p') - \Phi_j(N, v, p')\right)\\ &= |\mathcal{E}(N, v_{p'})|\left(\Psi_i(N, v, p') - \Psi_j(N, v, p')\right)\\ &= |\mathcal{E}(N, v_p)|\left(\Psi_i(N, v, p) - \Psi_j(N, v, p)\right)\end{aligned}$$

and, rearranging terms, it follows that, for every pair $i, j \in R$,

$$\Phi_i(N, v, p) - \Psi_i(N, v, p) = \Phi_j(N, v, p) - \Psi_j(N, v, p).$$

Note that this will remain true for any $R \subseteq N$ such that $p(R) > 0$. Therefore, it will still hold for any pair $i, j \in N$ such that $i \sim j$ (see page 7 for the definition of the relation $\sim$). To see this, given such a pair let $S_1, \ldots, S_r$ be a sequence of coalitions with positive index p such that $i \in S_1$, $j \in S_k$ and $S_r \cap S_{r+1} \neq \varnothing\ \forall r \in \{1, \ldots, k-1\}$. Thus, for each $r \in \{1, \ldots, k-1\}$ we can take $i_r \in S_r \cap S_{r+1}$. Now, by repeatedly applying our previous findings, we obtain

$$\begin{aligned}\Phi_i(N, v, p) - \Psi_i(N, v, p) &= \Phi_{i_1}(N, v, p) - \Psi_{i_1}(N, v, p)\\ &= \Phi_{i_2}(N, v, p) - \Psi_{i_2}(N, v, p)\\ &= \cdots = \Phi_{i_{k-1}}(N, v, p) - \Psi_{i_{k-1}}(N, v, p)\\ &= \Phi_j(N, v, p) - \Psi_j(N, v, p).\end{aligned}$$

To summarize, it is shown that $d(i) = \Phi_i(N, v, p) - \Psi_i(N, v, p)$ is constant for $i \in I$, where I is an island of p. Thus, for each island I we can define $\widetilde{d_p}(I) = d(i)$ with $i \in I$.

By (AvEE), for every island I of p, we also have $\sum_{i\in I} \Phi_i(N, v, p) = \sum_{i\in I} \Psi_i(N, v, p)$, so

$$0 = \sum_{i\in I}\left(\Phi_i(N, v, p) - \Psi_i(N, v, p)\right) = \sum_{i\in I} d(i) = |I|\widetilde{d_p}(I).$$

It follows that $\widetilde{d_p}(I) = 0$ for every island I, which contradicts our original assumptions, since it implies $\Phi(N, v, p) = \Psi(N, v, p)$. Thus, it must be the case that Φ and Ψ are equal everywhere, as desired. □

5 Examples and counterexamples

Now that we have completed the theoretical analysis of the new index, we will proceed to analyze its behavior on a series of numerical examples. In the first one, we will show that, as previously claimed, the essential coalitions index does not satisfy "essential stability". Recall that a value $\Psi : PI(N) \to \mathbb{R}^n$ is said to satisfy essential stability when, for every pair $(N, v, p_1), (N, v, p_2) \in PI(N)$ such that there is some $R \subseteq N$ for which $p_1(S) = p_2(S)\ \forall S \neq R$ and $p_2(R) = 0$, and every $i \in R$,

$$|\mathcal{E}(N, v_{p_2})|\, \Psi_i(N, v, p_2) \leqslant |\mathcal{E}(N, v_{p_1})|\, \Psi_i(N, v, p_2).$$

Example 5.1. Consider four (groups of) shareholders vying for control of a corporation. Suppose that none of them is able to run the company alone. Nonetheless, the biggest of them (henceforth, agent 1) can win joint control just by reaching an agreement with one of the two next biggest holders (call them agents 2 and 3). Agents 2 and 3 together do not hold a majority of the shares. Finally, agent 4 controls the smallest share of the company; for this agent to get into the board, they must form a coalition with two other shareholders. This situation can be described by the simple game (N, v) with

$$\mathcal{W}^m(N, v) = \{\{1, 2\},\ \{1, 3\},\ \{2, 3, 4\}\}.$$

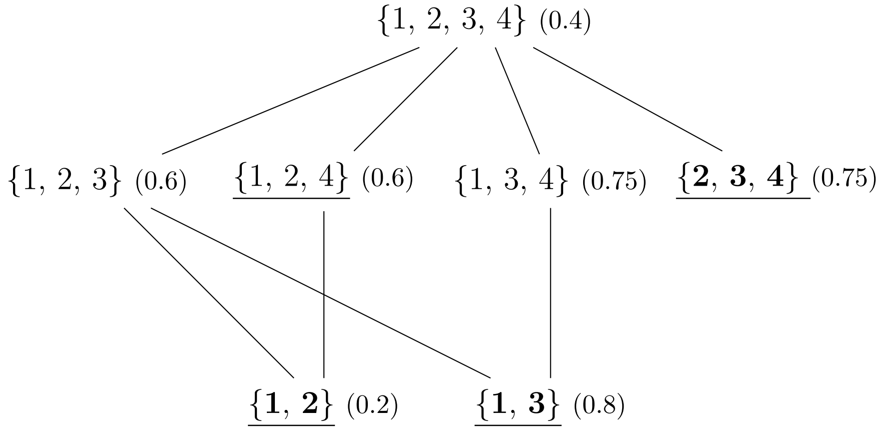
Before introducing a cooperation index that describes the relations between the agents, let us compute the unrestricted Deegan-Packel index, ρ, for this game. Agent 1 appears in two of the three minimal winning coalitions, both of size two; agent 4 on the other hand, only appears in one of them, which is of size three. Finally, agents 2 and 3 are symmetric; thus, since ρ is efficient and symmetric, we will have $\rho(N, v) = \left(\frac{1}{3}, \frac{5}{18}, \frac{5}{18}, \frac{1}{9}\right)$.

Now, when providing a cooperation index $p \in I(N)$ to restrict cooperation in this game, we are only concerned about $p(S)$ for $S \in \mathcal{W}(N, v)$. Indeed, due to Definition 3.1, the cooperation index on losing coalitions has no effect on this new game; thus, in turn, they do not affect the set of essential coalitions nor their associated power index.

That being said, consider a cooperation index p_1 such that $p_1(\{1, 2\}) = 0.2$ and $p_1(\{1, 3\}) = 0.8$, and, for every three player coalition S, let $p_1(S) = 0.6$ if it contains $\{1, 2\}$, and $p_1(S) = 0.75$ otherwise. Also take $p_1(N) = 0.4$.

In a loose sense, we may interpret the situation as follows. The preferred partner for agent 1 is agent 3; agreements between the two largest shareholders only are unlikely to be stable. However, in larger coalitions, size and diversity compel them to create a commission to keep each others' actions on check, thereby improving the working relation between agents 1 and 2. On the other hand, for coalitions including at most one of the two biggest shareholders, stability decreases with coalition size.

Figure 1: The lattice of winning coalitions of $(N,\, v)$ in Example 5.1, with the value of the cooperation index p_1 on each of them in parentheses. The minimal winning coalitions of $(N,\, v)$ are in **boldface**; the essential coalitions of $(N,\, v_{p_1})$ are underlined.



Be that as it may, by inspecting Figure 1, it is readily seen that

$$\mathcal{E}\,(N,\, v_{p_1}) = \{\,\{1,\, 2\},\ \{1,\, 3\},\ \{1,\, 2,\, 4\},\ \{2,\, 3,\, 4\}\,\}$$

is the set of essential coalitions of the restricted game. Therefore, the essential coalitions index for this game is $\mathfrak{e}\,(N,\, v,\, p_1) = \left(\frac{0.7}{4},\, \frac{0.55}{4},\, \frac{0.65}{4},\, \frac{0.45}{4}\right)$.

Observe that, while agents 2 and 3 were symmetric in the unrestricted game, the essential coalition index allocates a bigger payoff to the latter. In particular, despite agent 2 appearing in more essential coalitions (i.e. having more options to reach a stable pact), the difference does not compensate for the fact that an agreement only between agents 1 and 3 is significantly more stable than any of the options available to agent 2.

Consider now a cooperation index p_2 such that $p_2(S) = p_1(S)$ for every $S \neq \{1,\, 3\}$, and $p_2\,(\{1,\, 3\}) = 0$. Within the established narrative, the relation between agents 1 and 3 has broken down; however, broader agreements are assumed to be robust enough to withstand their mutual animosity. The essential coalitions of $(N,\, v_{p_2})$ are

$$\mathcal{E}\,(N,\, v_{p_2}) = \{\,\{1,\, 2\},\ \{1,\, 2,\, 3\},\ \{1,\, 2,\, 4\},\ \{1,\, 3,\, 4\},\ \{2,\, 3,\, 4\}\,\}\,.$$

Hence, the essential coalitions index is now $\mathfrak{e}\,(N,\, v,\, p_2) = \left(\frac{0.75}{5},\, \frac{0.75}{5},\, \frac{0.7}{5},\, \frac{0.7}{5}\right)$. Note that agents 1 and 2 appear in the same number of essential coalitions, and with the same cooperation indices; the same occurs for agents 3 and 4. As a consequence, agents 1 and 2 have the same essential coalition index in this game, and so do agents 3 and 4.

On the other hand, essential stability has been violated. Indeed,

$$|\mathcal{E}(N,\, v_{p_2})|\, \mathfrak{e}_1(N,\, v,\, p_2) = 0.75 > 0.7 = |\mathcal{E}(N,\, v_{p_1})|\, \mathfrak{e}_1(N,\, v,\, p_1)$$

and

$$|\mathcal{E}(N,\, v_{p_2})|\, \mathfrak{e}_3(N,\, v,\, p_2) = 0.7 > 0.65 = |\mathcal{E}(N,\, v_{p_1})|\, \mathfrak{e}_3(N,\, v,\, p_1)\,.$$

We do observe that, for agents 1 and 3, their essential coalition index has decreased due to having $p_2(\{1,\, 3\}) = 0$: we have $\mathfrak{e}_1(N,\, v,\, p_2) = \frac{0.75}{5} = 0.15 < 0.175 = \mathfrak{e}_1(N,\, v,\, p_1)$ and $\mathfrak{e}_3(N,\, v,\, p_2) = 0.14 < 0.1625 = \mathfrak{e}_3(N,\, v,\, p_1)$, respectively. In light of this, we may suspect that the essential coalitions index does satisfy the same notion of stability as the Shapley value on restricted games. The following example shows that this is not the case.

Example 5.2. Consider the six player simple game $(N,\, v)$ with

$$\mathcal{W}^m(N,\, v) = \{\{1,\, 2\},\; \{2,\, 3,\, 4,\, 5,\, 6\}\}$$

and the cooperation index $p_1(S) = 1\ \forall S \subseteq N$. Note that $\mathcal{E}(N,\, v_{p_1}) = \mathcal{W}^m(N,\, v)$; moreover, the essential coalitions index coincides with the Deegan-Packel index of the original game,

$$\mathfrak{e}(N,\, v,\, p_1) = \rho(N,\, v) = \left(\frac{0.5}{2},\, \frac{0.7}{2},\, \frac{0.2}{2},\, \frac{0.2}{2},\, \frac{0.2}{2},\, \frac{0.2}{2}\right) = \left(\frac{1}{4},\, \frac{7}{20},\, \frac{1}{10},\, \frac{1}{10},\, \frac{1}{10},\, \frac{1}{10}\right).$$

Consider now the cooperation index p_2 defined by $p_2(S) = p_1(S)\ \forall S \neq \{1,\, 2\}$ and $p_2(\{1,\, 2\}) = 0$. The essential coalitions of the restricted game $(N,\, v_{p_2})$ are

$$\mathcal{E}(N,\, v_{p_2}) = \{\{1,\, 2,\, 3\},\; \{1,\, 2,\, 4\},\; \{1,\, 2,\, 5\},\; \{1,\, 2,\, 6\},\; \{2,\, 3,\, 4,\, 5,\, 6\}\}\,.$$

Hence, the essential coalitions index is

$$\mathfrak{e}(N,\, v,\, p_2) = \left(\frac{1.33}{5},\, \frac{1.53}{5},\, \frac{0.53}{5},\, \frac{0.53}{5},\, \frac{0.53}{5},\, \frac{0.53}{5}\right) = \left(\frac{4}{15},\, \frac{23}{75},\, \frac{8}{75},\, \frac{8}{75},\, \frac{8}{75},\, \frac{8}{75}\right).$$

Even though the index for player 2 has decreased from $\mathfrak{e}_2(N,\, v,\, p_1) = \frac{7}{20} = 0.35$ to $\mathfrak{e}_2(N,\, v,\, p_2) = \frac{23}{75} \approx 0.307$, player 1 has actually improved from $\mathfrak{e}_1(N,\, v,\, p_1) = \frac{1}{4} = 0.25$ to $\mathfrak{e}_1(N,\, v,\, p_2) = \frac{4}{15} \approx 0.267$. To validate our calculations, note that essential fairness and average essential efficiency are satisfied.

Indeed, for the former we have

$$
\begin{aligned}
|\mathcal{E}(N, v_{p_1})|\left(\mathfrak{e}_2(N, v, p_1) - \mathfrak{e}_1(N, v, p_1)\right) &= 0.7 - 0.5 \\
&= 0.2 = 1.53 - 1.33 \\
&= |\mathcal{E}(N, v_{p_2})|\left(\mathfrak{e}_2(N, v, p_2) - \mathfrak{e}_1(N, v, p_2)\right).
\end{aligned}
$$

For (AvEE), note that for both p_1 and p_2, N is their only island; this follows from Lemma 3.1 and the fact that $p(N) > 0$. Thus, since all essential coalitions of (N, v_{p_1}) and (N, v_{p_2}) have cooperation index one, to show that the property is satisfied, it suffices to observe that both $\mathfrak{e}(N, v, p_1)$ and $\mathfrak{e}(N, v, p_2)$ add up to one.

The previous example hints at the following property of essential coalitions. For any simple game (N, v), if $p \in I(N)$ is monotonically non-increasing with respect to inclusion, then $E \in \mathcal{E}(N, v_p)$ if and only if $E \in \mathcal{W}^m(N, v)$ and $p(E) > 0$. In other words, the essential coalitions of the restricted game are those minimal winning coalitions of the original game with positive cooperation index.

This follows directly from Definition 3.2. We will now show that, if p is also symmetric with respect to cardinality, then the essential coalitions index and the probabilistic Deegan-Packel index only differ in their normalization.

Let $(N, v, p) \in PI(N)$, and suppose that $\mathcal{E}(N, v_p) = \mathcal{W}^m(N, v)$ and p is symmetric with respect to cardinality. Thus, so is the probability function $f(S) = \frac{p(S)}{\sum_{T \subseteq N} p(T)}$. Now, for each player $i \in N$, the probabilistic index satisfies

$$
\begin{aligned}
\rho_i^f(N, v) &= \sum_{\substack{S \subseteq N \\ i \in S}} \frac{v(S)}{|S|} P^f(S) = \sum_{\substack{S \in \mathcal{W}^m(N, v) \\ i \in S}} \frac{v(S)}{|S|} \cdot \frac{f(S)}{\sum_{W \in \mathcal{W}^m(N, v)} f(W)} \\
&= \left(\sum_{W \in \mathcal{W}^m(N,v)} f(W) \right)^{-1} \sum_{\substack{S \in \mathcal{W}^m(N,v) \\ i \in S}} \frac{f(S)}{|S|} \\
&= \left(\sum_{W \in \mathcal{W}^m(N,v)} \frac{p(W)}{\sum_{T \subseteq N} p(T)} \right)^{-1} \sum_{\substack{S \in \mathcal{W}^m(N,v) \\ i \in S}} \frac{p(S)}{\sum_{T \subseteq N} p(T)} \cdot \frac{1}{|S|} \\
&= \left(\sum_{W \in \mathcal{W}^m(N,v)} p(W) \right)^{-1} \sum_{\substack{S \in \mathcal{W}^m(N,v) \\ i \in S}} \frac{p(S)}{|S|}.
\end{aligned}
$$

Note that this implies that normalizing p to be a probability function over 2^N was irrelevant; the effect of this operation was cancelled out in the computation of the probabilistic index. Be that as it may, under our assumptions the essential coalitions index of i in (N, v, p) satisfies

$$\mathfrak{e}_i(N, v, p) = |\mathcal{E}(N, v_p)|^{-1} \sum_{\substack{S \in \mathcal{E}(N, v_p) \\ i \in S}} \frac{p(S)}{|S|} = |\mathcal{W}^m(N, v)|^{-1} \sum_{\substack{S \in \mathcal{W}^m(N, v) \\ i \in S}} \frac{p(S)}{|S|}.$$

In particular, the normalization $\widetilde{\mathfrak{e}}$ defined by

$$\widetilde{\mathfrak{e}}_i\,(N,\ v,\ p) = \frac{\mathfrak{e}_i\,(N,\ v,\ p)}{\sum\limits_{j \in N} \mathfrak{e}_j\,(N,\ v,\ p)}$$

coincides with the probabilistic index in this case. In other words, $\widetilde{\mathfrak{e}}$ generalizes ρ^f, since the probabilistic index is only applied when f defines a probability function over 2^N and is symmetric with respect to cardinality. We have already established that a cooperation index is not as constrained.

Many power indices are normalized to one, including the Shapley-Shubik index and the Deegan-Packel index. This feature is desirable in order to compare how different power indices distribute payoffs in the same scenario. Even though we will use $\widetilde{\mathfrak{e}}$ to compare the essential coalitions index to other indices, we will argue that it is meaningful for $\mathfrak{e}$ not to be efficient.

We will delve deeper into this in the next section. For now, we end this section with two more examples, in which the cooperation index is symmetric with respect to cardinality.

Example 5.3. Let $(N,\ v)$ be the five player simple game with

$$\mathcal{W}^m\,(N,\ v) = \{\,\{1,\ 2\},\ \{1,\ 3,\ 4\},\ \{1,\ 3,\ 5\},\ \{1,\ 4,\ 5\},\ \{2,\ 3,\ 4,\ 5\}\,\}$$

and consider the cooperation index defined as $p_1(S) = \frac{1}{|S|}$ for every non-empty $S \subseteq N$. Since p_1 is decreasing with respect to size, $\mathcal{E}\,(N,\ v_{p_1}) = \mathcal{W}^m\,(N,\ v)$. Hence, the essential coalitions index for $(N,\ v,\ p_1)$ is

$$\mathfrak{e}\,(N,\ v,\ p_1) = \left(\frac{7}{60}, \frac{1}{16}, \frac{41}{720}, \frac{41}{720}, \frac{41}{720}\right).$$

Now, if we denote by f_1 the normalization of p_1, i.e. $f_1(S) = \frac{p_1(S)}{\sum_{T \subseteq N} p_1(T)}$, then the probabilistic Deegan-Packel index $\rho^{f_1}\,(N,\ v)$ is

$$\rho^{f_1}\,(N,\ v) = \left(\frac{1}{3}, \frac{5}{28}, \frac{41}{252}, \frac{41}{252}, \frac{41}{252}\right).$$

One can verify that, as expected from our previous discussion, for each $i \in N$ we have

$$\widetilde{\mathfrak{e}}_i\left(N,\, v,\, p_1\right) = \frac{\mathfrak{e}_i\left(N,\, v,\, p_1\right)}{\sum\limits_{j \in N} \mathfrak{e}_j\left(N,\, v,\, p_1\right)} = \rho_i^{f_1}\left(N,\, v\right).$$

EXAMPLE 5.4. Consider the same game and cooperation index as in Example 5.3, and the new cooperation index p_2 defined as $p_2(S) = p_1(S)$ as long as $|S| \neq 2$, and let $p_2(S) = \frac{1}{5}$ if $|S| = 2$. Then,

$$\mathcal{E}\left(N,\, v_{p_2}\right) = \mathcal{W}^m\left(N,\, v\right) \cup \left\{\, \{1,\, 2,\, 3\},\ \{1,\, 2,\, 4\},\ \{1,\, 2,\, 5\} \,\right\}.$$

and so the essential coalitions index evaluates to

$$\mathfrak{e}\left(N,\, v,\, p_2\right) \approx \left(0.0958,\, 0.0620,\, 0.0495,\, 0.0495,\, 0.0495\right).$$

By taking $f_2(S) = \frac{p_2(S)}{\sum\limits_{T \subseteq N} p(T)}$ and computing the probabilistic Deegan-Packel index $\rho^{f_2}\left(N,\, v\right)$, we obtain

$$\rho^{f_2}\left(N,\, v\right) \approx \left(0.299,\, 0.112,\, 0.196,\, 0.196,\, 0.196\right).$$

For the sake of comparison, let us normalize the essential coalitions index,

$$\widetilde{\mathfrak{e}}\left(N,\, v,\, p_2\right) \approx \left(0.313,\, 0.202,\, 0.162,\, 0.162,\, 0.162\right).$$

As expected, now the normalization does not coincide with the probabilistic index. In fact, while they agree in that player 1 should be allocated the highest payoff and players 3 through 5 are symmetric, they differ on whether the latter should get more or less than player 2. Namely, the (normalized) essential coalitions index allots more to player 2 than to players 3 through 5.

This is explained by this player appearing in more essential coalitions. These have a positive effect in the namesake power index, and, in this case, compensate for the low cooperation index of $\{1,\, 2\}$. On the contrary, the probabilistic index only captures the negative consequences of player 2 being involved in this minimal winning coalition, compared to those with higher cooperation index in which players 3 through 5 participate.

6 Final remarks

Models to restrict cooperation are of great importance in game theory, as they provide a more realistic approach to analyze the results of the interactions between agents. In this work, we studied the broad restriction model of cooperation indices; more specifically, we focused on the study of power in so restricted games.

Our main contribution is the *essential coalitions index*, a new index fit to analyze several real-life situations. In particular, it is suitable to assess power in the formation of the board of a corporation, or a governing coalition in a parliamentary system. In such a process, the cohesion of the governing coalition is a relevant factor, and this is one feature the restriction model describes.

By design, the new index has similarities with the Deegan-Packel power index ([5]). Indeed, as motivation for the essential coalitions index, we argued that some of the assumptions that define the Deegan-Packel index were not realistic. For one, we claim that not only minimal winning coalitions will be formed; instead, in some occasions, having more agents involved in a coalition will make it more stable.

We also observed the need to generalize the probabilistic Deegan-Packel index ([6]), which aims to address the fact that "not all coalitions are equally likely to form"; but, arguably, uses a rather constrained model to do so. To be specific, in the probabilistic model, a probability function over the set of coalitions is given, but it is required to only depend on its size. This is a very strong and possibly unrealistic condition.

Indeed, it negates the importance of the identity of the members of a coalition in its probability of forming. All in all, the probabilistic index will satisfy a notion of symmetry with respect to the unrestricted game, but at the expense of the generality of the restriction model.

For its part, the cooperation index is a much more flexible restriction model: it is a function from the set of coalitions to the unit interval with the only condition of mapping singletons to the value one. Also, crucially, it encompasses other models of restriction, including Myerson's communication model ([9]), which has been more thoroughly studied (see [2] for more models that can be expressed as specific cases of the cooperation index).

And yet, the essential coalitions index retains a notion of symmetry, with respect to the restricted game. Namely, given $(N, v, p) \in PI(N)$, it can be shown that if two players $i, j \in N$ are symmetric in (N, v_p), then $\mathfrak{e}_i(N, v, p) = \mathfrak{e}_j(N, v, p)$. This follows from the definition of $\mathfrak{e}$ in (1) and the fact that two players $i, j \in N$ are symmetric in (N, v, p) if and only if for each essential coalition E_i containing i there is exactly one essential coalition E_j such that $j \in E_j$ and $p(E_j)$.

One key difference between the probabilistic index and the essential coalitions index, though, is in regards to efficiency. In general, the new index is not efficient; instead, it satisfies the property of *average essential efficiency*, introduced in Section 4. This is a consequence of the conceptual differences between the two indices.

As previously said, the probability function used for the probabilistic index is meant to measure the chance a given coalition has of forming. In contrast, the cooperation index conveys the stability of a coalition. For the sake of example, in the government formation framework the cooperation index of a coalition can be understood as the probability it has to remain united until the end of the term.

As such, we interpret the essential coalitions index of an agent as the expected fraction of the full term for which they lead the governing coalition. The "efficiency gap", that is, the quantity $1 - \sum_{i \in N} \mathfrak{e}_i(N, v, p)$ accounts for the possibility that the government collapses at some point, and a caretaker government is put in place. More specifically, this efficiency gap is interpreted as the expected time during which a caretaker government is running the country or institution at issue.

Nonetheless, we did see in Section 5 that the normalization of the essential coalitions index, $\widetilde{\mathfrak{e}}$, does generalize the probabilistic index. Accordingly, the characterization in Section 4 could be altered to apply to this normalization. Due to $\widetilde{\mathfrak{e}}$ being a generalization of the probabilistic index, we argue in favor of using it only when the cooperation index defines a probability function over the set of coalitions. In doing so, its interpretation remains intact: the normalized index conveys the expected fraction of the term for which an agent leads the governing coalition *provided that* it does not fall apart.

All of this is begging for the combination of the two approaches: a probability function assessing the chances each coalition has to emerge victorious, and a cooperation index indicating their stability. We hope to explore this in the future. To see how it might make for a more complete model, let us turn to the property of essential stability and Example 5.1.

There, we analyzed the consequences of the cooperation index of coalition $\{1, 3\}$ dropping to zero, with all other coalitions remaining as stable as they were. At first, this may seem an unlikely scenario; if $p(\{1, 3\})$ changes, it seems reasonable that so will the cooperation index of coalitions that contain $\{1, 3\}$. We argue that the probability that one such coalition is formed does decrease, but, with the right agreements in place, their stability (which is conditional on the coalition forming) does not change.

Now, in spite of its desirable features, the new index is not without its questionable assumptions. In acknowledging them, we open the possibility to amend them in future works; with this in mind, we dedicate the closing paragraphs to mentioning two of them.

The first one regards the equal division of payoffs. Within the framework of government formation, at the end of Section 3 we interpreted this hypothesis as the members of the governing coalition taking turns in leading it. The assumption is that at the end of the term each member has been the leader for the same amount of time.

In practice, even in coalition governments, this is hardly the case. Usually, the senior partner of the coalition serves as its leader, with others providing either external support or occupying secondary roles in government. Moreover, when there is a power-sharing agreement in place, it will likely be only the two biggest partners those who will share the position of leader of the government.

To solve this, we may justify our assumption as each member having equal influence on policy. While still debatable, it is closer to reality: even small partners will make demands in exchange for their support to the governing coalition. Either way, we point out that this issue already appeared in the definition of the original Deegan-Packel index. In other words, its authors did consider reasonable that members of a minimal winning coalition evenly divide their collective payoff.

The final issue we will address is how fit the essential coalitions are for the assumptions that define their namesake index. For the sake of example, let us turn to Figure 1. We showed there that $S = \{1, 2, 4\}$, with a cooperation index of 0.6, was an essential coalition, despite not being a minimal winning coalition.

This was reasonable in that example. Since $|S| = 3$, equally dividing $p(S) = 0.6$ between its members yields 0.2 to each of them. On the other hand, if agents 1 and 2 did not reach an additional agreement with agent 4, they would split $p(\{1, 2\}) = 0.2$, which yields 0.1 for each player. Thus, agents 1 and 2 have an incentive to enlarge their coalition.

But, if, for instance, $p(S) = 0.27$, then, equally dividing this among the three agents yields 0.09 to each of them. In other words, assuming they are rational, now agents 1 and 2 have no interest in reaching an agreement with agent 4. Nonetheless, our definition would still deem S an essential coalition.

This is arguably a more contentious point than the previous one. One could be in favor of using a family of coalitions that more accurately reflects the agents' rationality to define an analogous power index. There are possibly several alternatives in this direction.

All in all, it is worth noting that the essential coalitions were defined by extending to restricted games a property of minimal winning coalitions of a simple game. As such, in any case, we consider that the essential coalitions index is a reasonable extension of the Deegan-Packel index, although it is likely not the only one.

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