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LORENZ POPULATION MONOTONIC ALLOCATION SCHEMES FOR TU-GAMES

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Title: Lorenz Population Monotonic Allocation Schemes for TU-games

Abstract: Sprumont (1990) introduces Population Monotonic Allocation Scheme (PMAS) and proves that every assignment game with at least two sellers and two buyers, where each buyer-seller pair derives a positive gain from trade, lacks a PMAS. In particular glove games lacks PMAS. We propose a new cooperative TU-game concept, Lorenz-PMAS, which relaxes some population monotonicity conditions by requiring that the payoff vector of any coalition is Lorenz dominated by the corresponding restricted payoff vector of larger coalitions. We show that every TU-game having a Lorenz-PMAS is totally balanced, but the converse is not true in general. We obtain a class of games having a Lorenz-PMAS for every glove game and for every assignment game with at most five players. Additionally, we also introduce two new notions, Lorenz-PMAS-extendability and Lorenz-PMAS-exactness, and discuss their relationships with the convexity of the game.

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1 Introduction

Cooperative games (with transferable utility) mainly focus on how to share the profit derived from the cooperation of agents. Many allocation rules have been proposed and characterized from an axiomatic point of view.

Another important aspect of cooperative game theory focuses on justifying the allocation of profits from the point of view of the coalition formation process. In this line, Sprumont (1990) introduces the concept of Population Monotonic Allocation Schemes (PMAS): for accepting new members in each coalition of agents, we should take care not to harm its members by assigning them an amount smaller than what they have been promised before the entrance of the new members. An allocation is acceptable if we can propose an allocation for each subcoalition of agents showing that the entrance of new members from any starting coalition until forming the grand coalition of agents does not harm any agent. Sprumont characterizes the set of cooperative games having PMAS. However, many important games lack PMAS, even in cases where they are totally balanced.

In this paper, a new approach to support the final allocation is introduced. Instead of looking at individual incentives, we focus on social incentives. The entrance of new members is acceptable if the poorest agent (the one that receives the smallest amount) is richer than before; the sum of the payoffs of the first and second poorest agents is larger than before the entrance of the new members, and so on. This idea is based on making the payoffs of agents as egalitarian as possible, and thus using the criterion of Lorenz domination to compare the payoffs received by a set of agents.

In Section 2, we define the main concepts of cooperative games. In Section 3, we introduce the concept of Lorenz-PMAS and Lorenz monotonic core (the set of all the Lorenz-PMAS). We show that this set can be discrete, and thus a non-convex set (see Example 1), which makes it different from the case of PMAS. In Proposition 1, we point out that a PMAS can be reinterpreted as a Lorenz-PMAS, and thus the individual incentive point of view makes the allocation compatible with the social point of view. However, the converse is not true. In fact, in Example 3, we show a four-person game with Lorenz-PMAS, but without PMAS, demonstrating that there are cases where the social point of view is appropriate to justify allocations. Indeed, in Theorem 1, we introduce a sufficient condition for having Lorenz-PMAS that includes games with no PMAS. In Theorem 2, we discuss the case of glove games, a particular case of assignment games, and show that even though they do not have PMAS, any core allocation can be supported by a Lorenz-PMAS.

In Section 4, we discuss related concepts to Lorenz-PMAS. We state that convex games are Lorenz-PMAS-extendable (Theorem 3) and are the unique class of games that are Lorenz-PMAS-exact (Theorem 4). In Section 5 we conclude.

2 Notations

A cooperative game with transferable utility (a game) is a pair (N, v) (in short v), where $N = \{1, 2, \dots, n\}$ is a finite set of players and $v: 2^N \to \mathbb{R}$ is the characteristic function with $v(\emptyset) = 0$. A subset S of N, $S \in 2^N$, is a coalition of players, |S| denotes its cardinality, and v(S) is interpreted as the worth of coalition S. We denote by $P(N) = \{S \subseteq N \mid S \neq \emptyset\}$ the set of nonempty coalitions of N. Given $S \in P(N)$, we denote by (S, v_S) the subgame of (N, v) related to coalition S (i.e. $v_S(R) = v(R)$ for all $R \subseteq S$).

A payoff allocation is a vector $z = (z_i)_{i \in N} \in \mathbb{R}^N$, where z_i is the payoff to player *i*, and for $S \in P(N)$ we write $z(S) = \sum_{i \in S} z_i$, $z(\emptyset) = 0$ and $z|_S = (z_i)_{i \in S}$. The core of a game (N, v) is the set C(N, v) = 0 $\{z \in \mathbb{R}^N \mid z(N) = v(N), \ z(S) \ge v(S) \,\forall S \in P(N)\}.$

A game (N, v) is balanced if it has a nonempty core, it is totally balanced if the subgame (S, v_S) is balanced for all $S \in P(N)$, and it is convex (Shapley, 1971) if $v(S) + v(T) \le v(S \cup T) + v(S \cap T)$ for all $S,T \subseteq N.$

A Population Monotonic Allocation Scheme (PMAS) of a game (N, v) (Sprumont, 1990) is a vector $x = (x^S)_{S \in P(N)}$, where $x^S = (x_i^S)_{i \in S} \in \mathbb{R}^S$, that satisfies the following conditions:

- (i) Efficiency in each coalition: $\sum_{i \in S} x_i^S = v(S)$ for all $S \in P(N)$. (ii) Monotonicity: $x^S \leq x^T |_S$ ($x_i^S \leq x_i^T$ for all $i \in S$) for all $S, T \in P(N), S \subseteq T$.

We also use the notation $x = (x_i^S)_{S \in P(N), i \in S}$ to describe a PMAS. The above definition implies that a PMAS x selects a core allocation $x^{S} = (x_{i}^{S})_{i \in S} \in C(S, v_{S})$ for every subgame (S, v_{S}) in such a way that the payoff to any player cannot decrease when the coalition to which he/she belongs becomes larger. Thus every game having a PMAS is totally balanced. Sprumont shows that all convex games have a PMAS.

The monotonic core of a game $v \in G^N$, denoted by MC(N, v), is the set of all its PMAS (Moulin, 1990). This set always coincides with the core of a certain game associated to the initial game (Getán etal., 2009).

PMAS-extendability

A balanced game (N, v) is core-extendable (Kikuta and Shapley, 1986) when for every $S \in P(N)$ and $y \in C(S, v_S)$ there exists $z \in C(N, v)$ such that $z_i = y_i$ for all $i \in S$. Each convex game is core-extendable, but the converse is not necessarily true (Sharkey, 1982; Kikuta and Shapley, 1986).

A game (N, v) is *PMAS-extendable* (Getán *et al.*, 2014) if for every $S \in P(N)$ and for every y = $(y^R)_{R \in P(S)} \in MC(S, v_S)$ there exists $x = (x^R)_{R \in P(N)} \in MC(N, v)$ such that $y^R = x^R$ for all $R \in P(S)$. Notice that every PMAS-extendable game has at least one PMAS. Moreover, we know that a game (N, v)is convex if and only if it is PMAS-extendable (Getán et al., 2014). In particular, every PMAS-extendable game is core-extendable.

PMAS-exactness

A game (N, v) is called *exact* (Schmeidler, 1972) if for every $S \in P(N)$ there exists $z \in C(N, v)$ with z(S) = v(S). It is evident that all exact games are totally balanced. Additionally, it is easy to observe that every convex game is exact. However, in general, the converse statement does not hold.

A game (N, v) is *PMAS-exact* (Getán *et al.*, 2014) when for every $S \in P(N)$ there exists $x = (x^R)_{R \in P(N)} \in MC(N, v)$ such that $x^N(S) = v(S)$. It is important to note that every PMAS-exact game is also exact, and any subgame of a PMAS-exact game is also PMAS-exact. Moreover, it is known that a game (N, v) is convex if and only if it is PMAS-exact (Getán *et al.*, 2014).

The Lorenz domination

A standard of fairness is the one provided by the Lorenz domination criterion (Lorenz, 1905). To define it, consider a fix population of individuals denoted as $N = \{1, 2, ..., n\}$. Given a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^N$, we can interpret x_i as the income of individual $i \in N$ and we can order the individuals from the poorest to the richest to obtain $x_{(1)} \leq \ldots \leq x_{(n)}$. Now, given $x = (x_1, \dots, x_n) \in \mathbb{R}^N$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^N$, we say that y weakly Lorenz dominates x, and we denote it by $x \preccurlyeq_{\mathcal{L}} y$ or by $y \succcurlyeq_{\mathcal{L}} x$, if and only if:

$$\begin{array}{rcl} x_{(1)} & \leq & y_{(1)}, \\ & x_{(1)} + x_{(2)} & \leq & y_{(1)} + y_{(2)}, \\ & \dots & \dots & \\ & x_{(1)} + x_{(2)} + \dots + x_{(n)} & \leq & y_{(1)} + y_{(2)} + \dots + y_{(n)}. \end{array}$$

An equivalent way to express the Lorenz domination criterion is by means of a function $\varphi : \mathbb{R}^N \to \mathbb{R}^n$ (n = |N|), defined as follows. Let $x \in \mathbb{R}^N$ and $1 \le k \le n$, then we define the function $\varphi_k(x)$ as

$$\varphi_k(x) = \min \{x(S) | S \subseteq N \text{ and } |S| = k\} = x_{(1)} + \dots + x_{(k)}.$$

For $x, y \in \mathbb{R}^N$, we have that $x \preccurlyeq_{\mathcal{L}} y$ if and only if $\varphi_k(x) \leq \varphi_k(y)$ for all k = 1, ..., n. It is said that y Lorenz dominates x, denoted by $x \prec_{\mathcal{L}} y$, if and only if $x \preccurlyeq_{\mathcal{L}} y$ and $\varphi(x) \neq \varphi(y)$ (i.e. $\varphi_k(x) \neq \varphi_k(y)$ for some k = 1, ..., n).

The relation $\preccurlyeq_{\mathcal{L}}$ is a preorder on \mathbb{R}^N but not a partial order, as it satisfies the following properties:

- (i) Reflexivity: $x \preccurlyeq_{\mathcal{L}} x$ for all $x \in \mathbb{R}^N$.
- (ii) Transitivity: For $x, y, z \in \mathbb{R}^N$ with $x \preccurlyeq_{\mathcal{L}} y$ and $y \preccurlyeq_{\mathcal{L}} z$ we have $x \preccurlyeq_{\mathcal{L}} z$.

(iii) Non anti symmetry¹: For $x, y \in \mathbb{R}^N$ we have

 $x \preccurlyeq_{\mathcal{L}} y \text{ and } y \preccurlyeq_{\mathcal{L}} x$ $\iff x_{(k)} = y_{(k)} \text{ for all } k = 1, \dots, n$ $\iff x = y\Pi \text{ for some permutation matrix } \Pi.$

However, the relation $\preccurlyeq_{\mathcal{L}}$ is a partial order on the commutative monoid $\mathcal{D} = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^N \mid x_1 \leq \ldots \leq x_n\}$. Moreover, $\preccurlyeq_{\mathcal{L}}$ is compatible with the sum "+" of \mathcal{D} :

$$x \preccurlyeq_{\mathcal{L}} y \Longrightarrow x + z \preccurlyeq_{\mathcal{L}} y + z \text{ for all } x, y, z \in \mathcal{D}.$$

Notice that for $x,y\in \mathbb{R}^N$ we have the implications:

$$x \le y \quad \Rightarrow \quad x \preccurlyeq_{\mathcal{L}} y \quad \Rightarrow \quad x(N) \le y(N) \tag{1}$$

where $x \leq y$ means $x_i \leq y_i$ for all $i \in N$.

3 Lorenz-PMAS

In this section, we use the Lorenz domination criterion to introduce a new concept for a cooperative game known as Lorenz-PMAS. This concept aims to mimic and generalize the notion of PMAS. After providing the definition of Lorenz-PMAS, we present several results regarding Lorenz-PMAS for general cooperative games.

Definition 1. Let (N, v) be a cooperative game. We say that a vector $x = (x^S)_{S \in P(N)}$, where each $x^S = (x_i^S)_{i \in S} \in \mathbb{R}^S$, is a Lorenz Population Monotonic Allocation Scheme (Lorenz-PMAS or $\mathcal{L}PMAS$) if it satisfies the following conditions:

(i) Efficiency in each coalition: for all $S \in P(N)$, $\sum_{i \in S} x_i^S = v(S)$. (ii) Lorenz monotonicity: for all $S, T \in P(N)$, $S \subseteq T$,

$$x^{S} \preccurlyeq_{\mathcal{L}} x^{T}|_{S}, (i.e.\varphi_{k}(x^{S}) \leq \varphi_{k}(x^{T}|_{S}) \text{ for all } k = 1, \cdots, s)$$

 $^{^{1}}$ A square matrix Π is said to be a *permutation matrix* if each row and column has a single unit entry, and all other entries are zero (Marshall et al., 2011)

Notice that, by (1), the Lorenz monotonicity condition relaxes the monotonicity condition of Sprumont. The set of Lorenz-PMAS of the game (N, v) is denoted by

$$\mathcal{L}MC(N, v) = \{x \mid x \text{ is a } \mathcal{L}PMAS \text{ of } (N, v)\},\$$

and its projection to \mathbb{R}^N is denoted by

$$\mathcal{L}MC^{N}(N,v) = \left\{ x^{N} \mid x = \left(x^{S} \right)_{S \in P(N)} \in \mathcal{L}MC(N,v) \right\}.$$

Notice that the set $\mathcal{LMC}(N, v)$ is compact, but is not convex in general, as illustrated in the following example where the $\mathcal{LMC}(N, v)$ is a discrete set. This is a significant difference between Lorenz-PMAS and PMAS, which makes it difficult to state a general existence theorem for Lorenz-PMAS.

Example 1. Consider the three-player game (N, v) defined by:

$$v(S) = \begin{cases} 1 & \text{if } S = \{1, 2\}, \{1, 3\} & \text{or } N, \\ 0 & \text{otherwise}, \end{cases}$$

for all $S \subseteq N$. Then $|\mathcal{L}MC(N, v)| = 4$, since any $x \in \mathcal{L}MC(N, v)$ can be described as follows:

$$\begin{split} x^N &= (1,0,0); \\ x^{\{1,2\}} &= (1,0) \ or \ (0,1), \ x^{\{1,3\}} &= (1,0) \ or \ (0,1); \\ x^{\{2,3\}} &= (0,0); \\ x^{\{i\}} &= (0) \ for \ all \ i \in N. \end{split}$$

The four previous possibilities give rise to all $\mathcal{L}PMAS$ of (N, v).

We collect some basic facts about Lorenz-PMAS in the following proposition. Proofs are left to the reader.

Proposition 1. Let (N, v) be a cooperative game.

- (a) If $x = (x^S)_{S \in P(N)}$ is a $\mathcal{L}PMAS$ of (N, v), then $x^S \in C(S, v_S)$ for all $S \in P(N)$. In particular, $\mathcal{L}MC^N(N, v) \subseteq C(N, v)$.
- (b) Every PMAS of (N, v) is also a $\mathcal{L}PMAS$, i.e. $MC(N, v) \subseteq \mathcal{L}MC(N, v)$.
- (c) If (N, v) has a $\mathcal{L}PMAS$ and (N, v') is a game satisfying v'(S) = v(S) for all $S \subseteq N, S \neq N$ and $v'(N) \ge v(N)$, then (N, v') also has a $\mathcal{L}PMAS$.

Part (a) in Proposition 1 states that all cooperative games having a $\mathcal{L}PMAS$ are totally balanced. However, it is not true that all totally balanced games have a $\mathcal{L}PMAS$, as shown in the following example. Since every three-player totally balanced game has a PMAS (Sprumont, 1990), we need to consider games with at least four players.

Example 2. Fix a real number $a \ge 6$, and consider the four-player game (N, v) defined by:

$$\begin{split} v(N) &= a, \\ v(134) &= v(234) = 2, \ v(123) = 3, \ v(124) = a, \\ v(14) &= v(24) = 0, \\ v(12) &= v(13) = v(23) = v(34) = 2, \\ v(i) &= 0 \ for \ all \ i \in N. \end{split}$$

It is straightforward to see that this game is totally balanced. Its core is

$$C(N,v) = \{(\alpha,\beta,0,a-\alpha-\beta) \mid \alpha,\beta \ge 2 \text{ and } \alpha+\beta \le a-2\}.$$

Additionally, we have $C(R, v_R) = \{(1, 1, 1)\}$ for $R = \{1, 2, 3\}$. Moreover, (N, v) lacks a \mathcal{LPMAS} . To see this, suppose to the contrary that $x = (x^S)_{S \in P(N)}$ is a \mathcal{LPMAS} of (N, v). By part (a) in Proposition 1, we know that $x^N \in C(N, v)$, $x^R \in C(R, v_R)$. Therefore, we obtain

$$(1,1,1) = x^R \preccurlyeq_{\mathcal{L}} x^N \big|_R = (\alpha,\beta,0) \text{ for some } \alpha,\beta \ge 2.$$

This leads to a contradiction since $1 = \varphi_1(x^R) \le \varphi_1(x^N|_R) = 0.$

Note that for the game (N, v) in Example 2 every game (N, v') such that v'(S) = v(S) for all $S \subseteq N$, $S \neq N$, and $v'(N) \geq v(N)$ lacks a PMAS, as v(123) + v(134) < v(12) + v(13) + v(34) (Norde and Reijnierse, 2002). However, if we take $v'(N) \geq \frac{4}{3}a$ it can be shown that (N, v') has a $\mathcal{L}PMAS$. In fact, we can state a more general result for totally balanced four-player games.

Proposition 2. Let (N, v) be a totally balanced four-player game. Then there exists a real number $\nu' \ge v(N)$ such that the game (N, v') defined by v'(S) = v(S) for all $S \subseteq N, S \ne N$ and $v'(N) = \nu'$, has a \mathcal{LPMAS} .

Proof. Take $\nu' \in \mathbb{R}$ such that $\nu' \ge \max\left\{\frac{4v(S)}{|S|} \mid S \in P(N)\right\}$. Then, it is straightforward that the egalitarian allocation $\alpha^{v'} = \left(\frac{\nu'}{4}, \frac{\nu'}{4}, \frac{\nu'}{4}, \frac{\nu'}{4}\right)$ is in the core of (N, v'), i.e. $\alpha^{v'} \in C(N, v')$. Hence, define the

vector $x = (x^S)_{S \in P(N)}$ satisfying the following properties:

$$\begin{split} x^{N} &= \alpha^{v'}, \\ x^{S} \in C(S, v_{S}) \text{ for all } S \in P(N) \text{ with } |S| = 3, \\ x^{\{i,j\}} &= (v(i), v(ij) - v(i)) \\ \text{ for all } i, j \in N \text{ with } i < j \text{ and } v(i) \le v(j), \\ x^{\{i,j\}} &= (v(ij) - v(j), v(j)) \\ \text{ for all } i, j \in N \text{ with } i < j \text{ and } v(i) > v(j), \text{ and } \\ x^{\{i\}} &= (v(i)) \text{ for all } i \in N. \end{split}$$

It is straightforward to check that $x \in \mathcal{L}MC(N, v')$.

Now we introduce a class of games having Lorenz-PMAS, but not PMAS in general as illustrated in Example 3.

Theorem 1. Let (N, v) be a zero-normalized game (i.e. v(i) = 0 for all $i \in N$) satisfying the following properties:

(i) v(S)/s ≤ v(N)/n for all S ∈ P(N).
(ii) There exists a family {i_S}_{S∈P(N),S≠N}, with each i_S ∈ S, such that:

$$\frac{v(T)}{t-1} \ge \begin{cases} \frac{v(S)}{s} & \text{if } s = 1 \text{ or } i_T \notin S, \\ \frac{v(S)}{s-1} & \text{otherwise,} \end{cases}$$

for all $S, T \in P(N)$ with $S \subseteq T$, $S \neq T$ and $T \neq N$.

Then (N, v) has a \mathcal{LPMAS} .

Proof. First, notice that, by hypothesis, $v(S) \ge 0$ for all $S \in P(N)$. Then, define the vector $x = (x^S)_{S \in P(N)}$ as follows:

$$\begin{split} x_i^N &= \frac{v(N)}{n} \text{ for all } i \in N; \\ x_i^S &= \begin{cases} \frac{v(S)}{s-1} \text{ if } i \neq i_S, \\ 0 \quad \text{if } i = i_S, \end{cases} \end{split}$$

for all $S \in P(N), S \neq N, i \in S$. We next prove that $x \in \mathcal{LMC}(N, v)$. Indeed, it is clear that x satisfies efficiency in each coalition. By part (a) of Lemma 1 below and property (i) we have that for each $S \in P(N), S \neq N$ and $|S| \ge 2$ (case |S| = 1 is trivial), it holds:

$$x^{S} = \left(\frac{v(S)}{s-1}, \dots, \frac{v(S)}{s-1}, 0\right) \preccurlyeq_{\mathcal{L}} \left(\frac{v(N)}{n}, \dots, \frac{v(N)}{n}\right) = x^{N}|_{S}$$

Now, given two coalitions $S, T \in P(N)$ with $S \subseteq T$, $S \neq T$ and $T \neq N$, we want to argue that $x^S \preccurlyeq_{\mathcal{L}} x^T \mid_S$. If s = 1, it is clear since $x^S = (0)$. If s > 1 and $i_T \notin S$, then, by part (a) of Lemma 1 below and property *(ii)* we have:

$$x^{S} = \left(\frac{v(S)}{s-1}, \dots, \frac{v(S)}{s-1}, 0\right) \preccurlyeq_{\mathcal{L}} \left(\frac{v(T)}{t-1}, \dots, \frac{v(T)}{t-1}\right) = x^{T}|_{S}$$

Finally, if s > 1 and $i_T \in S$, by part (b) of Lemma 1 below and property (ii), we have:

$$x^{S} = \left(\frac{v(S)}{s-1}, \dots, \frac{v(S)}{s-1}, 0\right) \preccurlyeq_{\mathcal{L}} \left(\frac{v(T)}{t-1}, \dots, \frac{v(T)}{t-1}, 0\right) = x^{T}|_{S}.$$

Hence, we conclude x is a $\mathcal{L}PMAS$ of (N, v).

Lemma 1. Let $\nu, \nu' \in \mathbb{R}_+$ and $s \ge 1$ an integer. Then:

(a) $(\nu', \dots, \nu', 0) \preccurlyeq_{\mathcal{L}} (\nu, \dots, \nu)$ (in \mathbb{R}^s) if and only if $(s-1)\nu' \le s\nu$. (b) $(\nu', \dots, \nu', 0) \preccurlyeq_{\mathcal{L}} (\nu, \dots, \nu, 0)$ (in \mathbb{R}^s) if and only if s = 1 or $\nu' \le \nu$.

Proof. (a) Let $z := (\nu, \ldots, \nu)$ and let $z' := (\nu', \ldots, \nu', 0)$. Then we have:

$$z' \preccurlyeq_{\mathcal{L}} z \iff \varphi_k(z') \le \varphi_k(z) \text{ for all } k = 1, \dots, s$$
$$\iff (k-1)\nu' \le k\nu \text{ for all } k = 1, \dots, s$$
$$\iff (1-1/k)\nu' \le \nu \text{ for all } k = 1, \dots, s$$
$$\iff \max\{(1-1/k)\nu' \mid k = 1, \dots, s\} \le \nu$$
$$\iff (1-1/s)\nu' \le \nu.$$

(b) This part is straightforward.

In general, a zero-normalized game satisfying properties (i) and (ii) of Theorem 1 does not have a PMAS, as illustrated in the following example.

Example 3. Fix $a, b, c \in \mathbb{R}$ such that $2 \le a \le b \le c$, and consider the four-player game (N, v) defined by:

$$\begin{split} v(N) &= 4c, \\ v(123) &= 2b, \ v(124) = v(234) = 3c, \ v(134) = a, \\ v(12) &= 2b, \ v(13) = a, \\ v(14) &= v(23) = v(24) = 0, \ v(34) = 1, \\ v(i) &= 0 \ for \ all \ i \in N. \end{split}$$

It is straightforward to check that this game satisfies properties (i) and (ii) of Theorem 1 (taking $i_S := \max\{i \mid i \in S\}$ for all $S \in P'(N)$), and thus the game has a \mathcal{LPMAS} . However, this game lacks a PMAS since v(123) + v(134) < v(12) + v(13) + v(34) (Norde and Reijnierse, 2002).

In the next theorem we demonstrate the existence of Lorenz-PMAS for every glove game (Shapley, 1959). In fact, we show that every core allocation in a glove game can be reached by a $\mathcal{L}PMAS$.

Theorem 2. Let (N, v) be the glove game with respect to the disjoints sets L and R (i.e. $N = L \cup R, L \neq \emptyset$, $R \neq \emptyset$ and let $v(S) := min\{|S \cap L|, |S \cap R|\}$ for all $S \in P(N)$). Then $\mathcal{LMC}^N(N, v) = C(N, v)$. In particular, any glove game has a \mathcal{LPMAS} .

Proof. Without loss of generality, let us suppose that $|L| \leq |R|$ and let $z \in C(N, v)$. We next prove that there exists $x = (x^S)_{S \in P(N)} \in \mathcal{L}MC(N, v)$ such that $x^N = z$. To this aim, for every $S \in P(N)$ we denote $l_S = |S \cap L|$ and $r_S = |S \cap R|$. Then, define x^S as follows:

$$x^{S} = \begin{cases} z & \text{if } S = N, \\ (0, \dots, 0) & \text{if } S \subseteq L \text{ or } S \subseteq R, \\ (1, \dots, 1; 0, \dots, 0) & \text{if } S \neq N \text{ and } 1 \leq l_{S} \leq r_{S}, \\ (0, \dots, 0; 1, \dots, 1) & \text{if } 1 \leq r_{S} < l_{S}. \end{cases}$$

Notice that $x^N = z$ and $x^S \in C(S, v_S)$ for all $S \in P(N)$, Thus, $x = (x^S)_{S \in P(N)}$ satisfies efficiency in each coalition.

To prove that x is a $\mathcal{L}PMAS$ of (N, v), it remains only to check the Lorenz monotonocity of x. Let $S, T \in P(N)$ be two coalitions such that $S \subseteq T, S \neq T$. We claim that $x^S \preccurlyeq_{\mathcal{L}} x^T \mid_S$. Indeed, to prove it, we need to differentiate between several cases based on the previous definition of x^S :

Case 1. If $S \subseteq L$ or $S \subseteq R$, then it is straightforward since $x^S = (0, \ldots, 0)$.

Case 2. If T = N and |L| < |R|, then z = (1, ..., 1; 0, ..., 0) and $x^S \preccurlyeq_{\mathcal{L}} z|_S$ since the number of components equal to 1 in x^S is at most the number of components equal to 1 in $z|_S$.

Case 3. If T = N and |L| = |R|, then $z = (\lambda, ..., \lambda; 1 - \lambda, ..., 1 - \lambda)$ for some $0 \le \lambda \le 1$ and we must see that $x^S \preccurlyeq_{\mathcal{L}} z|_S$. We assume that $1 \le l_S \le r_S$, and thus $x^S = (1, ..., 1; 0, ..., 0)$. For $k = 1, ..., r_S$, we have $\varphi_k (x^S) = 0 \le \varphi_k (z|_S)$. For $k = r_S + 1, ..., r_S + l_S = s$, we have $\varphi_k (x^S) = k - r_S \le \varphi_k (z|_S)$ since $\varphi_k (z|_S) = (k - r_S) \lambda + r_S (1 - \lambda)$ when $\lambda \ge 1/2$, and $\varphi_k (z|_S) = l_S \lambda + (k - l_S) (1 - \lambda)$ when $\lambda \le 1/2$. Case 4. if $T \ne N$, $S \cap L \ne \emptyset$ and $S \cap R \ne \emptyset$, then $x^S \preccurlyeq_{\mathcal{L}} x^T|_S$, as the number of components equal to 1

Case 4. If $T \neq N$, $S \cap L \neq \emptyset$ and $S \cap R \neq \emptyset$, then $x^{S} \preccurlyeq_{\mathcal{L}} x^{T}|_{S}$, as the number of components equal to 1 in x^{S} is at most the number of components equal to 1 in $x^{T}|_{S}$.

Next we show the existence of Lorenz-PMAS for every assignment game (Shapley and Shubik, 1971) with at most five players.

Proposition 3. Every assignment game with at most five players has a *LPMAS*.

Proof. Consider a matrix $A = (a_{ij})_{i \in M, j \in M'} \in M_{m \times m'}(\mathbb{R}_+)$ where M (the set of buyers) and M' (the set of sellers) are two disjoints finite sets with respective cardinality $m, m' \ge 1$, and we assume each entry of the matrix to be positive, i.e. $a_{ij} \ge 0$.

We consider the assignment game (N, w_A) defined by the matrix A where $N = M \cup M'$ (Shapley and Shubik, 1971) and n = m + m'. For a coalition $S \in P(N)$, we denote $m_S = |S \cap M|$ as the number of buyers in $s, m'_S = |S \cap M'|$ as the number of sellers in S and $A_S = (a_{ij})_{i \in S \cap M, j \in S \cap M'} \in M_{m_S \times m'_S}(\mathbb{R}_+)$ denotes the corresponding submatrix of A at S. Without loss of geneality we assume $m \leq m'$.

If m = 1 or m = m' = 2, it is not difficult to observe that for each $z \in C(N, w_A)$ there exists an $x = (x^S)_{S \in P(N)} \in \mathcal{L}MC(N, v)$ such that $x^N = z$.

If m = 2 and m' = 3, we denote $z = (\underline{u}; \overline{v}) \in C(N, w_A)$ as the sellers-optimal ² core allocation of (N, w_A) . It is straightforward to check that $z(N \setminus \{j\}) = w_A(N \setminus \{j\})$ for all $j \in M'$ and thus we have $z|_S \in C(S, w_{A_S})$ for all $S \subset N$ with $m_S = m'_S = 2$. Proceeding as in the proof of Theorem 2, we observe that the vector $x = (x^S)_{S \in P(N)}$ defined by

$$x^{S} = \begin{cases} z & \text{if } S = N, \\ (0, \dots, 0) & \text{if } S \subseteq M \text{ or } S \subseteq M', \\ (w_{A}(S); 0, \dots, 0) & \text{if } m_{S} = 1 \leq m'_{S}, \\ (0, 0; w_{A}(S)) & \text{if } m_{S} = 2 \text{ and } m'_{S} = 1, \\ z|_{S} & \text{if } m_{S} = m'_{S} = 2, \end{cases}$$

for all $S \in P(N)$, is a $\mathcal{L}PMAS$ of (N, w_A) .

²For every seller $j \in M'$ is $\overline{v}_j := w_A(N) - w_A(N \setminus \{j\})$, and given an optimal matching μ w.r.t. the matrix A for every buyer $i \in M$ is $\underline{u}_i := \begin{cases} a_{i\mu(i)} + w_A(N \setminus \{\mu(i)\}) - w_A(N) & \text{if } i \text{ is matched by } \mu; \\ 0 & \text{if } i \text{ is not matched by } \mu. \end{cases}$

We would like to remark that in general, assignment games lack Lorenz-PMAS, as illustrated by the following example.

Example 4. Consider the following matrix $A = \begin{pmatrix} 9 & 7 & 5 & 3 \\ 7 & 5 & 3 & 1 \end{pmatrix}$. We claim the assignment game (N, w_A) relative to A lacks of Lorenz-PMAS. Indeed, let $M = \{1, 2\}$ be the set of buyers and $M' = \{3, 4, 5, 6\}$ be the set of sellers. If $x = (x^S)_{S \in P(N)} \in \mathcal{LMC}(N, v)$ was a Lorenz-PMAS, then we would necessarily have $x_5^N = x_6^N = 0$ and $\varphi_2(x^R) = 0$ for $R := \{1, 2, 5, 6\}$. However, this is impossible since

$$C(R, w_{A_R}) = \{ (\alpha, \alpha - 2; 5 - \alpha, 3 - \alpha) \mid 2 \le \alpha \le 3 \}$$

and therefore $\varphi_2(z) > 0$ for all $z \in C(R, w_{A_R})$.

Now we show the existence of Lorenz-PMAS in another interesting model. Shapley and Shubik (1967) introduces a model of a production economy involving a landowner and $m \ge 1$ peasants. The profit that arises if p peasants work for the landowner is denoted by f(p), where $f : \{0, 1, 2, ..., m\} \to \mathbb{R}$ is a production function such that:

$$f(0) = 0$$

if $0 \le p_1 < p_2 \le m$, then $f(p_1) \le f(p_2)$ (increasing function),

if $0 \le p_1 < p_2 < p_3 \le m$, then $f(p_2) - f(p_1) \le f(p_3) - f(p_2)$ (concavity).

Then, the associated cooperative game between the landowner (player 0) and the *m* peasants is defined as follows: for any coalition $\emptyset \neq S \subseteq N := \{0, 1, 2, \dots, m\},\$

$$v(S) := \begin{cases} f(|S| - 1) \text{ if } 0 \in S \\ 0 & \text{otherwise} \end{cases}$$

In this model, the marginal productivity of any peasant when working for the landowner is equal to $\Delta := f(m) - f(m-1) \ge 0$. The allocation $z \in \mathbb{R}^N$ such that $z_0 := f(m) - m\Delta$ and $z_i := \Delta$ for all $i = 1, \ldots, m$ is a core allocation since f is a concave function. In next proposition we prove this core allocation z is supported by a Lorenz-PMAS. **Proposition 4.** Under the previous notations and hypotheses, it holds that $z \in \mathcal{LMC}^{N}(N, v)$.

Proof. Consider the vector $x = (x^S)_{S \in P(N)}$ defined by:

$$x^{S} = \begin{cases} (f(m) - m\Delta, \Delta, \dots, \Delta) & \text{if } S = N, \\ (f(|S| - 1), 0, \dots, 0) & \text{if } 0 \in S \text{ and } S \neq N, \\ (0, \dots, 0) & \text{if } 0 \notin S. \end{cases}$$

Observe that $x^N = z$. We next prove that $x \in \mathcal{LMC}(N, v)$. Indeed, it is straightforward that x satisfies efficiency in each coalition. Now, given two coalitions $S, T \in P(N)$ with $S \subseteq T, S \neq T$, we want to see that $x^S \preccurlyeq_{\mathcal{L}} x^T \mid_S$. If $0 \notin S$, it is clear since $x^S = (0, \ldots, 0)$. If $0 \in S$ and T = N, then $x^S = (f(|S|-1), 0, \ldots, 0)$ and we have $x^S \preccurlyeq_{\mathcal{L}} z \mid_S = x^N \mid_S$ since $z(S) \ge v(S) = x^S(S)$. Finally, if $0 \in S$ and $T \neq N$ then $x^T = (f(|T|-1), 0, \ldots, 0)$ and $x^S \preccurlyeq_{\mathcal{L}} x^T \mid_S$ since f is an increasing function. Hence, we conclude x is a Lorenz-PMAS of (N, v).

We finish this section with another interesting example. Moretti and Norde (2021) analyze *weighted multi-glove games*. They generalize the model of glove markets, a two-sector production economy, by introducing several sectors, all of which are necessary to extract some positive profit. Each member of a sector has a certain number of units of an input. The production process requires using one unit of input from each sector to obtain one unit of output.

Formally, given a player set N and a partition of N into k sectors, $P = \{P_1, P_2, \dots, P_k\}$, each member *i* of sector P_j is endowed with w_i units of input. $w \in \mathbb{N}^N$ is the vector of inputs. Then, the worth of a coalition $S \subseteq N, S \neq \emptyset$ (the amount of output), is given by

$$v^{P,w}(S) = \min\left\{\sum_{i\in S\cap P_j} w_i : j=1,\ldots,k\right\}.$$

The authors demonstrate that the corresponding game is totally balanced and provide a characterization of when the game admits PMAS. However, in Example 3.6 of their paper, they present an example of a five-player game with a core element that cannot be extended by a PMAS. The specific game is as follows.

Example 5. (Example 3.6 of Moretti and Norde (2021)) Let $N = \{1, 2, 3, 4, 5\}$ be the set of agents, let $P = \{\{1, 2\}, \{3, 4\}, \{5\}\}$ be the partition of N that defines three sectors, and let w = (1, 1, 1, 1, 2) be the vector of inputs. The allocation x = (0.5, 0.5, 0.5, 0.5, 0) is in the core of the game, but the authors prove that there is no PMAS that extends this core allocation. However, it is easy to check that the following

Lorenz PMAS extends this allocation:

$$x^{S} = \begin{cases} (0.5, 0.5, 0.5, 0.5, 0) & \text{if } S = N, \\ (0, \dots, 0, v^{P, w}(S)) & \text{if } 5 \in S \neq N, \\ (0, \dots, 0) & \text{if } 5 \notin S. \end{cases}$$

The proof is left to the reader.

4 LPMAS-extendability and LPMAS-exactness

In this section, we provide a characterization of the convexity of a game in terms of Lorenz-PMAS. To do this, we introduce two new concepts related to the Lorenz-monotonic core: *Lorenz-PMAS-extendability* and *Lorenz-PMAS-exactness*. These notions are inspired by the concepts of PMAS-extendability and PMAS-exactness introduced by Getán *et al.* (2014).

Definition 2. A game (N, v) is Lorenz-PMAS-extendable if for every $S \in P(N)$ and for every $y = (y^R)_{R \in P(S)} \in \mathcal{L}MC(S, v_S)$ there exists $x = (x^R)_{R \in P(N)} \in \mathcal{L}MC(N, v)$ such that $y^R = x^R$ for all $R \in P(S)$.

It is worth noting that every Lorenz-PMAS-extendable game possesses at least one Lorenz-PMAS, as every game contains subgames with Lorenz-PMAS. For example, one can consider the restriction of the game to individual coalitions.

The following theorem proves that Lorenz-PMAS-extendability is implied by the convexity of the game.

Theorem 3. Let (N, v) be a convex game. Then (N, v) is Lorenz-PMAS-extendable.

Proof. To show that (N, v) is Lorenz-PMAS-extendable we proceed by recurrence. We consider $S \in P(N)$, $j \in N \setminus S$, and $y \in \mathcal{LMC}(S, v_S)$. Then, we define $x = (x^R)_{R \in P(S \cup \{j\})}$ as follows: $x^R = y^R$, for $\emptyset \neq R \subseteq S$, and for $R \subseteq S \cup \{j\}$, with $j \in R$,

$$x_i^R = \begin{cases} y_i^{R \setminus \{j\}} & \text{if } i \neq j, \\ v(R) - v(R \setminus \{j\}) & \text{if } i = j, \end{cases} \quad \text{for all } i \in R.$$

First, by definition we have $(x^R)_{R \in P(S)} = y$. Let us see that x is a Lorenz-PMAS of $v_{S \cup \{j\}}$. Notice that for each coalition $R \in P(S \cup \{j\})$ we have $x^R(R) = y^R(R) = v(R)$ when $j \notin R$, and $x^R(R) = x^R(R \setminus \{j\}) + x_j^R = y^{R \setminus \{j\}}(R \setminus \{j\}) + [v(R) - v(R \setminus \{j\})] = v(R)$ when $j \in R$. Moreover, we claim that

for each $R, T \in P(S \cup \{j\})$ such that $R \subseteq T$ the Lorenz monotonicity property holds, i.e. $x^S \preccurlyeq_{\mathcal{L}} x^T |_S$. To prove it we must distinguish different cases:

Case 1. If $j \notin T$, then $j \notin R$ and $x^R = y^R \preccurlyeq_{\mathcal{L}} y^T |_R = x^T |_R$. Case 2. If $j \in T$ and $j \notin R$ then $R \subseteq T \setminus \{j\} \subseteq S$ and $x^R = y^R \preccurlyeq_{\mathcal{L}} y^{T \setminus \{j\}} |_R = x^T |_R$. Case 3. If $j \in R$ then

$$x^{R} = \left(y^{R \setminus \{j\}}, v\left(R\right) - v\left(R \setminus \{j\}\right)\right) \preccurlyeq_{\mathcal{L}} \left(y^{T \setminus \{j\}}\Big|_{R \setminus \{j\}}, v\left(T\right) - v\left(T \setminus \{j\}\right)\right) = x^{T}\Big|_{R \setminus \{j\}}$$

where the Lorenz domination follows from Lemma 2 below taking into account that $y^{R \setminus \{j\}} \preccurlyeq_{\mathcal{L}} y^{T \setminus \{j\}} |_{R \setminus \{j\}}$ and $v(R) - v(R \setminus \{j\}) \leq v(T) - v(T \setminus \{j\})$, due to the convexity of the game.

Therefore we conclude
$$x \in \mathcal{LMC}(S \cup \{j\}, v_{S \cup \{j\}})$$
.

Lemma 2. Let $x, y \in \mathbb{R}^N$ with $x \preccurlyeq_{\mathcal{L}} y$ and let $a, b \in \mathbb{R}$ with $a \leq b$. Then $(x, a) \preccurlyeq_{\mathcal{L}} (y, b)$.

Proof. Since $x \preccurlyeq_{\mathcal{L}} y$ it holds that $\varphi_k(x) \le \varphi_k(y)$, for all k = 1, ..., n. Hence, $\varphi_1(x, a) = \min\{\varphi_1(x), a\} \le \min\{\varphi_1(y), b\} = \varphi_1(y, b)$. Moreover, for all k = 2, ..., n, we have $\varphi_k(x, a) = \min\{\varphi_k(x), \varphi_{k-1}(x) + a\} \le \min\{\varphi_k(y), \varphi_{y-1}(x) + b\} = \varphi_k(y, b)$. Therefore, we conclude $(x, a) \preccurlyeq_{\mathcal{L}} (y, b)$.

Since a game is PMAS-extendable if and only if it is convex (Getán *et al.*, 2014) we obtain the following result.

Corollary 1. Every PMAS-extendable game is Lorenz-PMAS-extendable. \Box

It is generally not true that every Lorenz-PMAS-extendable game is convex.

Example 6. Fix a real number a with $1.5 \le a < 2$, and consider the three-player game (N, v) defined by:

$$v(N) = a,$$

 $v(12) = v(13) = v(23) = 1,$
 $v(i) = 0 \text{ for all } i \in N.$

Then (N, v) is not convex, but it is totally balanced and its core is

$$C(N, v) = \{ (\alpha, \beta, a - \alpha - \beta) \mid \alpha, \beta \le a - 1 \text{ and } \alpha + \beta \ge 1 \}.$$

 $\mathcal{LMC}(S, v_S)$. We must build an allocation scheme $x = (x^R)_{R \in P(N)} \in \mathcal{LMC}(N, v)$ such that $x^R = y^R$ for all $R \in P(S)$. If |S| = 1, then $S = \{i\}$, y = (0) and x := z extends y. If |S| = 2, then $S = \{i, j\}$, $y = (0, 0; (\lambda, 1 - \lambda))$ for some $0 \le \lambda \le 1$. Without loss of generality we can assume, by symmetry, that $S = \{1, 2\}$. Then $x := (0, 0, 0; (\lambda, 1 - \lambda), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}); (\frac{a}{3}, \frac{a}{3}, \frac{a}{3})) \in \mathcal{LMC}(N, v)$ extends y, since $\min\{\lambda, 1 - \lambda\} \le \frac{1}{2} \le \frac{a}{3}$ and so $(\lambda, 1 - \lambda) \preccurlyeq \mathcal{L}(\frac{a}{3}, \frac{a}{3})$. Finally, if |S| = 3, then S = N and x := z trivially extends y.

Next, we approach the notion of convexity from a different perspective by introducing the concept of Lorenz-PMAS-exactness. In simple terms, Lorenz-PMAS-exactness implies that the worth of any coalition of players is achieved in at least one Lorenz-PMAS of the entire game.

Definition 3. A game (N, v) is Lorenz-PMAS-exact when for every $S \in P(N)$ there exists $x = (x^R)_{R \in P(N)} \in \mathcal{L}MC(N, v)$ such that $x^N(S) = v(S)$.

It is evident that a game which is Lorenz-PMAS-exact is also exact. Furthermore, it can be easily demonstrated that any subgame of a Lorenz-PMAS-exact game is also Lorenz-PMAS-exact. Next theorem establishes that Lorenz-PMAS-exactness is a characterization of the convexity of the game.

Theorem 4. Let (N, v) be a game. The following statements are equivalent:

- (i) (N, v) is convex.
- (ii) (N, v) is Lorenz-PMAS-exact.

Proof. $(i) \Rightarrow (ii)$ It is known that a game is convex if and only if it is PMAS-exact (Getán *et al.*, 2014). Moreover, any PMAS-exact game is Lorenz-PMAS-exact since $MC(N, v) \subseteq \mathcal{L}MC(N, v)$ by part (b) of Proposition 1.

 $(ii) \Rightarrow (i)$ Assume that (N, v) is Lorenz-PMAS-exact and let $S, T \subseteq N$. Since the subgame $(S \cup T, v_{S \cup T})$ is Lorenz-PMAS-exact too, there exists $x = (x^R)_{R \in P(S \cup T)} \in \mathcal{L}MC \ (S \cup T, v_{S \cup T})$ such that $x^{S \cup T}(S \cap T) = v(S \cap T)$. Therefore, by the second implication in (1), we obtain

$$\begin{split} v\left(S\right) + v\left(T\right) - v\left(S \cap T\right) &= x^{S}\left(S\right) + x^{T}\left(T\right) - x^{S \cup T}\left(S \cap T\right) \\ &\leq x^{S \cup T}\left(S\right) + x^{S \cup T}\left(T\right) - x^{S \cup T}\left(S \cap T\right) \\ &= x^{S \cup T}\left(S \cup T\right) = v\left(S \cup T\right). \end{split}$$

This proves the convexity of (N, v).

5 Conclusion

Allocation schemes serve as a means to illustrate the benefits of forming larger coalitions. The PMAS concept primarily emphasizes individual incentives, whereas Lorenz-PMAS justifies the final allocation from a social standpoint. This concept holds particular relevance in cooperative scenarios where players are substitutable or symmetric, as demonstrated in the case of a production economy or market situation.

For future research, it would be valuable to characterize the games that admit Lorenz-PMAS and analyze other models where the Lorenz criterion offers fresh perspectives on allocation problems

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